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DIPLOMARBEIT

An Analysis of Initiation in a Two-Stage Optimal Control Model of Illicit Drug Consumption

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Contents

Chapter 1

Introduction

Modelling initiation into drug use has passed through an evolution since the first optimal control models of illicit drug consumption. In a series of models it was assumed that initiation is mainly influenced by the price of the drug. However, there are variations over the evolution of a drug epidemic that cannot be described entirely by price fluctuations. As Tragler [1998] points out, drug initiation is partly driven by feedback effects. Drug users are mostly introduced by existing users (recruitment effect), while on the other hand they might also be deterred by the negative effects drugs have on the users (*Musto effect*, cf. Musto [1987]). Thus, it is more reasonable to model initiation as a function of the population of users. Further, it was observed that the deterrence effect is enhanced in later stages of the epidemic, i.e. when the number of users is high.

This thesis is resuming the analysis of the feedback effect initiated by Tragler [1998] using a two-stage model approach (see also Bultmann [2007], Bultmann et al. [2008], Bultmann et al. [to appear]). Following previous theses (Tragler [1998], Moyzisch [2003], Bultmann [2007]), assumptions are made that may not seem realistic but allow actual analyses. Perfect knowledge of the parameter values in both stages will be assumed such as the information about the actual duration of the first stage and the initiation in the second stage.

However, this thesis deals with uncertainty in a way that a large part of it is devoted to the impacts of changes of some parameters influencing initiation on the optimal dynamic drug policy.

Moreover, when initiation declines because of the Musto effect it does not behave like a two-stage problem by suddenly jumping to a lower level, but rather shows a multi-stage or even continuously decreasing performance. On the other hand, Johnson et al. [1996] report a rather sudden decline of the initiation rate of the U.S. cocaine epidemic in the late 1980s. As a consequence, this sudden jump will be referred to as initiation shock, following the recent analysis of supply shocks by Bultmann [2007]. The first stage reflects the beginning of the epidemic with a high initiation rate. After a certain time, the model switches to the second stage with an infinite time horizon and a lower initiation rate.

This thesis is organized as follows. Chapter 2 presents the base model and the baseline parameter values that are used for this thesis. Further, the model is analysed and solved by using Pontryagin's Maximum Principle.

In Chaper 3 a first sensitivity analysis for the base model is carried out focusing on qualitative changes of the system dynamics (bifurcations). The social costs are varied in order to determine ranges with different stability properties of the equilibria. Then the same is done for the initiation parameter. As a result some knowlegde is obtained about where to find multiple optimal long-run solutions depending on the social costs and the initiation.

Chapter 4 then deals with initiation shocks. First, the two-stage model is presented, followed by a discussion of a certain shock for different scenarios that were revealed in the previous chapter. This part is closed by comparing the results of those scenarios and formulating some implications for the policy maker.

Chapter 5 analyses the impact of the intensity and the duration of the shock on the optimal policy. It starts with a sensitivity analysis for the shocks with respect to the intensity and to the duration, respectively. Finally, those results are combined in a two-dimensional sensitivity analysis. Different shocks are compared with each other and statements are made about whether a longer, less intense shock or a shorter, more intense one is favourable in terms of the equilibrium solutions.

This thesis closes with conclusions to be drawn from the analyses carried out in the previous chapters.

Chapter 2

The Base Model

2.1 Base Model Formulation

The model for the following analysis is taken from Bultmann [2007] where it was derived from a one-state three-control model presented in Moyzisch [2003].

The main objective is the minimization of social costs that result from illicit drug consumption. On the one hand social costs occur that are caused by drug consumption itself. On the other hand there are costs due to treatment which is the only control instrument in our model. The drug problem itself is described by the number of users of the drug that follow the dynamics of a differential equation. Costs are weighted by a discount factor e^{-rt} which makes economic sense and assures convergence of the integral.

The objective is described by

$$\max_{u(t)} J \tag{2.1}$$

with

$$J = -\int_0^\infty e^{-rt}\Big(\underbrace{\kappa p^{-\omega} A(t)}_{\text{costs due to consumption}} + \underbrace{u(t)}_{\text{costs due to treatment}} \Big) dt$$

subject to the system dynamics

$$\dot{A}(t) = \underbrace{kp^{-a}A(t)(\bar{A} - A(t))}_{\text{initiation}} - \underbrace{c\beta(A(t), u(t))A(t)}_{\text{outflow due to treatment}} - \underbrace{\mu p^b A(t)}_{\text{natural outflow}} \qquad (2.2)$$

with

$$\beta(A(t), u(t)) = \left(\frac{u(t)}{A(t)}\right)^z$$

characterising the effects of treatment and $u(t) \geq 0$ for all times t.

Initiation is described by a logistic function rather than by a power function that was used in earlier models. For a detailed discussion and comparison we refer to Bultmann [2007] and Grass et al. [2008].

The outflow of the system consists of two terms. There are always users that quit by themselves which is here referred to as natural outflow. Additionally, there are users that exit the system due to treatment. By spending more money on the control it is intended to increase the outflow due to treatment.

In the following chapters the parameter t will be omitted whenever this does not confusion.

2.2 Solving the Problem - Pontryagin's Maximum Principle

In order to analyse problem (2.1) subject to (2.2) Pontryagin's Maximum Principle is applied.

For the current value Hamiltonian

$$H(A, u, \lambda) = -(\kappa p^{-\omega} A + u) + \lambda A(kp^{-a}(\bar{A} - A) - c\beta(A, u) - \mu p^b)$$

we have the necessary first order condition

$$u = \arg\max_u H(A, u, \lambda),$$

where λ denotes the costate variable. It can easily be shown that this condition is also sufficient as $\beta(A, u)$ is strictly concave with respect to u and therefore so is $H(A, u, \lambda)$, where we make use of the fact that the costate is nonpositive in our formulation. Setting

$$H_u = -1 - \lambda c \beta_u A = 0$$

reveals

$$u(A, \lambda) = (-c\lambda z)^{\frac{1}{1-z}} A. \tag{2.3}$$

According to the maximum principle (see, e.g., Feichtinger and Hartl [1986], Grass et al. [2008]) we will solve the maximization problem (2.1) subject to (2.2) by determining the stable manifolds arising from the canonical system that consists of the system dynamics (2.2) and the costate equation

$$\dot{\lambda} = r\lambda - H_A(A, u(A, \lambda), \lambda).$$

For our model, the canonical system is given by

$$\dot{A} = A\Big(kp^{-a}(\bar{A} - A) - c(-c\lambda z)^{\frac{z}{1-z}} - \mu p^b\Big), \tag{2.4}$$

$$\dot{\lambda} = r\lambda + \kappa p^{-\omega} + (-c\lambda z)^{\frac{1}{1-z}} + \lambda\Big(c(-c\lambda z)^{\frac{z}{1-z}} + \mu p^b - kp^{-a}(\bar{A} - 2A)\Big). \tag{2.5}$$

The steady states of the canonical system are the solutions of $\{\dot{A} = 0, \dot{\lambda} = 0\}$. In order to allow for an analytical derivation of the steady states and their stability properties we set $z = 0.5$ for the following analyses. It can be immediately seen from (2.4) that $\hat{A} = 0$ is always a solution. The associated λ can be found by solving $\dot{\lambda} = 0$ and using the negative solution of the quadratic equation obtained:

$$\hat{\lambda} = \frac{2}{c^2}\Big((r + \mu p^b - kp^{-a}\bar{A}) - \sqrt{(r + \mu p^b - kp^{-a}\bar{A})^2 + c^2 \kappa p^{-\omega}}\Big).$$

Further, two more steady states may occur, provided that they are feasable:

$$\hat{A} = \frac{\frac{c^2\lambda}{2} + kp^{-a}\bar{A} - \mu p^b}{kp^{-a}},$$

$$\hat{\lambda} = -\frac{2}{3c^2}\Big(r + kp^{-a}\bar{A} - \mu p^b \pm \sqrt{(r + kp^{-a}\bar{A} - \mu p^b)^2 - 3c^2 \kappa p^{-\omega}}\Big).$$

In Bultmann [2007] and Grass et al. [2008] a more general and detailed analysis of the constellations of parameter values is given for which the existence of three feasable steady states can be assured.

Figure 2.1 shows the phase diagram for the base model with the parameters presented in Section 2.3. There are three steady states – two saddle points and one unstable focus – and the optimal paths are highlighted in blue. See Table 2.1 for the steady state levels of the number of users, shadow prices, and control spending.

	A	λ	u
$\hat{A}_l$	0	-648.84	0
$\hat{A}_f$	4.402634×10^6	-222.25	1.01597×10^8
$\hat{A}_h$	1.086781×10^7	-38.07	7.35889×10^6

Table 2.1: Coordinates and associated control spending levels of the steady states of the canonical system. The indices l,f,h refer to low saddle point, unstable focus, and high saddle point, respectively.

We see that a DNSS-point occurs at $A_{DNSS} = 7,546,903$. According to the Maximum Principle the stable manifolds of the saddle point steady states provide candidates for the optimal solutions. By comparing the costs along those trajectories the optimal paths can be determined. A DNSS threshold (named after *Dechert-Nishimura-Sethi-Skiba*, cf. Dechert and Nishimura [1983], Sethi [1977], Sethi [1979] and Skiba [1978]) is an initial value for which two different optimal solutions for the problem exist, i.e. a point where the costs along two different trajectories are equal. That implies that in this point the decision maker is indifferent between two long-term policies to follow.

In other words, both saddle points are optimal long-term solutions locally, i.e. their optimality depends on the initial value $A_0 = A(0)$. In the DNSS point both saddle point steady states are optimal long-term outcomes. For A_0 smaller than the DNSS threshold it is optimal to approach the lower saddle point steady state, while for an initial value larger than the DNSS point the optimal long-term solution lies in the higher saddle point.

Figure 2.2 shows the optimal control spending for this policy. When converging to the lower saddle point starting at the DNSS threshold note that at first the spending levels have to be raised in order to reach the saddle point although the number of users is already decreasing. Figure 2.3 shows the value of the objective as a function of the initial value A_0. It reflects the total costs caused by drug consmption for each A_0.

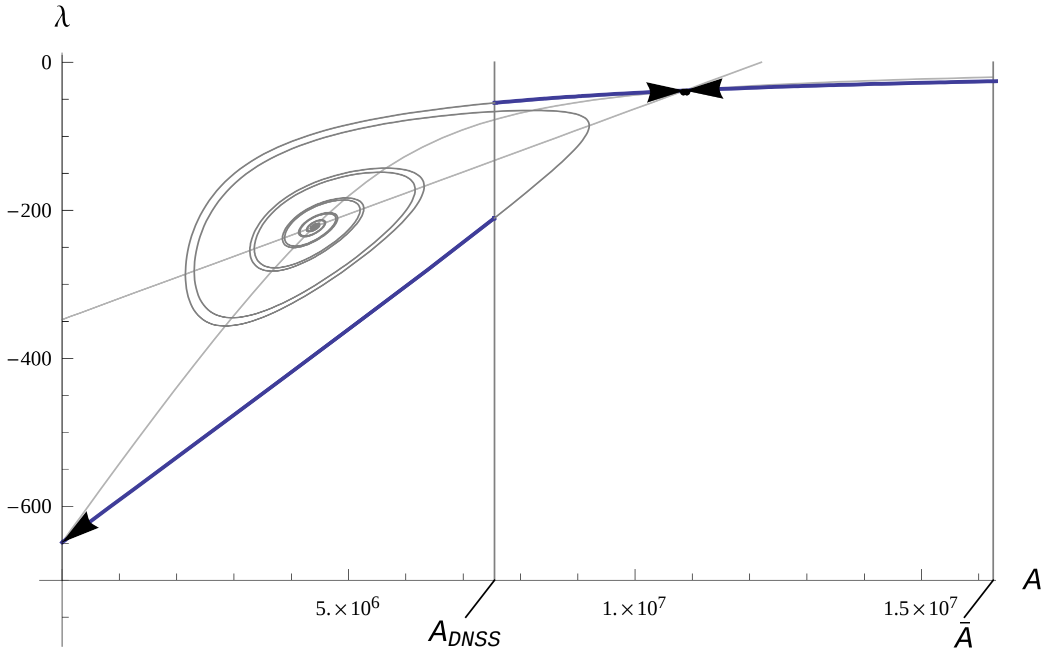

Figure 2.1: Phase diagram for the base model. Optimal paths are highlighed in blue.

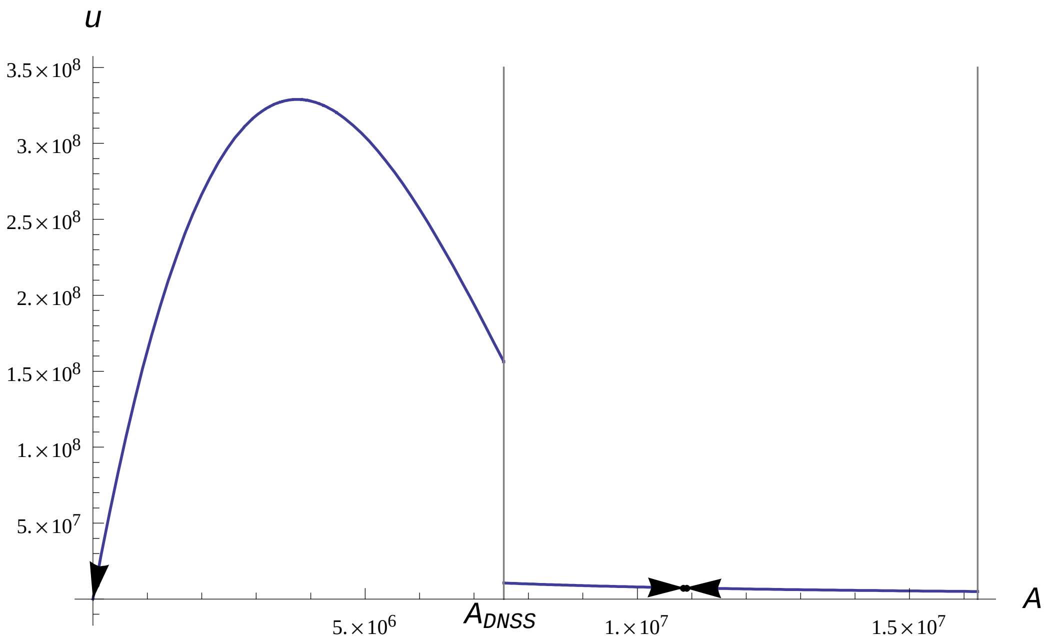

Figure 2.2: Base model: Optimal levels of control spending.

