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Realized Power Variation of some Fractional Stochastic Integrals

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Preface

Abstract

This thesis is concerned with the realized power variation (and some generalization thereof) of certain classes of stochastic processes (fractional stochastic integrals and integrated stable processes).

Chapter 1 introduces the concepts of fractional Brownian Motion (fBM) and realized power variation (r.p.v.) of a stochastic process.

In Chapter 2 we consider the asymptotic behaviour of the r.p.v of processes of the form

$$Z_t := \int_0^t u_s \, \mathrm{d}B_s^H \,,\ t \in [0,T]\,, \tag{1}$$

where $u = \{u_t \,,\, t \in [0,T]\}$ is a stochastic process with paths of finite q-variation, $0 < q < \frac{1}{1-H}$, $B^H = \left\{B_t^H \,,\, t \in [0,T]\right\}$ is an fBM with Hurst parameter $H \in (0,1)$ and the stochastic integral is a pathwise Riemann-Stieltjes integral. We will show the uniform convergence in probability of the r.p.v., properly normalized, to a stochastic process of the form $\mathbf{E}\left(|B_1^H|^p\right) \int_0^t |u_s|^p \, \mathrm{d}s$. The fluctuations of the normalized r.p.v. around this limit converge in distribution to a process of the form $\nu_1 \int_0^t |u_s|^p \, \mathrm{d}W_s$, where W denotes a Brownian Motion (BM) independent of B^H and ν_1 is a constant. The result holds for $H \in \left(0, \frac{3}{4}\right)$. For $H = \frac{3}{4}$ a similar result can be obtained by using an additional normalizing factor $(\log n)^{-1/2}$. For $H > \frac{3}{4}$ and u constant the limit will be the Rosenblatt process, i.e., a quadratic functional of BM. The discussion follows Corcuera *et al.* [CoNuWo06].

In Chapter 3 we consider the asymptotic behaviour of functionals of the form

$$F_{g,h}^{(n)}(Z)_t := \int_0^{\frac{\lfloor nt \rfloor}{n}} h\left(Z_s^{(n)}\right) g\left(\dot{Z}_s^{(n)} n^{H-1}\right) \mathrm{d}s\,,$$

where Z is given by (1), g, h are continuous functions, $Z^{(n)}$ denotes the broken line approximation of Z and $\dot{Z}_s^{(n)}$ is the derivative with respect to s. The r.p.v. considered in Chapter 1 is contained in this class. Hence, the results can be seen as a generalization of those in Chapter 1. The functionals converge uniformly in probability to a stochastic process $\int_0^t h(Z_s)\mathbf{E}^W(g(u_s W)) \, \mathrm{d}s$, where W is a standard normal random variable independent of B^H, and $\mathbf{E}^W$ denotes the expectation with respect to W. Whenever g is an even function and satisfies an additional condition, and for $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$, the fluctuations around the limit, properly normalized, converge in distribution to a process of the form $\int_0^t h(Z_s)\nu(u_s) \, \mathrm{d}W_s$, where W is a BM independent of B^H and ν is given by

$$\nu^2(x) := \lim_{n\to\infty} \mathbf{V}\left(\frac{1}{\sqrt{n}} \sum_{i=1}^n g\left(x\left(B_i^H - B_{i-1}^H\right)\right)\right).$$

This chapter is based on Corcuera *et al.* [CoNuWo08].

In Chapter 4 we follow Corcuera *et al.* [CoNuWo07] and consider the asymptotic behaviour of the r.p.v. of stochastic processes of the form

$$\int_0^t u_s \,\mathrm{d}S_s^\alpha, \; t \in [0,T], \tag{2}$$

where $S^\alpha = \{S^\alpha, t \in \mathbb{R}_{+,0}\}$ is an α-stable Lévy process with index of stability $\alpha \in (0,2)$, $u = \{u_t, t \in \mathbb{R}_{+,0}\}$ is a stochastic process with continuous paths and, if $\alpha \geq 1$, with finite q-variation on any finite interval for some $q < \frac{\alpha}{\alpha-1}$. The integral is a pathwise Riemann-Stieltjes integral for $\alpha \geq 1$ and a pathwise Lebesgue-Stieltjes integral for $\alpha < 1$. The normalized r.p.v. converges uniformly in probability to a process $\mathbf{E}(|S_1^\alpha|^p) \int_0^t |u_s|^p \,\mathrm{d}s$. The fluctuations around this limit, properly normalized and under certain conditions on u, converge in distribution to a process of the form $\nu_p \int_0^t |u_s|^p W_s$ with $\nu_p^2 := \mathbf{V}(|S_1^\alpha|^p)$.

The focus of thesis lies on Chapter 2, i.e., the r.p.v. of fractional stochastic integrals of the form (1). Chapters 3 and 4 can be seen as outlooks on similar results that can be obtained by modifying the setting accordingly.

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Chapter 1

Preliminaries

In this chapter we introduce the two key concepts of this thesis, namely fractional Brownian Motion and realized power variation, and provide the required theoretical foundations.

1.1 Fractional Brownian Motion

This section is mainly based on Mishura [Mish08], Brockwell and Davis [BrDa91] and Cheridito [Cher01a], [Cher01b].

1.1.1 Definition and Elementary Properties

In the following let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space[1].

Definition 1.1.1. A (two-sided, normalized) *fractional Brownian motion (fBM)* with *Hurst parameter* (*Hurst index* or *Hurst exponent*) $H \in (0, 1]$ is a continuous-time Gaussian process $B^H = \{B_t^H,\ t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, \mathrm{P})$, having the properties

(i) $B_0^H = 0$,

(ii) $m_{B^H}^{(1)}(t) : \mathbb{R} \to \mathbb{R} : t \mapsto \mathbf{E}\big(B_t^H\big) = 0$,

(iii) $m_{B^H}^{(2)}(s,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (s,t) \mapsto \mathbf{E}\big(B_s^H B_t^H\big) = \dfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$.

where $m_{B^H}^{(1)}$ and $m_{B^H}^{(2)}$ denote the first and second moment functions of B^H (see Definition A.1.1).

Remark 1.1.1. A special property of Gaussian processes (see Definition A.1.6) is inherited from the Gaussian distribution. Analogous to a Gaussian random variable whose distribution is uniquely defined by its first two moments, two Gaussian processes have the same distribution if their first and second moment functions coincide (hence, it suffices to specify these two functions to characterize a Gaussian process). As the finite-dimensional distributions of a Gaussian process are multivariate normal (therefore itself uniquely specified by the first and second moment functions of the process) and consistent (which can be

[1] Complete probability spaces are a convenient setting as some technical difficulties (like measurability problems) do not appear (e.g., functions which are defined only up to a subset of a (measurable nullset) may not necessarily be measurable in the incomplete setting). In fact, the assumption of completeness is not a real restriction as every probability space $(\Omega, \mathcal{F}, \mathrm{P})$ can be completed by replacing $\mathcal{F}$ by $\mathcal{F}^*$, the smallest σ-algebra containing $\mathcal{F}$ and all subsets of P-nullsets, and extending P to the new σ-algebra correspondingly (this construction gives the 'smallest' (in the sense of set inclusion, $\mathcal{F} \subseteq \mathcal{F}^*$) P-complete probability space containing $(\Omega, \mathcal{F}, \mathrm{P})$).

seen by analyzing the characteristic functions), Kolmogorov's extension theorem (see Theorem A.2.1) ensures the existence of a corresponding process (provided the given first and second moment functions specify positive semi-definite covariance matrices of the finite-dimensional distributions. According to Proposition 2.2 of [DoOpTa03], the function in item (iii) of Definition 1.1.1 is indeed positive semi-definite for $H \in (0,1]$.).

Remark 1.1.2. From items (ii) and (iii) of Definition 1.1.1 it follows that the covariance function of an fBM coincides with its second moment function, i.e., $\gamma_{B^H}(s,t) = m^{(2)}_{B^H}(s,t)$, $s,t \in \mathbb{R}$.

Remark 1.1.3. It is possible to consider fBM only on $\mathbb{R}_{+,0}$ (one-sided fBM) or on $[0,T]$ (with finite horizon $T \in \mathbb{R}_+$) with evident changes in Definition 1.1.1.

Remark 1.1.4. For $H = 1$, the simple construction $B^H_t = B^1_t := t\xi$, where ξ is a standard normal random variable (i.e., $\xi \sim \mathrm{N}(0,1)$), gives a process satisfying the properties of fBM. This simply corresponds to a line $B^1_t = tB^1_1$ with random slope B^1_1.

Remark 1.1.5. Since $\mathbf{E}\big((B^H_t B^H_s)^2\big) = |t-s|^{2H}$ (as a direct calculation or Lemma A.3.1 shows) and B^H is a Gaussian process, it has a continuous modification, according to the continuity theorem of Kolmogorov-Chentsov (see Theorem A.2.2). Furthermore, this modification can be chosen $(H-\varepsilon)$ - Hölder continuous for all $\varepsilon \in (0,H)$ (see Definition A.1.13 and Corollary A.3.1). If the fBM is defined on a compact parameter space (e.g., $[0,T]$), ε may be chosen from $(0,H]$.

Remark 1.1.6. It follows directly from Definition 1.1.1 that B^H_t is normally distributed with $\mathbf{E}\big(B^H_t\big) = 0$ and $\mathbf{V}\big(B^H_t\big) = \mathbf{E}\big((B^H_t)^2\big) = |t|^{2H}$, $t \in \mathbb{R}$ [2]. As any linear combination of components of a multivariate normal distribution is normal, the increments $(B^H_t - B^H_s)$ are normally distributed with $\mathbf{E}\big(B^H_t - B^H_s\big) = 0$ and $\mathbf{V}\big(B^H_t - B^H_s\big) = |t-s|^{2H}$, $t,s \in \mathbb{R}$.

Definition 1.1.2. A stochastic process $X = \{X_t,\ t \in \mathbb{R}\}$ is called *self-similar* if for any $a \in \mathbb{R}_+$ there exists $b_a \in \mathbb{R}_+$ such that

$$\{X_{at},\ t \in \mathbb{R}\} \stackrel{d}{=} \{b_a X_t,\ t \in \mathbb{R}\}$$

in the sense of finite-dimensional distributions. It is called *b-self-similar* (with $b \in \mathbb{R}_+$) if for any $a \in \mathbb{R}_+$ it holds that

$$\{X_{at},\ t \in \mathbb{R}\} \stackrel{d}{=} \left\{a^b X_t,\ t \in \mathbb{R}\right\},$$

i.e., the constant $b_a \in \mathbb{R}_+$ is a^b.

Self-similarity refers to invariance in distribution under an appropriate change of scale (thus, the sometimes used terms *stochastic* or *statistical self-similarity* would be more precise). Interpreting the index set of a stochastic process as time, self-similarity implies that a change of the time scale is equivalent to a (suitable) change in the state space scale (this definition does of course require that the process be defined in continuous time since we must be able to scale the time axis by any positive factor a).[3]

[2] The property "normalized" in Definition 1.1.1 means that $B^H_t \sim \mathrm{N}(0,|t|^{2H})$, $t \in \mathbb{R}$. A "non-normalized" fBM can be defined by replacing item (iii) with

$$\text{(iii)* } m^{(2)}_{B^H}(s,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R} : (s,t) \mapsto \mathbf{E}(X_s X_t) = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

for (fixed) $\sigma^2 \in \mathbb{R}_{+,0}$. In this case $B^H_t \sim \mathrm{N}(0,\sigma^2|t|^{2H})$, $t \in \mathbb{R}$.

[3] The term self-similar was coined by Mandelbrot (in the 1960s) and is now standard. Mandelbrot himself though prefers to use the term "self-affinity" now because time and space are scaled differently.

Lemma 1.1.1. *Any fBM* $B^H = \left\{B_t^H,\ t \in \mathbb{R}\right\}$ *is* H*-self-similar.*

Proof. To show the self-similarity property we have to proof that the finite-dimensional distributions of $\left\{B_{\alpha t}^H,\ t \in \mathbb{R}\right\}$ and $\left\{\alpha^H B_t^H,\ t \in \mathbb{R}\right\}$ are identical for any $\alpha \in \mathbb{R}_+$. This can be done by proofing that the characteristic functions (see Definition A.1.12) of the finite-dimensional distributions of the two processes are identical (hence, the distributions itself must be identical since a characteristic function determines the corresponding distribution uniquely). As fBM is a Gaussian process, all the finite-dimensional distributions are mulivariate normal. For a finite collection of indices $t = (t_1, \ldots, t_n) \in \mathbb{R}^n, n \in \mathbb{N}$, the characteristic function of the (finite-dimensional) distribution of the corresponding multivariate random variable $(B_{t_1}, \ldots, B_{t_n})$ has the form

$$\varphi_t(\lambda) = \mathbf{E}\left(\exp\left\{i \sum_{k=1}^n \lambda_k B_{t_k}^H\right\}\right) = \exp\left\{-\frac{1}{2}\left\langle C_t\lambda\,,\,\lambda\right\rangle\right\},\ \lambda \in \mathbb{R}^n, \tag{1.1}$$

where $C_t := \left(\mathbf{E}\left(B_{t_k}^H B_{t_l}^H\right)\right)_{1\le k,l\le n}$ denotes the covariance matrix of the (zero-mean) random variable $(B_{t_1}, \ldots, B_{t_n})$ and $\langle\,.\,,\,.\,\rangle$ is the (standard) inner product on $\mathbb{R}^n$. From item (iii) of Definition 1.1.1 and the linearity of the inner product it follows that for any $\alpha \in \mathbb{R}_+$ the characteristic function of the finite-dimensional distributions of the process $\left\{B_{\alpha t}^H,\ t \in \mathbb{R}\right\}$ has the form

$$\varphi_{\alpha t}(\lambda) = \exp\left\{-\frac{1}{2}\alpha^{2H}\left\langle C_t\lambda\,,\,\lambda\right\rangle\right\},\ \lambda \in \mathbb{R}^n. \tag{1.2}$$

As $\mathbf{E}(.)$ is a linear operator the covariance matrix of $\left\{\alpha^H B_t^H,\ t \in \mathbb{R}\right\}$ has the form $\alpha^{2H} C_t$. Consequently, the characteristic functions of the finite-dimensional distributions of this process have the same form as (1.2), which yields the desired result. ■

It follows from Remark 1.1.6 that an fBM has stationary increments (weakly and strictly, see Definitions A.1.3 and A.1.2) as their distribution only depends on the "lag", but is not stationary itself (neither strictly nor weakly). Strict stationarity is violated as the variance of the individual random variables of fBM is not constant. Weak stationarity is violated as the covariance function does not have the property $\gamma_{B^H}(s+h, t+h) = \gamma_{B^H}(s,t),\ t, s, h \in \mathbb{R}$.

As mentioned above, fBM can be seen as generalization of classical Brownian Motion (BM, or Wiener process). For $H = \frac{1}{2}$ the second moment function of an fBM has the form

$$m^{(2)}_{B^{1/2}}(s,t) = \frac{1}{2}(|t| + |s| - |t-s|) = \begin{cases} \min(|t|, |s|), & \operatorname{sign}(t) = \operatorname{sign}(s), \\ 0, & \text{otherwise}, \end{cases} \qquad s, t \in \mathbb{R}.$$

which coincides with the second moment function of a (two-sided, normalized) BM[4] With Remark 1.1.1 and Definition 1.1.1 it follows that $B^{\frac{1}{2}} = W$, where $W = \{W_t,\, t \in \mathbb{R}\}$ denotes a (normalized, two-sided) BM.

[4] A one-sided BM $W = \{W_t,\, t \in \mathbb{R}_{+,0}\}$ is said to be normalized, if $W_t \sim \mathrm{N}(0,t)\,, t \in \mathbb{R}_{+,0}$. A (normalized) two-sided BM $W = \{W_t,\, t \in \mathbb{R}\}$ can be constructed by taking two independent (normalized) one-sided Bms $W^1 = \left\{W_t^1,\, t \in \mathbb{R}_{+,0}\right\}$, $W^2 = \left\{W_t^2,\, t \in \mathbb{R}_{+,0}\right\}$ and setting

$$W_t := \begin{cases} W_t^1\,, & t < 0\,, \\ W_t^2\,, & t \ge 0\,. \end{cases}$$

Since $W_t \sim \mathrm{N}(0, |t|)\,, t \in \mathbb{R}$, W is said to be normalized.

For $H \in (0,1]$ it holds that

$$\mathbf{E}\big((B_t^H - B_s^H)(B_v^H - B_u^H)\big) = \frac{1}{2}(|s-u|^{2H} + |t-v|^{2H} - |t-u|^{2H} - |s-v|^{2H}),\ s,t,u,v \in \mathbb{R},$$

which in the case of $t_1 < t_2 < t_3 < t_4$ and $\alpha := H - \frac{1}{2}$ can be written as

$$\mathbf{E}\big((B_{t_4}^H - B_{t_3}^H)(B_{t_2}^H - B_{t_1}^H)\big) = 2\alpha H \underbrace{\int_{t_1}^{t_2} \underbrace{\int_{t_3}^{t_4} (u-v)^{2\alpha-1}\,\mathrm{d}u}_{>0\ \forall v \in [t_1,t_2]}\,\mathrm{d}v}_{>0}\,. \tag{1.3}$$

The sign of (1.3) only depends on α (the appearing double integral is strictly positive). As (1.3) is the covariance of two increments of an fBM over two successive (non-overlapping) time intervals, these increments are negatively correlated for $H \in (0, \frac{1}{2})$, uncorrelated (which is for Gaussian random variables equivalent of being independent) for $H = \frac{1}{2}$ (which is a known property of BM) and positively correlated for $H \in (\frac{1}{2}, 1]$. Consequently, fBM is a Lévy process (see Definition A.1.10) only for $H = \frac{1}{2}$ (for any sequence of increments of $B^{\frac{1}{2}}$ the pairwise uncorrelatedness implies the mutual independence of the whole sequence, as it is Gaussian), whereas for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ the independence of the increments is violated.

1.1.2 Long and Short Memory

The covariance function[5] γ_X of a weakly stationary stochastic process $X = \{X_t,\, t \in \mathbb{Z}\}$ satisfies $\gamma_X(s,t) = \gamma_X(s-t, 0)$, $s,t \in \mathbb{Z}$, and can therefore be redefined as a function of one variable

$$\hat{\gamma}_X : \mathbb{Z} \to \mathbb{R} \,:\, k \mapsto \hat{\gamma}_X(k) := \gamma_X(k, 0),$$

where $\hat{\gamma}_X(k)$ is referred to as the correlation function of X at 'lag' k. The correlation function[6] of such a process is defined by

$$\hat{\rho}_X : \mathbb{Z} \to \mathbb{R} \,:\, k \mapsto \hat{\rho}_X(k) := \frac{\hat{\gamma}_X(k)}{\hat{\gamma}_X(0)}\,.$$

Definition 1.1.3. A weakly stationary stochastic process $X = \{X_k,\, k \in \mathbb{Z}\}$ is said to have *long memory* or *long-range dependence* if there exist real constants $C \neq 0$ and $0 \leq d < \frac{1}{2}$ such that

$$\hat{\rho}_X \sim C k^{2d-1} \text{ as } k \to \infty, \tag{1.4}$$

where the symbol '$\sim$' denotes 'the same asymtotic behaviour'[7]. This asymptotic behaviour of the correlation function is equivalent to $\sum_{k\in\mathbb{Z}} |\hat{\rho}_X(k)| = \infty$. The process is said to have *short memory* or *short-range dependence* if there exist real constants $C \neq 0$ and $d < 0$ such that (1.4) holds, which is equivalent to $\sum_{k\in\mathbb{Z}} |\hat{\rho}_X(k)| < \infty$.

Remark 1.1.7. This is not the only possible definition for long and short range memory. Alternative ones can be found in the literature which are not all exactly equivalent (e.g., cf. [DoOpTa03, pp. 16-18].

