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Monte Carlo Valuation of American Options

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durch

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Chapter 1

Introduction

My diploma thesis which I wrote under the supervision of Prof. Teichmann in the summer term 2007/2008 covers the pricing of American options by simulation. This diploma thesis is based on a paper from L.C.G. Rogers (2002). In his paper he investigates a new approach using Monte Carlo techniques. He makes no attempt to determine an approximate exercise policy, but instead gives an upper bound for the true price. The payoff of an American option depends in a highly complex path-dependent fashion on the underlying, which means that the computing of the value and the optimal exercise time is very difficult.

The overwhelming majority of traded options are of American style. In general it is not possible to find explicit formulae for American option prices, and numerical techniques or approximation schemes are required for option evaluation. For pricing European style derivatives simulation has been used extensively, but for American style claims there have been only a few attempts to use simulation techniques for pricing. The problem lies in the estimation of the exercise boundary; the Monte Carlo method entails the simulation of the evolution of the asset prices forward in time, but the determination of the optimal exercise policy requires a backward style algorithm. Monte Carlo simulation is the most popular approach in computational finance for determining the price of financial options. The accurate calculation of prices is only one objective of Monte Carlo simulation.

To discuss this pricing process in detail, the diploma thesis is structured as followed: In a first step the essential mathematic definitions and basis are explained. For this reason the second and the third chapter are dedicated to an introduction to the basicly financial model. In the second chapter the discrete time model is presented. Basic contracts like European options are introduced, also the No Arbitrage Theory is discussed. The main point of the third chapter is the step from discrete time to continuous time. In addition to it this part includes hedging methods for European options. All important stochastic preliminaries like stochastic processes, filtrations, stopping times, adapted processes and martingales are summarized in the fourth chapter.

After an introduction to the Monte Carlo method in chapter five, the American options are discussed. It starts with the main properties of American options, followed by the Snell envelope, a backward recursion for pricing American options, and optimal stopping times.

The main chapter of this diploma thesis is chapter seven. After explanations about price and a hedging method for American options the way of constructing a good martingale for the Monte Carlo simulation is discussed.

In chapter eight the Monte Carlo method for pricing an American option is demonstrated on a numerical example. For the implementation Scilab was used (the source code is quoted in the appendix), followed by some concluding words.

Chapter 2

Discrete Time Models

2.1 First examples of pricing

In this section the main concept is introduced with a simple example. The startpoint are *one period models*, i.e. models for asset prices such that $S_0 > 0$ is the price at $t = 0$ and S is the price at some later point in time denoted by 1. Furthermore we assume a bank account (B_0, B_1) and for simplicity we take $B_0 = B_1 = 1$ (no interest rates) in this chapter. We are allowed to

- borrow and invest money in the bank account at time $t = 0$ in arbitrary portions,
- sell and buy stocks at time $t = 0$ in arbitrary portions.

A portfolio is an amount of money invested in the bank account and an amount of money invested in stocks at $t = 0$ and wait for $t = 1$ to see the result after on "tick in time". Formally this means that a *portfolio* is a vector $(\psi, \phi) \in \mathbb{R}^2$ of real numbers, where ψ denotes the amount of money in the bank account and ϕ denotes the number of shares in the stock. The value of portfolio V_0 at time $t = 0$ is given through

$$V_0 = \psi + \phi S_0,$$

and at time $t = 1$ through

$$V_1 = \psi + \phi S_1,$$

which is already a random variable.

Criterion 2.1.1 (No Arbitrage Criterion). *We call a one-period model arbitrage-free if there does not exist an (admissible) portfolio (ψ, ϕ) with initial value V_0 and non-zero final value $V_1 \stackrel{>}{\neq} 0$. If we can construct such a strategy in a one-period model, we call this strategy an arbitrage (strategy).*

The requirement of a model to be arbitrage-free is minimal for models of financial markets.

The 1-period model, i.e. an asset has the value x_0 at time 0 and x_1 with probability p_1 and x_2 with probability p_2 at time 1 ($p_1 + p_2 = 1$; $p_1 p_2 > 0$; $x_1 > x_0 > x_2$). A sample space

$\Omega := \{\omega_1, \omega_2\}$ with σ -algebra 2^Ω and probability measure $P(\omega_1) = p_1$ and $P(\omega_2) = p_2$. Furthermore a stochastic process $S_0(\omega_1 = x_0)$, $S_0(\omega_2 = x_0)$ and $S_1(\omega_1 = x_1)$, $S_1(\omega_2 = x_2)$. This represents the price of a traded asset in a stock market.

Remark 2.1.1. *The above described model is arbitrage-free since the outcomes of strategies with $V_0 = 0$ are given through*

$$V_1 = \phi(S_1 - S_0),$$

which is a positive random variable if and only if $\phi = 0$. Then in turn $\psi = 0$ and $V_1 = 0$

Let (S_0, S_1) be a general one-period model, i.e. S_0 is constant and $S_1 : (\Omega, \mathcal{F}, P) \rightarrow \mathbb{R}$ is a random variable on a finite probability space. We assume zero interest rates, i.e. a bank account process $B_0 = B_1 = 1$. Derivatives (Claims, Contingent Claims) \mathbf{X} on the asset are measurable functions $\mathbf{X} : \Omega \rightarrow \mathbb{R}$ which are the outcomes of contracts at time $t = 1$. These derivatives (claims) should as equally traded objects have a price at $t = 0$, i.e. buying the derivative at the fair price at $t = 0$ gives the buyer the outcome \mathbf{X} at $t = 1$.

Criterion 2.1.2 (No Arbitrage Pricing Rule). *Given a one-period model and a derivative (claim) \mathbf{X} at $t = 1$, an arbitrage-free price in the model (S_0, S_1) at $t = 0$ is a number π such that a portfolio consisting of an investment in the bank account, an investment in the stock and an investment in the derivative \mathbf{X} does not yield a value, which is 0 at $t = 0$ and positive at $t = 1$. Formally speaking there is no $(\psi, \phi, \eta) \in \mathbb{R}^3$ such that*

$$V_0 = \psi + \phi S_0 + \eta \pi = 0$$

and $V_1 = \psi + \phi S_1 + \eta \mathbf{X} \geq 0$ with $V_1 \neq 0$. This principle should hold for any finite number of derivatives.

Fair prices of derivatives need not exist and if they exist they need not be unique.

- given a derivative \mathbf{X} and a fair price π for \mathbf{X} , for any portfolio (ψ, ϕ) in bank account and asset we have that $V_1 \geq \mathbf{X} \Rightarrow V_1 \geq \pi$.
- prices for derivatives are additive, the no arbitrage pricing rule leads to an additive pricing system $\mathbf{X} \mapsto \pi(\mathbf{X})$ for derivatives \mathbf{X} with prices $\pi(\mathbf{X})$.
- if there are strategies ψ (units in bank account) and ϕ (units in stock) at time 0, such that at time 1 we obtain $\psi B_1 + \phi S_1 = V_1 = \mathbf{X}$ almost surely, then $\psi B_0 + \phi S_0 = V_0$ is the only fair price. There is a unique arbitrage-free price for all replicable portfolios.
- in particular the derivative S_1 is replicable (the forward contract), so all possible arbitrage-free prices $\pi(S_1)$ have to agree and

$$q(S_1) = V_0 = S_0$$

- the derivative $\mathbf{X} = 1_\Omega$ is also replicable (with the bond) and therefore $\pi(1_\Omega) = 1$.
- given a positive derivative $\mathbf{X} \geq 0$, then the price has to be positive, which is seen by taking the strategy $(0,0)$ for the derivative $-\mathbf{X}$ which yields that $0 \geq -\pi(\mathbf{X})$.

- given a linear map $\mathbf{X} \mapsto \pi(\mathbf{X})$ such that $V_1 \geq \mathbf{X} \Rightarrow V_1 \geq \pi$ holds, then it is an arbitrage-free pricing rule. Since then $\pi(S_1) = S_0$ and $q(1) = 1$. For any portfolio $(\psi, \phi_1, \dots, \phi_k)$ in derivatives S_1, \dots, X_{k-1} the relation $V_0 = 0$ leads to

$$\phi_1(S_1 - S_0) \geq 0 \quad \phi_2(\mathbf{X}_1 - \pi(\mathbf{X}_1)) \geq 0, \dots,$$

which in turn leads to $(\psi, \phi_1, \dots, \phi_k) = 0$.

- $\Omega = \{\omega_1, \dots, \omega_M\}$. Given a pricing rule $\mathbf{X} \mapsto \pi(\mathbf{X})$, one can calculate the Arrow-Debreu prices, i.e. the prices of $1_{\{\omega_i\}}$, which gives 1 if ω_i appears and 0 else. These prices are denoted by $q_i = \pi(1_{\{\omega_i\}})$ and every other price of a claim $\mathbf{X} = \sum_{i=1}^M \mathbf{X}(\omega_i) 1_{\omega_i}$ can be written as

$$\pi(\mathbf{X}) = \sum_{i=1}^M q_i \mathbf{X}(\omega_i)$$

by linearity. In particular $\sum_{i=1}^M q_i = 1$ and a measure on Ω can be defined.

In the first example, where we have two states of the world, every derivative is uniquely replicable and therefore a unique arbitrage-free pricing rule for derivatives exists. Given a derivative \mathbf{X} , this yields the system of linear equations

$$\begin{aligned} \psi + \phi x_1 &= \mathbf{X}(\omega_1) = c_1 \\ \psi + \phi x_2 &= \mathbf{X}(\omega_2) = c_2. \end{aligned}$$

which has a unique solution

$$\begin{aligned} \psi &= \frac{x_2 c_1 - x_1 c_2}{x_2 - x_1} \\ \phi &= \frac{c_1 - c_2}{x_1 - x_2}. \end{aligned}$$

This means there is "always" a fair price, namely

$$\begin{aligned} V_0 &= \frac{x_2 c_1 - x_1 c_2}{x_2 - x_1} + \frac{c_1 - c_2}{x_1 - x_2} x_0 \\ &= c_1 \left(\frac{x_2}{x_2 - x_1} - \frac{x_0}{x_2 - x_1} \right) + c_2 \left(-\frac{x_1}{x_2 - x_1} - \frac{x_0}{x_2 - x_1} \right) \\ &= \mathbb{E}_Q(\mathbf{X}). \end{aligned}$$

\mathbb{E}_Q denotes the expectation with respect to a new measure Q and is the unique pricing rules for derivatives \mathbf{X} .

$$\begin{aligned} Q(\omega_1) &= \frac{x_0 - x_2}{x_1 - x_2}, \\ Q(\omega_2) &= \frac{x_1 - x_0}{x_1 - x_2}. \end{aligned}$$

This measure is equivalent under our conditions and the most important property is

$$\begin{aligned} \mathbb{E}_Q(S_1) &= x_1 \frac{x_0 - x_2}{x_1 - x_2} + x_2 \frac{x_1 - x_0}{x_1 - x_2} \\ &= x_0 = S_0 \end{aligned}$$

so $(S_t)_{0 \leq t \leq 1}$ is a Q -martingale. Furthermore we obtain the equation

$$V_0 + \phi(S_1 - S_0) = \mathbf{X}.$$

To determine unique pricing rule we first calculate q_1, q_2 such that $q_1 + q_2 = 1$ and the associated probability measure on Ω which satisfies

$$\mathbb{E}_Q(S_1) = S_0$$

This means

$$q_1 x_1 + q_2 x_2 = x_0.$$

Here we have a unique solution

$$\begin{aligned} Q_{x_1, x_2}(\omega_1) &= \frac{x_0 - x_2}{x_1 - x_2}, \\ Q_{x_1, x_2}(\omega_2) &= \frac{x_1 - x_0}{x_1 - x_2}. \end{aligned}$$

With the measure Q we have found the unique arbitrage free pricing rule, i.e.

$$\pi(\mathbf{X}) = \mathbb{E}_Q(\mathbf{X}).$$

The case, where not all claims can be replicated (linear equations are not always solvable!): We take a one-period model (S_0, S_1) where Ω has three elements and P is some probability assigning positive values to all three states of the world. We assume $S_1(\omega_1) = x_1, S_1(\omega_2) = x_2, S_1(\omega_3) = x_3, S_0 = x_0$ with the relations

$$x_1 > x_2 > x_0 > x_3.$$

The model is arbitrage-free, since for a trading strategy $(\psi, \phi) \in \mathbb{R}^2$ with

$$\psi + \phi S_0 = 0$$

we obtain

$$\phi(S_1 - S_0) \geq 0$$

if and only if $\phi = 0$, which means that the portfolio vanishes.

We have three linear equations with two variables (the portfolio), we cannot hope for exact replication. We can calculate all no-arbitrage pricing rules. First we calculate all $q_1, q_2, q_3 > 0$ with $q_1 + q_2 + q_3 = 1$ such that the associated probability on Ω , which is denoted by Q satisfies

$$\mathbb{E}_Q(S_1) = S_0$$

So we have to solve the equation (*)

$$\begin{aligned} q_1 x_1 + q_2 x_2 + q_3 x_3 &= x_0, \\ q_1 + q_2 + q_3 &= 1, \\ q_1, q_2, q_3 &> 0 \end{aligned}$$

which can be solved by the previous knowledge. We define the measure

$$\begin{aligned} Q_{x_1, x_3}(\omega_1) &= \frac{x_0 - x_3}{x_1 - x_3}, \\ Q_{x_1, x_3}(\omega_2) &= 0, \\ Q_{x_1, x_3}(\omega_3) &= \frac{x_1 - x_0}{x_1 - x_3}. \end{aligned}$$

and the measure

$$\begin{aligned} Q_{x_2, x_3}(\omega_2) &= \frac{x_0 - x_3}{x_2 - x_3}, \\ Q_{x_2, x_3}(\omega_1) &= 0, \\ Q_{x_2, x_3}(\omega_3) &= \frac{x_2 - x_0}{x_2 - x_3}. \end{aligned}$$

Any convex combination of these two measures is a solution (with condition $q_i \geq 0$). Furthermore any solution is a convex combination of these two measures. The solutions of (*) can be written

$$Q = tQ_{x_1, x_3} + (1 - t)Q_{x_2, x_3}$$

for $t \in]0, 1[$. Then we know that for any derivative \mathbf{X} the map

$$\mathbf{X} \mapsto \mathbb{E}_Q(\mathbf{X})$$

is an arbitrage-free pricing rule and these are all arbitrage-free pricing rules.

We can also do the whole calculation with interest rates without any problems. The only difference is discounting. A one-period model (S_0, S_1) with bank account (B_0, B_1) , where $B_0 = 1$. Then the definitions of arbitrage-free models and arbitrage-free pricing rules remain unchanged. A one-period model with interest rates is arbitrage-free if the model $(S_0, \frac{S_1}{B_1})$ is arbitrage-free with zero interest rates. This is called the discounted model. For pricing rules all relations hold true. In particular arbitrage-free pricing rules are given by probability measure Q such that

$$\mathbb{E}_Q\left(\frac{S_1}{B_1}\right) = S_0$$

and an arbitrage-free price of a derivative \mathbf{X} is given through the price of the discounted derivative

$$\mathbb{E}_Q\left(\frac{S_1}{B_1}\right).$$

The steps to calculate all arbitrage-free pricing rules are the same like above.

2.2 Basic Contracts

$(S_t)_{t \in I}$ is the asset price on some interval I . $B_t = \exp(rt)$ is a risk-free bank account on I , which means continuous compounding. We assume the basic no-arbitrage pricing rule for the one-period model (S_t, S_T) for $t \leq T$ in I .

2.2.1 Forward contracts

A *forward contract* is the right and the obligation to buy one unit of the stock $(S_t)_{t \geq 0}$ at time $T > 0$ for an amount K , which is fixed at $t \geq 0$. We have the linear payoff-scenario

$$(S_T - K).$$

We denote the price of a forward contract of this type by F_t . We shall calculate the strike price K such that the contract can be entered today at $t = 0$ with zero premium $F_0 = 0$ and obtain

$$K = \exp(r(T - t))S_t.$$

If somebody entered the contract with $F_0 = 0$ and $K = \exp(r(T - t))S_t$, we would buy one unit of stock for S_t , which we have to borrow from the bank. Therefore at T we have debts $S_t \exp(r(T - t))$, but receive K in exchange for the stock. Hence a net gain of $K - \exp(r(T - t))S_t$. If we write a forward contract with $K < \exp(r(T - t))S_t$ with some other person, then we sell a unit of stock at t and receive S_t , which is put on the bank account. At T we receive a unit of stock in exchange for $K < \exp(r(T - t))S_t$. We clear the short amount of stock and have a net gain of $\exp(r(T - t))S_t - K$. Therefore the price of a forward with strike price K and maturity T is given at t through

$$F_t = S_t - \exp(-r(T - t))K.$$

2.2.2 European Call contracts (European Call Option)

A *European call* is the right but not the obligation to buy one unit of stock at time $T > 0$ for an amount K , which is fixed at t . We have the (non-linear) payoff-scenario

$$(S_T - K)_+$$

at time $t = T$.

2.2.3 European Put contracts (European Put Option)

A *European put* is the right but not the obligation to sell one unit of stock at time $T > 0$ for an amount K , which is fixed at t . We have the (non-linear) payoff-scenario

$$(S_T - K)_-$$

at time $t = T$. We denote the put price by P_t . We obtain the put-call parity by observing that

$$C_t - P_t = S_t - K \exp(-r(T - t))$$

has to be the price of the forward contract with strike price K .