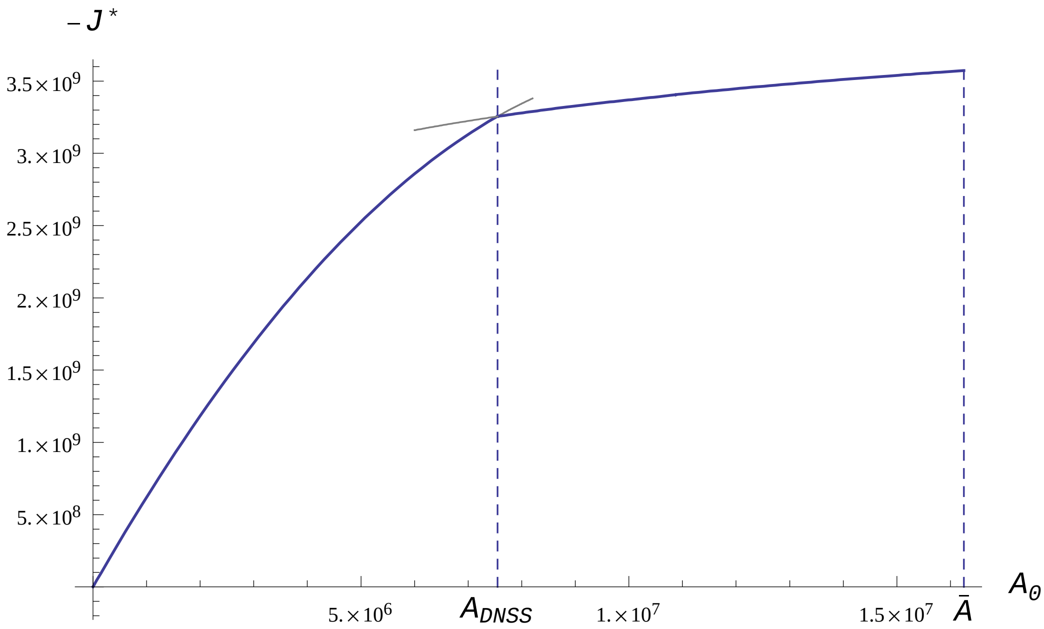

Figure 2.3: Base model: Optimal utility J^* as a function of the initial value A_0.

2.3 Parameters

The baseline parameters summarized in Table 2.3 for this thesis are taken from Bultmann et al. [to appear] for the most part. For more background information about the origin of these values we refer to there.

Parameter	Value	Description
r	0.04	annual discount rate
a	0.25	absolute value of the price elasticity of initiation
$\bar{A}$	$16,250,000$	maximum number of users in the uncontrolled model
b	0.25	price elasticity of desistance
c	0.043229	treatment efficiency constant
k	1.581272×10^{-8}	initiation rate constant
κ	4.185	proportionality constant for social costs
μ	0.181282758	natural outflow rate proportionality constant
p	0.12454	retail price per gram
ω	0.5	absolute value of the short-term price elasticity
z	0.5	treatment's diminishing returns

Table 2.2: Baseline Parameter Values

Chapter 3

Bifurcation Analysis

Bifurcations are local and global qualitative changes of a system that occur due to a variation of parameters. Here it is of special interest how the number and stability properties of steady states change when certain parameters (in this case the social costs κ and later the initiation constant k) change in order to determine the optimal policy. In what areas of the parameter space of κ and k do we find one or three steady states? In the latter case, are always two of them optimal? In fact, it will be demonstrated that the existence of three steady states does not guarantee multiple optimal long-term solutions.

Intuitively, one would expect that higher social costs cause the lower saddle point to be the optimal long-term solution. From a certain threshold of κ the policy of eradicating drug consumption may be more cost-effective than staying at a high long-term level. In fact, multiple steady states do appear just for κ smaller than a certain threshold, and that threshold is highly influenced by the initiation constant k.

In the following figures saddle points are always indicated by red lines while unstable focuses and nodes are displayed in green.

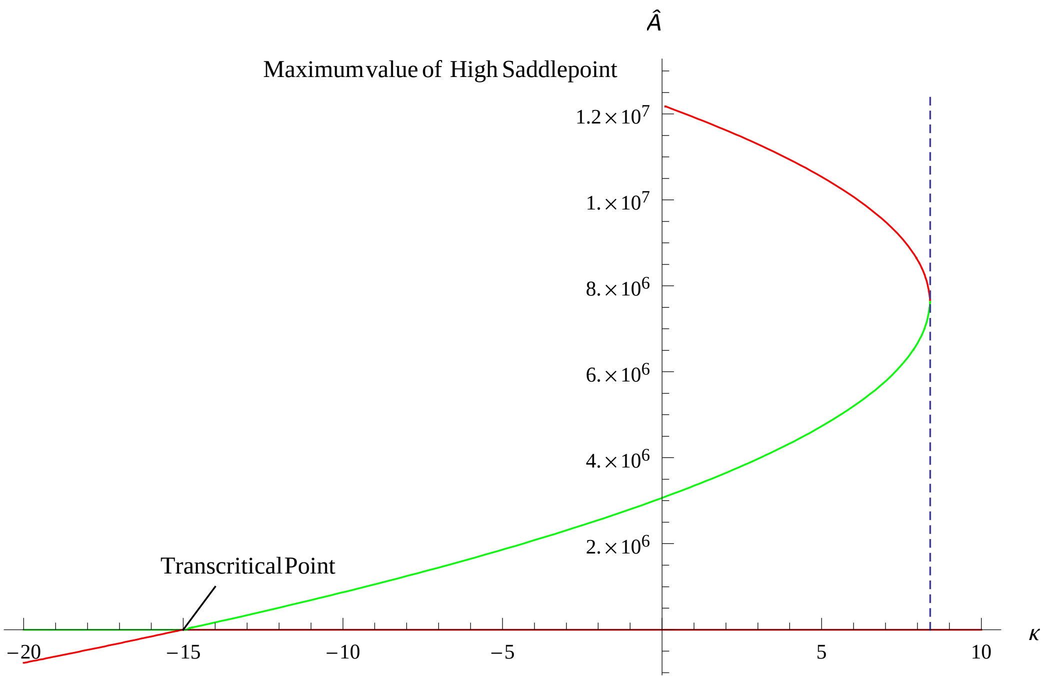

Figure 3.1: Bifurcation diagram for the base case.

3.1 Analytical Analysis

There are some critical points in the bifurcation diagram (Figure 3.1) that can be calculated analytically for this model.

Transcritical Point The value of κ for which the saddle point $\hat{A} = 0$ turns into an unstable vortex or node.

Maximum Value of the High Saddle Point The worst case scenario where no social costs due to drug consumption occur (i.e. $\kappa = 0$) and therefore no money is spent on the control.

Blue-Sky Point The largest value of κ with more than one steady state.

To determine the transcritical point the threshold where the sign of one of the eigenvalues of the steady state at $\hat{A} = 0$ switches from positive to negative has to be calculated.

For that purpose one has to determine the Jacobian matrix evaluated at the steady state and compute its eigenvalues. One of them is always positive, the other can be positive or negative depending on the chosen parameter values.

For the steady state

$$\hat{A} = 0$$

and

$$\hat{\lambda} = \frac{2}{c^2}\Big(\mu p^b + r - kp^{-a}\bar{A} - \sqrt{(\mu p^b + r - kp^{-a}\bar{A})^2 + c^2\kappa p^{-\omega}}\Big)$$

the eigenvalues

$$\sqrt{c^2\kappa p^{-\omega} + (\bar{A}kp^{-a} - \mu p^b - r)^2} \qquad\qquad r - \sqrt{c^2\kappa p^{-\omega} + (\bar{A}kp^{-a} - \mu p^b - r)^2}$$

are revealed. The first eigenvalue is always positive as all of the parameter values are positive and negative social costs κ are not considered for this model in general. The second one can be negative, zero or positive. Setting it to zero and solving this equation for κ reveals the transcritical point

$$\kappa_{Transcrit} = \frac{r^2 - (\bar{A}kp^{-a} - \mu p^b - r)^2}{c^2 p^{-\omega}}.$$

For the maximal value of the upper saddlepoint the utility functional has to be inspected. As $\kappa = 0$ the objective is reduced to

$$J = -\int_0^\infty e^{-rt}u(t)dt.$$

The maximum of this function is obviously $J = 0$ with $u(t) = 0$, $\forall t$, as $u(t)$ is nonnegative. In this scenario the optimal policy is not to spend anything on treatment as no costs for society are caused by drug consumption.

Hence, the system dynamics are reduced to

$$\dot{A}(t) = kp^{-a}A(t)(\bar{A} - A(t)) - \mu p^b A(t).$$

The steady states of this uncontrolled system are given by $\hat{A} = 0$ and

$$\hat{A}_{Max} = \bar{A} - \frac{\mu}{k}p^{a+b}.$$

As a result, it can be seen that the saddle point at $\hat{A}_{Max}$ is influenced by the initiation constant k. The higher k, the higher the upper long-term solution, which makes sense.

In order to calculate the blue-sky value of κ one needs to determine the point where the upper saddle point and the unstable steady state are equal. As already discussed in Chapter 2.2 positive steady states are given by

$$\hat{A} = \frac{1}{kp^{-a}}\Big(\frac{c^2\lambda}{2} + \bar{A}kp^{-a} - \mu p^b\Big)$$

and

$$\lambda_{1,2} = -\frac{2(r + \bar{A}kp^{-a} - \mu p^b)}{3c^2} \pm 2\sqrt{\left(\frac{r + \bar{A}kp^{-a} - \mu p^b}{3c^2}\right)^2 - \frac{\kappa p^{-\omega}}{3c^2}}.$$

It is easy to see that the two steady states are equal if λ_1 and λ_2 are equal. Obviously, this occurs when the square root term above is zero. Therefore, setting the root to zero and solving this equation for κ reveals the blue-sky value

$$\kappa_{BS} = \frac{p^{\omega}(r + \bar{A}kp^{-a} - \mu p^b)^2}{3c^2}.$$

Note that this is a quadratic function in k with its minimum in $k_{min} = \frac{\mu p^b - r}{\bar{A}p^{-a}}$.

3.2 Variation of κ for the Base Model

Application of the results of the previous section leads to the critical points of the bifurcation diagram for the base model as summarized in Table 3.1.

Transcritical Point	$\kappa = -15.02097406$
Maximum Value of the Upper Saddle Point	$\hat{A} = 1.22042 \times 10^7$
Blue-Sky Point	$\kappa = 8.3795245893$

Table 3.1: Critical points of the bifurcation diagram for the base model

As shown above in Figure 3.1, three feasible steady states exist for $-15.021 < \kappa < 8.3795$. For higher values of κ only the lower saddle at $\hat{A} = 0$ serves as a long-term optimal solution. For $\kappa < -15.021$ one steady state becomes negative and hence infeasible.

Some of these thoughts are only academic, however. As κ and A are parameters and variables representing social costs and numbers of people, respectively, note that the bifurcation diagram for $\kappa < 0$ and $A < 0$ is of no practical relevance for this model. Therefore, further analysis will be restricted to the area where κ and A are nonnegative. As a consequence, Figure 3.2 focuses only on the positive quadrant of the preceding bifurcation diagram.

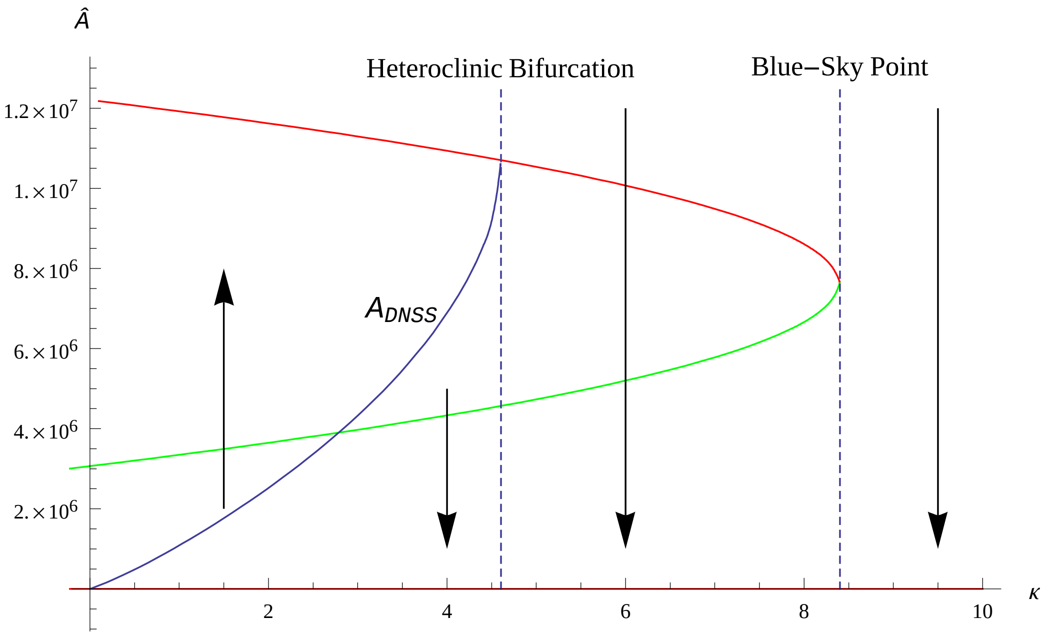

Figure 3.2: Detail of the bifurcation diagram and the DNSS threshold for the baseline parameters. The arrows indicate in which area it is optimal to converge to the upper or the lower saddle point, respectively.

In order to complete this discussion, the DNSS threshold is included. Here, it is represented by the blue curve as a function of κ. The function obviously has to start at the origin. In the case of no social costs ($\kappa = 0$) it is optimal for that specific utility functional not to spend anything on the control as no costs occur that have to be avoided. The uncontrolled number of users would therefore move towards the upper saddle point whenever $A(t) > 0$.