[5] The covariance function of a process is sometimes, especially in time series analysis, referred to as autocovariance function.

[6] or autocorrelation function.

[7] i.e., $f \sim g$ as $x \to \infty :\Leftrightarrow \dfrac{|f(x)|}{|g(x)|} = 1$ as $x \to \infty$.

Remark 1.1.8. The equivalence in Definition 1.1.3 can be seen by noting that (1.4) implies $\sum_{k\in\mathbb{Z}} |\hat{\rho}_X(k)| \sim \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2d-1}$ (in the sense of same convergence behaviour as $k \to \infty$) and by recalling that the generalized harmonic series $\sum_{n\in\mathbb{N}} n^{-p}$, $p \in \mathbb{R}_+$, convergences for $p > 1$ and diverges for $p \leq 1$ (as can be simply proofed by Cauchy's condensation test).

By taking the correlation $\hat{\rho}_X(k)$ of a weakly stationary process $X = \{X_k,\, k \in \mathbb{Z}\}$ as measure of dependence between any two of its random variables at lag k (i.e., X_t, X_{t+k}, for some $t \in \mathbb{Z}$), the properties of short resp. long memory get an intuitive interpretation. For short memory processes the dependence declines rapidly (which ensures the convergence of $\sum_{k\in\mathbb{Z}} |\hat{\rho}_X(k)|$) whereas long memory processes exhibit dependence also for large lags (thus, leading $\sum_{k\in\mathbb{Z}} |\hat{\rho}_X(k)|$ to diverge). This can be used to model phenomena exhibiting dependence upon larger time-scale. The divergence of the covariance series captures the intuition behind long memory; even though the high-lag correlations are individually small, their cumulative effect is of importance, thus giving rise to a behavior which is markedly different from that of processes with short memory.

As fBM has stationary increments, the stochastic process $X^H = \left\{X_k^H,\, k \in \mathbb{Z}\right\}$, defined by $X_k^H := B_k^H - B_{k-1}^H$, $k \in \mathbb{Z}$, is weakly stationary. In the literature this process is referred to as *fractional Gaussian noise* (fGn), see Section 1.1.4. From (1.3) it follows that the covariance function of X^H has the form

$$\hat{\gamma}_{X^H}(k) = \mathbf{E}\big(B_1^H(B_{k+1}^H - B_k^H)\big) = 2\alpha H \int_0^1 \int_k^{k+1} (u-v)^{2\alpha-1}\, \mathrm{d}u\, \mathrm{d}v,\ k \in \mathbb{Z}, \tag{1.5}$$

where $\alpha := H - \frac{1}{2}$. As $\hat{\gamma}_{X^H}(0) = \mathbf{V}\big(B_1^H\big) = 1$ the covariance function $\hat{\gamma}_{X^H}$ coincides with the correlation function $\hat{\rho}_{X^H}$. By taking a look at the exponent of the integrand in (1.5) it follows that

$$\hat{\rho}_{X^H}(k) \sim 2\alpha H |k|^{2\alpha-1} \quad \text{as } |k| \to \infty\,. \tag{1.6}$$

From (1.6) and Definition 1.1.3 it follows that X^H has long memory for $\alpha \in (0, \frac{1}{2}]$ (which is equivalent to $H \in (\frac{1}{2}, 1])$ and short memory for $\alpha < 0$ (equivalent to H $\in (0, \frac{1}{2})$). Since for $H = \frac{1}{2}$ the increments over non-overlapping time intervals of fBM are independent (and therefore uncorrelated), $X^{\frac{1}{2}}$ is said to have no memory.

The concept of short and long memory (defined in 1.1.3 for weakly stationary processes in discrete time) can be extended to processes with stationary increments in continuous time (although the process itself need not be weakly stationary, as required in Definition 1.1.3) by saying that a process $X = \{X_t,\, t \in \mathbb{R}\}$ has long resp. short memory if the increment processes of X have the corresponging properties, i.e., for any $h \in \mathbb{R}_+$ the process

$$X^{(h)} = \left\{X_k^{(h)} := X_{kh} - X_{(k-1)h},\, k \in \mathbb{Z}\right\}$$

exhibits long resp. short memory. Analogously to (1.5) it can be derived from (1.3) that for any $h \in \mathbb{R}_+$

$$\mathrm{Cov}(B_h^H, B_{k+h}^H - B_k^H) \sim 2\alpha H h^2 |k|^{2\alpha-1} \quad \text{as } |k| \to \infty. \tag{1.7}$$

This implies that the increments of fBM have long memory for $H \in \big(\frac{1}{2}, 1\big]$ and short memory for $H \in \big(0, \frac{1}{2}\big)$. Due to this we say that fBM itself has long memory for $H \in \big(\frac{1}{2}, 1\big]$ and short memory for $H \in \big(0, \frac{1}{2}\big)$.

1.1.3 Spectral Density and Spectral Representation of fBM

Two other important concepts in the theory of stochastic processes (especially in time series analysis) are the spectral distribution function and the spectral density function of a stochastic process (see Theorem A.2.3 for a definition). For the spectral density function of X^H, which we denote by $f_H(\lambda)$, it holds that (see [BiGu96])

$$f_H(\lambda) = C_H^{(0)} |e^{i\lambda} - 1|^2 \sum_{k\in\mathbb{Z}} |\lambda + 2\pi k|^{-2H-1} \, , \, \lambda \in [-\pi, \pi],$$

where $C_H^{(0)}$ is some constant depending on H. It holds that

$$f_H(\lambda) \sim C_H^{(0)} |\lambda|^2 |\lambda|^{-2H-1} = C_H^{(0)} |\lambda|^{-2H+1} \quad \text{as } \lambda \to 0.$$

Therefore, for $H \in (\frac{1}{2}, 1)$ it holds that $f_H(\lambda) \to \infty$ as $\lambda \to 0$, and, for $H \in (0, \frac{1}{2})$ it holds that $f_H(\lambda) \to 0$ as $\lambda \to 0$. Since this asymptotic behaviour for $\lambda \to 0$ is a characteristic of long and short memory processes, i.e., spectral densities are unbounded at $\lambda = 0$ for long memory and bounded for short memory, this could be used as alternative (and equivalent) frequency-domain definition of long and short memory.

According to [PiTa00] and [SaTa94], any fBM B^H admits a spectral representation of the form

$$\left\{B_t^H, \, t \in \mathbb{R}\right\} \stackrel{d}{=} \left\{C_H^{(1)} \int_{\mathbb{R}} (e^{itx} - 1)(ix)^{-1} |x|^{-H+\frac{1}{2}} \, \mathrm{d}\tilde{B}(x), \, t \in \mathbb{R}\right\},$$

where $\tilde{B} = B_1 + iB_2$ is a complex Gaussian measure with $B_1(A) = B_1(-A)$, $B_2(A) = -B_2(-A)$ and $\mathbf{E}\big((B_1(A))^2\big) = \mathbf{E}\big((B_2(A))^2\big) = \frac{\text{mesh}(A)}{2}$ for any Borel set $A \in \mathcal{B}$ of finite Lebesgue measure mesh(A) and $C_H^{(1)} := \left(\frac{\Gamma(2H+1)\sin(\pi/2(H+1/2))}{2\pi}\right)^{\frac{1}{2}}$.

1.1.4 Fractional Gaussian Noise

Recall that fGn is the process $X^H = \left\{X_k^H, \, k \in \mathbb{Z}\right\}$ defined by $X_k^H := B_k^H - B_{k-1}^H$, $k \in \mathbb{Z}$, and that it has long memory for $H \in \left(\frac{1}{2}, 1\right]$ and short memory for $H \in \left(0, \frac{1}{2}\right)$.

A striking feature of fGn with $H \neq \frac{1}{2}$ is that it provides a counterexample to the usual central limit theorem. Indeed, for $d_n^{-1} \sum_{k=1}^n X_k^H$ to converge in distribution, as $n \to \infty$, to a non-trivial limit, one cannot choose $d_n \sim n^{\frac{1}{2}}$ but rather $d_n \sim n^H$: Since fGn is the increment process of fBM, it holds that $n^{-H} \sum_{k=1}^n X_k^H = n^{-H} B_n^H$ which has the same distribution as B_1^H due to self-similarity of $\left\{B_t^H, \, t \in \mathbb{R}\right\}$. To ensure convergence, for fGn with long memory (i.e., $H > \frac{1}{2}$) the norming sequence d_n^{-1} has to decrease more rapidly than in the usual central limit theorem (whereas in the short memory case the norming sequence may even decrease more slowly). For practical purposes this means that the variance of time averages decreases far less rapidly than observed in usual (short memory) models. This property is not only exhibited by fGn but is a characteristic feature of long memory processes in general.

1.1.5 Mandelbrot-van Ness Representation of fBM

Let $W = \{W_t, \, t \in \mathbb{R}\}$ be a (two-sided, normalized) BM and denote $k_H(t, u) := (t-u)_+^{\alpha} - (-u)_+^{\alpha}$, where $\alpha = H - \frac{1}{2}$. The following (more constructive) representation of fBM is due to Mandelbrot and van Ness (see [MavN68]).

Theorem 1.1.1 (Theorem 1.3.1 of [Mish08])**.** *The process* $\overline{B}^H = \left\{\overline{B}_t^H,\, t \in \mathbb{R}\right\}$, $H \in (0,1)$, *defined by*

$$\overline{B}_t^H := C_H^{(2)} \int_{\mathbb{R}} k_H(t,u)\,\mathrm{d}W_u\,,\ t \in \mathbb{R}, \tag{1.8}$$

$$\text{where } C_H^{(2)} := \left(\int_{\mathbb{R}_+} ((1+s)^\alpha - s^\alpha)^2\,\mathrm{d}s + \frac{1}{2H}\right)^{-\frac{1}{2}} = \frac{\left(2H\sin(\pi H)\Gamma(2H)\right)^{\frac{1}{2}}}{\Gamma(H+1/2)}, \tag{1.9}$$

has a P*-a.s.continuous modification which is a normalized two-sided fBM.*

Remark 1.1.9. $C_H^{(2)}$ in (1.8) is a normalizing constant to ensure that $\overline{B}_t^H \sim \mathrm{N}(0, |t|^{2H})$ (i.e., that $\overline{B}^H$ is a normalized fBM).

Proof. For $H = \frac{1}{2}$ it is clear that $C_H^{(2)} = 1$ and

$$\overline{B}_t^H = \int_{\mathbb{R}} k_H(t,u)\,\mathrm{d}W_u = W_t,\ t \in \mathbb{R}.$$

For $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ the intergal in (1.8) can be understood as an L^2-limit of linear combinations of random variables from $\{W_t,\, t \in \mathbb{R}\}$ (see Chapter 1 of [Cher01b] for a detailed discussion). Hence, $\overline{B}^H$ is a Gaussian process with $\overline{B}_0^H = 0$ and $\mathbf{E}\left(\overline{B}_t^H\right) = 0$. Furthermore, it holds (by L^2-isometry) that for $t \geq 0$

$$\begin{aligned}
\mathbf{E}\left(\left(\int_{\mathbb{R}} k_H(t,u)\,\mathrm{d}W_u\right)^2\right) &= \int_{-\infty}^0 k_H^2(t,u)\,\mathrm{d}u + \int_0^t (t-u)^{2\alpha}\,\mathrm{d}u \\
&= \underbrace{\int_0^\infty \left((t+u)^\alpha - (u)^\alpha\right)^2 \mathrm{d}u}_{= \left|\begin{smallmatrix} s := \frac{u}{t} \\ \frac{\mathrm{d}s}{\mathrm{d}u} = \frac{1}{t}\end{smallmatrix}\right| = \int_0^\infty t^{2\alpha+1}\left((1+s)^\alpha - (s)^\alpha\right)^2\,\mathrm{d}u} + \underbrace{\frac{1}{2\alpha+1}\left((t-u)^{2\alpha+1}\Big|_{u=0}^t\right)}_{= \frac{t^{2H}}{2H}} \\
&= t^{2H}\left(\int_{\mathbb{R}_+} ((1+s)^\alpha - s^\alpha)^2\,\mathrm{d}s + \frac{1}{2H}\right).
\end{aligned} \tag{1.10}$$

Analogous, for $t < 0$ we have that

$$\begin{aligned}
\mathbf{E}\left(\left(\int_{\mathbb{R}} k_H(t,u)\,\mathrm{d}W_u\right)^2\right) &= \int_{-\infty}^t k_H^2(t,u)\,\mathrm{d}u + \int_t^0 (-u)^{2\alpha}\,\mathrm{d}u \\
&= (-t)^{2H}\left(\int_{\mathbb{R}_+} ((1+s)^\alpha - s^\alpha)^2\,\mathrm{d}s + \frac{1}{2H}\right).
\end{aligned} \tag{1.11}$$

From (1.10) and (1.11) the normalizing constant $C_H^{(2)}$ can be derived. This yields $\mathbf{E}\left(\left(\overline{B}_t^H\right)\right) = |t|^{2H}$, $t \in \mathbb{R}$. Furthermore, for $h > 0$ it holds that

$$\begin{aligned}
\overline{B}_{s+h}^H - \overline{B}_s^H = C_H^{(2)} &\left(\int_{-\infty}^s k_H(s+h,u) - k_H(s,u)\,\mathrm{d}W_u\right) + \\
&C_H^{(2)} \int_s^{s+h} k_H(s+h,u)\,\mathrm{d}W_u =: C_H^{(2)}(I_1 + I_2)\,.
\end{aligned} \tag{1.12}$$

As the BM W has independent and stationary increments, it follows that the terms I_1 and I_2 on the right-hand side of (1.12) are independend, that

$$I_1 \stackrel{d}{=} \int_{-\infty}^{0} (k_H(s,u) - k_H(0,u))\,\mathrm{d}W_u\,,\ I_2 \stackrel{d}{=} \int_0^h k_H(h,u)\,\mathrm{d}W_u\,,$$

and $\mathbf{E}\Big(\big(\overline{B}^H_{s+h} - \overline{B}^H_s\big)^2\Big) = \mathbf{E}\Big(\big(\overline{B}^H_h\big)^2\Big) = h^{2H}$. By combining these results, we obtain that

$$\begin{aligned}\mathbf{E}\Big(\overline{B}^H_s\overline{B}^H_t\Big) = \frac{1}{2}\left(\mathbf{E}\Big(\big(\overline{B}^H_s\big)^2\Big) + \mathbf{E}\Big(\big(\overline{B}^H_t\big)^2\Big) - \mathbf{E}\Big(\big(\overline{B}^H_t - \overline{B}^H_s\big)^2\Big)\right)\\ = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)\,. \quad (1.13)\end{aligned}$$

The proof follows from Definition 1.1.1 and Remark 1.1.5. For the second equality in (1.9) see Mishura [Mish08, pp. 363-364]. ■

Remark 1.1.10. According to [Cher01b], for $H \in \big(0,\frac{1}{2}\big) \cup \big(\frac{1}{2},1\big)$ the integral in (1.8) can also be understood as P-a.s. limit in the sense that $\int_{\mathbb{R}} k_H(t,u)\,\mathrm{d}W_u(\omega)$ exists as improper Riemann-Stieltjes integral for P-a.a. $\omega \in \Omega$ (see Proposition 1.3 of [Cher01b]).

Remark 1.1.11. As stated in the first paragraph of [MavN68], Mandelbrot and van Ness' idea behind (1.8) was to represent fBM with parameter $H \in (0,1)$ as moving average of $\mathrm{d}W_s$ in which past increments of W are weighted by the kernel $(t-s)^{2H-1}$. Due to this, (1.8) is also referred to in the literature as 'moving average representation' of fBM.

It arises the question whether any fBM B^H with $H \in (0,1)$ can be presented in the form (1.8). This is indeed the case and is stated in the next theorem. To simplify the notation we use $\alpha := H - \frac{1}{2}$ and define the operator

$$M^H_- f := \begin{cases} C^{(3)}_H I^\alpha_- f\,, & H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1),\\ f\,, & H = \frac{1}{2}, \end{cases} \qquad (1.14)$$

where $C^{(3)}_H := C^{(s)}_H \Gamma(H+\frac{1}{2})$ and I^α_- denotes the Riemann-Liouville left-sided fractional integral on $\mathbb{R}$ of order α, i.e.,

$$(I^\alpha_- f)(x) := \frac{1}{\Gamma(a)} \int_x^\infty f(t)(t-x)^{(\alpha-1}\,\mathrm{d}t\,.$$

As is proofed in Lemma 1.1.3 of [Mish08], for $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ it holds for all $t \in \mathbb{R}$ that

$$(I^\alpha_- \mathbf{1}_{(0,t)})(x) = \frac{1}{\Gamma(1+\alpha)}\left((t-x)^\alpha_+ - (-x)^\alpha_+\right)\,,$$

where $\mathbf{1}_{(.,.)}$ denotes the general indicator function, which is given by $(a,b \in \mathbb{R})$

$$\mathbf{1}_{(a,b)} := \begin{cases} 1, & a \le t < b,\\ -1, & b \le t < a,\\ 0, & \text{otherwise.}\end{cases}$$

With this notation (1.8) can be written as

$$\overline{B}^H_t := \int_{\mathbb{R}} (M^H_- \mathbf{1}_{(0,t)})(u)\,\mathrm{d}W_u\,,\ t \in \mathbb{R}. \qquad (1.15)$$

Theorem 1.1.2 (Corollary 1.6.11 of [Mish08]). *Any fBM B^H with $H \in (0,1)$ admits a Mandelbrot-van Ness representation of the form (1.15) with respect to a suitable BM W.* (For a proof see [Mish08, pp. 9-23].)

Remark 1.1.12. As suitable BM for the Mandelbrot-van Ness representation of any fBM B^H the process W defined by

$$W_t := C_H^{(4)} \int_{\mathbb{R}} M_-^{1-H} \mathbf{1}_{(0,t)}(u)\, \mathrm{d}B_u^H \,,\; t \in \mathbb{R},$$

where $C_H^{(4)} := \left(C_H^{(3)} C_{1-H}^{(3)} \right)^{-1}$, which is indeed a BM, can be used. The integral is a Wiener integral with respect to B^H (see [Mish08, pp. 16-23] for a construction).

Remark 1.1.13. Norros *et al.* [NoVaVi99] have shown that fBM also admits an integral representation analogous to (1.8) over finite intervals. Any fBM of the form $B^H = \left\{ B_t^H \,,\, t \in \mathbb{R}_{+,0} \right\}$ can be written as

$$B_t^H = \int_0^t l(t,u) \mathrm{d}\, B_u ,\;\; t \in \mathbb{R}_{+,0}\,, \tag{1.16}$$

with

$$l(t,u) := C_H^{(5)} \left[\left(\frac{t}{u} \right)^{\alpha} (t-u)^{\alpha} - \alpha u^{-\alpha} \int_u^t s^{\alpha-1} (s-u)^{\alpha} \mathrm{d}\, s \right]$$

with $\alpha = H - \frac{1}{2}$ and $C_H^{(5)} := \sqrt{\dfrac{2H\Gamma(1-\alpha)}{\Gamma(\alpha+1)\Gamma(1-2\alpha)}}$, where $\Gamma(.)$ denotes the Gamma function and $B = \{B_t \,,\, t \in \mathbb{R}_{+,0}\}$ denotes a standard BM. The integral in (1.16) can again be understood as L^2-limit.

1.1.6 Some Limit Results for fBM

In the next lemma we collect some limit results on the increments of fBM that we will partly need in this thesis.

