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THE MEHLER KERNEL

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Abstract

In the framework of studying noncommutative gauge field theory a model with an additional oscillator-like term in the action is discussed, which already like Grosse and Wulkenhaar showed is renormalizable to all orders in the case of a scalar $\phi{*}{*}4$ model. To be able to write down the propagator of such models one needs roughly speaking the inverse of the operator "$p^2 + x^2$". This is the Mehler kernel, which will be discussed in this diploma thesis. We will not only give a full derivation and different properties of the latter, but we will also explicitly execute some loop calculations in a special model where the Mehler kernel is needed.

Zusammenfassung

Im Rahmen der Untersuchung von BRST - invarianten Eichfeldmodellen, die einmal ultimativ zu einer zu allen Ordnungen renormierbaren nichtkommutativen Eichfeldtheorie führen sollen, wird unter anderem ein Modell mit einem Oszillator - ähnlichen Zusatzterm diskutiert, welcher bereits wie von Grosse und Wulkenhaar gezeigt im skalaren Fall zu einer vollständig renormierbaren Theorie geführt hat. Um den Propagator solcher Modelle anschreiben zu können braucht man grob gesprochen das Inverse des Operators "p^2+x^2". Das ist der Mehler Kern, welcher in dieser Diplomarbeit untersucht werden soll. Es werden nicht nur eine vollständige Herleitung und diverse Eigenschaften desselben gegeben, sondern auch in einem speziellen Modell explizite Loop Rechnungen durchgeführt wo der Mehler Kern gebraucht wird.

Contents

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Chapter 1

Introduction

1.1 Introduction to Noncommutative Quantum Field Theory (NCQFT)

The world we know extends over 61 scales if we use powers of ten, going from the Planck length $1.6 * 10^{-35}$m to the radius of the visible universe which is about $4.4 * 10^{26}$m. Altough we do not observe galaxies that far away, the WMAP data indicates that the universe is really at least 80% that big [1][2].

Among those 61 scales of the universe, only about 11 were relatively known to ancient Greeks and Romans 2000 years ago [2]. The largest colliders can nowadays reach the energy of 1 TeV, which corresponds to a length of $2 * 10^{-19}$m. This length will be reduced by ≈2 more scales by the data we will hopefully receive from the LHC in CERN in 2-3 years. Up from this length to the size of the universe we can observe our world. Unfortunately the very small scales are not observable yet, and anything could happen between $1.6 * 10^{-35}$m and $2 * 10^{-19}$m. The tasks of the theoreticians is to construct models which possibly are capable of giving us an insight into this unknown world.
Serious effort has been made and I would like to mention some of the theories which claim to explain the world:
1) String theory
2) Quantum loop gravity
3) Noncommutative field theory (NCQFT)[3][4][5]
4) Others

ad 1) String theory is indeed the most popular field in theoretical physics and the leading candidate for a quantum theory of gravitation, most of the

theoreticians (excluding the solid state physicists) work on it.
In this theory the fundamental objects are the so called strings, whose vibration modes create the known particles. All 4 fundamental forces (electromagnetic, weak, strong and gravity) are explained in a particular way, altough (or fortunately?) supersymmetry (SUSY) is needed in this context. Some problems arise when looking at the vacuum of the theory, the so-called vacuum landscape, where the real vacuum level is hidden somewhere. Furthermore, the mathematical structure of string theory is complicated up to the point where it may become depressing. For instance great effort is needed to put the string theory at two loops on some rigorous footing, and three loops seem almost hopeless.
From the NCQFT point of view string theory has a nice aspect in a certain framework, namely when one goes into a special coordinate system (we sit down on a D-brane) with a background B-field present, noncommutativity is suddenly encountered [6][7]. For interested readers in String theory we recommend to read [8] or [9] as an introduction, or [10][11] for experienced readers.

ad 2) Quantum loop gravity treats gravitation via second quantization, not via the standard path integral formalism. At the core of loop quantum gravity is a framework for nonperturbative quantization of diffeomorphism-invariant gauge theories, which one might call loop quantization. While originally developed in order to quantize vacuum general relativity in 3+1 dimensions, the formalism can accommodate arbitrary spacetime dimensionalities, fermions, an arbitrary gauge group (or even quantum group), and supersymmetry, and results in a quantization of the kinematics of the corresponding diffeomorphism-invariant gauge theory. Much work remains to be done on the dynamics, the classical limit and the correspondence principle, all of which are necessary in one way or another to make contact with experiment. Nevertheless, like in every other new theory there are many open problems and the calculations are difficult. For readers interested in this matter we recommend [12].

ad 3) In NCQFT space and time are expected to not commute any more. Namely, while in commutative QFT

$$[x^\mu, x^\nu] = 0, \tag{1.1}$$

in NCQFT one introduces a certain "twist" on the right hand side of this expression. Whether this twist is constant or a function of x^μ, depends on the model. In our case, we will take

$$[x^\mu, x^\nu] = \Theta^{\mu\nu} \quad \text{with} \quad \Theta^{\mu\nu} \text{ constant and antisymmetric.} \tag{1.2}$$

In the next paragraph we will explain and justify this ansatz:

-) Historically the motivation for NCQFT was the hope that one could get rid of UV divergences [13][14], but as we nowadays know it became even worse and we are now confronted with the UV/IR mixing problem.

-) It is a rather natural idea that, since gravity alters the very geometry of the ordinary space, any quantum theory of gravity should quantize ordinary space, not just the phase space of mechanics, as quantum mechanics does. Hence, at some point at or before the Planck scale we should expect the algebra of ordinary coordinates or observables to be generalized to a noncommutative algebra [15].

-) As mentioned above, in a certain frame in String theory, noncommutativity naturally comes out.

In the next paragraph we will show up recent developments and the progress in NCQFT:

Some years ago most people did not believe in NCQFT to be a serious model, because the calculations showed that the theory suffers from a new phenomenon called UV/IR mixing which leads to even more divergences than commutative QFT. UV/IR mixing means that not even high momenta but also very small momenta lead to a nonrenormalizability of the theory.
The first real breakthrough was the Grosse-Wulkenhaar model which is a fully renormalizable NC model to all orders [16]. The trick here is that an oscillator like term $\Omega^2 x_\mu x_\mu$ is added to the action to compensate the "bad" behaviour of the kinetic term $\partial_\mu\phi\partial_\mu\phi$. Grosse and Wulkenhaar did the proof in the matrix basis. This very nice behaviour of the model inspired the renormalization group specialist Rivasseau to work with this model [2], and indeed he also proved the renormalizability, but in coordinate space. In this context it was very important that the Landau-ghost problem was solved, which means that the Borel summability of the theory is guaranteed. For this task he used the so-called multiscale analysis, which is an alternative to the BPHZ-scheme. There will be a small section about this method in this diploma thesis.

Nowadays there already exist three renormalizable noncommutative scalar theories [16][17][18]. What is still missing is a renormalizable NC gauge field theory. That is what this diploma thesis is about. Two models [19] [20] seem to be promising candidates. One of them [20] will be presented and discussed. Since this is an application of the Mehler kernel, we will calculate this mathematical object explicitly and give some important properties of it.

1.2 Conventions

Of course we use the Einstein summation convention, and since we are in Euclidian space,

$$\sum_{\mu=1}^{D} A_\mu B_\mu = A_\mu B_\mu = A^\mu B^\mu \tag{1.3}$$

where D denotes the dimension number.
Furthermore, whenever a D-vector is multiplied with another one in an exponential, the indices will be left out, that means

$$e^{A_\mu B_\mu} = e^{AB}. \tag{1.4}$$

Further conventions concerning the noncommutative calculations will be made in the following chapter.

1.3 Mathematical framework of NCQFT

In this chapter we will mainly follow the NCQFT lecture of Prof. M Schweda ([21]) as well as the diploma thesis of S. Hohenegger ([22]).

At very small distances time and space do not commute any more. The geometry itself must therefore become noncommutative. One needs a general mathematical concept to express this. 3 possibilities are

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu} \qquad \text{simplest natural extension} \tag{1.5}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_{\rho}\hat{x}^\rho \qquad \text{the Lie case} \tag{1.6}$$

$$[\hat{x}^\mu, \hat{x}^\nu] = i\hat{R}^{\mu\nu}_{\rho\sigma}\hat{x}^\rho\hat{x}^\sigma \qquad \text{the quantum group space.} \tag{1.7}$$

In the following we will restrict ourselves to the case (1.5).
In contrast to the emergent gravity theory by Harold Steinacker [23] we will take our $\Theta_{\mu\nu}$ as a <u>constant</u>, antisymmetric deformation parameter ($\dim(\Theta_{\mu\nu}) = -2$ for $\hbar$=c=1).

$$\Theta^{\mu\nu} = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{1.8}$$

with $\theta \in \mathbb{R}$, which is the simplest possible choice.
As you've possibly already noticed in the notation, space and time variables as well as in consequence the fields themselves are denoted by a hat, which shall point out that these quantities are operator valued objects. However, we want to handle the fields in the usual way. We therefore define a new multiplication law, the so called Groenewold-Weyl-Moyal star product. In order to define this modified product we use the Fourier transformation to work out a relation between the noncommutative operator valued fields and the corresponding ordinary commuting fields [1]

$$\begin{aligned} \hat{\phi}(\hat{x}) &= \int dk\, e^{ik_\mu \hat{x}^\mu} \phi(k) \\ \phi(k) &= \int dx\, e^{-ikx} \phi(x) \end{aligned} \tag{1.9}$$

where k and x are 4-dimensional real variables. The operator in the exponential is to be understood as a formal Taylor series expansion.
The product of 2 operator valued fields then reads

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) = \int dk \int dk'\, e^{ik\hat{x}} e^{ik'\hat{x}} \phi(k)\phi(k'). \tag{1.10}$$

We can now use the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \tag{1.11}$$

which holds if

$$[A,[A,B]] = [B,[A,B]] = 0. \tag{1.12}$$

This is always 0 in our case because $[A, \Theta_{\mu\nu}] = 0$.
Applying the BCH-formula yields

$$\begin{aligned} \hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) &= \int dk \int dk'\, e^{ik\hat{x}} e^{ik'\hat{x}} \phi_1(k)\phi_2(k').+ \\ &= \int dk \int dk'\, e^{i(k+k')\hat{x} - \frac{i}{2} k_\mu k'_\nu \Theta^{\mu\nu}} \phi_1(k)\phi_2(k'). \end{aligned} \tag{1.13}$$

[1] Due to the importance of this step some people give the operator $\hat{T}(k) = e^{ik_\mu \hat{x}^\mu}$ even an extra name, it is called the "Weyl-operator"

This relation defines the new **star product**

$$\hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) \longleftrightarrow (\phi_1 \star \phi_2)(x). \tag{1.14}$$

This is a non-trivial relation: on the left hand side we have operator valued fields $\in \mathbb{R}^{NC}$ while we have on the right hand side well known commutative fields but linked with a star product.

In x-space this formula reads [2]

$$(\phi_1 \star \phi_2)(x) = e^{\frac{i}{2}\Theta^{\mu\nu}\partial^x_\mu \partial^y_\nu} \phi_1(x)\phi_2(y)\Big|_{x=y}, \tag{1.15}$$

which we will prove:

$$\begin{aligned}
(\phi_1 \star \phi_2)(x) &= e^{\frac{i}{2}\Theta^{\mu\nu}\partial^x_\mu \partial^y_\nu} \int dk \int dk'\, e^{i(k+k')x} \phi_1(x)\phi_2(y) \\
= \int dk \int dk' & \left(1 + \frac{i}{2}\Theta^{\mu\nu}\partial^x_\mu \partial^y_\nu + \ldots\right) e^{i(k+k')x}\phi_1(k)\phi_2(k') \\
= \int dk \int dk' & \, e^{i(k+k')x - \frac{i}{2}\Theta^{\mu\nu}k_\mu k'_\nu} \phi_1(k)\phi_2(k'),
\end{aligned} \tag{1.16}$$

which completes the prove.
With this definition of the Groenewold-Moyal-Weyl star product [24][25] we can make a second-guess of the algebra we started from.
Using (1.15) we conclude

$$\begin{aligned}
x^\mu \star x^\nu &= e^{\frac{i}{2}\Theta^{\mu'\nu'}\partial^x_{\mu'}\partial^{x'}_{\nu'}} x^\mu x'^\nu \Big|_{x=x'} \\
&= x^\mu x^\nu + \frac{i}{2}\Theta^{\mu'\nu'}\partial^x_{\mu'}\partial x'_{\nu'} x^\mu x'^\nu \Big|_{x=x'} \\
&= x^\mu x^\nu + \frac{i}{2}\Theta^{\mu'\nu'}\delta^\mu_{\mu'}\delta^\nu_{\nu'} \\
&= x^\mu x^\nu + \frac{i}{2}\Theta^{\mu\nu},
\end{aligned} \tag{1.17}$$

and therefore

$$x^\mu \star x^\nu - x^\nu \star x^\mu = [x^\mu \stackrel{\star}{,} x^\nu] = i\Theta^{\mu\nu}, \tag{1.18}$$

the relation we started from.

[2] Some people write it as $(\phi_1 \star \phi_2)(x) = \phi_1(x) e^{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu}\phi_2(x)$

Summary of the Weyl-Moyal correspondence:

$$\boxed{\begin{gathered}\hat{\phi}(\hat{x}) \Longleftrightarrow \phi(x) \\ \hat{\phi}_1(\hat{x})\hat{\phi}_2(\hat{x}) \Longleftrightarrow (\phi_1 \star \phi_2)(x) \\ \hat{\phi}(\hat{x})\hat{\phi}(\hat{x}) = \int dk \int dk'\, e^{i(k+k')\hat{x} - \frac{i}{2}k_\mu k'_\nu \Theta^{\mu\nu}} \phi_1(k)\phi_2(k') \\ [\hat{x}^\mu, \hat{x}^\nu] \Longleftrightarrow [x^\mu \overset{\star}{,} x^\nu]\end{gathered}} \tag{1.19}$$

Now we need to compute some <u>*important properties of the star product*</u>:

•) The star product of 2 exponentials:

$$\begin{aligned} e^{ikx} \star e^{ik'x} &= e^{i\frac{i}{2}\Theta^{\mu\nu}\partial_\mu^x \partial_\nu^y} e^{ikx} e^{ik'y}\Big|_{x=y} \\ &= \left(1 + \frac{i}{2}\Theta^{\mu\nu}\partial_\mu^x\partial_\nu^y + \frac{1}{2}\left(\frac{i}{2}\right)^2 \Theta^{\mu\nu}\partial_\mu^x\partial_\nu^y\Theta^{\rho\sigma}\partial_\rho^x\partial_\sigma^y + \ldots\right) e^{ikx}e^{ik'y}\Big|_{x=y} \\ &= e^{i(k+k')x}\left(1 - \frac{i}{2}k\Theta k' + \frac{1}{2}\left(\frac{i}{2}\right)^2 (k\Theta k')^2 \ldots\right) = e^{i(k+k')x}e^{-\frac{i}{2}k\Theta k'}, \end{aligned} \tag{1.20}$$

where we have used the short form $k\Theta k' = k_\mu \Theta_{\mu\nu} k'_\nu$.