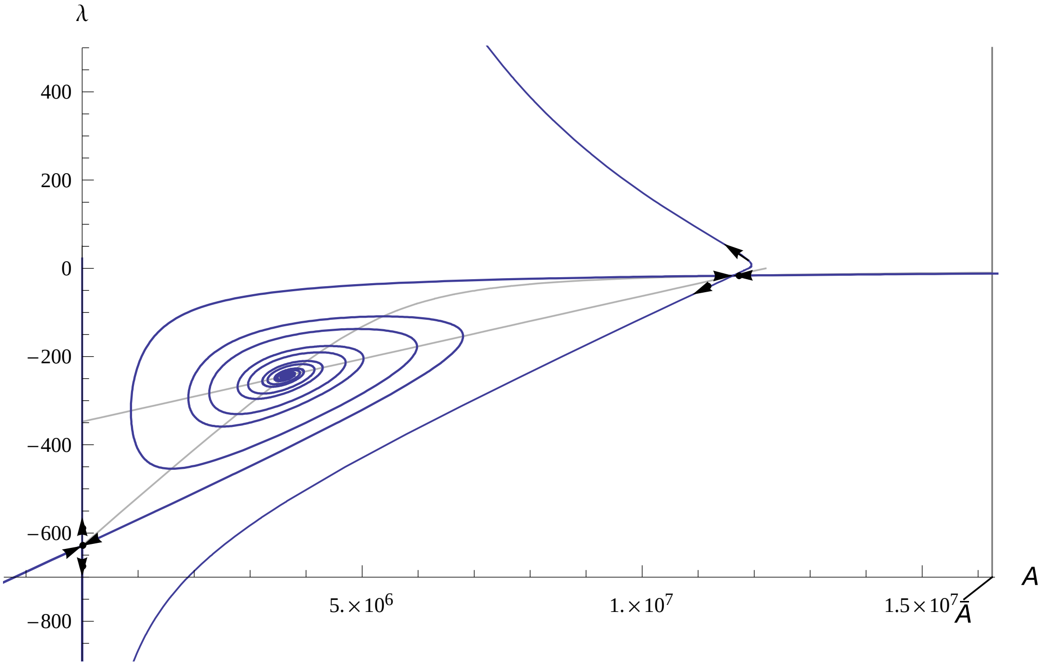

Figure 3.3: Phase diagram for the base case with $\kappa = 2$ including the unstable manifolds. Arrows pointing towards a steady state indicate stable manifolds, arrows pointing away unstable manifolds.

Further, a monotonically increasing behaviour of the DNSS threshold can be observed. As the social costs per gram consumed rise, more and more initial values A_0 require eradication of drug consumption in order to be cost-effective in the long run. Figure 3.3 shows the phase diagram for $\kappa = 2$. In comparison to the base model discribed in Chapter 2.2 it can be clearly seen that the larger κ the more the stable manifold of the lower saddle point tends towards the upper saddle point. At the same time, the stable manifolds of both saddle points approximate each other when circling around the unstable focus.

At $\kappa = 4.60557596885$ the DNSS threshold has the same level of A as the higher saddle point. In this situation a so-called heteroclinic bifurcation

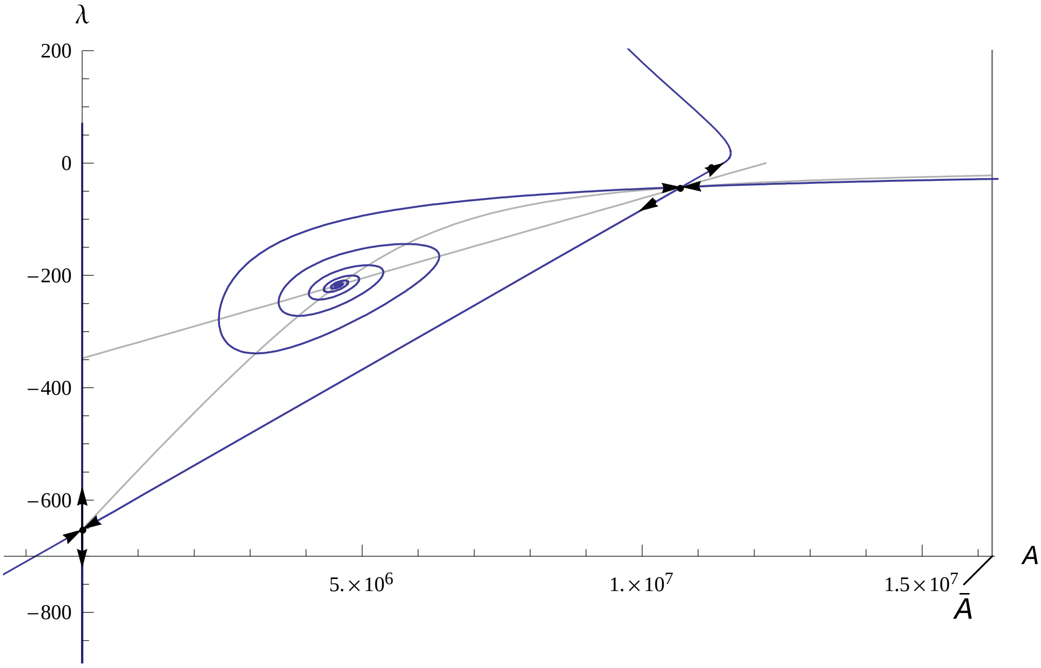

Figure 3.4: Phase diagram of the base model for $\kappa = 4.60557596885$ showing a heteroclinic bifurcation.

occurs. Figure 3.4 shows the phase diagram for this case. Note that the unstable manifold of the upper saddle point is at the same time the stable manifold of the lower saddle point (*heteroclinic connection*). Further, the unstable focus now only serves as a source for the upper saddle point.

For a more detailed discussion on heteroclinic bifurcations and implications for DNSS thresholds also refer to Wagener [2003].

For κ higher than the heteroclinic bifurcation point no DNSS threshold occurs despite the existence of three steady states. Figure 3.5 shows the phase diagram for $\kappa = 6.5$. For κ bigger than the heteroclinic bifurcation value the higher saddle point is dominated by the lower one, i.e. the lower saddle point is always cost-efficient. As a consequence, from now on there is only one optimal long-term solution. The optimal policy now always consists of eradication of drug consumption. The higher saddle point still is a long-term solution but it is no longer optimal.

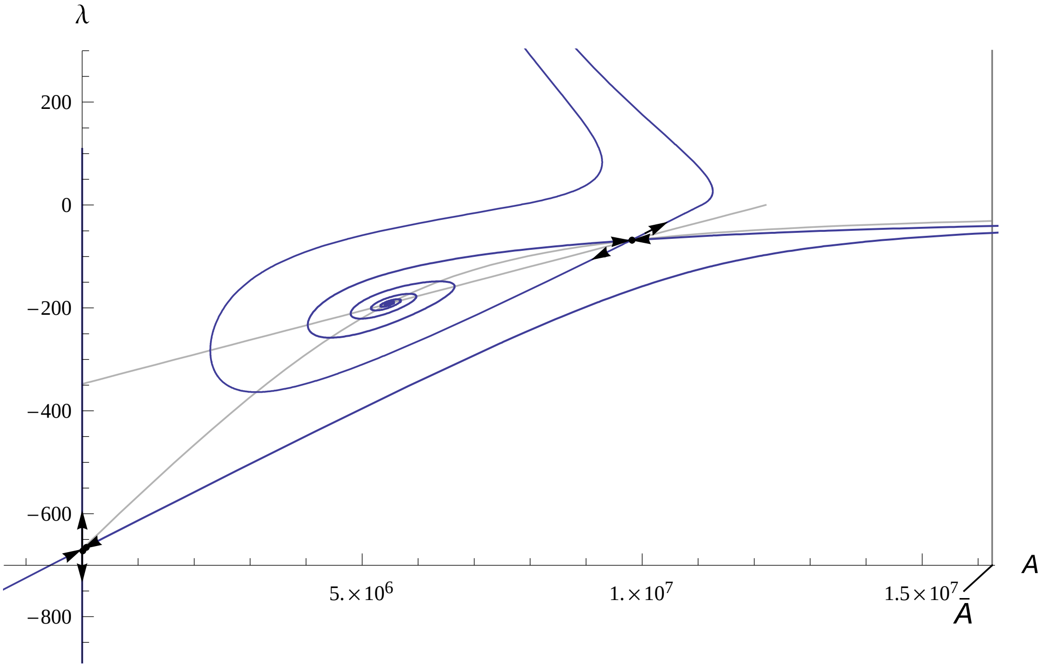

Figure 3.5: Phase diagram of the base model with $\kappa = 6.5$.

In the bifurcation diagram of Figure 3.2 and also in the previous phase diagrams it can be noticed that the unstable focus and the upper saddle point are approaching each other for increasing values of κ. The point where they finally meet is called blue-sky or Saddle-Node bifurcation (see also again Wagener [2003]). Two separate but neighbouring steady states collide and vanish.

For κ larger than the blue-sky threshold only one steady state - the lower saddle point - continues to exist. Therefore, only one long-term solution exists and clearly it is optimal to approach it.

3.3 Variation of Initiation

In the previous section bifurcations were analysed where only one parameter – the social costs κ – changed. Now a second parameter, the initiation constant k has its values changed in the analysis. This will be of special interest in the next chapter when discontinuities of the initiation constant are studied. In this thesis, changes of the parameter k are indicated by using a scaling factor.

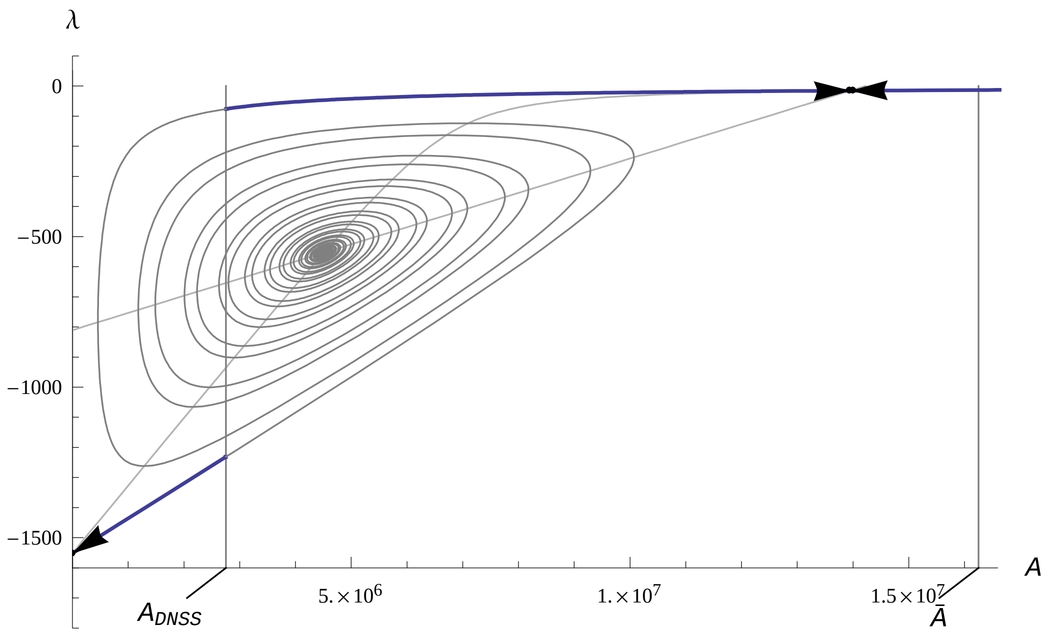

Figure 3.6: Phase diagram for the base model with initiation constant of $2k$.

Figure 3.6 shows the phase diagram of the basemodel with a higher initiation constant of $2k$. Comparing this scenario with the baseline case (see Chapter 2.2) note that the stable manifolds of the saddle point steady states are circulating the unstable focus more often. The whole system reacts much faster to changes, i.e. it is more volatile. A direct consequence is the shift of the DNSS threshold from 7.546903×10^6 to 2.754878×10^6. Also note that the lower saddle point now has a costate of -1500. An additional user in this scenario causes around twice the costs compared to an additional user in the base model. This is due to the fact that a new user now causes twice as many new users as in the base case.

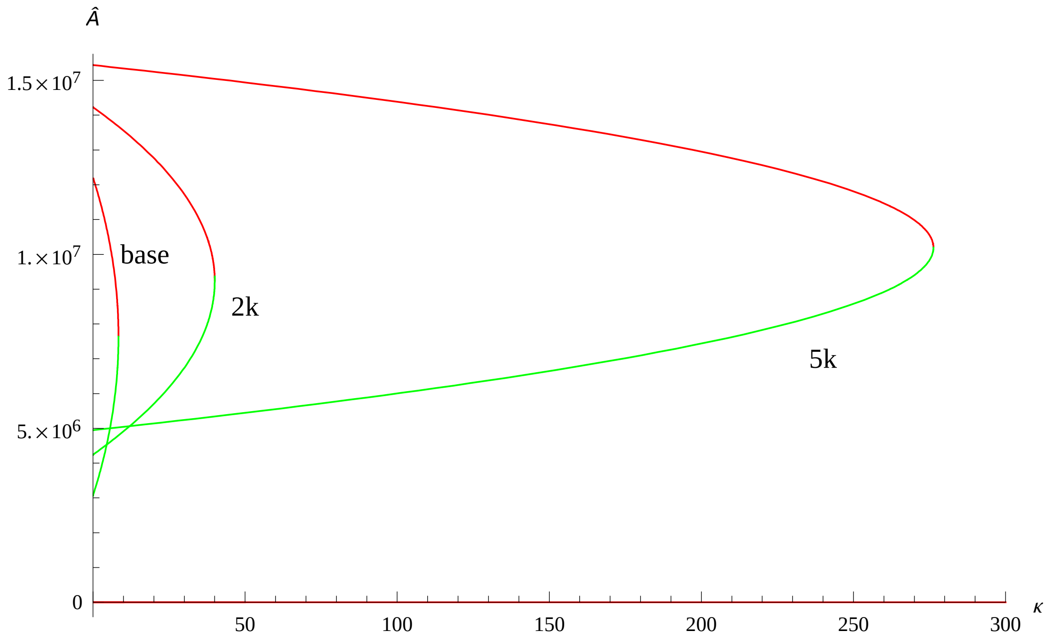

Figure 3.7: Comparison of bifurcation diagrams for the base case initiation and initiation at the twofold and fivefold levels, respectively.

Figure 3.7 shows the bifurcation diagram for different initiation constants. Focusing on the comparison with larger initiation constants than the base case the figure indicates a monotonically increasing behaviour of the Blue-Sky point with respect to both A and κ. First, when k increases the range with three steady states also increases. In the case of initiation twice the size of the baseline case the feasible range of three steady states, i.e. where $\kappa \geq 0$, has almost grown by the factor five. Taking an initiation constant even five times higher leads to an area that is 33 times larger than in the base case. Therefore, a nonlinear monotonically increasing behaviour of the range with three steady states with respect to κ can be observed. However, as was pointed out in Section 3.1 the blue-sky threshold is a quadratic function with a minimum at $k_{min} = \frac{\mu p^b - r}{Ap^{-a}}$. For the base model this minimum is at a scaling factor of 0.156497. As the main focus of this thesis is mainly on scenarios where k is larger than in the base case, a monotonic increasing function may be assumed.