Lemma 1.1.2. *Let B^H be an fBM with $H \in (0,1]$, and $T, p, q \in \mathbb{R}_+$. Then:*

(i) $n^{-1+pH} \sum_{j=0}^{n-1} \left| B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T} \right|^p \xrightarrow{a.s.} \boldsymbol{E}\big(|B_T^H|^p\big)$ *as $n \to \infty$ and the convergence is also in L^1.*

(ii) $n^{-1+pH-q} \sum_{j=0}^{n-1} \left| B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T} \right|^p \xrightarrow{a.s.} 0$ *as $n \to \infty$ and the convergence is also in L^1.*

(iii) $n^{-1+pH+q} \sum_{j=0}^{n-1} \left| B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T} \right|^p \xrightarrow{p} \infty$ *as $n \to \infty$, i.e., for all $L \in \mathbb{R}_+$ there exists an n_0 such that for all $n \geq n_0$:* $\mathrm{P}\left(n^{-1+pH+q} \sum_{j=0}^{n-1} \left| B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T} \right|^p < L \right) < \frac{1}{L}$.

Proof. (i) The sequence $\left\{ B^H_{(j+1)T} - B^H_{jT} \,,\, j \in \mathbb{N}_0 \right\}$ is strictly stationary (see Remark 1.1.6). Since it is Gaussian and (see (1.7))

$$\mathrm{Cov}\left(B_T^H - B_0^H, B^H_{(j+1)T} - B^H_{jT} \right) \to 0\,,\;\text{as } j \to \infty\,,$$

it is also mixing, hence ergodic. The Ergodic theorem (see Theorem A.2.4 and Remark A.2.2) then implies

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|B^H_{(j+1)T}-B^H_{jT}\right|^p \xrightarrow{a.s.} \mathbf{E}\big(|B^H_T|^p\big)\ ,\ \text{as } n\to\infty\,. \tag{1.17}$$

From the stationarity of the above sequence follows that $\mathbf{E}(|\,B^H_{(j+1)T}-B^H_{jT}\,|^p)$ is constant for all $j\in\mathbb{N}_0$, which, by Scheffé's lemma, implies the convergence of (1.17) in L^1. It follows from the self-similarity of B^H that for all $n\in\mathbb{N}$

$$n^{-1+pH}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T}-B^H_{\frac{j}{n}T}\right|^p$$

has the same distribution as

$$\sum_{j=0}^{n-1}\left|B^H_{(j+1)T}-B^H_{jT}\right|^p.$$

This completes the proof of (i).

(ii) Follows immediately from (i).

(iii) Choose $L\in\mathbb{R}_+$. It follows from (ii) that there exists an $n_1\in\mathbb{N}$ such that

$$\mathrm{P}\left(\left|\mathbf{E}\Big(\big|B^H_T\big|^p\Big)-n^{-1pH}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T}-B^H_{\frac{j}{n}T}\right|^p\right|>\frac{1}{2}\mathbf{E}\Big(\big|B^H_T\big|^p\Big)\right)<\frac{1}{L}$$

for all $n\geq n_1$. This implies that for all $n\geq n_1$

$$\mathrm{P}\left(n^{-1pH}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T}-B^H_{\frac{j}{n}T}\right|^p<\frac{1}{2}\mathbf{E}\Big(\big|B^H_T\big|^p\Big)\right)<\frac{1}{L}$$

or, equivalently,

$$\mathrm{P}\left(n^{-1pH+q}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T}-B^H_{\frac{j}{n}T}\right|^p<n^q\frac{1}{2}\mathbf{E}\Big(\big|B^H_T\big|^p\Big)\right)<\frac{1}{L}\,.$$

This shows the existence of an $n_0\in\mathbb{N}$ such that

$$\mathrm{P}\left(n^{-1pH+q}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T}-B^H_{\frac{j}{n}T}\right|^p<L\right)<\frac{1}{L}$$

for all $n\geq n_0$ and (iii) is proved.

■

1.1.7 The Hurst Parameter H

The Hurst parameter H divides fBM into three different classes. $B^{\frac{1}{2}}$ is a two-sided (normalized) BM (which has independent increments). For $H \in \left(0, \frac{1}{2}\right)$ the covariance between two increments over non-overlapping time intervals is negative and the process has short memory, whereas for $H \in \left(\frac{1}{2}, 1\right]$ this covariance is positive and the process has long memory. Due to this correlation structure the paths of fBM get smoother (less zigzagged) as H goes from 0 to 1. For $H \in \left(0, \frac{1}{2}\right)$ any two consecutive increments of fBM tend to have opposite signs and thus be more zigzagging because of their negative covariance (this tendency increases as H approaches 0)[8]. For $H \in \left(\frac{1}{2}, 1\right]$ the covariance of two consecutive increments is positive. Hence, the increments tend to have the same signs (with increasing tendency as H approaches 1), which leads to smoother paths[9]. For the case of $H = 1$ the paths of fBM become lines with random slope (see Remark 1.1.4). Due to this path behaviour fBM is called *antipersistent* for $H \in \left(0, \frac{1}{2}\right)$, *chaotic* for $H = \frac{1}{2}$ and *persistent* for $H \in \left(\frac{1}{2}, 1\right]$. Figures 1.1 to 1.3 show simulated sample paths of fBM with varied parameter H to illustrate its influence on the path appearance (see Appendix B for details).

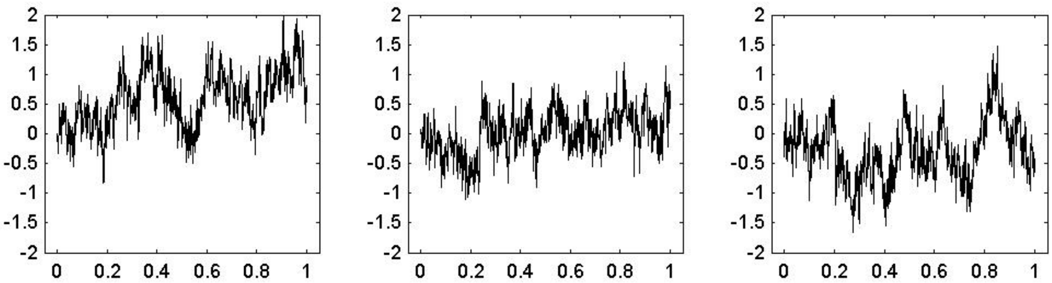

Figure 1.1: Simulated sample paths (each 1000 points) of fBM with $H = 0.2$.

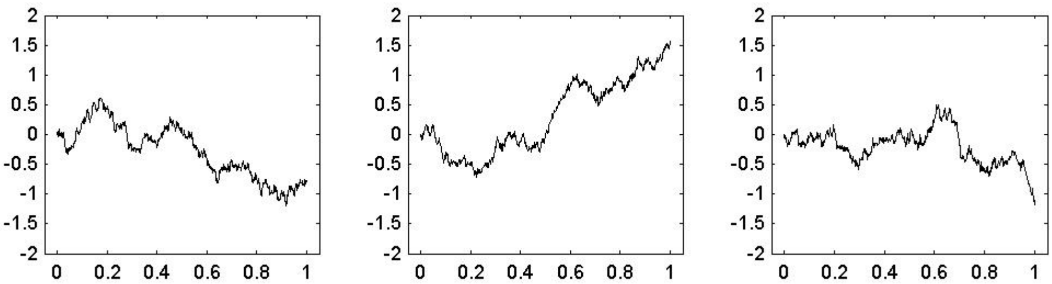

Figure 1.2: Simulated sample paths (each 1000 points) of fBM with $H = 0.5$.

1.1.8 FBM and the Martingale Property

Since fBM can be seen as a generalizaton of BM it arises the question whether fBM inherits the martingale property. As it turns out, for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$ this is not the case. As has been shown by several authors, fBM is not even a semimartingale (w.r.t. its natural

[8] Because of this behaviour, fBM with $H \in \left(0, \frac{1}{2}\right)$ has been used as a model of turbulence (see [Shir99] and references therein)

[9] As this positive covariance of consecutive increments persists over arbitrary long time intervals, this leads to a rather natural and smooth appreance of sample paths which can be used in fractal landscape generation (especially for $H \in (\frac{3}{4}, 1)$, see [Fesz05] for further information).

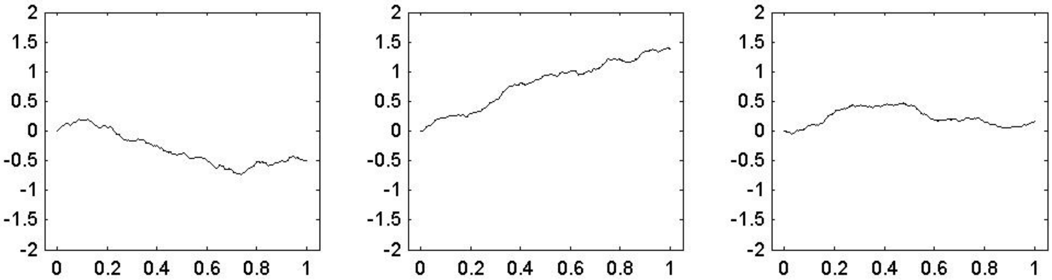

Figure 1.3: Simulated sample paths (each 1000 points) of fBM with $H = 0.8$.

filtration, see Definition A.1.4) for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. Furthermore, fBM is not a weak semimartingale[10] (which is a sligthly stronger statement since the natural filtration of an fBM does not satisfy the usual conditions). That B^H is not a weak semimartingale for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ can be derived from the fact that in this case B^H does not have the 'right' variation. For a detailed discussion of this matter I would like to refer to [Cher01a] where the author examines the semimartingale property of 'mixed fractional Brownian Motion', i.e., a linear combination of different fractional Brownian Motions, of the special form

$$M^{H,\alpha} := W + \alpha B^H ,$$

where W is a BM, B^H is a fBM with $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]$ and $\alpha \in \mathbb{R} \setminus \{0\}$. The main result of the cited paper is

Theorem 1.1.3 (Theorem 1.7 of [Cher01a]). *$(M^{H,\alpha})_{t\in[0,1]}$ is not a weak semimartingale if $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{3}{4}\right]$, it is equivalent to $\sqrt{1+\alpha^2}$ times Brownian Motion if $H = \frac{1}{2}$ and equivalent to Brownian Motion if $H \in \left(\frac{3}{4}, 1\right]$.*
(For a proof see [Cher01a].)

Remark 1.1.14. For simplicity (to avoid localization arguments) the processes considered in the cited paper are of the form $M^{H,\alpha} = (M^{H,\alpha})_{t\in[0,T]}$ with $T < \infty$. Due to self-similarity of fBM there is no loss of generality in assuming $T = 1$.

Remark 1.1.15. Semimartingale processes were for several decades the best model to implement many ideas. For example, they provide a convenient setting to consider financial markets as was demonstrated by various authors (see for example Delbaen and Schachermayer [DeSa06]). However, in recent years it turns out that the theory of semimartingales is insufficient to describe many phenomena. Empirical data indicates that various objects like telecommunication connections or asset prices exhibit long memory (see Section 1.1.2). This effect cannot be described satisfyinly by BM-type processes as they have independent increments and therefore no memory. As an alternative, fBM could be used as a relatively simple non-semimartingale model that allows to account for dependencies such as long memory. The study of non-semimartigale processes and their use as models (e.g., in mathematical finance) is currently an active field of research.

1.1.9 FBM as Financial Market Model

Bachelier was the first to propose a continuous-time stochastic model to describe the price evolution of a financial asset. The model that he suggested in his thesis (1900) for the price of a stock is as follows:

[10] Weak semimartingales are stochastic processes that, in contrast to semimartingales, are not required to be a.s. right-continuous and the filtration w.r.t. which they are considered need not fulfill the usual conditions, see Definition A.1.5. For a rigorous definition of weak semimartingales see [Cher01b].

$$S_t = S_0 + \mu t + \sigma B\,, \tag{1.18}$$

where S_0, μ and σ are real constants and B is a Brownian Motion. Samuelson (1965) introduced the model

$$S_t = S_0 \exp\left(\left\{\mu - \frac{\sigma^2}{2}\right\} t + \sigma B\right)\,, \tag{1.19}$$

where again S_0, μ and σ are real constants and B is a Brownian Motion. The Samuelson model is often regarded as a more realistic description of the price evolution of a financial asset since, in contrast to the Bachelier model, it does not admit negative asset prices[11].

Black and Scholes (1973) considered a simple financial market model of the form $(S^0, S) = \big((S^0_t)_{[0,T]}, (S_t)_{[0,T]}\big)$, consisting of a bond whose evolution is described by S^0 and a stock whose evolution S is described by (1.19). They noticed that if there exists a positive real constant r such that $S^0_t = \exp(rt)$ the pay-off of a European call option on S can be replicated by continuous trading in S^0 and S. This model is often referred to as Black-Scholes or Black-Scholes-Merton model.

Since the Samuelson model also has some deficiencies there have been many efforts to build better models. For example, Cutland *et al.* [CuKoWi95] discuss the empirical evidence that suggests that long-range dependence should be accounted for when modelling stock price evolutions and present a fractional version of the Samuelson model.

Possibilities to define fractional versions (and therefore to account for long-range dependencies) of both the Bachelier and the Samuelson model are

$$S^0_t = 1\,,\;\; S_t = S_0 + \mu t + \sigma B^H_t\,,\; t \in [0,T]\,, \tag{1.20}$$

and

$$S^0_t = \exp(rt)\,,\;\; S_t = S_0 \exp\big(\{r+\mu\}t + \sigma B^H_t\big)\,,\; t \in [0,T]\,, \tag{1.21}$$

where S_0, μ, $\sigma \in \mathbb{R}_+$ and $r \in \mathbb{R}$ are constants and B^H denotes a fBM. The model described by (1.20) is called fractional Bachelier model and the model described by (1.21) is called fractional Samuelson model or, alternatively, fractional Black-Scholes model.

A significant deficiency of the above stated fractional models is that they admit arbitrage. As stated in Section 1.1.8 fBM is not a weak semimartingale for $H \in \big(0, \frac{1}{2}\big) \cup \big(\frac{1}{2}, 1\big)$. Particularly, it is not a semimartingale w.r.t. $\overline{\mathcal{F}}^H = \Big\{\overline{\mathcal{F}}^H_t,\ t \in [0,T]\Big\}$, where $\mathcal{F}^H = \{\mathcal{F}_t,\ t \in [0,T]\}$ denotes the natural filtration of $B^H = \big\{B^H_t,\ t \in [0,T]\big\}$ and $\overline{\mathcal{F}}^H$ is the smallest σ-algebra that contains $\mathcal{F}^H$ and satisfies the usual conditions[12]. It follows that the discounted stock price $\tilde{S} := S/S^0$ (with S^0 as numeraire) in the models (1.20) and (1.21) is not a semimartingale either. Therefore, it follows immediately from Theorem 7.2 of [DeSa94] that (1.20) and (1.21) admit a 'free lunch with vanishing risk' consisting of simple predictable integrands adapted to $\overline{\mathcal{F}}^H$.

[11]For many practical purposes the possibility of negative asset prices in the Bachelier model can be neglected as the maturity of derivatives on the corresponding asset are short and the underlying Gaussian distribution has light tails. However, in the long run the potential occurrence of negative prices is a drawback of the Bachelier model. For a comparison of the Bachelier and the Samuelson model see [ScTe08].

[12]The natural filtration of an fBM does not satisfy the usual conditions.

Remark 1.1.16. For a detailled discussion of arbitrage in fractional Brownian Motion models see Cheridito [Cher01b]. The author discusses fBM, models of the structure (1.20) and (1.21) and ways to regularize them, i.e., to make them arbitrage-free. Some heuristic arguments are given that indicate why for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ the behaviour of the function $\phi_H(x) := (x)_+^{H-\frac{1}{2}}$ used in the evolution kernel $k_H(.,.)$ of equation (1.8) (the Mandelbrot-van Ness representation of fBM) near zero is responsible for the existence of arbitrage in the fractional models (1.20) and (1.21) and how ϕ_H can be regularized to yield a process which can be used to construct an arbitrage-free stock price model (which still allows for taking long-range dependence into account). As an example the pricing of a European call option in a regularized fractional Samuelson model of type (1.21) is discussed.

1.1.10 Some Notes on the History of fBM

The history of fBM goes back to 1940 when Kolmogorov [Kolm40] was the first to study Gaussian processes in continuous time with stationary increments and the stochastic self-similarity property (see Definition 1.1.2) in a Hilbert space framework. It can be shown that such processes $X = \{X_t\,,\, t \in \mathbb{R}\}$ with the additional property of zero mean have a special correlation structure of the form

$$m_X^{(2)}(s,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (s,t) \mapsto \mathbf{E}(X_s X_t) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})\,, \tag{1.22}$$

where $H \in (0,1)$ (for a proof see Doukhan *et al.* [DoOpTa03, pp. 7-8]). Obviously, (1.22) is the second moment function of an fBM. Kolmogorov called such zero-mean Gaussion processes “Wiener Spirals” or “Wiener screw-lines”. Later, when papers by the British hydrologist H. E. Hurst and other authors, devoted to long-term storage capacity in reservoirs, were published, the parameter H got the name “Hurst Parameter”. In their pioneering work (from which the stochastic calculus for processes with the above mentioned structure originated) Mandelbrot and van Ness [MavN68] considered the intergal moving average representation of X via Brownian Motion on an infinite intervall (see Section 1.1.5) and called this process fractional Brownian Motion.
An intense wave of interest in fBM (and other long-memory processes) arose in the 1990s due to various applications in teletraffic, finance, climate and weather derivatives. The stochastic calculus for fBM was developed further continuously (which was necessary since fBM is neither a semimartingale, except for $H = 1/2$, nor a Markov process) mainly based on the “fractional integral” representation of fBM via BM (on finite and infinite intervals, see equations (1.8) and (1.16)). This, together with the Gaussian property and the Hölder continuity of the trajectories of fBM permits to develop an interesting and rich calculus (as part of the theory of long-memory processes).

Remark 1.1.17. For a detailed account of the historical development of fBM and an exhaustive list of references to related material see the preface of Mishura [Mish08].

1.2 Realized Power Variation

This section is mainly based on Barndorf-Nielson and Shephard [BaSh02] and Corcuera *et al.* [CoNuWo06].

1.2.1 Definition and Elementary Properties

In this section let $X = \{X_t\,,\, t \in \mathbb{R}_{+,0}\}$ be an arbitrary (real-valued) stochastic process.

For any $\delta \in \mathbb{R}_+$ and $t \in \mathbb{R}_{+,0}$ define

$$X_t^{(\delta)} := X_{\lfloor t/\delta \rfloor \delta}\,, \tag{1.23}$$

where $\lfloor a \rfloor$ for any $a \in \mathbb{R}$ denotes the largest integer less than or equal to a. The such defined process $X^{(\delta)} = \left\{X_t^{(\delta)}\,,\, t \in \mathbb{R}_{+,0}\right\}$ is a discrete approximation to X.

Definition 1.2.1. For any $p, \delta \in \mathbb{R}_+$ the *realized power variation of order* p or *realized* p*-tic variation (with mesh* δ*)* (r.p.v.) of $X = \{X_t\,,\, t \in \mathbb{R}_{+,0}\}$ is defined as

$$V_p^\delta(X)_t := \sum_{j=1}^{\lfloor t/\delta \rfloor} |X_{j\delta}^{(\delta)} - X_{(j-1)\delta}^{(\delta)}|^p = \sum_{j=1}^{\lfloor t/\delta \rfloor} |X_{j\delta} - X_{(j-1)\delta}|^p\,. \tag{1.24}$$

Remark 1.2.1. Definition 1.2.1 is based on Barndorf-Nielson and Shephard [BaSh02] in which the concept of r.p.v. of order $p \in \mathbb{R}_+$ was introduced as generalization of the realized quadratic variation of a random process. Although it could simply have been introduced by the second expression of (1.24) the given definition stresses that r.p.v. is based on the discrete approximation (1.23) of the underlying process.