•) Associativity (the proof is elementary):

$$\Big[(f \star g) \star h\Big] = \Big[f \star (g \star h)\Big] \tag{1.21}$$

•)Star product of higher orders:

The star product can be easily generalized to higher orders,

$$
\begin{aligned}
& f_1(x) \star f_2(x) \star \ldots \star f_m(x) \\
= & \int \frac{d^4k_1}{(2\pi^4)} \int \frac{d^4k_2}{(2\pi^4)} \cdots \int \frac{d^4k_m}{(2\pi^4)} e^{i \sum\limits_{i=1}^{m} k_{i\mu} x_\mu} e^{-\frac{i}{2} \sum\limits_{i<j}^{m} k_i \times k_j} \tilde{f}_1(k_1) \tilde{f}_2(k_2) \ldots \tilde{f}_m(k_m)
\end{aligned} \tag{1.22}
$$

where the first exponential is just the usual Fourier transformation and the second exponential is the noncommutative factor over all permutations.

•)Cyclic permutation under the integral:

$$
\int d^4x f_1(x) \star f_2(x) \star \ldots \star f_m(x) = \int d^4x f_2(x) \star \ldots \star f_m(x) \star f_1(x) \tag{1.23}
$$

We will show this property for $m = 3$

$$
\begin{aligned}
& \int d^4x f_1(x) \star f_2(x) \star f_3(x) \\
= & \int \frac{d^4k_1}{(2\pi^4)} \int \frac{d^4k_2}{(2\pi^4)} \int \frac{d^4k_3}{(2\pi^4)} e^{i(k_1+k_2+k_3)x} \tilde{f}_1(k_1) \tilde{f}_2(k_2) \tilde{f}_3(k_3) e^{-\frac{i}{2} k_1 \times k_2 + k_1 \times k_3 + k_2 \times k_3}
\end{aligned} \tag{1.24}
$$

We will now rename the momenta

$$
\begin{aligned}
k_1 &\rightarrow k_2 \\
k_2 &\rightarrow k_3 \\
k_3 &\rightarrow k_1,
\end{aligned} \tag{1.25}
$$

then our expression becomes

$$
\begin{aligned}
& \int d^4x\, f_1(x) \star f_2(x) \star f_3(x) \\
= & \int \frac{d^4k_2}{(2\pi^4)} \int \frac{d^4k_3}{(2\pi^4)} \int \frac{d^4k_1}{(2\pi^4)} e^{i(k_2+k_3+k_1)x} \tilde{f}_1(k_2) \tilde{f}_2(k_3) \tilde{f}_3(k_1) e^{-\frac{i}{2} k_2 \times k_3 + k_2 \times k_1 + k_3 \times k_1} \\
= & \int \frac{d^4k_1}{(2\pi^4)} \int \frac{d^4k_2}{(2\pi^4)} \int \frac{d^4k_3}{(2\pi^4)} e^{i(k_1+k_2+k_3)x} \tilde{f}_1(k_1) \tilde{f}_2(k_2) \tilde{f}_3(k_3) e^{-\frac{i}{2} k_2 \times k_3 + k_2 \times k_1 + k_3 \times k_1} \\
= & \int d^4x\, f_2(x) \star f_3(x) \star f_1(x) \qquad \text{q.e.d.}
\end{aligned} \tag{1.26}
$$

•) In bilinear expressions under an integral the star drops out:
Using the above cyclic permutation relation we can show that

$$\int d^4x\, f_1(x) \star f_2(x) = \int d^4x\, f_2(x) \star f_1(x) \stackrel{!}{=} \int d^4x f_1(x) f_2(x), \tag{1.27}$$

which means that the star product then becomes a normal commutative product.
In particular this leads to the conclusion that the bilinear part of the action (=and therefore also the free fields) is completely free of noncommutative effects, only the interaction part gives rise to further structure hidden beyond!

1.4 Why can't we derive the Mehler kernel with the ordinary Green functions method?

If one wants to invert an operator

$$L\phi(x) = \rho(x) \tag{1.28}$$

with the method of Green functions, one has to solve the problem

$$LG(x - x') = \delta(x - x'). \tag{1.29}$$

To perform this task, one usually makes an ansatz for the Green function

$$G(x, x') = \int \tilde{G}(k) e^{ik(x-x')} dk. \tag{1.30}$$

Letting the operator L act on this ansatz is basically equivalent to Fourier transform the operator. The remaining problem would then just be to solve the integral over k.
Now comes the point: Fourier transforming a harmonic oscillator like operator

$$\mathcal{F}\Big(\Delta_4 + \Omega^2 \tilde{x}^2\Big) \cong k^2 + \partial_k^2 \tag{1.31}$$

yields another partial derivative in momentum space, which is no improvement, and one is confronted again with the same problem.... Therefore one has to search for another possibility to get the Green function for our operator.

Chapter 2

Mehler kernel

2.1 Time dependence of operators

In this section we will mainly follow the work of B. Thaller [26]. The time evolution of an observable is given by

$$A(t) = e^{iHt} A e^{-iHt}, \tag{2.1}$$

where $A(t)$ is the observable in the Heisenberg picture and A is the observable in the Schrödinger picture. This is a solution of the Heisenberg equation of motion

$$\frac{dA(t)}{dt} = i[H, A(t)] \tag{2.2}$$

with the initial condition $A(0) = A$. In the special case when the observable commutes with the Hamiltonian, the Heisenberg equation of motion has the trivial solution $A(t) = A(0) = A$. Such an observable A is called a *constant of motion.*
For the harmonic oscillator we can calculate the time evolution explicitly. Our Hamiltonian becomes in this case

$$H\big(x(t), p(t)\big) = \frac{1}{2}\big(x(t)^2 + p(t)^2\big), \tag{2.3}$$

where we have set the mass and ω equal 1. This will make calculations easier and there will be no error implemented due to the fact that m and ω are only constants.

For this topic we use (2.2) and insert $p(t)$ for the observable

$$\begin{aligned}
&i\Big[H\big(x(t),p(t)\big),p(t)\Big]\\
=&\frac{i}{2}\big[x(t)^2,p(t)\big]=\frac{i}{2}\Big(x(t)\big[x(t),p(t)\big]+\big[x(t),p(t)\big]x(t)\Big)=-x(t)=\frac{dp(t)}{dt},
\end{aligned}\tag{2.4}$$

where we have used

$$[x(t),p(t)]=i \tag{2.5}$$

with $\hbar=c=1$. Now we calculate the same expression with the observable $x(t)$

$$\begin{aligned}
&i\Big[H\big(x(t),p(t)\big),x(t)\Big]\\
=&\frac{i}{2}\big[p(t)^2,x(t)\big]=\frac{i}{2}\Big(p(t)\big[p(t),x(t)\big]+\big[p(t),x(t)\big]p(t)\Big)=p(t)=\frac{dx(t)}{dt}.
\end{aligned}\tag{2.6}$$

The coupled differential system from the equations (2.4) and (2.6)

$$\begin{aligned}
-x(t)&=\frac{dp(t)}{dt}\\
p(t)&=\frac{dx(t)}{dt}
\end{aligned}\tag{2.7}$$

can be solved explicitly:

$$\begin{aligned}
x(t)&=\sin(t)p+\cos(t)x\\
p(t)&=\cos(t)p-\sin(t)x,
\end{aligned}\tag{2.8}$$

where $p=p(0)$ and $x=x(0)$. This can be verified by directly inserting this solution into the system of differential equations.
The reader is recommended to note that the *Hamiltonian itself is not time dependent* because if we insert (2.8) into H we realize that the mixed terms cancel and since $\sin^2 A+\cos^2 A=1$ with some arbitrary $A\in\mathbb{R}$ we get

$$H\big(x(t),p(t)\big)=x(t)^2+p(t)^2=x^2+p^2. \tag{2.9}$$

Therefore we are allowed to leave out the time dependence of the Hamiltonian in the following chapters.

2.2 Time evolution and translation

Consider the time evolution of the momentum operator

$$p(t) = e^{iHt} p \; e^{-iHt} \tag{2.10}$$

which follows from relation (2.1). This can be written in the following form

$$e^{-ix_0 p(t)} = e^{iHt} e^{-ix_0 p} e^{-iHt}. \tag{2.11}$$

One can show this by expanding the exponential function in a Taylor series. The 0. order cancels cause $e^{-iHt}e^{iHt} = 1$. The first order just gives (2.10) and all higher orders cancel too as is now shown.
Consider for example the second order:

$$\begin{aligned}
\Big(-ix_0 p(t)\Big)^2 &= e^{-iHt}\Big(-ix_0 p\Big)^2 e^{iHt} \\
-x_0^2 p(t)^2 &= e^{-iHt}\left(-x_0^2 p^2\right) e^{iHt} \\
-x_0^2 p(t)^2 &= e^{-iHt}\left(-x_0^2\right) p e^{iHt} e^{-iHt} p e^{iHt} \\
-x_0^2 p(t)^2 &= e^{-iHt}\left(-x_0^2\right) p e^{iHt} p(t) \\
-x_0^2 p(t)^2 &= -x_0^2 e^{-iHt} p e^{iHt} p(t) \\
-x_0^2 p(t)^2 &= -x_0^2 p(t)^2.
\end{aligned} \tag{2.12}$$

You see, that by inserting the complete 1 you can eliminate all higher orders.

Now we need the fact that a momentum operator shifts the state in space, namely it describes a translation:

$$e^{-x_0 p}\psi(x) = \psi(x - x_0), \tag{2.13}$$

which we will prove now. For that we expand $\psi(x - x_0)$ in a Taylor series

$$\psi(x - x_0) = \psi(x) - \psi'(x)x_0 + \psi''(x)x_0^2 - ... \tag{2.14}$$

and we also expand the exponential function in a Taylor series (and apply it to $\psi(x)$)

$$e^{-ix_0 p}\psi(x) = e^{-x_0 \frac{d}{dx}}\psi(x) = \psi(x) - x_0\frac{d}{dx}\psi(x) + x_0^2\frac{d^2}{dx^2}\psi(x) - ... \tag{2.15}$$

which is exactly the same as (2.14). For further development in this section we need the Weyl relation

$$e^{i(ax+bp)} = e^{iax}e^{ibp}e^{iab/2} \tag{2.16}$$

which follows from the Baker-Campbell-Hausdorff formula and from (2.5). Now we insert the time evolution of the harmonic oscillator (2.8) into this formula

$$\begin{aligned} e^{-iHt}e^{-ix_0p}e^{iHt} &= e^{-ix_0p(t)} \\ = e^{-ix_0(-\sin(t)x+\cos(t)p)} &= e^{ix_0\sin(t)x}e^{-ix_0\cos(t)p}e^{-i\frac{x_0^2}{4}\sin(2t)}, \end{aligned} \tag{2.17}$$

where we have used the BCH-formula as well as relation (2.5). We now may change $t \to -t$ in this formula

$$e^{-iHt}e^{-ix_0p}e^{iHt} = e^{-ix_0\sin(t)x}e^{-ix_0\cos(t)p}e^{i\frac{x_0^2}{4}\sin(2t)} \tag{2.18}$$

and apply it to a state $\phi(x,t) = e^{-iHt}\phi(x)$:

$$e^{-iHt}e^{-ix_0p}\phi(x) = e^{-ix_0\sin(t)x}e^{-ix_0\cos(t)p}e^{i\frac{x_0^2}{4}\sin(2t)}e^{-iHt}\phi(x). \tag{2.19}$$

Now we may bring the phase term to the beginning of the expression

$$e^{-iHt}e^{-ix_0p}\phi(x) = e^{i\frac{x_0^2}{4}\sin(2t)}e^{-ix_0\sin(t)x}e^{-ix_0\cos(t)p}e^{-iHt}\phi(x). \tag{2.20}$$

Due to relation (2.13) we can write this as

$$e^{-iHt}\phi(x-x_0) = e^{i\frac{x_0^2}{4}\sin(2t)}e^{-ix_0\sin(t)x}\phi(x-x_0\cos(t),t). \tag{2.21}$$

The term $e^{-ix_0\sin(t)x}$ shifts the function ϕ by $x_0\sin(t)$ in momentum space. The term $e^{i\frac{x_0^2}{4}\sin(2t)}$ is just a phase factor.

We now want to calculate the probability density of ϕ.
Since we all know that in the theory of the harmonic oscillator $p \propto (A - A^\dagger)$ and $x \propto (A + A^\dagger)$ (where A is the annihilation and $A^\dagger$ is the creation operator), one easily sees that

$$\begin{aligned} p^\dagger &= -p \\ x^\dagger &= x. \end{aligned} \tag{2.22}$$

The time is of course the hermitian conjugate to itself as well as the Hamiltonian and so the absolute value of $\phi(x-x_0)$ gives

$$|\phi(x-x_0)|^2 = |\phi(x-x_0\cos(t))|^2. \tag{2.23}$$

We can draw some interesting conclusions from this result: Assume that $\phi = \phi_n$ is an eigenstate of the harmonic oscillator. The eigenstates have a trivial time evolution, they always remain centred at the origin, the expectation values for the position x and the momentum p are zero for all times. Now,the eigenfunction is shifted to a new position x_0, that is, the new initial state is given by

$$\psi(x,0) = \phi_n(x - x_0). \tag{2.24}$$

From formula (2.21) we can now conclude how ψ evolves in time

$$\psi(x,t) = \exp\Big(i\frac{x_0^2}{4}\sin(2t) - iE_n - i(x_0\sin(t))x\Big)\phi(x - x_0\cos(t)). \tag{2.25}$$

The probability density of this expression is given by

$$|\psi(x,t)|^2 = |\phi_n(x - x_0\cos(t)|^2. \tag{2.26}$$

The conclusion of this is:
Time evolution does not change the shape of the function, it just translates it to its classical position $x_0\cos(t)$.

2.3 Motion of Gaussian Wave Packets

Here we apply the results of the previous section to the ground state

$$\Omega(x) = \pi^{-\frac{1}{4}}e^{-\frac{1}{2}x^2}. \tag{2.27}$$

Among the eigenstates it is distinguished also by the property that it is optimal with respect to the uncertainty relation

$$\triangle x \triangle p \geq \frac{1}{2}. \tag{2.28}$$

The ground state satisfies

$$\triangle x \triangle p = \frac{1}{2}. \tag{2.29}$$

Now we can apply the results obtained in the previous section (2.25). We insert the energy of the ground state $E_n = n + 1/2$ for $n = 0$ and obtain

$$\psi(x,t) = \Big(\frac{1}{\pi}\Big)^{1/4}\exp\Big(i\frac{x_0^2}{4}\sin(2t) - i\frac{t}{2}\Big)\exp\Big(ip_t x - \frac{(x - x_t)^2}{2}\Big) \tag{2.30}$$

with $x_t = x_0 \cos(t)$ and $p_t = -x_0 \sin(t)$. This is a normalized Gaussian function centred at the average position x_t with average momentum p_t. Here (x_t, p_t) describes the classical oscillation of a particle with initial position x_0. Since we already know that the ground state is optimal with respect to the uncertainty relation, the state $\psi(x,t)$ hence satisfies equation (2.30) for all times. Those states with minimal uncertainty are called *coherent states.* The maximum of a coherent state always follows the trajectory of the classical-mechanical particle that starts at x_0 with zero initial momentum. The wavelength of the phase always corresponds to the momentum of the classical particle.
One can even state (without proof): ψ is a coherent state if and only if

$$\frac{1}{2}\Big(< p >^2_{\psi(t)} + < x >^2_{\psi(t)} \Big) = \frac{1}{2} < p^2 + x^2 >_{\psi(t)}, \tag{2.31}$$

where $< x >$ and $< p >$ means taking the mean value which corresponds to the classical values.