The number of users in the blue-sky point is also a monotonically increasing, but it is concave and in k and has an upper bound. It is given by

$$\hat{A}^{BS} = \frac{5(\bar{A}kp^{-a} - \mu p^b) - r}{6kp^{-a}}.$$

This can be obtained by substituting the value κ_{BS} from Section 3.1 for κ in the formula for the positive steady states $\hat{A} > 0$ from Chapter 2.2. It can be shown easily that $\frac{\partial \hat{A}^{(BS)}}{\partial k} > 0$ and $\frac{\partial^2 \hat{A}^{(BS)}}{\partial k^2} < 0$. The limit for $k \to \infty$ may be obtained by

$$\lim_{k\to\infty} \hat{A}^{(BS)} = \frac{5}{6}\bar{A}.$$

For very small values of k additional observations can be made. For a scaling factor $k < 0.433924$ the transcritical point moves into the positive range of κ. Therefore, for small k there are cases where only the upper saddlepoint serves as a long-term solution. Additionally, the blue-sky point may move into the negative range of A. This implies that in those cases the upper saddle point becomes the only optimal long-term solution.

Furthermore, the properties of the unstable steady state may change for very small k. As we have seen in the discussion of the base model in Chapter 2.2, the unstable steady state in the middle is a focus, i.e. its eigenvalues are complex conjugates. For small k the imaginary part vanishes and the steady state becomes an unstable node.

Finally, statements about the behaviour of the steady states themselves can be made. It can easily be seen that the upper saddle point is an increasing function of k for all κ. Consequently, so is the maximum value for this saddle point as it was shown in Section 3.1. However, the same statement cannot be made for the unstable focus/node.

3.4 Variation of κ and k

After having considered analyses for changes of κ and then of k separately and their impacts on the qualitative properties of the system, the results are now combined. Again, it is of special interest in which areas of the (k, κ)-space the existence and optimality of multiple steady states can be assured. With regard to the following two chapters the focus lies on the part of the k-space where the scaling factor of k is larger than 1.

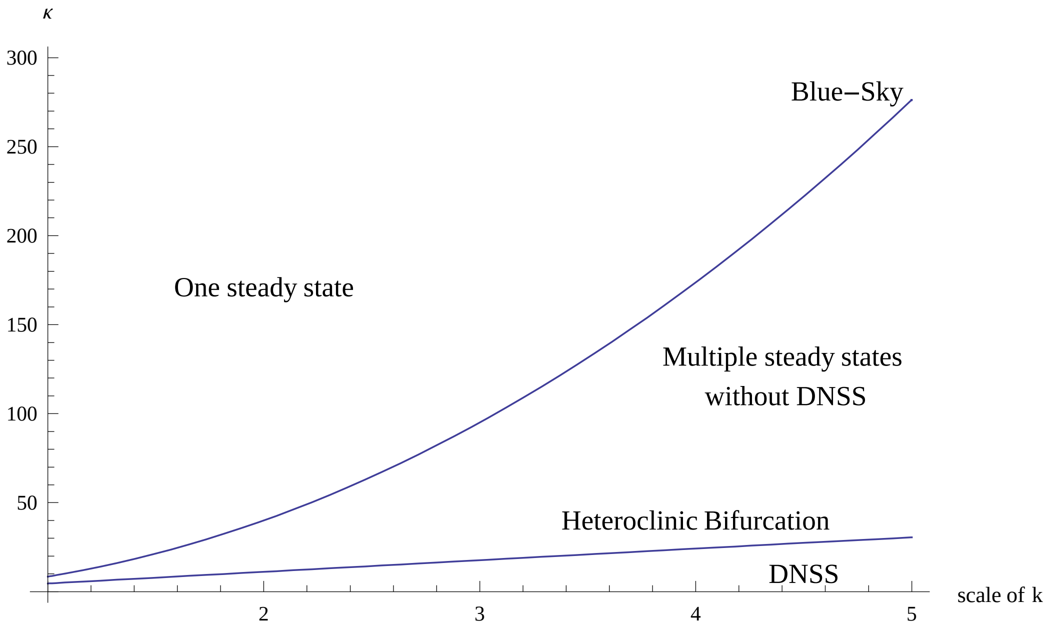

Figure 3.8: Bifurcation diagram: (k, κ)-space with focus on the properties of steady states.

As the final result of this chapter, Figure 3.8 displays the (k, κ)-space for scales of k between 1 and 5. Again, it can be clearly seen that the blue-sky threshold is an increasing function of both the social costs κ and the scale of the initiation constant k. This implies that for either of those parameters increasing, the range with multiple steady states increases as well.

However, it is of greater interest for the policy maker in which areas of the parameter space the existence of not only multiple but also multiple optimal steady states can be assured. This threshold is displayed by the heteroclinic bifurcation curve in Figure 3.8. Only in the area below that line a DNSS threshold occurs. It is astonishing that this area shows a seemingly linear increasing behaviour at the best whereas the area with three steady states itself is growing convexly.

The pure existence of three steady states after the change of a parameter does not necessarily imply a qualitative change in policy as it will be demonstrated in the next chapter. It may, however, have consequences if the changes happen along the heteroclinic bifurcation curve.

Chapter 4

Analysis of Initiation Shocks

In this chapter, initiation shocks and their consequences on the optimal dynamic drug policy are analysed.

An initiation shock is a sudden change of the initiation constant k. Here, we only discuss shocks where initiation in the shock stage is higher than initiation in the base model. The idea is that initiation is higher at the beginning of a drug epidemic, i.e. when the drug is new and fancy, and therefore more people are willing to try it.

4.1 The Two-Stage Model

Suppose that there is a stage with a higher initiation rate and at time T the initiation constant shifts back to a lower level. Then the problem can be divided into two stages. The first stage has a finite time horizon of duration T whereas the second one is a problem with an infinite time horizon. For similar applications of this procedure see also Tragler [1998] and Bultmann [2007].

Starting with the first – finite – stage the objective function is given by

$$J_T = \int_0^T e^{-rt}(\kappa p^{-\omega} A(t) + u(t))dt + e^{-rT} S(A(T)) \tag{4.1}$$

with $S(A(T))$ describing the value of the state $A(T)$ at time T. In accordance to the maximum principle the transversaltiy condition

$$\lambda(T) = S_A(A^*(T))$$

must hold at time T, with $A^*(T)$ being the optimal value of A at time T (see Feichtinger and Hartl [1986]). In order to connect the two problems reasonably now a salvage function $S(A(T))$ has to be chosen. Following Tragler [1998] an appropriate choice is the optimal value J^* of the utility functional of the second stage using $A^*(T)$ as the initial value A_0.
As this thesis is dealing with a problem with an infinite time horizon the optimal utility functional is given by

$$\frac{1}{r}H(A(0), u(0), \lambda(0))$$

(see Feichtinger and Hartl [1986]). Application to our model reveals the following transversality condition

$$\lambda^{(k_S)}(T) = \frac{1}{r}H_A^{(k_b)}(A^*(T), u^{(k_b)}(0), \lambda^{(k_b)}(0))$$

where k_S denotes the initiation constant during the shock stage and k_b the parameter for the base case, i.e. the second stage.

4.2 Initiation shocks

We start the analysis with an initiation in the shock stage being at twice the level of the base initiation, i.e. $k_S = 2k_b$. After two time units it switches to the base level, i.e. the duration of the shock is $T = 2$. For most of the following scenarios a rather low initial value of $A_0 = 250,000$ is chosen. This may be motivated by the fact that stages of higher initiation are more likely to happen when a new drug enters the market and thus, not that many people have already used it. Additionally, the value of κ is varied in order to cover the different steady state scenarios discussed in Chapter 3. Table 4.1 summarizes the additional parameters for the scenarios discussed in the subsections below.

Initial value A_0	social costs κ
$250,000$	4.185
5×10^6	4.185
$250,000$	7
$250,000$	15
$250,000$	2

Table 4.1: Parameters for the different scenarios of initiation shocks.

4.2.1 DNSS Threshold and a Low Initial Value

In this first scenario, the baseline parameter values and a low initial value of $A_0 = 250,000$ are assumed.

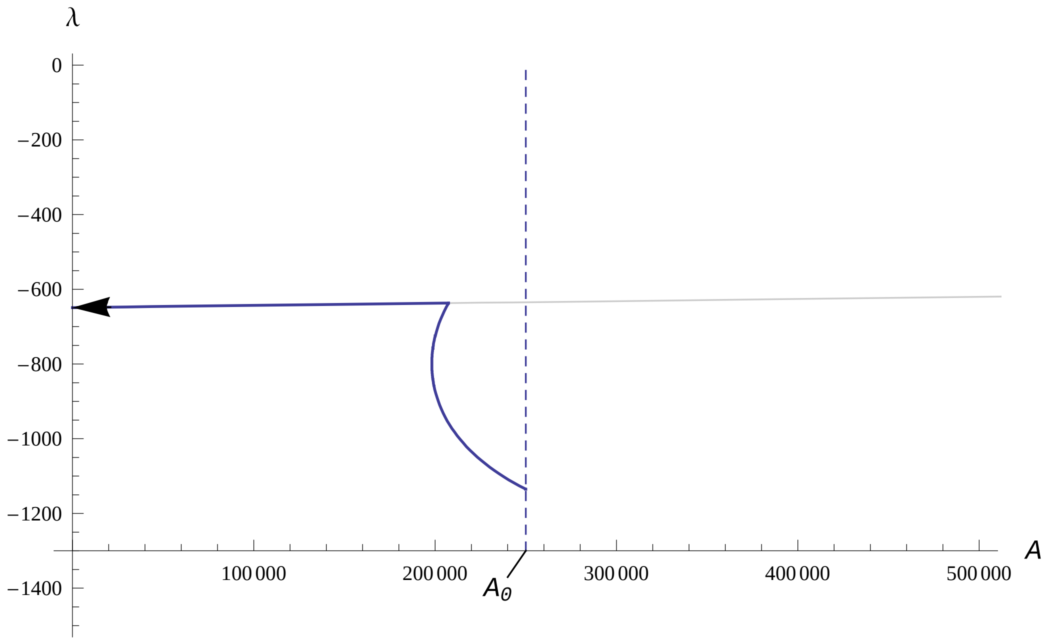

Figure 4.1: Two-stage model: Detail of phase diagram for shock of intensity $2k_b$ and duration $T = 2$. Optimal path without shock gray, optimal path with shock blue.

Figure 4.1 shows the relevant detail of the phase diagram. The blue curve represents the optimal path in case of a shock, the gray line the base model.

Due to the shock with higher initiation it starts at a lower value of λ. This makes sense as the absolute value of the costate represents the costs that are added by an additional user. Due to the higher initiation constant a new user causes more new entries into the system and therefore causes additional costs.

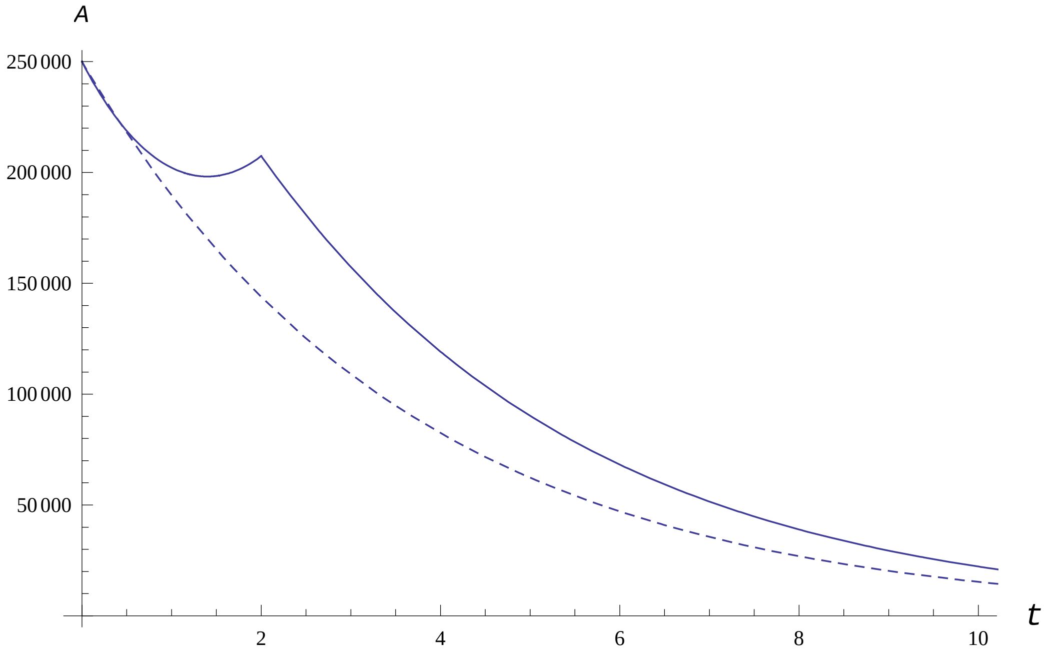

Figure 4.2: Two-stage model: Number of users as a function of time for $A_0 = 250,000$. The blue line are users in case of a shock, the dashed line refers to the base model, i.e. without shock.

Similar to the one-stage model a DNSS point occurs but this time at a lower level of users. As costs are accumulated it is more often a cost-optimal policy to tolerate a high long-term level of users rather than to eradicate them by spending huge amounts of money on the control. The threshold for a shock of intensity $k_s = 2k_b$ and duration $T = 2$ is found at $A_{DNSS} = 4.111040 \times 10^6$.

Interestingly, the number of users show a nonmonotonic behaviour over time (see Figure 4.2). At first, a decrease can be observed, but as control spending decreases as well (see Figure 4.3), the number of users start to rise again until the initiation constant jumps to the lower base level. From there it declines to the lower saddle point.

