DIPLOMARBEIT

REALIZATIONS AND PARAMETERIZATIONS OF STRUCTURAL ARMAX- AND STATE SPACE SYSTEMS

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DIPLOMA THESIS

REALIZATIONS AND PARAMETERIZATIONS OF STRUCTURAL ARMAX- AND STATE SPACE SYSTEMS

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Preface

The recent years have seen a surge in the use of state space models in macroeconomics. These linear models have been most predominantly studied and used by control engineers. Nowadays they are often used by macroeconomists as a (log-) linear approximation to stochastic steady-states in theoretical macro-models. The main idea of state space models is to seperate the *state* of the model from the *observation*. This property, as well as the fact that the theory has already been well developed, renders those models ideal for use in economics.
A different – yet related – representation for a linear system is that of an ARMAX system. The properties of these systems, regarding estimation, identifiability and the topological structure have been well studied (see, among others, Brockwell and Davis (1991), Hannan and Deistler (1988), Glover and Willems (1974), Deistler (1983)). Application to macroeconomics started in the 80's, following the demise of the Cowles Commission approach to econometric modelling and policy evaluation.
The first chapter, based on the book by Hannan and Deistler (1988), presents a short overview of the realization theory of linear systems. Starting from the transfer function of a linear system, we consider minimal state space realizations and irreducible ARMAX realizations. Regarding identification we discuss echelon forms and the relation between echelon state space and ARMAX realizations from a topological point of view.
The second chapter considers conditions under which a linear system has a purely (finite) autoregressive representation. With the work done in the first chapter, this is easy to answer for ARMA systems. For state space systems, two lines of thought are followed. The first is that when a system is supposed to have a finite autoregressive representation, its infinite AR representation has to stop at some point. Under a certain regularity condition, this yields a nice necessary and sufficient condition in terms of the eigenvalues of a certain matrix. The second approach uses the beautiful relationship between echelon ARMAX and echelon state space realizations to adress the state space case by reducing the problem to the easily soluble ARMAX case.
Chapter three presents the use of state space and autoregressive systems in macroeconomic modelling. We give an overview of the VAR approach and the problems associated with it. We especially consider the problem of invertibility of a system, following the work by Fernandez-Villaverde et al. (2005). The diploma thesis is concluded by an appendix reviewing the mathematical tools used in our analysis.

A student writing a thesis usually owes a lot to his supervisor, and I am no exception. From the time on when I first sat in one of his lectures in econometrics, Prof. Deistler has supported me in every possible way, and I would like to thank him wholeheartedly. His constant enthusiasm and motivation is truly inspiring.

The work has greatly benefited from comments by B. Böhm and G. M. Görg, and I am very grateful to them. Finally, I would like to express my gratitute to my parents for their constant support over the years.

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CHAPTER 1

Realizations of Linear Systems

1.1. Linear System Representations

We consider here *causal linear time-invariant systems in discrete time.* Linear in the sense that the s-dimensional stochastic output process $y(t)$ is determined by the m-dimensional stochastic input process $z(t)$ and residual influences (or, errors) $\varepsilon(t)$ through a linear relationship. Both inputs and outputs are observed; the unobserved residual influences $\varepsilon(t)$ are modeled as s-dimensional random variables which we assume to be uncorrelated over time, that is, $\mathbb{E}(\varepsilon(t_1)\varepsilon(t_2)') = 0$ for all $t_1 \neq t_2$. All these random variables are defined over the same underlying probability space $(\Omega, \mathcal{F}, P)$, so $y(t), \varepsilon(t) : \Omega \to \mathbb{R}^s$ and $z(t) : \Omega \to \mathbb{R}^m$. Furthermore, we assume that they have finite second moments

$$\mathbb{E}(y(t)'y(t)) < \infty, \qquad E(z(t)'z(t)) < \infty, \qquad \mathbb{E}(\varepsilon(t)'\varepsilon(t)) < \infty$$

thus their elements are in the Hilbert space of square integrable real random variables over $(\Omega, \mathcal{F}, P)$ with the inner product $< x, y >\equiv \mathbb{E}x'y$. Since we work in discrete time, all these processes are discrete and $t \in \mathbb{Z}$ (or $t \in \mathbb{N}$ when we explicitly say so). Causality means that output at time t does not depend on future inputs $z(s)$, $s > t$ and future errors $\varepsilon(s)$, $s > t$. Finally, by assuming that the system is time invariant we guarantee that the internal characteristics of the system do not change over time.
Such systems can be represented in different forms. First, the *input-output representation*,

$$y(t) = \sum_{j=-\infty}^{\infty} K(j)\varepsilon(t-j) + \sum_{j=-\infty}^{\infty} L(j)z(t-j)$$

where $K(j) \in \mathbb{R}^{s\times s}, L(j) \in \mathbb{R}^{s\times m}$ and causality implies that $K(j) = L(j) = 0$ for all $j < 0$. By the assumption of time invariance, the $K(j), L(j)$ do not depend on t. The function

$$\mathbb{C} \mapsto \mathbb{C} \times \mathbb{C} : z \to (k(z), l(z)) \equiv \left(\sum_{j=-\infty}^{\infty} K(j)z^j, \sum_{j=-\infty}^{\infty} L(j)z^j \right)$$

is called the *transfer function* of the linear system. We see that the system is causal if and only if each component of the transfer function has a power series expansion in a neighborhood of zero. If the transfer function (k, l) can be written as $(k, l) = a^{-1}(b, d)$ with a, b and d polynomial matrices[1], and $\det a(z)^{-1} \not\equiv 0$, we say that the transfer function is *rational* and the corresponding system is *finite dimensional.*

[1]See Appendix A.1 for an overview of the properties of polynomial matrices.

From now on we will consider only white noise errors, that is, the $\varepsilon(t)$ suffice

$$\mathbb{E}(\varepsilon(t)) = 0, \qquad \mathbb{E}(\varepsilon(t_0)\varepsilon(t_1)') = \delta_{t_0t_1}\Sigma, \qquad \mathbb{E}(z(t_0)\varepsilon(t_1)') = 0 \tag{1.1.1}$$

for all $t_0, t_1 \in \mathbb{Z}$, where $\Sigma \in \mathbb{R}^{s\times s}$, $\Sigma > 0$ and $\delta_{t_0t_1}$ denotes the Kronecker delta.

1.1.1. ARMAX-Systems. Let $A(0), A(1), \ldots, A(p) \in \mathbb{R}^{s\times s}$, $B(0), B(1), \ldots, B(q) \in \mathbb{R}^{s\times s}$, $D(0)$, $D(1), \ldots,$ $D(r) \in \mathbb{R}^{s\times m}$ and $y(t), \varepsilon(t)$ and $z(t)$ as above, with $\varepsilon(t)$ white noise as in (1.1.1). Assume furthermore that $\det A(0) \neq 0$, then

$$\sum_{j=0}^{p} A(j)y(t-j) = \sum_{j=0}^{q} B(j)\varepsilon(t-j) + \sum_{j=0}^{r} D(j)z(t-j) \tag{1.1.2}$$

is called an *ARMAX-System*, short for autoregressive moving-average system with exogenous variables. If $D(j) = 0$, $j \geq 0$, then (1.1.2) is called an ARMA-System. Likewise, if $B(j) = D(j) = 0$, $j \geq 0$, then (1.1.2) is called an AR-System. We will concern ourselves with those systems in more detail in chapter 2. Let

$$a(z) \equiv \sum_{j=0}^{p} A(j)z^j, \qquad b(z) \equiv \sum_{j=0}^{q} B(j)z^j, \qquad d(z) \equiv \sum_{j=0}^{r} D(j)z^j, \tag{1.1.3}$$

then the transfer function of (1.1.2) is

$$(k(z), l(z)) = a(z)^{-1}(b(z), d(z)). \tag{1.1.4}$$

Note that the analyticity of the transfer function in a neighbourhood of zero – and thus causality – is ensured by the assumption that $\det A(0) \neq 0$. We also write z for the backward shift operator on stochastic processes on $\mathbb{Z}$:

$$z : (y(t)|t \in \mathbb{Z}) \mapsto (y(t-1)|t \in \mathbb{Z}), \tag{1.1.5}$$

this enables us to write the ARMAX-system (1.1.2) as

$$a(z)y(t) = b(z)\varepsilon(t) + d(z)z(t) \qquad \text{or} \qquad y(t) = k(z)\varepsilon(t) + l(z)z(t), \tag{1.1.6}$$

the latter being the input-output representation.

Before we proceed to defining a third representation of a linear sytem, we occupy ourselves with the concept of *solutions* of ARMAX-systems. We say that a process $y(t)$ is a solution of a system allowing for an ARMAX representation $(a(z), b(z), d(z))$ if for a given process $z(t)$, the process $y(t)$ suffices

$$a(z)y(t) = b(z)\varepsilon(t) + d(z)z(t). \tag{1.1.7}$$

Imposing the so-called *stability condition*

$$\det a(z) \neq 0, \quad \text{for all} \quad |z| \leq 1 \tag{1.1.8}$$

on the matrix polynomial $a(z)$, we notice that, since $a(z)^{-1} = (\det a(z))^{-1} \operatorname{adj}(a(z))$, the transfer function $(k(z), l(z)) = a(z)^{-1}(b(z), d(z)$ has a power series expansion that converges in an open set containing the closed unit disc. We thus arrive at a causal solution

$$y(t) = \sum_{j=0}^{\infty} K(j)\varepsilon(t-j) + \sum_{j=0}^{\infty} L(j)z(t-j) \tag{1.1.9}$$

that converges in mean-square sense if $z(t)$ is stationary with finite second moments. This can be seen by using the triangle inequality and, subsequently, interchanging norm and infinite series due to the fact that the norm is continuous. Since (1.1.9) is (weak-sense) stationary and the homogeneous part $a(z)y(t) = 0$ of (1.1.7) has no nontrivial stationary solution, (1.1.9) is the unique stationary solution to (1.1.7).

1.1.2. State space Systems. We now turn to the definition of state space systems:

$$x(t+1) = Fx(t) + Lz(t) + K\varepsilon(t) \tag{1.1.10}$$

$$y(t) = Hx(t) + \varepsilon(t) \tag{1.1.11}$$

where $F \in \mathbb{R}^{n\times n}, L \in \mathbb{R}^{n\times m}, K \in \mathbb{R}^{n\times s}, H \in \mathbb{R}^{s\times n}$, $y(t)$ is s-dimensional and $x(t)$ is n-dimensional, is called a *state space representation*[2]. $x(t)$ is called the *state* of the system, $y(t)$ the *observation*.
Rearranging (1.1.10) yields

$$x(t) = z(I_n - Fz)^{-1}(Lz(t) + K\varepsilon(t)) \tag{1.1.12}$$

and, together with (1.1.11),

$$y(t) = Hz(I_n - Fz)^{-1}(Lz(t) + K\varepsilon(t)) + \varepsilon(t) \tag{1.1.13}$$

where $(I - Fz)^{-1} = \sum_{j=0}^{\infty} F^j z^j$, whenever the sum converges. These two equations give a causal solution to the state space system (where solutions to state space systems are defined in an analogous way to ARMAX solutions). Since $\det(I - Fz) = 1$ for $z = 0$, the transfer function has a power series expansion that converges in a neighbourhood of zero.

$$(k(z), l(z)) = \left(\sum_{j=1}^{\infty} HF^{j-1}Kz^j + I_s, \sum_{j=1}^{\infty} HF^{j-1}Lz^j\right) \tag{1.1.14}$$

In order to ensure that the output process is stationary whenever the input process is, we again need to impose a stability condition. The power series expansion (1.1.14) of the transfer function needs to converge in the closed unit disc, so we need that

$$\rho(F) < 1, \tag{1.1.15}$$

i.e. the spectral radius and thus the modulus of the maximum eigenvalue of F needs to be smaller than unity.

1.1.3. Existence of Realizations.

THEOREM 1.1. (i) *Every ARMAX-system (1.1.2) with* $\det a(0) \neq 0$ *and every state space system (1.1.10),(1.1.11) has a causal, rational transfer function.*

(ii) *For every causal, rational transfer function with* $(k(0), l(0)) = (I_s, 0)$*, there exists an ARMAX-system with* $\det(a(0)) \neq 0$ *and a state space system that have* (k, l) *as its transfer function. (Such systems are called ARMAX realizations, resp. state space realizations of* (k, l)*.)*

[2]It is possible to define state space representations in a more general form. However, one can show that for a suitably chosen $x(t)$ these systems always have a representation of the form (1.1.10), (1.1.11). For the sake of brevity we omit the slightly more general formulation.

(iii) *If, in addition to (ii), the transfer function is analytic on the closed unit disc, then there exists a stable ARMAX realization and a stable state space realization of* (k, l).

(iv) *The transfer function of every stable ARMAX system and of every stable state space system is analytic on the closed unit disc.*

PROOF. (i) For ARMAX-systems with $\det a(0) \neq 0$, the transfer function is $(k(z), l(z)) = a(z)^{-1}(b(z), d(z))$ (see above). This function has a power series expansion that converges in a neighborhood of zero, thus is causal. Rationality follows immediately from the above representation. For state space representations, the transfer function is (1.1.14). As we have already observed, it is causal. Clearly, all entries of (k, l) are fractions of polynomials. We can therefore extract the least common denominator c of the entries of (k, l), yielding $(k, l) = c^{-1}(N, M)$, where N and M are polynomial matrices. Thus the transfer function is rational.

(ii) From the rationality of the transfer function, we have $(k, l) = a^{-1}(b, d)$ for certain polynomial matrices a, b and d. Since $(k(0), l(0)) = (I_s, 0)$, we have $A(0) = B(0)$ nonsingular and $D(0) = 0$. Clearly then, (a, b, d) is an ARMAX realization of (k, l). We now construct a state space system, that has the same transfer function as (a, b, d). Observe that $(\tilde{a}, \tilde{b}, \tilde{d}) = A(0)^{-1}(a, b, d)$ is also an ARMAX realization of (k, l), with $\tilde{A}(0) = I$. Let

$$F = \begin{pmatrix}
-A(1) & -A(2) & \cdots & -A(p-1) & -A(p) & D(1) & \cdots & D(r-1) & D(r) & B(1) & \cdots & B(q-1) & B(q) \\
I_s & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & I_s & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & I_s & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & I_m & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_m & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & I_s & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_s & 0
\end{pmatrix} \tag{1.1.16}$$

$$K = (I_s, 0, \ldots, 0, I_s, 0, \ldots, 0)' \tag{1.1.17}$$

$$L = (0, \ldots, 0, I_m, 0, \ldots, 0)' \tag{1.1.18}$$

$$H = (-A(1), -A(2), \ldots, -A(p), D(1), \ldots, D(r), B(1), \ldots, B(q)) \tag{1.1.19}$$

then (F, H, K, L) is such that (1.1.10), (1.1.11) is equivalent to $\tilde{a}(z)y(t) = \tilde{b}(z)\varepsilon(t) + \tilde{d}(z)z(t)$.

(iii) (k, l) is rational, therefore we can extract the least common denominator,

$$(k, l) = c^{-1}(N, M), \tag{1.1.20}$$

where c is a least common denominator polynomial and N and M are polynomial matrices. Now if (k, l) is supposed to be analytic on the unit disc $\bar{B}(0,1)$, the zeros of c must lie outside $\bar{B}(0,1)$, else (k, l) would not be analytic on that point, thus contradicting our assumption. Now take $a = c \cdot I, b = N, d = M$ and we obtain a stable ARMAX realization of (k, l). To obtain a suitable state space realization, assume without loss of generality, that $a(0) = I$. We again employ (1.1.16) - (1.1.19). Observe that

$$\det(F - zI) = z^{p+r+q} \det a(z^{-1}). \tag{1.1.21}$$

Since $a(z)$ has no zeros on $\bar{B}(0,1)$ and $z \to z^{-1}$ maps $\{z \in \mathbb{C} : |z| > 1\}$ bijectively onto $B(0,1)\backslash\{0\}$, we conclude that all eigenvalues of F must lie within the complex unit disc, thus establishing that (F, H, K, L) is stable.

(iv) Again, from $(k, l) = a(z)^{-1}(b(z), d(z) = (\det a(z))^{-1} \operatorname{adj} a(z)(b(z), d(z))$, we see that (k, l) is analytic on $\bar{B}(0,1)$ if (a, b, d) is stable. Regarding state space systems, the infinite series in (1.1.14) converge if (1.1.15) is met.