Remark 1.2.2. The use of the term power variation is not consistent throughout the literature. Some authors refer to r.p.v. simply as power variation. Others denote with power variation the limit (in probability) of r.p.v. as the mesh of the partition tends to zero (where they consider all partitions of a given (finite) interval, not only equidistant ones as used in Definition 1.2.1), i.e., taking $\sum_i |X_{t_i} - X_{t_{i-1}}|^p$ as $\max_i |t_i - t_{i-1}| \to 0$. This two concepts are, obviously, closely related. For $p = 2$, r.p.v. is referred to as quadratic r.p.v. and the described limit concept is referred to as quadratic variation. For a process X it holds that

$$V_2^\delta(X)_t \xrightarrow{p} [X]_t\,,\; t \in \mathbb{R}_{+,0}\,,\; \text{as } \delta \to 0\,,$$

where $[X]_t$ denotes the quadratic variation process of X and the convergence is in probability. Note also that

$$V_2^\delta(X) = \left[X^{(\delta)}\right]\,,$$

i.e., the two processes coincide.

Remark 1.2.3. In recent years, in finance the concept of power variation (in the above described limit sense, where X_t denotes the log-price process), as an estimate for the integrated volatility, became popular as a measure for the change in the volatility, because stochastic volatility models play an important role in overcoming the problems of the Black-Scholes world, especially being able to fit skews and smiles (see, e.g., [Woer05]).

1.2.2 p-Variation of a real-valued Function

The concept of r.p.v. should not be confused with the similarly-named p-variation (or *strong* variation) of a function which will also play an important role in this thesis.

Definition 1.2.2. For a a real-valued function f on an interval $[a, b]$ the p-variation, $p \in \mathbb{R}_+$, is defined by

$$\mathrm{var}_p(f; [a, b]) := \sup_{\pi} \left(\sum_{i=1}^{n} |f(t_i) - f(t_{(i-1)}|^p \right)^{1/p} ,$$

where the supremum is taken over all partitions$\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$. If this expression is finite then f is said to have bounded p-variation on $[a, b]$.

Remark 1.2.4. The case of $p = 1$ in Definition 1.2.2 gives the usual definition of bounded variation.

Remark 1.2.5. The concept of p-variation applied to stochastic processes has been studied in probability literature by various authors, see, for example, the work of Lyons [Lyons94] and Mikosch and Norvaisa [MiNo00]. In the course of this thesis we will also consider stochastic processes with paths of finite p-variation.

Remark 1.2.6. As can be checked easily, α-Hölder continuity, $\alpha \in \mathbb{R}_+$, of a real-valued function f on a finite interval $[a, b]$ implies its finite $(1/\alpha)$-variation on $[a, b]$ (see Lemma A.3.2).

Chapter 2

Realized Power Variation of some Fractional Stochastic Integrals

This chapter is concerned with the asymptotic behaviour of the r.p.v of some stochastic processes of the form $\int_0^t u_s \, \mathrm{d}B_s^H$, i.e., fractional stochastic integrals[1] w.r.t. fBM. The discussion follows Corcuera *et al.* [CoNuWo06] and presents the main results therein.

2.1 The Setting

We consider fractional stochastic integrals of the form

$$\int_0^t u_s \, \mathrm{d}B_s^H, \; t \in [0,T], \tag{2.1}$$

where $T \in \mathbb{R}_+$ is fixed, $u = \{u_t \, , \, t \in [0,T]\}$ is a stochastic process with paths of finite q-variation, $0 < q < \frac{1}{1-H}$, $B^H = \left\{B_t^H \, , \, t \in [0,T]\right\}$ is an fBM with Hurst parameter $H \in (0,1)$ and the stochastic integral is a pathwise Riemann-Stieltjes integral.
Young [Young36] proved that it is sufficient for the existence of the Riemann-Stieltjes integral $\int_a^b f \, \mathrm{d}g$ that f and g have finite p-variation and finite q-variation, respectively, in the interval $[a,b]$ and $1/p + 1/q > 1$. Furthermore, the following inequality holds:

$$\left| \int_a^b f \, \mathrm{d}g - f(a)(g(b) - g(a)) \right| \leq c_{p,q} \operatorname{var}_p(f;[a,b]) \operatorname{var}_q(g;[a,b]), \tag{2.2}$$

where $c_{p,q} := \zeta(1/q + 1/p)$, with $\zeta(s) := \sum_{n \geq 1} n^{-s}$. Since the trajectories of B^H have finite $1/(H - \varepsilon)$-variation, $\varepsilon \in \mathbb{R}_+$, on any finite interval (see Lemma A.3.2 and Corollary A.3.1), Young's result ensures the existence of the pathwise integrals $\int_0^t u_s \, \mathrm{d}B_s^H$ provided the trajectories of the process $u = \{u_t \, , \, t \in [0,T]\}$ have finite q-variation on $[0,T]$ for some $q < 1/(1-H)$.
In the following we are interested in the asymptotic behaviour of a renormalized version of the r.p.v. of the above specified fractional stochastic integrals, i.e., the behaviour of the process $\xi^{(n)} = \left\{\xi_t^{(n)} \, , \, t \in [0,T]\right\}$, $n \in \mathbb{N}$, defined by ($p \in \mathbb{R}_+$)

$$\xi_t^{(n)} := n^{-1+pH} \, V_p^{1/n}(u)_t = n^{-1+pH} \sum_{i=1}^{\lfloor nt \rfloor} \left| \int_{(i-1)/n}^{1/n} u_s \, \mathrm{d}B_s^H \right|^p, \; t \in [0,T], \tag{2.3}$$

[1] In the probability literature stochastic integrals w.r.t. fBM are usually called fractional stochastic integrals.

as n tends to infinity. The additional renormalizing constant n^{-1+pH} in (2.3) is required to obtain non-trivial limit theorems (i.e., with other limits than 0 or ∞). The types of convergence used in the asymptotic analysis of (2.3) are uniform convergence in probability[2](denoted by u.c.p.) in the interval $[0,T]$ and convergence in law. For a function $f:[a,b]\to\mathbb{R}$ and $\alpha\in\mathbb{R}_{+,0}$ we denote by $\|f\|_\alpha$ the expression

$$\|f\|_\alpha := \sup_{a\le s<t\le b} \frac{|f(t)-f(s)|}{|t-s|^\alpha}\,. \tag{2.4}$$

2.2 A Law of Large Numbers for the Realized Power Variation

The following theorem is one of the main results in Corcuera et al. [CoNuWo06] and can be understood as a law of large numbers for the r.p.v. of the stochastic integrals (2.1). The essential tool for the proof of the theorem will be the Ergodic theorem (which itself can be interpreted as a law of large numbers for dependent random variables).

Theorem 2.2.1 (Theorem 1 of [CoNuWo06])**.** *Suppose that $u=\{u_t\,,\,t\in[0,T]\}$ is a stochastic process with paths of finite q-variation, where $0<q<\frac{1}{1-H}$, and $B^H=\{B^H_t\,,\,t\in[0,T]\}$ is an fBM with Hurst parameter $H\in(0,1)$. Define a stochastic process $Z=\{Z_t\,,\,t\in[0,T]\}$ by*

$$Z_t := \int_0^t u_s\,\mathrm{d}B^H_s\,,\ t\in[0,T]. \tag{2.5}$$

Then, for $p\in\mathbb{R}_+$,

$$\xi^{(n)}_t = n^{-1+pH}\,V^{1/n}_p(Z)_t \overset{u.c.p.}{\longrightarrow} c_p\int_0^t |u_s|^p\,\mathrm{d}s\,,$$

as $n\to\infty$, where

$$c_p := \boldsymbol{E}\big(|B^H_1|^p\big) = \frac{2^{p/2}\Gamma((p+1)/2)}{\Gamma(1/2)}.$$

Remark 2.2.1. For the equality in the definition of c_p see Lemma A.3.1.

Proof. We first consider the case $p\le 1$. For any $m,n\in\mathbb{N}$ with $m\ge n$ we can write

[2] A sequence of random variables $(X_n)_{n\in\mathbb{N}}$ on $(\Omega,\mathcal{F},\mathrm{P})$ is said to converge in probability to a random variable X on $(\Omega,\mathcal{F},\mathrm{P})$, $X_n\overset{p}{\to}X$, if $\lim_{n\to\infty}\mathrm{P}\left(|X_n-X|\ge\varepsilon\right)\to 0\ \forall\varepsilon\in\mathbb{R}_+$. A sequence of stochastic processes $(X_n)_{n\in\mathbb{N}}$, where each X_n is a stochastic process $X_n=\{X_{n,t}\,,\,t\in[a,b]\}$ with some common parameter space $[a,b]$, is said to converge uniformely in probability to a stochastic process $X=\{X_t\,,\,t\in[a,b]\}$ in $[a,b]$ if $\|X_{n,t}-X_t\|_\infty\overset{p}{\to}0$, where $\|.\|_\infty$ denotes the supremum norm on the interval $[a,b]$.

$$
\begin{aligned}
& m^{-1+pH}\, V_p^{1/m}(Z)_t - c_p \int_0^t |u_s|^p \,\mathrm{d}s \\
= \quad & m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left(\left| \int_{(j-1)/m}^{j/m} u_s \,\mathrm{d}B_s^H \right|^p - \left| u_{(j-1)/m} (B_{j/m}^H - B_{(j-1)/m}^H) \right|^p \right) \\
& + m^{-1+pH} \left(\sum_{j=1}^{\lfloor mt \rfloor} \left| u_{(j-1)/m} (B_{j/m}^H - B_{(j-1)/m}^H) \right|^p - \sum_{i=1}^{\lfloor nt \rfloor} \left| u_{(j-1)/n} \right|^p \sum_{j \in I_n(i)} \left| B_{j/m}^H - B_{(j-1)/m}^H \right|^p \right) \\
& + m^{-1+pH} \sum_{i=1}^{\lfloor nt \rfloor} \left| u_{(j-1)/n} \right|^p \sum_{j \in I_n(i)} \left| B_{j/m}^H - B_{(j-1)/m}^H \right|^p - c_p n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} \left| u_{(i-1)/n} \right|^p \\
& + c_p \left(n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} \left| u_{(i-1)/n} \right|^p - \int_0^t |u_s|^p \,\mathrm{d}s \right) \\
=: \quad & A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(n)} \,,
\end{aligned}
$$

where

$$
I_n(i) := \left\{ j \in \mathbb{N} \,\middle|\, \frac{j}{m} \in \left(\frac{i-1}{n}, \frac{i}{n} \right] \right\}, \qquad 1 \le i \le \lfloor nt \rfloor \,.
$$

We obtain

$$
\left\| m^{-1+pH}\, V_p^{1/m}(Z)_t - c_p \int_0^t |u_s|^p \,\mathrm{d}s \right\|_\infty \le ||A_t^{(m)}||_\infty + ||B_t^{(n,m)}||_\infty + ||C_t^{(n,m)}||_\infty + ||D_t^{(n)}||_\infty \,.
$$

For any fixed n the term $C_t^{(n,m)}$ converges in probability in zero, uniformly in t, as $m \to \infty$. In fact, by the triangle inequality,

$$
||C_t^{(n,m)}||_\infty \le \sum_{i=1}^{\lfloor nT \rfloor} \left| u_{(j-1)/n} \right|^p \left| m^{-1+pH} \sum_{j \in I_n(i)} \left| B_{j/m}^H - B_{(j-1)/m}^H \right|^p - c_p n^{-1} \right|
$$

and by the self-similarity of the fBM, the term

$$
\left| m^{-1+pH} \sum_{j \in I_n(i)} \left| B_{j/m}^H - B_{(j-1)/m}^H \right|^p - c_p n^{-1} \right|
$$

has the same distribution as

$$
\left| \frac{1}{m} \sum_{j \in I_n(i)} \left| B_j^H - B_{j-1}^H \right|^p - c_p n^{-1} \right| ,
$$

which by the Ergodic theorem converges to zero in L^1 (hence in probability) as $m \to \infty$ (see Lemma 1.1.2).

For the term $B^{(n,m)}$ we have the upper estimate

$$
\begin{aligned}
||B_t^{(n,m)}||_\infty \le\; & m^{-1+pH} \sum_{i=1}^{\lfloor nT \rfloor} \sum_{j \in I_n(i)} \big|\, |u_{(i-1)/n}|^p - |u_{(j-1)/m}|^p \big|\, |B^H_{j/m} - B^H_{(j-1)/m}|^p \\
& + \|\, |u|^p \,\|_\infty \sup_{0\le t\le T} m^{-1+pH} \sum_{mn^{-1}\lfloor nt \rfloor \le j \le mn^{-1}(\lfloor nt \rfloor + 1)} |B^H_{j/m} - B^H_{(j-1)/m}|^p \\
\le\; & m^{-1+pH} \sum_{i=1}^{\lfloor nT \rfloor} \sup_{s \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)} \big|\, |u_{(i-1)/n}|^p - |u_s|^p \big| \sum_{j \in I_n(i)} |B^H_{j/m} - B^H_{(j-1)/m}|^p \\
& + \sup_{0\le t\le T} \|\, |u|^p \,\|_\infty \; m^{-1+pH} \sum_{mn^{-1}\lfloor nt \rfloor \le j \le mn^{-1}(\lfloor nt \rfloor + 1)} |B^H_{j/m} - B^H_{(j-1)/m}|^p \,,
\end{aligned}
$$

where we denote

$$
\mathcal{I}_n(i) := \left(\frac{i-1}{n}, \frac{i}{n} \right] , \qquad 1 \le i \le \lfloor nt \rfloor \,.
$$

As $m \to \infty$, again by the Ergodic theorem, this converges in probability to

$$
E_n := \frac{c_p}{n} \left(\sum_{i=1}^{\lfloor nT \rfloor} \sup_{s \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)} \big|\, |u_{(i-1)/n}|^p - |u_s|^p \big| + \|\, |u|^p \,\|_\infty \right) .
$$

The term E_n, in turn, tends to zero almost surely (and therefore in probability) as $n \to \infty$. In fact, since the trajectories of the process $|u|^p$ are regulated (as they have finite q-variation[3]) they admit right and left limits at each point of the interval $[0, T]$. Hence, for any $\varepsilon \in \mathbb{R}_+$, there exists n_0 such that for all $n > n_0$ and $1 \le i \le [nT]$

$$
\sup_{s \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)} \big|\, |u_{(i-1)/n}|^p - |u_s|^p \big| < \varepsilon + \big|\, |u_{(i-1)/n}|^p - |u_{((i-1)/n)-}|^p \big| + \big|\, |u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p \big| ,
$$

where $u_{(.)-}$ and $u_{(.)+}$ denote the left and right limits, respectively. Also because $|u|^p$ has regulated trajectories, by an application of the Bolzano-Weierstrass theorem, the number of their jumps bigger than ε is finite. Therefore,

$$
\begin{aligned}
E_n \le & \Bigg(3T\varepsilon + \frac{1}{n} \sum_{|\, |u_{(i-1)/n}|^p - |u_{((i-1)/n)-}|^p | > \varepsilon} \big|\, |u_{(i-1)/n}|^p - |u_{((i-1)/n)-}|^p \big| \\
& + \frac{1}{n} \sum_{|\, |u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p | > \varepsilon} \big|\, |u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p \big| + \frac{\|\, |u|^p \,\|_\infty}{n} \Bigg) ,
\end{aligned}
$$

which implies

$$
\limsup_{n \to \infty} E_n \le 3 c_p T \varepsilon \,,
$$

and the convergence of E_n follows by letting ε tend to zero.

[3]If a function $f : [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{R}$, $a < b$, has bounded p-variation for some $p \in \mathbb{R}_+$ then it is regulated on $[a, b]$. That is, there exist the limits $f(t+) := \lim_{u \downarrow t} f(u)$ for each $t \in [a, b)$ and $f(t-) := \lim_{u \uparrow t} f(u)$ for each $t \in (a, b]$.

For the term $D_t^{(n)}$ we have $\lim_{n\to\infty} ||D_t^{(n)}||_\infty = 0$, which implies its convergence in probability to zero. In fact,

$$||D_t^{(n)}||_\infty \le c_p n^{-1} \sum_{i=1}^{\lfloor nT \rfloor} \sup_{s\in\mathcal{I}_n(i)} \big| |u_{(i-1)/n}|^p - |u_s|^p \big| + c_p \frac{\| \, |u|^p \, \|_\infty}{n} .$$

For the term $A_t^{(m)}$, and for $p \le 1$ (hence $||x|^p - |y|^p| \le |x-y|^p$, $\forall x, y \in \mathbb{R}$), we can write by Young's inequality (2.2)

$$\begin{aligned}
|A_t^{(m)}| &= m^{-1+pH} \left| \sum_{j=1}^{\lfloor mt \rfloor} \left(\left| \int_{(j-1)/m}^{j/m} u_s \, \mathrm{d}B_s^H \right|^p - \left| u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H) \right|^p \right) \right| \\
&\le m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left| \int_{(j-1)/m}^{j/m} u_s \, \mathrm{d}B_s^H - u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H) \right|^p \\
&\le c_{p^*,q} \underbrace{\sum_{j=1}^{\lfloor mt \rfloor} \left(\mathrm{var}_q(u; \mathcal{I}_m(j)) \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j)) \right)^p}_{=: F_m} \\
&= c_{p^*,q} F_m ,
\end{aligned}$$

where $p^* := 1/(H-\varepsilon), 0 < \varepsilon < H$. Fix $\delta \in \mathbb{R}_+$ and consider the decomposition

$$\begin{aligned}
F_m \le m^{-1+pH} & \sum_{j \,:\, \mathrm{var}_q(u;\mathcal{I}_m(j)) > \delta} \left(\mathrm{var}_q(u; \mathcal{I}_m(j)) \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j)) \right)^p \\
& + \delta^p m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left(\mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j)) \right)^p .
\end{aligned}$$

It holds that

$$\sum_{j=1}^{\lfloor mt \rfloor} (\mathrm{var}_q(u; \mathcal{I}_m(j)))^q \le (\mathrm{var}_q(u; [0,T]))^q < \infty ,$$

and, as a consequence, the number of indices j for which $\mathrm{var}_q(u; \mathcal{I}_m(j)) > \delta$ is bounded by $(\mathrm{var}_q(u; [0,T]))^q / \delta^q =: M$. Hence,

$$\begin{aligned}
F_m \le M m^{-1+pH} & \max_{1\le j \le \lfloor mT \rfloor} \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))^p (\mathrm{var}_q(u; [0,T]))^p \\
& + \delta^p m^{-1+pH} \sum_{j=1}^{\lfloor mT \rfloor} \left(\mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j)) \right)^p .
\end{aligned}$$

The first summand in the above estimate goes to zero when $m \to \infty$ if $\varepsilon < 1/p$ since

$$m^{-1+pH} \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))^p \le m^{-1+pH} ||B^H||_{H-\varepsilon}^p m^{-p(H-\varepsilon)} = m^{-1+\varepsilon p} ||B^H||_{H-\varepsilon}^p ,$$

where $||B^H||_{H-\varepsilon}$ denotes (see (2.4))

$$||B^H||_{H-\varepsilon} := \sup_{0\le s<t\le T} \frac{\left|B_t^H - B_s^H\right|}{|t-s|^{H-\varepsilon}},$$

which is finite (by the Kolmogorov-Chentsov continuity theorem and the compactness of $[0,T]$) and independent of m.

For the second summand we use the fact that it has the same law (by the self-similarity of fBM) as

$$\delta^p\, m^{-1} \sum_{j=1}^{\lfloor mT \rfloor} \left(\mathrm{var}_{1/(H-\varepsilon)}(B^H;[j-1,j])\right)^p$$

which, by the Ergodic theorem, converges almost surely and in L^1 to

$$\delta^p\, T\, \mathbf{E}\Big(\left(\mathrm{var}_{1/(H-\varepsilon)}(B^H;[0,1])\right)^p\Big) < \infty$$

as $m \to \infty$. In fact, the functional $\left(\mathrm{var}_{1/(H-\varepsilon)}(B^H;[0,1])\right)$ is a seminorm on the trajectories of the fBM which is finite almost surely. Hence, we have that

$$\mathbf{E}\Big(\left(\mathrm{var}_{1/(H-\varepsilon)}(B^H;[0,1])\right)^p\Big) < \infty$$

for any $p \in \mathbb{R}_+$ by Fernique's theorem (see Fernique [Fern75]), and we can indeed apply the Ergodic theorem. Finally, it suffices to let δ tend to zero.