2.3 No Arbitrage Theory for discrete models

A *discrete model for a financial market* is an adapted $(d+1)$ -dimensional stochastic process S with $S_n := (S_n^0, \dots, S_n^d)$ for $n = 0, \dots, N$ on a finite probability space (Ω, \mathcal{F}, P) with filtration $\mathcal{F}_0 \subset \dots \subset \mathcal{F}_N = \mathcal{F}$. We shall always assume that all σ -algebras contain all P -nullsets. The price process $(S_n^0)_{n=0, \dots, N}$ is assumed to be strictly positive and called the *riskless asset* (even if it is stochastic) and we define $S_0^0 = 1$. The coefficients $\beta_n := \frac{1}{S_n^0}$ for $n = 0, \dots, N$ are called *discount factors*. The assets S^1, \dots, S^d are called *risky assets*. A *trading strategy* is a predictable stochastic process ϕ with $\phi_n = (\phi_n^0, \dots, \phi_n^d)$ for $n = 0, \dots, N$. We think of a portfolio formed by an amount of ϕ_n^0 in the bank account and ϕ_n^i units of risky assets, at time n . The value or wealth at time n of such a portfolio is

$$V_n(\phi) = \phi_n S_n := \sum_{i=0}^d \phi_n^i S_n^i$$

for $n = 0, \dots, N$. The *discounted value process* is given through

$$\tilde{V}_n(\phi) = \beta_n(\phi_n S_n) = \phi_n \tilde{S}_n$$

for $n = 0, \dots, N$, where $\tilde{S}_n = \beta_n S_n$ denotes the *discounted price process*. A trading strategy ϕ is called *self-financing* if

$$\phi_n S_n = \phi_{n+1} S_n$$

for $n = 0, \dots, N-1$. We interpret this condition that the readjustment at time n to new prices S_n is done without bringing in or consuming any wealth. This condition is obviously equivalent to

$$\phi_{n+1}(S_{n+1} - S_n) = \phi_{n+1} S_{n+1} - \phi_n S_n$$

for $n = 0, \dots, N-1$, which means that the changes of the value process are due to changes in the stock prices.

Proposition 2.3.1. *Let $S = (S^0, \dots, S^d)$ be a discrete model of a financial market and ϕ a trading strategy, then the following assertions are equivalent:*

1. *The strategy ϕ is self-financing*
2. *For $n = 1, \dots, N$ we have*

$$V_n(\phi) = V_0(\phi) + (\phi \cdot S)_n.$$

3. *For $n = 1, \dots, N$ we have*

$$\tilde{V}_n(\phi) = V_0(\phi) + (\phi \cdot \tilde{S})_n.$$

A self-financing trading strategy ϕ can also be given through the initial value $V_0(\phi)$ and (ϕ^1, \dots, ϕ^d) , which is proved in the following proposition:

Proposition 2.3.2. *For any predictable process (ϕ^1, \dots, ϕ^d) and for any value V_0 there exists a unique predictable process ϕ^0 such that (ϕ^0, \dots, ϕ^d) is a self-financing trading strategy with $V_0(\phi) = V_0$ such that $\tilde{V}_n(\phi) = V_0 + (\phi \cdot \tilde{S})_n$ for $n = 0, \dots, N$.*

A trading strategy ϕ is called *admissible* if there is $C \geq 0$ such that $V_n(\phi) \geq -C$ for $n = 0, \dots, N$, which is not a restriction for discrete models.

Definition 2.3.1. Let $S = (S^0, \dots, S^d)$ be a discrete model for a financial market, then the model is called *arbitrage-free* if for any trading strategy ϕ the assertion

$$V_0(\phi) = 0 \text{ and } V_N(\phi) \geq 0, \text{ then } V_N(\phi) = 0$$

holds true. We call a trading strategy ϕ an *arbitrage opportunity* (arbitrage strategy) if $V_0(\phi) = 0$ and $V_N(\phi) \stackrel{\geq}{\neq} 0$.

Definition 2.3.2. A contingent claim (derivative) is an element of $L^2(\Omega, \mathcal{F}, P)$. We denote by $\tilde{\mathbf{X}}$ the discounted price at time N , i.e. $= \frac{1}{S_N^0} \mathbf{X}$. We call the subspace of $\mathcal{K} \subset L^2(\Omega, \mathcal{F}, P)$

$$\begin{aligned} \mathcal{K} &:= \left\{ \tilde{V}_N(\phi) \mid \phi \text{ self-financing trading strategy, } \tilde{V}_0(\phi) = 0 \right\} \\ &= \left\{ (\phi \cdot \tilde{S})_N \mid \phi \text{ predictable} \right\} \end{aligned}$$

the space of contingent claims attainable at price 0 (see Proposition 2.3.1). We call the convex cone

$$C = \{Y \in L^2(\Omega, \mathcal{F}, P) \mid \text{there is } \mathbf{X} \in \mathcal{K} \text{ such that } X \geq Y\} = \mathcal{K} - L_{\geq 0}^2(\Omega, \mathcal{F}, P)$$

the cone of claims super-replicable at price 0 or the outcomes with zero investment and consumption. A contingent claim \mathbf{X} is called *replicable at price x and at time N* if there is a self-financing trading strategy ϕ such that

$$\tilde{\mathbf{X}} = x + (\phi \cdot \tilde{S})_N \in x + \mathcal{K}.$$

A contingent claim \mathbf{X} is called *super-replicable at price x and at time N* if there is a self-financing trading strategy ϕ such that

$$\tilde{\mathbf{X}} \leq x + (\phi \cdot \tilde{S})_N \in x + \mathcal{K}.$$

in other words if $\tilde{\mathbf{X}} \in C$.

Remark 2.3.1. The set \mathcal{K} is a supspace of $L^2(\Omega, \mathcal{F}, P)$ and the positive cone $L_{\geq 0}^2(\Omega, \mathcal{F}, P)$ is polyhedral, therefore C is closed. We see immediately that a discrete model for a financial market is arbitrage-free if

$$\mathcal{K} \cap L_{\geq 0}^2(\Omega, \mathcal{F}, P) = \{0\},$$

which is equivalent to

$$C \cap L_{\geq 0}^2(\Omega, \mathcal{F}, P) = \{0\}.$$

Given a discrete model for a financial market, then we call a measure Q equivalent to P an *equivalent martingale measure* with respect to the numeraire S^0 if the discounted price process \tilde{S}^i are Q -martingales for $i = 0, \dots, N$. We denote the set of equivalent martingale measures with respect to the numeraire S^0 by $\mathcal{M}^e(S, S^0)$. If the numeraire satisfies $S^0 = 1$ we shall write $\mathcal{M}^e(S)$. In particular $\mathcal{M}^e(S, S^0) = \mathcal{M}^e(\tilde{S})$. We denote the absolutely continuous martingale measures with respect to the numeraire S^0 by $\mathcal{M}^a(S, S^0)$. If the numeraire satisfies $S^0 = 1$ we shall write $\mathcal{M}^a(S)$. In particular $\mathcal{M}^a(S, S^0) = \mathcal{M}^a(\tilde{S})$.

Theorem 2.3.1. *Let S be a discrete model for financial market, then the following two assertions are equivalent:*

1. *The model is arbitrage-free.*
2. *The set of equivalent martingale measures is non-empty, $\mathcal{M}^a(\tilde{S}) \neq \emptyset$.*

Definition 2.3.3. *A pricing rule for contingent claims $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$ at time N is a map*

$$\mathbf{X} \mapsto \pi(\mathbf{X})$$

where $\pi(\mathbf{X}) = (\pi(\mathbf{X})_n)_{n=0, \dots, N}$ is an adapted stochastic process, which determines the price of the claim at time N at time $n \leq N$. In particular $\pi(\mathbf{X})_N = \mathbf{X}$ for any $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$. A pricing rule is arbitrage-free if for any finite set of claims $\mathbf{X}_1, \dots, \mathbf{X}_k$ the discrete time model of a financial market

$$(S^0, S^1, \dots, S^d, \pi(\mathbf{X}_1)), \dots, \pi(\mathbf{X}_k))$$

is arbitrage-free.

The next Lemma is an argument for super-replication where trading is involved.

Lemma 2.3.1 (superreplication principle). *Let π be an arbitrage-free pricing rule, then π is linear, positive and for any self-financing trading strategy $\phi = (\phi^0, \dots, \phi^d)$ the assertion*

$$V_N(\phi) \geq \mathbf{X} \Rightarrow V_n(\phi) \geq \pi(\mathbf{X})_n$$

for $n = 0, \dots, N$ holds.

Corollar 2.3.1 (perfect replication). *Let \mathbf{X} be a replicable claim, i.e. there is a portfolio ϕ such that $V_N(\phi) = \mathbf{X}$, then for all arbitrage-free pricing rules q we have*

$$\pi(\mathbf{X})_n = V_n(\mathbf{X})$$

for $n = 0, \dots, N$.

Lemma 2.3.2 (arbitrage-free prices). *Let π be an arbitrage-free pricing rule for each contingent claims, then the discrete model (S^0, S^1, \dots, S^d) is arbitrage-free and there is $Q \in \mathcal{M}^e(\tilde{S})$ such that*

$$\pi(\mathbf{X})_n = \mathbb{E}_Q\left(\frac{S^0_n}{S^0_N} \mathbf{X} \mid \mathcal{F}_n\right).$$

If the discrete time model S is arbitrage-free, then

$$\pi(\mathbf{X})_n = \mathbb{E}_Q\left(\frac{S_n^0}{S_0^0} \mathbf{X} | \mathcal{F}_n\right).$$

is an arbitrage-free pricing rule for any contingent claims $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$. Hence the only arbitrage-free prices are conditional expectation of the discounted claims with respect to Q .

Remark 2.3.2. Taking not an equivalent but an absolutely continuous martingale measure $Q \in \mathcal{M}^a(\tilde{S})$ means that there is at least one state ω_i such that $Q(\omega_i = 0) > 0$. Hence the claim 1_A with $P(A) > 0$ would have price 0, which immediately leads to arbitrage by entering this contract. Therefore only equivalent martingale measures are possible for pricing.

The set of equivalent martingale measures $\mathcal{M}^e(\tilde{S})$ is convex and the set $\mathcal{M}^a(\tilde{S})$ is compact and convex. Therefore the analysis of the extreme points of $\mathcal{M}^a(\tilde{S})$ is of particular importance.

Remark 2.3.3. Given an arbitrage-free financial market such that $\mathcal{M}^e(\tilde{S})$ contains more than one measure. Then an equivalent martingale measure $Q \in \mathcal{M}^e(\tilde{S})$ can never be an extreme point of $\mathcal{M}^a(\tilde{S})$. Assume that there were an extreme point $Q \in \mathcal{M}^e(\tilde{S})$ of $\mathcal{M}^a(\tilde{S})$ and take $Q_0 \neq Q$ with $Q_0 \in \mathcal{M}^e(\tilde{S})$. Then we know that the segment $tQ + (1-t)Q_0 \in \mathcal{M}^e(\tilde{S})$. For t near 1 we also have equivalent martingale measures. We also know that C is finitely generated by $\langle h_1, \dots, h_M, -h_1, \dots, -h_M, -e_1, \dots, -e_k \rangle_{\text{con}}$, where h_1, \dots, h_M is a basis of \mathcal{H} and e_1, \dots, e_k generates the non-negative cone $L_{\geq 0}^2(\Omega, \mathcal{F}, P) = \{0\}$. An equivalent martingale measure Q_t is defined via the relation $\mathbb{E}_{Q_t}(C) \leq 0$. The Q_t are equivalent martingale measures since $\mathbb{E}_{Q_t}(h_i) = 0$ and $\mathbb{E}_{Q_t}(e_i) < 0$. So we can continue a little bit in t -direction beyond 1 and obtain again equivalent martingale measures, too. Hence Q cannot be an extreme point, since it is a middle point of two martingale measures. Therefore an extreme point is necessarily absolutely continuous and not equivalent to P .

Theorem 2.3.2. Let S be a discrete model for a financial market and assume $\mathcal{M}^e(\tilde{S}) \neq \emptyset$ and $\mathcal{M}^a(\tilde{S}) = \langle Q_1, \dots, Q_M \rangle$. Then for all $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$ the following assertions are equivalent:

1. $\mathbf{X} \in \mathcal{K}(\mathbf{X} \in C)$.
2. For all $Q \in \mathcal{M}^e(\tilde{S})$ we have $\mathbb{E}_Q(\mathbf{X}) = 0$ (for all $Q \in \mathcal{M}^e(\tilde{S})$ we have $\mathbb{E}_Q(\mathbf{X}) \leq 0$).
3. For all $Q \in \mathcal{M}^a(\tilde{S})$ we have $\mathbb{E}_Q(\mathbf{X}) = 0$ (for all $Q \in \mathcal{M}^a(\tilde{S})$ we have $\mathbb{E}_Q(\mathbf{X}) \leq 0$).
4. For all $i = 1, \dots, m$ we have $\mathbb{E}_{Q_i}(\mathbf{X}) = 0$ (For all $i = 1, \dots, m$ we have $\mathbb{E}_{Q_i}(\mathbf{X}) \leq 0$).

Definition 2.3.4. Let S be a discrete model for a financial market and assume $\mathcal{M}^e(\tilde{S}) \neq \emptyset$. The financial market is called complete if $\mathcal{M}^e(\tilde{S}) = \{Q\}$, i.e. the equivalent martingale measure is unique. The financial market is called incomplete if $\mathcal{M}^e(\tilde{S})$ contains more than one element. In this case $\mathcal{M}^a(\tilde{S}) = \langle Q_1, \dots, Q_M \rangle_{\text{convex}}$ for linearly independent measures Q_i , $i = 1, \dots, m$ and $m \geq 2$.

Theorem 2.3.3 (complete markets). Let S be a discrete model of a financial market with $\mathcal{M}^e(\tilde{S}) \neq \emptyset$. Then the following assertions are equivalent:

1. S is complete financial market.
2. For every claim \mathbf{X} there is a self-financing trading strategy ϕ such that the claim can be replicated, i.e.

$$V_N(\phi) = \mathbf{X}$$

3. For every claim \mathbf{X} there is a predictable process ϕ and a unique number x such that the discounted claim can be replicated, i.e.

$$\tilde{\mathbf{X}} = \frac{1}{S_N^0} \mathbf{X} = x + (\phi \cdot \tilde{S}).$$

4. There is a unique pricing rule for every claim \mathbf{X} .

The Cox-Ross-Rubinstein model is a complete financial market model: The CRR-model is defined by the following relations

$$S_n^0 = (1 + r)^n$$

for $n = 0, \dots, N$ and $r \geq 0$ is the bond-process

$$S_{n+1} := \begin{cases} S_n(1 + a) \\ S_n(1 + b) \end{cases}$$

for $-1 < a < b$ and $n = 0, \dots, N$. The σ -algebras \mathcal{F}_n are given by $\sigma(S_0, \dots, S_n)$, which means that atoms of \mathcal{F}_n are of type

$$\{(x_1, \dots, x_n, y_{n+1}, \dots, y_N) \text{ for all } y_{n+1}, \dots, y_N \in \{1 + a, 1 + b\}\}$$

with $x_1, \dots, x_n \in \{1 + a, 1 + b\}$ fixed. The returns $(T_i)_{i=1, \dots, N}$ are well-defined by

$$T_i := \frac{S_i}{S_{i-1}}$$

for $i = 1, \dots, N$. This process is adapted and each T_i can take two values

$$T_i := \frac{1+a}{1+b}$$

with some specified probabilities depending on $i = 1, \dots, N$. We also note the following formula

$$S_n \prod_{i=n+1}^m T_i = S_m$$

for $m \geq n$. Hence it is sufficient for the definition of the probability on (Ω, \mathcal{F}, P) to know the distribution of (T_1, \dots, T_N) , i.e.

$$(T_1 = x_1, \dots, T_N = x_N)$$

has to be known for each $x_i \in \{1 + a, 1 + b\}$.

Proposition 2.3.3. *Let $-1 < a < b$ and $r \geq 0$, then the CRR-Model is arbitrage-free if and only if $r \in]a, b[$. If this condition is satisfied, then the martingale measure Q for the discounted price process $\left(\frac{S_n}{(1+r)^n}\right)_{n=0,\dots,N}$ is unique and characterized by the fact that $(T_i)_{i=1,\dots,N}$ are independent and identically distributed and*

$$T_i := \begin{cases} 1 + a & \text{with probability } 1 - q \\ 1 + b & \text{with probability } q \end{cases}$$

for $q = \frac{r-a}{b-a}$.

Theorem 2.3.4 (incomplete markets). *Let S be a discrete model of a financial market with $\mathcal{M}^e(\tilde{S}) \neq \emptyset$. Then the following assertions are equivalent:*

1. S is an incomplete financial market.
2. For every claim \mathbf{X} there is a self-financing trading strategy ϕ such that the claim can be super-replicated, i.e.

$$V_N(\phi) \geq \mathbf{X}$$

and there is at least one claim \mathbf{X} , which cannot be replicated.

3. For every claim \mathbf{X} there is a predictable process ϕ and a unique number x such that the discounted claim can be super-replicated, i.e.

$$\tilde{\mathbf{X}} = \frac{1}{S_N^0} \mathbf{X} \leq x + (\phi \cdot \tilde{S}).$$

and there is at least one claim, which cannot be replicated.

In particular we have that the no arbitrage prices at time 0 form an open interval $]\pi_\downarrow(\mathbf{X}), \pi_\uparrow(\mathbf{X})[$ if $\pi_\downarrow(\mathbf{X}) < \pi_\uparrow(\mathbf{X})$ with

$$\begin{aligned} \pi_\downarrow(\mathbf{X}) &= \inf \left\{ \mathbb{E}_Q\left(\frac{\mathbf{X}}{S_N^0}\right) \text{ for } Q \in \mathcal{M}^e(\tilde{S}) \right\}, \\ \pi_\uparrow(\mathbf{X}) &= \sup \left\{ \mathbb{E}_Q\left(\frac{\mathbf{X}}{S_N^0}\right) \text{ for } Q \in \mathcal{M}^e(\tilde{S}) \right\}. \end{aligned}$$

The case $\pi_\downarrow(\mathbf{X}) = \pi_\uparrow(\mathbf{X})$ (there is only one no-arbitrage price for the claim \mathbf{X}) occurs if and only if \mathbf{X} can be replicated.