□

In the literature, the term *(left) matrix fraction description* (MFD) of a transfer function (k, l) is used when

$$(k, l) = a^{-1}(b, d) \tag{1.1.22}$$

with $\det a \not\equiv 0$. This is equivalent to saying that (a, b, d) is an ARMAX realization of (k, l). From the theorem above we know that every rational causal transfer function has a MFD. Clearly, the use of left MFD's is arbitrary; one could as well use right MFD's (see Kailath (1980), Rugh (1993)).

1.2. Realization Theory

We now concern ourselves in much more detail with the relation between the transfer function and its (state space and ARMAX-) realizations. Our starting point is always a causal, rational transfer function $(k(z), l(z)) = (\sum_{j=0}^{\infty} K(j)z^j, \sum_{j=0}^{\infty} L(j)z^j)$ with $K(0) = I$ and $L(0) = 0$. This is no restriction of generality, if we arrive at the transfer function by factorizing a spectral density of a stationary process.

THEOREM 1.2. *Let f be a rational and almost everywhere nonsingular spectral density of a stationary stochastic process. Then f may be uniquely factorized as*

$$f(\omega) = \frac{1}{2\pi} k(z) \Sigma k(z)^* \tag{1.2.1}$$

where Σ is a positive definite square matrix and $k(z)$ is a rational transfer function that is analytic in a circle containing the closed complex unit disc. Additionally, $k(0) = I$.

PROOF. See Hannan and Deistler (1988), Chapter 1.3. □

Denote by T_A the set of all polynomial matrices (a, b, d) such that s and m are fixed, $\det a(z) \not\equiv 0$, and $a^{-1}(b, d)$ suffices the conditions for the transfer function given above. Define $\pi : T_A \to U_A, (a, b, d) \mapsto a^{-1}(b, d)$ the mapping attaching to every ARMAX-system in T_A the corresponding transfer function. U_A is the set of all $s \times (s + m)$ rational and causal transfer functions where $(k(0), l(0)) = (I, 0)$. From

Theorem 1.1 we know that π is surjective. Two ARMAX-systems $(a,b,d),(\tilde{a},\tilde{b},\tilde{d})$ that have the same transfer function are called *observationally equivalent*, in other words, $(a,b,d)\sim(\tilde{a},\tilde{b},\tilde{d})$ if and only if $\pi(a,b,d)=\pi(\tilde{a},\tilde{b},\tilde{d})$. It is easy to check that $\sim$ is an equivalence relation.

Usually when one considers a linear system and wants to find an ARMAX representation for it, restrictions on certain parameters are imposed, describing what is a priori known about the system. The class of ARMAX systems under consideration is therefore reduced to a *model class* $\hat{T}\subset T_A$ of ARMAX systems that are of interest to the researcher. One desirable property of the model class is identifiability. The set $\tilde{T}\subset T_A$ is called *identifiable* if $\pi|_{\tilde{T}}$ is injective, or, in other words, if any two observationally equivalent systems in $\tilde{T}$ are identical. Often, this is not the case. Consider the set $\tilde{T}/\pi=\{\pi^{-1}(k,l)\cap\tilde{T}\,|\,(k,l)\in U_A\}$ of equivalence classes with respect to observational equivalence. Taking a representative of each equivalence class gives us an identifiable set of ARMAX systems. This leads to the concept of canonical forms.[3]

In some cases it is possible to distill the information required to describe all ARMAX systems in the model class $\hat{T}$. A bijective mapping $\psi_{\hat{T}}:\hat{T}\to T\subset\mathbb{R}^d$, where d is as small as possible, is called a *parameterization of* $\hat{T}$. The set T is called a *parameter space* for $\hat{T}$ and its elements are called vectors of *free parameters*. Analogously, one defines parameterizations of sets of transfer functions; a parameterization of $U\subset U_A$ is a bijective mapping $\psi_U:U\to T_U\subset\mathbb{R}^{d'}$, d' minimal[4]. Thus, every vector of free parameters contains the whole information of the corresponding ARMAX system (respectively, the transfer function) in a most compressed way, since the parameter space has minimal dimension. Clearly, in the desirable case that $\hat{T}$ is identifiable, a parameter space $T=\psi_{\hat{T}}(\hat{T})$ for $\hat{T}$ is also a parameter space for $\pi(\hat{T})$, since $\pi|_{\hat{T}}$ is bijective itself.

Clearly, what has been said above regarding observational equivalence, identifiability and parameterizations is defined for state space systems in an analogous way. Denote by Δ_A the set of all state space systems (F,H,K,L) for given s and m and denote by $\rho:\Delta_A\to U_A,\rho(F,H,K,L)\mapsto Hz(I_n-Fz)^{-1}(K,L)+(I,0)$ the mapping that attaches to each state space system in Δ_A its transfer function. The rest of the definitions for state space systems are completely analogous to the ones given above, and will be omitted here.

1.3. ARMAX realizations

We proceed to characterizing the observational equivalence classes of the set of all ARMAX systems T_A with given dimension parameters s and m. However, since $\hat{T}/\pi=\{\overline{(k,l)}\cap\hat{T}\,|\,\overline{(k,l)}\in T_A/\pi\}$, even if the model class is not T_A due to restrictions (e.g. stability condition, linear restrictions on the parameters), the following theorem (Hannan and Deistler (1988)) is very useful in identification.

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[3]See Denham (1974).

[4]Clearly, if one starts from an arbitrary causal and rational transfer function, it cannot be taken for granted that the dimension d' of the parameter space is finite. However, if we assume that the transfer function comes from an ARMAX system of given order, then a parameterization can be found such that d' is bounded by the number of elements in the system's representation as matrix polynomials.

THEOREM 1.3. *The ARMAX systems $(\tilde{a}, \tilde{b}, \tilde{d})$ and (a, b, d), the latter being left coprime, are observationally equivalent if and only if there exists a polynomial matrix u such that*

$$(\tilde{a}, \tilde{b}, \tilde{d}) = u(a, b, d). \tag{1.3.1}$$

If $(\tilde{a}, \tilde{b}, \tilde{d})$ is left coprime as well, u is unimodular.

PROOF. See Corollary A.7 to Theorem A.6 in the appendix. □

If (a, b, d) is left coprime, the corresponding MFD $a^{-1}(b, d)$ is called *irreducible.* One can always obtain an irreducible MFD from any MFD by applying elementary column operations (see Lemma A.4). From the above theorem we know that irreducible MFD's are unique up to multiplication by unimodular matrices.

1.4. State space realizations

We first consider properties of the state space system

$$x(t+1) = Fx(t) + Lz(t) + K\varepsilon(t) \tag{1.4.1}$$

$$y(t) = Hx(t) + \varepsilon(t) \tag{1.4.2}$$

where $F \in \mathbb{R}^{n\times n}, L \in \mathbb{R}^{n\times m}, K \in \mathbb{R}^{n\times s}, H \in \mathbb{R}^{s\times n}$.
We call the system (F, H, K, L) or the matrix pair (F, H) *observable* if the matrix

$$\mathcal{O} = \left(H', F'H', F'^2H', \ldots, F'^{n-1}H'\right)' \tag{1.4.3}$$

has full column rank n. Note that including other elements $F'^jH', j \geq n$ to the observability matrix $\mathcal{O}$ will not increase its rank, since by the Cayley-Hamilton theorem, F suffices its own characteristic equation and thus there is a linear combination of $F, F^2, \ldots, F^n$ that is zero.
The name "observability" comes from the fact that if we assume the absence of noise $\varepsilon(t)$, and zero-input, $z(t) = 0$, we can determine the initial state $x(0)$ (and thus the subsequent states $x(t)$) from the observed variables $y(t)$. Suppose that $\varepsilon(t) = z(t) = 0$, then

$$\begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(n-1) \end{pmatrix} = \begin{pmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{pmatrix} x(0). \tag{1.4.4}$$

Now if $\mathcal{O}$ does not have rank n, $x(0)$ is not uniquely determined by (1.4.4). On the other hand, if $\mathcal{O}$ does have rank n, we have

$$x(0) = (\mathcal{O}'\mathcal{O})^{-1}\mathcal{O}'(y(0)', y(1)', \ldots, y(n-1)')'. \tag{1.4.5}$$

Likewise, in the presence of stochastic noise $\varepsilon(t)$ with $\mathbb{E}\varepsilon(t) = 0$ and $z(t) = 0$, formula (1.4.5) gives an unbiased estimator for $x(0)$ (Aoki (1990)).
The system (F, H, K, L) (or, the matrix pair $(F, (K, L))$) is called *controllable* or *reachable* if the matrix

$$\mathcal{C} = \left((K, L), F(K, L), F^2(K, L), \ldots, F^{n-1}(K, L)\right) \tag{1.4.6}$$

has full row rank n. From (1.4.1) we have

$$x(t+1) - F^{t+1}x(0) = \sum_{j=0}^{t} F^j(K,L)(\varepsilon(t-j), z(t-j)) = \mathcal{C}\left(\varepsilon(t)', z(t)', \varepsilon(t-1)', z(t-1)', \ldots, \varepsilon(0), z(0)\right)'. \tag{1.4.7}$$

Thus, if reachability holds, any state $x(t+1), t \geq n-1$ can be reached from any $x(0)$ by certain combination of $\varepsilon(t), z(t), \ldots, \varepsilon(0), z(0)$.
Assumptions of observability and reachability do not restrict generality, as the following theorem shows.

THEOREM 1.4. *Let $(k,l) \in U_A$. Then there exists a state space realization (F,H,K,L) of (k,l) that is both observable and reachable.*

PROOF. From Theorem 1.1 we know that there is a state space realization (F,H,K,L) of (k,l). Suppose it is not observable, and $\mathcal{O}$ has rank $n_0 < n$. Then a $T \in GL(n)$ exists such that the last $n - n_0$ rows of $T'^{-1}\mathcal{O}'$ are zero. Transforming the system and partitioning,

$$(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L}, \tilde{x}(t)) = (TFT^{-1}, HT^{-1}, TK, TL, Tx(t)), \tag{1.4.8}$$

$$\tilde{F} = \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix}, \qquad \tilde{H} = (\tilde{H}_1, 0), \tag{1.4.9}$$

we have

$$T'^{-1}\mathcal{O}' = \begin{pmatrix} \tilde{H}_1' & \tilde{F}_{11}'\tilde{H}_1' & \tilde{F}_{11}'\tilde{F}_{11}'\tilde{H}_1' & \ldots \\ 0 & \tilde{F}_{12}'\tilde{H}_1' & \tilde{F}_{12}'\tilde{F}_{11}'\tilde{H}_1' & \ldots \end{pmatrix}, \tag{1.4.10}$$

the last $n - n_0$ rows being zero by construction of T. This implies $\tilde{H}_1\tilde{F}_{11}\tilde{F}_{12} = 0$. Since $\tilde{H}_1\tilde{F}_{11}$ has full row rank, we have $\tilde{F}_{12} = 0$. Partition the state vector $\tilde{x}(t) = (\tilde{x}_1(t)', \tilde{x}_2(t)')'$. Since

$$y(t) = \tilde{H}_1\tilde{x}_1(t) + \varepsilon(t) \tag{1.4.11}$$

and

$$\begin{pmatrix} \tilde{x}_1(t+1) \\ \tilde{x}_2(t+1) \end{pmatrix} = \begin{pmatrix} \tilde{F}_{11} & 0 \\ \tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{pmatrix} + \begin{pmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{pmatrix} z(t) + \begin{pmatrix} \tilde{K}_1 \\ \tilde{K}_2 \end{pmatrix} \varepsilon(t), \tag{1.4.12}$$

$\tilde{x}_2(t)$ does neither enter the state equation for $\tilde{x}_1(t+1)$, nor the output equation (1.4.11), and can therefore be omitted. The system $(\tilde{F}_{11}, \tilde{H}_1, \tilde{K}_1, \tilde{L}_1)$ is thus observationally equivalent to (F,H,K,L) and observable by equation (1.4.10).
The proof for controllability is similar. Again, suppose that (F,H,K,L) is not controllable. Let $T \in GL(n)$ such that the last rows of $T\mathcal{C}$ are zeros. Transform the system as in (1.4.8) and partition $\tilde{F}$ as in (1.4.9). Then

$$T\mathcal{C} = \begin{pmatrix} (\tilde{K}_1, \tilde{L}_1) & \tilde{F}_{11}(\tilde{K}_1, \tilde{L}_1) & \tilde{F}_{11}\tilde{F}_{11}(\tilde{K}_1, \tilde{L}_1) & \ldots \\ 0 & \tilde{F}_{21}(\tilde{K}_1, \tilde{L}_1) & \tilde{F}_{21}\tilde{F}_{11}(\tilde{K}_1, \tilde{L}_1) & \ldots \end{pmatrix}, \tag{1.4.13}$$

and, by the same argument as above, $\tilde{F}_{21} = 0$. The system $(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L})$ is observationally equivalent to (F,H,K,L). Apply the formula for the inverse of a partitioned matrix to obtain

$$(k,l) = \tilde{H}_1 z(I - \tilde{F}_{11}z)^{-1}(\tilde{K}_1, \tilde{L}_1) + (I, 0). \tag{1.4.14}$$

This is due to the fact that $\tilde{F}_{21} = 0$. The transfer function (k,l) depends only on $\tilde{F}_{11}, \tilde{H}_1, \tilde{K}_1$, and $\tilde{L}_1$ and the corresponding state space realization $(\tilde{F}_{11}, \tilde{H}_1, \tilde{K}_1, \tilde{L}_1)$ is thus observationally equivalent to (F,H,K,L), albeit its state vector has lower dimension. Finally, note that $(\tilde{F}_{11}, \tilde{H}_1, \tilde{K}_1, \tilde{L}_1)$ is controllable, since the first rows of $T\mathcal{C}$ are linearly independent (eq. (1.4.13)).
If the original system (F,H,K,L) was observable, then $(\tilde{F}_{11}, \tilde{H}_1, \tilde{K}_1, \tilde{L}_1)$ is also observable, since the first rows of the observability matrix are of full rank as well. □

1.5. The Hankel Matrix and System Minimality

Let $(k,l) \in U_A$, $(k,l) = \sum_{j=1}^{\infty}(K(j), L(j))z^j + (I, 0)$. We define

$$\mathscr{H}_i^j = \begin{pmatrix} K(1), L(1) & K(2), L(2) & \dots & K(j), L(j) \\ K(2), L(2) & K(3), L(3) & \dots & K(j+1), L(j+1) \\ \vdots & \vdots & \ddots & \vdots \\ K(i), L(i) & K(i+1), L(i+1) & \dots & K(i+j-1), L(i+j-1) \end{pmatrix} \tag{1.5.1}$$

where $i, j \in \mathbb{N} \cup \{\infty\}$. The matrix $\mathscr{H} := \mathscr{H}_\infty^\infty$ is called the *block Hankel matrix corresponding to* (k,l). Denote the j-th row in the i-th block of rows by $h(i,j)$. Clearly, $\mathscr{H}$ contains exactly the information of the transfer function; in other words, there is a bijection that attaches to each $(k,l) \in U_A$ the corresponding Hankel matrix. For state space systems, we have

$$\mathscr{H}_n^n = \mathcal{O}\mathcal{C}, \qquad \mathscr{H} = \left(H', F'H', F'^2H', \dots\right)' \left((K,L), F(K,L), F^2(K,L), \dots\right). \tag{1.5.2}$$

The Hankel matrix is, as we shall see later, at the very heart of the theory of state space realizations. However, it is also very convenient when dealing with observability and controllability matrices. In order to find sensible realizations, we need to define the properties we desire. One of them is that we do not want redundant information in the state vector. More precisely, we want the dimension of the state vector (and thus the dimension of the transition matrix F) of a state space system (F,H,K,L) to be minimal among all realizations of the transfer function. If this property is fulfilled, we call the system *minimal.*

THEOREM 1.5. *Let (F,H,K,L) be a state space realization of (k,l) with $F \in \mathbb{R}^{n \times n}$. Then the following statements are equivalent:*

(i) *(F,H,K,L) is minimal.*
(ii) *(F,H,K,L) is both observable and reachable.*
(iii) *$\mathscr{H}_n^n = \mathcal{O}\mathcal{C}$ has rank n.*

PROOF. *(ii) ⇒ (iii).* $\mathcal{O}$ and $\mathcal{C}$ have full rank n, thus $\mathcal{O}'\mathcal{O}$ and $\mathcal{C}\mathcal{C}'$ are $n \times n$ and nonsingular. Therefore, $\mathcal{O}'\mathcal{O}\mathcal{C}\mathcal{C}'$ is also $n \times n$ and nonsingular, and so must be $\mathscr{H}_n^n = \mathcal{O}\mathcal{C}$.
(iii) ⇒ (ii). Let (F,H,K,L) be either nonobservable or nonreachable. Then either $\mathcal{O}$ or $\mathcal{C}$ has rank less than n, and so must have $\mathscr{H}_n^n = \mathcal{O}\mathcal{C}$.
(i) ⇒ (ii). If (F,H,K,L) is nonobservable or nonreachable, the dimension of the state vector may be reduced, as shown in the proof of Theorem 1.4.