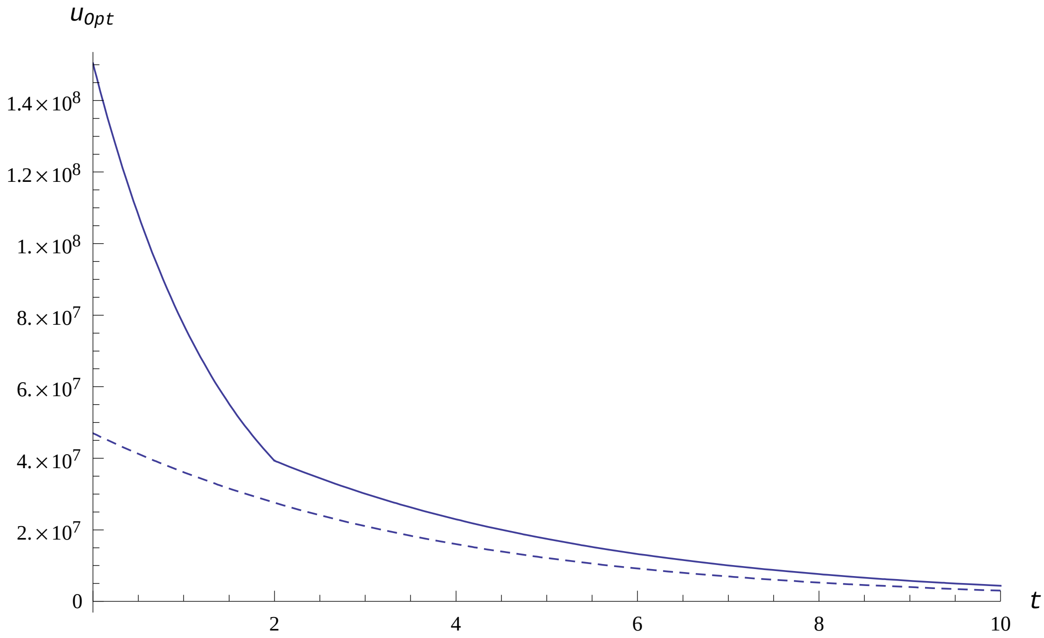

Figure 4.3: Two-stage model: Optimal spending on the control u as a function of time. The blue line represents spendings with a stage of higher initiation, the dashed line allows a comparison with the base model.

Comparison of the total costs reveals the dimension of the costs added by the higher initiation stage. In a one-stage model with initial value $A_0 = 250,000$ total costs sum up to $-J_b^* = 1.60419284 \times 10^8$. Due to the shock costs rise to $-J_s^* = 2.89067 \times 10^8$. This is an increase of more than 80%.

4.2.2 DNSS Threshold and a High Initial Value

In this scenario, a higher initial value of $A_0 = 5 \times 10^6$ is chosen. Additionally, a second scenario with a longer shock period $T = 5$ is added in order to compare the effects of shocks of a different duration. Implications for the optimal dynamic policy are completely different, though. It was already mentioned in the previous section that due to the shock a shift of the DNSS threshold occurs. In the previous scenario that shift did not have any qualitative implications for the optimal policy. Here, the optimal solution consists of moving to the upper steady state instead. A completely different policy is required.

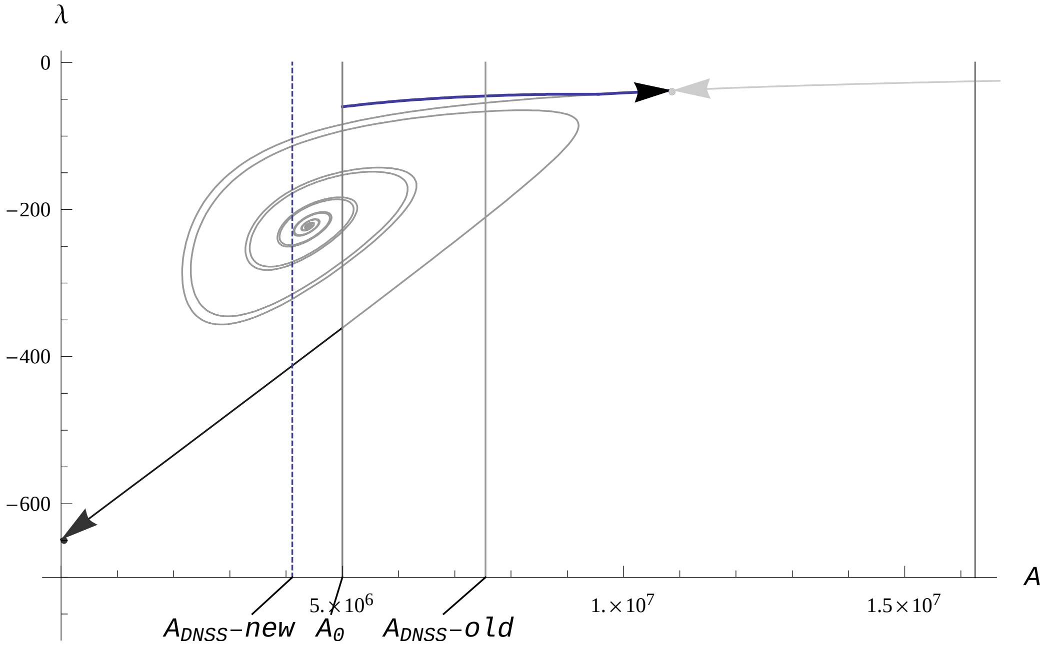

Figure 4.4: Two-stage model with high initial value and $T = 2$: Phase diagram showing the optimal path with shock in blue, without shock in dark gray.

Figure 4.4 shows the phase diagram for the scenario $T = 2$ including the DNSS point with and without shock ("new" DNSS and "old" DNSS, respectively). Note that the chosen initial value lies between the original and the new DNSS point. In fact, a qualitative change of the optimal dynamic drug policy only occurs if the initial value happens to be inbetween those two thresholds. For $T = 5$ the shift of the DNSS threshold is even stronger ($A_{DNSS-5} = 2.414707 \times 10^6$). Chapter 5 will focus on the impacts of a shock's properties (i.e. intensity vs. duration) on the DNSS threshold. Also note that the new optimal path lies above the original one and therefore has a lower costate in absolute value terms. This indicates that the shock supports convergence towards the equilibrium as an additional user causes less incremental costs than in the no-shock scenario.

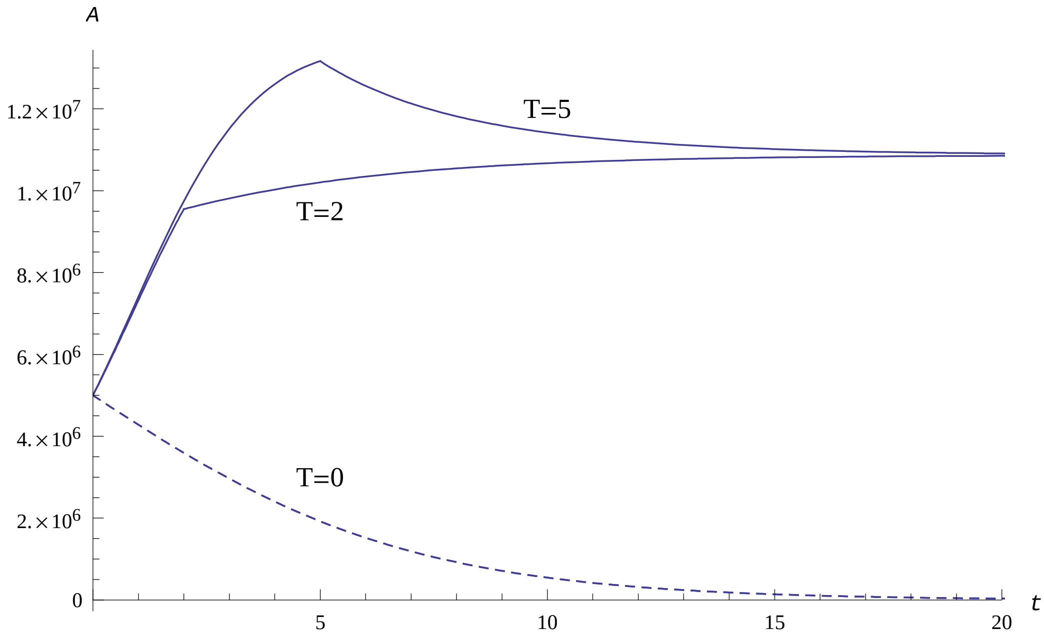

Figure 4.5: Two-stage model with high initial value: number of users as a function of time.

Considering the number of users over time (see Figure 4.5) a sudden, almost vertical increase in the number of users during the shock stage can be observed for both shock scenarios. However, note that in the $T = 5$-case the number of users overshoots during the shock stage showing a nonmonotonic behaviour whereas for $T = 2$ the convergence to the higher saddle point is monotonically increasing.

A impressive result is given by looking at the optimal control spending. Figure 4.6 reveals for the non-shock scenario massive spendings in the first 15 time periods and convergence to zero later on. In the case of a shock optimal control spendings seem to stay at almost the same level all the time. A closer look on spending levels during the shock (see Figure 4.7) allows us to observe a nonmonotonic behaviour during the shock period can be noticed: decreasing at first, then increasing towards the end of the first stage. Note

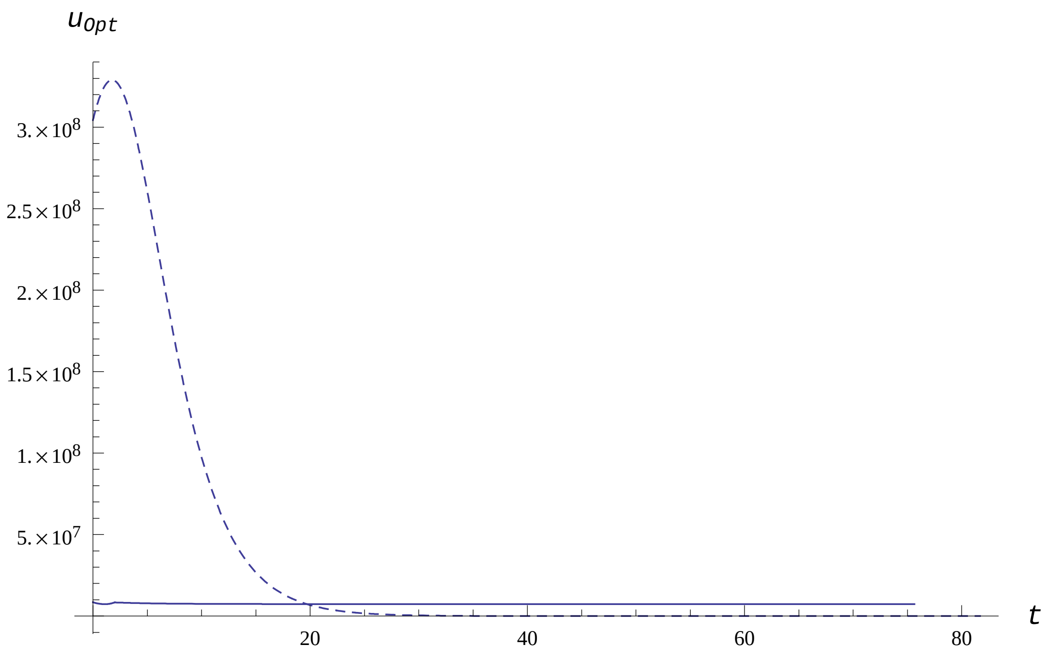

Figure 4.6: Two-stage model with high initial value: optimal control spending.

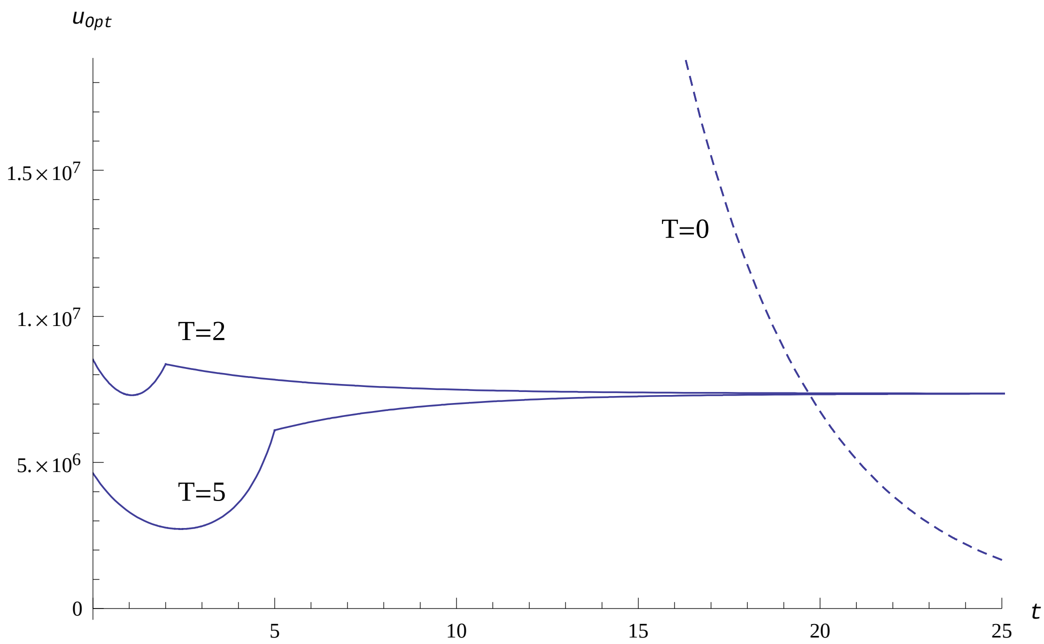

Figure 4.7: Two-stage model with high initial value: detail of optimal control spending focusing on the shock.

that although the number of users is rapidly increasing during the shock, optimal control spending is dropping. Also note, that spending levels during the shorter shock are higher than in the case of the longer shock.

A possible explaination is that due to the sudden and strong increase of users and the resulting additional costs caused by the former it wouldn't make sense to spend even more money on the control as it wouldn't have a sufficient impact. Towards the end of the shock, however, higher spending makes sense again to keep control after the high initiation stage. If the shock is longer, even more users occur and the less efficient are spendings on the control as users are tending towards the high equilibrium anyway. However, due to the overshooting higher spending levels are required later in order to decrease the number of users again once the shock is over.

4.2.3 Multiple Equilibria without DNSS Point

For the remaining scenarios an initial value of $A_0 = 250,000$ is assumed again. Here, however, the social costs parameter is raised to $\kappa = 7$.

Figure 4.8 shows the corresponding phase diagram. As already discussed in Chapter 3 only the stable manifold of the higher saddle point has its origin in the unstable vortex. Further, no DNSS threshold occurs as the lower saddle point dominates the higher equlibrium. It is now always optimal to move to the lower saddle point even though multiple candidates do exist. This is caused by the higher social costs κ: permanent drug consumption by a high long-term level of users becomes too expensive in comparison with the costs of eradication.

Further, when looking at the number of users (see Figure 4.9) a nonmonotonic behaviour similar to the first scenario can be observed: a decrease at first, then an increase with a local maximum at T, finally again a decreasing behaviour. Control spending levels, however, are much more increased during the high stage (see Figure 4.10). After the shock, optimal control spending levels are almost equal to the case without a shock.