Given a financial market $(S_n^0, S_n^1, \dots, S_n^d)_{n=0,\dots,N}$ on a probability space (Ω, \mathcal{F}, P) with filtration $(\mathcal{F}_n)_{n=0,\dots,N}$. Without restriction we assume $d = 1$ and $S_n^0 = 1$ for $n = 0, \dots, N$, since

$$\mathcal{M}^a\left(\frac{S^1}{S^0}, \dots, \frac{S^d}{S^0}\right) = \cap_{i=1}^d \mathcal{M}^a\left(\frac{S^i}{S^0}\right).$$

So if we are able to calculate the martingale measures for a one asset model, we can do it in general easily for an \mathbb{R}^d -valued process.

The defining definition for a martingale $(S_n)_{n=0,\dots,N}$, namely

$$\mathbb{E}_Q(S_n | S_{n-1} = x) = x$$

for all values x of S_{n-1} on Ω . We define for all values x of S_{n-1} the conditional probabilities

$$\mathbb{E}_Q(1_{\{A^n\}} | S_{n-1} = x) = q^x(A^n),$$

which is 0 if $S_{n-1}(A^n) \neq \{x\}$ for the atoms A^n of \mathcal{F}_n . Then we get the equations

$$\begin{aligned} \sum_{A^n \in \mathcal{A}(\mathcal{F}_n)} q^x(A^n) S_n(A^n) &= x, \\ \sum_{i=1}^{|\Omega|} q^x(A^n) &= 1, \\ q^x(A^n) &\geq 0, \end{aligned}$$

which can be solved if the model is arbitrage-free. The martingale measures Q is then given through

$$Q(A^n) = \mathbb{E}_Q(1_{A^n} | S_{n-1} = x) Q(S_{n-1} = x)$$

if $S_{n-1}(A^n) = \{x\}$. $\mathcal{M}^a(S^1) = \langle Q_1, \dots, Q_m \rangle$ for $m \geq 1$ (both cases are included, complete or incomplete), then we want to calculate (super)replicating strategies. Given a claim \mathbf{X} there is one Q_i for some $i \in \{1, \dots, m\}$ such that

$$\pi_{\uparrow}(\mathbf{X}) = \mathbb{E}_{Q_i}(\mathbf{X}),$$

which is trivial in the complete case and requires some reasoning in the incomplete one. Then calculate the conditional expectations of \mathbf{X} with respect to Q_i

$$X_n := \mathbb{E}_{Q_i}(\mathbf{X} | \mathcal{F}_n)$$

for $n = 0, \dots, N$. The difference $\mathbf{X}_n - \mathbf{X}_{n-1}$ for $n = 1, \dots, N$ is then

$$\mathbf{X}_n - \mathbf{X}_{n-1} = \phi_n(S_n - S_{n-1})$$

for some predictable process ϕ , which can be easily calculated from this equation for $n = 1, \dots, N$.

Remark 2.3.4. The wealth process $(V_n/\phi)_{0 \leq n \leq N}$ of every portfolio ϕ , which finally produces the attainable claims $V_N(\phi)$, is equal to the price process $(q(V_n)_n)_{0 \leq n \leq N}$, which in turn is a Q -martingale for each $Q \in \mathcal{M}^a(\tilde{S})$ if we discount with respect to S^0 . If the portfolio is strictly positive we can take as new unit of calculation and calculate prices with respect to this portfolio. Certainly the prices should remain unchanged, since they should not depend on the currency unit. For a discrete model of a financial market S

- the discounted processes $(\frac{S_n^1}{S_n^0}, \dots, \frac{S_n^d}{S_n^0})_{0 \leq n \leq N}$ have to satisfy the condition that there is an equivalent martingale measure.

- calculating the price of a claim \mathbf{X} at time n amounts to calculating the conditional expectation of the discounted claim $\frac{\mathbf{X}S_n^0}{S_N^0}$, discounted for time n . If one could discount this price to time 0 one obtains that the discounted price process is again a martingale.
- hence calculations are always done with respect to a numeraire and a discounting procedure to some fixed time 0. This is economically obvious since one cannot compare amounts of money at different times without taking care of the time changes in value of the money

Definition 2.3.5. Let S be a discrete model of a financial market. A numeraire process $(C_n)_{0 \leq n \leq N}$ is a strictly positive, adapted stochastic process, which has the property that there is a self-financing trading strategy ϕ such that $C_n = V_n(\phi)$ for $0 \leq n \leq N$.

Theorem 2.3.5. Let S be a discrete model of a financial market and C a numeraire process. If S is arbitrage-free, then also the market

$$S_n^C = (1, \frac{S_n^0}{C_n}, \frac{S_n^1}{C_n}, \dots, \frac{S_n^d}{C_n})$$

for $n = 0, \dots, N$ is arbitrage-free and we have

$$\mathcal{M}^e(S^C, 1) = \left\{ Q^C \mid \frac{dQ^C}{dP} = \frac{1}{S_N^0} \frac{C_N}{C_0} \frac{dQ}{dP} \text{ for } Q \in \mathcal{M}^e(S, S^0) \right\}$$

Every pricing rule π^C for contingent claims \mathbf{X} is of the form

$$\pi^C(\mathbf{X})_n = \mathbb{E}_Q\left(\frac{C_n}{C_N} \mathbf{X} \mid \mathcal{F}_n\right)$$

for any $Q \in \mathcal{M}^e(S^C, 1)$ and the discounted processes with respect to the numeraires C are martingales.

In the sequel we shall formulate most of the assertions with respect to a basis in $L^2(\Omega, \mathcal{F}, P)$. We shall assume (not a real restriction), that $\mathcal{F} = 2^\Omega$ and $P(\omega_i) > 0$ for $i = 1, \dots, |\Omega|$. We choose $(1_{\{\omega\}})_{\omega \in \Omega}$ and identify $L^2(\Omega, \mathcal{F}, P)$ with some $\mathbb{R}^{|\Omega|}$. Hence we can apply our duality theory for cones.

Proposition 2.3.4. Let S be a discrete model of a financial market and assume $\mathcal{M}^e(\tilde{S}) \neq \emptyset$. Then there are linearly independent measures Q_1, \dots, Q_n such that

$$\mathcal{M}^a(\tilde{S}) = \langle Q_1, \dots, Q_n \rangle_{\text{convex}},$$

the polar cone C^0 equals

$$C^0 = \left\langle \frac{dQ_1}{dP}, \dots, \frac{dQ_n}{dP} \right\rangle_{\text{cone}}.$$

Furthermore the Q_i have at least $n - 1$ zeros, where n equals the codimension of \mathcal{K} .

2.4 Optimization

2.4.1 First examples of optimization

Definition 2.4.1. A real valued function $u : I \rightarrow \mathbb{R}$ is called *utility function* if $I =]0, \infty[$ or $I =]-\infty, \infty[$ and u is an increasing, strictly concave function. We shall denote $\text{dom}(u) := I$ and we define $u(x) = -\infty$ for $x \notin \text{dom}(u)$. Furthermore we shall assume that $\lim_{x \downarrow 0} u(x) = -\infty$ if $\text{dom}(u) =]0, \infty[$.

We consider a financial market $(S_n^0, \dots, S_n^d)_{n=0,1}$ on (Ω, \mathcal{F}, P) with one period and aim to solve the following optimization problem for a given utility function $u : \text{dom}(u) \rightarrow \mathbb{R}$ and $x \in \text{dom}(u)$.

$$\begin{aligned} \mathbb{E}_P \left(u \left(\frac{1}{S_1^0} V_1(\phi) \right) \right) &\rightarrow \max, \\ V_0(\phi) &= x, \end{aligned}$$

where ϕ is running over all self-financing trading strategies. This leads to the following one dimensional optimization problem

$$a \mapsto \mathbb{E}_P \left(u(x + a(\widetilde{S}_1 - \widetilde{S}_0)) \right),$$

which can be solved by classical analysis. We see immediately that the existence of an optimal strategy $\widehat{a}(x)$ for a fixed $x \in \text{dom}(u)$ leads to

$$\mathbb{E}_P \left(u'(x + \widehat{a}(x)(\widetilde{S}_1 - \widetilde{S}_0))(\widetilde{S}_1 - \widetilde{S}_0) \right) = 0$$

This in turn means that the vector can be normalized to a probability measure Q , i.e.

$$\frac{dQ}{dP} = \frac{1}{\lambda} u' \left(x + \widehat{a}(x)(\widetilde{S}_1 - \widetilde{S}_0) \right)$$

which is a martingale measure since $\mathbb{E}_Q(\widetilde{S}_1 - \widetilde{S}_0) = 0$. Therefore the existence of an optimizer leads to arbitrage-free markets.

2.4.2 Basic concepts of utility optimization

Given a financial market $(S_n^0, \dots, S_n^d)_{n=0,1}$ on (Ω, \mathcal{F}, P) and a utility function u , then we define the *utility optimization problem* as determination of $U(x)$ for $x \in \text{dom}(u)$, i.e.

$$\sup_{\substack{\phi \text{ trading strategy} \\ \phi \text{ self-financing} \\ V_0(\phi)=x}} \mathbb{E} \left(u \left(\frac{1}{S_N^0} V_N(\phi) \right) \right) =: U(x).$$

We say that the utility optimization problem at $x \in \text{dom}(u)$ is *solvable* if $U(x)$ is finitely valued and if we find an optimal self financing trading strategy $\widetilde{\phi}(x)$ for $x \in \text{dom}(u)$ such that

$$\begin{aligned} U(x) &= \mathbb{E} \left(u \left(\frac{1}{S_N^0} V_N(\widetilde{\phi}) \right) \right), \\ V_0(\widetilde{\phi}(x)) &= x. \end{aligned}$$

We assume that $\mathcal{F} = 2^\Omega$ and $P(\omega) > 0$ for $\omega \in \Omega$. We then have three characteristic dimensions: the dimension of all random variables $|\Omega|$ (the number of paths), then the dimension of discounted outcomes at initial wealth 0, denoted by $\dim \mathcal{K}$, and the number of martingale measures m . We have the basic relation

$$m + \dim \mathcal{K} = |\Omega|.$$

- the pedestrian method is an unconstraint extremal value problem in $\dim \mathcal{K}$ variables.
- the Lagrangian method yields an unconstraint extremal value problem in $|\Omega| + m$ variables.
- the duality method (martingale approach) yields an unconstraint extremal value problem in m variables. Additionally one has to transform the dual value function to the original, which is a one dimensional extremal value problem.

In financial mathematics usually $\dim \mathcal{K} \gg m$, which means that the duality method is of particular importance.

Pedestrian's method

We can understand utility optimization as unrestricted optimisation problem. Define \mathbf{S} the vector space of all predictable strategies $(\phi_n)_{n=0,\dots,N}$, then the utility optimization problem for $x \in \text{dom}(u)$ is equivalent to solving the following problem

$$\begin{aligned} F_x &: \begin{cases} \mathbf{S} \rightarrow \mathbb{R} \cup \{-\infty\} \\ (\phi_n)_{n=0,\dots,N} \mapsto \mathbb{E} \left(u(x + (\phi \cdot \tilde{S})_N) \right) \end{cases} \\ \sup_{\phi \in \mathbf{S}} F_x(\phi) &= U(x) \end{aligned}$$

This is an ordinary extremal value problem for every $x \in \text{dom}(u)$. We introduce a basis on \mathbf{S} , namely

$$(1_A)_{A \in \mathcal{A}(\mathcal{F}_{i-1})}$$

is a basis for the i -th component of a predictable strategy. With respect to this basis we can calculate gradients. Let $(\widehat{\phi}_n)_{n=0,\dots,N}$ be an optimal strategy, then necessarily

$$\text{grad } F_x((\widehat{\phi}_n)_{n=0,\dots,N}) = 0$$

and therefore we can in principle calculate the optimal strategy. From this formulation we take one fundamental conclusion.

Theorem 2.4.1. *Let the utility optimization problem at $x \in \text{dom}(u)$ be solvable and let $(\widehat{\phi}_n)_{n=0,\dots,N}$ be an optimal strategy, so*

$$\sup_{\phi \in \mathbf{S}} F_x(\phi) = U(x) = F_x(\widehat{\phi}),$$

then $\mathcal{M}^e(\tilde{S}) \neq \emptyset$.

Remark 2.4.1. We shall always assume $\mathcal{M}^e(\tilde{S}) \neq \emptyset$.

Proposition 2.4.1. Assume $\mathcal{M}^e(\tilde{S}) \neq \emptyset$ and $\lim_{x \rightarrow \infty} u'(x) = 0$ if $\text{dom}(u) = (R)$, then the utility optimization problem for $x \in \text{dom}(u)$ has a unique solution $\hat{\mathbf{X}}(x)$ is C^1 on $\text{dom}(u)$. If $x \notin \text{dom}(u)$, then $\sup_{\phi \in S} F_x(\phi) = -\infty$.

Duality methods

Since we have a dual relation between the set of martingale measures and the set \mathcal{K} of claims attainable at price 0, we can formulate the optimization problem as constraint problem: for any $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$

$$\mathbf{X} \in \mathcal{K} \iff \mathbb{E}_Q(\mathbf{X}) = 0$$

for $Q \in \mathcal{M}^a(\tilde{S})$ and for any probability measure Q

$$Q \in \mathcal{M}^a(\tilde{S}) \iff \mathbb{E}_Q(\mathbf{X}) = 0$$

for all $\mathbf{X} \in \mathcal{K}$. Therefore we can formulate the problem as constraint optimization problem and apply the method of Langrangian multipliers. First we define a function $H : L^2(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R}$ via

$$H(\mathbf{X}) := \mathbb{E}_P(u(\mathbf{X}))$$

for a utility function u . For $x \in \text{dom}(u)$ we can formulate the constraints

$$U_x := \mathcal{K} + x = \{ \mathbf{X} \in L^2(\Omega, \mathcal{F}, P) \text{ such that } \mathbb{E}_Q(\mathbf{X}) = x \text{ for } Q \in \mathcal{M}^a(\tilde{S}) \}.$$

Consequently the utility optimization problem reads

$$\sup_{\mathbf{X} \in U_x} \mathbb{E}_P(u(\mathbf{X})) = U(x)$$

for $x \in \text{dom}(u)$. Hence we can treat the problem by Langragian multipliers, i.e. if $\hat{\mathbf{X}} \in U_x$ is an optimizer, then

$$(LM) \quad \begin{aligned} u'(\hat{\mathbf{X}}) - \sum_{i=1}^m \hat{\eta}_i \frac{dQ_i}{dP} &= 0 \\ \mathbb{E}_{Q_i}(\hat{\mathbf{X}}) &= 0 \end{aligned}$$

for $i = 1, \dots, m$, $\mathcal{M}^a(\tilde{S}) = \langle Q_1, \dots, Q_m \rangle$ and some values $\hat{\eta}_i$. This result is obtained by taking the gradient of the function

$$\mathbf{X} \mapsto \mathbb{E}_P(u(\mathbf{X})) - \sum_{i=1}^m \hat{\eta}_i \left(\frac{dQ_i}{dP} \mathbf{X} - x \right)$$

with respect to some basis. We can choose $\hat{\eta}_i$ positive, since $u'(\hat{\mathbf{X}})$ represents a positive multiple of an equivalent martingale measure. With assumption $u'(x) > 0$ for all $x \in \text{dom}(u)$, and $u'(\hat{\mathbf{X}})$ is finitely valued.

Lemma 2.4.1. *If $(\widehat{\mathbf{X}}, \widehat{\eta}_1, \dots, \widehat{\eta}_m)$ is a solution of the Lagrangian multiplier equation (LM), then the multipliers $\widehat{\eta}_i > 0$ are uniquely determined and $\sum_{i=1}^m \widehat{\eta}_i > 0$. Given $x \in \text{dom}(u)$, the map $x \mapsto (\widehat{\eta}_i(x))_{i=1, \dots, m}$ is C^1 .*

The Lagrangian \widetilde{L} is given through

$$\widetilde{L}(\mathbf{X}, \eta_1, \dots, \eta_m) = \mathbb{E}_P(u(\mathbf{X})) - \sum_{i=1}^m \eta_i (\mathbb{E}_{Q_i}(\mathbf{X}) - x)$$

for $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$ and $\eta_i \geq 0$. We introduce $y := \eta_1 + \dots + \eta_m$ and $\mu_i := \frac{\eta_i}{y}$ (we can assume $y > 0$ since the value for η_i we are looking for has to satisfy $y > 0$). Therefore

$$L(\mathbf{X}, y, Q) = \mathbb{E}_P(u(\mathbf{X})) - y(\mathbb{E}_Q(\mathbf{X}) - x)$$

for $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$, $Q \in \mathcal{M}^a(\widetilde{S})$ and $y > 0$. We define

$$\Phi(\mathbf{X}) := \inf_{y>0, Q \in \mathcal{M}^a(\widetilde{S})} L(\mathbf{X}, y, Q)$$

for $\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)$ and

$$\psi(y, Q) = \sup_{\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)} L(\mathbf{X}, y, Q)$$

for $y > 0$ and $Q \in \mathcal{M}^a(\widetilde{S})$. We can hope for

$$\sup_{\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)} \Phi(\mathbf{X}) = \inf_{y>0} \inf_{Q \in \mathcal{M}^a(\widetilde{S})} \psi(y, Q) = U(x).$$

by a mini-max consideration.

Lemma 2.4.2. *Let u be a utility function and $(S_n^0, S_n^1, \dots, S_n^d)_{n=0, \dots, N}$ be a financial market, which is arbitrage-free, then*

$$\sup_{\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)} \Phi(\mathbf{X}) = U(x).$$

For the application of the minimax theorem we need to calculate ψ .

Lemma 2.4.3. *Given an arbitrage-free financial market (S^0, \dots, S^d) , the function*

$$\psi(y, Q) = \sup_{\mathbf{X} \in L^2(\Omega, \mathcal{F}, P)} L(\mathbf{X}, y, Q)$$

can be expressed by the conjugate function v of u ,

$$\psi(y, Q) = \mathbb{E}_P(v(y \frac{dQ}{dP})) + yx.$$

Definition 2.4.2. *Given the above setting we call the optimization problem*

$$V(y) := \inf_{Q \in \mathcal{M}^a(\widetilde{S})} \mathbb{E}_P(v(y \frac{dQ}{dP}))$$

the dual problem and V the dual value function for $y > 0$.