For $p > 1$ we can proceed similarly (using Minkowski's inequality) by the following upper estimate:

$$\begin{aligned}
&\left|\left(m^{-1+pH}\, V_p^{1/m}(Z)_t\right)^{1/p} - \left(c_p \int_0^t |u_s|^p\, \mathrm{d}s\right)^{1/p}\right| \\
\le\;& m^{-1/p+H}\left(\sum_{j=1}^{\lfloor mt \rfloor}\left|\int_{(j-1)/m}^{j/m} u_s\, \mathrm{d}B_s^H - u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)\right|^p\right)^{1/p} \\
&+ m^{-1/p+H}\left(\sum_{i=1}^{\lfloor nt \rfloor}\sum_{j\in I_n(i)}\left|(u_{(j-1)/m} - u_{(i-1)/m})(B_{j/m}^H - B_{(j-1)/m}^H)\right|^p\right)^{1/p} \\
&+ \left| m^{-1/p+H}\left(\sum_{i=1}^{\lfloor nt \rfloor}\left|u_{(i-1)/n}\right|^p \sum_{j\in I_n(i)}\left|B_{j/m}^H - B_{(j-1)/m}^H\right|^p\right)^{1/p}\right. \\
&\left.- \left(c_p n^{-1}\sum_{i=1}^{\lfloor nt \rfloor}\left|u_{(i-1)/n}\right|^p\right)^{1/p}\right| \\
&+ c_p^{1/p}\left|\left(n^{-1}\sum_{i=1}^{\lfloor nt \rfloor}\left|u_{(i-1)/n}\right|^p\right)^{1/p} - \left(\int_0^t |u_s|^p\, \mathrm{d}s\right)^{1/p}\right|.
\end{aligned}$$

■

The previous theorem can be generalized in the following way.

Corollary 2.2.1 (Corollary 2 of [CoNuWo06]). *Assume the same conditions as in Theorem 2.2.1. Consider a stochastic process* $Y = \{Y_t\,,\, t \in [0,T]\}$ *such that*

$$n^{-1+pH} V_p^{1/n}(Y)_t \xrightarrow{u.c.p.} 0 \tag{2.6}$$

as $n \to \infty$. *Then, for* $p \in \mathbb{R}_+$,

$$n^{-1+pH}\, V_p^{1/n}(Z+Y)_t \xrightarrow{u.c.p.} c_p \int_0^t |u_s|^p \, \mathrm{d}s\,,$$

as $n \to \infty$.

Remark 2.2.2. The process Y can be interpreted as additional "noise" that is added to the fractional integral process Z. Under condition (2.6) this does not disturb the asymptotic behaviour of the r.p.v. of Z. Condition (2.6) is for instance satisfied if Y is a process whose trajectories are γ-Hölder for some $\gamma \in (H,1]$, i.e., a process which possesses slightly more regularity than fBM.

Proof. Again, we first consider the case $p \le 1$. By the triangle inequality we obtain

$$\begin{aligned}
&\left| n^{-1+pH}\, V_p^{1/n}(Z+Y)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \right| \\
\le\; & \left| n^{-1+pH}\, V_p^{1/n}(Z+Y)_t - n^{-1+pH}\, V_p^{1/n}(Z)_t \right| \\
& + \left| n^{-1+pH}\, V_p^{1/n}(Z)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \right| \\
\le\; & n^{-1+pH}\, V_p^{1/n}(Y)_t + \left| n^{-1+pH}\, V_p^{1/n}(Z)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \right| .
\end{aligned}$$

The first summand tends to zero by the assumption and the second by Theorem (2.2.1). For $p > 1$ the proof can be done similarly using Minkowski's inequality instead. ■

2.3 Central Limit Theorems for the Realized Power Variation

In this section we will analyze the asymptotic fluctuations of the r.p.v. of the intergals (2.1) around its limit, i.e., we will derive central limit theorems. For $H \in (0, \frac{3}{4}]$ these fluctuations, properly normalized, have Gaussian asymptotic distributions. For $H > \frac{3}{4}$ the problem is more complicated and only the special case where the integrand process u is constant will be considered. In this case the the limit of the fluctuations will be a quadratic functional of BM (Rosenblatt process).
In the beginning we have to introduce some notation.
For any $p \in \mathbb{R}_+$, we set

$$\delta_p := 2^p \left(\frac{1}{\sqrt{\pi}} \Gamma\left(p + \frac{1}{2}\right) - \frac{1}{\pi} \Gamma\left(\frac{p+1}{2}\right)^2 \right)$$

and

$$\nu_1^2 := \delta_p + 2 \sum_{j \ge 1} \left(\gamma_p(\rho_H(j)) - \gamma_p(0) \right)\,,$$

where $\Gamma(.)$ denotes the Gamma function, $\gamma_p(x)$ for any $x \in (-1, 1)$ is given by

$$\gamma_p(x) := (1-x^2)^{p+1/2} 2^p \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{\pi(2k)!} \Gamma\left(\frac{p+1}{2}+k\right)^2 ,$$

and

$$\rho_H(n) := \frac{1}{2}\left((n+1)^{2H} - (n-1)^{2H} - 2n^{2H}\right) .$$

Remark 2.3.1. As can be shown (see [CoNuWo06, Lemma 9]), for $(U\,,\,V) \sim N_2\left(0, \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix}\right)$ with $|\sigma| < 1$ it holds that

$$\mathbf{E}(|U|^p |V|^p) = \gamma_p(\sigma).$$

By $\xrightarrow{\mathcal{L}}$ we denote convergence in law and by $\mathcal{D}([0,T])$ the Skorohod space on $[0,T]$ (i.e., the space of all functions $f : [0,T] \to \mathbb{R}$ that are right-continuous with left limits), equipped with the Skorohod topology (see Bilingsley [Bill68] for a definition). $\mathcal{F}_t^H, t \in [0,T]$, denotes the σ-algebra generated by the random variables $\{B_s^H\,,\, s \in [0,t]\}$ and the null sets.

We will first consider a functional limit theorem for the r.p.v. of fBM (which can be interpreted as stochastic integral of the form (2.1) with $u_t = 1, t \in [0,T]$).

Theorem 2.3.1 (Theorem 3 of [CoNuWo06]). *Fix $p \in \mathbb{R}_+$ and assume $0 < H < \frac{3}{4}$. Then*

$$\left(B_t^H\,,\,\left(n^{-1/2+pH} V_p^{1/n}(B^H)_t - c_p t n^{1/2}\right)\right) \xrightarrow{\mathcal{L}} \left(B_t^H\,,\,\nu_1 W_t\right) , \tag{2.7}$$

as $n \to \infty$, where $W = \{W_t\,,\, t \in [0,T]\}$ is a BM independent of the process B^H (i.e., independent of $\mathcal{F}_T^H$), and the convergence is in the product space $\mathcal{D}([0,T])^2$ equipped with the Skorohod topology.

Proof. The proof will be done in two steps. Set

$$Z_t^{(n)} := n^{-1/2+pH} V_p^{1/n}(B^H)_t - c_p t n^{1/2} .$$

Step 1. We first show the convergence of the finite-dimensional distributions. Let $J_k = (a_k, b_k]$, $k = 1, \dots, N$, be pairwise disjoint intervals contained in $[0,T]$. Define the random vectors $B := (B_{b_1}^H - B_{a_1}^H, \dots, B_{b_N}^H - B_{a_N}^H)$ and $X^{(n)} := (X_1^{(n)}, \dots, X_N^{(n)})$, where

$$X_k^{(n)} := n^{-1/2+pH} \sum_{\lfloor na_k \rfloor < j \le \lfloor nb_k \rfloor} |B_{j/n}^H - B_{(j-1)/n}^H|^p - n^{1/2} c_p |J_k| ,$$

$k = 1, \dots, N$, and $|J_k| = b_k - a_k$. We claim that

$$(B\,,\,X^{(n)}) \xrightarrow{\mathcal{L}} (B\,,\,V) , \tag{2.8}$$

where B and V are independent and V is a Gaussian random vector with zero mean and independent components of variance $\nu_1^2 |J_k|$.
By the self-similarity of fBM, the sequence $\left(n^{pH}|B_{j/n}^H - B_{(j-1)/n}^H|^p - c_p\right)_{1\le j \le n}$ has the same law as $\left(|B_j^H - B_{j-1}^H|^p - c_p\right)_{1\le j \le n}$. Set $X_j := B_j^H - B_{j-1}^H$ and $H(x) := |x|^p - c_p$. Then $\{X_j\,,\, j \in \mathbb{N}\}$ is a stationary Gaussian sequence with zero mean, unit variance and $\mathbf{E}(X_j\, X_{j+n}) = \rho_H(n)$.

Thus, the convergence (2.8) is equivalent to the convergence in distribution of $(B^{(n)}\,,\, Y^{(n)})$ to $(B\,,\,V)$, where

$$B_k^{(n)} := n^{-H} \sum_{\lfloor na_k \rfloor < j \le \lfloor nb_k \rfloor} X_j\,, \qquad 1 \le k \le N,$$

and

$$Y_k^{(n)} := n^{-1/2} \sum_{\lfloor na_k \rfloor < j \le \lfloor nb_k \rfloor} H(X_j)\,, \qquad 1 \le k \le N.$$

For a proof of the convergence $(B^{(n)}\,,\, Y^{(n)}) \xrightarrow{\mathcal{L}} (B\,,\,V)$ I would like to refer, for example, to Proposition (10) in Corcuera *et al.* [CoNuWo06] where the authors use a direct argument based on a recent central limit theorem for stochastic integrals (see Nualart and Peccati [NuPe05]; Peccati and Tudor [PeTu]; Hu and Nualart [HuNu05]).

Remark 2.3.2. By taking into account that $H(x) = |x|^p - c_p$ has Hermite rank[4] 2, and that

$$\sum_{n=1}^{\infty} \rho_{_H}^2(n) < \infty$$

due to $\rho_{_H}(n) = O(n^{2H-2})$, the (one-dimensional) convergence of the sequence of vectors $Y^{(n)}$ to the vector V would also follow from Breuer and Major [BrMa83, Theorem 1] or Giraitis and Surgailis [GiSu85, Theorem 5].

Step 2. To establish the convergence in $\mathcal{D}([0,T])$ we have to show that the sequence of processes $Z^{(n)}$ is tight in this space. To do so, we calculate, for $s < t$,

$$\mathbf{E}\Big(|Z_t^{(n)} - Z_s^{(n)}|^4\Big) = n^{-2}\mathbf{E}\left(\left|\sum_{j=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} H(X_j)\right|^4\right).$$

By Taqqu [Taqqu77, Proposition 4.2] we know that for all $N \ge 1$

$$\frac{1}{N^2}\mathbf{E}\left(\left|\sum_{j=1}^{N} H(X_j)\right|^4\right) \le K \left(\sum_{u=0}^{\infty} \rho_{_H}^2(u)\right)^2.$$

As a consequence,

$$\sup_{n\in\mathbb{N}} \mathbf{E}\Big(|Z_t^{(n)} - Z_s^{(n)}|^4\Big) \le C|t-s|^2,$$

and by Billingsley [Bill68, Theorem 15.6] we obtain the desired tightness property.

■

Remark 2.3.3. Theorem 2.3.1 can easily be adopted to derive limit results for any constant integrand process of the form $u_t = c$, $t \in [0,T]$, for some fixed $c \in \mathbb{R}$.

Remark 2.3.4. The convergence established in Theorem 2.3.1 can also be expressed in terms of the concept of *stable convergence* (see Aldous and Eagleson [AlEa78] for a discussion). In fact, for any bounded random variable X measurable with respect to the σ-algebra $\mathcal{F}_T^H$ and for any continuous and bounded function ϕ on the skorohod space $\mathcal{D}([0,T])$ we have

$$\lim_{n\to\infty} \mathbf{E}\Big(X\,\phi(Z^{(n)})\Big) = \mathbf{E}(X)\,\mathbf{E}(\phi(W))\,.$$

[4] The Hermite rank of a function f is the index of the first non-zero coefficient in the expansion of f in Hermite polynomials.

If X is a continuous functional of $B^H = \left\{B_t^H\,,\, t\in[0,T]\right\}$ this convergence is an immediate consequence of Theorem 2.3.1. In the general case the convergence follows by an approximation argument.

As a consequence of Theorem 2.3.1 we can derive the following central limit theorem for the r.p.v. of the stochastic integrals studied in this chapter. Here an additional Hölder continuity condition on the trajectories and the measurability of the integrand process u are required.

Theorem 2.3.2 (Theorem 4 of [CoNuWo06]). *Fix $p\in\mathbb{R}_+$. Let $B^H = \left\{B_t^H\,,\, t\in[0,T]\right\}$ be an fBM with $H\in(0,\frac{3}{4})$. Suppose that $u=\{u_t\,,\, t\in[0,T]\}$ is a stochastic process measurable with respect to $\mathcal{F}_T^H$ and with Hölder continuous trajectories of order $a>1/(2(p\wedge 1))$. Then, for $Z=\{Z_t\,,\, t\in[0,T]\}$, defined by $Z_t := \int_0^t u_s\,\mathrm{d}B_s^H$, it holds that*

$$\left(B_t^H\,,\, n^{-1/2+pH}V_p^{1/n}(Z)_t - c_p\sqrt{n}\int_0^t |u_s|^p\,\mathrm{d}s\right) \xrightarrow{\mathcal{L}} \left(B_t^H\,,\, \nu_1\int_0^t |u_s|^p\,\mathrm{d}W_s\right),$$

as $n\to\infty$, where $W=\{W_t\,,\, t\in[0,T]\}$ is a BM independent of $\mathcal{F}_T^H$, and the convergence is in $\mathcal{D}([0,T])^2$.

Proof. The proof will be based on Theorem 2.3.1. For any $m\geq n$ and with the same notation as in Theorem 2.3 we can write

$$m^{-1/2+pH}V_p^{1/m}(Z)_t - c_p\sqrt{m}\int_0^t |u_s|^p\,\mathrm{d}s = A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(m)}\,,$$

where

$$A_t^{(m)} = m^{-1/2+pH}\sum_{j=1}^{\lfloor mt\rfloor}\left(\left|\int_{(j-1)/m}^{j/m} u_s\,\mathrm{d}B_s^H\right|^p - \left|u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)\right|^p\right),$$

$$\begin{aligned}
B_t^{(n,m)} &= m^{-1/2+pH}\sum_{j=1}^{\lfloor mt\rfloor}\left|u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)\right|^p - m^{-1/2}c_p\sum_{j=1}^{\lfloor mt\rfloor}|u_{(j-1)/m}|^p \\
&\quad -\sum_{i=1}^{\lfloor nt\rfloor}|u_{(i-1)/n}|^p\sum_{j\in I_n(i)} m^{-1/2+pH}|B_{j/m}^H - B_{(j-1)/m}^H|^p + \frac{\sqrt{m}}{n}c_p\sum_{i=1}^{\lfloor nt\rfloor}|u_{(i-1)/n}|^p, \\
C^{(n,m)} &= \sum_{i=1}^{\lfloor nt\rfloor}|u_{(i-1)/n}|^p\sum_{j\in I_n(i)} m^{-1/2+pH}|B_{j/m}^H - B_{(j-1)/m}^H|^p - \frac{\sqrt{m}}{n}c_p\sum_{i=1}^{\lfloor nt\rfloor}|u_{(i-1)/n}|^p
\end{aligned}$$

and

$$D^{(m)} = m^{-1/2}c_p\sum_{j=1}^{\lfloor mt\rfloor}|u_{(j-1)/m}|^p - \sqrt{m}c_p\int_0^t |u_s|\,\mathrm{d}s\,.$$

First we show that $||D^{(m)}||_\infty \xrightarrow{a.s.} 0$ as $m\to\infty$, where $\xrightarrow{a.s.}$ denotes convergence almost surely. Using the Hölder continuity of u we can write

$$\begin{aligned}
|D_t^{(m)}| &\le c_p m^{-1/2} \sum_{j=1}^{\lfloor mt \rfloor} \big|\, |u_{(j-1)/m}|^p - |u_{\tilde{t}_{j-1}^m}|^p \,\big| + \frac{c_p}{\sqrt{m}} \, \|\, |u|^p \,\|_\infty \\
&\le c_p m^{-1/2} (p \vee 1) \, \|u\|_\infty^{(p-1)_+} \sum_{j=1}^{\lfloor mt \rfloor} |u_{(j-1)/m} - u_{\tilde{t}_{j-1}^m}|^{p \wedge 1} + \frac{c_p}{\sqrt{m}} \, \|\, |u|^p \,\|_\infty \\
&\le c_p T (p \vee 1) \, \|u\|_a^{p \wedge 1} \, \|u\|_\infty^{(p-1)_+} \, m^{-1/2 - a(p \wedge 1)} + \frac{c_p}{\sqrt{m}} \, \|\, |u|^p \,\|_\infty
\end{aligned}$$

where $\tilde{t}_{j-1}^m \in \mathcal{I}_m(j)$ and $(x)_+$ denotes the positive part of any $x \in \mathbb{R}$. Hence $\left\| D^{(m)} \right\|_\infty \xrightarrow{a.s.} 0$ as $m \to \infty$ because $a(p \wedge 1) > \frac{1}{2}$.

Let us now consider the term $C_t^{(n,m)}$. Set

$$Y_{n,m}^i := \sum_{j \in I_n(i)} m^{-1/2 + pH} |B_{j/m}^H - B_{(j-1)/m}^H|^p - \frac{\sqrt{m}}{n} c_p \, .$$

By Theorem 2.3.1 and by taking into account that it implies the stable convergence of $\left\{ Y_{n,m}^1, Y_{n,m}^2, \ldots, Y_{n,m}^n \right\}_{m \ge 1}$ for any n (see Remark 2.3.4 and Aldous and Eagleson [AlEa78, Proposition 1], we have that for any $\mathcal{F}_T^H$-measurable random variable $|u_{(i-1)/n}|^p$, as $m \to \infty$,

$$\left(|u_{(i-1)/n}|^p \, , \, Y_{n,m}^i \right)_{a \le i \le \lfloor nt \rfloor} \xrightarrow{\mathcal{L}} \left(|u_{(i-1)/n}|^p \, , \, \nu_1 (W_{i/n} - W_{(i-1)/n}) \right)_{1 \le i \le \lfloor nt \rfloor} \, ,$$

where W is a BM independent of $\mathcal{F}_T^H$. Hence,

$$C_t^{(n,m)} \xrightarrow{\mathcal{L}} \nu_1 \sum_{i=1}^{\lfloor nt \rfloor} |u_{(i-1)/n}|^p (W_{i/n} - W_{(i-1)/n})$$

as $m \to \infty$, and this convergence is also stable (see Aldous and Eagleson [AlEa78, Theorem 1']). On the other hand,

$$\sum_{i=1}^{\lfloor nt \rfloor} |u_{(i-1)/n}|^p (W_{i/n} - W_{(i-1)/n}) \xrightarrow{u.c.p.} \int_0^t |u_s|^p \, \mathrm{d}W_s \, ,$$

as $n \to \infty$. This implies, by first letting m and then n to infinity, that

$$C_t^{(n,m)} \xrightarrow{\mathcal{L}} \nu_1 \int_0^t |u_s|^p \, \mathrm{d}W_s$$

in $\mathcal{D}([0,T])$.