2.4 The Hermite polynomials

2.4.1 The generating equation

We now want to calculate the generating function of the Hermite Polynomials. I want to remind the reader that we have set $\hbar = c = 1$ and now in addition we set $m = \omega_0 = 1$, where m is the mass and ω_0 is the angular frequency. The potential then simply becomes $V(x) = \frac{1}{2}m\omega_0^2 x^2 = \frac{1}{2}x^2$. The Schrödinger equation then reads

$$(-\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2)\psi(x) = E\psi(x). \tag{2.32}$$

We now want to have a solution which is zero at infinity and which behaves like a wave packet, so we try some sort of Gaussian ansatz:

$$\psi(x) = e^{\frac{-x^2}{2}} H(x). \tag{2.33}$$

For our calculation we need the 2^{nd} derivative of ψ

$$\begin{aligned} \frac{d^2}{dx^2}\psi(x) =& \frac{d}{dx}\Big(- xe^{\frac{-x^2}{2}} H(x) + e^{\frac{-x^2}{2}} H'(x)\Big) \\ =& e^{\frac{-x^2}{2}} \Big(- H(x) + x^2 H(x) - xH'(x) - xH'(x) + H''(x)\Big). \end{aligned} \tag{2.34}$$

So, equation (2.32) gives (we have divided out the factor $e^{\frac{-x^2}{2}}$)

$$
\begin{aligned}
-\frac{1}{2}\Big(-H(x)+x^2H(x)-2xH'(x)+H''(x)-x^2H(x)\Big) &= EH(x) \\
\frac{1}{2}\Big(-H(x)+2EH(x)-2xH'(x)+H''(x)\Big) &= 0 \\
H''(x)-2xH'(x)+2\underbrace{(E-\frac{1}{2})}_{n}H(x) &= 0. \qquad (2.35)
\end{aligned}
$$

This is exactly the generating differential equation of the Hermite polynomials.

It's solutions are the Hermite polynomials, namely

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \tag{2.36}$$

which can be directly verified by inserting.
For example one can now calculate the first few Hermite polynomials:

$$
\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \qquad (2.37)
\end{aligned}
$$

2.4.2 The generating function

We now want to prove that

$$w(x,t) = e^{2xt-t^2} \tag{2.38}$$

is the generating function for the Hermite polynomials. This means that the Hermite polynomials are the coefficients of the Taylor series expansion of $w(x,t)$:

$$w(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n. \tag{2.39}$$

For the proof we first need to expand $w(x,t)$ in a Taylor series.

$$w(x,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial t^n}\Big(w(x,0)\Big)\Big|_{t=0} t^n \tag{2.40}$$

for $t \epsilon \mathbf{R}$. The n^{th} derivative of ω is given by

$$\frac{\partial^n}{\partial t^n}\Big(w(x,0)\Big)\Big|_{t=0} = \frac{\partial^n}{\partial t^n}\Big(e^{2xt-t^2}\Big)\Big|_{t=0} = e^{x^2}\frac{\partial^n}{\partial t^n}\Big(e^{-(x-t)^2}\Big)\Big|_{t=0}. \tag{2.41}$$

We now substitute $u = x - t$ and get

$$(-1)^n e^{x^2}\frac{\partial^n}{\partial u^n}\Big(e^{-u^2}\Big)\Big|_{u=x} = (-1)^n e^{x^2}\frac{\partial^n}{\partial x^n}\Big(e^{-x^2}\Big) = H_n(x), \tag{2.42}$$

which is exactly the same as (2.36), q.e.d.

2.4.3 Recursion formula for the Hermite polynomials

We now want to prove the recursion formula for the Hermite polynomials which is given by

$$H_{n+1}(x) = -\frac{d}{dx}H_n(x) + 2xH_n(x). \tag{2.43}$$

By differentiating the generating function we get the following identity:

$$\frac{d}{dt}w(x,t) = (2x - 2t)w(x,t). \tag{2.44}$$

Now we inject the Taylor expansion of $w(x,t)$ (2.39) into the identity:

$$\frac{d}{dt}\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n - 2x\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n + 2t\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n = 0. \tag{2.45}$$

We execute the differentiation with respect to t (we can differentiate every term of the Taylor series expansion separately) and get

$$\sum_{n=1}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1} - 2x\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n + 2t\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n = 0. \tag{2.46}$$

Note that the first sum now begins at $n = 1$ because the differentiation eliminated the 0^{th} order.
We can rewrite the last term

$$2t\ w(x,t) = 2te^{2xt-t^2} = \frac{d}{dx}e^{2xt-t^2} = \frac{d}{dx}w(x,t) \tag{2.47}$$

and get

$$\sum_{n=1}^{\infty}\frac{H_n(x)}{(n-1)!}t^{n-1} - 2x\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n + 2\frac{d}{dx}\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n = 0. \tag{2.48}$$

Now we need to shift the index in the first term and get

$$\sum_{n=0}^{\infty} \frac{H_{n+1}(x)}{n!} t^n - 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2\frac{d}{dx} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = 0. \tag{2.49}$$

By comparing the coefficients follows for all $n \in \mathbb{R}$:

$$\boxed{H_{n+1}(x) = -\frac{d}{dx} H_n(x) + 2xH_n(x)} \tag{2.50}$$

which we wanted to prove.

We now want to show that the Hermite polynomials are indeed the eigenfunctions of the harmonic oscillator (apart from numerical factors). That means we want to show that

$$(A^\dagger)^n \Omega(x) = \frac{1}{\sqrt{2^n}} H_n(x) \Omega(x) \tag{2.51}$$

with the ground state $\Omega(x)$ (2.27). For this proof we need formula (2.50) and the fact that $A^\dagger$ can be represented in terms of position and momentum operators

$$A^\dagger = \frac{1}{\sqrt{2}} \Big(-\frac{d}{dx} + x \Big). \tag{2.52}$$

The proof is best done by induction: Equation (2.51) is true for $n = 0$ because $H_0(x) = 1$. Assuming that it is valid for $n = k$, let us prove the validity for $n = k + 1$:

$$\begin{aligned}
(A^\dagger)^{k+1} \Omega(x) =& A^\dagger (A^\dagger)^k \Omega(x) \\
=& A^\dagger \frac{1}{\sqrt{2^k}} H_k(x) \Omega(x) \\
=& \frac{1}{\sqrt{2^{k+1}}} (-H_k' \Omega - H_k \Omega' + x H_k \Omega) \\
=& \frac{1}{\sqrt{2^{k+1}}} (-H_k' \Omega + 2x H_k \Omega) \\
=& \frac{1}{\sqrt{2^{k+1}}} H_{k+1} \Omega(x).
\end{aligned} \tag{2.53}$$

This completes the proof of the assertion.

2.5 The propagator and the kernel

In this section we want to enlight how a kernel is related to the propagator and why it is so important. Our main task in calculating a propagator is to invert an operator. That means, we want to solve

$$H_x \Delta(x, y) = \delta^D(x - y). \tag{2.54}$$

x and y are some vectors $\in \mathbb{R}^D$, and H_x is a positive definite operator which we have denoted by a little index x to emphasize that this is an operator depending only on one variable (coordinate) and derivatives with respect to x. Now, consider a "time evolution" of a state $\psi(x)$, defined by

$$\psi(x, t) \equiv e^{-H_x t}\psi(x). \tag{2.55}$$

Note that t does not represent a time with physical meaning. It is just a mathematical parameter. The physical time, as far as one can say time in the Euclidian space, is already contained in the D-dimensional vector x. Clearly,

$$\psi(x, 0) = \psi(x) \tag{2.56}$$

and since H_x is assumed to be positive definite, we have

$$\lim_{t\to\infty} \psi(x, t) = 0. \tag{2.57}$$

By differentiating eq. (2.55) with respect to t, one sees that $\psi(x, t)$ satisfies the equation

$$\frac{d}{dt}\psi(x, t) = -H_x \psi(x, t). \tag{2.58}$$

Besides the factor $-i$ missing on the left hand side, this corresponds to the Schrödinger equation of quantum mechanics with Hamiltonian H_x.
The effect of time evolution (2.55) can usually be described by a kernel K as

$$\psi(x, t) = \int d^D y \; K(x, y, t)\psi(y), \tag{2.59}$$

and if we are lucky, K is even known. By acting with a derivative with respect to t on (2.59), one can easily show that K satisfies the relation

$$\frac{d}{dt}K(x, y, t) = -H_x K_h(x, y, t). \tag{2.60}$$

We now integrate out time in eq. (2.58) to get rid of the time dependence which the Green function we are looking for is not dependent of, and obtain

$$\psi(x,\infty) - \psi(x,0) = -\int_0^\infty dt\ H_x\psi(x,t) \tag{2.61}$$

$$= -\int d^D y\psi(y) H_x \int_0^\infty dt\ K(x,y,t). \tag{2.62}$$

Using eq. (2.56) and (2.57), we get

$$\psi(x) = \int d^D y\ \psi(y) H_x \int_0^\infty dt\ K(x,y,t). \tag{2.63}$$

This must be valid for any test function ψ. Thus we have

$$H_x \int_0^\infty dt\ K(x,y,t) = \delta^D(x-y). \tag{2.64}$$

Comparing this with (2.54), we conclude

$$\boxed{\int_0^\infty dt\ K(x,y,t) = \Delta(x,y)} \tag{2.65}$$

2.6 The Mehler kernel in one dimension

2.6.1 Derivation

In this chapter we will switch to Minkowski space to underline the analogue with the time evolution operator. As justified in the previous chapter we can make the following ansatz for the Mehler kernel:

$$e^{-iHt}G_q(x) = \int_{-\infty}^{\infty} K_{osc}(x,y,t) G_q(y) dy. \tag{2.66}$$

The goal is now to extract the Mehler kernel on the right hand side. We can easily see that for this task some sort of Fourier transformation will be

needed as well as a test function G_q that comes up to our expectations. It is sufficient to do this calculation with a dense set of wave functions because the action of a unitary operator can always be extended by continuity to the whole Hilbert space.
We chose the dense set spanned by the (finite) linear combinations of the functions

$$G_q(x) = e^{iqx}\Omega(x), \qquad q \in \mathbf{R}, \tag{2.67}$$

where $\Omega(x)$ is the ground state (2.27).

For the harmonic oscillator the time evolution operator is given by

$$\exp(-iHt) = \exp(-\frac{i}{2}(p^2 + x^2)t) = \exp(-i(A^\dagger A + \frac{1}{2})t) \tag{2.68}$$

which we want to apply to this dense set of wave functions. For an arbitrary function in this set we obtain, using the known temporal behaviour of the position observable (2.8) and (2.1),

$$\begin{aligned} e^{-iHt}G_q =& e^{-iHt}e^{iqx}e^{iHt}e^{-iHt}\Omega \\ =& e^{iqx(-t)}e^{-iHt}\Omega \\ =& e^{iqx(-t)}e^{-it/2}\Omega \\ =& e^{iq(x\cos t - p\sin t)}e^{-it/2}\Omega, \end{aligned} \tag{2.69}$$

where we have used the fact that the annihilation operator applied onto the ground state gives zero.
With the explicit representation of x and p in terms of A and $A^\dagger$ we can show that the following formula holds:

$$x\cos t - p\sin t = (e^{-it}A^\dagger + e^{it}A)/\sqrt{2}. \tag{2.70}$$

Using the Weyl relation (2.16) we obtain

$$\begin{aligned} e^{-iHt}G_q =& e^{-it/2}\exp\Big(\frac{iq}{\sqrt{2}}e^{-it}A^\dagger\Big)\exp\Big(\frac{iq}{\sqrt{2}}e^{it}A\Big)\exp\Big(-\frac{q^2}{4}\Big)\Omega \\ =& e^{-it/2}\exp\Big(-\frac{q^2}{4}\Big)\exp\Big(\frac{iq}{\sqrt{2}}e^{-it}A^\dagger\Big)\exp\Big(\frac{iq}{\sqrt{2}}e^{-it}A\Big)\Omega. \end{aligned} \tag{2.71}$$

Here the factor in the exponential of the function A has changed its sign, which does not matter because it is only applied to Ω. Now one can apply the Weyl relation again to conclude

$$
\begin{aligned}
e^{-iHt}G_q =& e^{-it/2}\exp\Big(-\frac{q^2}{4}\Big)\exp\Big(\frac{q^2}{4}e^{-2it}\Big)\exp\Big(\frac{iq}{\sqrt{2}}e^{-it}(A^\dagger + A)\Big)\Omega \\
=& e^{-it/2}\exp\Big(-\frac{q^2}{4}(1-e^{-2it})\Big)\exp\Big(iqe^{-it}x\Big)\Omega. \qquad (2.72)
\end{aligned}
$$

We can now insert this into our ansatz for the Mehler kernel (2.66). Then our equation becomes

$$
\int_{-\infty}^{\infty} K_{osc}(x,y,t)e^{-y^2/2}e^{iqy}dy = \exp\Big(-i\frac{t}{2} - \frac{q^2}{4}(-e^{-2it}) + iqe^{-it}x - \frac{x^2}{2}\Big), \qquad (2.73)
$$

where we have divided out the factor $\pi^{-1/4}$. Now we can calculate the Mehler kernel by an inverse Fourier transformation with respect to y:

$$
\begin{aligned}
&K_{osc}(x,y,t)e^{-y^2/2} = \\
=&\frac{1}{2\pi}\exp\Big(-i\frac{t}{2} - \frac{x^2}{2}\Big)\int_{-\infty}^{\infty} e^{-iqy}\exp\Big(-\frac{q^2}{4}(1-e^{-2it}) + iqe^{-it}x\Big)dq. \qquad (2.74)
\end{aligned}
$$

Because this is an integral over a Gaussian function, it can be calculated explicitly. One obtains

$$
K_{osc}(x,y,t) = \frac{1}{\sqrt{\pi}}\frac{e^{-it/2}}{(1-e^{-2it})^{1/2}}\exp\Big(-\frac{(e^{-it}x-y)^2}{1-e^{-2it}} - \frac{x^2}{2} + \frac{y^2}{2}\Big). \qquad (2.75)
$$

A little trigonometry converts this expression into (will be carried out in appendix (A.1.1))

$$
K_{osc}(x,y,t) = \frac{1}{\sqrt{2\pi i\ \sin(t)}}\exp\Big(i\frac{x^2+y^2}{2}\cot(t) - i\frac{xy}{\sin(t)}\Big). \qquad (2.76)
$$

It is very interesting to note that at time $t = 0, \pm\pi, \pm 2\pi, ...$ the Mehler kernel must behave like a delta distribution which one can easily see looking at formula (2.66). For example at $t = 0$ the following relation holds

$$
\psi(x) = \int_{-\infty}^{\infty} K_{osc}(x,y,0)\psi(y)dy, \qquad (2.77)
$$

and from that one can see that

$$K_{osc}(x, y, 0) = \delta(x - y). \tag{2.78}$$

From now on we will turn to Euclidian space, that means it has to be replaced by t, hence the Mehler kernel becomes

$$K_{osc}(x, y, t) = \frac{1}{\sqrt{\pi}} \frac{e^{-t/2}}{(1 - e^{-2t})^{1/2}} \exp\Big(- \frac{(e^{-t}x - y)^2}{1 - e^{-2t}} - \frac{x^2}{2} + \frac{y^2}{2} \Big). \tag{2.79}$$

Upon bringing the exponent of the Mehler kernel to the same denominator, we get

$$K_{osc}(x, y, t) = \frac{1}{\sqrt{\pi}} \frac{e^{-t/2}}{(1 - e^{-2t})^{1/2}} \exp\Big(- \frac{(x^2 + y^2)(1 + e^{-2t})/2 - 2e^{-t}xy}{(1 - e^{-2t})} \Big). \tag{2.80}$$

2.7 The Euclidian Mehler kernel in higher dimensions

2.7.1 Target space

We can now use the results of the previous section to calculate the Mehler kernel in higher dimensions. In this chapter we will come back to Euclidian space because only in this context we are able to compute the Mehler kernel in higher dimensions. We will write the index i instead of μ to underline that we are in Euclidian space. The dimension number will be noted with D. The Minkowskian Mehler kernel in higher dimensions is still a task of research.