Lemma 2.4.4. *Let u be a utility function under the above assumptions and assume $\mathcal{M}^a(\widetilde{S}) \neq \emptyset$, then there is a unique optimizer $\widehat{Q}(y)$ such that*

$$V(y) = \inf_{Q \in \mathcal{M}^a(\tilde{S})} \mathbb{E}_P(v(y \frac{dQ}{dP})) = \mathbb{E}_P(v(y \frac{d\hat{Q}(y)}{dP})).$$

Furthermore

$$\inf_{y>0} (V(y) + xy) = \inf_{y>0, Q \in \mathcal{M}^a(\tilde{S})} \mathbb{E}_P(v(y \frac{dQ}{dP}) + xy).$$

Theorem 2.4.2. *Let (S^0, \dots, S^d) be an arbitrage-free market and u a utility function with the above properties, then*

$$U(x) = \inf_{y>0, Q \in \mathcal{M}^a(\tilde{S})} \mathbb{E}_P(v(y \frac{dQ}{dP}) + xy)$$

and the mini-max assertion holds.

This theorem enables us to formulate the following duality relation. Given a utility optimization problem for $x \in \text{dom}(u)$

$$\sup_{Y \in \mathcal{K}} \mathbb{E}_P(u(x + Y)) = U(x),$$

then we can associate a dual problem, namely

$$\inf_{Q \in \mathcal{M}^a(\tilde{S})} \mathbb{E}_P(v(y \frac{dQ}{dP})) = V(y)$$

for $y > 0$. The main assertion of the minimax considerations is that

$$\inf_{y>0} (V(y) + xy) = U(x),$$

so the concave conjugate of V is U and since V shares the same regularity as U , also U is the convex conjugate of V . First we solve the dual problem (which is much easier) and obtain $y \mapsto \hat{Q}(y)$. For given $x \in \text{dom}(u)$ we can calculate $\hat{y}(x)$ and obtain

$$\begin{aligned} V(\hat{y}(x)) + x\hat{y}(x) &= U(x) \\ u'(\hat{\mathbf{X}}(x)) &= \hat{y}(x) \frac{d\hat{Q}(\hat{y}(x))}{dP}. \end{aligned}$$

Chapter 3

Continuous Time Models

3.1 From discrete to continuous time

The next step is the intuition from discrete models for the pricing and hedging of contingent claims in continuous time models. Brownian motion plays a very important role in continuous time models.

Definition 3.1.1. Let (Ω, \mathcal{F}, P) be a probability space and $(\mathcal{F}_t)_{t \geq 0}$ a filtration of σ -algebras which satisfies the usual conditions, i.e.

- the σ -algebra \mathcal{F}_t contains all P -nullsets
- right continuity holds, $\bigcap_{t > s} \mathcal{F}_t = \mathcal{F}_s$ for $s \geq 0$.

Brownian motion then is a stochastic process $(B_t)_{t \geq 0}$ such that

- B_t is \mathcal{F}_t -measurable for $t \geq 0$ (the process is adapted to the filtration).
- $B_t - B_s$ is independent of \mathcal{F}_s for $t \geq s \geq 0$.
- $B_t - B_s$ is normally distributed $N(0, t - s)$ for $t \geq s \geq 0$.
- $B_0 = 0$.

Furthermore we assume that the paths of Brownian motion are continuous, i.e. for all $\omega \in \Omega$ the curve

$$t \mapsto B_t(\omega)$$

is continuous. The same definition can be done on $[0, T]$ and yields to a Brownian motion on $[0, T]$.

Lemma 3.1.1. Let $(B_t)_{t \geq 0}$ be a Brownian motion on (Ω, \mathcal{F}, P) , then

1. Brownian motion is a martingale, i.e. $\mathbb{E}(B_t | \mathcal{F}_s) = B_s$ for $t \geq s$.
2. the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent for $0 \leq t_1 \leq t_2 \leq \dots \leq t_n$ and $n \geq 1$.

Now the basic definition of a financial market with finite time horizon $T > 0$, such that second moments exist and interest rates are constant.

Definition 3.1.2. Let $(\Omega, \mathcal{F}_T, P)$ be a probability space and $(\mathcal{F}_t)_{0 \leq t \leq T}$ a filtration of σ -algebras which satisfies the usual conditions. A financial market is given by a bank account process $S_t^0 = \exp(rt)$, where $r \geq 0$ and $0 \leq t \leq T$ denotes the interest rate and an adapted process $(S_t^1)_{0 \leq t \leq T}$ with continuous paths. We assume that $S_t^1 \in L^2(\Omega, \mathcal{F}_T, P)$ and $S_0 > 0$ is a constant. A simple portfolio $(\psi_t, \phi_t)_{0 \leq t \leq T}$ is given by stochastic processes $(\psi_t, \phi_t)_{0 \leq t \leq T}$ such that there is $0 = t_0 < t_1 < t_2 < \dots < t_n$ and $F_i, G_i \in L^\infty(\Omega, \mathcal{F}_t, P)$ for $i = 0, \dots, n-1$ such that

$$\begin{aligned}\psi_t &= \sum_{i=0}^{n-1} G_i 1_{]t_i, t_{i+1}]}(t), \\ \phi_t &= \sum_{i=0}^{n-1} F_i 1_{]t_i, t_{i+1}]}(t),\end{aligned}$$

where $\psi_0 = G_0$ and $\phi_0 = F_0$ by definition. The value process is given by

$$V_t(\psi, \phi) = \psi_t S_t^0 + \phi_t S_t^1$$

for $0 \leq t \leq T$. The discounted value process is given by

$$\tilde{V}_t(\psi, \phi) = \psi_t + \phi_t \tilde{S}_t^1$$

with $\tilde{S}_t^1 = \exp(-rt) S_t^1$ for $0 \leq t \leq T$. A simple portfolio is called self-financing if for $i = 0, \dots, n-1$ we have

$$\psi_{t_i} S_{t_i}^0 + \phi_{t_i} S_{t_i}^1 = \psi_{t_{i+1}} S_{t_{i+1}}^0 + \phi_{t_{i+1}} S_{t_{i+1}}^1.$$

We denote by \mathcal{K} the space of all discounted outcomes at initial investment 0.

Lemma 3.1.2. Given a financial market, then for every self-financing portfolio $(\psi_t, \phi_t)_{0 \leq t \leq T}$ we obtain

$$\tilde{V}_t(\psi, \phi) = V_0(\psi, \phi) + \sum_{i=0}^{n-1} \phi_{t_i} (\tilde{S}_{t_{i+1} \wedge t}^1 - \tilde{S}_{t_i \wedge t}^1) = V_0(\psi, \phi) + (\phi \cdot \tilde{S})_t,$$

hence

$$\mathcal{K} = \left\{ (\phi \cdot \tilde{S})_t \text{ for } \phi \text{ a simple, self-financing trading strategy} \right\}$$

Condition 3.1.1. We shall assume that the L^2 -closure of \mathcal{K} can be described by

$$\overline{\mathcal{K}} = \{ \mathbf{X} \in L_{\geq 0}^2(\Omega, \mathcal{F}_T, P) \text{ such that } \mathbb{E}_Q(\mathbf{X}) = 0 \}$$

for some equivalent measure $Q \sim P$. We call this market complete.

Lemma 3.1.3. Given a complete financial market, the measure Q is the unique absolutely continuous martingale measure for the process $(\tilde{S}_t^1)_{0 \leq t \leq T}$. Furthermore

$$\overline{\mathcal{K}} \cap L_{\geq 0}^2(\Omega, \mathcal{F}_T, P) = \{0\}.$$

Now the first main example of a continuous time model, known as Bachelier model. We assume zero interest rates $r = 0$ (for equally that the discounted price process equals S_t^B). Let $(B_t)_{0 \leq t \leq T}$ be a Brownian motion on $(\Omega, \mathcal{F}_T, P)$ and let $S_0 > 0$ and $\sigma > 0$ be constants, then

$$S_t^B := S_0(1 + \sigma B_t)$$

for $0 \leq t \leq T$

Theorem 3.1.1. *For the Bachelier model we have $\overline{\mathcal{K}} = \{\mathbf{X} \in L^2(\Omega, \mathcal{F}_T, P) \text{ such that } \mathbb{E}_P(\mathbf{X}) = 0\}$, so in particular $(S_t^B)_{0 \leq t \leq T}$ is a martingale.*

Given a derivative $Y \in L^2(\Omega, \mathcal{F}_T, P)$, we know from finite dimensional theory that the only arbitrage-free prices are given through

$$\mathbb{E}(Y | \mathcal{F}_t) = \pi(Y)_t$$

for $0 \leq t \leq T$. In the Bachelier framework this can be easily calculated, which is the "main advantage" of continuous time models.

Theorem 3.1.2. *Let $S_0, \sigma > 0$ be given, then the price of a European call with strike price $K > 0$ and maturity T at time $t = 0$ is given through*

$$C(S_0, T, K) = (S_0 - K)\Phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right) + S_0\sigma\sqrt{T}\phi\left(\frac{S_0 - K}{S_0\sigma\sqrt{T}}\right)$$

with

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \\ \Phi(x) &= \int_{-\infty}^x \phi(x) dx.\end{aligned}$$

The second important example is the Black-Scholes model. Given $\mu \geq 0$ and $S_0, \sigma > 0$, then

$$S_t^{BS} := S_0 \exp\left(\mu t - \frac{\sigma^2}{2}t + \sigma B_t\right)$$

for $0 \leq t \leq T$. The process is adapted and has continuous paths. Furthermore it is a martingale with respect to the following measure.

Proposition 3.1.1. *Given the Black-Scholes model S^{BS} on $[0, T]$, the measure Q on $(\Omega, \mathcal{F}_T, P)$ by*

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu}{\sigma} B_T - \frac{\mu^2}{2\sigma^2} T\right)$$

is an equivalent martingale measure for S^{BS} .

Theorem 3.1.3. *For the Black-Scholes model we have $\overline{\mathcal{K}} = \{\mathbf{X} \in L^2(\Omega, \mathcal{F}_T, P) \text{ such that } \mathbb{E}_{Q_T}(\mathbf{X}) = 0\}$, so in particular $(S_t^{BS})_{0 \leq t \leq T}$ is a Q_T -martingale.*

Theorem 3.1.4. *Given the Black-Scholes model $(S_t^{BS})_{0 \leq t \leq T}$, a maturity time $T_0 \leq T$ and a strike price $K \geq 0$, the unique price of the European call $(S_{T_0} - K)_+$ without interest rates is given through*

$$C(S_0, K, T_0) = S_0 \Phi\left(\frac{\ln \frac{S_0}{K} + \frac{1}{2}\sigma^2 T_0}{\sigma \sqrt{T_0}}\right) - K \Phi\left(\frac{\ln \frac{S_0}{K} + \frac{1}{2}\sigma^2 T_0}{\sigma \sqrt{T_0}}\right).$$

Through replacement from the strike K with the discounted value Ke^{-rT_0} , we get the price with interest rate r

$$C(S_0, K, T_0) = S_0 \Phi\left(\frac{\ln \frac{S_0}{K} + (\frac{1}{2}\sigma^2 + r)T_0}{\sigma \sqrt{T_0}}\right) - e^{-rT_0} K \Phi\left(\frac{\ln \frac{S_0}{K} + (\frac{1}{2}\sigma^2 + r)T_0}{\sigma \sqrt{T_0}}\right).$$

Theorem 3.1.5. *Let $t \geq 0$ be a fixed point in time and $(B_s)_{s \geq 0}$ a Brownian motion, then*

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{2^n-1} (B_{\frac{t(i+1)}{2^n}} - B_{\frac{ti}{2^n}})^2 = t$$

almost surely.

Now we turn to the construction of the Ito-Integral. Given a standard Brownian motion $(B_t)_{t \geq 0}$ on \mathbb{R}^d . We denote by $L^2(\mathbb{R}_{\geq 0} \times \Omega, \mathcal{F}_p, dt \otimes P)$ the set of all progressively measurable processes, i.e. the set of

$$\phi : \mathbb{R}_{\geq 0} \times \Omega \rightarrow \mathbb{R},$$

which are measurable with respect to the σ -algebra \mathcal{F}_p , i.e. the σ -algebra generated by $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$ for $t \geq 0$ and square-integrable. These all such that the restriction $\phi 1_{[0, t]}$ lies in $L^2([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_p, dt \otimes P)$ and

$$\mathbb{E}(\int_0^\infty \phi(s)^2 ds) = \int_\Omega \int_0^\infty \phi(s, \omega)^2 ds P(d\omega) < \infty.$$

The subspace of simple predictable processes, i.e.

$$u(t) = \sum_{i=0}^{n-1} F_i 1_{]t_i, t_{i+1}]}(t)$$

with F_i a F_{t_i} -measurable and $\mathbb{E}(F_i^2) < \infty$ (hence $F_i \in L^2(\Omega, \mathcal{F}_{t_i}, P)$, $n \geq 0$) and $0 = t_0 \leq t_1 \leq \dots \leq t_n$, is denoted by \mathcal{E} . On \mathcal{E} we define the Ito-Integral by

$$I(u) = \int_0^\infty u(t) dB_t := \sum_{i=0}^{n-1} F_i (B_{t_{i+1}} - B_{t_i})$$

Theorem 3.1.6. *The mapping $I : \mathcal{E} \rightarrow L^2(\Omega, \mathcal{F}, P)$ is a well defined isometry and $\mathbb{E}(I(u)) = 0$ for all $u \in \mathcal{E}$, i.e.*

$$\mathbb{E}(I(u)I(v)) = \mathbb{E}(\int_0^\infty u(t)v(t)dt).$$

Definition 3.1.3. *The closure of \mathcal{E} in $L^2(\mathbb{R}_{\geq 0} \times \Omega, \mathcal{F}_p, dt \otimes P)$ is denoted by $L^2(B)$. The unique continuous extension $I : L^2(B) \rightarrow L^2(\Omega)$ is called the stochastic integral with respect to Brownian motion or the Ito integral, we denote*

$$\int_0^\infty u(t) dB_t := I(u).$$

In particular we have for all $u, v \in L^2(B)$

$$\begin{aligned} \mathbb{E}(\int_0^\infty u(t)dB_t) &= 0 \\ \mathbb{E}(\int_0^\infty u(t)dB_t \int_0^\infty v(t)dB_t) &= \mathbb{E}(\int_0^\infty u(t)v(t)dB_t) \end{aligned}$$

The definite integral is defined in the following way

$$\int_0^t u(s)dB_s := \int_0^t u(s)1_{[0,t]}(s)dB_s$$

for $t \geq 0$, which is well defined since the processes u are progressively measurable.

Theorem 3.1.7. *The vector space \mathcal{E} is dense in $L^2(\mathbb{R}_{\geq 0} \times \Omega, \mathcal{F}_p, dt \otimes P)$.*

Corollar 3.1.1. *The process $M_t := \int_0^t u(s)dB_s$ has a version with continuous paths.*

Remark 3.1.1. *All simple processes $u \in \mathcal{E}$ are progressively measurable by definition. Given $u \in L^2(\mathbb{R}_{\geq 0} \times \Omega, \mathcal{F}_p, dt \otimes P)$ with continuous paths, then we can approximate the process $(u_s 1_{[0,t]})_{s \geq 0}$ by elements in \mathcal{E} of the form*

$$u_s^n := \sum_{i=0}^{2^n-1} u_{\frac{ti}{2^n}} 1_{\frac{ti}{2^n}, \frac{t(i+1)}{2^n}}(s),$$

which have the property

$$u_s^n \rightarrow u_s$$

surely by continuity and $u^n \rightarrow u 1_{[0,t]}$ in $L^2(\mathbb{R}_{\geq 0} \times \Omega, \mathcal{F}_p, dt \otimes P)$ by dominated convergence. Therefore we can calculate the Ito integral for processes with continuous paths by

$$\lim_{n \rightarrow \infty} \int_0^t u_s^n dB_s = \lim_{n \rightarrow \infty} \sum_{i=0}^{2^n-1} u_{\frac{ti}{2^n}} (B_{\frac{t(i+1)}{2^n}} - B_{\frac{ti}{2^n}}).$$

Theorem 3.1.8. *Let $f \in C_b^2([0, T] \times \mathbb{R}, \mathbb{R})$ (bounded with bounded derivatives) be given. Suppose $u, v \in L^2(\mathbb{R}_{\geq 0} \times \Omega, \mathcal{F}_p, dt \otimes P)$. Let \mathbf{X} be the continuous process*

$$\mathbf{X}_t := \mathbf{X}_0 + \int_0^t u(s)dB_s + \int_0^t v(s)ds,$$

then

$$\begin{aligned} f(t, \mathbf{X}_t) &= f(0, \mathbf{X}_0) + \int_0^t \frac{\partial}{\partial x} f(s, \mathbf{X}_s) ds + \int_0^t \frac{\partial}{\partial x} f(s, \mathbf{X}_s) u(s) dB_s + \int_0^t \frac{\partial}{\partial x} f(s, \mathbf{X}_s) v(s) ds + \\ &\quad \frac{1}{2} \int_0^t \frac{\partial^2}{\partial x^2} f(s, \mathbf{X}_s) u^2(s) ds. \end{aligned}$$

In short notation this is written as,

$$\begin{aligned} d\mathbf{X}_t &:= u(t)dB_t + v(t)dt, \\ df(t, \mathbf{X}_t) &= \frac{\partial}{\partial x} f(t, \mathbf{X}_t) dt + \frac{\partial}{\partial x} f(t, \mathbf{X}_t) d\mathbf{X}_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} f(t, \mathbf{X}_t) dt, \end{aligned}$$

where the process is given through $d\langle \mathbf{X} \rangle_t := u^2(t)dt$.