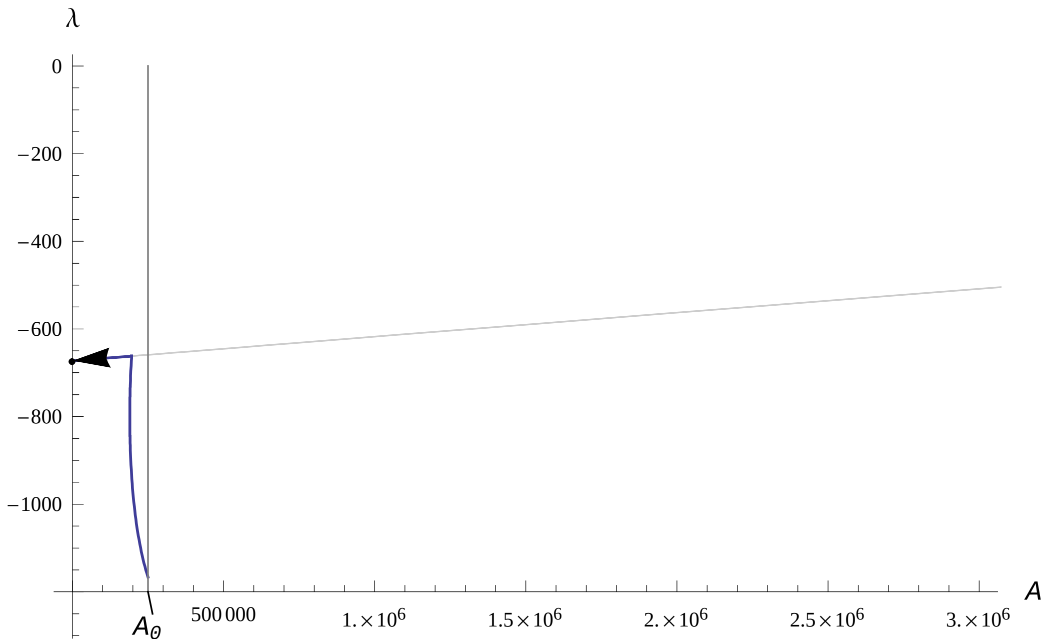

Figure 4.8: Phase diagram of the two-stage model for the case of three steady states without a DNSS threshold. Optimal paths are highlighted in blue.

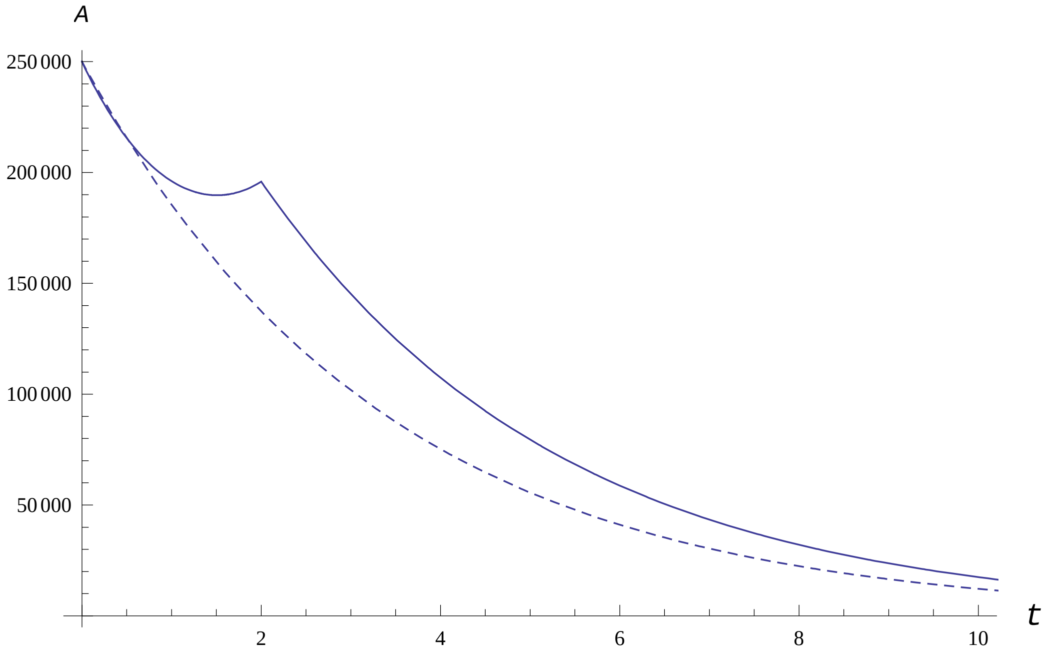

Figure 4.9: Two-stage model with three steady states without a DNSS point: Number of users as a function of time.

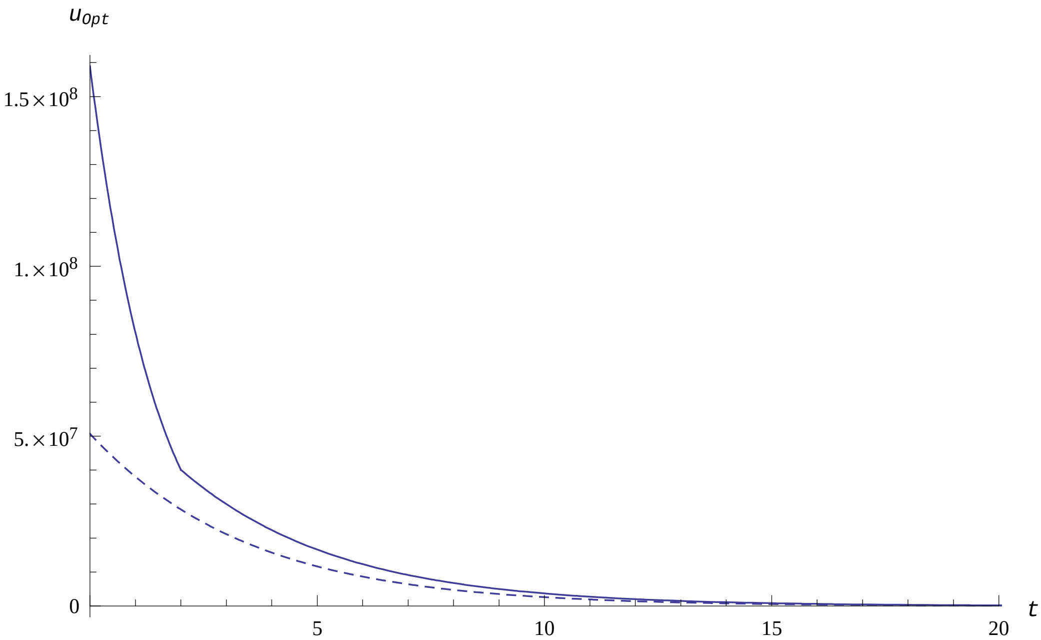

Figure 4.10: Two-stage model with three steady states without a DNSS point: Optimal control spending as a function of time.

4.2.4 Lower Saddle Point

Raising social costs even more than in the last scenario to $\kappa = 15$ extinguishes the upper saddle point and the unstable vortex (see Figure 4.11).

Impacts of the shock on the number of users and optimal control spending are almost equal to the previous scenario. The number of users show only a slight nonmonotonic behaviour during the time of the shock, then converge to the lower equilibrium slower than in the base model. Optimal control spending levels are very high in the shock stage and behave similarly to the case without shock afterwards (see Figures 4.12 and 4.13, respectively). Note, however, that the optimal spending starts at a higher level of 1.8×10^8 at $t = 0$ than in the previous scenarios with convergence to the lower saddle point (1.55×10^8 and 1.4×10^8, respectively).

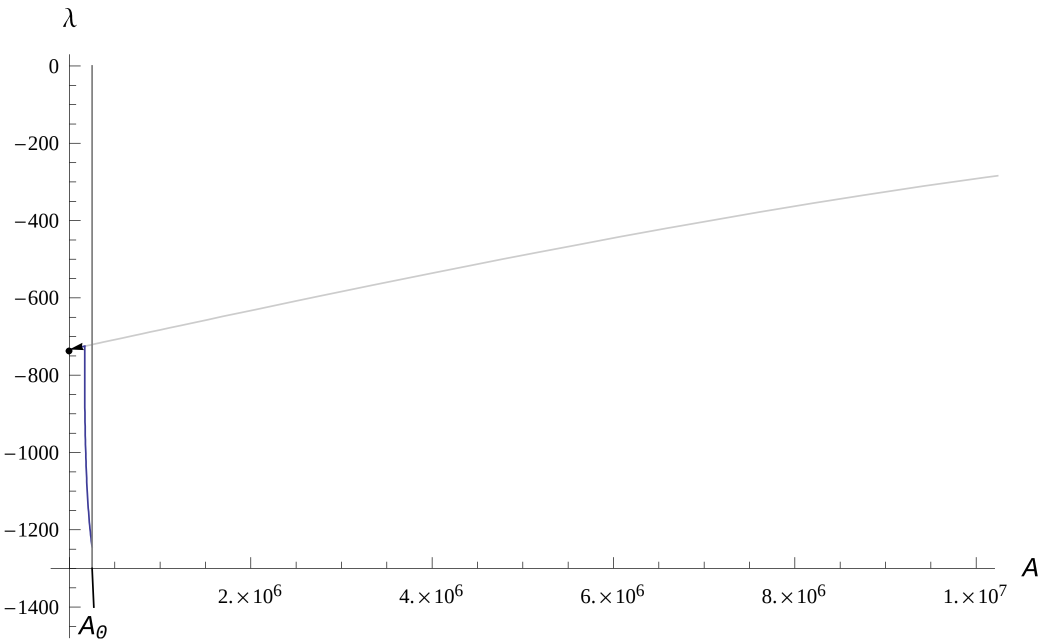

Figure 4.11: Two-stage model: Phase diagram for the case of only one steady state. The optimal path is highlighted in blue.

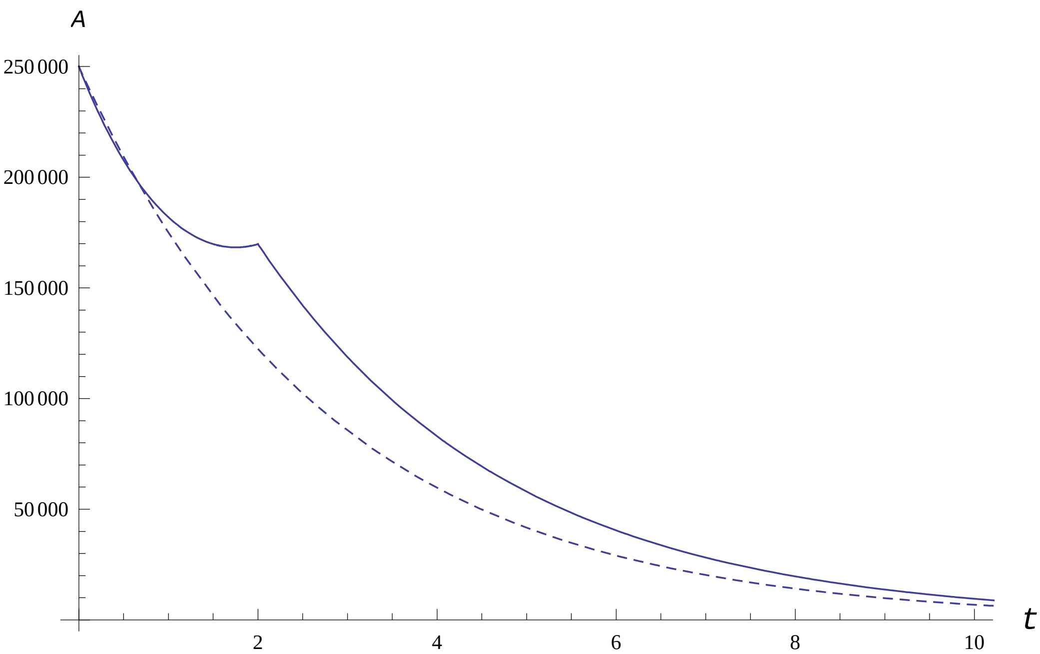

Figure 4.12: Two-stage model, lower saddle point: Number of users as a function of time.

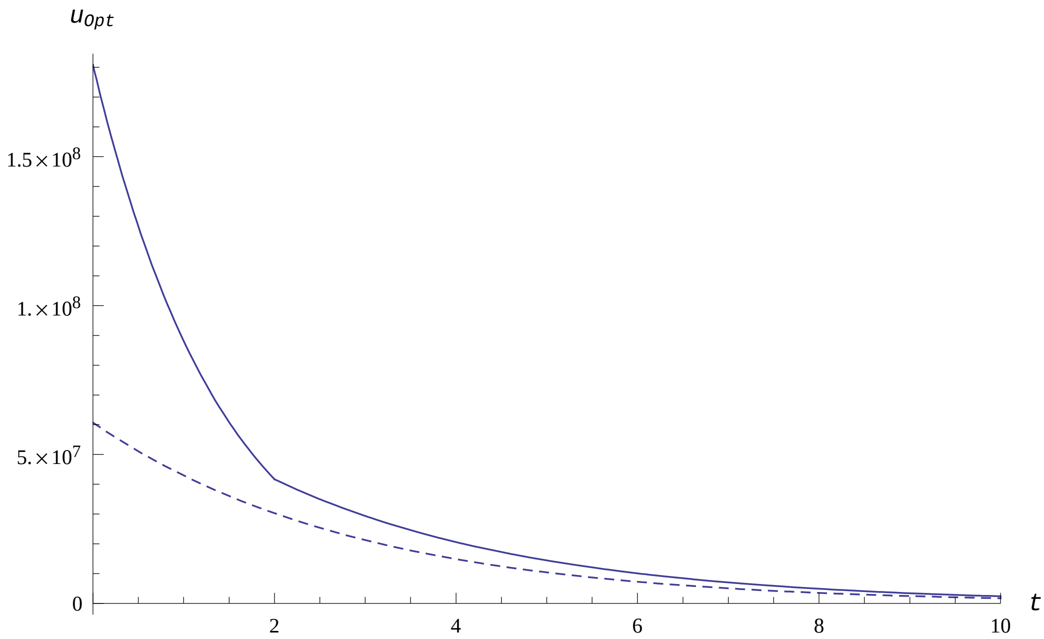

Figure 4.13: Two-stage model, lower saddle point: Optimal control spendings as a function of time.

4.2.5 Higher Saddle Point

In order to analyse the impacts of an initiation shock in a model where only the higher saddle point serves as an optimal long-term solution, more than one parameter has to be changed. In Chapter 3 it was shown that for the base model this scenario only occurs for $\kappa < -15$, a case which is not feasible here. Generally speaking, the parameters have to be modified in a way such that the transcritical value of κ is positive and then take a value that is smaller than that point but still positive. One possible parameter setting is displayed in Table 4.2.