In the paper of D.N. Blaschke et. al. [20] propagators like

$$(-\Delta_4 + \Omega^2 \tilde{x}_\mu \tilde{x}_\mu)^{-1} \quad (\text{with } \Delta_4 = \partial_i \partial_i \text{ and } i = 1, \dots 4) \tag{2.81}$$

arise, and one needs to invert the expression in the brackets. The result would be the Mehler kernel in higher dimensions which we want to calculate now.

One needs a Hamiltonian to start from and as we want to invert the expression in the brackets in formula (2.81) the natural choice for our starting Hamiltonian is

$$H = \frac{1}{2} \left[-\partial_i \partial_i + \frac{\Omega^2}{\Theta^2} x_i x_i \right]. \tag{2.82}$$

The main idea of this section is now to rewrite this Hamiltonian so that we can compare it with the Hamiltonian in the one-dimensional case[1]

$$h = \frac{1}{2}(p^2 + x^2) = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2). \tag{2.83}$$

This leads to the following Mehler kernel (compare: (2.75))

$$K_h(x, y, t) = \frac{1}{\sqrt{\pi}} \frac{e^{-it/2}}{(1 - e^{-2it})^{1/2}} \exp\Big(- \frac{(e^{-it}x - y)^2}{1 - e^{-2it}} - \frac{x^2}{2} + \frac{y^2}{2}\Big). \tag{2.84}$$

This means we can calculate the time evolution by

$$e^{-ht}\psi(x) = \int dy\ K_h(x, y, t)\psi(y). \tag{2.85}$$

So let's now rewrite our Hamiltonian H by the trick of substitution: We'll substitute x_i:

$$x_i = x_i' \sqrt{\frac{\Theta}{\Omega}}. \tag{2.86}$$

Our Hamiltonian then becomes

$$\begin{aligned} H &= \frac{1}{2}\left[-\partial^2 + \frac{\Omega^2}{\Theta^2}x^2\right] \\ &= \frac{1}{2}\left[-\frac{\Omega}{\Theta}\partial'^2 + \frac{\Omega^2}{\Theta^2}\frac{\Theta}{\Omega}x'^2\right] \\ &= \frac{\Omega}{\Theta}\left[\underbrace{-\frac{1}{2}\partial'^2 + \frac{1}{2}x'^2}_{\sum_i h_i}\right]. \end{aligned} \tag{2.87}$$

Our formula for the Mehler kernel in higher dimensions then turns to

$$e^{-Ht}\psi(x) = \int dy\ K_H(x_i, y_i, t)\psi(y)$$

$$e^{-\frac{\Omega}{\Theta}\left[\underbrace{-\frac{1}{2}\partial'^2 + \frac{1}{2}x'^2}_{\sum_i h_i}\right]t}\psi(x) = \int dy\ K_H(x_i, y_i, t)\psi(y). \tag{2.88}$$

[1]Simon B. used a slightly different Hamiltonian $h = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2 - 1)$ and gets therefore an additional factor $e^{\frac{1}{2}t}$ in his Mehler kernel.

We can now construct our full Mehler kernel K_H out of the single Mehler kernels K_h

$$K_H\left(x_i, y_i, t\right) = \exp\left(\sqrt{\frac{\Theta}{\Omega}}\right)^D \prod_{i=1}^{D} K_h(x_i\sqrt{\frac{\Omega}{\Theta}}, y_i\sqrt{\frac{\Omega}{\Theta}}, \underbrace{t\frac{\Omega}{\Theta}}_{\tau}). \tag{2.89}$$

Note the additional factor $\exp\left(\sqrt{\frac{\Theta}{\Omega}}\right)^D$. This factor is due to the fact that one must substitute the variable y also on the right hand side of the equation to get the same wave function on both sides. Also notice the additional substitution of the time $\tau = t\frac{\Omega}{\Theta}$, which has been computed to get rid of the prefactor $\frac{\Omega}{\Theta}$ in front of the single Hamiltonians h_i. By integrating the Mehler kernel $\int_0^\infty dtK_H\left(x_i, y_i, t\right)$ one can now get the propagator corresponding to the Hamiltonian H. The reason why this is the case has been discussed in detail in the chapter "The propagator and the kernel"

We can now insert the Mehler kernel (2.84) into the formula to give the full Mehler kernel a face. We know that integrating out the Mehler kernel gives the propagator. So here it is:

$$\begin{aligned}\Delta(x, y) &= \int_0^\infty dt\ K_H(x, y, t) \qquad (2.90)\\ &= \frac{\left(\frac{\Theta}{\Omega}\right)^{1-\frac{D}{2}}}{\pi^{\frac{D}{2}}} \int_0^\infty d\tau \left(\frac{e^{-\tau}}{1-e^{-2\tau}}\right)^{\frac{D}{2}} \exp\left(-\frac{(x^2+y^2)(1+e^{-2\tau})/2 - 2e^{-\tau}xy}{\frac{\Theta}{\Omega}(1-e^{-2\tau})}\right)\end{aligned}$$

with $\tau = t\frac{\Omega}{\Theta}$.

Note that in contrast to (2.84) xy now means $\sum_{i=1}^{D} x_i y_i$.

2.7.2 The Mehler kernel in terms of long and short variables

From papers of Rivasseau and his team, e.g. ([2]), we know that we can write the Mehler kernel in terms of long and short variables. We'll explicitly show this: By choosing the dimension $D = 4$, (2.90) becomes:

$$K_{osc}(x, y, t) = \frac{2}{\pi^2}\frac{\Omega}{\Theta}\frac{e^{-2\tau}}{(1-e^{-2\tau})^2} \exp\Big(-\frac{(x^2+y^2)(1+e^{-2\tau})/2 - 2e^{-\tau}xy}{\frac{\Theta}{\Omega}(1-e^{-2\tau})}\Big). \tag{2.91}$$

We will now expand the exponential by a factor e^{τ} and the prefactor by a factor $e^{2\tau}$. With the abbreviation $\omega = \frac{\Theta}{\Omega}$ our expression becomes

$$= \frac{1}{\pi^2\omega}\frac{1}{(e^{\tau}-e^{-\tau})^2}\exp\Big(-\frac{(x^2+y^2)(e^{\tau}+e^{-\tau})/2-2xy}{\omega(e^{\tau}-e^{-\tau})}\Big). \tag{2.92}$$

Surprisingly we see that all expressions can be written in terms of hyperbolic functions

$$\begin{aligned}
&= \frac{1}{\pi^2\omega}\frac{1}{(2\sinh\tau)^2}\exp\Big(-\frac{(x^2+y^2)(2\cosh\tau)/2-2xy}{\omega(2\sinh\tau)}\Big)\\
&= \frac{1}{\pi^2\omega}\frac{1}{(2\sinh\tau)^2}\exp\Big(-\frac{(x^2+y^2)\cosh\tau-2xy}{2\omega\sinh\tau}\Big).
\end{aligned}$$

Using the relation $\cosh\tau = \cosh^2(\frac{\tau}{2}) + \sinh^2(\frac{\tau}{2})$ for the first term in the exponential, $\cosh^2(\frac{\tau}{2}) - \sinh^2(\frac{\tau}{2}) = 1$ for the second term in the exponential and $\sinh\tau = 2\sinh\frac{\tau}{2}\cosh\frac{\tau}{2}$ in the denominator, we get

$$\begin{aligned}
&= \frac{1}{\pi^2\omega}\frac{1}{(2\sinh\tau)^2}\exp\left[-\frac{(x^2+y^2)}{4\omega}(\coth\frac{\tau}{2}+\tanh\frac{\tau}{2})+\frac{2xy}{4\omega}(\coth\frac{\tau}{2}-\tanh\frac{\tau}{2})\right]\\
&= \frac{1}{\pi^2\omega}\frac{1}{(2\sinh\tau)^2}\exp\left[-\frac{1}{4\omega}\Big((x-y)^2\coth\frac{\tau}{2}+(x+y)^2\tanh\frac{\tau}{2}\Big)\right], \qquad (2.93)
\end{aligned}$$

which is an expression only in terms of long $(x+y)$ and short $(x-y)$ variables.

2.7.3 Comparison with the heat kernel

The heat kernel is given by

$$K(t,x,y) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}} \tag{2.94}$$

for a Hamiltonian

$$H = \partial^2. \tag{2.95}$$

As one can easily see, this has a very similar form to the Mehler kernel (2.93). Nevertheless, the main difference one should notice is that the heat kernel depends only on the differences in space $(x-y)$, whereas the Mehler kernel depends also on $(x+y)$. This means that in contrast to the heat kernel, the Mehler kernel is <u>not translation invariant</u>.

2.7.4 Momentum space

Next, we want to know the Mehler kernel (2.90) in momentum space. This is a little bit different to the usual case since our propagator now is not translation invariant. Thus, we have to transform Δ with respect to both arguments. The Fourier transform is thus given by

$$\tilde{\Delta}(p,q) = (2\pi)^{-D} \int d^D x \; d^D y \; e^{-ipx-iqy} \Delta(x,y). \tag{2.96}$$

This task can easily be accomplished since only Gaussian integrals are involved and will be carried out in appendix (A.1.2). The result is:

$$\tilde{\Delta}(p,q) = \frac{\omega^3}{2\pi^2} \int_0^\infty d\tau \frac{e^{-2\tau}}{(1-e^{-2\tau})^2} \exp\left(- \frac{\frac{\omega}{2}(p^2+q^2)(1+e^{-2\tau}) + 2\omega e^{-\tau} pq}{1-e^{-2\tau}} \right) \tag{2.97}$$

with $\omega = \frac{\Theta}{\Omega}$ and D=4. Note that this expression looks very similar to (2.90), except that coordinates and momenta are exchanged. This relation is based on the fundamental observation that both coordinates and momenta appear (only) squared in the harmonic oscillator. The fact that $K(x,y)$ and $K(p,q)$ have the same form is called the *Langmann - Szabo - duality.*
I want to point out that in the literature the propagator, which is the integral over the Mehler kernel, itself is again called “The Mehler kernel”. From now on we will stick to this formulation and call (2.99) “The Mehler kernel”.
You may wonder that the sign of the mixed momenta term in the exponent of the Mehler kernel changed its sign. To understand this we first look at an easier example, the propagator in the commutative case (1-dimensional), which looks like $\frac{1}{\Box}\delta(x-y)$ in target space. Let’s Fourier transform the δ-function:

$$\int d^4x d^4y \, e^{-ipx} e^{-iqy} \delta(x-y) = \int d^4 y e^{-iy(p+q)} = \delta(p+q). \tag{2.98}$$

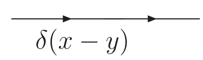


Figure 2.1: Propagator with the right convention

As you can see, the argument in the δ-function has changed the inner sign. <u>But</u> now comes the point: the conventions of the momenta flow of the propagator has changed too. To understand this point we may plot the propagator. Previously we had figure 2.1, but now we have figure 2.2, which we

$\delta(p+q)$

Figure 2.2: Propagator with the wrong convention

don't want. To rearrive at our convention we change the second momentum to get $\delta(p-q)$.
The Mehler kernel behaves the same way, and to be able to keep our convention we change the momentum $q \to -q$ by force. The Mehler kernel then has the right behaviour 2.1 due to our convention and looks like

$$\tilde{\Delta}(p,q) = \frac{\omega^3}{2\pi^2} \int_0^\infty d\tau \frac{e^{-2\tau}}{(1-e^{-2\tau})^2} \exp\left(- \frac{\frac{\omega}{2}(p^2+q^2)(1+e^{-2\tau}) - 2\omega e^{-\tau} pq}{1-e^{-2\tau}} \right). \tag{2.99}$$

Certainly, the Mehler kernel in momentum space can also be written in terms of long and short variables, which works in the same way as derived in subsection (2.7.2):

$$\tilde{\Delta}(p,q) = \frac{\omega^3}{8\pi^2} \int_0^\infty d\tau \frac{1}{(\sinh(\tau))^2} \exp\left[-\frac{\omega}{4} \left(\coth\left(\frac{\tau}{2}\right)(p-q)^2 + \tanh\left(\frac{\tau}{2}\right)(p+q)^2 \right) \right]. \tag{2.100}$$

2.7.5 Multiscale analysis

For the calculation of loop graphs of higher order one has to deal with numerous parameter integrals (one per Mehler kernel). To deal with this one must find some sort of power counting. For this task Rivasseau et. al. invented the Multiscale analysis (see e.g. [2], page 60).
(2.93) can be divided into a sum $\sum\limits_{i=1}^{\infty}$ of slices

$$C^i(x-y, x+y) = \int_{M^{-2i}}^{M^{-2(i-1)}} d\alpha \frac{4}{\omega[2\pi\sinh(\alpha)]^2} e^{-\frac{1}{\omega}\left(\coth(\frac{\alpha}{2})(x-y)^2 + \tanh(\frac{\alpha}{2})(x+y)^2\right)} \tag{2.101}$$

for some arbitrary constant M, which "slice" the interval $[0,1]$ whereas the interval $[1,\infty]$ gives a finite contribution.