We now show that $\left\| B^{(n,m)} \right\|_\infty \xrightarrow{p} 0$ as $n, m \to \infty$. We may rewrite $B^{(n,m)}$ as

$$\begin{aligned} B^{(n,m)} &= \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j\in I_n(i)} |u_{(j-1)/m}|^p \left(m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p \right) \\ &\quad - \sum_{i=1}^{\lfloor nt \rfloor} |u_{(i-1)/n}|^p \left(\sum_{j\in I_n(i)} m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - \frac{\sqrt{m}}{n} c_p \right) \\ &\quad + \sum_{j \geq \frac{m}{n}\lfloor nt \rfloor}^{\lfloor mt \rfloor} |u_{(j-1)/m}|^p \left(m^{-1/2+pH} \left| B^H_{j/m} - B^H_{(j-1)/m} \right|^p - m^{-1/2} c_p \right). \end{aligned}$$

As a consequence, by the mean value theorem, we get

$$\begin{aligned} |B^{(n,m)}| &\leq \left| \sum_{i=1}^{\lfloor nt \rfloor} |u_{\tilde{s}}|^p \sum_{j\in I_n(i)} \left(m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p \right) \right. \\ &\quad \left. - \sum_{i=1}^{\lfloor nt \rfloor} |u_{(i-1)/n}|^p \left(\sum_{j\in I_n(i)} m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - \frac{\sqrt{m}}{n} c_p \right) \right| \\ &\quad + \sup_{0\leq t\leq T} \sum_{\frac{m}{n}\lfloor nt \rfloor \leq j \leq \lfloor mt \rfloor}^{\lfloor mt \rfloor} \left| |u_{(j-1)/m}|^p \left(m^{-1/2+pH} \left| B^H_{j/m} - B^H_{(j-1)/m} \right|^p - m^{-1/2} c_p \right) \right| \\ &\leq \sum_{i=1}^{\lfloor nT \rfloor} \sup_{s\in \mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \left| |u_s|^p - |u_{(i-1)/n}|^p \right| \, |Y^i_{n,m}| + \frac{c_p}{\sqrt{m}} \| \, |u|^p \, \|_\infty \\ &\quad + \sup_{0\leq t\leq T} \left| \sum_{\frac{m}{n}\lfloor nt \rfloor \leq j \leq \lfloor mt \rfloor}^{\lfloor mt \rfloor} |u_{(j-1)/m}|^p \left(m^{-1/2+pH} \left| B^H_{j/m} - B^H_{(j-1)/m} \right|^p - m^{-1/2} c_p \right) \right| \end{aligned}$$

where $\tilde{s}(\omega) \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)$. Then, by Theorem 2.3.1, for any $\varepsilon \in \mathbb{R}_+$ we obtain

$$\begin{aligned} \limsup_{m\to\infty} \mathrm{P}\left(\left\| B^{(n,m)} \right\|_\infty > \varepsilon \right) &\leq \mathrm{P}\left(\nu_1 \sum_{i=1}^{\lfloor nT \rfloor} \sup_{s\in \mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \left| |u_s|^p - |u_{(i-1)/n}|^p \right| \, |W_{i/n} - W_{(i-1)/n}| \right. \\ &\qquad \left. + \nu_1 \| \, |u|^p \, \|_\infty \frac{1}{n} \sup_{0\leq t\leq T} |W_t - W_{\lfloor nt \rfloor/n}| > \varepsilon \right). \end{aligned}$$

The Hölder continuity of the trajectories of u and the condition $a(p \wedge 1) > \frac{1}{2}$ imply

$$\begin{aligned} &\sum_{i=1}^{\lfloor nT \rfloor} \sup_{s\in \mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \left| |u_s|^p - |u_{(i-1)/n}|^p \right| \, |W_{i/n} - W_{(i-1)/n}| \\ &\quad \leq (p \vee 1)\, T \, \|u\|_a^{p\wedge 1} \, \|u\|_\infty^{(p-1)_+} \, 2^{a(p\wedge 1)} n^{-a(p\wedge 1)+1/2-\varepsilon}, \end{aligned}$$

which converges to zero as $n \to \infty$. Moreover

$$\frac{1}{n} \sup_{0\leq t\leq T} |W_t - W_{\lfloor nt \rfloor/n}| \xrightarrow{a.s.} 0$$

as $n \to \infty$, and we deduce the desired result.

Finally, we have to show that $\left\| A^{(m)} \right\|_{\infty} \xrightarrow{p} 0$ as $m \to \infty$. From

$$\begin{aligned}
|A_t^{(m)}| \leq & m^{-1/2+pH}(p \vee 1) 2^{(p-2)_+} \sum_{j=1}^{\lfloor mt \rfloor} \left| u_{(j-1)/m}(B^H_{j/m} - B^H_{(j-1)/m}) \right|^{(p-1)_+} \\
& \times \left| \int_{(j-1)/m}^{j/m} u_s \, \mathrm{d}B_s^H - u_{(j-1)/m}(B^H_{j/m} - B^H_{(j-1)/m}) \right|^{p \wedge 1} \\
& m^{-1/2+pH}(p \vee 1) 2^{(p-2)_+} \sum_{j=1}^{\lfloor mt \rfloor} \left| \int_{(j-1)/m}^{j/m} u_s \, \mathrm{d}B_s^H - u_{(j-1)/m}(B^H_{j/m} - B^H_{(j-1)/m}) \right|^{p}
\end{aligned}$$

and by using Young's inequality (2.2), as in Theorem 2.2.1, we get

$$\begin{aligned}
|A_t^{(m)}| \leq & (p \vee 1) 2^{(p-2)_+} c^p_{1/(H-\varepsilon),1/a} \left\| B^H \right\|_{H-\varepsilon}^{(p-1)_+} \|u\|_{\infty}^{(p-1)_+} m^{-1/2+pH-(H-\varepsilon)(p-1)_+} \\
& \times \sum_{j=1}^{\lfloor mT \rfloor} \left(\mathrm{var}_{1/a}(u; \mathcal{I}_m(j)) \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j)) \right)^{p \wedge 1} \\
& + (p \vee 1) 2^{(p-2)_+} c^p_{1/(H-\varepsilon),1/a} m^{-1/2+pH} \\
& \times \sum_{j=1}^{\lfloor mT \rfloor} \left(\mathrm{var}_{1/a}(u; \mathcal{I}_m(j)) \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j)) \right)^{p} \\
\leq & (p \vee 1) 2^{(p-2)_+} c^p_{1/(H-\varepsilon),1/a} T \left\| B^H \right\|_{H-\varepsilon}^{p} \|u\|_a^{p \wedge 1} \|u\|_{\infty}^{(p-1)_+} m^{1/2 - a(p\wedge 1) + p\varepsilon} \\
& + (p \vee 1) 2^{(p-2)_+} c^p_{1/(H-\varepsilon),1/a} T \left\| B^H \right\|_{H-\varepsilon}^{p} \|u\|_a^{p} m^{1/2-ap+p\varepsilon} ,
\end{aligned}$$

which converges to zero as $m \to \infty$, provided $\varepsilon < p^{-1}(a(p \wedge 1) - \frac{1}{2})$. This completes the proof.

■

Analogously to Corollary 2.2.1 the previous theorem can be generalized by adding an additional noise process satisfying a regularity condition to the integral process Z.

Corollary 2.3.1 (Corollary 5 of [CoNuWo06]). *Assume the same conditions as in Theorem 2.3.2. Consider a stochastic process $Y = \{Y_t \, , \, t \in [0,T]\}$ such that*

$$n^{-1/2+pH} V_p^{1/n}(Y)_t \xrightarrow{p} 0 \, ,$$

as $n \to \infty$. Then

$$\left(B_t^H \, , \, n^{-1/2+pH} V_p^{1/n}(Z+Y)_t - c_p \sqrt{n} \int_0^t |u_s|^p \, \mathrm{d}s \right) \xrightarrow{\mathcal{L}} \left(B_t^H \, , \, \nu_1 \int_0^t |u_s|^p \, \mathrm{d}W_s \right) ,$$

as $n \to \infty$, where $W = \{W_t \, , \, t \in [0,T]\}$ is a BM independent of $\mathcal{F}_T^H$, and the convergence is in $\mathcal{D}([0,T])^2$.

For $H = \frac{3}{4}$ the fluctuations of the r.p.v. still converge to a Gaussian process, but with a different normalization. First we will again consider a functional limit theorem for the r.p.v. of fBM.

Theorem 2.3.3 (Theorem 6 of [CoNuWo06]). *Suppose that $H = \frac{3}{4}$. Then*

$$\left(B_t^H\,,\,(\log n)^{-1/2}\left(n^{-1/2+pH}V_p^{1/n}(B^H)_t - c_p t\sqrt{n}\right)\right) \xrightarrow{\mathcal{L}} \left(B_t^H\,,\,\nu_2 W_t\right)\,,$$

as $n \to \infty$, where $W = \{W_t\,,\,t \in [0,T]\}$ is a BM independent of $\mathcal{F}_T^H$ and ν_2 is given by

$$\nu_2 := \lim_{n\to\infty} \frac{2}{\log n} \sum_{j=1}^{n} \frac{n-j}{n} \gamma_p(\rho_H(j))\,. \tag{2.9}$$

Proof. In this case we have

$$\sum_{j=1}^{n} \rho_H^2(j) \sim c\sum_{j=1}^{n} j^{-1} \sim c\log n\,.$$

As a consequence, we can apply the same arguments as in the proof of Theorem 2.3.1. For example, the convergence of the finite-dimensional distributions of the process $(\log n)^{-1/2} Z_t^{(n)}$ would follow from Breuer and Major [BrMa83, Theorem 1']. ∎

For the fluctuations of the r.p.v. of stochastic integrals (2.1) with $u_t = 1$, $t \in [0,T]$, the following (one-dimensional) convergence in law holds true in the case $H = \frac{3}{4}$.

Theorem 2.3.4 (Theorem 7 of [CoNuWo06]). *Suppose that $B^H = \{B_t^H\,,\,t \in [0,T]\}$ is an fBM with $H = \frac{3}{4}$ and $u = \{u_t\,,\,t \in [0,T]\}$ is a stochastic process measurable with respect to $\mathcal{F}_T^H$ and with Hölder continuous trajectories of order $a > 1/(2(p \wedge 1))$. Consider a stochastic process $Y = \{Y_t\,,\,t \in [0,T]\}$ such that*

$$n^{-1/2+pH}V_p^{1/n}(Y)_t \xrightarrow{p} 0\,,$$

as $n \to \infty$. Then, for $Z = \{Z_t\,,\,t \in [0,T]\}$, defined by $Z_t := \int_0^t u_s\,\mathrm{d}B_s^H$, it holds that

$$(\log n)^{-1/2}\left(n^{-1/2+pH}V_p^{1/n}(Z+Y)_t - c_p\sqrt{n}\int_0^t |u_s|^p\,\mathrm{d}s\right) \xrightarrow{\mathcal{L}} \nu_2 \int_0^t |u_s|\,\mathrm{d}W_s\,,$$

as $n \to \infty$, where $W = \{W_t\,,\,t \in [0,T]\}$ is a BM independent of $\mathcal{F}_T^H$ and ν_2 is given by (2.9).

In the case $H > \frac{3}{4}$ the fluctuations of the r.p.v. converge to a process which is called the Rosenblatt process. In fact, we have the following result.

Theorem 2.3.5 (Theorem 8 of [CoNuWo06]). *Fix $p > 0$ and assume that $\frac{3}{4} < H < 1$. Then*

$$n^{2-2H}(n^{-1+pH}V_p^{1/n}(B^H)_t - c_p t) \xrightarrow{\mathcal{L}} Z_t\,,$$

where

$$\begin{aligned} Z_t := &\frac{1}{\Gamma(2-2H)\cos((1-H)\pi)} d_p \\ &\times \int_0^\infty \int_0^\infty \frac{e^{i(x_1+x_2)t}-1}{i(x_1+x_2)} |x_1|^{1/2-H}|x_2|^{1/2-H}\,\mathrm{d}W_{x_1}\,\mathrm{d}W_{x_2}\,, \end{aligned}$$

is the Rosenblatt process, $W = \{W_t\,,\,t \in [0,T]\}$ is a standard BM,

$$d_p := \left(\boldsymbol{E}(|B_1^H|^{2+p}) - \boldsymbol{E}(|B_1^H|^p)\right)\,,$$

and the convergence is in $\mathcal{D}([0,T])$.

Proof. Since $\rho_H(n) = O(n^{2h-2})$, $\frac{3}{4} < H < 1$ and $\left(|B_j^H - B_{j-1}^H|^p - c_p\right)_{1\le j\le n}$ is an L^2-functional, with Hermite rank 2, of a stationary zero mean Gaussian sequence, we can apply Taqqu [Taqqu79, Theorem 5.6]. ∎

Chapter 3

Convergence of Some Functionals of Fractional Stochastic Integrals

This chapter is concerned with the asymptotic behaviour of functionals of fractional stochastic integrals of the form (2.1). The r.p.v. discussed in the last chapter is contained in the considered class of functionals, i.e., we will generalize the results of Chapter 2 by considering a greater class of integrand processes. The discussion follows Corcuera *et al.* [CoNuWo08] and presents the main results therein.

Remark 3.0.5. The theorems in this chapter are given without proof (or just with a short sketch of the proof) as they are extensive and (structurally) quite analogous to the corresponding proofs given in Chapter 2. The presented material therefore has to be understood simply as an outlook on the generalization of the previous results.

3.1 The Setting

Again we consider fractional stochastic integrals of the form $\int_0^t u_s \, \mathrm{d}B_s^H$, $t \in [0,T]$, where $T \in \mathbb{R}_+$ is fixed, $u = \{u_t \,,\, t \in [0,T]\}$ is a stochastic process with paths of finite q-variation, $0 < q < \frac{1}{1-H}$, $B^H = \left\{B_t^H \,,\, t \in [0,T]\right\}$ is an fBM with Hurst parameter $H \in (0,1)$ and the stochastic integral is a pathwise Riemann-Stieltjes integral. Hence, the setting and the notation of Section 2.1 apply throughout this chapter as well. Additionally, we have to introduce some new notation.
For any real-valued stochastic process $X = \{X_t \,,\, t \in \mathbb{R}_{+,0}\}$ and for each $n \in \mathbb{N}$ we denote by

$$X_t^{(n)} := X_{\frac{i-1}{n}} + n\left(t - \frac{i-1}{n}\right)\left(X_{\frac{i}{n}} - X_{\frac{(i-1)}{n}}\right) \,, \quad \frac{i-1}{n} \leq t < \frac{i}{n} \,, \tag{3.1}$$

the broken line approximation of X of order n.
Instead of the r.p.v. we are now interested in the asymptotic behaviour of functionals of the form

$$\begin{aligned} F_{g,h}^{(n)}(Z)_t :=& \int_0^{\frac{\lfloor nt \rfloor}{n}} h\left(Z_s^{(n)}\right) g\left(\dot{Z}_s^{(n)} n^{H-1}\right) \mathrm{d}s \\ =& \sum_{i=1}^{\lfloor nt \rfloor} g\left(n^H \int_{\frac{i-1}{n}}^{\frac{i}{n}} u_\nu \, \mathrm{d}B_\nu^H\right) \int_{\frac{i-1}{n}}^{\frac{i}{n}} h\left(Z_s^{(n)}\right) \mathrm{d}s \,, \end{aligned} \tag{3.2}$$

where $Z = \{Z_t\,,\, t \in [0,T]\}$ is given by $Z_t := \int_0^t u_s \,\mathrm{d}B_s^H$, $Z^{(n)}$ is the broken line approximation (3.1) of Z and g, h are continuous functions. $\dot{Z}_s^{(n)}$ denotes the (pathwise) right-sided derivative of the broken line approximation $Z_s^{(n)}$ with respect to s, i.e.,

$$\dot{Z}_s^{(n)} = n\left(Z_{\frac{i}{n}} - Z_{\frac{(i-1)}{n}}\right), \quad \frac{i-1}{n} \le s < \frac{i}{n}\,.$$

Remark 3.1.1. The right-sided derivative of $Z_s^{(n)}$ is used in (3.2) to avoid gaps in the definition, since $Z_s^{(n)}$ is not differentiable at $\frac{k}{n}, k \in \mathbb{N}$. Equivalently, the left-sided derivative could be used since we consider (path-wise) Riemann integrals which are invariant to changes of the integrand at finitely many points.

Remark 3.1.2. In the particular case $g(x) = |x|^p$, $p \in \mathbb{R}_+$, and $h \equiv 1$, $F_{g,h}^{(n)}(Z)_t$ is the (properly normalized) r.p.v. of order p considered in Chapter 2. In fact,

$$\begin{aligned}
\int_0^{\frac{\lfloor nt \rfloor}{n}} \left|\dot{Z}_s^{(n)} n^{H-1}\right|^p \mathrm{d}s =& \sum_{i=1}^{\lfloor nt \rfloor} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \left|n\left(Z_{\frac{i}{n}} - Z_{\frac{(i-1)}{n}}\right) n^{H-1}\right|^p \mathrm{d}s = \\
=& \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} \left|\left(Z_{\frac{i}{n}} - Z_{\frac{(i-1)}{n}}\right) n^{H}\right|^p = \\
=& n^{-1+pH} \sum_{i=1}^{\lfloor nt \rfloor} \left|\left(Z_{\frac{i}{n}} - Z_{\frac{(i-1)}{n}}\right)\right|^p\,,
\end{aligned}$$

and the results in this chapter have already been established in Chapter 2 in this case.

Remark 3.1.3. Functionals of the form (3.2) have been studied by León and Ludeña [LeLu04] assuming that Z is the solution of a stochastic differential equation driven by an fBM with $H > 1/2$ of the form:

$$Z_t = z_o + \int_0^t b(Z_s)\,\mathrm{d}s + \int_0^t \sigma(Z_s)\,\mathrm{d}B^H\,,$$

under certain restrictions on b and σ.

In the following let $c_g(z) = \mathbf{E}^W(g(zW))$ for any $z \in \mathbb{R}$, where W is a $\mathrm{N}(0,1)$ random variable and $\mathbf{E}^W(.)$ denotes the expectation w.r.t. W.

3.2 A Functional Law of Large Numbers

Let us impose the following condition on the function g:

(H) *There exist constants* $\alpha \in (0,1]$, $a, b \in \mathbb{R}_{+,0}$ *and* $p \in [0,2)$ *such that for all* $x < y$ *we have*

$$|g(y) - g(x)| \le C(\xi)|y-x|^\alpha,$$

where $\xi \in [x,y]$ *and the function* C *satisfies* $0 \le C(u) \le ae^{b|u|^p}$.

Then we have the following result, which can be interpreted as generalization of Theorem 2.2.1 (and again can be understood as a law of large numbers).

Theorem 3.2.1 (Theorem 1 of [CoNuWo08]). *Assume (H), suppose that $u = \{u_t\,,\, t \in [0,T]\}$ is a stochastic process with paths of finite q-variation, $0 < q < \frac{1}{1-H}$, and $B^H = \{B^H_t\,,\, t \in [0,T]\}$ is an fBM with Hurst parameter $H \in (0,1)$. Define a stochastic process $Z = \{Z_t\,,\, t \in [0,T]\}$ by*

$$Z_t := \int_0^t u_s \,\mathrm{d}B^H_s\,,\ t \in [0,T].$$

Then,

$$F^{(n)}_{g,h}(Z)_t \xrightarrow{u.c.p.} \int_0^t h(Z_s)c_g(u_s)\,\mathrm{d}s$$

as $n \to \infty$.

Remark 3.2.1. The proof of Theorem 3.2.1 follows the proof of Theorem 2.2.1, i.e., using an upper estimate of the form

$$\left|F^{(n)}_{g,h}(Z)_t - \int_0^t h(Z_s)c_g(u_s)\right| \le A^{(m)}_t + B^{(n,m)}_t + C^{(n,m)}_t + D^{(n)}_t$$

and showing that the terms on the right side converge to zero in probability, uniformly in t. Again, the Ergodic theorem (in Banach spaces) plays a central role. For details see [CoNuWo08, Theorem 1].