3.2 Bachelier Hedging

In order to come up with a Hedging formula we need to redefine our model. From now on we call - given a Brownian motion $(B_t)_{0 \leq t \leq T}$ - the (discounted) price process

$$\begin{aligned} S_t &= S_0 + \sigma^B B_t, \\ dS_t &= \sigma^B dB_t \end{aligned}$$

for $0 \leq t \leq T$, where we call σ^B the absolute Bachelier volatility. We can calculate - by the previous methods - the price of a European Call Option in this model

$$\begin{aligned} C^B(S_0, T) &:= \mathbb{E}((S_T - K)_+) \\ &= \int_{\frac{K-S_0}{\sigma^B \sqrt{T}}}^{\infty} (S_0 + \sigma^B \sqrt{T}x - K) \phi(x) dx \\ &= (S_0 - K) \Phi\left(\frac{S_0 - K}{\sigma^B \sqrt{T}}\right) + \sigma^B \sqrt{T} \phi\left(\frac{S_0 - K}{\sigma^B \sqrt{T}}\right). \end{aligned}$$

By simple differentiation we check that

$$\frac{\partial}{\partial T} C^B(S_0, T) = \frac{(\sigma^B)^2}{2} \frac{\partial^2}{\partial S_0^2} C^B(S_0, T)$$

for $T > 0$ and $S_0 \in \mathbb{R}$. Ito's Formula for the stochastic process $(C^B(S_t, T - t))_{0 \leq t \leq T}$ then yields the following result:

$$\begin{aligned} C^B(S_T, 0) &= C^B(S_0, T) - \int_0^T \frac{\partial}{\partial T} C^B(S_t, T - t) dt \\ &\quad + \int_0^T \frac{\partial}{\partial S_0} C^B(S_t, T - t) dS_t \\ &\quad + \frac{1}{2} \int_0^T \frac{(\sigma^B)^2}{2} \frac{\partial^2}{\partial S_0^2} C^B(S_t, T - t) dt \\ &= C^B(S_0, T) + \int_0^T \frac{\partial}{\partial S_0} C^B(S_t, T - t) dS_t. \end{aligned}$$

Consequently we can build a self-financing portfolio at initial wealth $C^B(S_0, T)$, which replicates the European Call Contract.

3.3 Black-Scholes Hedging

We take a Black-Scholes model with volatility $\sigma > 0$, drift μ and today's price S_0 ,

$$S_t = S_0 \exp(\mu t - \frac{\sigma^2}{2} t + \sigma B_t)$$

for $0 \leq t \leq T$. Furthermore we assume an interest rate $r \geq 0$, we obtain the discounted price process

$$\begin{aligned} \tilde{S}_t &= S_0 \exp(\mu t - rt - \frac{\sigma^2}{2} t + \sigma B_t), \\ d\tilde{S}_t &= \tilde{S}_t(\mu - r)dt + \tilde{S}_t \sigma dB_t. \end{aligned}$$

We calculate - like in the Bachelier model - the price of a European Call Option, hence

$$C(S_0, T, r) = S_0 \Phi\left(\frac{\ln \frac{S_0}{K} + (\frac{1}{2}\sigma^2 + r)T}{\sigma\sqrt{T}}\right) - e^{-rT} K \Phi\left(\frac{\ln \frac{S_0}{K} - (\frac{1}{2}\sigma^2 - r)T}{\sigma\sqrt{T}}\right).$$

As before we see that for $T > 0$ and $S_0 > 0$,

$$\frac{\partial}{\partial T} C(S_0, T, r) = \frac{\sigma^2 S_0^2}{2} \frac{\partial^2}{\partial S_0^2} C^B(S_0, T, r).$$

In order to calculate the Hedging Portfolio, we apply Ito's Formula to the process $(C(\tilde{S}_t, T-t))_{0 \leq t \leq T}$,

$$\begin{aligned} C(S_T, 0, r) &= C(S_0, T, r) - \int_0^T \frac{\partial}{\partial T} C(\tilde{S}_t, T-t, r) dt \\ &\quad + \int_0^T \frac{\partial}{\partial S_0} C(\tilde{S}_t, T-t, r) d\tilde{S}_t \\ &\quad + \frac{1}{2} \int_0^T \frac{\sigma^2 S_t^2}{2} \frac{\partial^2}{\partial S_0^2} C(\tilde{S}_t, T-t, \tilde{S}_t) dt \\ &= C^B(S_0, T) + \int_0^T \frac{\partial}{\partial S_0} C(\tilde{S}_t, T-t) d\tilde{S}_t. \end{aligned}$$

Chapter 4

Stochastic Preliminaries

This chapter is meant to recall the basic ideas and definitions of stochastic processes (in particular martingales, filtrations,...) which are needed for better understanding of the following chapters. The most is taken from [Teichmann] and [Wertz].

4.1 Stochastic Processes

Ω is a finite, non-empty set. A subset $\mathcal{F} \subset 2^\Omega$ of the power set is called a σ - *algebra* if it is closed under countable unions, closed under taking complements and contains Ω . A *probability* measure is a map

$$P : \mathcal{F} \rightarrow \mathbb{R}$$

such that

- for all mutually disjoint sequences $(A_n)_{n \geq 0} \in \mathcal{F}$ we have $P(\bigcup_{n \geq 0} A_n) = \sum_{n \geq 0} P(A_n)$.
- $P(\Omega) = 1$.

The set of all probability measures on (Ω, \mathcal{F}) is $\mathbb{R}(\Omega)$. For a probability space (Ω, \mathcal{F}, P) we shall always assume that \mathcal{F} is complete with respect to P , i.e. for every set $B \subset \Omega$, such that $B \subset A$ with $A \in \mathcal{F}$ and $P(A) = 0$, we have $B \in \mathcal{F}$. Such sets are called *P-nullsets*. The *P-completeness* assumption allows to deal with maps, which are defined up to sets of probability 0. A random variable $X : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is a measurable map, i.e. the inverse image of Borel measurable sets is measurable in \mathcal{F} . The set of measurable maps is denoted by $L^0(\Omega, \mathcal{F}, P)$, a measurable map takes constant values on each atom of the measurable space ($X(A)$ for A an atom in \mathcal{F}). Given a set $M \subset 2^\Omega$, there is a smallest σ -algebra containing M denoted by $\sigma(M)$. If the set M is given as inverse image of Borel subsets from \mathbb{R} via a map $\mathbf{X} : \Omega \rightarrow \mathbb{R}$, then we write for the σ -algebra $\sigma(\mathbf{X})$. This is the smallest σ -algebra such that \mathbf{X} is measurable $\mathbf{X} : (\Omega, \sigma(\mathbf{X})) \rightarrow \mathbb{R}$. On

$$L^p(\Omega, \mathcal{F}, P) = \{ \mathbf{X} \in L^0 \text{ such that } \mathbb{E}(|\mathbf{X}|^p) < \infty \}$$

we consider L^p - *convergence* due to $\mathbf{X}_n \rightarrow \mathbf{X}$ if $\mathbb{E}(|\mathbf{X}_n - \mathbf{X}|^p) \rightarrow 0$ as $n \rightarrow \infty$ for each $p \geq 1$, which coincides with L^0 on finite probability spaces. $L^2((\Omega, \mathcal{F}, P))$ is an euclidean vector space with scalar product

$$\langle X, Y \rangle = \mathbb{E}(XY)$$

for $X, Y \in L^2(\Omega, \mathcal{F}, P)$ and $L^\infty(\Omega, \mathcal{F}, P)$ is the set of bounded random variables with the *supremum norm*, which is equal to L^0 . A sequence $(X_n)_{n \geq 0}$ is said to converge P -almost surely to X if $X_n \rightarrow X$ outside a null set as $n \rightarrow \infty$.

Lemma 4.1.1. *Let (Ω, \mathcal{F}, P) be a probability space and $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ be subalgebras, then*

- *for all $X \in L^1(\Omega, \mathcal{F}, P)$ we have $\mathbb{E}(X|\mathcal{G}) = X$.*
- *the conditional expectation $\mathbb{E}(\cdot|\mathcal{G})$ is a linear map on $L^p(\Omega, \mathcal{F}, P)$ and an orthogonal projection as map from $L^2(\Omega, \mathcal{F}, P)$ to $L^2(\Omega, \mathcal{G}, P)$.*
- *the conditional expectation is a positive map, i.e. $\mathbb{E}(X|\mathcal{G}) \geq 0$ if $X \geq 0$.*
- *the tower law holds, $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ for all $X \in L^1(\Omega, \mathcal{F}, P)$.*
- *Jensen's inequality holds, i.e. for convex $\phi : \mathbb{R} \rightarrow \mathbb{R}$ we have $\phi(\mathbb{E}(X|\mathcal{G})) \leq \mathbb{E}(\phi(X)|\mathcal{G})$ for $X \in L^1(\Omega, \mathcal{F}, P)$.*
- *for all $Z \in L^1(\Omega, \mathcal{G}, P)$ we have*

$$\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$$

$$X \in L^1(\Omega, \mathcal{F}, P).$$

- *If X is independent of \mathcal{G} then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.*
- *Let $X, Y \in L^1(\Omega, \mathcal{F}, P)$ be given and take σ -algebras $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}$. Assume $A \in \mathcal{G}_1 \cap \mathcal{G}_2$ such that $X = Y$ on A and $A \cap \mathcal{G}_1 = A \cap \mathcal{G}_2$ (in this case the σ -algebras $\mathcal{G}_1, \mathcal{G}_2$ are called *locally on A equal σ -algebras*). Then $\mathbb{E}(X|\mathcal{G}_1) = \mathbb{E}(Y|\mathcal{G}_2)$ on A .*
- *We denote the atoms of \mathcal{G} by $A(\mathcal{G})$, then we have*

$$\mathbb{E}(X|\mathcal{G}) = \sum_{A \in \mathcal{A}(\mathcal{G}), P(A) > 0} \frac{\mathbb{E}(1_A X)}{P(A)} 1_A.$$

Consequently the conditional expectation is well-defined up to sets of probability 0.

4.2 Filtrations, Stopping Times, Adapted Processes and Martingales

A *filtration* on (Ω, \mathcal{F}, P) is a finite sequence of σ -algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_N \subset 2^\Omega$, where $\mathcal{F} = \mathcal{F}_N$ for $N \geq 1$. Filtrations represent increasing degrees of information on probability space.

- A *stochastic process* on (Ω, \mathcal{F}, P) is a sequence of \mathbb{R}^d -valued random variables $(X_n)_{0 \leq n \leq N}$.
- A stochastic process $(X_n)_{0 \leq n \leq N}$ is called *adapted* to a filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$ if X_n is \mathcal{F}_n -measurable for $0 \leq n \leq N$. In this case we shall often speak of an adapted process if there is no doubt about the filtration.

- A stochastic process $(H_n)_{0 \leq n \leq N}$ is called *predictable* if H_0 is constant and H_n is \mathcal{F}_{n-1} -measurable for $1 \leq n \leq N$. A predictable process is certainly adapted.
- Let $(H_n)_{0 \leq n \leq N}, (X_n)_{0 \leq n \leq N}$ be stochastic processes, then the Riemannian sum for $0 \leq n \leq N$ is

$$(H \cdot X)_n := \sum_{i=1}^n H_i(X_i - X_{i-1}).$$

$$(H \cdot X)_n = H_N X_N - H_0 X_0 - (X_{*-1} \cdot H)_n,$$

where $(X_{*-1})_n := X_{n-1}$ for $1 \leq n \leq N$ and $(X_{*-1})_0 = X_0$.

Definition 4.2.1. Let (Ω, \mathcal{F}, P) be a probability space and $(\mathcal{F}_n)_{0 \leq n \leq N}$ a filtration, then a sequence of \mathbb{R}^d -valued random variables $(M_n)_{0 \leq n \leq N}$ is called a *martingale* if

$$\mathbb{E}(M_n | \mathcal{F}_m) = M_m$$

for $0 \leq m \leq n \leq N$. The sequence is called a *submartingale* (*supermartingale*) if $\mathbb{E}(M_n | \mathcal{F}_m) \geq M_m$ ($\mathbb{E}(M_n | \mathcal{F}_m) \leq M_m$ respectively) for $0 \leq m \leq n \leq N$.

Definition 4.2.2. Let (Ω, \mathcal{F}, P) be a probability space and $(\mathcal{F}_n)_{0 \leq n \leq N}$ a filtration, then a random variable $\tau : \Omega \rightarrow \mathbb{N}_{\geq 0}$ is called a *stopping time* if

$$\{\tau \leq n\} \in \mathcal{F}_n$$

for $0 \leq n \leq N$. Let M be an adapted process and τ a stopping time with $\tau \leq N$ almost surely, then

$$M_\tau(\omega) := M_{\tau(\omega)}(\omega)$$

for $\omega \in \Omega$. The stopped process M^τ is defined for any stopping time τ

$$M_n^\tau := M_{\tau \wedge n}$$

for $0 \leq n \leq N$. The stopped σ -algebra

$$\mathcal{F}_\tau := \{A \in \mathcal{F} \text{ such that } A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for } 0 \leq n \leq N\}$$

contains all informations from the stopping time τ .

Important examples of stopping times are **hitting times** τ_B^X of a set $B \in \mathcal{B}$ by a stochastic process X :

$$\tau_B^X = \inf \{t \geq 0 : X_t \in B\}.$$

Lemma 4.2.1. Let $\tau, \eta, \eta_1, \eta_2, \dots$ be stopping times, then

- $\sum_{i=1}^k \eta_i, \inf \eta_i, \sup \eta_i, \limsup \eta_i, \liminf \eta_i$ are stopping times.
- If $\tau \leq \eta$ bounded by N , then $\mathcal{F}_\tau \subset \mathcal{F}_\eta$ and the sets $\{\tau \leq \eta\}$ and $\{\eta \leq \tau\}$ lie in $\mathcal{F}_{\tau \wedge \eta} = \mathcal{F}_\tau \cap \mathcal{F}_\eta$.

- If τ, η bounded by N , then $\{\tau \leq \eta\} \cap \mathcal{F}_\tau \subset \mathcal{F}_{\tau \wedge \eta}$.
- If τ bounded by N , then $\mathcal{F}_\tau = \mathcal{F}_n$ on $\{\tau = n\}$, i.e. $\{\tau = n\} \cap \mathcal{F}_\tau = \{\tau = n\} \cap \mathcal{F}_n$.
- Let τ be bounded by N . If $A \in \mathcal{F}_\tau$, then $\tau_A = \tau 1_A + N 1_{A^c}$ is a stopping time.
- Given an adapted sequence of random variables M and τ, η stopping times bounded by N , M_τ is \mathcal{F}_τ -measurable and $\mathbb{E}(M_\tau | \mathcal{F}_n)$ is $\mathcal{F}_{\tau \wedge \eta}$ -measurable.

Theorem 4.2.1 (Doob's optional sampling). *Let (Ω, \mathcal{F}, P) be a finite probability space and $(\mathcal{F}_n)_{n \leq n \leq N}$ a filtration. Let $(M_n)_{n \leq n \leq N}$ be an adapted process.*

1. *If M is a martingale, then for every predictable process $(H_n)_{n \leq n \leq N}$ the stochastic integral $(H \cdot M)$ is a martingale. In particular $\mathbb{E}((H \cdot M)_N) = 0$.*
2. *If the stochastic integral $(H \cdot M)$ satisfies*

$$\mathbb{E}((H \cdot M)_N) = 0$$

for every predictable process H , then M is a martingale.

3. *If for all stopping times $\tau \leq N$*

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$$

holds, then M is a martingale, and if M is a martingale, then

$$\mathbb{E}(M_\tau) = \mathbb{E}(M_0).$$

4. *If M is a martingale, then for all stopping times $\tau \leq \eta \leq N$ almost surely we have*

$$\mathbb{E}(M_\tau | \mathcal{F}_\eta) = M_\eta.$$

More generally we have that for any two stopping times $\tau, \eta \leq N$

$$\mathbb{E}(M_\tau | \mathcal{F}_\eta) = M_{\tau \wedge \eta}.$$

A equivalent measure $Q \tilde{P}$ is a measure such that for all $A \in \mathcal{F}$, $P(A) = 0$ if and only if $Q(A) = 0$. Given any measure Q on Ω the Radon-Nikodym derivative $\frac{dQ}{dP}$ is a random variable, such that for all $Z \in L^0(\Omega, \mathcal{F}, P)$,

$$\begin{aligned} \mathbb{E}_Q(Z) &= \mathbb{E}_P(Z \frac{dQ}{dP}). \\ \frac{dQ}{dP}(A) &= \frac{Q(A)}{P(A)} \end{aligned}$$

for all atoms $A \in \mathcal{A}(\mathcal{F})$ with $P(A) > 0$. A measure Q is called absolutely continuous with respect to P if for all $A \in \mathcal{F}$ with $P(A) = 0$, $Q(A) = 0$. In the generic case of $P(\omega_i) > 0$ for all $i = 1, \dots, |\Omega|$ every measure Q is absolutely continuous with respect to P .

Lemma 4.2.2 (change of measure). *Let (Ω, \mathcal{F}, P) be a probability space with filtration $(\mathcal{F}_n)_{0 \leq n \leq N}$ and Q be an equivalent probability measure such that*

$$\frac{dQ}{dP} = X$$

for some $X \in L^1(\Omega, \mathcal{F}, P)$. Then $Q|_{\mathcal{F}_n}$ are equivalent probability measures on $(\Omega, \mathcal{F}_n, P|_{\mathcal{F}_n})$ for $n = 0, \dots, N$ and

$$\frac{dQ_n}{dP_n} =: X_n$$

is a P -martingale. Here P_n denotes the restriction of P to \mathcal{F}_n . Furthermore

$$\mathbb{E}_P(X|\mathcal{F}_n) = X_n$$

and

$$\mathbb{E}_Q(Y|\mathcal{F}_n) = \frac{1}{X_n} \mathbb{E}_P(YX|\mathcal{F}_n)$$

for all $Y \in L^1(\Omega, \mathcal{F}, Q)$. In particular $X_n > 0$ almost surely with respect to P .