(iii) $\Rightarrow$ *(i)*. Let $(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L})$, $\tilde{F} \in \mathbb{R}^{\tilde{n}\times\tilde{n}}, \tilde{n} < n$ be another realization of (k,l). By the Cayley-Hamilton Theorem, $\tilde{F}$ satisfies its own characteristic equation and thus $\left(\tilde{H}, \tilde{F}'\tilde{H}', \ldots, \tilde{F}'^n\tilde{H}'\right)'$ has rank less or equal $\tilde{n}$. Hence, $\mathscr{H}_n^n$ has rank less or equal $\tilde{n}$, contradicting (iii). $\square$

The next theorem (Kailath (1980)) completely characterises the class of minimal state space realizations.

THEOREM 1.6. *Let (F, H, K, L) be a minimal realization of the transfer function $(k, l) \in U_A$. Then the system $(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L})$ is observationally equivalent and minimal if and only if there exists a $T \in GL(n)$ such that*

$$(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L}) = (TFT^{-1}, HT^{-1}, TK, TL). \tag{1.5.3}$$

PROOF. One direction is simple and has been already mentioned several times. If (1.5.3) holds, $(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L})$ has the same transfer function (k,l) as (F, H, K, L) and is minimal.
Let $(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L})$ be observationally equivalent to (F, H, K, L) and minimal. Then $\mathscr{H}_n^n = \mathcal{O}\mathcal{C} = \tilde{\mathcal{O}}\tilde{\mathcal{C}}$. Define

$$T = \mathcal{C}\tilde{\mathcal{C}}'(\tilde{\mathcal{C}}\tilde{\mathcal{C}}')^{-1}, \qquad \tilde{T} = (\tilde{\mathcal{O}}'\tilde{\mathcal{O}})^{-1}\tilde{\mathcal{O}}'\mathcal{O}, \tag{1.5.4}$$

we then have

$$\tilde{\mathcal{O}} = \mathcal{O}T, \qquad \tilde{\mathcal{C}} = \tilde{T}\mathcal{C}. \tag{1.5.5}$$

Furthermore, we obtain

$$\tilde{T}T = (\tilde{\mathcal{O}}'\tilde{\mathcal{O}})^{-1}\tilde{\mathcal{O}}'\mathcal{O}\mathcal{C}\tilde{\mathcal{C}}'(\tilde{\mathcal{C}}\tilde{\mathcal{C}}')^{-1} = I \tag{1.5.6}$$

since $\mathcal{O}\mathcal{C} = \tilde{\mathcal{O}}\tilde{\mathcal{C}}$. Thus, $\tilde{T} = T^{-1}$. We have

$$\mathcal{O}F\mathcal{C} = \begin{pmatrix} K(2), L(2) & K(3), L(3) & \ldots & K(n+1), L(n+1) \\ K(3), L(3) & K(4), L(4) & \ldots & K(n+2), L(n+2) \\ \vdots & \vdots & \ddots & \vdots \\ K(n+1), L(n+1) & K(n+2), L(n+2) & \ldots & K(2n+1), L(2n+1) \end{pmatrix} = \tilde{\mathcal{O}}\tilde{F}\tilde{\mathcal{C}} \tag{1.5.7}$$

and thus

$$\tilde{F} = (\tilde{\mathcal{O}}'\tilde{\mathcal{O}})^{-1}\tilde{\mathcal{O}}'\mathcal{O}F\mathcal{C}\tilde{\mathcal{C}}'(\tilde{\mathcal{C}}\tilde{\mathcal{C}}')^{-1} = T^{-1}FT. \tag{1.5.8}$$

From (1.5.5) we have from the first rows (columns)

$$\tilde{H} = HT, \qquad (\tilde{K}, \tilde{L}) = T^{-1}(K, L), \tag{1.5.9}$$

which proves the theorem. Note that the transformation matrix T is unique: let $\tilde{T}$ be another transformation matrix relating (F, H, K, L) and $(\tilde{F}, \tilde{H}, \tilde{K}, \tilde{L})$, then from (1.5.5) we have

$$\mathcal{O}(T - \tilde{T}) = 0 \tag{1.5.10}$$

and since $\mathcal{O}$ has full rank, we have $T = \tilde{T}$. $\square$

1.6. The Forward Transfer Function and Structure Indices

As we have seen above, the relationship between minimal state space realizations of a transfer function (k,l) is a very simple one. We now want to have a nice representation of the order of minimal systems in terms of (k,l). Unfortunately, this is not easily possible. We need to define an associated transfer function to (k,l) whose input-output behaviour is the same as for (k,l). Let $(k,l) = \sum_{j=1}^{\infty}(K(j),L(j))z^j + (I,0)$, then define the *forward transfer function* as

$$(\tilde{k}(z),\tilde{l}(z)) = \sum_{j=1}^{\infty}(K(j),L(j))z^{-j} = (k(z^{-1}),l(z^{-1}) - (I,0). \tag{1.6.1}$$

The forward transfer function has a number of advantages, most of which will become evident later. However, note that $(\tilde{k},\tilde{l})$ is *strictly proper*: all degrees of the denominator polynomials are higher than the degrees of the numerators.

Let us concern ourselves with the relationship between MFD's of (k,l) and of $(\tilde{k},\tilde{l})$. Let $(\tilde{k},\tilde{l}) = \tilde{a}^{-1}(\tilde{b},\tilde{d})$ with $\tilde{a},\tilde{b},\tilde{d}$ matrix polynomials of degrees $\tilde{p},\tilde{q},\tilde{r}$, respectively. Then $(k(z^{-1}),l(z^{-1})) = \tilde{a}^{-1}(z)(\tilde{b}(z),\tilde{d}(z))+ (I,0)$ and $(\tilde{k},\tilde{l})$ realizes the vector difference equation

$$\tilde{a}(z^{-1})y(t) = \tilde{d}(z^{-1})z(t) + (\tilde{a}(z^{-1}) + \tilde{b}(z^{-1}))\varepsilon(t). \tag{1.6.2}$$

Thus, a MFD for (k,l) is given by

$$(a(z),b(z),d(z)) = \operatorname{diag}(z^{n_i})(\tilde{a}(z^{-1}),\tilde{a}(z^{-1}) + \tilde{b}(z^{-1}),\tilde{d}(z^{-1})) \tag{1.6.3}$$

where n_i denotes the degree of the i-th row of $\tilde{a}(z)$. Since $(\tilde{k},\tilde{l})$ ought to be stricly proper, n_i is greater than the corresponding row degrees of $\tilde{b}$ and $\tilde{d}$. Hence, (1.6.3) defines polynomial matrices (a,b,d).

LEMMA 1.7. *Let $\tilde{a}$ be row reduced.*[5] *Then $\tilde{a}^{-1}(\tilde{b},\tilde{d})$ is strictly proper if and only if the rows of $\tilde{b}$ and $\tilde{d}$ are of degree less than those of $\tilde{a}$.*

PROOF. As we have already observed, if $(\tilde{k},\tilde{l})$ is strictly proper, since $\tilde{a}(z)\left(\sum_{j=1}^{\infty}(K(j),L(j))z^{-j}\right) = (\tilde{b},\tilde{d})$, the row degrees of $\tilde{b}$ and $\tilde{d}$ are less than those of $\tilde{a}$.

As regards the other direction, apply Cramer's rule to the equation $(\tilde{k},\tilde{l}) = \tilde{a}^{-1}(\tilde{b},\tilde{d})$. The (i,j)-entry of $\tilde{k}$ is given by $\tilde{k}_{ij} = (\det\tilde{a})^{-1}\det(\tilde{a}^{ij}_{\tilde{b}})$, where $\tilde{a}^{ij}$ is the matrix $\tilde{a}$ with the j-th column of $\tilde{b}$ instead of the i-th column. The rows of $\tilde{b}$ have degrees less than the rows of $\tilde{a}$, so the column end matrix $[\tilde{a}^{ij}_{\tilde{b}}]_r$ has zeros in the j-th column and is singular. Thus the degree of $\det(\tilde{a}^{ij}_{\tilde{b}})$ is lower than the sum of the row degrees of $\tilde{a}$ (see Lemma A.1). However, due to the assumption that $\tilde{a}$ is row reduced, the degree of its determinant is the sum of its row degrees, establishing that $\tilde{k}$ is strictly proper. The proof for $\tilde{l}$ is analogous. □

Denote by $M(n)$ the set of all strictly proper forward transfer functions $(\tilde{k},\tilde{l})$. Since there is a one-to-one relation between the $(\tilde{k},\tilde{l})$ and the (k,l), we identify those two with each other and write $M(n) \subset U_A$.

We now return to the question of the dimension of minimal state space realizations. Consider the set of all irreducible MFDs $(\tilde{a},\tilde{b},\tilde{d})$ of $(\tilde{k},\tilde{l})$. The degree of $\det\tilde{a}$ is an invariant for all $\tilde{a}$ in this set (Theorem A.6, (iv)), and we call it the *order* or *McMillan degree* of $(\tilde{k},\tilde{l})$. We will see in a moment that n equals the dimension of the row space of $\mathscr{H}$, which itself will turn out to be the dimension of minimal state space realizations of (k,l).

First a useful lemma for dealing with the rows of $\mathscr{H}$. Recall that we have denoted the j-th row in the i-th block of rows of $\mathscr{H}$ by $h(i,j)$.

[5]See Appendix A or Kailath (1980), Section 6.3, for a review of row reduced matrices.

LEMMA 1.8. *Let $h(i,j)$ be a linear combination of $h(i_1,j_1), h(i_2,j_2), \dots, h(i_k,j_k)$. Then $h(i+1,j)$ is a linear combination of $h(i_1+1,j_1), h(i_2+1,j_2), \dots, h(i_k+1,j_k)$ with the same coefficients.*

PROOF. For any row $h(i,j)$ of $\mathscr{H}$, we have $h(i,j) = (K(i), L(i), h(i+1,j))$ due to the block Hankel structure. The claim follows immediately. ☐

THEOREM 1.9. (i) *The Hankel matrix $\mathscr{H}$ has finite rank if and only if (k,l) is rational.*
(ii) *The rank of $\mathscr{H}$ equals the McMillan degree of $(\tilde{k},\tilde{l})$.*

PROOF. (i) Suppose that (k,l) is rational. Then its associated $(\tilde{k},\tilde{l})$ is rational as well. Let $(\tilde{a},\tilde{b},\tilde{d})$ be an irreducible MFD of $(\tilde{k},\tilde{l})$ and compare the coefficients of the negative powers of z in

$$\left(\tilde{A}(0) + \tilde{A}(1)z + \tilde{A}(2)z^2 + \cdots\right)\left(\sum_{j=1}^{\infty}(K(j),L(j))z^{-j}\right) = \left((\tilde{B}(0),\tilde{D}(0)) + (\tilde{B}(1),\tilde{D}(1))z + \cdots\right) \quad (1.6.4)$$

to obtain

$$\left(\tilde{A}(0), \tilde{A}(1), \dots\right)\mathscr{H} = 0. \quad (1.6.5)$$

From Theorem 1.1 we know that (k,l) has a state space realization. Its infinite observability and reachability matrices

$$\left(H', F'H', F'^2H', \dots\right)', \qquad \left((K,L), F(K,L), F^2(K,L), \dots\right) \quad (1.6.6)$$

must have finite rank (Cayley-Hamilton Theorem), thus also their product $\mathscr{H}$.

Conversely, let $\mathscr{H}$ be of finite rank n. Then the matrix $\mathscr{H}_n^{\infty}$ is also of rank n. To prove this, assume the contrary: let $\mathscr{H}_n^{\infty}$ have rank $n_0 < n$. We now construct a basis[6] for the row space of $\mathscr{H}_n^{\infty}$. Starting from the first row, we select a row to be in the basis if it is not in the span of the previously selected rows. This way, every row of $\mathscr{H}_n^{\infty}$ is either in the basis, or is a linear combination of basis rows *above* itself. By Dirichlet's principle, there is a block of s rows such that none of them is in the basis, thus each of them can be expressed as a linear combination of previous basis rows (unless the block is the first one, in which case $(k,l) = 0$). By Lemma 1.8, all rows below the block can also be expressed as linear combinations of previous rows. Therefore $\mathscr{H}$ has rank n_0, which contradicts our assumption. Thus, $\mathscr{H}_n^{\infty}$ has rank n, and the row $(K(n+1), L(n+1), K(n+2), L(n+2), \dots)$ is a linear combination of the rows of $\mathscr{H}_n^{\infty}$,

$$(\tilde{A}(0), \tilde{A}(1), \dots, \tilde{A}(n))\mathscr{H}_{n+1}^{\infty}. \quad (1.6.7)$$

This way, we have defined a matrix polynomial $\tilde{a}$ that suffices (1.6.5). Define $(\tilde{b},\tilde{d})$ from equation (1.6.4) and we have obtained an MFD $\tilde{a}^{-1}(\tilde{b},\tilde{d})$ of $(\tilde{k},\tilde{l})$. Clearly, (k,l) is also rational.

(ii) Construct a basis of the row space of $\mathscr{H}$ as in (i). Denote by n_i the number of basis rows that are on the i-th position within a block. Clearly then, the dimension of the row space of $\mathscr{H}$ is $n = n_1 + \cdots + n_s$. Let $(\tilde{a},\tilde{b},\tilde{d})$ be left prime MFD of $(\tilde{k},\tilde{l})$ and in polynomial Echelon form (see

[6]This basis will play an important role later on, when we define echelon realizations. It is the basis selected by the Kronecker indices (see section 1.8).

Theorem A.9). Assume that $\nu(\det(\tilde{a})) < n$, then there must be a row i in $\tilde{a}$ where $\nu(\tilde{a}_{ii}) < n_i$. However, equation (1.6.5) then shows that the corresponding basis rows on the i-th positions within blocks are linearly dependent, a contradiction. Analogously, if $\nu(\det(\tilde{a})) > n$, $(\tilde{a}, \tilde{b}, \tilde{d})$ cannot be minimal.

□

1.7. Minimal state space realizations

The following derivation of minimal state space realizations is based on the work by Akaike (1974). In the case of white noise input, it provides a nice interpretation of the state vector as a basis of the predictor space for future output.

Consider the case where there is no observed input z, thus $l = 0$ (or, equivalently, z is white noise, rendering it indistinguishable from ε). For simplicity, let $\Sigma = \mathbb{E}\varepsilon_t\varepsilon_t' = I$. We then have

$$\mathbb{E}y(t+l)\varepsilon(t)' = \mathbb{E}\left(\sum_{j=0}^{\infty} K(j)\varepsilon(t+l-j)\right)\varepsilon(t) = K(l) \tag{1.7.1}$$

Denote by $\mathcal{H}(\varepsilon(t-))$ the Hilbert space generated by $\varepsilon(t-1), \varepsilon(t-2), \dots$, that is, the closure (with respect to the mean-square norm) of the linear space generated by the finite linear combinations of those vectors' components. Likewise, denote by $\mathcal{H}(y(t+))$ the Hilbert space generated by the components of $y(t), y(t+1), \dots$. Let $y(t+l|t-)$ be the l-step predictors[7] of $y(t+l)$ from $\mathcal{H}(\varepsilon(t-))$, that is, the orthogonal projection of $y(t+l)$ on the latter space. From the projection theorem (see any textbook on functional analysis, e.g. Yoshida (1995)) we have

$$\mathbb{E}(y(t+l) - y(t+l|t-))\varepsilon(t-j)' = 0 \tag{1.7.2}$$

for all $j > 0$; and, together with (1.7.1),

$$\mathbb{E}y(t+l|t-)\varepsilon(t-j)' = K(l+j). \tag{1.7.3}$$

Due to the whiteness of the components of $\varepsilon(t)$, the predictors are given by

$$y(t+l|t-) = \sum_{j=1}^{\infty} K(j+l)\varepsilon(t-j). \tag{1.7.4}$$

We now want to select a basis for the predictor space $\mathcal{H}(y(t+)|\varepsilon(t-))$, the Hilbert space generated by projecting the l-step predictors for $y(t+l)$, where $l = 0, 1, \dots$, onto the space $\mathcal{H}(\varepsilon(t-))$. Due to (1.7.3), we have

$$\begin{pmatrix} y(t|t-) \\ y(t+1|t-) \\ \vdots \end{pmatrix} = \begin{pmatrix} K(1) & K(2) & K(3) & \dots \\ K(2) & K(3) & K(4) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \varepsilon(t-1) \\ \varepsilon(t-2) \\ \vdots \end{pmatrix} = \mathscr{H}\varepsilon(t-) \tag{1.7.5}$$

where $\varepsilon(t-) := (\varepsilon(t-1)', \varepsilon(t-2)', \dots)'$. Thus, selecting a basis of the predictor space is equivalent to selecting a basis for the row space of $\mathscr{H}$. From Lemma (1.8) we know that the row space is finite dimensional if and only if the transfer function is rational, which we assume throughout. Let $x(t)$ be a

[7]Clearly, this l has nothing to do with the l from the transfer function (k, l).

basis of $\mathcal{H}(y(t+)|\varepsilon(t-))$, consisting of n elements. Such a basis could be constructed by selecting rows from $\mathscr{H}$ by means of a selector matrix S such that $S\mathscr{H}$ is a matrix containing the basis rows (as is used, for example, by Hannan and Deistler (1988)). However, for the sake of generality, we will in this construction abstain from limiting ourselves to selecting rows from $\mathscr{H}$. Suffice to say that all basis vectors are related by a nonsingular transformation.