Since $\tanh(x) \approx x$ and $\coth(x) \approx \frac{1}{x}$ in the interval of $\left[-\frac{1}{2}, \frac{1}{2}\right]$, one can approximate the expressions in the exponential. For some constants K (large) and c (small) we get:

$$C^i(x-y, x+y) \leq KM^{2i} e^{-c\left(M^i||x-y||+M^{-i}||x+y||\right)}. \tag{2.102}$$

With these steps one is able to simplify the calculations a lot, because instead of complicated integrals one has now sums of simpler expressions. By looking at certain representative subgraphs (hence the name multiscale, because one looks at different scales) and applying such estimations one can win a powerful "power counting" formula with which one can prove the renormalizability of the theory.

2.8 The Mehler kernel in the limit $\Omega \to 0$

We now want to compare the Mehler kernel with the ordinary result for the Propagator

$$\Delta(p, q) = \frac{1}{p^2}\delta^4(p-q). \tag{2.103}$$

To simplify calculations (and to avoid calculations involving test functions) we will compare the two results where we have already integrated out one momentum

$$\int d^4p \Delta(p, q) = \frac{1}{q^2}. \tag{2.104}$$

We now integrate out one momentum of the Mehler kernel

$$\begin{aligned}
&\int K_M(p, q) d^4p \\
=&\frac{\omega^3}{2\pi^2}\int_0^\infty d\alpha\ \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}\int d^4p \exp\left[-\frac{\frac{\omega}{2}(p^2+q^2)(1+e^{-2\alpha}) - 2\omega e^{-\alpha}pq}{1-e^{-2\alpha}}\right] \\
=&\frac{\omega^3}{2\pi^2}\int_0^\infty d\alpha\ \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}\int d^4p \exp\left[-\frac{\frac{\omega}{2}p^2(1+e^{-2\alpha}) - 2\omega e^{-\alpha}pq + \frac{\omega}{2}q^2(1+e^{-2\alpha})}{1-e^{-2\alpha}}\right].
\end{aligned} \tag{2.105}$$

Quadratic completition gives

$$
\begin{aligned}
=&\frac{\omega^3}{2\pi^2}\int\limits_0^\infty d\alpha\ \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}\int d^4p \\
&\exp\left[-\left(p\underbrace{\sqrt{\frac{(1+e^{-2\alpha})\frac{\omega}{2}}{1-e^{-2\alpha}}}}_{A}-\frac{\frac{\omega e^{-\alpha}q}{1-e^{-2\alpha}}}{A}\right)^2+\left(\frac{\frac{\omega e^{-\alpha}q}{1-e^{-2\alpha}}}{A}\right)^2-\frac{\frac{\omega}{2}q^2(1+e^{-2\alpha})}{1-e^{-2\alpha}}\right].
\end{aligned}
\tag{2.106}
$$

We now solve the Gauß integral and get

$$
=\frac{\omega^3}{2\pi^2}\int\limits_0^\infty d\alpha\ \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}\frac{\pi^2}{A^4}\exp\left[\left(\frac{\sqrt{2\omega}e^{-\alpha}q}{\sqrt{(1+e^{-2\alpha})(1-e^{-2\alpha})}}\right)^2-\frac{\frac{\omega}{2}q^2(1+e^{-2\alpha})}{1-e^{-2\alpha}}\right].
\tag{2.107}
$$

Bringing the exponentiated terms to the same denominator and inserting back the variable A gives

$$
\begin{aligned}
=&\frac{\omega^3}{2}\int\limits_0^\infty d\alpha\ \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}\left(\frac{1-e^{-2\alpha}}{(1+e^{-2\alpha})\frac{\omega}{2}}\right)^2 \\
&\qquad\exp\left[\frac{2\omega e^{-2\alpha}q^2}{(1-e^{-2\alpha})(1+e^{-2\alpha})}-\frac{\frac{\omega}{2}q^2(1+e^{-2\alpha})}{1-e^{-2\alpha}}\right] \\
=&\frac{\omega^3}{2}\int\limits_0^\infty d\alpha\ \frac{e^{-2\alpha}}{[\frac{\omega}{2}(1+e^{-2\alpha})]^2}\exp\left[q^2\omega\left(\frac{2e^{-2\alpha}-\frac{1}{2}(1+e^{-2\alpha})^2}{(1+e^{-2\alpha})(1-e^{-2\alpha})}\right)\right].
\end{aligned}
\tag{2.108}
$$

You may have realized the numerous appearances of the term $e^{-2\alpha}$. Indeed, we can substitute this term to simplify our expression, so we substitute

$$
\begin{aligned}
\lambda &= e^{-2\alpha} \\
\Rightarrow d\alpha &= -\frac{1}{2\lambda}d\lambda.
\end{aligned}
\tag{2.109}
$$

Then we get

$$
\begin{aligned}
&\frac{\omega^3}{2}\int\limits_0^1 d\lambda\left(-\frac{1}{2\lambda}\right)\frac{\lambda}{(\frac{\omega}{2})^2(1+\lambda)^2}\exp\left(q^2\omega\frac{2\lambda-(1+\lambda)^2\frac{1}{2}}{1-\lambda^2}\right)\\
=&\frac{\omega^3}{2}\int\limits_0^1 d\lambda\left(-\frac{1}{2}\right)\frac{1}{(\frac{\omega}{2})^2(1+\lambda)^2}\exp\left(\frac{q^2\omega}{2}\frac{1-2\lambda+\lambda^2}{1-\lambda^2}\right)\\
=&\omega\int\limits_0^1 d\lambda\frac{1}{(1+\lambda)^2}\exp\left(\frac{q^2\omega}{2}\frac{1-\lambda}{1+\lambda}\right).
\end{aligned}
\tag{2.110}
$$

Now, any mathematics program does the job for us to integrate this out. The result is

$$
=\frac{1-e^{-\frac{q^2\omega}{2}}}{q^2}. \tag{2.111}
$$

Now we take the limit $\Omega \to 0$ ($\Rightarrow \omega \to \infty$), which corresponds to ordinary noncommutative quantum field theory without the oscillator term. We get as promised

$$
=\frac{1}{q^2}. \tag{2.112}
$$

2.9 Verifying that the Mehler kernel is indeed the inverse of $p^2 - \frac{\partial^2}{\omega^2\partial p^2}$

Letting $p^2 - \frac{\partial^2}{\omega^2\partial p^2}$ act on the Mehler kernel (2.99) we get

$$
\begin{aligned}
&\frac{\omega^3}{2\pi^2}\left(p^2-\frac{\partial^2}{\omega^2\partial p^2}\right)\int\limits_0^\infty dt\frac{e^{-2t}}{(1-e^{-2t})^2}\exp\left(-\frac{\frac{\omega}{2}(p^2+q^2)(1+e^{-2t})-2\omega e^{-t}pq}{1-e^{-2t}}\right)\\
=&\frac{\omega^3}{2\pi^2}\int\limits_0^\infty dt\left(\frac{4}{\omega}\frac{1+e^{-2t}}{1-e^{-2t}}+4pq\frac{e^{-t}(1+e^{-2t})}{(1-e^{-2t})^2}-4(p^2+q^2)\frac{e^{-2t}}{(1-e^{-2t})^2}\right)\\
&\qquad\frac{e^{-2t}}{(1-e^{-2t})^2}\exp\left(-\frac{\frac{\omega}{2}(p^2+q^2)(1+e^{-2t})-2\omega e^{-t}pq}{1-e^{-2t}}\right)\\
\overset{!}{=}&\delta^{(4)}(p-q).
\end{aligned}
\tag{2.113}
$$

We set $q = 0$ and integrate over p:

$$= \int d^4p \frac{\omega^3}{2\pi^2} \int\limits_0^\infty dt \left(\frac{4}{\omega} \frac{1+e^{-2t}}{1-e^{-2t}} - 4p^2 \frac{e^{-2t}}{(1-e^{-2t})^2} \right)$$

$$\frac{e^{-2t}}{(1-e^{-2t})^2} \exp\left(-\frac{\frac{\omega}{2}p^2(1+e^{-2t})}{1-e^{-2t}} \right)$$

$$= \int d^4p' \frac{\omega^3}{2\pi^2} \int\limits_0^\infty dt \left(\frac{4}{\omega} \frac{1+e^{-2t}}{1-e^{-2t}} - 4p'^2 \left(\frac{2}{\omega} \frac{1-e^{-2t}}{1+e^{-2t}} \right) \frac{e^{-2t}}{(1-e^{-2t})^2} \right)$$

$$\frac{e^{-2t}}{(1-e^{-2t})^2} \left(\frac{2}{\omega} \frac{1-e^{-2t}}{1+e^{-2t}} \right)^2 e^{-p'^2}$$

$$= \int d^4p' \frac{8}{\pi^2} \int\limits_0^\infty dt \left(\frac{1+e^{-2t}}{1-e^{-2t}} - 2p'^2 \frac{e^{-2t}}{1-e^{-4t}} \right) \frac{e^{-2t}}{(1+e^{-2t})^2} e^{-p'^2}. \qquad (2.114)$$

Using the formulas

$$\int d^4p'\, e^{-p'^2} = \pi^2$$

$$\int d^4p'\, p'^2 e^{-p'^2} = 2\pi^2 \qquad (2.115)$$

we arrive at

$$= 8 \int\limits_0^\infty dt \left(\frac{1+e^{-2t}}{1-e^{-2t}} - 4 \frac{e^{-2t}}{1-e^{-4t}} \right) \frac{e^{-2t}}{(1+e^{-2t})^2}$$

$$= 8 \int\limits_0^\infty dt \frac{1-e^{-2t}}{1+e^{-2t}} \frac{e^{-2t}}{(1+e^{-2t})^2}$$

$$= 8 \int\limits_0^\infty dt \frac{(1-e^{-2t})(e^{-2t})}{(1+e^{-2t})^3}. \qquad (2.116)$$

Upon substituting $e^{-2t} = \lambda$ we get

$$= -4 \int\limits_1^0 dt \frac{1-\lambda}{(1+\lambda)^3} = 4 \int\limits_0^1 dt \frac{1-\lambda}{(1+\lambda)^3} = 1, \qquad (2.117)$$

which we of course expected because

$$\int d^4p \delta^{(4)}(p-q)\Big|_{q=0} = \int d^4p \delta^{(4)}(p) = 1 \quad \text{q.e.d.} \qquad (2.118)$$

Chapter 3

Feynman rules

3.1 General remarks on Feynman rules

The general question is how to calculate propagators and vertices.
To show how that works we will mainly follow ([21]). For further details look up books about the path integral formalism, e.g. [27].
The free propagator (often called the free 2-point Green function) is defined as the time ordered expectation value of the free fields

$$\Delta_{ab}(x,y) =< 0|T\Psi_a(x)\Psi_b(y)|0>_{(0)} . \tag{3.1}$$

Here Ψ_a stands for any field and the (0) denotes that we are looking at the free fields. The symbol T stands for the time ordering operator

$$TA(t_1)B(t_2) = A(t_1)B(t_2)\Theta(t_1 - t_2) + B(t_2)A(t_1)\Theta(t_2 - t_1) \tag{3.2}$$

for arbitrary operators A, B. In the path integral formalism, which is an alternative formalism to quantum field theory, one introduces the generating functional Z for all Green functions in an Euclidian space as the vacuum to vacuum transition amplitude

$$Z[J] =< 0|Te^{-\int d^4xJ_a(x)\Psi_a(x)}|0>= \frac{\int \mathcal{D}[\psi]e^{-\int d^4x(S+J_a(x)\psi_a(x))}}{\int \mathcal{D}[\psi]e^{-\int d^4xS}}, \tag{3.3}$$

where J_a are the sources of the fields ψ_a. This is known as the Gell-Mann-Low formula. The denominator is a normalization factor which kills all vacuum graphs, i.e. graphs without external legs. The field $\Psi_a(x)$ on the left hand side of the formula denotes the full field with all the quantum corrections, while the field $\psi_a(x)$ on the right hand side is the free field.

By varying $Z[J]$ twice with respect to the sources and by then setting them equal to zero one gets

$$\frac{\delta^2 Z[J]}{\delta J_a(x)\delta J_b(y)}\Big|_{J=0} =< 0|T\Psi_a(x)\Psi_b(y)|0 > . \tag{3.4}$$

This looks formally like (3.3), but here the fields $\psi_a(x)$ are not the free fields. Nevertheless, since it's a 2 point function, the propagator must be contained in this expression. Indeed, the propagator is its 0^{th} order, that means without any quantum corrections. Later on we will consider only the 0^{th} order because we have to work on the tree level.
The next step is to specify on the generating functional for the connected Green functions Z^c because disconnected graphs are physically uninteresting for us and would furthermore lead to divergences because there is an infinite number of possibilities that particles do not interact. Therefore we define Z^c

$$Z[J] = e^{-Z^c[J]}. \tag{3.5}$$

We now perform a Legendre-transformation of Z^c given by

$$\Gamma[\psi^{cl}] := Z^c[J] - \int d^4x J_a(x)\psi_a^{cl}(x), \tag{3.6}$$

where the classical fields ψ^{cl} (which are Schwartz fast decreasing test functions) are defined as

$$\psi_a^{cl} = \frac{\delta Z^c[J]}{\delta J_a(x)}. \tag{3.7}$$

They are called classical because they are the vacuum expectation values of the field operators. Γ is the full action plus quantum corrections. That this is indeed the case is a rather long proof which I don't want to give, but you can find it in several quantum field theory books like e.g. [28].
Furthermore, for renormalizable theories it can be shown that $\Gamma_0[\psi^{cl}]$ reduces to $S_0[\psi^{cl}]$, that means for the effective bilinear action (and we are just interested in this expression for the derivation of the propagators) we get the classical action S_0. This means that from now on we only look at the 0^{th} order of (3.4).

Obviously, this Legendre transformation allows us to express the fields by variation of Z^c and the sources by variations of S_0.

$$\begin{aligned}\frac{\delta Z^c[J]}{\delta J_a(x)} &= \psi_a^{cl} \\ \frac{\delta S_0}{\delta \psi_a^{cl}} &= -J_a(x).\end{aligned} \tag{3.8}$$

With these ingredients we can now show that we can write the propagator as

$$-\frac{\delta^2 Z^c}{\delta J_a(x)\delta J_b(y)} = -\frac{\delta \psi_b^{cl}(y)}{\delta J_a(x)}. \tag{3.9}$$

We'll prove this. Looking at the first order of (3.4) we conclude

$$\begin{aligned}\Delta_{ab}(x,y) = < 0|T\psi_a(x)\psi_b(y)|0 >_{(0)} &= \frac{\delta^2 Z_{(0)}[J]}{\delta J_a(x)\delta J_b(y)}\Big|_{J=0} \\ =\frac{\delta^2 e^{-Z^c_{(0)}[J]}}{\delta J_a(x)\delta J_b(y)}\Big|_{J=0} &= \frac{\delta}{\delta J_a(x)}\left(-\frac{\delta Z^c_{(0)}[J]}{\delta J_b(y)}e^{-Z^c_{(0)}[J]}\right)\Big|_{J=0} \\ = -\frac{\delta^2 Z^c_{(0)}[J]}{\delta J_a(x)\delta J_b(y)}e^{-Z^c_{(0)}[J]}\Big|_{J=0} &+ \frac{\delta Z^c_{(0)}[J]}{\delta J_a(x)}\frac{\delta Z^c_{(0)}[J]}{\delta J_b(y)}e^{-Z^c_{(0)}[J]}\Big|_{J=0} \\ = -\frac{\delta^2 Z^c_{(0)}[J]}{\delta J_a(x)\delta J_b(y)}\Big|_{J=0} &+ \frac{\delta Z^c_{(0)}[J]}{\delta J_a(x)}\frac{\delta Z^c_{(0)}[J]}{\delta J_b(y)}\Big|_{J=0}.\end{aligned} \tag{3.10}$$

The Index (0) denotes that we are looking at the 0^{th} order, that means for the action we take only the bilinear part S_0.
The 2^{nd} term in this expression corresponds to the unphysical one-point Green functions, also called tadpoles. Those are unphysical contributions which normally turn out to be zero, or at least can be normalized to zero. We shall see later on that we indeed are able to achieve this aim.
Following these statements we can leave out the 2^{nd} term and have therefore verified formula (3.9).