Let M be an adapted process, then the set of measures Q equivalent to P such that M is a Q -martingale by $\mathcal{M}^e(M)$. The set of measures Q absolutely continuous with respect to P such that M is a Q -martingale is denoted by $\mathcal{M}^a(M)$. The set $\mathcal{M}^a(M)$ is always a closed set and it is the convex hull of linearly independent measures Q_1, \dots, Q_m , since it is polygonal as intersection of hyperlanes. If $\mathcal{M}^a(M)$ contains more than one element, the measures Q_i are not equivalent to the measure P if $P(\omega_i) > 0$ for $i = 1, \dots, |\Omega|$.

Lemma 4.2.3. *Let M be a d -dimensional adapted process with $\mathcal{M}^e(M) \neq \emptyset$, then for any $Q \in \mathcal{M}^a(M)$ and $A \in \mathcal{F}_k$, we can define a probability measure Q^A on (A, \mathcal{F}_k) for $Q(A) \neq 0$ via*

$$Q^A(B) = \frac{Q(B)}{Q(A)}$$

for $B \in \mathcal{F}^A = \{B \in \mathcal{F}, B \subset A\}$. The process $M^A := (M_n|_A)_{n=k, \dots, N}$ is a Q^A -martingale with respect to the filtration (\mathcal{F}_k^A) . Given a martingale measure R on (Ω, \mathcal{F}_k) for $M^k := (M_n)_{n=0, \dots, k}$ and S_A martingale measure for M^A for every $A \in \mathcal{A}(\mathcal{F}_k)$, the probability measure

$$Q^{R, (S_A)}(B) = \sum_{A \in \mathcal{A}(\mathcal{F}_k)} R(A) S^A(B \cap A)$$

is a martingale measure for M .

Corollar 4.2.1. *Let M be a d -dimensional adapted process with $\mathcal{M}^e(M) \neq \emptyset$, then for $0 \leq k \leq N$ and $A \in \mathcal{A}(\mathcal{F}_k)$*

$$\mathcal{M}^a(M^A) = \{Q^A \text{ for } Q \in \mathcal{M}^a(M)\}$$

and

$$\mathcal{M}^a(M^k) = \{Q_k \text{ for } Q \in \mathcal{M}^a(M)\}.$$

Chapter 5

Monte Carlo Valuation

5.1 Basics

The Monte Carlo method is based on the law of large numbers. We consider a random variable with law $\mu(dx)$ and we generate a sequence of independent trials, X_1, \dots, X_n, \dots with common distribution μ . Applying the law of large numbers, we can assert that if f is a μ -integrable function, i.e. the integral $\int f(x)\mu(dx)$ exists,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{1 \leq n \leq N} f(X_n) = \int f(x)\mu(dx).$$

- **Strong law of Large Numbers:** This law shows that the mean of $f(X_i)$ for a large sample converges to the expected value of f under an integrability condition. If X_i are *i.i.d.* (independent and identically distributed) and if $\int_{\mathbb{R}} f(x)\mu(dx) < \infty$ then

$$\frac{1}{N} \sum_{i=1}^N f(X_i) \xrightarrow{a.s.} \int_{\mathbb{R}} f(x)\mu(dx)$$

- **Central Limit Theorem:** We note for $d = 1$, $\int f(x)\mu(dx) = m$ and $\text{var}(f) = \int_{\mathbb{R}} (f - m)^2 \mu(dx) < \infty$ we have

$$\frac{1}{\sqrt{\text{var}(f)}} \frac{1}{\sqrt{N}} \sum_{i=1}^N (f(X_i) - m) \xrightarrow{\text{in distribution}} \mathcal{N}(0, 1)$$

for $N \rightarrow \infty$.

$$\frac{1}{N} \sum_{i=1}^N (f(X_i) - m) \approx \mathcal{N}\left(0, \frac{1}{N} \text{var}(f)\right).$$

5.2 Overview of Monte Carlo Simulation in finance

5.2.1 Stochastic differential equations

A stochastic differential equation (SDE) with general drift and volatility terms has the form

$$dS_t = a(S, t)dt + b(S, t)dW_t,$$

which is simply a shorthand for the more formal integral equation

$$S_t = S_0 + \int_0^t a(S_{t'}, t') dt' + \int_0^t b(S_{t'}, t') dW_{t'}.$$

The stochastic term $b(S_t, t)dW_t$ models the uncertain, unpredictable events which influence asset price, interest rates, exchange rates and other financial variables. Monte Carlo simulation estimates the expectation $\mathbb{E}[f(S_T)]$ by simulationg a finite number of future paths, and averaging over that finite set.

The scalar SDE which underlies the Black-Scholes model is geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where r is the constant risk-free interest rate and σ is a constant volatility. Using Itô calculus, the corresponding SDE for $X \equiv \log S$ is

$$dX_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t,$$

which may be integrated subject to initial conditions $X(0) = X_0 = \log S_0$ to give

$$X_t = X_0 + (r - \frac{1}{2}\sigma^2)T + \sigma W_T,$$

and hence

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W_T\right).$$

Performing a change of variables, the expected value of some financial payoff $P = f(S_T)$ can be expressed as

$$V \equiv \mathbb{E}[f(S_T)] = \int f(S) p_s(S) dS,$$

where

$$p_s(S) = \left(\frac{\partial S}{\partial W}\right)^{-1} p_w = \frac{1}{S\sigma\sqrt{2\pi T}} \exp\left(-\frac{1}{2}\left(\frac{\log(S/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)^2\right)$$

is the log-normal probability density function for S_T .

5.2.2 Monte Carlo sampling and numerical solution of SDEs

The Monte Carlo estimate for the same case of geometric Brownian motion is

$$\hat{V} = M^{-1} \sum_m f(S^{(m)}),$$

where

$$S^{(m)} = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma W^{(m)}\right),$$

with the M values $W^{(m)}$ being independent samples from the probability distribution for W_T . The expected value for the Monte Carlo estimate \hat{V} is equal to the true expected

value V . Because the samples are independent, the variance of the estimate is equal to $M^{-1}\mathbb{V}[f(S_T)]$, where $\mathbb{V}[f(S_T)]$ is the variance of a single sample. Thus the root-mean-square sampling error is proportional to $M^{-1/2}$.

In the general case in which the SDE can not be explicitly integrated, the time interval $[0, T]$ is split into N timesteps of size $h = T/N$, and $S^{(j)}$ is replaced by the approximation $\hat{S}_N^{(j)}$, the value at the end of the N^{th} timestep in a numerical approximation of the SDE. The simplest approximation is the Euler discretisation,

$$\hat{S}_{n+1} = \hat{S}_n + a(\hat{S}_n, t_n)h + b(\hat{S}_n, t_n)\Delta W_n$$

in which the Brownian increments ΔW_n are all independent Normal variables with zero mean and variance h . Each path involves N random inputs W_n , to produce the one random output \hat{S}_N .

5.2.3 Evaluating sensitivities

If $V(\theta)$ represents the expected value of the payoff $f(S_T)$ for a particular value of one of the input parameters (e.g. S_0 , r or σ in the case of geometric Brownian motion) then for the purposes of hedging and risk analysis one often wants to evaluate $\partial V/\partial\theta$ and $\partial^2 V/\partial^2\theta$. The simplest approach is to use a finite difference approximation,

$$\begin{aligned}\frac{\partial V}{\partial\theta} &\approx \frac{V(\theta + \Delta\theta) - V(\theta - \Delta\theta)}{2\Delta\theta}, \\ \frac{\partial^2 V}{\partial^2\theta} &\approx \frac{V(\theta + \Delta\theta) - 2V(\theta) + V(\theta - \Delta\theta)}{(\Delta\theta)^2}.\end{aligned}$$

The drawback is, that it is computationally expensive. Since we divide by a small quantity $2\Delta\theta$ the error can increase, so care must be taken in the choice of $\Delta\theta$. In the case of a scalar SDE for which one can compute a terminal probability distribution, the second approach, the Likelihood Ratio Method (LRM), obtains

$$\frac{\partial V}{\partial\theta} = \int f(S) \frac{\partial p_S}{\partial\theta} dS = \int f(S) \frac{\partial(\log p_S)}{\partial\theta} p_S dS,$$

where p_S is known. That it does not require the differentiation of $f(S)$ is the great advantage. Second derivatives can also be computed using the LRM approach. Differentiating twice leads to

$$\frac{\partial^2 V}{\partial\theta^2} = \int f(S) \frac{\partial^2 p_S}{\partial\theta^2} dS,$$

where the so-called "score" g is defined as

$$g = p_s^{-1} \frac{\partial^2 p_s}{\partial\theta^2} = \frac{\partial^2 \log p_s}{\partial\theta^2} + \left(\frac{\partial \log p_s}{\partial\theta} \right)^2.$$

In the same case of a scalar SDE with a terminal probability distribution, the third approach of pathwise sensitivities gives

$$\frac{\partial V}{\partial\theta} = \int \frac{\partial f}{\partial S} \frac{\partial S_T}{\partial\theta} p_w dW = \mathbb{E} \left[\frac{\partial f}{\partial S} \frac{\partial S_T}{\partial\theta} \right],$$

with the partial derivative $\partial S_T / \partial \theta$ being evaluated at fixed W . It yields to

$$\frac{\partial \hat{S}_{n+1}}{\partial \theta} = \left(1 + \frac{\partial a}{\partial S} h + \frac{\partial b}{\partial S} \Delta W_n \right) \frac{\partial \hat{S}_n}{\partial \theta} + \frac{\partial a}{\partial \theta} h + \frac{\partial b}{\partial \theta} \Delta W_n.$$

Solving this gives $\partial \hat{S}_N / \partial \theta$ from which we get the Monte Carlo estimate for the first order sensitivity as the average of the sensitivity of M independent paths,

$$\frac{\partial \hat{V}}{\partial \theta} = M^{-1} \sum_m \frac{\partial f}{\partial S}(\hat{S}_N^{(m)}) \frac{\partial \hat{S}_N^{(M)}}{\partial \theta}.$$

The second order is easily obtained by differentiating a second time.

The key limitation of the pathwise sensitivity approach is the differentiability required of the drift and volatility functions, and the payoff function $f(S)$. If the payoff function is suitable, then the pathwise sensitivity estimator has a much lower variance than the LRM estimator, and so it is computationally more efficient.

Chapter 6

American Options

6.1 Basic information on American options

We consider discrete time, i.e. $t = 0, \dots, T$ with $T \in \mathbb{N}$, $T \geq 0$, $\mathbb{T} = \{0, \dots, T\}$.

We have a model $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{T}}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, for a financial market with primary securities $S^i = (S_t^i)_{t \in \mathbb{T}}$, $i = 1, \dots, d$. We suppose that there exist a numeraire pair (N, Q_N) , which ensures that the primary securities are free of arbitrage. We also assume that the model is complete.

A European contingent claim with maturity T is a non-negative random variable X_T , which is \mathcal{F}_T -measurable. In cases of practical relevance, $X_T = F(S_u^i, u \leq T, i = 1, \dots, d)$. Let $X = (X_t)_{t \leq T, t \in \mathbb{T}}$ be a family of non-negative random variables with each X_t being \mathcal{F}_t -measurable with $\mathbb{E}_{Q_N}(X_t/N_t) < \infty$. We say X is an **American option with maturity T** if the holder has the right to receive the non-negative payoff X_t if she decides to exercise the option at time $t \leq T$. The holder is allowed to exercise once at any time $t \in \mathbb{T}$ before or equal maturity T .

Definition 6.1.1. *An American contingent claim is a non-negative adapted process $X = (X_t)_{t=0, \dots, T}$ on the filtered space $(\Omega, (\mathcal{F}_t)_{t=0, \dots, T})$.*

An exercise strategy for an American contingent claim X is an \mathcal{F} -measurable variable τ taking values in $\{0, \dots, T\}$. The payoff obtained by using τ is equal to

$$X_\tau(\omega) := X_{\tau(\omega)}(\omega), \omega \in \Omega.$$

An American put option on the i^{th} asset and with strike $c > 0$ pays the amount

$$C_t^{put} := (K - S_t^i)^+,$$

if it is exercised at time t . The payoff at time t of the corresponding American call option is given by

$$C_t^{call} := (S_t^i - K)^+.$$

We are interested in the fair price, $V_t(X)$, of the American option X at time t provided the option has not been exercised yet. Obviously, we have the inequality

$$V_t(X) \geq X_t, t \in \mathbb{T}, t \leq T.$$

We start by approaching the problem from an intuitive point of view by restricting ourselves to the situation of discrete time, $\mathbb{T} = \{0, 1, \dots\}$. Clearly, at maturity T , we have $V_T(X) = X_T$. At time $T - 1$ we can either decide to exercise and receive X_{T-1} , or we wait until maturity T to get the final pay-off X_T . At time $T - 1$ the value of the European option to receive X_T at time T is given by

$$N_{T-1} \mathbb{E}_{Q_N}(X_T/N_T | \mathcal{F}_{T-1}).$$

This implies

$$\begin{aligned} V_{T-1}(X) &= \max(X_{T-1}, N_{T-1} \mathbb{E}_{Q_N}(X_T/N_T | \mathcal{F}_{T-1})) \\ &= \max(X_{T-1}, N_{T-1} \mathbb{E}_{Q_N}(V_T(X)/N_T | \mathcal{F}_{T-1})). \end{aligned}$$

Going further backwards, at time $T - 2$ the holder can decide to exercise and receive X_{T-2} or, alternatively, to wait until $T - 1$ to "get something worth" $V_{T-1}(X)$. Finally we end up with the recursion

$$V_T(X) = X_T \tag{6.1}$$

$$V_t(X) = \max(X_t, N_t \mathbb{E}_{Q_N}(V_{t+1}(X)/N_{t+1} | \mathcal{F}_t)), t = T - 1, \dots, 0. \tag{6.2}$$

The main result of this section justifies that, in discrete time, the fair price of the American option X at time 0 can be defined as the quantity $V_0(X)$, which we obtain as a result of the above backward induction algorithm.

Corollar 6.1.1 (American option price as optimal stopping value). *We have*

$$V_0(X) = N_0 \sup_{\tau \leq T} \mathbb{E}_{Q_N} \left(\frac{X_\tau}{N_\tau} \right),$$

where the supremum is taken over all stopping times τ bounded by T .

Definition 6.1.2. Any stopping time $\tau_0 \leq T$, satisfying

$$\mathbb{E}_{Q_N} \left(\frac{X_{\tau_0}}{N_{\tau_0}} \right) = \sup_{\tau, \tau \leq T} \mathbb{E}_{Q_N} \left(\frac{X_\tau}{N_\tau} \right)$$

is called an **optimal stopping time** for the interval $[0, T]$.

Theorem 6.1.1 (Snell envelope). *Let $\tilde{V}(X)$ be defined by the backward recursion (6.1), (6.2):*

$$\begin{aligned} \tilde{V}_T(X) &= \tilde{X}_T \\ \tilde{V}_t(X) &= \max \left(\tilde{X}_t, \mathbb{E}_{Q_N}(\tilde{V}_{t+1}(X) | \mathcal{F}_t) \right), t = T - 1, \dots, 0. \end{aligned}$$

1. $\tilde{V}(X)$ is a Q_N -supermartingale, with $\tilde{V}(X) \geq \tilde{X}$. It is even the smallest supermartingale with this property, the so-called **Snell envelope** of \tilde{X} .
2. Define $\tau_0 = \inf \left\{ t : \tilde{V}_t(X) = \tilde{X}_t \right\} = \inf \{ t : V_t(X) = X_t \}$. Then τ_0 is a stopping time and $\left(\tilde{V}_{t \wedge \tau_0}(X) \right)_{t \in \mathbb{T}}$ is a Q_N -martingale.

3. A stopping time $\tau_0, \tau_0 \leq T$, is optimal for the interval $[0, T]$ if and only if $V_{\tau_0}(X) = X_{\tau_0}$ and $\left(\tilde{V}_{t \wedge \tau_0}(X)\right)_{t \in \mathbb{T}}$ is a Q_N -martingale.

4. The snell envelope $\tilde{V}(X)$ of \tilde{X} is characterized by the equality

$$\tilde{V}_t(X) = \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbb{E}_{Q_N}(\tilde{X}_\tau | \mathcal{F}_t),$$

where the supremum is over all stopping times τ taking values in $[t, T]$.

Proof. 1. The definition of $\tilde{V}(X)$ directly yields $\tilde{V}(X) \geq \tilde{X}$ and $\tilde{V}_t(X) \geq \mathbb{E}_{Q_N}(\tilde{V}_{t+1}(X) | \mathcal{F}_t)$, i.e., \tilde{V} is a supermartingale. Let Y be another supermartingale with $Y \geq \tilde{X}$. Then

$$\begin{aligned} Y_T &\geq \tilde{X}_T = \tilde{V}_T(X) \\ Y_{T-1} &\geq \mathbb{E}_{Q_N}(Y_T | \mathcal{F}_{T-1}) \geq \mathbb{E}_{Q_N}(\tilde{V}_T(X) | \mathcal{F}_{T-1}) \\ Y_{T-1} &\geq \max\left(\tilde{X}_{T-1}, \mathbb{E}_{Q_N}(\tilde{V}_T(X) | \mathcal{F}_{T-1})\right) = \tilde{V}_{T-1}(X) \end{aligned}$$

Induction finally proves the assertion.