From the transfer function we have, since $K(0) = I$,

$$y(t) = y(t|t-) + \varepsilon(t). \tag{1.7.6}$$

Clearly, $y(t|t-) \in \mathcal{H}(y(t+)|\varepsilon(t-))$, therefore, there is a unique $H \in \mathbb{R}^{s\times n}$ such that $y(t|t-) = Hx(t)$. Then equation (1.7.6) is equivalent to

$$y(t) = Hx(t) + \varepsilon(t). \tag{1.7.7}$$

Define $x(t+1)$ as the same linear combination of $\varepsilon(t), \varepsilon(t-1), \ldots$ that was used to define $x(t)$, with the exception that the indices are incremented by one; i.e. the coefficient of $\varepsilon(t-1)$ is now the coefficient of $\varepsilon(t)$, the coefficient of $\varepsilon(t-2)$ is now the coefficient of $\varepsilon(t-1)$ and so on. Clearly, $x(t+1) \in \mathcal{H}(y((t+1)+)|\varepsilon((t+1)-)) \subset \mathcal{H}(\varepsilon((t+1)-))$. Since $\mathcal{H}(\varepsilon(t-)) \subset \mathcal{H}(\varepsilon((t+1)-))$ and $\varepsilon(t+1)$ is orthogonal to $\mathcal{H}(\varepsilon(t-))$, $x(t+1)$ can be uniquely decomposed as

$$x(t+1) = x_1(t+1) + x_2(t+1) \tag{1.7.8}$$

where $x_1(t+1) \in \mathcal{H}(y(t+)|\varepsilon(t-))$ and $x_2(t+1)$ is in the Hilbert space spanned by the components of $\varepsilon(t)$. Thus, there are unique $F \in \mathbb{R}^{n\times n}, K \in \mathbb{R}^{n\times s}$ such that

$$x(t+1) = Fx(t) + K\varepsilon(t). \tag{1.7.9}$$

The equations (1.7.7) and (1.7.9) together form a state space realization of the transfer function k, in which the state vector $x(t)$ contains all information about the future output $y(t+l), k = 0, 1, \ldots$ that is available at time t, in terms of the past inputs, $\varepsilon(t-l), l = 1, 2, \ldots$. Clearly, the dimension of $x(t)$ is the rank of $\mathscr{H}$, thus it is minimal. Having defined the state as a basis of the predictor space, all other possible basis selections are related by nonsingular transformations – exactly those matrices T in Theorem 1.6 that link minimal state space representations. It follows that the construction shown above characterises *all* minimal state space realizations of the transfer function.

This construction also works if there is an exogenous input process $z(t)$; however, due to the $z(t)$ being nonwhite, the interpretation of the state vector as a basis of the predictor space is lost. For notational convenience, we now use a selector matrix S to represent the chosen basis of $\mathscr{H}$. Each element in S is zero, save for a unit in the column we choose to have in the basis. Additionally, let S have n rows, where n is the row rank of $\mathscr{H}$, such that

$$\mathscr{H}_\alpha^\infty := S\mathscr{H} \tag{1.7.10}$$

consists of the n chosen basis rows. Following Hannan and Deistler (1988), define the state as

$$x(t) = \mathscr{H}_\alpha^\infty \left(\varepsilon(t-1)', z(t-1)', \varepsilon(t-2)', z(t-2)', \ldots\right)' \tag{1.7.11}$$

and the system matrices (F, H, K, L) from the equations

$$F\mathscr{H}_\alpha^\infty = S\begin{pmatrix} K(2) & K(3) & \dots \\ K(3) & K(4) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.7.12}$$

and

$$(K, L) = S\left((K(1), L(1))', (K(2), L(2))', \dots\right)', \qquad H\mathscr{H}_\alpha^\infty = (K(1), L(1), K(2), L(2), \dots). \tag{1.7.13}$$

Note that F and H are well defined due to the fact that – by construction – $\mathscr{H}_\alpha^\infty$ has full row rank n. It is then easy to verify that (F, H, K, L) satisfies

$$x(t+1) = Fx(t) + K\varepsilon(t) + Lz(t) \tag{1.7.14}$$

$$y(t) = Hx(t) + \varepsilon(t). \tag{1.7.15}$$

Again, (F, H, K, L) is minimal due to the dimension of $x(t)$ being n, the rank of $\mathscr{H}$.

1.8. Irreducible ARMAX realizations

Although the concepts of minimality of state space systems and irreducibility of MFD's are somewhat similar, we will not be able to extend the idea of choosing a basis of the row space of $\mathscr{H}$ to obtain a unique realization to ARMAX systems, since basis vectors (and thus minimal state space realizations) are related by nonsingular matrix transformations, whereas irreducible ARMAX realizations are unique up to premultiplication by unimodular polynomial matrices – the former being a proper subset of the latter. However, by virtue of Theorem 1.3, if we find a canonical form that gives us one minimal ARMAX realization, all others are then given by premultiplying it by unimodular matrices.
Starting again from the forward transfer function $(\tilde{k}, \tilde{l})$, we seek a MFD such that

$$\tilde{a}(\tilde{k}, \tilde{l}) = (\tilde{b}, \tilde{d}) \tag{1.8.1}$$

holds. Comparing the coefficients on the left and on the right side yield the two equations

$$(\tilde{A}(0), \tilde{A}(1), \dots)\mathscr{H} = 0 \tag{1.8.2}$$

and

$$(\tilde{A}(0), \tilde{A}(1), \dots)\begin{pmatrix} 0 & 0 & 0 & \cdots \\ K(1), L(1) & 0 & 0 & \cdots \\ K(2), L(2) & K(1), L(1) & 0 & \cdots \\ K(3), L(3) & K(2), L(2) & K(1), L(1) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = (\tilde{B}(0), \tilde{D}(0), \tilde{B}(1), \tilde{D}(1), \dots). \tag{1.8.3}$$

In order to construct a MFD of $(\tilde{k}, \tilde{l})$, we search for an $\tilde{a}$ such that (1.8.2) holds and then determine $\tilde{b}, \tilde{d}$ from (1.8.3). We will do this for the special case of *echelon ARMAX realizations* (see, for example, Dickinson et al. (1974)).

We first describe a basis for the row space of $\mathscr{H}$ (however – as mentioned above – this does not uniquely determine an ARMAX realization), the so-called *Kronecker basis*. We start from the first row, $h(1,1)$ of $\mathscr{H}$, selecting it for inclusion in the basis if it is not the row constisting solely of zeros. In each step, consider the next row and select it if it is not in the linear span of the previously selected rows (or, equivalently, if it is not in the linear span of *all* rows above itself). This procedure yields a unique basis of the row space of $\mathscr{H}$. Due to the structure of the Hankel matrix, Lemma 1.8, the rows selected for inclusion in the basis have the following property: for each i, if $h(n_i, i)$ is in the basis, so is $h(j,i)$, for all $j < n_i$. Thus, we can describe the selection by a *structure index* $(n_1, n_2, \dots, n_s)$, which means that the basis consists of the rows $h(1,1), \dots, h(n_1,1), h(1,2), \dots, h(n_2,2), \dots, h(1,s), \dots, h(n_s,s)$. If $n_i = 0$, no $h(j,i), j > 0$ are included in the basis. Clearly, we have $n_1 + n_2 + \cdots + n_s = n$, where n denotes the rank of $\mathscr{H}$. We call the structure index $(n_1, n_2, \dots, n_s)$ corresponding to the Kronecker basis the *Kronecker indices* of $\mathscr{H}$.

For echelon ARMAX-realizations, we express $-h(n_i+1, i)$ as a linear combination of the preceding basis rows of the Kronecker basis. The unique coefficients are then used as elements of $\tilde{A}(0), \tilde{A}(1), \dots$ in (1.8.2), with all other entries set to zero. Writing what has just been said in equation form, we obtain unique $\tilde{a}_{ij}(u)$ from

$$-h(n_i+1, i) = \sum_{j=1}^{s} \sum_{u=1}^{n_{ij}} \tilde{a}_{ij}(u-1) h(u,j), \qquad i = 1, \dots, s \tag{1.8.4}$$

where

$$n_{ij} = \begin{cases} \min(n_i+1, n_j) & \text{for } j \le i \\ \min(n_i, n_j) & \text{for } j > i. \end{cases} \tag{1.8.5}$$

Use these $\tilde{a}_{ij}(u)$ together with $\tilde{a}_{ii}(n_i) = 1$, $i = 1, \dots, s$ to define a matrix polynomial $\tilde{a} = (\tilde{A}(0), \tilde{A}(1), \dots)$ where all other entries are zero. Then $\tilde{a}$ satisfies equation (1.8.2). Calculating $\tilde{b}, \tilde{d}$ from (1.8.3) yields a MFD $(\tilde{k}, \tilde{l}) = \tilde{a}^{-1}(\tilde{b}, \tilde{d})$. Use then formula (1.6.3) to obtain the realization (a,b,d) of (k,l), the so-called *reversed echelon ARMAX realization.* We call the realization $(\tilde{a}(z^{-1}), \tilde{a}(z^{-1}) + \tilde{b}(z^{-1}), \tilde{d}(z^{-1}))$ the *echelon ARMAX realization* of (k,l). Since the Kronecker indices corresponding to a transfer function are unique, the echelon ARMAX realization is also unique, as shown during the construction. We have (Dickinson et al. (1974)):

Theorem 1.10. *Let $(k,l) \in U_A$ with Kronecker indices $(n_1, n_2, \dots, n_s)$. Then the realization*

$$(\tilde{a}(z^{-1}), \tilde{a}(z^{-1}) + \tilde{b}(z^{-1}), \tilde{d}(z^{-1})) \tag{1.8.6}$$

is the echelon ARMAX realization defined by (1.8.4) and (1.8.2) if and only if it satisfies the following properties:

(i) *$(\tilde{a}, \tilde{b}, \tilde{d})$ is left coprime.*

(ii) *The $\tilde{a}_{ii}$ are monic polynomials. Furthermore,*

$$\nu(\tilde{a}_{ij}) \leq \nu(\tilde{a}_{ii}) = n_i, \quad j \leq i \tag{1.8.7}$$

$$\nu(\tilde{a}_{ij}) < \nu(\tilde{a}_{ii}), \qquad j > i \tag{1.8.8}$$

$$\nu(\tilde{a}_{ji}) < \nu(\tilde{a}_{ii}), \qquad j \neq i \tag{1.8.9}$$

$$\nu(\tilde{b}_{ij}) < \nu(\tilde{a}_{ii}), \quad \nu(\tilde{d}_{ij}) < \nu(\tilde{a}_{ii}), i, j = 1, \dots, s. \tag{1.8.10}$$

Proof. Let $\tilde{a}, \tilde{b}, \tilde{d}$ be the matrix polynomials corresponding to the echelon ARMAX realization. The first two sets of inequalities in (ii) follow directly from $\tilde{a}_{ii}$ being the rightmost element in each row of $(\tilde{A}(0), \tilde{A}(1), \dots)$. Likewise, the third set of inequalities also follows from equation (1.8.4) and the fourth and fifth set follow from (1.8.3). Thus, the column end matrix of $\tilde{a}$ is a lower diagonal matrix and $\tilde{a}$ is row reduced. By Lemma A.1, $n_1 + n_2 + \dots + n_s = \nu(\det \tilde{a})$ holds and thus $(\tilde{a}, \tilde{b}, \tilde{d})$ is left coprime (Theorem A.6).

Let $(\tilde{a}, \tilde{b}, \tilde{d})$ satisfy (i) and (ii). Then from $\tilde{a}$ being row reduced, $\nu(\det \tilde{a}) = \sum_i n_i = n$ and by Theorem A.6, $(\tilde{k}, \tilde{l})$ has McMillan degree n. Hence, by Theorem 1.9, $\mathscr{H}$ has rank n. From the first two sets of inequalities in (ii) follows that in the i-th row of $(\tilde{A}(0), \tilde{A}(1), \dots)$ the rightmost element is a unity in column $s(\nu(\tilde{a}_{ii}) + 1) + i$. From (1.8.2) follows that the rows $h(j, i), j \leq \nu(\tilde{a}_{ii}), i = 1, \dots, s$ span the row space of the Hankel matrix. Let $(n_1, n_2, \dots, n_s)$ be the Kronecker indices, then we must have $\nu(\tilde{a}_{ii}) \geq n_i$. However, since $\sum n_i = \sum \nu(\tilde{a}_{ii}) = n$ we have $(\nu(\tilde{a}_{11}), \nu(\tilde{a}_{22}), \dots, \nu(\tilde{a}_{ss})) = (n_1, n_2, \dots, n_s)$ and $(\tilde{a}, \tilde{a} + \tilde{b}, \tilde{d})$ is the echelon ARMAX realization. □

The following corollary summarizes the properties of reversed echelon ARMAX realizations.

Corollary 1.11. *Let $(k, l) \in U_A$ with Kronecker indices $(n_1, n_2, \dots, n_s)$. Then the reversed echelon ARMAX realization (a, b, d) defined by (1.8.4),(1.8.5),(1.8.3),(1.6.3) has the following properties*

(i) *(a, b, d) is left coprime.*

(ii) *$A(0)$ is lower triangular with a main diagonal consisting of unit entries. Furthermore, $D(0) = 0$. The degree of the i-th row of (a, b, d) is n_i.*

Proof. (a, b, d) is defined from $(\tilde{a}, \tilde{b}, \tilde{d})$ via (1.6.3). From Theorem 1.10 (ii) we have that the row end matrix of $\tilde{a}$ is lower triangular. Thus, $A(0) = \mathrm{diag}(z^{n_i})[\tilde{a}]_r$ is also lower triangular. In the same way, the property of $A(0)$ having unit entries in the main diagonal follows from the the $\tilde{a}_{ii}$ being monic polynomials. Since $(\tilde{a}, \tilde{b}, \tilde{d})$ is left coprime, so is $(\tilde{a}(z^{-1}), \tilde{b}(z^{-1}), \tilde{d}(z^{-1}))$. From Theorem A.6 we know that it has row rank s for $z \neq 0$. Hence, $(\tilde{a}(z^{-1}), \tilde{a}(z^{-1}) + \tilde{b}(z^{-1}), \tilde{d}(z^{-1}))$ also has row rank s for $z \neq 0$. Clearly, $\mathrm{diag}(z^{n_i})$ has also full row rank. Since $(a(0), b(0), d(0)) = (A(0), A(0), 0)$, we have by Theorem A.6 that (a, b, d) is left coprime. $D(0) = 0$ follows immediately from the fourth and fifth set of inequalities of Theorem 1.10 (ii). □

1.9. Echelon state space realizations

The Kronecker basis used to define echelon ARMAX realizations may also be used to define a unique state space realization, the *echelon state space realization* corresponding to a given transfer function. As we will notice, it has some very convenient properties. Following the notation by Hannan and Deistler (1988), we denote the Hankel matrix constisting of the Kronecker basis rows $h(1,1),\dots,h(n_1,1),\dots,h(1,s),\dots,$ $h(n_s,s)$ by $\mathscr{H}_\alpha^\infty$ if $\alpha=(n_1,n_2,\dots,n_s)$ are the Kronecker indices. Let S be the corresponding $\mathbb{R}^{|\alpha|\times\infty}$ selector matrix such that

$$\mathscr{H}_\alpha^\infty = S\mathscr{H}, \tag{1.9.1}$$

where $|\alpha| = n_1+n_2+\cdots+n_s$ denotes the McMillan degree of the transfer function. We then have:

THEOREM 1.12. *Let $(k,l)\in U_A$ with Kronecker indices $\alpha=(n_1,n_2,\dots,n_s)$ and corresponding selector matrix S defined as in (1.9.1). Then a unique state space realization (F,H,K,L), the echelon state space realization, is given by*

$$F\mathscr{H}_\alpha^\infty = S\begin{pmatrix} K(2) & K(3) & \dots \\ K(3) & K(4) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.9.2}$$

$$(K,L) = S\left((K(1),L(1))',(K(2),L(2))',\dots\right)' \tag{1.9.3}$$

$$H\mathscr{H}_\alpha^\infty = (K(1),L(1),K(2),L(2),\dots)\,. \tag{1.9.4}$$

The matrices (F,H,K,L) have the following properties:

(i) *(F,H,K,L) is minimal,*

(ii) *The matrix F has the structure $F=(F_{ij})_{i,j=1,\dots,s}\in\mathbb{R}^{n\times n}$, where $F_{ij}\in\mathbb{R}^{n_i\times n_j}$ is given by*

$$F_{ii} = \begin{pmatrix} 0 & & \\ \vdots & I_{n_i-1} & \\ 0 & & \\ -\tilde{a}_{ii}(0) & \cdots & -\tilde{a}_{ij}(n_i-1) \end{pmatrix} \tag{1.9.5}$$

$$F_{ij} = \begin{pmatrix} & & & 0 & & & \\ -\tilde{a}_{ij}(0) & \cdots & -\tilde{a}_{ij}(n_{ij}) & 0 & \cdots & 0 \end{pmatrix}, \quad i\neq j. \tag{1.9.6}$$

In equation (1.9.6) the n_{ij} are given by (1.8.5).