What's left to mention is that the practical application of formula (3.9) is performed due to the chain rule, that means following formula (3.7) we first differentiate $Z_0 = S_0 + J_a\psi_a$ with respect to the fields and then with respect to the sources, not forgetting to add the minus sign, and voilà, we get the propagator.

3.2 Propagators

We start at the action taken from [20]. As mentioned in the introduction we have here an oscillator like term, inspired by the success of the Grosse-Wulkenhaar model. Apart from this we have of course ghost fields c and $\bar{c}$ with an oscillator term too to improve the behaviour of the ghost propagator. Furthermore we have a multiplier field $\tilde{c}_\mu$ implementing BRST-invariance.

$$
\begin{aligned}
\Gamma^{(0)} &= \Gamma_{\mathrm{inv}} + \Gamma_{\mathrm{m}} + \Gamma_{\mathrm{gf}}, \\
\Gamma_{\mathrm{inv}} &= \frac{1}{4}\int d^4x\, F_{\mu\nu} \star F_{\mu\nu}, \\
\Gamma_{\mathrm{m}} &= \frac{\Omega^2}{4}\int d^4x \left(\frac{1}{2}\left\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\right\} \star \left\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\right\} + \left\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\right\} \star \left\{\tilde{x}_\mu \stackrel{\star}{,} c\right\} \right) \\
&= \frac{\Omega^2}{8}\int d^4x\, (\tilde{x} \star \mathcal{C}_\mu), \\
\Gamma_{\mathrm{gf}} &= \int d^4x \left[B \star \partial_\mu A_\mu - \frac{1}{2} B \star B - \bar{c} \star \partial_\mu s A_\mu - \frac{\Omega^2}{8}\tilde{c}_u \star s\, \mathcal{C}_\mu \right]
\end{aligned} \tag{3.11}
$$

with

$$
\mathcal{C}_\mu = \Big(\left\{\left\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\right\} \stackrel{\star}{,} A_\nu\right\} + \left[\left\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\right\} \stackrel{\star}{,} c\right] + \left[\bar{c} \stackrel{\star}{,} \left\{\tilde{x}_\mu \stackrel{\star}{,} c\right\}\right] \Big). \tag{3.12}
$$

B is the multiplier field implementing a non-linear gauge fixing

$$
\frac{\delta\Gamma^{(0)}}{\delta B} = \partial_\mu A_\mu - B + \frac{\Omega^2}{8}\left(\left[\left\{\tilde{x}_\mu \stackrel{\star}{,} c\right\} \stackrel{\star}{,} \tilde{c}_\mu\right] - \left\{\tilde{x}_\mu \stackrel{\star}{,} \left[\tilde{c}_\mu \stackrel{\star}{,} c\right]\right\} \right). \tag{3.13}
$$

The action is invariant under the BRST transformations given by

$$
\begin{aligned}
&sA_\mu = D_\mu c = \partial_\mu c - ig\left[A_\mu \stackrel{\star}{,} c\right], && s\bar{c} = B, \\
&sc = igc \star c, && sB = 0, \\
&s\tilde{c}_\mu = \tilde{x}_\mu && s^2\phi = 0 \forall \phi \in \{A_\mu, B, c, \bar{c}\tilde{c}_\mu\}.
\end{aligned} \tag{3.14}
$$

B is the multiplier field implementing the gauge fixing, which for $\tilde{c}_\mu \to 0$ reduces to the usual covariant Feynman gauge $\partial_\mu A_\mu - B = 0$. Ω is a constant parameter and c, $\bar{c}$ are the ghost/antighost, respectively. The "mass" term for the ghosts (cf. second term in Γ_m) has been introduced in order to have a Mehler kernel also for the ghost propagator. The field $\tilde{c}_\mu$ is a multiplier field with mass dimension 1 and ghost number -1, which imposes a constraint, namely on-shell BRST invariance of $\mathcal{C}_\mu$. In fact, because of $s\tilde{x}_\mu = 0$, this constraint also implies on-shell BRST invariance of the mass terms Γ_m. Furthermore, $s^2\mathcal{C}_\mu = 0$ vanishes identically, i.e. off-shell.

The bilinear part of the action is given by

$$S_{\rm Bi} = \int d^4x \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{3.15}$$
$$+ \frac{\Omega^2}{8}\tilde{x}_\mu \Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}] \Big)$$
$$+ B\partial_\mu A_\mu - \frac{1}{2}B^2 - \bar{c}\partial_\mu\partial_\mu c + j^A_\mu A_\mu + j_{\bar{c}} c + j_B B + j_c \bar{c},$$

where we've already eliminated one star due to the property that one star can be left out under an integral. Varying this action with respect to the fields gives

$$\frac{\delta S_{\rm Bi}(x)}{\delta B(y)} = \int d^4x (\partial_\mu A_\mu(x) - B(x) + j_B(x))\delta(x-y) = 0 \tag{3.16}$$
$$\Rightarrow B = \partial_\mu A_\mu + j_B$$

for the B-field and

$$\frac{\delta S_{\rm Bi}}{\delta A_\nu} =$$
$$= -\Box A_\nu + \partial_\mu\partial_\nu A_\mu + \frac{\Omega^2}{8} 2\left(\tilde{x}_\mu\tilde{x}_\mu \star A_\nu + \tilde{x}_\mu \left\{A_\nu \stackrel{\star}{,} \tilde{x}_\mu\right\} + \tilde{x}_\mu A_\nu \star \tilde{x}_\mu\right) - \partial_\nu B + j^A_\nu$$
$$= -\Box A_\nu + \partial_\mu\partial_\nu A_\mu + \frac{\Omega^2}{4}\left(2\tilde{x}_\mu\tilde{x}_\mu \star A_\nu + 2\tilde{x}_\mu A_\nu \star \tilde{x}_\mu\right) - \partial_\nu B + j^A_\nu$$
$$= -\Box A_\nu + \partial_\mu\partial_\nu A_\mu + \frac{\Omega^2}{2}\tilde{x}_\mu \left\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\right\} - \partial_\nu B + j^A_\nu$$
$$= -\Box A_\nu + \partial_\mu\partial_\nu A_\mu + \frac{\Omega^2}{2}\tilde{x}_\mu \left\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\right\} - \partial_\nu(\partial_\mu A_\mu + j_B) + j^A_\nu$$
$$= -\Box A_\nu + \frac{\Omega^2}{2}\tilde{x}_\mu\{\tilde{x}_\mu, A_\nu\} - \partial_\nu j_B + j^A_\nu$$
$$= -\Box A_\nu + \Omega^2\tilde{x}_\mu\tilde{x}_\mu A_\nu - \partial_\nu j_B + j^A_\nu$$

$$\Rightarrow \Box A_\nu + \Omega^2\tilde{x}_\mu\tilde{x}_\mu A_\nu - \partial_\nu = \partial_\nu j_B - j^A_\nu$$
$$\Rightarrow A_\nu = \frac{1}{-\Box + \Omega^2\tilde{x}_\mu\tilde{x}_\mu}(\partial_\nu j_B - j^A_\nu) \tag{3.17}$$

for the A_ν-field where we have inserted the expression for the B-field from above and where we have used $\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} = \{\tilde{x}_\mu, A_\nu\}$. Also we have eliminated the δ-function via integration already in the first line and left out the arguments of the fields to avoid blowing up the formulas.

The photon propagator is given by

$$G^A_{\mu\nu}(x,y) = -\frac{\delta A_\nu(x)}{\delta j^A_\mu(y)} = -\frac{1}{-\Box + \Omega^2\tilde{x}^2}\delta^4(x-y)g_{\mu\nu}. \tag{3.18}$$

The photon-B-field propagator is

$$G^{BA}_\nu(x,y) = -\frac{\delta A_\nu}{\delta j_B} = -\frac{1}{-\Box + \Omega^2\tilde{x}^2}\partial_\nu\delta^4(x-y). \tag{3.19}$$

We now want to calculate the propagator for the B-field. In expression (3.16) the B-field still depends on A_μ. Therefore we have to insert the expression for the A_μ-field into (3.16):

$$B = \partial_\nu A_\nu + j_B = \partial_\nu \frac{1}{-\Box + \Omega^2\tilde{x}_\mu\tilde{x}_\mu}(\partial_\nu j_B - j^A_\nu) + j_B. \tag{3.20}$$

The propagator for the B-field is therefore

$$G^B(x,y) = -\frac{\delta B(x)}{\delta j_B(y)} = \left[-\partial_\nu \frac{1}{-\Box + \Omega^2\tilde{x}_\mu\tilde{x}_\mu}\partial_\nu - 1\right]\delta^4(x-y). \tag{3.21}$$

Let's now look at the ghost part:

$$\begin{aligned} &S_{\mathrm{Bi}}(\text{ghost part}) = \\ &= \int d^4x\, \frac{\Omega^2}{8}\tilde{x}_\mu \star \Big([\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}] \Big) - \bar{c} \star \partial_\mu\partial_\mu c + j_{\bar{c}}c + j_c\bar{c}. \end{aligned} \tag{3.22}$$

Expanding the commutators and the anticommutators gives

$$\begin{aligned} = \int d^4x\, \frac{\Omega^2}{8}\tilde{x}_\mu \star \Big(&\tilde{x}_\mu \star \bar{c} \star c + 2\bar{c} \star \tilde{x}_\mu \star c - 2c \star \tilde{x}_\mu \star \bar{c} \\ &- c \star \bar{c} \star \tilde{x}_\mu + \bar{c} \star c \star \tilde{x}_\mu - \tilde{x}_\mu \star c \star \bar{c}\Big) \\ &- \bar{c} \star \Box c + j_{\bar{c}}c + j_c\bar{c}. \end{aligned} \tag{3.23}$$

We now use the property of the star product that we can cyclic permutate the fields under the integral. We now have to be very careful: when we pull a fermion through another fermion the grading gives a minus sign. So when we want to cyclic permutate a c-field from the end of a term to the beginning, we'll get an extra minus sign (because we only have bilinear terms). We also partially integrate the quabla term

$$\begin{aligned} = \int d^4x\, \frac{\Omega^2}{8} c \star \Big(&- 2\tilde{x}_\mu \star \tilde{x}_\mu \star \bar{c} - 4\tilde{x}_\mu \star \bar{c} \star \tilde{x}_\mu - 2\bar{c} \star \tilde{x}_\mu \star \tilde{x}_\mu\Big) \\ &+ c \star \Box\bar{c} + j_{\bar{c}}c + j_c\bar{c}. \end{aligned} \tag{3.24}$$

Varying the action with respect to c gives the corresponding field equation, where we've again already eliminated the integral with the delta function.

$$\begin{aligned}\frac{\delta S_{\text{Bi}}(x)}{\delta c(y)} =& \frac{\Omega^2}{8}\Big(- 2\tilde{x}_\mu \star \tilde{x}_\mu \star \bar{c} - 4\tilde{x}_\mu \star \bar{c} \star \tilde{x}_\mu - 2\bar{c} \star \tilde{x}_\mu \star \tilde{x}_\mu\Big) \\ &+ \Box\bar{c} - j_{\bar{c}},\end{aligned} \tag{3.25}$$

where the current $j_{\bar{c}}$ has changed its sign because in order to vary $\bar{c}$ we had to permutate the variation through $j_{\bar{c}}$ and this gave a minus sign due to the fermionic character of $j_{\bar{c}}$. We can now rewrite the terms in the bracket in terms of anticommutators:

$$\begin{aligned}&= \frac{\Omega^2}{8}\Big(- 2\tilde{x}_\mu \star \{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} - 2\,\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \star \tilde{x}_\mu\Big) + \Box\bar{c} - j_{\bar{c}} \\ &= \frac{\Omega^2}{8}(-2)\,\{\tilde{x}_\mu \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\}\} + \Box\bar{c} - j_{\bar{c}}.\end{aligned} \tag{3.26}$$

Now we use another property of the star product, namely $\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} = \{\tilde{x}_\mu, \bar{c}\}$. Thus we get

$$\begin{aligned}&= -\frac{\Omega^2}{4}\{\tilde{x}_\mu, \{\tilde{x}_\mu, \bar{c}\}\} + \Box\bar{c} - j_{\bar{c}} \\ &= -\frac{\Omega^2}{4}\Big(4\tilde{x}_\mu\tilde{x}_\mu\bar{c}\Big) + \Box\bar{c} - j_{\bar{c}} \\ &= -\Omega^2\tilde{x}_\mu\tilde{x}_\mu\bar{c} + \Box\bar{c} - j_{\bar{c}} \\ &= [-\Omega^2{\tilde{x}_\mu}^2 + \Box]\bar{c} - j_{\bar{c}} = 0.\end{aligned} \tag{3.27}$$

To understand this I want to remind the reader that at this stage we have no stars any more and therefore only commuting fields.
The propagator is now

$$G^{\bar{c}c}(x,y) = -\frac{\delta\bar{c}(y)}{\delta j_{\bar{c}}(x)} = -\frac{1}{-\Box + \Omega^2\tilde{x}^2}\delta^4(x-y). \tag{3.28}$$

Derivating the action with respect to the ghost and the antighost in reversed order compared to what we have calculated yet works in the same way and yields only a minus sign:

$$G^{c\bar{c}}(x,y) = -\frac{\delta c(y)}{\delta j_c(x)} = \frac{1}{-\Box + \Omega^2{\tilde{x}_\mu}^2}\delta^4(x-y). \tag{3.29}$$

3.3 Which building blocks for loop graphs do we have?

Let's first look at the gauge invariant part of the action:

$$S_{\text{inv}} = \int d^4x \frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu - ig \left[A_\mu \stackrel{\star}{,} A_\nu\right])^2 \frac{1}{4}. \tag{3.30}$$

We pick out only the terms of order 3 or higher and ignore the bilinear terms which we have already treated in the previous section to calculate the propagators:

$$\begin{aligned}
&= \int d^4x \, \frac{-g^2}{4} \left[A_\mu \stackrel{\star}{,} A_\nu\right] \star \left[A_\mu \stackrel{\star}{,} A_\nu\right] - \frac{ig}{2} \partial_\mu A_\nu \star \left[A_\mu \stackrel{\star}{,} A_\nu\right] - \frac{ig}{2} \left[A_\mu \stackrel{\star}{,} A_\nu\right] \star \partial_\mu A_\nu \\
&= \int d^4x \, \frac{-g^2}{2} A_\mu \star A_\nu \star \left[A_\mu \stackrel{\star}{,} A_\nu\right] - ig \partial_\mu A_\nu \star \left[A_\mu \stackrel{\star}{,} A_\nu\right], \qquad (3.31)
\end{aligned}$$

where we have renamed indices and used the property that we can cyclic permutate fields under the integral to be able to sum up terms. We can see that we therefore have a *4 photon vertex* and a *3 photon vertex*.