2. Since $\tau_0 = \inf\{t : \tilde{V}_t(X) - \tilde{X}_t \in \{0\}\}$, it is a stopping time. For the martingale property of $\left(\tilde{V}_{t \wedge \tau_0}(X)\right)_{t \geq 0}$ we have to show that for $A \in \mathcal{F}_t$

$$\int_A \tilde{V}_{t \wedge \tau_0}(X) dQ_N = \int_A \tilde{V}_{(t+1) \wedge \tau_0}(X) dQ_N.$$

This is equivalent to

$$\int_{A \cap \{t < \tau_0\}} \tilde{V}_t(X) dQ_N = \int_{A \cap \{t < \tau_0\}} \tilde{V}_{t+1}(X) dQ_N.$$

But on $\{t < \tau_0\}$ we have $\tilde{V}_t(X) > \tilde{X}_t$ and thus $\tilde{V}_t(X) = \mathbb{E}_{Q_N}(\tilde{V}_{t+1}(X) | \mathcal{F}_t)$ on this set, which finishes the proof of (2).

3. From $\tilde{X}_t \leq \tilde{V}_t(X)$ and the fact that $\tilde{V}(X)$ is a supermartingale, for every stopping time τ_0 we get the chain of inequalities

$$\mathbb{E}_{Q_N} \tilde{X}_{\tau_0} \leq \mathbb{E}_{Q_N} \tilde{V}_{\tau_0}(X) \leq \mathbb{E}_{Q_N} \tilde{V}_{\tau_0 \wedge t}(X) \leq \tilde{V}_0(X) = \sup_{\tau, \tau \leq T} \mathbb{E}_{Q_N} \tilde{X}_\tau.$$

Now, if τ_0 is optimal for $[0, T]$ all " \leq " signs reduce to equalities. This implies that $V_{\tau_0}(X) = X_{\tau_0}$ and $(\tilde{V}_{\tau_0 \wedge t})$ is a martingale. The opposite direction is straightforward. \square

$\tau_0 = \inf\{t : V_t(X) = X_t\}$ is the smallest optimal stopping time.

Lemma 6.1.1 (Largest optimal stopping time). *The largest optimal stopping time is given by*

$$\tau_m = \max\{t \leq T : \tilde{A}_t = 0\},$$

where $\tilde{V}(X) = \tilde{M} - \tilde{A}$ is the Doob decomposition of $\tilde{V}(X)$.

The majority of models for the primary securities S^1, \dots, S^n are such that $(\tilde{S}^1, \dots, \tilde{S}^n)$ is an n -dimensional Markov process under the measure Q_N . In this situation the pricing problem of an American option $X = (X_t)$ often reduces to a backward induction on the "states" of $(\tilde{S}^1, \dots, \tilde{S}^n)$.

If, at each time t , the payoff X_t of the American option is of the form

$$\tilde{X}_t = P(t, \tilde{S}_t^1, \dots, \tilde{S}_t^n)$$

with some payoff function $P(t, x_1, \dots, x_n)$ then by Snell Envelope (Point 4) and the Markov Property

$$\begin{aligned} \tilde{V}_t(X) &= \operatorname{ess\,sup}_{\tau, t \leq \tau \leq T} \mathbb{E}_{Q_N}(\tilde{X}_\tau | \mathcal{F}_t) \\ &= \operatorname{ess\,sup}_{\tau, t \leq \tau \leq T} \mathbb{E}_{Q_N}(\tilde{X}_\tau | \sigma(\tilde{S}_t^1, \dots, \tilde{S}_t^n)) \\ &=: F(t, \tilde{S}_t^1, \dots, \tilde{S}_t^n). \end{aligned}$$

This means that the value $\tilde{V}_t(X)$ at time t is some measurable function of the "state" of $(\tilde{S}^1, \dots, \tilde{S}^n)$ at time t . This yields an interesting insight. From the Snell Envelope we know that the smallest optimal stopping time is

$$\tau_0 = \inf \left\{ t : \tilde{V}_t(X) = \tilde{X}_t \right\} = \inf \left\{ t : F(t, \tilde{S}_t^1, \dots, \tilde{S}_t^n) = P(t, \tilde{S}_t^1, \dots, \tilde{S}_t^n) \right\}.$$

The set

$$D = \{(t, x_1, \dots, x_n) : F(t, x_1, \dots, x_n) > P(t, x_1, \dots, x_n)\}$$

is called the continuation region and its complement, D^c , is the exercise region. As long as the process $(t, \tilde{S}_t^1, \dots, \tilde{S}_t^n)$ stays in the continuation region D it is not optimal to exercise. The first time this process enters the exercise region is the smallest optimal stopping time. If there is just one stochastic security, as is the case in the Black & Scholes or in the binomial model, the exercise region can often be characterized by a so-called exercise boundary $B(t, x)$, meaning that D is of the form $D = \{(t, x) : x < B(t, x)\}$.

Example: In case of the Binomial tree model,

$$S_t^1 = (1+r)^t = N_t, S_t^2 = S_{t-1}^2 \xi_t, \text{ with } \xi_t \in \{u, d\},$$

and

$$Q_N(\xi_t = u) = p, Q_N(\xi_t = d) = 1 - p,$$

the backward induction algorithm for pricing an American option with payoff $X_t = P(t, S_t^2)$ is

$$\begin{aligned} V_T(X) &= P(T, S_T^2) =: F(T, S_T^2) \\ V_t(X) &= \max\left(P(t, S_t^2), \frac{1}{1+r} (F(t+1, S_t^2 \cdot u)p + (F(t+1, S_t^2 \cdot d)(1-p)))\right) \\ &=: F(t, S_t^2), t = T-1, \dots, 0. \end{aligned}$$

6.2 Hedging American options

We assume the market to be complete.

(V_t) is the value process of an American option described by the sequence (X_t) , by the system

$$\begin{cases} V_T = X_T \\ V_t = \max(X_t, S_t^0 \mathbb{E}^*(V_{t+1}/S_{t+1}^0 | \mathcal{F}_t)) \quad \forall t \leq T-1. \end{cases}$$

Thus, the sequence (\tilde{V}_t) defined by $\tilde{V}_t = V_t/S_t^0$ (discounted price of the option) is the snell envelope, under P^* , of the sequence \tilde{X}_t . We deduce that

$$\tilde{V}_t = \sup_{t \leq \tau \leq T} \mathbb{E}^*(\tilde{X}_\tau | \mathcal{F}_t)$$

and consequently

$$V_t = S_t^0 \sup_{t \leq \tau \leq T} \mathbb{E}^*\left(\frac{X_\tau}{S_\tau^0} | \mathcal{F}_t\right).$$

We can write

$$\tilde{V}_t = \tilde{M}_t - \tilde{A}_t,$$

where (\tilde{M}_t) is a P^* -martingale and (\tilde{A}_t) is an increasing predictable process, null at 0. Since the market is complete, there is a self-financing strategy ϕ such that

$$V_t(\phi) = S_t^0 \tilde{M}_t,$$

for $t = 0, \dots, T$. For the sequence $(\tilde{V}_T(\phi))$ is a P^* -martingale, we have

$$\begin{aligned} \tilde{V}_T(\phi) &= \mathbb{E}^*(\tilde{V}_T(\phi) | \mathcal{F}_t) \\ &= \mathbb{E}^*(\tilde{M}_T | \mathcal{F}_t) \\ &= \tilde{M}_t, \end{aligned}$$

and consequently

$$\tilde{V}_t = \tilde{V}_t(\phi) - \tilde{A}_t$$

Therefore

$$V_t = V_t(\phi) - A_t,$$

where $A_t = S_t^0 \tilde{A}_t$. From the previous equality, it is obvious that the writer of the option can hedge herself perfectly: once he receives the premium $V_0 = V_0(\phi)$, she can generate a wealth equal to $V_t(\phi)$ at time t which is bigger than V_t and a fortiori X_t . What is the optimal date to exercise the option? The date of exercise is to be chosen among all the stopping times. For the buyer of the option, there is no point in exercising at time t when $V_t > X_t$, because he would trade an asset worth V_t (the option) for an amount X_t (by exercising the option). Thus an optimal date τ of exercise is such that $V_\tau = X_\tau$. On the other hand, there is no point in exercising after the time

$$\tau_{max} = \inf \{j, A_{j+1} \neq 0\}$$

(which is equal to $\inf \{j, A_{j+1} \neq 0\}$) because, at that time, selling the option provides the holder with a wealth $V_{\tau_{max}} = V_{\tau_{max}}(\phi)$ and, following the strategy ϕ from that time, he

creates a portfolio whose value is strictly bigger than the option's at time $\tau_{max} + 1, \tau_{max} + 2, \dots, T$. Therefore we set as a second condition, $\tau \leq \tau_{max}$, which allows us to say that \tilde{V}^τ is a martingale. As a result, optimal dates of exercise are optimal stopping times for the sequence (\tilde{X}_t) , under probability P^* . (If he hedges himself using the strategy ϕ as defined above and if the buyer exercises at time τ which is not optimal, then $V_\tau > X_\tau$ or $A_\tau > 0$. In both cases, the writer makes a profit $V_\tau(\phi) - X_\tau = V_\tau + A_\tau - X_\tau$, which is positive.)

Chapter 7

Monte Carlo Valuation of American Options

7.1 Price of an American option

We consider discrete time, i.e. $t = 0, \dots, T$ with $T \in \mathbb{N}$, $T \geq 0$, $\mathbb{T} = \{0, \dots, T\}$.

As seen in chapter 6, an American option is a contract, which can be exercised at any time prior the expiration date T . We have the expiration date $T > 0$, and suppose two adapted processes $(r_t)_{0 \leq t \leq T}$ and $(X_t)_{0 \leq t \leq T}$, which are defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$. The first process is the spot rate of interest, and the second defines the amount paid to the holder of an American option at the moment of exercise, the payoff $X_t = \max(K - S_t, 0)$ and K is the strike price. The discounted exercise value of the option is $\tilde{X}_t = X_t/B_t$, where B_t is the value at time t of 1 unit invested in a riskless money market account at $t = 0$. The time 0 value of an American option is given by

$$Y_0^* \equiv \sup_{0 \leq \tau \leq T} \mathbb{E}(\tilde{X}_\tau),$$

where \tilde{X}_t is the discounted exercise value of the option. The supremum is taken over all the possible stopping times τ less then the expiration date T , and the expectation is taken over the risk-neutral probability density. (We assume that $Y_0^* < \infty$, also that for some $p > 1$, $\sup_{0 \leq t \leq T} |\tilde{X}_t| \in L^p$, and also that the paths of X are right continuous). This assumptions lead to the *Snell envelope* process

$$Y_t^* \equiv \operatorname{ess\,sup}_{t \leq \tau \leq T} \mathbb{E} \left[\tilde{X}_\tau | \mathcal{F}_t \right],$$

where Y_0^* is a supermartingale, and has a Doob-Meyer decomposition

$$Y_t^* = Y_0^* + M_t^* - A_t^*,$$

where M^* is a martingale vanishing at zero, and A^* is a previsible integrable increasing process, also vanishing at zero.

Theorem 7.1.1. $Y_0^* = \inf_{M \in H_0^1} \mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t) \right]$, where H_0^1 is the space of martingales M for which $\sup_{0 \leq t \leq T} |M_t| \in L^1$, and such that $M_0 = 0$. The infimum is attained by taking $M = M^*$.

Proof. We assume that $Y_0^* < \infty$, also that for some $p > 1$, $\mathbb{E} \left[\left(\sup_s |\tilde{X}_s| \right)^p \right] < \infty$.

First, we note that Y^* is dominated by the L^p -bounded martingale $\tilde{x}_t \equiv \mathbb{E}(\sup_s |\tilde{X}_s| | \mathcal{F}_t)$, and so $\sup_{0 \leq t \leq T} |M_t^*| \leq \sup_{0 \leq t \leq T} \tilde{x}_t + |Y_0^*| + A_T$, proving that M^* is indeed in H_0^1 . Returning to the definition of Y_0^* in the theorem, we have for any $M \in H_0^1$ that

$$\begin{aligned} Y_0^* &= \sup_{0 \leq \tau \leq T} \mathbb{E} \tilde{X}_\tau \\ &= \sup_{0 \leq \tau \leq T} \mathbb{E} [\tilde{X}_\tau - M_\tau] \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t) \right]; \end{aligned}$$

taking the infimum over all $M \in H_0^1$ proves that Y_0^* is bounded by the right-hand side of the expression in our theorem. On the other hand, since $\tilde{X}_t \leq Y_t^* = Y_0^* + M_t^* - A_t^*$,

$$\begin{aligned} \inf_{M \in H_0^1} \mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t) \right] &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t^*) \right] \\ &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} (Y_t^* - M_t^*) \right] \\ &= \mathbb{E} \left[\sup_{0 \leq t \leq T} (Y_0^* - A_t^*) \right] \\ &= Y_0^*, \end{aligned}$$

as claimed. \square

Remark 7.1.1. *Davis and Karatzas (1994) proved that $\mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t + M_T^* - M_t^*) \right] = Y_0^*$ in the present notation.*

Remark 7.1.2. *Of course a conditional form of Theorem 7.1.1 holds too.*

Starting from this theorem we can describe a method of pricing the American option: We pick a suitable martingale M , and evaluate by simulation the expectation $\mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t) \right]$. Obtaining the optimal martingale is of course a task of a similar complexity to finding the optimal exercise policy, but often simple martingales can be found, that provide remarkably good and quick bounds.

7.2 Hedging

As seen in the previous theorem, it is necessary to find a good martingale $M \in H_0^1$ for a good approximation of the price Y_0^* of the American option.

Holding M fixed, we have an upper bound for Y_0^* - namely, the mean of the random variable

$$\eta \equiv \sup_{0 \leq t \leq T} (\tilde{X}_t - M_t). \quad (7.1)$$

Let us set $\eta_t \equiv \mathbb{E}(\eta | \mathcal{F}_t)$ for the martingale closed on the right by η , so that $\eta \equiv \eta_T$. We now think of the martingale M as the discounted gains-from-trade process of some portfolio; thus if we start with wealth η_0 and used this portfolio, our discounted wealth at time t would just be $\eta_0 + M_t$. Now (7.1) implies the inequality for any $t \in [0, T]$

$$\tilde{X}_t \leq \eta + M_t$$

and taking conditional expectation given \mathcal{F}_t and rearranging gives the key inequality

$$\tilde{X}_t \leq \mathbb{E}[\eta_T - \eta_0 | \mathcal{F}_t] + (M_t + \eta_0).$$

The interpretation is illuminating: the (discounted) amount \tilde{X}_t , which has to be paid out to the holder of the option if exercised at time t , is almost hedged by the (discounted) value of our portfolio. The shortfall is at worst

$$\mathbb{E}[\eta_T - \eta_0 | \mathcal{F}_t]^+ \leq \mathbb{E}[(\eta_T - \eta_0)^+ | \mathcal{F}_t].$$

It will be highly desirable that the quantity $\mathbb{E}[|\eta_T - \eta_0|]$, which bounds the mean of the shortfall should be small.

(The perfect situation is $M = M^*$, where we have a zero bound on the shortfall). It could be that a given martingale M gives a good bound on the price of the option (i.e. $\mathbb{E}(\eta) - Y_0^*$ is small), while having a large shortfall, and therefore being less desirable for hedging.

The dual problem can be interpreted in a very concrete way: we are trying to choose the hedging strategy to minimize the lookback value of $\tilde{X} - M$. In any Markovian example, we would have that \tilde{X} is a function time and a Markov process Z , and the solution to be such that at any time the optimal hedging portfolio should be a function of t, Z_t , and $\sup_{u \leq t} (\tilde{X}_u - M_u)$.

We may also use a candidate martingale M to suggest an exercise policy, namely, to stop when first \tilde{X} exceeds the value of hedging policy:

$$\tau_M \equiv \inf \left\{ t \in [0, T] : M_t + \eta_0 \leq \tilde{X}_t \right\} \wedge T.$$

In the case where the hedging policy was optimal, this stopping rule would also be optimal.

7.3 Arbitrary martingale - Theory

For an arbitrary martingale M_t , we define a dual function F_t^M as

$$\frac{F_t^M}{B_t} = \mathbb{E} \left[\sup_{t \leq \tau \leq T} (\tilde{X}_\tau - M_\tau) \right] + M_t.$$

The dual problem is to minimise the dual function at time 0 over all martingales M_t . Let U_0 denote the optimal value of the dual problem, so that

$$U_0 = \inf_M F_0^M = \inf_M \mathbb{E} \left[\sup_{0 \leq \tau \leq T} (\tilde{X}_\tau - M_\tau) \right] + M_0.$$

The main result is that the optimal values of the dual and the primal problems coincide.

Proof. For an arbitrary (adapted) martingale M_t , we have

$$\begin{aligned} Y_0 &= \sup_{0 \leq \tau \leq T} \mathbb{E} [\tilde{X}_\tau] = \sup_{0 \leq \tau \leq T} \mathbb{E} [\tilde{X}_\tau - M_\tau + M_\tau] \\ &= \sup_{0 \leq \tau \leq T} \mathbb{E} [\tilde{X}_\tau - M_\tau] + M_0 \\ &\leq \mathbb{E} \left[\sup_{0 \leq \tau \leq T} (\tilde{X}_\tau - M_\tau) \right] + M_0 \end{aligned}$$

where the second equality follows from the optional sampling theorem. Since M_t was an arbitrary martingale, the inequality will hold after taking the infimum, implying $Y_0 \leq U_0$. The supermartingale property of Y_t/B_t allows a Doob-Meyer decomposition of the form

$$\frac{Y_t}{B_t} = M_t - A_t,$$

where M_t is a martingale, and A_t is a predictable integrable increasing process with $A_0 = 0$. Using this martingale in the dual problem gives

$$U_0 \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\frac{X_t}{B_t} - \frac{Y_t}{B_t} - A_t \right) \right] + Y_0.$$

Since $Y_t \geq X_t$ for all t , we conclude that $Y_0 \geq U_0$. Therefore $Y_0 = U_0$ when M_t is taken to be the martingale component of the discounted American option price process Y_t/B_t . When the optimal martingale is used, both the expectation and the variance of the lookback option are equal to zero, i.e.

$$\mathbb{E} \left[\sup_{0 \leq \tau \leq T} (\tilde{X}_\tau - M_\tau) \right] = 0$$

and

$$\text{var} \left[\sup_{0 \leq \tau \leq T} (\tilde{X}_\tau - M_\tau) \right] = 0.$$

□

7.4 Implementation

An upper bound on the price of an American option can be constructed by evaluating the dual function using an arbitrary martingale M_t ,

$$F_0^M = \mathbb{E} \left[\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t) \right] + M_0 \geq Y_0.$$

The choice of the martingale is very important, because the tightness of the upper bound will depend on this. A suitable choice of M_t is one that approximates the martingale component of the discounted price of the American option. A good place to start is to consider the martingale part of the corresponding European option. L. C. G. Rogers

reports in his paper from results with errors in the region of 1-2 percent. The martingale can be refined by including a weighting coefficient, which is determined by a numerical optimisation procedure on an initial subsample of size N_1 , followed by a simulation of N_2 paths. To choose an appropriate martingale is the main difficulty in this approach. The Rogers' approach is quick to calculate, but requires a careful choice of martingale.