(iii) *If $n_i>0$ for all $i=1,\dots,s$, then $H=(h_{ij})_{i,j=1\dots s}$ where $h_{ij}\in\mathbb{R}^{1\times n_j}$ and*

$$h_{ij} = \begin{cases} (1,0,\dots,0) & (i=j)\wedge(n_j>0) \\ 0 & \textit{else.} \end{cases} \tag{1.9.7}$$

Conversely, every (F,H,K,L) that satisfies (i)-(iii) is an echelon state space realization of its transfer function.

PROOF. The uniqueness of the echelon state space realization follows directly from the construction; see equations (1.7.10)-(1.7.13) and the preceding paragraph. Clearly, it is the same construction as in these equations, with a specific basis of the row space of $\mathscr{H}$ – namely, the Kronecker basis. Since there is a unique F that suffices (1.9.2) and F as specified by (1.9.5),(1.9.6) does indeed, property (ii) follows. Note that the last row in each block of rows in F gives $h(n_i+1,i)$ as a linear combination of the preceding Kronecker basis rows. Therefore, the $\tilde{a}_{ij}(u)$ are exactly the same as in the echelon ARMAX realization. Property (iii) is shown exactly as (ii).

Now let (F, H, K, L) be a state space realization satisfying (i)-(iii). From (ii) we see that the $h(1,1), \ldots,$ $h(n_1), \ldots,$ $h(1,s), \ldots, h(n_s, s)$ span the whole row space of $\mathscr{H}$, therefore, if $(m_1, \ldots, m_s)$ are the Kronecker indices of the transfer function, $m_i \leq n_i$ must hold for every $i = 1, \ldots, s$. But from (i), $\sum m_i = \sum n_i = n$ and equality must hold. Thus $(m_1, \ldots, m_s)$ are the Kronecker indices and (F, H, K, L) is the echelon state space realization. □

1.10. Parameterizations and topological issues related to echelon realizations

We first introduce a topology for the set U_A of rational and causal transfer functions (k, l) where $(k(0), l(0)) = (I, 0)$, as in section 1.2. Since $(k(z), l(z)) = \sum_{j=0}^{\infty}(K(j), L(j))z^j$ and $(K(j), L(j)) \in \mathbb{R}^{s\times(s+m)}$ for every $j \in \mathbb{N}$, we can identify each $(k, l) \in U_A$ with an element in $(\mathbb{R}^{s\times(s+m)})^{\mathbb{N}}$. We endow the latter space with the product topology of the euclidean space[8] $(\mathbb{R}^{s\times(s+m)})$ and call it the *pointwise topology* T_{pt}. The name is due to the implication that convergence of (k_t, l_t) to (k, l) is equivalent to pointwise convergence of the power series coefficients

$$(K_t(i), L_t(i)) \to (K(i), L(i)), \qquad i \in \mathbb{N}. \tag{1.10.1}$$

If we consider a subset of U_A, we endow it with the corresponding relative topology.

We now use echelon realizations to define parameterizations of the transfer functions. For each set of Kronecker indices $\alpha = (n_1, \ldots, n_s)$, let $V_\alpha \subseteq U_A$ be the set of transfer functions that have α as Kronecker indices. Since every $(k, l) \in U_A$ has exactly one set of Kronecker indices, the V_α constitute a partition of U_A.

For a given Kronecker index $\alpha = (n_1, \ldots, n_s)$, let $(\tilde{a}(z^{-1}), \tilde{a}(z^{-1}) + \tilde{b}(z^{-1}), \tilde{d}(z^{-1}))$ be the reversed echelon ARMAX realization corresponding to a transfer function $(k, l) \in V_\alpha$. Construct a vector τ consisting of all entries in the parameter matrices $A(j), B(j), D(j)$ that are not restriced to zero or unity by the structural properties of the echelon realization, Theorem 1.10 (ii). Clearly the dimension of this vector τ is independent from the chosen $(k, l) \in V_\alpha$. Denote by $\psi_\alpha : V_\alpha \to T_\alpha$ the mapping attaching to every $(k, l) \in V_\alpha$ the corresponding vector τ (where the entries in the parameter matrices of the reversed echelon ARMAX realization are arranged in a certain predetermined order) and $T_\alpha := \psi_\alpha(V_\alpha)$.

The same procedure can be done for state space systems. Define a mapping $\phi_\alpha : V_\alpha \to \Delta_\alpha$ that attaches to every $(k, l) \in V_\alpha$ a parameter vector ϑ that contains (1) the unrestricted parameters in F, (2) the entries of the matrices K, L, where (F, H, K, L) is the echelon state space realization corresponding to (k, l). Again, Δ_α is the set of all vectors ϑ obtained that way, $\Delta_\alpha = \phi_\alpha(V_\alpha)$.

By Theorems 1.10 and 1.12, the mappings ϕ_α and ψ_α are bijective. We partition the vectors $\tau = (\tau_1', \tau_2')'$, where τ_1 contains the $\tilde{a}_{ij}(u)$, and $\vartheta = (\vartheta_1', \vartheta_2')'$, where ϑ_1 contains the unrestricted elements in F. By Theorem 1.12, these unrestricted elements in F are the $\tilde{a}_{ij}(u)$ from the echelon ARMAX realization. Therefore, $\vartheta = (\tau_1', \vartheta_2')'$.

We first give a preliminary result, which will be refined later.

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[8]Since $\mathbb{R}^{s\times(s+m)}$ is a finite dimensional linear space over $\mathbb{R}$, all norms defined on it are equivalent and there is only one norm induced topology.

LEMMA 1.13. *The sets T_α and Δ_α are subsets of $\mathbb{R}^{d_\alpha}$, where*

$$d_\alpha = |\alpha|(s+m) + \sum_{i,j=1,\dots,s} n_{ij} \tag{1.10.2}$$

with n_{ij} defined as in (1.8.5).

PROOF. The claim follows from counting the elements not restricted by Theorems 1.10, respectively 1.12. In the $A(j)$, there are $\sum n_{ij}$, and in the $B(j)$ and $D(j)$ there are $n \cdot s$ and $n \cdot m$, respectively. Analogous for F, K, L. □

It is easy to see that the set of $\tau \in \mathbb{R}^{d_\alpha}$ whose corresponding $(\tilde{a}, \tilde{b}, \tilde{d})$ do not fulfill (i) of Theorem 1.10 is exactly the set $\mathbb{R}^{d_\alpha} \backslash T_\alpha$. More about that later.

THEOREM 1.14. *The mappings $\psi_\alpha : V_\alpha \to T_\alpha$ and $\phi_\alpha : V_\alpha \to \Delta_\alpha$ are T_{pt}-homeomorphisms.*

PROOF. As discussed above, both mappings are bijective. Let $A(j), B(j), D(j)$ be the matrices of the echelon ARMAX realization (the mapping that yields these matrices from τ are the canonical injections, which are continuous). Comparing the coefficients of the positive powers of z in $a(k,l) = (b,d)$ yields the equations

$$(K(1), L(1)) = (A(0))^{-1}((B(1), D(1)) - (A(1), 0)) \tag{1.10.3}$$

$$(K(2), L(2)) = (A(0))^{-1}((B(2), D(2)) - A(1)(K(1), L(1)) - A(2)(K(2), L(2))) \tag{1.10.4}$$

$$\vdots$$

and since $\det A(0) = 1$ (Corollary 1.11) we have, by induction, that π restricted to $\mathbb{R}^{d_\alpha}$ is continuous. Conversely, let (k_t, l_t) be a sequence of transfer functions in V_α converging to $(k,l) \in V_\alpha$. This is equivalent to the $(K_t(j), L_t(j))$ converging to $(K(j), L(j))$. Thus, the Hankel matrices converge element-wise, $\mathscr{H}_t \to \mathscr{H}$, and, if α is the set of Kronecker indices for (k,l), the corresponding submatrices also converge, $\mathscr{H}^\infty_{\alpha,t} \to \mathscr{H}^\infty_\alpha$. The regular matrices are an open set; therefore from a certain t_0 onwards, the $\mathscr{H}^\infty_{\alpha,t}$ have full rank n. The coefficients expressing $h_t(n_i+1, i)$ as a linear combination of preceding basis rows in $\mathscr{H}_t$ also converge, $\tilde{a}_{ij,t} \to \tilde{a}_{ij}$. Thus, the $\tilde{a}_{ij}$ continuously depend on the transfer function (k,l). Equation (1.8.3) then shows that the other parameters τ_2 depend continuously on the $\tilde{a}_{ij}$, thus establishing that ψ_α is continuous, hence a homeomorphism. One proves the claim for ϕ_α in an analogous fashion. □

One beautiful property of echelon realizations is the following corollary:

COROLLARY 1.15. *The mapping $\phi_\alpha \circ \psi_\alpha^{-1}$ that attaches to each $\tau = (\tau_1', \tau_2')' \in T_\alpha$ the $\vartheta = (\tau_1', \vartheta_2')'$ of the echelon state space realization corresponding to the same transfer function is a homeomorphism on $\mathbb{R}^{d_\alpha}$.*

PROOF. ψ_α and ϕ_α are both T_{pt}-homeomorphisms, hence $\phi_\alpha \circ \psi_\alpha^{-1}$ is a homeomorphism on $\mathbb{R}^{d_\alpha}$. □

We now present a stronger version of Lemma 1.13.

THEOREM 1.16. *T_α and Δ_α are open and dense subsets of $\mathbb{R}^{d_\alpha}$.*

PROOF. By Lemma 1.13, T_α and Δ_α are subsets of $\mathbb{R}^{d_\alpha}$. For any $\vartheta \in \Delta_\alpha$, the mappings

$$\vartheta \mapsto (H', F'H', \ldots, F'^{n-1}H') = \mathcal{O}' \tag{1.10.5}$$

$$\vartheta \mapsto ((K, L), F(K, L), \ldots, F^{n-1}(K, L)) = \mathcal{C}, \tag{1.10.6}$$

where (F, H, K, L) is the echelon state space system corresponding to ϑ, are continuous. Let $\vartheta_0 \in \Delta_\alpha$, then the corresponding $\mathcal{O}$ and $\mathcal{C}$ have rank $|\alpha|$ and (F, H, K, L) is minimal. There are neighbourhoods of $\mathcal{O}$ and $\mathcal{C}$, respectively, consisting entirely of matrices (of the same dimensions) that have rank $|\alpha|$ as well. Thus, by the continuity of the mappings above, there is a neighbourhood of ϑ_0 where the corresponding echelon state space systems are minimal. Hence, by Theorem 1.12, Δ_α is open in $\mathbb{R}^{d_\alpha}$. From Corollary 1.15 we deduce that T_α is also open. Consider now the set $\mathbb{R}^{d_\alpha}\backslash\Delta_\alpha$. These are the parameters corresponding to state space systems that satisfy (ii) and (iii) of Theorem 1.12, but fail property (i), minimality. Thus, for a $\vartheta_0 \in \mathbb{R}^{d_\alpha}\backslash\Delta_\alpha$, at least one of the matrices $\mathcal{O}$ and $\mathcal{C}$ does not have full rank $|\alpha|$. Clearly though, since in every neighbourhood of a singular matrix there is a nonsingular one, in every neighborhood of ϑ_0, there is a ϑ such that the corresponding observability and reachability matrices have full rank. Therefore, Δ_α is dense in $\mathbb{R}^{d_\alpha}$. Again, the fact that T_α is dense in $\mathbb{R}^{d_\alpha}$ follows from an analogous argument ($(\tilde{a}, \tilde{b}, \tilde{d})$ are left coprime if and only if the matrix has full row rank s; the matrices that have full row rank are dense in the space of all matrices of the given dimensions). □

The preceding theorem establishes that the τ and ϑ are free parameters and ψ_α and ϕ_α are ARMAX resp. state space parameterizations of V_α. Let $\alpha = (n_1, \ldots, n_s)$ and $\beta = (m_1, \ldots, m_s)$. We introduce the partial order $\leq$ on Kronecker indices, where $\alpha \leq \beta$ means that $n_i \leq m_i$ for all $i = 1, \ldots, s$. If at least one strict inequality holds, we write $\alpha < \beta$. We have (Deistler et al. (1978), Hannan and Deistler (1988), Deistler (1985))

THEOREM 1.17. (i) *The following equality holds:*

$$\pi(\overline{T_\alpha}) = \bigcup_{\beta \leq \alpha} V_\beta. \tag{1.10.7}$$

(ii) *Let $(k, l) \in V_\beta$, where $(m_1, \ldots, m_s) = \beta \leq \alpha(n_1, \ldots, n_s)$. Then the (k, l)-equivalence class in $\overline{T_\alpha}$ is an affine subspace of dimension*

$$\sum_{i=1}^{s}\sum_{j=1}^{s}(n_{ij} - n'_{ij}), \tag{1.10.8}$$

where $n_{ii} = n_i$, n_{ij} is given by (1.8.5) corresponding to α, and

$$n'_{ij} = \begin{cases} \min(n_i + 1, m_j), & \text{if } j < i \\ \min(n_i, m_j), & \text{if } j \geq i \end{cases} \tag{1.10.9}$$

(iii) *V_α is T_{pt}-open in $\overline{V_\alpha}$ and $\pi(\overline{T_\alpha}) \subset \overline{V_\alpha}$.*

PROOF. (i) (based on Hannan and Kavalieris (1984)) Let $(k, l) \in V_\beta, \beta \leq \alpha$, then the parameter vector $\tau \in T_\beta \subset \mathbb{R}^{d_\beta}$ from the reversed echelon realization of (k, l) can be embedded in R^{d_α} by filling the missing elements in the parameter vector with zeros.

Conversely, let $\tau \in \overline{T_\alpha}$, then there exists a sequence $\tau_t \to \tau$, $\tau_i \in T_\alpha$. Let $\gamma = (m_1, \dots, m_s)$ be the Kronecker index of τ. Since π is continuous, $\mathscr{H}_t$ converges element-wise to $\mathscr{H}$, using an obvious notation. Consider the submatrix $\mathscr{H}_{\delta,t}$ of $\mathscr{H}_t$ consisting of all α-Kronecker basis rows up to $h(n_i+1, i)$ and $h(n_i+1, i)$ itself. Since $\mathscr{H}_{\delta,t} \to \mathscr{H}_\delta$ element-wise and the determinant being continuous, $h(n_i+1, i)$ is linearly dependent on the rows above itself in $\mathscr{H}_\delta$. Hence $m_i \leq n_i$ for all $i = 1, \dots, s$, and $\pi(\tau) \in V_\gamma$, $\gamma \leq \alpha$.