Now let's look at the gauge fixing part (where we have to leave out the bilinear part). On the one hand we have

$$\begin{aligned}
\bar{c} \star \partial_\mu s A_\mu &= \bar{c} \star \partial_\mu (\partial_\mu c - ig \left[A_\mu \stackrel{\star}{,} c\right]) \qquad (3.32) \\
&= (\text{only trilinear}) = -ig \bar{c} \partial_\mu \left[A_\mu \stackrel{\star}{,} c\right]
\end{aligned}$$

which leads to the *2 ghosts 1 photon vertex*, and on the other hand we have

$$\tilde{c} \star s\mathcal{C}_\mu = \tilde{c} \star s\Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}] \Big), \tag{3.33}$$

but these terms always lead to vertices with a $\tilde{c}$-leg and since we have no $\tilde{c}$ propagator we can't construct a Feynman graph with such vertices.

We'll now sum up the results of this chapter in the following table:

3 photon vertex	$-ig\partial_\mu A_\nu \star \left[A_\mu \stackrel{\star}{,} A_\nu\right]$
4 photon vertex	$\frac{-g^2}{2} A_\mu \star A_\nu \star \left[A_\mu \stackrel{\star}{,} A_\nu\right]$
2 ghosts 1 photon vertex	$-ig\bar{c}\partial_\mu \left[A_\mu \stackrel{\star}{,} c\right]$

3.4 3-photon vertex

The relevant term in the action is

$$S_{\text{int}} = \int d^4x \; -ig\partial_\mu A_\nu(x) \star [A_\mu(x) \stackrel{\star}{,} A_\nu(x)]\,. \tag{3.34}$$

As a first step we try to evaluate the commutator of a star product.

$$\begin{aligned}
A_\mu(x) \star A_\nu(x) &= \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} e^{i(k_1+k_2)x} \tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2) e^{-\frac{i}{2}k_1\times k_2} \\
\Rightarrow [A_\mu(x) \stackrel{\star}{,} A_\nu(x)] &= \\
= \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} & e^{i(k_1+k_2)x} \tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2) \left(\underbrace{e^{-\frac{i}{2}k_1\times k_2} - e^{+\frac{i}{2}k_1\times k_2}}_{-2i\sin(\frac{1}{2}k_1\times k_2)} \right),
\end{aligned}$$

where we have used the abbreviation $k_1 \times k_2 = k_1\Theta k_2 = k_{1\mu}\Theta_{\mu\nu}k_{2\nu}$.
After this calculation we can easily conclude what (3.34) is:

$$\begin{aligned}
S_{\text{int}} =& \int d^4x \; -ig\partial_\mu A_\nu(x) \star [A_\mu(x) \stackrel{\star}{,} A_\nu(x)] \qquad (3.35) \\
=& -ig \int d^4x \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} e^{i(k_1+k_2+k_3)x} (ik_1^\mu)\tilde{A}_\nu(k_1)\tilde{A}_\mu(k_2)\tilde{A}_\nu(k_3) \\
& \left[e^{-\frac{i}{2}(k_1\times k_2 + k_2\times k_3 + k_1\times k_3)} - e^{-\frac{i}{2}(k_1\times k_2 + k_3\times k_2 + k_1\times k_3)} \right].
\end{aligned}$$

We'll now use that $k_3 \times k_2 = -k_2 \times k_3$ and therefore get

$$\begin{aligned}
=& ig \int d^4x \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} e^{i(k_1+k_2+k_3)x} (ik_1^\mu)\tilde{A}_\nu(k_1)\tilde{A}_\mu(k_2)\tilde{A}_\nu(k_3) \\
& 2i\sin\left(\frac{k_2\times k_3}{2}\right) \qquad (3.36) \\
=& -2ig(2\pi)^4 \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \delta(k_1+k_2+k_3) k_1^\mu \tilde{A}_\nu(k_1)\tilde{A}_\mu(k_2)\tilde{A}_\nu(k_3) \\
& \sin\left(\frac{k_2\times k_3}{2}\right). \qquad (3.37)
\end{aligned}$$

The rule for evaluating this vertex is

$$V^{3A} = -(2\pi)^{12} \frac{\delta}{\delta\tilde{A}^\rho(-k_1)} \frac{\delta}{\delta\tilde{A}^\sigma(-k_2)} \frac{\delta}{\delta\tilde{A}^\tau(-k_3)} S_{\text{int}}. \tag{3.38}$$

We'll use the delta function to rewrite the sine: $\sin(q_2 \times q_3) = -\sin(q_1 \times q_3)$. Applying the rule for calculating the vertex to (3.37) gives

$$
\begin{aligned}
&\frac{\delta}{\delta \tilde{A}^{\sigma}(-k_2)} \frac{\delta}{\delta \tilde{A}^{\tau}(-k_3)} \frac{\delta}{\delta \tilde{A}^{\rho}(-k_1)} (-2ig)(2\pi)^4 \int d^4q_1 \int d^4q_2 \int d^4q_3 \\
&\delta(q_1 + q_2 + q_3) q_1^{\mu} \tilde{A}_{\nu}(q_1) \tilde{A}_{\mu}(q_2) \tilde{A}_{\nu}(q_3) \sin\left(\frac{q_1 \times q_3}{2}\right) \\
=&\frac{\delta}{\delta \tilde{A}^{\sigma}(-k_2)} \frac{\delta}{\delta \tilde{A}^{\tau}(-k_3)} (-2ig)(2\pi)^4 \int d^4q_1 \int d^4q_2 \int d^4q_3 \\
&\delta(q_1 + q_2 + q_3) q_1^{\mu} \sin\left(\frac{q_1 \times q_3}{2}\right) \\
&\Big[\delta^{\rho}_{\nu} \delta(k_1 + q_1) \tilde{A}_{\mu}(q_2) \tilde{A}_{\nu}(q_3) + \delta^{\rho}_{\mu} \delta(k_1 + q_2) \tilde{A}_{\nu}(q_1) \tilde{A}_{\nu}(q_3) \\
&\quad + \delta^{\rho}_{\nu} \delta(k_1 + q_3) \tilde{A}_{\nu}(q_1) \tilde{A}_{\mu}(q_2)\Big] \\
=&\frac{\delta}{\delta \tilde{A}^{\sigma}(-k_2)} (-2ig)(2\pi)^4 \int d^4q_1 \int d^4q_2 \int d^4q_3 \delta(q_1 + q_2 + q_3) q_1^{\mu} \sin\left(\frac{q_1 \times q_3}{2}\right) \\
&\Big[\delta^{\rho}_{\nu} \delta(k_1 + q_1) \left(\delta^{\sigma}_{\mu} \delta(k_2 + q_2) \tilde{A}_{\nu}(q_3) + \delta^{\sigma}_{\nu} \delta(k_2 + q_3) \tilde{A}_{\mu}(q_2)\right) \\
&+ \delta^{\rho}_{\mu} \delta(k_1 + q_2) \left(\delta^{\sigma}_{\nu} (k_2 + q_1) \tilde{A}_{\nu}(q_3) + \delta^{\sigma}_{\nu} \delta(k_2 + q_3) \tilde{A}_{\nu}(q_1)\right) \\
&+ \delta^{\rho}_{\nu} \delta(k_1 + q_3) \left(\delta^{\sigma}_{\nu} \delta(k_2 + q_1) \tilde{A}_{\mu}(q_2) + \delta^{\sigma}_{\mu} \delta(k_2 + q_2) \tilde{A}_{\nu}(q_1)\right) \Big] \\
=&(-2ig)(2\pi)^4 \int d^4q_1 \int d^4q_2 \int d^4q_3 \delta(q_1 + q_2 + q_3) q_1^{\mu} \sin\left(\frac{q_1 \times q_3}{2}\right) \\
&\big[\delta^{\rho}_{\nu} \delta^{\sigma}_{\mu} \delta^{\tau}_{\nu} \delta(k_1 + q_1) \delta(k_2 + q_2) \delta(k_3 + q_3) + \delta^{\rho}_{\nu} \delta^{\sigma}_{\nu} \delta^{\tau}_{\mu} \delta(k_1 + q_1) \delta(k_2 + q_3) \delta(k_3 + q_2) \\
&+ \delta^{\rho}_{\mu} \delta^{\sigma}_{\nu} \delta^{\tau}_{\nu} \delta(k_1 + q_2) \delta(k_2 + q_1) \delta(k_3 + q_3) + \delta^{\rho}_{\mu} \delta^{\sigma}_{\mu} \delta^{\tau}_{\nu} \delta(k_1 + q_2) \delta(k_2 + q_3) \delta(k_3 + q_1) \\
&+ \delta^{\rho}_{\nu} \delta^{\sigma}_{\nu} \delta^{\tau}_{\mu} \delta(k_1 + q_3) \delta(k_2 + q_1) \delta(k_3 + q_2) + \delta^{\rho}_{\nu} \delta^{\sigma}_{\mu} \delta^{\tau}_{\nu} \delta(k_1 + q_3) \delta(k_2 + q_2) \delta(k_3 + q_1)\big].
\end{aligned}
$$

We'll now eliminate the delta functions by summation over indices and then by solving integrals with the delta functions.

$$
\begin{aligned}
&(-2ig)(2\pi)^4 \int d^4q_1 \int d^4q_2 \int d^4q_3 \delta(q_1+q_2+q_3) q_1^\mu \sin\left(\frac{1}{2}q_1 \times q_3\right) \\
&\big[\delta^\sigma_\mu g^{\rho\tau}\delta(k_1+q_1)\delta(k_2+q_2)\delta(k_3+q_3) + g^{\rho\sigma}\delta^\tau_\mu\delta(k_1+q_1)\delta(k_2+q_3)\delta(k_3+q_2) \\
&+ \delta^\rho_\mu g^{\sigma\tau}\delta(k_1+q_2)\delta(k_2+q_1)\delta(k_3+q_3) + \delta^\rho_\mu g^{\sigma\tau}\delta(k_1+q_2)\delta(k_2+q_3)\delta(k_3+q_1) \\
&+ \delta^\tau_\mu g^{\rho\sigma}\delta(k_1+q_3)\delta(k_2+q_1)\delta(k_3+q_2) + \delta^\sigma_\mu g^{\rho\tau}\delta(k_1+q_3)\delta(k_2+q_2)\delta(k_3+q_1)\big]
\end{aligned}
\tag{3.39}
$$

$$
\begin{aligned}
=&(-2ig)(2\pi)^4\delta(k_1+k_2+k_3) \\
&\Bigg[g^{\rho\tau}\left(k_1^\sigma \sin\left(\frac{k_1 \times k_3}{2}\right) + k_3^\sigma \sin\left(\frac{k_3 \times k_1}{2}\right)\right) \\
&+ g^{\rho\sigma}\left(k_2^\tau \sin\left(\frac{k_2 \times k_1}{2}\right) + k_1^\tau \sin\left(\frac{k_1 \times k_2}{2}\right)\right) \\
&+ g^{\sigma\tau}\left(k_2^\rho \sin\left(\frac{k_2 \times k_3}{2}\right) + k_3^\rho \sin\left(\frac{k_3 \times k_2}{2}\right)\right)\Bigg] \\
=&(-2ig)(2\pi)^4\delta(k_1+k_2+k_3) \\
&\Bigg[g^{\rho\tau} \sin\left(\frac{k_1 \times k_3}{2}\right)\left(k_1^\sigma - k_3^\sigma\right) \\
&+ g^{\rho\sigma} \sin\left(\frac{k_1 \times k_2}{2}\right)\left(k_1^\tau - k_2^\tau\right) \\
&+ g^{\sigma\tau} \sin\left(\frac{k_2 \times k_3}{2}\right)\left(k_2^\rho - k_3^\rho\right)\Bigg] \\
=&(-2ig)(2\pi)^4\delta(k_1+k_2+k_3) \sin\left(\frac{k_1 \times k_2}{2}\right) \\
&\Big[-g^{\rho\tau}(k_1^\sigma - k_3^\sigma) + g^{\rho\sigma}(k_1^\tau - k_2^\tau) + g^{\sigma\tau}(k_2^\rho - k_3^\rho)\Big].
\end{aligned}
\tag{3.40}
$$

A nice observation is that this can be written in terms of a cross product

$$
= (-2ig)(2\pi)^4\delta(k_1+k_2+k_3)\sin(\frac{1}{2}k_1 \times k_2)(\vec{k}_\mu \times \vec{e}_\mu)\begin{pmatrix} g^{\mu\rho}g^{\sigma\tau} \\ g^{\mu\sigma}g^{\rho\tau} \\ g^{\mu\tau}g^{\rho\sigma}\end{pmatrix}, \tag{3.41}
$$

where the vector arrow denotes the name of the momentum, $\vec{k}^\mu = \begin{pmatrix} k_1^\mu \\ k_2^\mu \\ k_3^\mu \end{pmatrix}$.