We assume that the asset price S satisfies the lognormal risk-neutral process $dS = rSdt + \sigma SdW$, where σ is the volatility. For each sample path, the process is simulated at Q equally spaced discrete times. There are two natural ways of defining a martingale based on the European put:

$$M_t^A = B_t^{-1} V_{euro} [S_t, K, \sigma, T - t, r] - V_{euro} [S_0, K, \sigma, T, r], \quad (7.2)$$

$$M_t^B = V_{euro} [S_t, K, \sigma, T - t, r] - B_t V_{euro} [S_0, K, \sigma, T, r]. \quad (7.3)$$

where V_{euro} is the Black-Scholes value of the European put, and we have assumed a deterministic short-rate r . David Lamper and Sam Howison show in their paper [11], that M^A is a better martingale than M^B , because M^B has a greater value of the mean pathwise maximum (pwm).

7.4.1 Adding additional European martingales

This section shows the way, how to create the martingale from multiple European contracts. We consider

$$M = \sum_{i=0}^n \lambda_i \pi_i^{euro}$$

where π_i^{euro} is the martingale part of a European option calculated using expression (7.2). We take π_0^{euro} to be a European option with the same contract parameters as the American option, and by adding n extra martingales we seek to improve the martingale and reduce the upper bound. The λ_i 's are found by numerical optimisation to minimise the sum of the pathwise maximum.

The addition of an extra contract within the martingale leads to an improved upper bound for the price of an American option (see [11]).

7.4.2 Analytic approximation

As discussed in 7.3, the value of the optimal martingale at $t = 0$ is equal to the value of the American option itself. This implies if an optimal martingale (or one close to it) has been found, then evaluating this at time zero will provide a good approximation to the American option price (the values obtained using this method are no longer upper bounds, since we are just evaluating the martingale at $t = 0$). Having determined a martingale suitable for a specific value of S_0 using Monte-Carlo simulation, we then have an analytic expression for the martingale at $t = 0$ and can calculate M as a function of S . This provides an approximation to the American option value at asset prices near S_0 , without having to perform a full Monte-Carlo simulation each time the asset price changes slightly. In this manner, it is possible to provide a very quick approximation to the American option price over a range of S values once the martingale has been determined.

Chapter 8

Numerical Example

8.1 American Put on a Single Asset

In this section, an numerical example, the standard American put, is discussed.

The calculations were performed throughout in Scilab (www.scilab.org, Scilab is a free Matlab "clone"), for the code take a look to the appendix.

The first step is to simulate sample paths for the Brownian motion.

To get a good result, the choice of the martingale is very important, in this example, it is the martingale part of the corresponding European option.

For the implementation mentioned in the appendix following martingale was used:

$$M_t^A = B_t^{-1} V_{euro} [S_t, K, \sigma, T - t, r] - V_{euro} [S_0, K, \sigma, T, r],$$

the discounted value of the corresponding European put, started when the option goes in the money, at the first time that S_t falls below the strike K .

The example handles the American put on a single log-Brownian asset, whose price process is given by

$$S_t = S_0 \exp(\sigma W_t + (r - \sigma^2/2)t),$$

with r denoting as usual the riskless rate of interest, assumed constant, and σ denoting the constant volatility.

Holding M fixed, we have an upper bound for the price of the American option:

$$\sup_{0 \leq t \leq T} (\tilde{X}_t - M_t)$$

The Monte Carlo values, calculated with this method, are within 1% to the true American price. The true American prices are taken from the paper of Ait-Sahalia and Carr (1997).

S_0	European (true)	American (true)	American (MC)	Error	Error %
80	20.6893	21.6059	21.6782	0.0723	0.33%
85	17.3530	18.0374	18.0553	0.0179	0.10%
90	14.4085	14.9187	14.9600	0.0413	0.28%
95	11.8516	12.2314	12.2179	0.0135	0.11%
100	9.6642	9.9458	9.9374	0.0084	0.08%
105	7.8183	8.0281	8.0329	0.0048	0.06%
110	6.2797	6.4352	6.4097	0.0255	0.40%
115	5.0113	5.1265	5.1236	0.0029	0.06%
120	3.9759	4.0611	4.0487	0.0124	0.31%
Average Error					0.19%

Table 8.1: Simulation Prices of Standard American Puts Using the Discounted Value of the Corresponding European Put Parameter values: $K = 100, r = 0.06, T = 0.5, \sigma = 0.4$

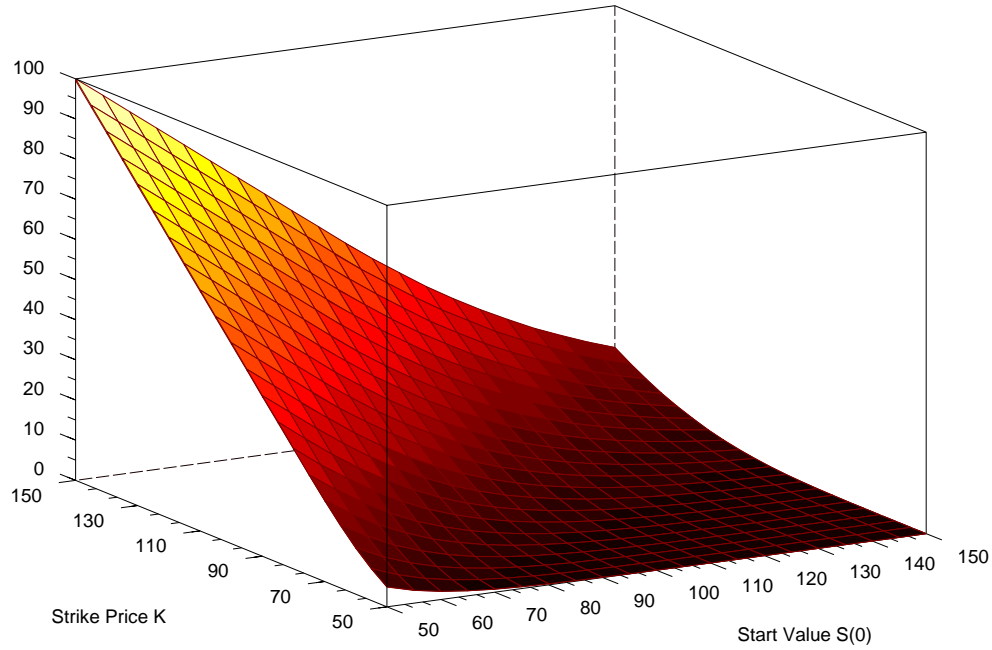


Figure 8.1: American Put Price calculated with Monte Carlo method, Parameter values: $r = 0.06, T = 0.5, \sigma = 0.4$

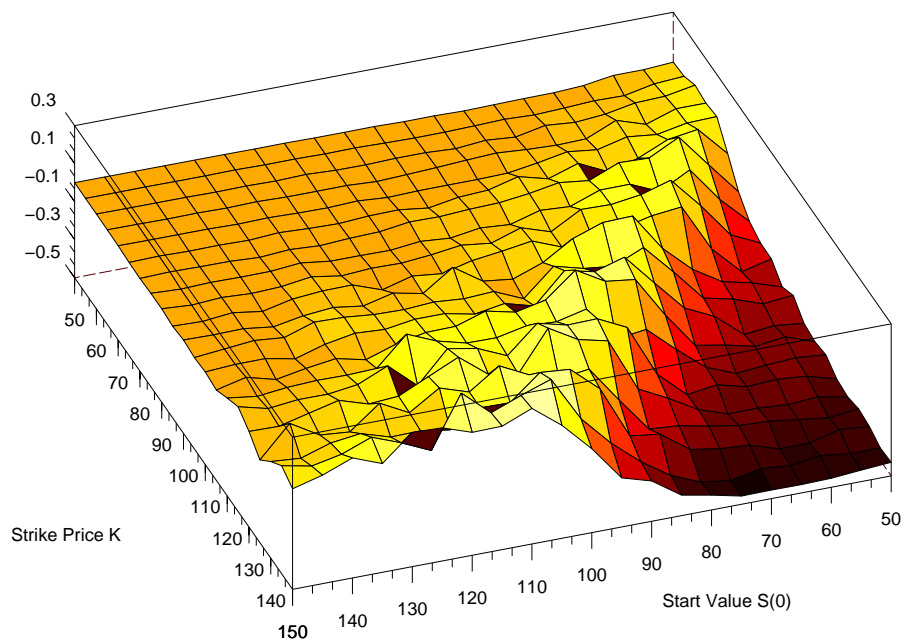


Figure 8.2: Absolute Error between Monte Carlo method and true American Put Price

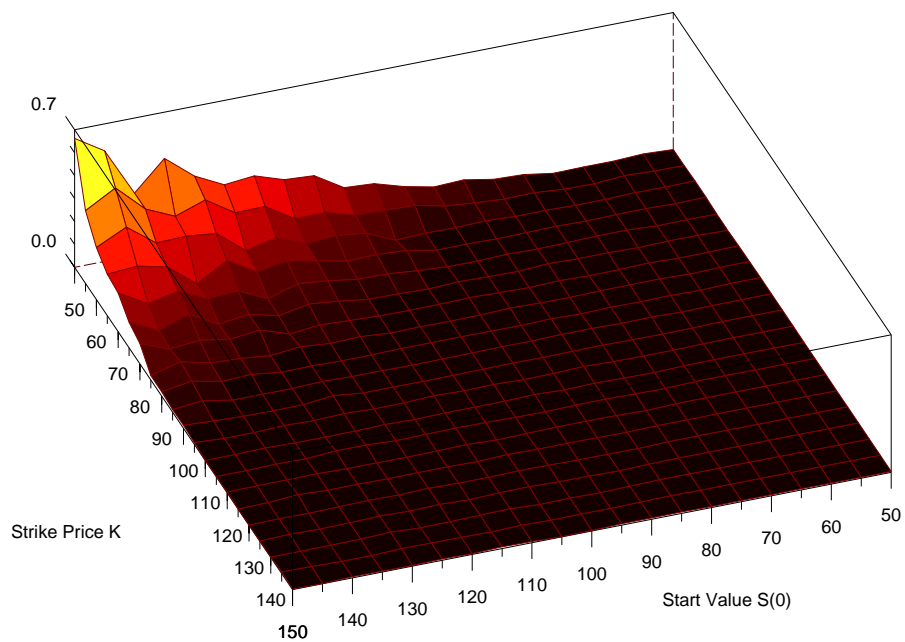


Figure 8.3: Error in % between Monte Carlo method and true American Put Price

Chapter 9

Conclusions

In my Diploma thesis I presented a method for evaluating the prices of American-style options by a direct simulation approach, based on a dual characterization of the optimal exercise problem.

The method involves the choice of a suitable Lagrangian hedging martingale, which can be thought of as a hedging strategy designed to minimize the lookback value of the excess of the option exercise value over the chosen hedging strategy. A choice of the hedging strategy gives bounds on expected shortfall (evaluated through simulation).

The quality of the upper bound depends on the martingale chosen. Even using very primitive choices for the hedging martingales the results of the Monte Carlo simulation are remarkably good, usually in range 1-2%.

This shows that the Monte Carlo method is very efficient and reliable for pricing American-style options, under the prediction to have a "good" martingale.

Chapter 10

Appendix

10.1 Scilab Code

```
//-----european_put_price.sci-----
//Input:  start value S0e, strike price Ke, sigma, time Te, interest rate re,
//        time steps Ne
//Return: price of the european put calculated with an Brownian motion model
//-----

function [p] = european_put_price(S0e,Ke,sigmae,Te,re,Ne)

    s = S0e * exp((re-sigmae*sigmae/2)*Te + sigmae * sqrt(Te)
        * rand(1,Ne, 'normal'))
    payoff = (-1)*min(s - Ke, 0)
    p = exp(-re*Te)*sum(payoff)/Ne

endfunction

//-----american_put_price.sci-----
//Input:  start value S0, strike price K, sigma, time T, interest rate r,
//        time steps N
//Return: price of the American put calculated with an Brownian motion model
//-----

function [ap] = american_put_price(S0,K,sigma,T,r)

    NumberofPaths = 200;
    N = 10; //NumberofSteps

    nue_Summe=0;
    e_p_S0=european_put_price(S0,K,sigma,T,r,1000000);
```

```
for i=1:NumberofPaths,

nue=0;
St=S0;
Wt=0;
for t=(T/N):(T/N):T,
    if t==0 then
        Wt=0
    else
        Wt=Wt + sqrt(T/N)* rand(1,'normal');
    end
    St=S0 * exp((r-(sigma*sigma)/2)*t + sigma * Wt);
    Mt=(exp(-r*t)*european_put_price(St,K,sigma,T-t,r,10000)-e_p_S0);
    Zt=exp(-r*t)*(-1)*min(St -K, 0);
    nue=max(Zt-Mt,nue);
end

nue_Summe=nue_Summe + nue;

end

ap=(nue_Summe/NumberofPaths);

endfunction

//american_put_price(80,100,0.4,0.5,0.06);
//american_put_price(85,100,0.4,0.5,0.06);
//american_put_price(90,100,0.4,0.5,0.06);
//american_put_price(95,100,0.4,0.5,0.06);
//american_put_price(100,100,0.4,0.5,0.06);
//american_put_price(105,100,0.4,0.5,0.06);
//american_put_price(110,100,0.4,0.5,0.06);
//american_put_price(115,100,0.4,0.5,0.06);
//american_put_price(120,100,0.4,0.5,0.06);

//-----American_European_put_price.sci-----
//Input:  start value S0, strike price K, sigma, time T, interest rate r,
//        time steps N, steps in stock price M
//Return: price of the American put
//-----

function[A_E_p] = American_European_put(S,K,v,T,r)

//----- Enter parameters
M=50; //no. of steps in stock price
N=100; //no of time steps
```

```
tol=0.00000001; //tolerance for SOR convergence
ExerciseType="a"; //set 'a' for American / 'e' for European
//-----

dt=T/N; //time step size
smax=S*2;
ds=smax/M; //stock price step size

payoffvec=zeros(M+1,1); //vector to store option price grid

for j=0:M,
    payoffvec(j+1)=max(K-j*ds,0);
end

payvec=zeros(M-1,1);
for j=1:M-1,
    payvec(j)=K-j*ds;
end

svec=0:ds:smax;
svec=svec';

vnew=payoffvec;
ds2=ds*ds;

//Start rollback of time steps
for ti=1:N,
    vold=vnew;
    v_previteration=vnew*0;
    err=1;
    nits=1;
    while (err > tol)
        nits=nits+1;
        for i=2:M,
            si=svec(i);
            ai=0.5*v*v*si*si;
            bi=r*si;
            ci=-r;
            denom=1/dt+2*ai/ds2-ci;
            tmp1=vold(i)/dt;
            tmp2=ai*(vnew(i+1)+vnew(i-1))/ds2;
            tmp3=bi*(vnew(i+1)-vnew(i-1))*0.5/ds;
            vnew(i)=(tmp1+tmp2+tmp3)/denom;
        end
        err=vnew-v_previteration;
        err=err'*err;
        v_previteration=vnew;
    end
end
```

```
end
if ExerciseType=="a",
    vnew(2:M)=max(vnew(2:M),payvec);
end;
end

svec=0:ds:smax;
option_price_sor=interp1(svec',vnew,S)
A_E_p=option_price_sor

endfunction

//-----Plot_in_3d_S0_K_MC.sce-----
// Plots the American Put Price, calculated with Monte Carlo
//-----
Steps=5;

for S0=50:Steps:150
    for K=50:Steps:150
        price(S0,K)=american_put_price(S0,K,0.4,0.5,0.06);
    end
end

S0=[50:Steps:150];
K=[50:Steps:150];

C=hotcolormap(32);
xset("colormap",C)
xset("hidden3d",30)
xbasc()

plot3d1(S0,K,price(S0,K),leg="Start value@Strike price@American Put Price");

//-----Plot_in_3d_S0_K_Diff.sce-----
// Plots absolute error between Monte Carlo method
// and true American Price
//-----
Steps=5;

for S0=50:Steps:150
    for K=50:Steps:150
        price(S0,K)=american_put_price(S0,K,0.4,0.5,0.06)
            - American_European_put(S0,K,0.4,0.5,0.06);
    end
end

S0=[50:Steps:150];
```

```
K=[50:Steps:150];

C=hotcolormap(32);
xset("colormap",C)
xset("hidden3d",30)
xbasc()

plot3d1(S0,K,price(S0,K),leg="Start value@Strike price@American Put Price");

//-----Plot_in_3d_S0_K_Diff_Proz.sce-----
// Plots error in % between Monte Carlo method
// and true American Price
//-----
Steps=5;

for S0=50:Steps:150
    for K=50:Steps:150
        Ap=American_European_put(S0,K,0.4,0.5,0.06)
        price(S0,K)=(abs(american_put_price(S0,K,0.4,0.5,0.06) - Ap))/Ap;
    end
end

S0=[50:Steps:150];
K=[50:Steps:150];

C=hotcolormap(32);
xset("colormap",C)
xset("hidden3d",30)
xbasc()

plot3d1(S0,K,price(S0,K),leg="Start value@Strike price@American Put Price");
```


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