(ii) Let $(k, l) \in V_\beta$, where $(m_1, \dots, m_s) = \beta \leq \alpha(n_1, \dots, n_s)$. The parameters τ of the echelon realizations are obtained by expressing $h(n_i+1, i)$ as a linear combination of its preceding basis rows, equation (1.8.4). However, since $\beta \leq \alpha$, the coefficients of these linear combinations are not unique anymore. Counting the linearly independent rows for each i in (1.8.4), we have $h(1, j), \dots, h(m_j, j)$ for $j = i$; $h(1, j), \dots, h(\min(m_j+1, n_j), j)$ for $j < i$ and $h(1, j), \dots, h(\min(m_j, n_j), j)$ for $j > i$. Using the notation from the hypothesis, we observe that the space of solutions of (1.8.4) is an affine space of dimension $\sum_{i=1}^{s} \sum_{j=1}^{s} (n_{ij} - n'_{ij})$. For any given solution τ we obtain a unique τ_2 by means of (1.8.3), thus the final parameter space is of the same dimension as the parameter space for τ.

(iii) The first statement is an immediate consequence of ψ_α being a homeomorphism and thus an open mapping, and Theorem 1.16: for any $(k_0, l_0) \in V_\alpha$, the preimage of a neighbourhood of $\psi_\alpha(k_0, l_0) \subset T_\alpha$ is open and subset of V_α. The second statement follows from the continuity of π.

□

CHAPTER 2

Autoregressive Processes and AR-pure Systems

Autoregressive processes are those stochastic processes y_t that, for all t, satisfy a vector difference equation of the type

$$a(z)y_t = \varepsilon_t, \tag{2.0.10}$$

where $a(z)$ is a matrix polynomial of degree p. Concerning properties of y_t and ε_t, we continue to use the assumptions and notations from the previous discussions. In this sense, an AR-system is a special type of ARMAX-system (1.1.2) where $b(z) = I$ and $d(z) = 0$. In absence of the latter condition, we call the system an *ARX-system*. The following discussion is easily extensible to ARX-systems; however, for the sake of simplicity we concern ourselves with systems where exogenous variables are absent. Regarding notation, we add an asterisk to a set if we only consider elements that correspond to systems where no exogenous input processes are present, e.g. $U_A^* = \{(k,l) \in U_A \mid l = 0\}$.
The following definition will be of paramount importance to what follows: we call a linear system (given either by the transfer function, by an ARMA-realization (a,b) or by a state space representation (F,H,K)) *AR-pure* if its transfer function has an ARMA-realization (a,I), that is, iff there exists an AR-system that has the same transfer function. Clearly, the systems that are AR-pure are exactly those whose transfer functions are inverses of polynomial matrices.
Concerning ARMA-systems, we have the following result, which is essentially a corollary to Theorem 1.3:

PROPOSITION 2.1. *Let (a,b) be an ARMA-system where a and b are left coprime. Then (a,b) is AR-pure if and only if b is unimodular.*

PROOF. Let (a,b) be AR-pure. Then there exists a system $(\tilde{a},I)$ which is observationally equivalent to (a,b). By Theorem 1.3 there exists a u unimodular such that $(a,b) = u(\tilde{a},I)$. Thus, $b = u$. Conversely, if b is unimodular, then $b^{-1}(a,b)$ is an observationally equivalent realization that defines an AR-system. □

The following proposition is based on an idea by M. Watson[1] (Fernandez-Villaverde et al. (2007)).

PROPOSITION 2.2. *A minimal state space system (F,H,K) is AR-pure if and only if there exists a $j_0 \geq 0$ such that $H(F-KH)^j = 0$ for all $j \geq j_0$.*

PROOF. Starting from

$$x(t+1) = Fx(t) + K\varepsilon(t) \tag{2.0.11}$$

$$y(t) = Hx(t) + \varepsilon(t), \tag{2.0.12}$$

[1]In the cited paper, the authors include AR(∞) processes in their definition of autoregressive processes, resulting in a much bigger class of processes under consideration.

rearranging and substituting yields

$$(I-(F-KH)z)x(t) = Kzy(t) \tag{2.0.13}$$

where z again denotes the backward shift operator. In a neighborhood of zero, the matrix on the left-hand side is nonsingular; we thus obtain after substituting into (2.0.11)

$$y(t) = \sum_{j=0}^{\infty} H(F-KH)^j Ky(t-1-j) + \varepsilon(t), \tag{2.0.14}$$

an AR(∞) representation that converges in a neighborhood of zero. Due to minimality of (F, H, K), the matrix K has full row rank and the proposition follows. ☐

The preceding proposition yields a nice necessary and sufficient condition for AR-purity of those systems that exhibit a certain kind of regularity – that is, those systems that have $n_i > 0, i = 1, \ldots, s$ where $(n_1, \ldots, n_s)$ are the Kronecker indices.

Corollary 2.3. *Let (F, H, K) be a minimal state space realization of the transfer function $k(z)$ with Kronecker indices $\alpha = (n_1, \ldots, n_s)$, where $n_i > 0$ for all $i = 1, \ldots, s$. Then $k(z)$ is AR-pure if and only if all eigenvalues of $(F - KH)$ are zero.*

Proof. If $n_i > 0$ for all $i = 1, \ldots, s$, the matrix H must have full row rank, since the corresponding H of the echelon state space realization has full row rank and is related by a nonsingular transform, Theorem 1.6. Thus, by Proposition 2.2 $k(z)$ is AR-pure if and only if $(F - KH)^j = 0$ for all $j > j_0$, i.e. $(F - KH)$ is nilpotent. From Theorem A.10 the claim then follows. ☐

In the case of echelon state space realizations (and, of course, assuming $n_i > 0$ for all i), we can state the matrix $(F - KH)$ explicitly. We have $(F - KH) = (N_{ij})_{i,j=1,\ldots,s} \in \mathbb{R}^{n \times n}$, where $N_{ij} \in \mathbb{R}^{n_i \times n_j}$ is given by

$$N_{ii} = \begin{pmatrix} -h^{\alpha}_{(i-1)s+1,i} & & & \\ \vdots & & I_{n_i-1} & \\ -h^{\alpha}_{(i-1)s+s-1,i} & & & \\ -h^{\alpha}_{(i-1)s+s,i} - \tilde{a}_{ii}(0) & -\tilde{a}_{ii}(1) & \cdots & -\tilde{a}_{ij}(n_i-1) \end{pmatrix} \tag{2.0.15}$$

$$N_{ij} = \begin{pmatrix} -h^{\alpha}_{(i-1)s+1,j} & & & & & & \\ \vdots & & & & 0 & & \\ -h^{\alpha}_{(i-1)s+s-1,j} & & & & & & \\ -h^{\alpha}_{(i-1)s+s,j} - \tilde{a}_{ij}(0) & -\tilde{a}_{ij}(1) & \cdots & -\tilde{a}_{ij}(n_{ij}) & 0 & \cdots & 0 \end{pmatrix}, \quad i \neq j \tag{2.0.16}$$

where $h^{\alpha}_{i,j}$ are the (i,j)-elements of $\mathscr{H}^{\infty}_{\alpha}$. From the corollary above follows that in the case of strictly positive Kronecker indices, the system is AR-pure if and only if $(N_{ij})_{ij}$ is nilpotent, i.e. has only zeros as eigenvalues.

We now follow a different line of thought to obtain a nice necessary and sufficient condition for AR-purity of echelon state space realizations. We make use of Proposition 2.1 and the fact that echelon ARMAX and echelon state space realizations are related by simple transformations. Using the notation from the previous chapter, we have:

LEMMA 2.4. *Let $\mathcal{E} : (F, H, K) \mapsto \tilde{a}(z)$ be the mapping attaching to every echelon state space realization (F, H, K) of a transfer function $k(z) \in V_\alpha^*$ the corresponding echelon ARMAX realization $(\tilde{a}, \tilde{b})$. If $\alpha = (n_1, \dots, n_s)$ with $n_i > 0$, $i = 1, \dots, s$, then $\mathcal{E}$ is given by*

$$\mathcal{E}(F, H, K) = (\tilde{a}, \tilde{b}) \tag{2.0.17}$$

where

$$\tilde{a}(z) = \Phi^{(1)}\zeta^{(1)}F\Phi^{(2)} + \operatorname{diag}(z^{n_i}) \tag{2.0.18}$$

$$\tilde{b}(z) = \Phi^{(1)}F\Phi^{(3)}(I_{n+1} \otimes H)\Psi(F)(I_{n+1} \otimes K)\Phi^{(4)} \tag{2.0.19}$$

$$\Phi^{(1)} = \operatorname{diag}(\Phi_i^{(1)})_{i=1,\dots,s}, \qquad \Phi_i^{(1)} = (0, \dots, 0, 1) \in \mathbb{R}^{1 \times n_i} \tag{2.0.20}$$

$$\Phi^{(2)} = \operatorname{diag}(\Phi_i^{(2)})_{i=1,\dots,s}, \qquad \Phi_i^{(2)} = (1, \dots, 1) \in \mathbb{R}^{n_i \times 1} \tag{2.0.21}$$

$$\Phi^{(3)} = (\Phi_{ij}^{(3)})_{i=1,\dots,s;j=1,\dots,n}, \qquad \Phi_{ij}^{(3)} = (\phi_{kl})_{k=1,\dots,n_i;l=1,\dots,s}, \qquad \phi_{kl} = \begin{cases} 1 & \textit{if } (k = j) \wedge (l = i) \\ 0 & \textit{else} \end{cases} \tag{2.0.22}$$

$$\Psi^{(4)} = (I_s, zI_s, \dots, z^n I_s)' \tag{2.0.23}$$

$$\Psi(F) = (\Psi_{ij})_{i,j=1,\dots,n+1}, \qquad \Psi_{ij} = \begin{cases} 0 \in \mathbb{R}^{n \times n} & \textit{if } j \geq i \\ F^{i-j} & \textit{else} \end{cases} \tag{2.0.24}$$

$$\zeta^{(1)} = \operatorname{diag}(\zeta_i^{(1)})_{i=1,\dots,s}, \qquad \zeta_i^{(1)} = \operatorname{diag}(z^j)_{j=1,\dots,n_i-1} \tag{2.0.25}$$

If $n_i = 0$ for some i, then $\tilde{a}$ is obtained by the formula above, and, subsequently, inserting rows $(0, \dots, 0, 1, 0, \dots, 0)$ at the appropriate places. Likewise, $\tilde{b}$ is obtained by inserting rows into $\Phi^{(1)}F\Phi^{(3)}$.

PROOF. The claim can easily be verified by checking the equations. Equation (2.0.18) holds due to the structure of echelon ARMAX realizations, equation (2.0.19) is derived from equation (1.8.3). □

PROPOSITION 2.5. *Let (F, H, K) be an echelon state space realization of $k(z) \in V_\alpha^*$ where $\alpha = (n_1, \dots, n_s)$. Then (F, H, K) is AR-pure if and only if the matrix*

$$\operatorname{diag}(z^{-n_i})(\mathcal{E}(F, H, K)(1, 1)'), \tag{2.0.26}$$

where $\mathcal{E}$ is given by Lemma 2.4, is unimodular.

PROOF. By Lemma 2.4, $\operatorname{diag}(z^{-n_i})(\mathcal{E}(F, H, K)(1, 1)')$ is, in negative powers of z, the second component of the reversed echelon ARMAX realization of $k(z)$. Hence, from Proposition 2.1 we know that $k(z)$, and thus (F, H, K), is AR-pure if and only if $\operatorname{diag}(z^{-n_i})(\mathcal{E}(F, H, K)(1, 1)')$ is unimodular. □

We now investigate the subset of $M(n)$ that is AR-pure. Recall that the sets V_α with $|\alpha| = n$ partition $M(n)$. We have (Deistler (1985))

LEMMA 2.6. *$\{V_\alpha \,|\, |\alpha| = n\}$ is a disjoint partition of $M(n)$ containing $\begin{pmatrix} n+s-1 \\ s-1 \end{pmatrix}$ sets.*

PROOF. The V_α are disjoint because for each (k, l) the Kronecker indices are unique. It is easy to check that for $|\alpha| = n$, there are $\begin{pmatrix} n+s-1 \\ s-1 \end{pmatrix}$ different possible sets of indices. $\square$

Clearly, the lemma is also valid for $M(n)^*$ and V_α^*. Denote by $T_{\alpha,AR}^*$ the subset of T_α^* that corresponds to AR-pure systems.

PROPOSITION 2.7. *$T_{\alpha,AR}^*$ is a closed subset of T_α^* and its complement in $\overline{T_\alpha^*}$ is dense.*

PROOF. T_α^* is an open subset of $\mathbb{R}^{|\alpha|s+\sum_{i,j} n_{ij}}$. The parameter vectors corresponding to AR-pure echelon realizations are those whose $\operatorname{diag}(z^{n_i})(\tilde{a}(z)+\tilde{b}(z))$ is unimodular, i.e. has a constant real nonzero determinant. Clearly, the entries of this matrix are a continuous transformation of the entries of $\tilde{a}$ and $\tilde{b}$. Regard the determinant as a continuous mapping from $\mathbb{R}^{sn}$ to $\mathbb{R}^{|\alpha|-s+1}$, the latter space being identified with the set of polynomials of degree less or equal $|\alpha| - s$. Partition $\tau = (\tau_1, \tau_2)$ as above, we have that $\{\det(\tau_2) \,|\, (\tau_1, \tau_2) \in T_{\alpha,AR}^*\} = \{(c, 0, \dots, 0) \,|\, c \in \mathbb{R}\}$. Since the latter set is closed and the determinant is continuous, we have that $T_{\alpha,AR}^*$ is closed.
We prove the second statement by showing that in each neighborhood of $\tau \in T_{\alpha,AR}^*$ there is an element of $\mathbb{R}^{s|\alpha|+\sum_{i,j} n_{ij}} \backslash T_{\alpha,AR}^*$. For given $\tau \in T_{\alpha,AR}^*$, consider the row end matrix of the matrix polynomial b of the corresponding echelon ARMA-realization. Since the nonsingular matrices are dense in the space of all matrices of given dimension, there is a row reduced matrix in every neighborhood of τ (actually, in a neighborhood of a continuous transformation of τ since we do not regard the entries of $\tilde{a}$ and $\tilde{b}$ but the entries of $\operatorname{diag}(z^{n_i})(\tilde{a}(z) + \tilde{b}(z))$. However, since this transformation is continuous, the topological properties of the preimage are contained). By Lemma A.1 we conclude that this row reduced matrix is nonunimodular and the hypothesis follows. $\square$

Likewise, by Theorem 1.14, the set of $k \in V_\alpha^*$ that are AR-pure is closed in V_α^*, and its complement is dense in V_α^*.

PROPOSITION 2.8. *The set of $\tau \in T_{\alpha,AR}^*$ whose corresponding AR system is stable is an open subset of $T_{\alpha,AR}^*$.*

PROOF. Consider the set $T_{\alpha,AR}^*$. For any τ in this set, the mapping that attaches to τ the elements of the corresponding reversed echelon ARMA realization's $a(z)$ and $b(z)$ is continuous. Since $b(z)$ is unimodular, the mapping attaching to τ the elements of $b(z)^{-1}a(z)$ is continuous as well. Furthermore, note that the determinant is continuous. The mapping attaching to a polynomial of given degree its complex roots is bijective and its inverse is holomorphic, thus biholomorphic, thus continuous. Finally, the minimum of continuous functions is continous itself. Summarizing, we have shown that the mapping $\tau \mapsto \min_i |\lambda_i|$ where λ_i are the roots of the determinant of the AR system's $a(z)$ matrix polynomial that corresponds to τ, is continuous. Finally, note that the set $\mathbb{C} \backslash \overline{B(0,1)}$ is open in $\mathbb{C}$, and its preimage under the aforementioned continuous mapping is thus open as well. $\square$

Likewise, we have

COROLLARY 2.9. *The set of all $k \in V^*_{\alpha,AR}$ that have stable AR realizations is open in $V^*_{\alpha,AR}$.*

PROOF. The echelon ARMA parameterization ψ is continuous by Theorem 1.14. The rest follows from Proposition 2.8. $\square$

CHAPTER 3

Invertibility in Macroeconomic Models

This chapter discusses applications of the preceding material to questions in macroeconomic modeling. Predominantly, we discuss two approaches to macroeconomic modelling. The first approach is to construct theoretical models that resist the Lucas critique (Lucas (1976)), deduce a stochastic steady state that can be represented as a state space system and then adress policy questions via this framework. The second approach is to estimate a relatively unresticted vector autoregression containing the relevant variables. This is known as the VAR approach (see Favero (2001) for an overview and application to the monetary transmission mechanism). We discuss a problem – namely, the invertibility problem – that may arise when estimating VARs. This discussion is based on the work of, among others, Thomas J. Sargent (Hansen and Sargent (2005), Fernandez-Villaverde et al. (2007)).