Remark 3.2.2. For $g(x) = |x|^p$, $p \in \mathbb{R}_+$, and $h \equiv 1$ (see Remark 3.1.2) we get indeed the result of Theorem 2.2.1. In this case we have

$$\int_0^t h(Z_s)c_g(u_s) = \int_0^t \mathbf{E}^W(|u_s W|^p)\,\mathrm{d}s = \underbrace{\mathbf{E}^W(|W|^p)}_{=\mathbf{E}(|B^H_1|^p)}\int_0^t |u_s|^p\,\mathrm{d}s\,,$$

since W is independent of u.

3.3 Functional Central Limit Theorems

In this section we take a look at the asymptotic fluctuations of the functionals (3.2) around their limit accorting to Theorem 3.2.1.
We first consider the case where the process u takes a constant value $z \in \mathbb{R}$, i.e., $Z_t = zB^H_t$, and $h \equiv 1$. In this case

$$F^{(n)}_{g,1}(zB^H)_t = \sum_{i=1}^{\lfloor nt\rfloor} g\left(zn^H\left(B^H_{\frac{i}{n}} - B^H_{\frac{i-1}{n}}\right)\right)$$

and it suffices to consider the process

$$\sum_{i=1}^{\lfloor nt\rfloor} g\left(z\left(B^H_i - B^H_{i-1}\right)\right)$$

which has the same distribution, due to the self-similarity of fBM.
Let $\nu(x), x \in \mathbb{R}$, be defined by

$$\nu^2(x) := \lim_{n\to\infty} \mathbf{V}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} g\left(x\left(B^H_i - B^H_{i-1}\right)\right)\right).$$

We have the following theorem.

Theorem 3.3.1 (Theorem 2 of [CoNuWo08]). *Fix $T \in \mathbb{R}_+$. Assume that $H \in \left(0, \frac{3}{4}\right)$, g is even, i.e., $g(x) = g(-x), x \in \mathbb{R}$, and satisfies condition (H). Then*

$$\left(B_t^H\,,\, \sqrt{n}\left(F_{g,1}^{(n)}(zB^H)_t - c_g(z)\,t\right)\right) \xrightarrow{\mathcal{L}} \left(B_t^H\,,\, \nu(z)W_t\right),$$

as $n \to \infty$, where $W = \{W_t\,,\, t \in \mathbb{R}_{+,0}\}$ is a BM independent of the process B^H, and the convergence is in the product space $\mathcal{D}([0,T])^2$ equipped with the Skorohod topology.

Remark 3.3.1. The proof goes along the same lines as in Theorem 2.3.1. The first step is to show the convergence of the finite-dimensional distributions. By the self-similarity property of fBM it suffices to prove the convergence of $\left(B^{(n)}\,,\, Y^{(n)}\right) \xrightarrow{\mathcal{L}} \left(B^H\,,\, V\right)$, where the components of the first vector are

$$B_k^{(n)} := n^{-H} \sum_{\lfloor na_k \rfloor < j \leq \lfloor nb_k \rfloor} X_j\,, \qquad 1 \leq k \leq N,$$

and

$$Y_k^{(n)} := n^{-1/2} \sum_{\lfloor na_k \rfloor < j \leq \lfloor nb_k \rfloor} H(X_j, z)\,, \qquad 1 \leq k \leq N,$$

where $X_j := B_j^H - B_{j-1}^H$ and $H(x,y)$ is defined by $H(x,y) := g(yx) - c_g(y), x, y \in \mathbb{R}$. By $J_k = (a_k, b_k]$, $k = 1, \ldots, N$, we denote pair-wise disjount intervals contained in $[0,T]$ with $|J_k| := b_k - a_k$. The limiting vector is centered Gaussian, B^H is an fBM independent of V, and V has independent components with variances $\nu^2(z)|J_k|$. The finiteness of $\nu^2(z)$ is ensured by $H < \frac{3}{4}$. The proof for this convergence is analogous to Proposition 10 of [CoNuWo06]. The additional condition of g being even, together with (H), ensures that $\mathbf{E}^W\left(H(W,z)^2\right) < \infty$, which is required for this convergence result.
The second step is to show the tightnes of the sequence $Z_t^{(n)} := \sqrt{n}\left(F_{g,1}^{(n)}(zB^H)_t - c_g(z)\,t\right)$ in $\mathcal{D}([0,T])$. Again, the desired tightness property follows from Billingsley [Bill68, Theorem 15.6].

For a general process u the following theorem holds.

Theorem 3.3.2 (Theorem 4 of [CoNuWo08]). *Suppose that $u = \{u_t\,,\, t \in [0,T]\}$ is a stochastic process measurable with respect to $\mathcal{F}_T^H$ and with Hölder continuous trajectories of order $\alpha > \frac{1}{2}$ and that $B^H = \left\{B_t^H\,,\, t \in [0,T]\right\}$ is an fBM with $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$. Assume that g is even and satisfies condition (H) and that h is Hölder continuous of order $\beta \in \left(\frac{2}{3}, 1\right]$ and $\beta H > \frac{1}{2}$. Define $Z = \{Z_t\,,\, t \in [0,T]\}$ by $Z_t := \int_0^t u_s \,\mathrm{d}B_s^H\,,\, t \in [0,T]$. Then*

$$\left(B_t^H\,,\, \sqrt{n}\left(F_{g,h}^{(n)}(Z)_t - \int_0^t h(Z_s)c_g(u_s)\,\mathrm{d}s\right)\right) \xrightarrow{\mathcal{L}} \left(B_t^H\,,\, \int_0^t h(Z_s)\nu(u_s)\,\mathrm{d}W_s\right),$$

as $n \to \infty$, where $W = \{W_t\,,\, t \in \mathbb{R}_{+,0}\}$ is a BM independent of the process B^H, and the convergence is in the product space $\mathcal{D}([0,T])^2$.

Remark 3.3.2. The proof of Theorem 3.3.2 follows similar steps as in Theorem 2.3.2. For details see [CoNuWo08, Theorem 4].

Chapter 4

The Realized Power Variation of some Integrated Stable Processes

This chapter is concerned with the asymptotic behaviour of the r.p.v. of some stochastic processes of the form $\int_0^t u_s \, \mathrm{d}S_s^\alpha$ where S^α denotes an α-stable process with index of stability $\alpha \in (0,2)$. The discussion follows Corcuera *et al.* [CoNuWo07] and presents their main results.

Remark 4.0.3. As in Chapter 3, the theorems here are without proof (due to the same reasons, see Remark 3.0.5). Again, the presented material has to be understood as an outlook on results similar to those in Chapter 3 that can be obtained by replacing the stochastic integral $\int_0^t u_s \, \mathrm{d}B_s^H$ with $\int_0^t u_s \, \mathrm{d}S_s^\alpha$.

4.1 The Setting

We consider stochastic integrals of the form

$$\int_0^t u_s \, \mathrm{d}S_s^\alpha, \; t \in [0,T] \,, \tag{4.1}$$

where $T \in \mathbb{R}_+$ is fixed, $S^\alpha = \{S^\alpha \,,\, t \in \mathbb{R}_{+,0}\}$ is an α-stable Lévy process (see Definition-levy) with index of stability $\alpha \in (0,2)$, $u = \{u_t \,,\, t \in \mathbb{R}_{+,0}\}$ is a stochastic process with continuous paths and, if $\alpha \geq 1$, with finite q-variation on any finite interval for some $q < \frac{\alpha}{\alpha-1}$. The integral is a pathwise Riemann-Stieltjes integral for $\alpha \geq 1$ and a pathwise Lebesgue-Stieltjes integral for $\alpha < 1$.

Any α-stable Lévy process is a pure jump Lévy process and may be characterized by the Lévy-Khintchine formula

$$\mathbf{E}\big(e^{iuS_t^\alpha}\big) = \exp\left(t \int_{\mathrm{R}\setminus\{0\}} \big(e^{iux} - 1 - iuh(x)\big) \, \nu(\mathrm{d}x)\right) ,$$

where $h = 0$ if $\alpha < 1$, $h = 1$ if $\alpha > 1$ and $h = \mathbb{1}_{\{|x|<1\}}$ if $\alpha = 1$. The Lévy measure ν has the form

$$\nu(\mathrm{d}x) = rx^{-1-\alpha} \mathbb{1}_{\{x>1\}}(x) + q(-x)^{-1-\alpha} \mathbb{1}_{\{x<1\}}(x) \,,$$

with $r, p \geq 0$ and $r + p > 0$, and where $r = q$ for $\alpha = 1$. From the form of the Lévy measure we can deduce some properties that we will need in the following. Any α-stable

Lévy process is self-similar (see Definition 1.1.2) and satisfies a scaling relation of the form

$$t^{-1/\alpha} S_t^\alpha \stackrel{d}{=} S_1^\alpha. \tag{4.2}$$

The process possesses all moments of order less than α. For $\alpha < 1$ the trajectories are of bounded variation, whereas for $\alpha \geq 1$ they are of unbounded variation.

For pure jump Lévy processes without drift component it is known that the Blumenthal-Getoor index β, defined by

$$\beta := \inf_{\gamma \geq 0} \int_{\mathrm{R}\setminus\{0\}} (1 \wedge |x|^\gamma)\ \nu(\mathrm{d}x)\ < \infty\,,$$

determines the behaviour of the p-variation of the trajectories. It is finite if $p > \beta$. Hence, for α-stable Lévy processes it is finite for $p > \alpha$. By Young's result and the statments about the existence of Riemann-Stieltjes integrals in Section 2.1, for $\alpha \geq 1$ the Rieman-Stieltjes integrals 4.1 will exist if the process u has paths of finite q-variation on any finite interval for some $q < \frac{\alpha}{\alpha-1}$.

$\mathcal{F}_t^\alpha\,, t \in \mathbb{R}_{+,0}$, denotes the σ-algebra generated by the random variables $\{S_s^\alpha\,,\ s \in [0,t]\}$ and the null sets. The same notation as in Section 2.1 applies.

Remark 4.1.1. Corcuera *et al.* [CoNuWo07] consider a more general setting by replacing the integral process 4.1 with $\int_0^t u_{s-}\,\mathrm{d}S_s^\alpha$, where $u_{\cdot-}$ denotes the (pathwise) left sided limit of u. The integrand u is a stochastic process with càdlàg trajectories (and the same regularity conditions as stated above). For $\alpha \geq 1$ the intergal is a pathwise Refinement-Riemann-Stieltjes integral (see for a definition) which allows for common discontinuities, as long as they are not one-sided. To avoid such difficulties we consider only integrands with continuous paths here (and can use Riemann-Stieltjes integrals therefore).

4.2 A Law of Large Numbers for the Realized Power Variation

We now consider the asymptotic behaviour of the r.p.v. of $\int_0^t u_s\,\mathrm{d}S_s^\alpha$. In the following we will only consider the case $p < \alpha$, where the non-normalized r.p.v. tends to infinity and we therefore need a norming sequence that converges to zero in an appropriate way.

Remark 4.2.1. For $p > \alpha$ it is known that the non-normed r.p.v. tends to the p-th power of the absolute values of the jumps of the underlying integral process (4.3).

The following result is the analogon to Theorem 2.2.1 in the previous setting.

Theorem 4.2.1 (Theorem 1 of [CoNuWo07]). *Fix $T \in \mathbb{R}_+$. Suppose that $u = \{u_t\,,\ t \in \mathbb{R}_{+,0}\}$ is a stochastic process continuous trajectories and, if $\alpha \geq 1$, with finite q-variation on any finite interval, where $q < \frac{\alpha}{\alpha-1}$. Define a stochastic process $Z = \{Z_t\,,\ t \in [0,T]\}$ by*

$$Z_t := \int_0^t u_s\,\mathrm{d}S_s^\alpha\,,\ t \in [0,T]. \tag{4.3}$$

Assume that $Y = \{Y_t\,,\ t \in [0,T]\}$ is a stochastic process satisfying

$$n^{-1+p/\alpha} V_p^{1/n}(Y)_t \stackrel{u.c.p.}{\longrightarrow} 0 \tag{4.4}$$

as $n \to \infty$. Then, for any $p < \alpha$

$$n^{-1+p/\alpha}\,V_p^{1/n}(Z+Y)_t \xrightarrow{u.c.p.} c_p \int_0^t |u_s|^p \,\mathrm{d}s\,,$$

as $n \to \infty$, where $c_p := \boldsymbol{E}(|S_1^\alpha|^p)$.

Remark 4.2.2. The proof of Theorem 4.2.1 follows the same steps as in Theorem 2.2.1, using the scaling relation for stable Lévy processes (4.2) and the law of large numbers instead of the Ergodic theorem. For details see [CoNuWo07, Theorem 1].

Remark 4.2.3. The generalization of adding an additional "noise" process Y that satisfies a regularity condition (4.4) to Z (analogous to Theorem 2.2.1) is already included in Theorem 4.2.1.

4.3 Central Limit Theorems for the Realized Power Variation

For $p \in \left(0, \frac{\alpha}{2}\right]$ the fluctuations of the r.p.v., properly normalized, have Gaussian asymptotic distributions. For the formulation of the result we introduce some notation.

For any $p \in \left(0, \frac{\alpha}{2}\right]$ we set

$$\nu_p^2 := \mathbf{V}(|S_1^\alpha|^p)\,.$$

The following functional limit theorem is the analogon of Theorem 2.3.1 for the r.p.v. of an α-stable Lévy process S^α.

Theorem 4.3.1 (Theorem 2 of [CoNuWo07])**.** *Fix $p \in \left(0, \frac{\alpha}{2}\right]$ and assume $\alpha \in (0,2)$. Then*

$$\left(S_t^\alpha\,,\, \left(n^{-1/2+p/\alpha} V_p^{1/n}(S^\alpha)_t - c_p t n^{1/2}\right)\right) \xrightarrow{\mathcal{L}} \left(S_t^\alpha\,,\, \nu_p W_t\right)\,, \tag{4.5}$$

as $n \to \infty$, where $W = \{W_t\,,\, t \in [0,T]\}$ is a BM independent of the process S^α (i.e., independent of $\mathcal{F}_T^\alpha$), and the convergence is in the product space $\mathcal{D}([0,T])^2$ equipped with the Skorohod topology.

Remark 4.3.1. Analogous to Theorem 2.3.1 the result follows in two steps by first showing the convergence of the finite-dimensional distributions (using Theorem 2 in [AlEa78]) and a following tightness argument (using Theorem 16.1 in [Bill68]). For details see [CoNuWo07, Theorem 2].

From Theorem 4.3.1 a functional central limit theorem for the r.p.v of the stochastic integrals (4.1) can be derived. For this we need an additional condition on the integrand process u.

(K) *Assume that u satisfies: for $\gamma \in \mathbb{R}_+$*

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \big|\, |u|^\gamma(\eta_{n,j}) - |u|^\gamma(\chi_{n,j}) \big| \xrightarrow{a.s.} 0,$$

as $n \to \infty$, for any $\eta_{n,j}$ and $\chi_{n,j}$ such that

$$0 \le \chi_{n,1} \le \eta_{n,1} \le \frac{1}{n}\chi_{n,2} \le \eta_{n,2} \le \frac{2}{n} \le \cdots \le \chi_{n,n} \le \eta_{n,n} \le T.$$

We then have the following result (the analogon to Theorem 2.3.2).

Theorem 4.3.2. *Let S^α be an α-stable Lévy process with $\alpha \in (0,2)$. Fix $p \in \left(0, \frac{\alpha}{2}\right]$ and suppose that $u = \{u_t\,,\, t \in [0,T]\}$ is a stochastic process with continuous paths, measurable w.r.t. $\mathcal{F}_T^\alpha$, satisfying (K) with $\gamma = p$ and, if $\alpha \geq 1$, with finite q-variation on any finite interval for $q < 2p$. We also assume that the stochastic process $Y = \{Y_t\,,\, t \in [0,T]\}$ satisfies*

$$n^{-1+p/\alpha} V_p^{1/n}(Y)_t \xrightarrow{u.c.p.} 0 \tag{4.6}$$

as $n \to \infty$. Then, for $Z_t := \int_0^t u_s \,\mathrm{d}S_s^\alpha\,,\, t \in [0,T]$ we have

$$\left(S_t^\alpha\,,\, \left(n^{-1/2+p/\alpha} V_p^{1/n}(Z+Y)_t - c_p n^{1/2} \int_0^t |u_s|^p \,\mathrm{d}s\right)\right) \xrightarrow{\mathcal{L}} \left(S_t^\alpha\,,\, \nu_p \int_0^t |u_s|^p W_s\right), \tag{4.7}$$

as $n \to \infty$, where $W = \{W_t\,,\, t \in [0,T]\}$ is a BM independent of the process S^α (i.e., independent of $\mathcal{F}_T^\alpha$), and the convergence is in the product space $\mathcal{D}([0,T])^2$ equipped with the Skorohod topology.

Remark 4.3.2. For a proof see [CoNuWo07, Theorem 3].

Appendix A

Miscellaneous

A.1 Definitions

In the following definitions, if not stated otherwise, all random variables and stochastic processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$, are real-valued (i.e., with state space $(\mathbb{R}, \mathcal{B})$, where $\mathcal{B}$ denotes the Borel sigma-algebra on $\mathbb{R}$) with an arbitrary parameter space (index set) T.

Definition A.1.1. If $X = \{X_t,\ t \in T\}$ is a stochastic process such that $\mathbf{E}(X_t) < \infty$ $\forall t \in T$, then the *first moment function* (or *Expected value function*) $m_X^{(1)}(.)$ of X is defined by

$$m_X^{(1)}(t) : T \to \mathbb{R} : t \mapsto \mathbf{E}(X_t)\,.$$

If $\mathbf{E}\big(X_t^2\big) < \infty \quad \forall t \in T$ (which is equal to $\mathbf{V}(X_t) < \infty \quad \forall t \in T$ and yields, by Hölder's inequality, that $\mathbf{E}(X_s X_t) < \infty \quad \forall s,t \in T$), then the second moment function $m_X^{(2)}(.,.)$ of X is defined by

$$m_X^{(2)}(s,t) : T \times T \to \mathbb{R} : (s,t) \mapsto \mathbf{E}(X_s X_t)\,.$$

The (auto)covariance function $\gamma_X(.,.)$ of X is then defined by

$$\gamma_X(s,t) : \begin{cases} T \times T \to \mathbb{R}, \\ (s,t) \mapsto \mathrm{Cov}(X_s, X_t) = m_X^{(2)}(s,t) - m_X^{(1)}(s)\, m_X^{(1)}(t), \end{cases}$$

where $\mathrm{Cov}(X_s, X_t) := \mathbf{E}((X_s - \mathbf{E}(X_s))(X_t - \mathbf{E}(X_t)))\,,\ s,t \in T$.

Definition A.1.2. A stochastic process $X = \{X_t,\ t \in T\}$ with $T \subseteq \mathbb{R}$ is called *(weakly) stationary* if

(i) $\mathbf{E}\big(|X_t|^2\big) < \infty, \forall t \in T$

(ii) $m_X^{(1)(t)} = \mu,\ t \in T,$

(iii) $m_X^{(2)}(s,t) = m_X^{(2)}(s+h, t+h),\ \ s,t \in T,\ h \in \mathbb{R}$ such that $s+h,\ t+h \in T$.

Note: This type of stationarity is frequently referred to in the literature as covariance stationarity, stationarity in the wide sense, second-order stationarity or stationarity.

Definition A.1.3. A stochastic process $X = \{X_t,\ t \in T\}$ with $T \subseteq \mathbb{R}$ is called *(strictly) stationary* if the joint distributions of $(X_{t_1}, \ldots, X_{t_n})$ and $(X_{t_1+h}, \ldots, X_{t_n+h})$ are the same for all $t_1, \ldots, t_n \in T$, $n \in \mathbb{N}$ and all $h \in \mathbb{R}$ such that $t_1 + h, \ldots, t_n + h \in T$.