Figure 4.14 shows the relevant part of the phase diagram for this scenario. Again, we start at a lower level of λ than in a situation without a shock. That means that due to higher initiation an additional user causes much more costs than in a scenario of low initiation. Compared with the second scenario where the shock started at a lower costate this may be surprising. The question arises, however, whether those scenarios can be compared as a

Parameter	baseline value	new value
p	0.12454	1.2454
r	0.04	0.1
transcritical point	-15.021	4.58046
κ	4.185	2

Table 4.2: Parameters for a scenario with the high saddle point as unique optimal steady state solution.

different initial value is chosen and here no qualitative change of the policy occurs.

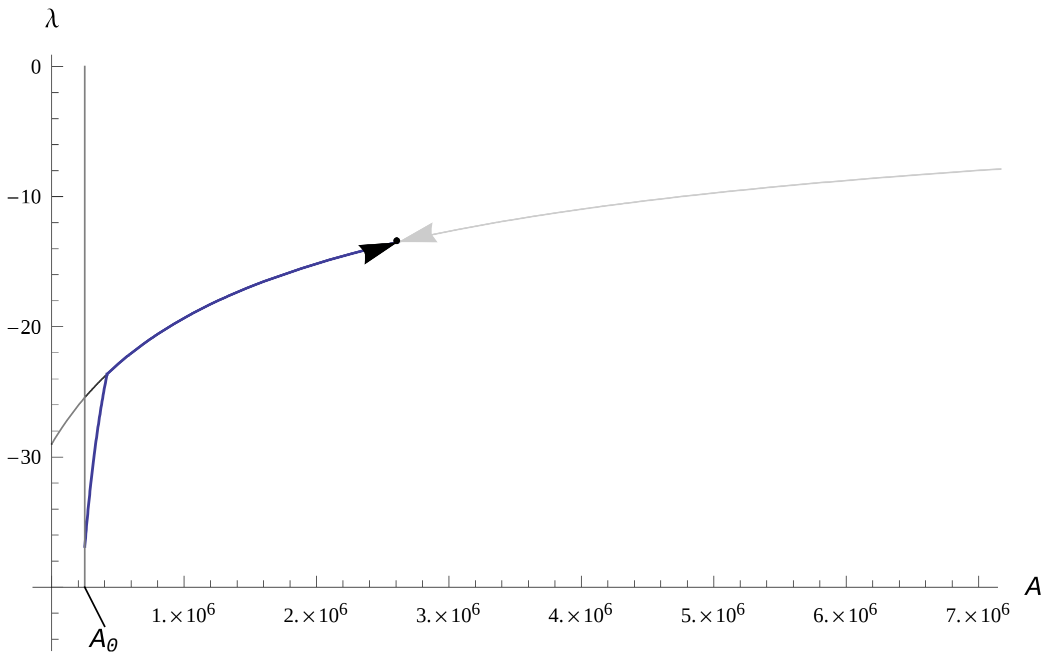

Figure 4.14: Two-stage model, high saddle point: Phase diagram. The optimal path is highlighted in blue, the case without shock in gray.

The number of users show a steep increase during the shock (see Figure 4.15) and afterwards converge to the high equilibrium. The high initiation stage accelerates the number of users reaching the long-term solution but doesn't change the qualitative behaviour. However, control spending levels

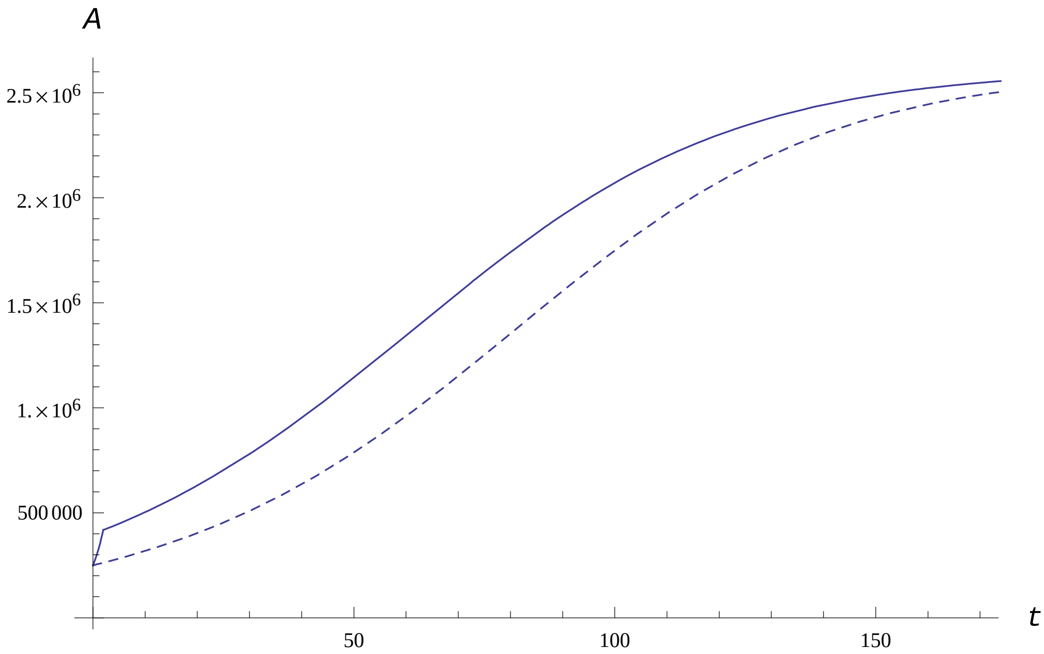

Figure 4.15: Two-stage model, high saddle point: Number of users as a function of time.

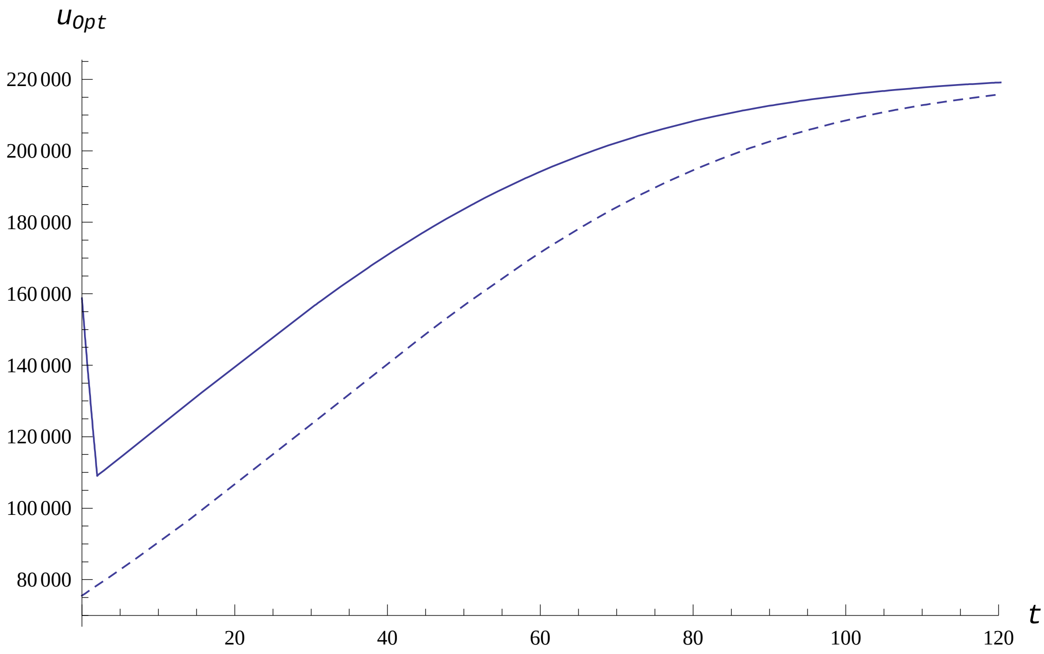

Figure 4.16: Two-stage model, high saddle point: Optimal control spending as a function of time.

are reacting in a different way than in the scenarios before (see Figure 4.16). Although the number of users are rapidly increasing optimal control spendings are decreasing at the same time, however starting at a much higher level than in the case without shock. A possible explanation may be obtained by looking at the cost-initiation relation. Due to the high initiation and the additional costs it would not be cost effective to spend even more on the control knowing that it will be necessary after the shock to spend more on the control.

4.3 Comparison and implications for optimal policy

Table 4.3 summarizes the effects of the high initiation stage on the costs and establishes a comparison with the base model. One important result is that irrespective of the qualitative impact on the optimal policy stages of high initiation cause a dramatic rise of total costs. In the analysed scenarios those increases range from 29% to more than 80%. So, generally speaking, initiation shocks with increasing rates of initiation are definitely bad for the policy maker.

T	A_0	κ	$-J_b^*$	$-J_s^*$	J_s^*/J_b^*
2	250000	4.185	1.604193×10^8	2.89067×10^8	1.80195
2	5×10^6	4.185	2.526012×10^9	3.275315×10^9	1.29663
5	5×10^6	4.185	2.526012×10^9	3.404600×10^9	1.34782
2	250000	7	1.66478×10^8	2.967590×10^8	1.78257
2	250000	15	1.818282×10^8	3.157118×10^8	1.73632
2	250000	2	6.776269×10^6	1.021725×10^7	1.5078

Table 4.3: Comparison of total costs for different scenarios.

Moreover, due to the discussion of different scenarios some further implications for optimal policies may be established.

Comparing the scenarios where it is optimal to converge to the lower saddle point, the same behaviour of the control can be observed. It always starts at a much higher level than in the base case, decreases sharply during the shock, flattens at the end of the shock stage and finally converges slowly towards zero. These scenarios only differ in the magnitude of the control spendings which correlates to the social cost parameter, respectively.

If nothing is spent on the control the users will certainly tend to the higher saddle point. High initiation implies that this convergence accelerates. However, this behaviour is not always optimal in a cost-efficient way. Therefore, in order to force the users converging to the lower equlibrium the usage of the control is necessary. In a situation with accelerated initiation even more spending is required. However, for the policy maker the additional spending might be a bitter pill to swallow as costs not only for the control itself but also for consumption tend to explode whenever initiation is high.

A different picture is drawn when observing the impacts on the number of users. A comparison of Figures 4.2, 4.9 and 4.12 reveals that the local maximum at T is most prominent in the one scenario where the social cost parameter κ has its lowest value. This indicates that the impact of the shock on the population of users is weaker when social costs are already high.

The same conclusion can be made when comparing the absolute increase of total costs due to the shock in Table 4.3. As κ is increasing, additional total costs due to the shock are slightly decrasing from 80% to 73%.

This might be a surprising result. However, large amounts of money are already required in the case without a shock due to the high social costs (see column $-J_b^*$ in Table 4.3). Therefore, impacts of additional costs due to the shock might not be as severe compared to the social costs.

If the high saddle point is the optimal long-term solution, the additional costs due to the shock don't seem so bad compared with other scenarios (51% compared to an increase of 80% in the latter case). This has several reasons. First, the high equilibrium only serves as an optimal solution when social costs are at a rather low level and a high long-term level of users is bearable. Second, the initiation shock strongly increases the number of users that cause additional costs. However, less spending on the control is possible as the shock may support the convergence towards the high saddle point.

Chapter 5

Sensitivity Analysis

In the last chapter it was shown that changes of the optimal dynamic policy are in part driven by the way the DNSS point reacts to disturbances or changes of the system. Depending on the initial value it may become optimal to move towards different steady states in situations with and without a shock. The DNSS thresholds are indicators for which initial values such a change in policy occurs. Therefore, part of this thesis analyses in what way the DNSS point is sensitive to changes of the intensity and/or the duration of the initiation shock. At first only one parameter is changed at a time, then a two-dimensional sensitivity analysis is carried out. Note that this sensitivity analysis is done for shocks concerning the baseline parameter settings (cf. scenarios 4.2.1 and 4.2.2).

5.1 Variation of Shock Intensity k

As in Chapter 4.1 an initiation shock of constant duration $T = 2$ is assumed. The initiation constant k is scaled from 1 to 6. As mentioned before, only shocks with a higher initiation rate in the first stage are discused here.

Figure 5.1 displays the DNSS threshold in case of a shock with duration $T = 2$ while the intensity of the shock is varied (abscissa). A monotonically decreasing behaviour of the DNSS threshold with respect to the initiation constant k can be observed. This result, of course, is not surprising at all. A higher initiation constant in the first stage causes an intensified increase of the number of users A (see Chapter 4) and/or a higher value of the costate λ during the shock (cf. Subsection 4.2.3). This results in increased costs during the shock period which causes higher total costs. Further, increased spending on the control during the shock could be observed in the previous chapter. As a consequence, the DNSS threshold is decreasing and for fewer initial values A_0 the optimal policy consists of eradicating the problem.

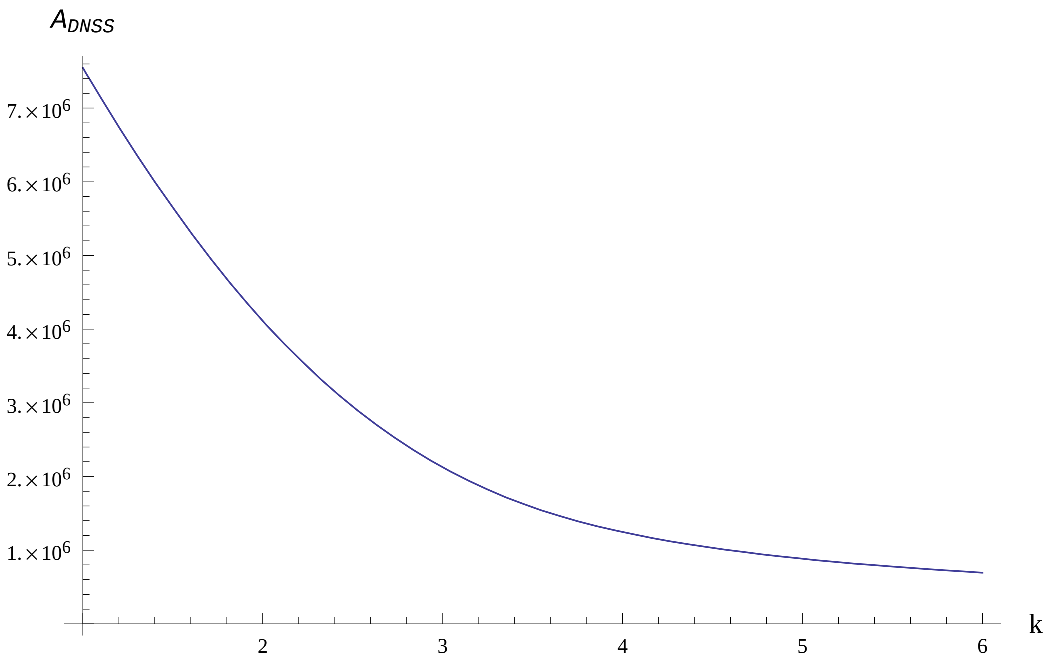

Figure 5.1: Variation of k: A_{DNSS} as a function of the scale of the initiation constant during the shock with duration $T = 2$.