3.1. Identification of Shocks

A steady state of a macroeconomic model – or a (log-)linearized version thereof – can often be represented in the form

$$x(t+1) = Ax(t) + Bw(t) \tag{3.1.1}$$

$$y(t) = Cx(t) + Dw(t), \tag{3.1.2}$$

where $x(t)$ is a n-dimensional process containing possibly unobserved economic variables, $y(t)$ is a s-dimensional process containing observed economic variables and w is a m-dimensional process of economic shocks modeled as a Gaussian white noise process, $w(t) \sim N(0, I)$, $\mathbb{E}w(t)w(t-j) = 0$ for $j \neq 0$. Economists who are interested in the consequences of shocks to certain variables attempt to seek answers in different ways. They may construct a theoretical model based on preferences and behaviours of economic agents, aggregating and describing the long-run behaviour of the system – often by means of a state space model like (3.1.1)-(3.1.2). In this case, the matrices A, B, C, D contain parameters that describe the details of the economic mechanisms – the so-called *deep parameters* that appear in the (possibly microeconomic) foundations of the theoretical model.

However, many economists (e.g. Sims (1980)) are sceptical of the assumptions made by theorists. They limit themselves to accepting that the economic variables can be modeled by an autoregressive process[1] of the form

$$\sum_{j=0}^{p} A(j)y(t-j) = B(0)w(t) \tag{3.1.3}$$

[1]It seems that the term *VAR* – for *vector autoregression* – as opposed to *AR* has stuck with macroeconomists.

where y denotes the process of observed economic variables and w denotes a vector process of unobserved structural disturbances (shocks) that are uncorrelated over time and $w(t) \sim N(0, I)$. Clearly, the parameters in (3.1.3) are not identifiable and the VAR researcher has to resort to estimating the equation

$$y(t) = -\sum_{j=1}^{p} \tilde{A}(j)y(t-j) + \varepsilon(t). \tag{3.1.4}$$

Here $\mathbb{E}\varepsilon(t)y(t-j) = 0$ for $j > 0$. This renders the $\varepsilon(t)$ being the *innovations* to the process y, i.e. $\varepsilon(t) = y(t) - \mathbb{E}(y(t)|y^{t-1})$ where y^{t-1} denotes the Hilbert space spanned by the components of the $y(t-j)$, $j > 0$. It is easy to see how these assumptions correspond to those in Chapter 1. Comparing equations (3.1.3 and (3.1.4) yields the relationship

$$\tilde{A}(j) = A(0)^{-1}A(j) \tag{3.1.5}$$

$$\varepsilon(t) = A(0)^{-1}B(0)w(t). \tag{3.1.6}$$

When describing economic models, macroeconomists are predominantly interested in the impulse responses of economic variables to shocks.[2] A VAR researcher must therefore factor the covariance matrix of the innovations ε,

$$\mathbb{E}\varepsilon(t)\varepsilon(t)' = A(0)^{-1}B(0)B(0)'A(0)^{-1\ \prime} \tag{3.1.7}$$

– or the corresponding sample moment – to infer the shocks $w(t)$ from the innovations. One method, proposed by C.A. Sims, is based on the Cholesky decomposition of a symmetric positive definite matrix. Let Σ be the sample covariance matrix of $\varepsilon(t)$, then there is a unique lower triangular matrix L such that

$$\Sigma = LL'. \tag{3.1.8}$$

The matrix L is then factorized as

$$L = A(0)^{-1}B(0), \qquad A(0)^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & & 0 \\ a_{31} & a_{32} & 1 & & 0 \\ \vdots & & & \ddots & \vdots \\ a_{s1} & a_{s2} & & \cdots & 1 \end{pmatrix}, \qquad B(0) = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_{ss} \end{pmatrix}. \tag{3.1.9}$$

For another popular identification method, see Blanchard and Quah (1989).

3.2. Invertibility

Implicitly the VAR researchers assume that the shocks are a linear combination of the innovations. However, this may not necessarily be the case: if the space spanned by the components of the present and the past of ε is strictly smaller than the space spanned by the present and past of w, a mapping that yields the latter as an expression of the former may fail to exist. In the words of Hansen and Sargent (1991), the process $\varepsilon(t)$ needs to be *fundamental* for the process $w(t)$. Henceforth, we say that a model satisfying this property is *invertible.*

[2]Indeed, much of the literature regarding the VAR approach has been developed to investigate the consequences of monetary policy shocks. See also Christiano et al. (1998).

The following analysis is based on the work by Fernandez-Villaverde et al. (2005) and Fernandez-Villaverde et al. (2007). Starting from a state space system (3.1.1) – (3.1.2) where the system matrices A, B, C, D are parameterized by the theory, we calculate an equivalent state space system using the Kalman filter.[3] It applies a Gram-Schmidt orthogonalization procedure to the state vector such that the new state vector $\hat{x}(t)$ is the orthogonal projection of $x(t)$ on the space y^{t-1} (following the notation of section 3.1). We obtain a representation

$$\hat{x}(t+1) = A\hat{x}(t) + K_t\varepsilon(t) \tag{3.2.1}$$

$$y(t) = C\hat{x}(t) + \varepsilon(t) \tag{3.2.2}$$

where $\varepsilon(t) = y(t) - \mathbb{E}(y(t)|y^{t-1})$ are the innovations to the process $y(t)$ and K_t is called the *Kalman gain* in step t. Under certain regularity conditions (see Anderson and Moore (1979)) – which we assume to hold – the Kalman gain converges to a *steady-state Kalman gain* K, yielding the system

$$\hat{x}(t+1) = A\hat{x}(t) + K\varepsilon(t) \tag{3.2.3}$$

$$y(t) = C\hat{x}(t) + \varepsilon(t). \tag{3.2.4}$$

Furthermore, we have $\mathbb{E}\varepsilon(t)\varepsilon(t)' = C\Sigma C' + DD'$, where $\Sigma = \mathbb{E}(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))'$ satisfies the *steady-state algebraic matrix Riccati equation*

$$\Sigma = A\Sigma A' + BB' - (A\Sigma C' + BD')(C\Sigma C' + DD')^{-1}(A\Sigma C' + BD')'. \tag{3.2.5}$$

The steady-state Kalman gain K satisfies the equation

$$K = (A\Sigma C' + BD')(C\Sigma C' + DD')^{-1}. \tag{3.2.6}$$

Under the conditions that the Kalman gain converges, the eigenvalues of the matrix $A - KC$ are less than unity in modulus (see Anderson and Moore (1979)). We need this result to describe the relationship between shocks and innovations. From (3.1.1) – (3.1.2) and (3.2.3) – (3.2.4) we obtain

$$\begin{pmatrix} x(t+1) \\ \hat{x}(t+1) \end{pmatrix} = \begin{pmatrix} A & 0 \\ KC & A-KC \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} + \begin{pmatrix} B \\ KD \end{pmatrix} w(t). \tag{3.2.7}$$

From (3.1.1) and (3.2.4) follows that

$$\varepsilon(t) = C(x(t) - \hat{x}(t)) + Dw(t). \tag{3.2.8}$$

Write

$$A^* = \begin{pmatrix} A & 0 \\ KC & A-KC \end{pmatrix} \tag{3.2.9}$$

then, if (A, B, C, D) is stable, we can write

$$\varepsilon(t) = \left(D + \begin{pmatrix} C & -C \end{pmatrix} z(I - A^*z)^{-1} \begin{pmatrix} B \\ KD \end{pmatrix} \right) w(t). \tag{3.2.10}$$

[3]The Kalman filter is well described in the literature on linear system theory and time series analysis. See, for example, Anderson and Moore (1979), Brockwell and Davis (1991), Hannan and Deistler (1988) or Boehm (2007).

Thus, if the eigenvalues of A are smaller than unity in modulus, $\varepsilon^t \subset w^t$ for all t. The converse, $\varepsilon^t \supset w^t$, is true if the mapping in (3.2.10) is invertible. We will return to this mapping later when we consider the case where the number of shocks equals the number of observables, i.e. where D is a square matrix.
We now turn to autoregressive representations of the state space system:

THEOREM 3.1 (Hansen and Sargent (2005)). *If the Kalman gain converges, $y(t)$ has an autoregressive representation in terms of $\varepsilon(t)$:*

$$y(t) = C(I - (A - KC)z)^{-1}Ky(t-1) + \varepsilon(t) \tag{3.2.11}$$

PROOF. From the time-invariant innovations representation (3.2.3) – (3.2.4) we derive a Wold moving average representation[4] of $y(t)$:

$$y(t) = (I + C(I - Az)^{-1}Kz)\varepsilon(t). \tag{3.2.12}$$

The operator on the right side of this equation has an inverse in nonnegative powers of z if and only if

$$\det(I + C(I - Az)^{-1}Kz) \neq 0 \quad \text{for all } |z| \leq 1. \tag{3.2.13}$$

This is equivalent to saying that

$$\det(I + C(I - Az)^{-1}Kz) = 0 \quad \text{implies } |z| > 1. \tag{3.2.14}$$

From linear algebra (matrix inversion lemma) we know that

$$\det(a)\det(d + ca^{-1}b) = \det(d)\det(a + bd^{-1}c) \tag{3.2.15}$$

and

$$(a - bd^{-1}c)^{-1} = a^{-1} + a^{-1}b(d - ca^{-1}b)^{-1}ca^{-1} \tag{3.2.16}$$

for appropriately sized and invertible matrices a, b, c, d. Applying this for $a = I, b = -C, c = K, d = (zI - A)$ yields

$$(I + C(zI - A)^{-1}K)^{-1} = I - C(zI - (A - KC))^{-1}K, \tag{3.2.17}$$

and

$$\det(I + C(zI - A)^{-1}K) = \frac{\det(zI - (A - KC))}{\det(zI - A)}. \tag{3.2.18}$$

Therefore, the zeros of $\det(I + C(zI - A)^{-1}K)$ are the eigenvalues of $A - KC$, which we know to be smaller than one in modulus. Rearranging (3.2.17) immediately yields equation (3.2.11). □

3.3. The square case

We now consider the case where there are as many economic shocks as there are observables; in other words, the dimension of $w(t)$ equals the dimension of $y(t)$ and D is a square matrix. Furthermore, we assume that D is nonsingular. Then, the invertibility of the mapping (3.2.10) is equivalent to

$$\det\left(I + \begin{pmatrix} C & -C \end{pmatrix}(I - A^* z)^{-1}\begin{pmatrix} BD^{-1} \\ K \end{pmatrix} z\right) \neq 0 \;\; \text{for} \;\; |z| \leq 1. \tag{3.3.1}$$

[4]Actually, it is Wold representation only if A is a stable matrix, i.e. has spectral radius less than unity in modulus.

Since the mapping $^{-1} : \mathbb{C}\backslash\{0\} \to \mathbb{C}, z \mapsto \frac{1}{z}$ bijectively maps the pointed closed complex unit disc $\overline{B(0,1)}\backslash\{0\}$ onto $\mathbb{C}\backslash B(0,1)$, this is equivalent to

$$\det\left(I + \begin{pmatrix} C & -C \end{pmatrix} (zI - A^*)^{-1} \begin{pmatrix} BD^{-1} \\ K \end{pmatrix}\right) = 0 \text{ implies that } |z| < 1. \tag{3.3.2}$$

The zeros of this polynomial can be easily computed in terms of A, B, C, D, as the following theorem shows.

Theorem 3.2. *Under the assumptions made hitherto, the zeros of*

$$\det\left(I + \begin{pmatrix} C & -C \end{pmatrix} (zI - A^*)^{-1} \begin{pmatrix} BD^{-1} \\ K \end{pmatrix}\right) \tag{3.3.3}$$

are the eigenvalues of $A - BD^{-1}C$ and the eigenvalues of A.

Proof. We again use formula (3.2.15), this time for $a = I, b = C^*, c = B^*, d = (zI - A^*)$, where

$$C^* = \begin{pmatrix} C & -C \end{pmatrix}, \qquad B^* = \begin{pmatrix} BD^{-1} \\ K \end{pmatrix}. \tag{3.3.4}$$

We obtain

$$\det(I + C^*(zI - A^*)^{-1}B^*) = \frac{\det(zI - A^* + B^*C^*)}{\det(zI - A^*)}. \tag{3.3.5}$$

However since

$$zI - A^* + B^*C^* = zI - \begin{pmatrix} A - BD^{-1}C & BD^{-1}C \\ 0 & A \end{pmatrix} \tag{3.3.6}$$

the set of zeros of $\det(I + C^*(zI - A^*)^{-1}B^*)$ is exactly the union of the spectra of A and $A - BD^{-1}C$. $\square$

Let us take a closer look at what happens in the square case. We again start from the state space representation (3.1.1) – (3.1.2). If D is nonsingular, we can write $w(t) = D^{-1}(y(t) - Cx(t))$ and substituting in the first equation yields

$$x(t+1) = (A - BD^{-1}C)x(t) + BD^{-1}y(t). \tag{3.3.7}$$

If $(A - BD^{-1}C)$ is stable, we can rewrite this equation as

$$x(t+1) = \sum_{j=0}^{\infty} (A - BD^{-1}C)^j BD^{-1}y(t-j). \tag{3.3.8}$$

This equation shows that the state vector can be expressed as a linear combination of the history of observables, i.e. no part of the state vector is hidden; the state is in fact observed. Insofar, it is not surprising that the innovations and the shocks span the same space and one can be expressed in terms of the other. How does the innovations representation look like? Clearly, $\Sigma = \mathbb{E}(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))' = 0$ and $\mathbb{E}\varepsilon(t)\varepsilon(t)' = DD'$. From equation (3.2.6) follows that $K = BD^{-1}$.

3.4. An Example

We briefly examine a permanent income model[5] presented by Fernandez-Villaverde et al. (2007). The stochastic steady state is represented by the following univariate state space system:

$$c(t) = c(t-1) + (1 - R^{-1})\sigma_w w(t) \tag{3.4.1}$$

$$y(t) - c(t) = -c(t-1) + R^{-1}\sigma_w w(t). \tag{3.4.2}$$

Here, the process $y(t) - c(t)$ is observed, whereas $c(t)$ is not directly observed. We assume $R = (1 + r)$ where $r > 0$ is the constant exogenous interest rate. $w(t)$ are serially uncorrelated economic shocks with $\mathbb{E}w(t)^2 = 1$. Since the system corresponds to the square case and D^{-1} exists, we may inspect the eigenvalues of $A - BD^{-1}C$ to check whether the innovations are fundamental for the shocks (Theorem 3.2). Since $A - BD^{-1}C = R > 1$ invertibility fails to exist. Note that this conclusion is independent from a specification of the deep parameters – in this case the interest rate r (as long as it is positive, that is). Calculating the time-invariant innovations representation from the steady-state Kalman filter equations (3.2.5) and (3.2.6) yields

$$\Sigma = (1 - R^{-2})\sigma_w^2, \qquad K = R^{-1} - 1, \tag{3.4.3}$$

and the time-invariant innovations representation is therefore

$$\hat{c}(t) = \hat{c}(t-1) + (R^{-1} - 1)\varepsilon(t) \tag{3.4.4}$$

$$y(t) - c(t) = -\hat{c}(t-1) + \varepsilon(t) \tag{3.4.5}$$

where $\mathbb{E}\varepsilon(t)\varepsilon(t)' = \sigma^2$.

3.5. Implications for the VAR Approach

We now discuss potential uses and pitfalls of the VAR approach. The main advantage is clear: the economist does not need a theoretical model to obtain quantitative and qualitative implications of changes in certain economic variables (mostly shocks). Therefore, the model is less prone to errors in specification – although the economist will usually restrict certain parameters in the coefficient matrices to be zero in order to reduce the number of estimated parameters, thereby including his a-priori information about the economic mechanisms in the model. The implicit assumption of linearity is not as restricting as it may seem at first glance, since even the linear case allows for some other specifications (e.g. the log-linear case). Furthermore, even models that depend on much theory may have to be linearized in order to be tractable.

However, as tempting as the VAR approach may be, several difficulties arise. One must keep in mind the following issues:

- Many restrictions that are sensible and are accepted even by VAR researchers may not take the form of locating zeros in the VAR's parameter matrices (Fernandez-Villaverde et al. (2005) explicitly mention the restictions of present value budget balance in a permanent income model). Therefore, these restrictions are difficult to account for in VAR models.

[5]Essentially, this model is a simplification of the quadratic preferences permanent income model in Hansen and Sargent (2005).