Definition A.1.4. For a stochastic process $X = \{X_t,\ t \in T\}$ with totally ordered index set $(T, \leq)$ and state space (S, Σ) the *natural filtration* $\mathcal{F} = \{\mathcal{F}_t,\ t \in T\}$ is defined by

$$\mathcal{F}_t := \sigma\left(X_s \middle| \, s \leq t\right) = \sigma\left\{X_s^{-1}(A) \middle| \, A \in \Sigma, s \leq t\right\} ,$$

where $\sigma(.)$ denotes the σ-operator.

Definition A.1.5. A filtration $\mathcal{F} = \{\mathcal{F}_t,\ t \in [0, T]\}$, where $T \in \mathbb{R}_+$, is said to satisfy the *usual conditions* if

(i) $\mathcal{F}$ is right-continuous, i.e., $\bigcap_{s>t} \mathcal{F}_s = \mathcal{F}_t\,,\ t \in [0, T)$,
(ii) $\mathcal{F}_T$ is complete w.r.t. the underlying measure and
(iii) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_T$.

Definition A.1.6. A stochastic process $X = \{X_t,\ t \in T\}$ is called a *Gaussian process* if for any finite subset of indices $\{t_1, \ldots, t_n\} \subseteq T$, $n \in \mathbb{N}$, the multivariate random variable

$$X_{t_1,\ldots,t_n} := (X_{t_1}, \ldots, X_{t_n})$$

has a multivariate normal distribution (sometimes also called multivariate Gaussian distribution). A Gaussian process is called *centered* (or *normalized*), if its first moment function is the zero function on T (i.e., $m_X^{(1)} \equiv 0$).

Definition A.1.7. A stochastic process $X = \{X_t, , t \in T\}$ is said to have *independent increments* if for all $t_1 < t_2 < \cdots < t_n \in T, n \in \mathbb{N}$, the corresponding increments of X, i.e., the random variables $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$, are independent.

A stochastic process $X = \{X_t, , t \in T\}$ is said to have *stationary increments* if for all $t_1 < t_2 < \cdots < t_n \in T, n \in \mathbb{N}$, the increments $X_{t_1+h} - X_{t_1}, \ldots, X_{t_n+h} - X_{t_n}$ have the same distribution for all $h \in \mathbb{R}_+$.

Definition A.1.8. A random variable X (or, more precisely, its distribution P_X) is called *infinitely divisible* if for any $n \in \mathbb{N}$ there exist independent and identically distributed random variables $X_1, \ldots, X_n$ such that

$$X_1 + \cdots + X_n \overset{d}{=} X\,,$$

where $\overset{d}{=}$ denotes "same distribution".

Definition A.1.9. An infinitely divisible random variable X (or, more precisely, its distribution P_X) is called *α-stable* if for any $n \in \mathbb{N}$ there exist independent random variables $X_1, \ldots, X_n$ such that $X_i \overset{d}{=} X$, $i = 1, \ldots, n$, and

$$X_1 + \cdots + X_n \overset{d}{=} n^{1/\alpha} X + d_n\,,$$

where $\overset{d}{=}$ denotes "same distribution", and $\alpha \in (0, 2]$ and $d_n \in \mathbb{R}$ are some constants. α is called the index of stability of X (or its distribution).

Definition A.1.10. A stochastic process $X = \{X_t,\ t \in T\}$ is called *Lévy process* if
(i) $X_0 = 0$,
(ii) X has independent and stationary increments (see Definition A.1.7)
(iii) and the trajectories of X are P-a.s. right continuous with left limits, i.e., càdlàg.

It is called α-stable Lévy process if X_1 has an α-stable distribution (see Definition A.1.9).

Definition A.1.11. The *Gamma function* (on the positive reals) is defined as

$$\Gamma : \mathbb{R}_+ \to \mathbb{R} : z \mapsto \int_0^\infty t^{z-1} e^{-t} \, \mathrm{d}t \,.$$

The domain of the Gamma function can be extended to the complex plane, excepting the non-positive integers.

Definition A.1.12. If X is a multivariate random variable taking values in $\mathbb{R}^n \,, n \in \mathbb{N}$, then its *characteristic function* is defined as

$$\varphi_X : \mathbb{R}^n \to \mathbb{C} : t \mapsto \mathbf{E}\Big(e^{i \langle t \,, X \rangle}\Big) \,,$$

where $\langle \,.\,, \,.\, \rangle$ denotes the (standard) inner product on $\mathbb{R}^n$ (i.e., $\langle x \,, y \rangle = \sum_{i=1}^n x_i y_i \,,$ $x, y \in \mathbb{R}^n$).

Definition A.1.13. Let $f : (\mathrm{X}, d_\mathrm{X}) \to (\mathrm{Y}, d_\mathrm{Y})$ be a mapping between any two metric spaces. $f(.)$ is *α-Hölder continuous* (or simply *α-Hölder*), or satisfies a *Hölder condition of order* α, if there are constants $\alpha, C \in \mathbb{R}_{+,0}$ such that

$$d_\mathrm{Y}(f(x), f(y)) \leq C \, d_\mathrm{X}(x, y)^\alpha \quad \forall x, y \in \mathrm{X},$$

where α is called the *Hölder exponent.*

A.2 Theorems

The following theorems are used in this thesis and are stated just for the sake of completeness (and therefore without proofs).

Theorem A.2.1 (Kolmogorov‘s extension theorem, see Def. 1.2.1 in [BrDa91])**.** *For an arbitrary index set T let $I(T)$ denote the set of all n-tupel $t = (t_1, \ldots, t_n)$, $n \in \mathbb{N}$, whose components are pairwisely distinct elements of T. The probability distribution functions $\{F_t(.),\ t \in I(T)\}$ are the finite-dimensional distribution functions of some (real-valued) stochastic process with index set T (i.e., there exists a process with these finite-dimensional distributions) if and only if for any $t \in I(T)$ and any $1 \le i \le n$*

$$\lim_{x_i \to \infty} F_t(x) = F_{t(i)}(x(i))\,, \tag{A.1}$$

where x_i denotes the i^{th} component of x and $t(i)$ and $x(i)$ are the $(n-1)$-component vectors obtained by deleting the i^{th} components of t and x respectively. By using the characteristic functions of the finite-dimensional distributions, (A.1) may be restated in the equivalent form

$$\lim_{x_i \to 0} \varphi_t(x) = \varphi_{t(i)}(x(i))\,. \tag{A.2}$$

Conditions (A.1) and (A.2) are simply the ‘consistency’ requirements that each function $F_t(.)$ should have marginal distributions which coincide with the specified lower dimensional distribution functions.

Theorem A.2.2 (Kolmogorov-Chentsov continuity theorem)**.** *Let $X = \{X_t,\ t \in T\}$ be a stochastic process whose index set T is dense in an open subset $D \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, with state space $S = (S, \rho)$, where S is a polish space (i.e., a separable, completely metrizable topological space) with metric ρ. Suppose that there are positive constants α, β, C such that*

$$\mathbf{E}(\,(\rho(X_s, X_t))^\alpha\,) \le C\,||s-t||_2^{\beta+d},\ \forall s, t \in T,$$

where $||.||_2$ denotes the Euclidean norm on $\mathbb{R}^d$. Then there exists a $(\mathrm{P}$ *-a.s.) continuous modification of X (i.e., a stochastic process $\tilde{X} = \{\tilde{X}_t,\ t \in T\}$, such that $\mathrm{P}(X_t = \tilde{X}_t) = 1$, $\forall t \in T$, for which the mapping $t \mapsto \tilde{X}_t(\omega)$ is continuous for P-a.a. $\omega \in \Omega$). Furthermore, this modification is γ-Hölder continuous P-a.s. for all $\gamma \in (0, \frac{\beta}{\alpha})$. For every compact subset of T this modification is even γ-Hölder continuous P-a.s. for all $\gamma \in [0, \frac{\beta}{\alpha})$.*

Theorem A.2.3 (Corollary 4.3.1 in [BrDa91])**.** *A real-valued function $\gamma : \mathbb{Z} \to \mathbb{R}$ is the (auto)covariance function of a weakly stationary stochastic process $X = \{X_t,\ t \in \mathbb{Z}\}$ if and only if either*

(i) $\gamma(k) = \int_{(-\pi,\pi]} \cos(kz)\,\mathrm{d}F_X(z)$, $\forall k \in \mathbb{Z},$, *where F is a right-continuous, non-decreasing, bounded function on $[-\pi, \pi]$ with $F(-\pi) = 0$ (i.e., a generalized distribution function assigning all its mass to $(-\pi, \pi]$), or (equivalently)*

(ii) $\sum_{i,j=1}^n a_i \gamma(i-j) a_j \ge 0\ \ \forall (a_1, \ldots, a_n) \in \mathbb{R}^n,\ \forall n \in \mathbb{N}.$

The equivalence of (i) and (ii) is asserted by Herglotz's theorem (e.g., see Theorem 4.3.1 in [BrDa91]). $F(.)$ is referred to as the spectral distribution function *of X (and of $\gamma(.)$ analogously). If there exists a function $f : [-\pi, \pi] \to \mathbb{R}_{+,0}$ such that $F(z) = \int_{-\pi}^{z} f(x)\,\mathrm{d}x$, $-\pi \le z \le \pi$, then $f(.)$ is referred to as the* spectral density function *of X (and of $\gamma(.)$ analogously).*

Theorem A.2.4 (Ergodic theorem (formulation of Birkhoff))**.** *Let X be a real-valued random variable on $(\Omega, \mathcal{F}, \mathrm{P})$ with $\boldsymbol{E}(X) < \infty$ and $T : (\Omega, \mathcal{F}) \to (\Omega, \mathcal{F})$ a measure-preserving mapping (i.e., $\mathrm{P}\left(T^{-1}(A)\right) = \mathrm{P}\left(A\right), \forall A \in \mathcal{F}$). Then*

$$\frac{1}{n}\sum_{i=1}^{n} X \circ T^{i-1}(\omega) \xrightarrow{a.s.} \boldsymbol{E}(X\,|\mathcal{T}\,)$$

as $n \to \infty$, where $\mathcal{T}$ denotes the σ-algebra generated by the T-invariant sets (i.e., the sets $A \in \mathcal{F}$ with $T^{-1}(A) = A$) and $\boldsymbol{E}(X\,|\mathcal{T}\,)$ denotes the conditional expectation of X w.r.t. $\mathcal{T}$. If T is ergodic then

$$\frac{1}{n}\sum_{i=1}^{n} X \circ T^{i-1}(\omega) \xrightarrow{a.s.} \boldsymbol{E}(X)$$

as $n \to \infty$.

Remark A.2.1. There are other theorems which are also referred to as Ergodic theorem in the literature. The formulation of Theorem A.2.4 is sometimes referred to as *strong Ergodic theorem* of *pointwise Ergodic theorem.*

Remark A.2.2. By chosing an appropriate Ω' and mappings X, T any strictly stationary sequence $Y = \{Y_n \; n \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \mathrm{P})$ can be represented in the form $Y_n = X \circ T^{n-1}$, $n \in \mathbb{N}$. If $\mathbf{E}(Y_n) < \infty$ and Y is ergodic then, by the Ergodic theorem,

$$\frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{a.s.} \mathbf{E}(Y_j)\ , \ \text{ for an arbitrtary } j \in \mathbb{N}\,.$$

A.3 Additional Results

This section contains some calculations and proofs for results that were used in this thesis (but were moved to the appendix to tighten the representation).

Lemma A.3.1. *Let $B^H = \left\{B_t^H,\ t \in \mathbb{R}\right\}$ be an* fBm. *Then, for all $p > 0$ it holds that*

$$\mathbf{E}\left(|B_t^H - B_s^H|^p\right) = \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}|t-s|^{pH}\Gamma(\frac{p+1}{2}),\ t,s \in \mathbb{R},$$

where $\Gamma(.)$ denotes the Gamma function (see Definition A.1.11).

Proof. As $(B_t^H - B_s^H) \sim \mathrm{N}(0, |t-s|^{2H})$, it follows that

$$\begin{aligned}
\mathbf{E}\big(|B_t^H - B_s^H|^p\big) &= \int\limits_{-\infty}^{\infty} |x|^p \frac{1}{\sqrt{2\pi}|t-s|^H} \exp\left\{-\frac{x^2}{2|t-s|^{2H}}\right\} \mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}|t-s|^H} \int\limits_{0}^{\infty} x^p \exp\left\{-\frac{x^2}{2|t-s|^{2H}}\right\} \mathrm{d}x \\
&= \left| \begin{array}{c} u := \frac{x^2}{2|t-s|^{2H}} \\ \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{|t-s|^{2H}} = \frac{\sqrt{2}u^{1/2}}{|t-s|^H} \end{array} \right| = \\
&= \frac{2}{\sqrt{2\pi}|t-s|^H} \frac{|t-s|^H}{\sqrt{2}} \int\limits_0^{\infty} \frac{\left(\sqrt{2}|t-s|^H u^{\frac{1}{2}}\right)^p}{u^{\frac{1}{2}}} \exp(-u)\ \mathrm{d}u \\
&= \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}|t-s|^{pH} \underbrace{\int\limits_0^{\infty} u^{\frac{p-1}{2}} \exp(-u)\ \mathrm{d}u}_{=\Gamma\left(\frac{p-1}{2}+1\right)=\Gamma\left(\frac{p+1}{2}\right)} = \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}|t-s|^{pH}\Gamma(\frac{p+1}{2})\,.
\end{aligned}$$

∎

Corollary A.3.1. *Any* fBm B^H *has a modification* $\tilde{B}^H$ *whose paths are* γ *- Hölder for all* $\gamma \in (0, H)$. *If* B^H *is defined on a compact index set,* γ *may be chosen from* $[0, H)$.

Proof. From Lemma A.3.1 and the Kolmogorov-Chentsov theorem (see Theorem A.2.2) it follows that there exists a modification $\tilde{B}^H$ whose paths are γ - Hölder for all $\gamma \in (0, H - \frac{1}{p})$ and all $p > 0$. Letting $p \to \infty$ yields the desired result. According to Kolmogorov-Chentsov γ may be chosen from $[0, H)$ for a compact index set (this is obvious, as every continuous function on a compact domain is bounded, which is equivalent to 0 - Hölder). ∎

Lemma A.3.2. *Any α-Hölder continuous function $f:\ [a,b] \to \mathbb{R}$ with $a, b \in \mathbb{R}$, $a < b$, and $\alpha \in \mathbb{R}_+$ has finite $(1/\alpha)$-variation on $[a,b]$.*

Proof. Since f is α-Hölder continuous it holds that (see Definition A.1.13)

$$|f(x) - f(y)|^{\frac{1}{\alpha}} \leq C^{\frac{1}{\alpha}}|x-y|\,,\ \forall x, y \in [a,b]\,,$$

for some fixed $C \in \mathbb{R}_+$. Therefore,

$$\begin{aligned}
\mathrm{var}_{1/\alpha}(f;[a,b]) &= \sup_\pi \left(\sum_{i=1}^{n} |f(t_i) - f(t_{(i-1)})|^{\frac{1}{\alpha}} \right)^\alpha \\
&\leq \sup_\pi \left(C^{\frac{1}{\alpha}} \sum_{i=1}^{n} |t_i - t_{(i-1)}| \right)^\alpha \\
&= \sup_\pi \left(C^{\frac{1}{\alpha}} (b-a) \right)^\alpha \\
&= C(b-a)^\alpha \ < \infty\,,
\end{aligned}$$

where the supremum is taken over all partitions of $[a,b]$ of the form $\pi = \{a = t_0 < t_1 < \cdots < t_n = b\}$, hence $\sum_{i=1}^{n} |t_i - t_{(i-1)}| = (b-a)$. ■

A.4 Notation

This section contains notation used in this thesis that was (usually) not defined explicitly in the text.

Symbol	Meaning
$\mathbb{N}$	the natural numbers (i.e., $\mathbb{N} = \{1, 2, 3, \dots\}$)
$\mathbb{N}_0$	the natural numbers including zero (i.e., $\mathbb{N} = \{0, 1, 2, 3, \dots\}$)
$\mathbb{Z}$	the integers (i.e., $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$)
$\mathbb{R}_+$	the reals
$\mathbb{R}_+$	the positive reals (i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} \vert\, x > 0\}$)
$\mathbb{R}_{+,0}$	the non-negative reals (i.e., $\mathbb{R}_{+,0} = \{x \in \mathbb{R} \vert\, x \geq 0\}$)
$\mathbb{C}$	the complex numbers
i	the imaginary unit (i.e., $i \in \mathbb{C}$ and $i^2 = -1$)
$\mathcal{B}$	the σ-algebra of Borel sets on R
$\mathbf{E}(.)$	Expectation
$\mathbf{V}(.)$	Variance
$\mathrm{N}(\mu, \sigma^2)$	Gaussian (or Normal) distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma \in \mathbb{R}_+$
$\stackrel{d}{=}$	equality in distribution
2^Ω	the power set of some set Ω
$\mathbb{1}_A$	the indicator function of a set A
$\sigma(.)$	the σ-operator (i.e., for $\mathcal{A} \subseteq 2^\Omega$ for some set Ω, $\sigma(\mathcal{A})$ denotes the smallest σ-algebra on Ω that contains $\mathcal{A}$)
a.a.	almust all
a.s.	almust surely
$\xrightarrow{a.s.}$	almost sure convergence
$\xrightarrow{p}$	convergence in probability
$\xrightarrow{u.c.p}$	uniform convergence in probability
w.r.t	with respect to

Appendix B

Simulating Sample Paths of fBM

Sample paths (or, more precisely, discrete approximations to sample paths) of an fBM can be obtained, for example, by simulating points of a realization of a process with the corresponding properties over a discrete grid with constant (and small) mesh. To simulate a number of $n \in \mathbb{N}$ (discrete) sample points $(b_1, \ldots, b_n)$ at times $t_1 < t_2 < \cdots < t_n$, $t_i \in \mathbb{R}$, of an fBM $B^H = \{B_t^H,\, t \in \mathbb{R}\}$ with Hurst parameter $H \in (0,1)$:

- Form the covariance matrix of $(B_{t_1}, \ldots, B_{t_n})$, i.e., the $n \times n$ matrix $\Sigma = (\sigma_{ij})_{i,j=1}^n$ where $\sigma_{ij} := \gamma_{B^H}(B_{t_i}^H, B_{t_j}^H) = \frac{1}{2}(|t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H}),\ 1 \le i, j \le n\,.$
- Compute a square root, say A, of Σ, i.e., a matrix A for which $\Sigma = AA$ holds. The existence of such a matrix is ensured by the positive semi-definiteness of Σ (see Remark 1.1.1).
- Construct a vector $r := (r_1, \ldots, r_n)$ whose components are n numbers drawn from a standard normal distribution.
- Applying A to this vector yields the desired sample points, i.e., $Ar = (b_1, \ldots, b_n)$.

Remark B.0.1. This algorithm can easily be implemented, e.g., using MATLAB. Figures 1.1 to 1.3 in Section 1.1.7 were generated by the following MATLAB function:

```
function fbm(H,k)

for i=1:k,
 for j=1:k,
  c(i,j)= 1/2 * ( ((i-1)/k)^(2*H) + ...
            ((j-1)/k)^(2*H) - abs( (i-1)/k - (j-1)/k)^(2*H) );
 end
end

subplot(1,3,1); plot((0:1/k:1-1/k), sqrtm(c) * randn(k,1));
subplot(1,3,2); plot((0:1/k:1-1/k), sqrtm(c) * randn(k,1));
subplot(1,3,3); plot((0:1/k:1-1/k), sqrtm(c) * randn(k,1));
```

Listing B.1: MATLAB function to generate sample paths of fBM

Remark B.0.2. Calling the above function with parameters $H = 0.5$ and $k = 1000$ would generate three times 1000 sample points (over $[0,1]$) of an fBM with $H = 0.5$.

Remark B.0.3. The described algorithm (slightly modified) can be used to simulate sample paths of any Gaussian process of known first and second moment functions.

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