5.2 Variation of T

Here, we assume an intensity of the shock of $2k$ as in the previous chapter while varying the duration T of the first stage from 0 to 10.

Figure 5.2 shows the DNSS threshold as a function of the duration of the shock T. Note that this time the shift of the DNSS point is nonmonotonic. A minimum of 2.35147×10^6 can be observed at $T = 7.23019$. As pointed out in Bultmann [2007] this may lead to fundamental changes in the optimal policy depending on the duration of the shock. Depending on the initial value it may be optimal for brief shocks to eradicte the problem, for longer shocks it may become optimal to converge to the high equilibrium and for even longer shocks the optimal policy consists of eradiction again.

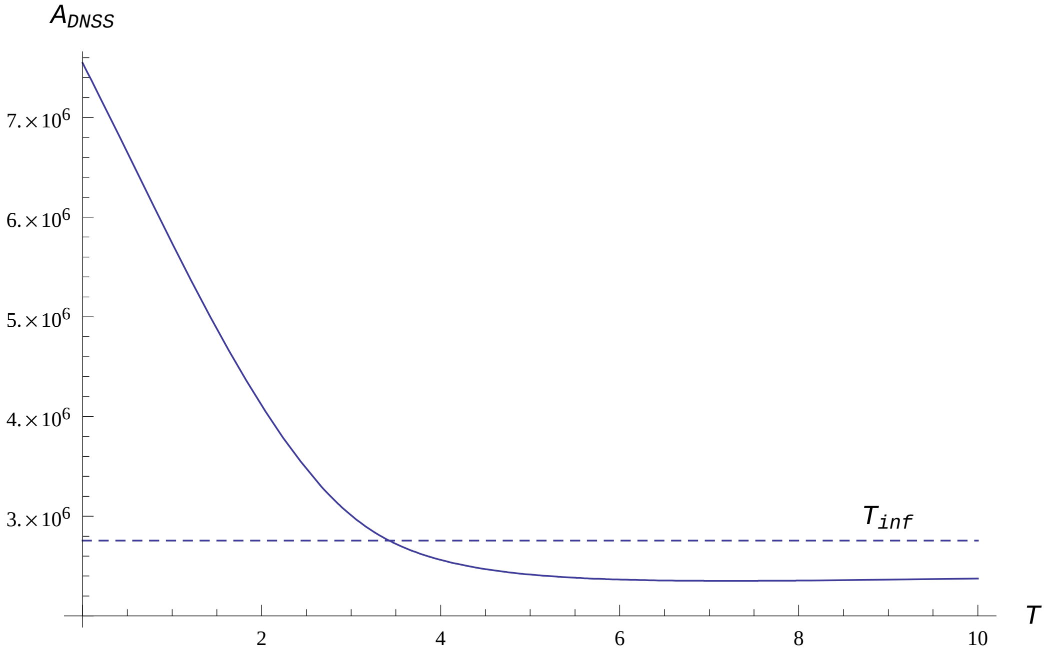

Figure 5.2: Variation of T: A_{DNSS} as a function of T for intensity $2k$ with a minimum at $T = 7.23019$. T_{inf} indicates the limit of A_{DNSS} for $T \to \infty$.

Further, a statement about the long-term behaviour of the DNSS threshold may be made. Let us look at the utility functional of the first stage (see eq. (4.1)) and let $T \to \infty$. The upper boundary of the integral changes to ∞ whereas the discounted salvage function converges to 0. As a result, the utility functional of a one-stage model is obtained, i.e. the base model. Hence, if the limit of $A_{DNSS}(T)$ for $T \to \infty$ is required, the base model with a higher initiation constant has to be considered. For $2k$ a DNSS threshold of $A_{DNSS} = 2.754878 \times 10^6$ is obtained. In Figure 5.2 this limit is displayed by the horizontal dashed line T_{inf}.

5.3 Two-Dimensional Sensitivity Analysis – Joint Variations of k and T

Having made observations about the influence of either k or T on the DNSS threshold, a comparison of their impacts is crucial. As a consequence, careful statements can be made about whether it is better to have a longer and less intense shock or to have a shorter and more intense one. Here, better refers to the shift of the DNSS point. The larger the shift, the greater the effect of the shock, the higher the additional costs, and thus, the worse.

Figure 5.3 shows contour lines for DNSS iso-levels, i.e. combinations of intensities k_s and durations T that result in the same DNSS levels. Therefore, they represent shocks that are equivalent in terms of qualitative changes of the optimal policy. The abscissa refers to the scale of the initiation constant k and the ordinate refers to the duration of the shock T. The axes themselves represent the base model as either the duration of the shock is zero or the intensity of the shock equals that of the second stage.

The contour lines displayed here are referring to the levels of the DNSS threshold in millions of users. The lowest curve is representing the DNSS thresholds of 7×10^6, the next upper one is 6×10^6 and so on. The axes themselves, i.e. $k_s = k$ and $T = 0$, represent the base model with a DNSS point at 7.5469×10^6. Narrow contours imply that impacts of changes of the parameters are strong because a very small change may cause a large shift. Broad contours indicate that small changes of the parameters are not likely to change a lot concerning the DNSS threshold.

The contours themselves are convex curves, which implies that that the intensity and the duration of the shock are complements. Therefore, they explain which combinations of shocks are equivalents with respect to the

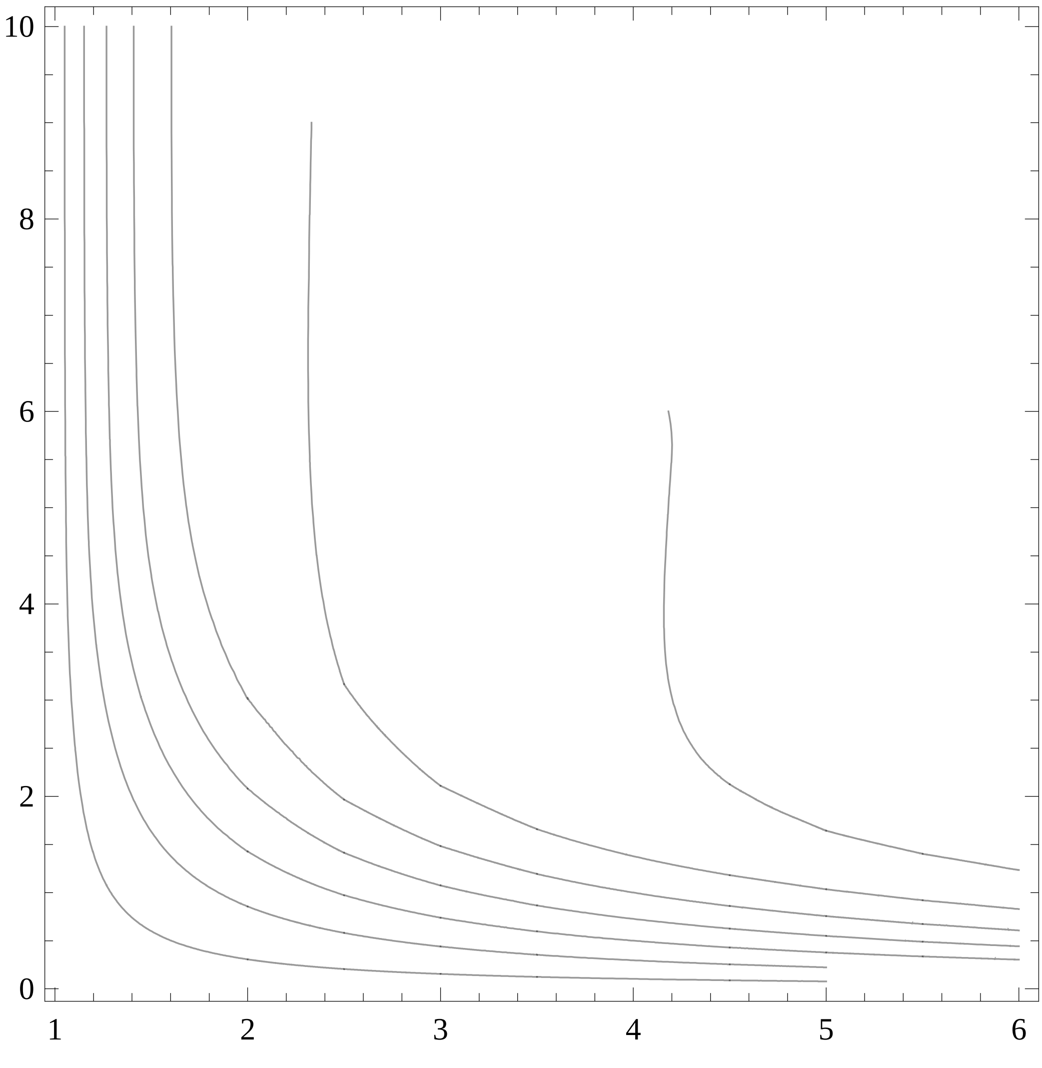

Figure 5.3: Contour plot showing changes of A_{DNSS} with respect to the intensity (abscissa) and the duration (ordinate) of the shock.

impact on the DNSS threshold.

Table 5.1 displays some values taken from the contour with level 3×10^6. For example, a shock of intensity 2 and duration 3 causes the same shift of the DNSS threshold as a shock of intensity 6 and duration 0.6.

intensity $\times k$	1.78847	2	2.47848	4	5.03075	6
duration T	4	3.0107	2	0.999925	0.75	0.607765

Table 5.1: Some combinations of intensity and duration of initiation shocks that result in a DNSS threshold at 3×10^6.

With that in mind, one may try to compare shocks and make statements about what kind of shock is better.

If the shock analysed in chapter 4 is taken, i.e. with intensity 2 and duration 2, would it be preferable to have a shock of half the duration and twice the intensity or half the intensity and twice the duration? Table 5.2 shows the DNSS thresholds for four different scenarios. A shock of intensity 2 and duration 2 causes a shift of the DNSS point from 7.546903×10^6 to 4.11104×10^6. A shock of twice the intensity, 4, and half the duration, 1, causes a shift to 2.99976×10^6. In the latter case the change of the DNSS threshold was more intense. Thus, a policy maker would prefer a longer but less intense shock. This developement continues if the shock becomes even more intense and shorter. For a shock of intensity 6 and a duration of 0.66667 the DNSS threshold is found at 2.69736×10^6. However, note that the shift for the last shock is not that much stronger in comparison to the second scenario. A declining deterioration may be noticed. When a shock becomes more intense and shorter the decreasing of the DNSS threshold is getting weaker.

Analogously, better results of the shock are obtained with a weaker intensity and a longer duration. The scale of the intensity and the duration of the shock were changed by 50% to a scale of 1.5 and a duration of 3, respectively. Such a shock results in a shift of the DNSS threshold to 4.77515×10^6, which is better than a shock of duration and intensity 2.

intensity $\times k$	duration T	DNSS
2	2	4.11104×10^6
4	1	2.99976×10^6
6	0.66667	2.69736×10^6
1.5	3	4.77515×10^6

Table 5.2: Comparison of the effects of different shock scenarios.

Note that qualitatively the same results are obtained for most analogous examples. A longer but less intense shock seems to be more favorable than a shorter but more intense one. That may also be seen in Figure 5.3. For high values of k the contours are slightly more narrow than for high values of T, which implies a dominating influence of the intensity of the shock. However, when focusing on the duration of the shock (see Figure 5.2) a nonmonotonic behaviour of the DNSS threshold as a function of T was noticed. Therefore, there are areas of the (k, T)-plane, where the intensity and the duration of the shock are not complements anymore. This may lead to scenarios where a shorter more intense shock might be favorable.

The results of this chapter may also used for direct policy guidance. If interpreted as initial values, the contour levels provide information about qualitative changes of the policy. For shock combinations below this level curve (including the base model) convergence is to the low-level saddle point and thus, no disruptive change in the optimal policy occurs. Shocks above the curve suggest a convergence to the high-level equilibrium instead and therefore a fundamental change in optimal policy.

Chapter 6

Summary and Conclusions

The subject of this thesis was to discuss the influence and role of initiation in a previously known one-state one-control model on optimal dynamic drug policy.

First, the base model per se was discussed and first implications for the optimal policy were stated. Then a closer look was taken on how the properties of the equilibria of the system change when either the social costs or the initiation parameter are modified. As a first result it was shown that the range with multiple equilibria increases quite strongly with growing initiation. However, the area with multiple optimal equlibria and DNSS points only slightly increases. This is good news for the policy maker as the system is rather insensitive to changes in qualitative matters when social costs are high. Attention must be paid, however, when social costs are low and multiple optimal long-term solutions exist. A slight change of one paramter might result in a fundamental change of the policy.

After that, a two-stage model was considered, for which initiation shocks were discussed for different equilibria and parameter settings. We learned that stages of high initiation are bad for the policy maker as they always result in a notable increase of costs. Intuitively, this result is what was expected, because an increase in users should lead to additional costs. Further, it was shown that shocks may result in a qualitative change of the optimal dynamic policy.

Finally, the impacts of the intensity and the duration of the shock on optimal dynamic policy were discussed. The indicator that was used here is the threshold where the optimal policy switches from one long-term solution to the alternative one, i.e. the DNSS point. Thus, a policy maker may conclude whether it is still optimal to follow the same policy as before the shock or whether it is time to change. Moreover, a device for comparing shocks is given. If a policy maker is facing a choice of realising shocks of different intensities and durations, strategic support is provided by the insights made here. For moderate intensities and durations it is preferable to have a longer but less intense shock.

Comparing this thesis with other works (e.g. Bultmann [2007]) some additional assumptions may be made. When analysing price shocks, a symmetric behaviour of droughts and gluts was observed. The former supports optimal dynamic policy whereas the latter rather interferes. This thesis focused on high initiation stages, because there is evidence that this situation has already occurred (see, e.g., Johnson et al. [1996]) and those scenarios have been declared as unfortunate for optimal policy. Reasoning by analogy would therefore mean that stages with low initiation are advantageous for the policy maker. Thus, further analysis in this direction would certainly be interesting.

The reader should also bear in mind that this mathematical model is highly abstract. Nevertheless, it provides some remarkable insights about the impacts of high initiation at the beginning of a drug epidemic and the related optimal dynamic policy. Thus, extensions of this model and related issues should be considered in future work.

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