- As already discussed, the model does not have to be invertible, that is, the shocks may not be deductible from the innovations of the VAR. Clearly, this is a property of the underlying theoretical model that the VAR tries to match – and, generally, the numerical specification of the deep parameters. Consequently, if the VAR researcher does not trust the theory and/or the parameter specification used to construct the theoretical model's state space representation, she will probably disregard its failure to provide invertibility. If, however, the model's lack of invertibility is structural, i.e. independent from the parameter choice (as in our permanent income model, section 3.4), and the researcher trusts the theory, it is not sensible to estimate a VAR model, since the impulse responses will differ. This is probably the case where checking invertibility is most valuable.
- Even when there is no invertibility problem, there is still the question of how to identify the shocks from the innovations, especially if there are measurement errors associated with the observed variables.
- In general, the VAR associated with a theoretical model is an infinite-order vector autoregression. Our analysis in Chapter 2 reveals that it is *almost certainly* an infinite-order VAR, thus we will almost certainly make an error due to truncating the VAR. Our results give suitable conditions for the existence of finite-order VARs. Note, however, that these do not constitute statistical tests for checking whether the null hypothesis of finite order is appropriate – in order to do that, one has to impose prior distributions on the model parameters.

APPENDIX A

Mathematical Preliminaries

A.1. Matrix Polynomials and Unimodular Matrices

Polynomial matrices play a crucial role in the theory of ARMAX realizations. We present a short overview of their properties that are relevant to the theory of linear systems (following Kailath (1980), Hannan and Deistler (1988), Rugh (1993)).

A *polynomial matrix of degree* q is a matrix whose elements are polynomials of degree q over a field $\mathbb{F}$. Alternatively, one can also describe them as polynomials over $\mathbb{F}$ with matrices as coefficients - those two views are interchangeable. We use the notation $\nu(p)$ to describe the degree of a polynomial p. Additionally, $\nu(0) := -1$.

However, polynomials form an algebraic ring and are not complete under inversion. We therefore often consider *rational matrices*, matrices whose entries are rational functions, i.e. ratios of polynomials. Rational functions form a field, thus, one can define all operations on rational matrices in the usual manner, and all results that do not depend on the entries being real or complex are still valid. For a square rational matrix $a(z)$ we see from

$$a(z)^{-1} = (\det a(z))^{-1} \operatorname{adj} a(z) \tag{A.1.1}$$

that $a(z)$ is invertible if and only if $\det a(z) \not\equiv 0$.

First, some definitions regarding the degrees of polynomial matrices. Let u be a $s \times n$ polynomial matrix, and denote the i-th row of u by $u_i = \sum_{j=0}^{\nu(u_i)} u_i(j) z^j$, the coefficient vector of z^j in the i-th row being $u_i(j)$. Then the $\mathbb{F}$-matrix that consists of the rows corresponding to the highest degree in each row is called the *row end matrix* $[u]_r = (u_1(\nu(u_1))', \dots, u_s(\nu(u_s))')'$. If $[u]_r$ is of full rank $\min(s, n)$ (or, equivalently, if $\det([u]_r'[u]_r) \neq 0$) then we call u *row reduced.* The definitions for column end matrices and column reduced matrices are analogous. We have the following lemma:

Lemma A.1. *Let $u(z)$ be a nonsingular $n \times n$ polynomial matrix with row degrees $\nu(u_1), \dots, \nu(u_n)$. Then*

$$\nu(\det u(z)) = \sum_{j=1}^{n} \nu(u_n) \tag{A.1.2}$$

if and only if $u(z)$ is row reduced.

Proof. By Leibniz' formula for determinants, we have

$$\det u(z) = (\det[u(z)]_r) z^{\sum_n \nu(u_n)} + r(z) \tag{A.1.3}$$

where $r(z)$ is a polynomial of degree less than $\sum_n \nu(u_n)$. Thus, (A.1.2) holds if and only if $\det[u(z)]_r \neq 0$, which is per definitionem equivalent to $u(z)$ being row reduced. $\square$

A.1.1. Unimodular Matrices. We now investigate which matrices represent elementary row (or column) operations on polynomial matrices. A polynomial matrix u is called *unimodular*, if $\det u = c$, with $c \in \mathbb{R}\backslash\{0\}$. Unimodular matrices are exactly those polynomial matrices that are complete under inversion.

LEMMA A.2. *Let u be a polynomial matrix. Then u is unimodular if and only if u^{-1} exists and is a polynomial matrix itself.*

PROOF. If u is unimodular, by definition $\det u = c \in \mathbb{R}\backslash\{0\}$, whose inverse is a polynomial. $\operatorname{adj} u$ is a polynomial matrix, from (A.1.1) we see that u^{-1} exists and is a polynomial matrix, which proves the first direction.
Regarding the other direction, observe that $u \cdot u^{-1} = I$, thus $(\det u)(\det u^{-1}) = 1$. Both factors must be polynomials; in order for the product to be unity, they must both be constant and nonzero. □

Left-multiplicaton by a unimodular matrix is equivalent to applying a series of elementary row operations: (i) interchanging two rows, (ii) multiplying a row by a nonzero real scalar, (iii) adding the polynomial multiple of a row to another. Note that multiplying a row by a nonconstant polynomial is *not* an elementary row operation, as it cannot be represented as a left-multiplication by a unimodular matrix.
Let a, b be appropriately sized polynomial matrices. If polynomial matrices u and $(\tilde{a}, \tilde{b})$ exist such that

$$(a, b) = u(\tilde{a}, \tilde{b}) \tag{A.1.4}$$

holds, then u is called a *common left divisor* of a and b, or a *left divisor* of the matrix (a, b). If u is a common left divisor of a and b, and every other common left divisor of a and b is a left divisor of u, we call u a *greatest common left divisor* of a and b.

LEMMA A.3. *Let q be a common left divisor of (a, b),*

$$(a, b) = q(\tilde{a}, \tilde{b}), \tag{A.1.5}$$

then for every unimodular matrix u, $q \cdot u$ is also a common left divisor of (a, b).

PROOF. Clearly, $(a, b) = q(\tilde{a}, \tilde{b}) = q \cdot u \cdot u^{-1}(\tilde{a}, \tilde{b})$. The matrix $u^{-1}(\tilde{a}, \tilde{b})$ is polynomial by Lemma A.2, and thus $q \cdot u$ is a common left divison of (a, b). □

We draw two conclusions: (i) the greatest common left divisor of two polynomial matrices a and b is unique up to right-multiplication by unimodular matrices; (ii) since the identity matrix I is a common left divisor of every two polynomial matrices, so is every unimodular matrix. The latter motivates the following definition: if all common left divisors of two polynomial matrices a and b are unimodular, we call them *left (co-)prime*. As the following lemma shows, one can find a greatest common left divisor by applying elementary column operations.

LEMMA A.4. *Let a be a $s \times s$ and b be a $s \times k$ polynomial matrix. Apply elementary column operations corresponding to postmultiplication by a unimodular matrix u such that the last k columns of the resulting matrix are zero,*

$$(a, b) \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = (r(z), 0), \tag{A.1.6}$$

then the polynomial matrix $r(z)$ in (A.1.6) is a greatest common left divisor of a and b.

PROOF. Since u is unimodular, the matrix

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}^{-1} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \tag{A.1.7}$$

is a polynomial matrix itself. From (A.1.6) we then have

$$(a, b) = r(z)(v_{11}, v_{12}) \tag{A.1.8}$$

and $r(z)$ is a common left divisor of a and b. It remains to show that any other common left divisor $r_1(z)$ is a left divisor of $r(z)$. From $(a, b) = r_1(z)(\tilde{a}, \tilde{b})$ for certain polynomial matrices $(\tilde{a}, \tilde{b})$, we have from (A.1.6) that

$$r(z) = a \cdot u_{11} + b \cdot u_{21} = r_1(z)(\tilde{a}u_{11} + \tilde{b}u_{12}), \tag{A.1.9}$$

thus establishing that $r_1(z)$ is a left divisor of $r(z)$. □

COROLLARY A.5. *(Bezout Identity) The $s \times s$ and $s \times k$ polynomial matrices a and b are left coprime if and only if polynomial matrices g, h exist such that*

$$ag + bh = I_s. \tag{A.1.10}$$

PROOF. Let a and b be left coprime. Then all greatest common left divisors of a and b are unimodular, in particular, $r(z)$ in (A.1.6). Postmultiply equation (A.1.6) by the polynomial matrix $r(z)^{-1}$, then the first s columns are of the form (A.1.10).
Conversely, assume that (A.1.10) holds. Let $r(z)$ be a greatest common left divisor of a and b. Write $(a, b) = r(z)(\tilde{a}, \tilde{b})$. Then, from (A.1.10),

$$r(z)(\tilde{a}g + \tilde{b}h) = I_s \tag{A.1.11}$$

the polynomial matrix $\tilde{a}g + \tilde{b}h$ must be the inverse of $r(z)$. Thus, $r(z)$ is unimodular by Lemma A.2. □

Following Hannan and Deistler (1988) and drawing from Kailath (1980), the next theorem gives us a characterisation of left coprime matrices. For the sake of applicability to the problem of ARMAX realizations, we state it here using three matrices instead of two. For convenience, we also include the Bezout Identity, Corollary A.5.

THEOREM A.6. *For polynomial matrices (a, b, d) with $\det a \not\equiv 0$, the following statements are equivalent:*

(i) *The matrices a, b and d are left coprime.*
(ii) $(a(z), b(z), d(z))$ *has full row rank s for all $z \in \mathbb{C}$.*
(iii) *There exist appropriately sized polynomial matrices g, h such that*

$$ag + (b, d)h = I_s. \tag{A.1.12}$$

(iv) $\det a$ *has minimal degree among all $(\tilde{a}, \tilde{b}, \tilde{d})$ with $\tilde{a}^{-1}(\tilde{b}, \tilde{d}) = a^{-1}(b, d)$.*

PROOF. (i) ⇒ (ii) Suppose that there is a $z_0 \in \mathbb{C}$ such that $(a(z_0), b(z_0), d(z_0))$ has rank $t < s$. By elementary row operations, we can transform $(a(z_0), b(z_0), d(z_0))$ into a matrix whose first $s - t$

rows are zeros. Since elementary row operations can be represented by unimodular matrices, there is a u unimodular, such that

$$u \cdot (a(z_0), b(z_0), d(z_0)) = \begin{pmatrix} 0 \\ K \end{pmatrix} = \begin{pmatrix} zI_t & 0 \\ 0 & I_{s-t} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ K \end{pmatrix}. \tag{A.1.13}$$

Leftmultiplying this equation by u^{-1} shows that $(a(z_0), b(z_0), d(z_0))$ has a nonunimodular left divisor.

(ii) $\Rightarrow$ (i) Suppose that (a, b, d) is not left coprime. Then there is a nonunimodular common left divisor p of a, b, d. By the fundamental theorem of algebra, there is a $z_0 \in \mathbb{C}$ such that $\det p(z_0) = 0$. Thus, the matrix $(a(z_0), b(z_0), d(z_0))$ cannot have full row rank.

(iii) $\Rightarrow$ (iv) From $\tilde{a}^{-1}(\tilde{b}, \tilde{d}) = a^{-1}(b, d)$ we have $(a, b, d) = a\tilde{a}^{-1}(\tilde{a}, \tilde{b}, \tilde{d})$. Now if (A.1.12) holds, we obtain

$$a\tilde{a}^{-1}\tilde{a}g + a\tilde{a}^{-1}(\tilde{b}, \tilde{d})h = I_s, \tag{A.1.14}$$

equivalently,

$$\tilde{a}g + (\tilde{b}, \tilde{d})h = \tilde{a}a^{-1}. \tag{A.1.15}$$

Thus we have established that $\tilde{a}a^{-1}$ is a polynomial matrix. Since

$$(\tilde{a}, \tilde{b}, \tilde{d}) = \tilde{a}a^{-1}(a, b, d), \tag{A.1.16}$$

we have that $\nu(\det \tilde{a}) = \nu(\det(\tilde{a}a^{-1})) + \nu(\det a) \geq \nu(\det a)$.

(iv) $\Rightarrow$ (i) Assume that (a, b, d) is not coprime, that is, there is a nonunimodular polynomial matrix r and $(\tilde{a}, \tilde{b}, \tilde{d})$, such that $(a, b, d) = r \cdot (\tilde{a}, \tilde{b}, \tilde{d})$. Clearly, $a^{-1}(b, d) = (r \cdot \tilde{a})^{-1}(r\tilde{b}, r\tilde{d}) = \tilde{a}^{-1}(\tilde{b}, \tilde{d})$. However, $\det a = \det r \cdot \det \tilde{a}$ and since $\det r$ is nonconstant, $\nu(\det \tilde{a}) < \nu(\det a)$, which contradicts (iv).

□

A most useful byproduct of the proof of (iii) $\Rightarrow$ (iv) is the following corollary:

COROLLARY A.7. *Let (a, b, d) and $(\tilde{a}, \tilde{b}, \tilde{d})$ be polynomial matrices of the same dimensions, respectively. Furthermore, let (a, b, d) be left coprime. Then there exists a polynomial matrix u such that*

$$(\tilde{a}, \tilde{b}, \tilde{d}) = u(a, b, d) \tag{A.1.17}$$

if and only if $\tilde{a}^{-1}(\tilde{b}, \tilde{d}) = a^{-1}(b, d)$. If $(\tilde{a}, \tilde{b}, \tilde{d})$ is also coprime, u is unimodular.

PROOF. If $\tilde{a}^{-1}(\tilde{b}, \tilde{d}) = a^{-1}(b, d)$, then from equation (A.1.16) we know that $u \equiv \tilde{a}a^{-1}$ is polynomial. The other direction holds since $\tilde{a}^{-1}(\tilde{b}, \tilde{d}) = (u \cdot a)^{-1}(u \cdot b, u \cdot d) = a^{-1}(b, d)$.
If $(\tilde{a}, \tilde{b}, \tilde{d})$ is coprime, then, analogous to the proof above, $a\tilde{a}^{-1} = (\tilde{a}a^{-1})^{-1}$ is polynomial and thus unimodular (Lemma A.2). □

A.1.2. Certain Forms of Polynomial and Rational Matrices.

THEOREM A.8 (Hermite form of a polynomial matrix). *Let a be a $s \times s$ polynomial matrix, $\det a(z) \not\equiv 0$. Then a can be transformed by elementary row operations (or, equivalently, premultiplication by a unimodular matrix) to a matrix $\bar{a}$ with the following properties:*

(i) $\bar{a}$ *is lower triangular,*

(ii) $\bar{a}_{ii}, i = 1, \ldots, s$ *are polynomials with leading coefficients of* 1 *(monic polynomials),*

(iii) $\nu(\bar{a}_{ji}) < \nu(\bar{a}_{ii})$ *for all* $j \neq i$.

PROOF. See, for example, Kailath (1980) or Hannan and Deistler (1988). □

THEOREM A.9 (Polynomial Echelon form of a polynomial matrix). *Let* a *be a* $s \times s$ *polynomial matrix,* $\det a(z) \not\equiv 0$. *Then* a *can be transformed by elementary row operations (or, equivalently, premultiplication by a unimodular matrix) to a matrix* $\tilde{a}$ *with the following properties:*

(i) *The* $\tilde{a}_{ii}$ *are monic polynomials.*

(ii) $\nu(\tilde{a}_{ij}) \leq \nu(\tilde{a}_{ii}), \quad j \leq i$

(iii) $\nu(\tilde{a}_{ij}) < \nu(\tilde{a}_{ii}), \quad j > i$

(iv) $\nu(\tilde{a}_{ji}) < \nu(\tilde{a}_{ii}), \quad j \neq i$.

PROOF. See Kailath (1980), Section 6.7. □

For further properties of polynomial matrices, we refer to Kailath (1980).

A.2. Nilpotent Matrices

A $n \times n$ square matrix M over $\mathbb{R}$ or $\mathbb{C}$ is called *nilpotent* iff there exists a $j \in \mathbb{N}$ such that $M^j = 0$. The following theorem characterises nilpotent matrices.

THEOREM A.10. *Let* M *be a square matrix over* $\mathbb{R}$ *or* $\mathbb{C}$. *Then the following statements are equivalent:*

- M *is nilpotent.*
- *All eigenvalues of* M *are zero.*
- M *is similar to a block diagonal matrix* $N = \operatorname{diag}(N_i)$ *where each* N_i *is of the form*

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.2.1}$$

PROOF. (i) $\Rightarrow$ (ii) Let $j \in \mathbb{N}$ be such that $M^j = 0$. Let $\lambda \neq 0$ be an eigenvalue of M, then there exists an eigenvector $x \neq 0$ such that $Mx = \lambda x$. Therefore, $M^j x = \lambda^j x \neq 0$ which is a contradiction.

(ii) $\Rightarrow$ (iii) This implication follows immediately from the Jordan representation theorem.

(iii) $\Rightarrow$ (i) As is easily shown, every square matrix that is similar to a nilpotent matrix is nilpotent. Since N is obviously nilpotent, the claim follows.

□

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