3.5 4-photon vertex

The relevant term in the action this time is

$$S_{\text{int}} = \int d^4x \; -\frac{g^2}{2} A_\mu \star A_\nu \star [A_\mu \overset{\star}{,} A_\nu] \,. \tag{3.42}$$

With our familiar formula for the star product in momentum space this gives us

$$\begin{aligned}
=&\frac{-g^2}{2} \int d^4x \int \frac{d^4k_1 ... k_4}{(2\pi)^{4*4}} e^{i(k_1+k_2+k_3+k_4)x} \tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2)\tilde{A}_\mu(k_3)\tilde{A}_\nu(k_4) \\
&\Big[e^{-\frac{i}{2}(k_1\times k_2 + k_1\times k_3 + k_1\times k_4 + k_2\times k_3 + k_2\times k_4 + k_3\times k_4)} \\
&- e^{-\frac{i}{2}(k_1\times k_2 + k_1\times k_3 + k_1\times k_4 + k_2\times k_3 + k_2\times k_4 + k_4\times k_3)} \Big] .
\end{aligned} \tag{3.43}$$

To simplify the exponentials we use the following relations:

$$\begin{aligned}
k_1 \times k_3 &= k_1 \times (-k_2 - k_4) \\
k_2 \times k_3 &= k_2 \times (-k_1 - k_4) \\
k_4 \times k_3 &= -k_3 \times k_4 .
\end{aligned} \tag{3.44}$$

We also use the delta function to solve the integral over x and get

$$\begin{aligned}
S_{\text{int}} =& \frac{-2ig^2}{2} \int \frac{d^4k_1 ... k_4}{(2\pi)^{16}} \delta(\sum_i k_i)(2\pi)^4 \tilde{A}_\mu(k_1)\tilde{A}_\nu(k_2)\tilde{A}_\mu(k_3)\tilde{A}_\nu(k_4) \\
& e^{-\frac{i}{2}(k_1\times k_2)} \sin\left(\frac{k_3 \times k_4}{2}\right) .
\end{aligned} \tag{3.45}$$

The formula for the vertex is now

$$V^{4A} = -(2\pi)^{16} \frac{\delta}{\delta\tilde{A}_\lambda(k_4)} \frac{\delta}{\delta\tilde{A}_\tau(k_3)} \frac{\delta}{\delta\tilde{A}_\sigma(k_2)} \frac{\delta}{\delta\tilde{A}_\rho(k_1)} S_{\text{int}} . \tag{3.46}$$

Calculating this is a lengthy procedure. We'll skip it at this stage and only present the result, but give the full calculation for this in appendix A.2.1 for the interested reader.

$$
\begin{aligned}
V^{4A} = &-4g^2\delta\left(\sum_i k_i\right)(2\pi)^4 \\
&\left[\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)(g^{\rho\tau}g^{\sigma\lambda}-g^{\rho\lambda}g^{\sigma\tau})\right. \\
&+\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)(g^{\rho\sigma}g^{\tau\lambda}-g^{\rho\lambda}g^{\sigma\tau}) \\
&\left.+\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_2\times k_3}{2}\right)(g^{\rho\sigma}g^{\tau\lambda}-g^{\rho\tau}g^{\sigma\lambda})\right].
\end{aligned}
\tag{3.47}
$$

3.6 2 ghost 1 photon vertex

The relevant term in the action is

$$
-ig\int d^4x\,\bar{c}\partial_\mu\left[A_\mu \overset{\star}{,} c\right]. \tag{3.48}
$$

Using our formula for the star product, we get

$$
\begin{aligned}
=&-ig\int d^4x\int\frac{d^4k_1...k_3}{(2\pi)^{4*3}}e^{i(k_1+k_2+k_3)x}(-ik_{1\mu})\tilde{\bar{c}}\tilde{A}_\mu(k_2)\tilde{c}(k_3) \\
&\left[e^{-\frac{i}{2}(k_1\times k_2+k_1\times k_3+k_2)\times k_3)}-e^{-\frac{i}{2}(k_1\times k_2+k_1\times k_3+k_3\times k_2)}\right],
\end{aligned}
\tag{3.49}
$$

where the minus sign in front of the $k_{1\mu}$ comes from partial integration. We now, like with the other vertices, use the fact that $k_3\times k_2=-k_2\times k_3$, as well as the δ-function to simplify the exponentials. We get

$$
\begin{aligned}
=&ig(2\pi)^4\int\frac{d^4k_1...k_3}{(2\pi)^{12}}\delta(k_1+k_2+k_3) \\
&(-k_{1\mu})\tilde{\bar{c}}(k_1)\tilde{A}_\mu(k_2)\tilde{c}(k_3)\left(-2i\sin\left(\frac{k_2\times k_3}{2}\right)\right) \\
=&2g(2\pi)^4\int\frac{d^4k_1...k_3}{(2\pi)^{12}}\delta(k_1+k_2+k_3)k_{1\mu}\tilde{\bar{c}}(k_1)\tilde{A}_\mu(k_2)\tilde{c}(k_3)\sin\left(\frac{k_2\times k_3}{2}\right) \\
=&S_{\text{int}}(k_1,k_2,k_3).
\end{aligned}
\tag{3.50}
$$

Our formula for the vertex is

$$
V^{\gamma 2c}=-(2\pi)^{12}\frac{\delta}{\delta\tilde{\bar{c}}(-k_1)}\frac{\delta}{\delta\tilde{A}_\rho(-k_2)}\frac{\delta}{\delta\tilde{c}(-k_3)}S_{\text{int}}(q_1,q_2,q_3). \tag{3.51}
$$

Since we don't have to apply the product rule here because all involved fields are different, the application of the variations is rather simple. As a result we get

$$
\begin{aligned}
V^{\gamma 2c} =& -2ig(2\pi)^4 \int d^4 q_1 ... q_3 \delta(q_1 + q_2 + q_3) q_{1\mu} \\
& \delta(q_1 + k_1) \delta^{\rho}_{\mu} \delta(q_2 + k_2) \delta(q_3 + k_3) \sin\left(\frac{q_2 \times q_3}{2}\right) \\
=& -2ig(2\pi)^4 \delta(k_1 + k_2 + k_3) k_{1\mu} \delta^{\rho}_{\mu} \sin\left(\frac{k_2 \times k_3}{2}\right) \\
=& -2ig(2\pi)^4 \delta(k_1 + k_2 + k_3) k_1^{\rho} \sin\left(\frac{k_1 \times k_2}{2}\right). \qquad (3.52)
\end{aligned}
$$

Chapter 4

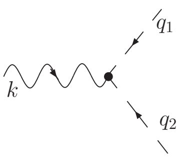
Loop Calculations

4.1 Building Feynman graphs

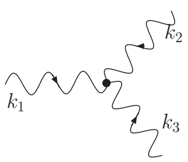
We already know how the vertices and how the propagators look like:

$$= K_M(p,q)$$

$$= K_M(p,q)\delta_{\mu\nu}$$



$$= -2ig(2\pi)^4\delta^4(q_1+q_2+k)q_1^\mu \sin\left(\tfrac{q_1\times q_2}{2}\right)$$



$$\begin{aligned} &= 2ig(2\pi)^4\delta^4(k_1+k_2+k_3) \\ &\quad [(k_3^\rho - k_2^\rho)\delta^{\sigma\tau} + (k_1^\sigma - k_3^\sigma)\delta^{\rho\tau} + (k_2^\tau - k_1^\tau)\delta^{\rho\sigma}]\sin\left(\tfrac{k_1\times k_2}{2}\right). \end{aligned}$$

Thus we can now build Feynman graphs. Let's start with the 1-loop level:

4.2 The Tadpole

4.2.1 General considerations

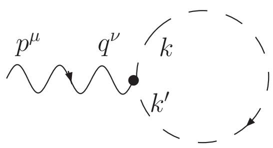


Figure 4.1: Ghost loop tadpole

The ghost loop Tadpole (figure 4.1) is

$$\begin{aligned} &= -\int d^4k \int d^4k' \int d^4q K_M(k,k')K_M(p,q)\delta_{\mu\nu}(-2ig)(2\pi)^4 \\ &\qquad \delta^4(q-k+k')(-k^\nu)\sin\left(\frac{-k\times k'}{2}\right) \\ &= \int d^4k \int d^4k' \int d^4q K_M(k,k')K_M(p,q)\delta_{\mu\nu}(-2ig)(2\pi)^4 \\ &\qquad \delta^4(q-k+k')k^\nu \sin\left(\frac{k\times q}{2}\right), \end{aligned} \tag{4.1}$$

where the momentum k has to be taken negative because it points out of the vertex. Notice also that an additional overall minus sign has been added to the graph according to the Feynman rules ($\rightarrow$ add a global minus sign for every closed fermion loop).
The next 1-loop graph is the photon loop tadpole

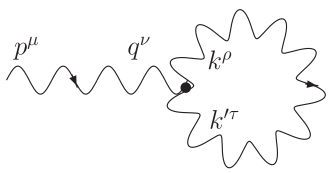


Figure 4.2: Photon loop tadpole

$$= \int d^4k \int d^4k' \int d^4q K_M(k,k') K_M(p,q) \delta^{\mu\nu} \delta^{\rho\tau} (2ig)(2\pi)^4 \delta^4(q-k+k')$$
$$[(-k-q)_\tau \delta_{\rho\nu} + (k'+k)_\nu \delta_{\rho\tau} + (q-k')_\rho \delta_{\nu\tau}] \sin\left(\frac{-k\times k'}{2}\right)$$
$$= \int d^4k \int d^4k' \int d^4q K_M(k,k') K_M(p,q) (2ig)(2\pi)^4 \delta^4(q-k+k')$$
$$\left[(-k-q)_\mu + (k+k')_\mu \underbrace{\delta_{\rho\rho}}_{4} + (q-k')_\mu\right] \sin\left(\frac{k\times q}{2}\right)$$
$$= \int d^4k \int d^4k' \int d^4q K_M(k,k') K_M(p,q) (2ig)(2\pi)^4 \delta^4(q-k+k')$$
$$\left[3k_\mu + 3k'_\mu\right] \sin\left(\frac{k\times q}{2}\right). \tag{4.2}$$

Now we can sum up the 2 graphs which is in perfect agreement to the results of D.N. Blaschke et. al. ([19]):

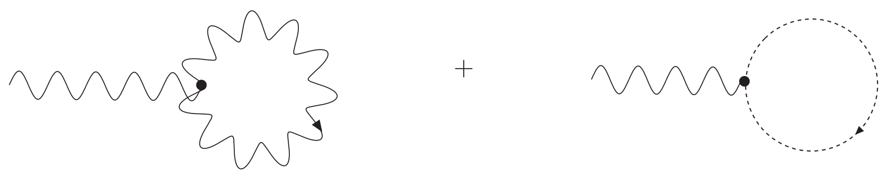


Figure 4.3: Sum of tadpole graphs

$$\Pi_\mu(p) = \int d^4k \int d^4k' \int d^4q (2ig)(2\pi)^4 \delta^4(q-k+k')[2k_\mu + 3k'_\mu]$$
$$\sin\left(\frac{k\times q}{2}\right) K_M(k,k') K_M(p,q). \tag{4.3}$$

We can now rewrite the expression in terms of long and short variables

$$S = k - k', \quad L = k + k' \quad \Rightarrow \quad k = \frac{L+S}{2}, \quad k' = \frac{L-S}{2}. \tag{4.4}$$

This is a two-dimensional variable substitution, and therefore we get a Jacobian:

$$J = \begin{pmatrix} \frac{\partial k}{\partial L} & \frac{\partial k}{\partial S} \\ \frac{\partial k'}{\partial L} & \frac{\partial k'}{\partial S} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$
$$\Rightarrow \det J = -\frac{1}{2} \xrightarrow{4-dim} \left(-\frac{1}{2}\right)^4 = \frac{1}{16} \tag{4.5}$$

Our expression hence becomes

$$
\begin{aligned}
&= \int d^4L \int d^4S \int d^4q (2ig)\pi^4 \delta^4(q-S)[L_\mu + S_\mu + \frac{3}{2}(L_\mu - S_\mu)] \\
&\qquad\qquad \sin\left(\frac{(L+S)\times q}{4}\right) K_M(\frac{L+S}{2}, \frac{L-S}{2}) K_M(p,q) \\
&= \int d^4L \int d^4S \int d^4q (2ig)\pi^4 \delta^4(q-S)[\frac{5}{2}L_\mu - \frac{1}{2}S_\mu] \sin\left(\frac{L\times q}{4}\right) \\
&\qquad\qquad K_M(L,S) K_M(p,q).
\end{aligned} \tag{4.6}
$$

4.2.2 The Tadpole without amputated external legs

We start with (4.6) and insert into this expression the Mehler kernel in terms of long and short variables. We have already calculated this. The result is

$$
K_M(L,S) = \frac{\omega^3}{8\pi^2} \int_0^\infty d\alpha \frac{1}{\sinh^2\alpha} \exp\left[-\frac{\omega}{4}S^2 \coth(\frac{\alpha}{2}) - \frac{\omega}{4}L^2 \tanh(\frac{\alpha}{2})\right]. \tag{4.7}
$$

We want to get rid of the ugly hyperbolic functions, which we can realize by a substitution:

$$
y = \tanh(\frac{\alpha)}{2}) \Rightarrow \alpha = 2\text{arctanh}(y) \Rightarrow \frac{d\alpha}{dy} = \frac{2}{1-y^2}. \tag{4.8}
$$

Then our Mehler kernel becomes

$$
K_M(L,S) = \frac{\omega^3}{8\pi^2} \int_0^1 dy \frac{y^2}{2(1-y^2)^3} \exp\left[-\frac{\omega}{4}S^2\frac{1}{y} - \frac{\omega}{4}L^2 y\right], \tag{4.9}
$$

where we have used

$$
\frac{1}{\sinh^2(\alpha)} = \frac{1}{4\sinh^2(\frac{\alpha}{2})\cosh^2(\frac{\alpha}{2})} \tag{4.10}
$$

as well as

$$
\begin{aligned}
\sinh^2[\text{arctanh}(y)] &= \frac{y^2}{1-y^2} \\
\cosh^2[\text{arctanh}(y)] &= \frac{1}{1-y^2}.
\end{aligned} \tag{4.11}
$$

Now we can insert this kernel into our tadpole

$$\int d^4L \int d^4S \int d^4q(2ig)\pi^4\delta^4(q-S)[\frac{5}{2}L_\mu - \frac{1}{2}S_\mu]\sin\left(\frac{L\times q}{4}\right) \tag{4.12}$$
$$\frac{\omega^3}{8\pi^2}\int_0^1 dy\frac{y^2}{2(1-y^2)^3}\exp\left[-\frac{\omega}{4}S^2\frac{1}{y}-\frac{\omega}{4}L^2y\right]K_M(p,q).$$

We solve the delta function

$$\int d^4L \int d^4S(2ig)\pi^4[\frac{5}{2}L_\mu - \frac{1}{2}S_\mu]\sin\left(\frac{L\times S}{4}\right) \tag{4.13}$$
$$\frac{\omega^3}{8\pi^2}\int_0^1 dy\frac{y^2}{2(1-y^2)^3}\exp\left[-\frac{\omega}{4}S^2\frac{1}{y}-\frac{\omega}{4}L^2y\right]K_M(p,S).$$

We write the sine in terms of exponentials

$$\int d^4L \int d^4S(2ig)\pi^4[\frac{5}{2}L_\mu - \frac{1}{2}S_\mu]\sum_{\eta=-1}^{1}\frac{\eta}{2i}\exp\left(\frac{i\eta}{4}L\times S\right) \tag{4.14}$$
$$\frac{\omega^3}{8\pi^2}\int_0^1 dy\frac{y^2}{2(1-y^2)^3}\exp\left[-\frac{\omega}{4}S^2\frac{1}{y}-\frac{\omega}{4}L^2y\right]K_M(p,S)$$
$$=\int d^4L \int d^4S\frac{\pi^2 g\omega^3}{32}[5L_\mu - S_\mu]\sum_{\eta=-1}^{1}\eta\exp\left(\frac{i\eta}{4}L\times S\right)$$
$$\int_0^1 dy\frac{y^2}{(1-y^2)^3}\exp\left[-\frac{\omega}{4}S^2\frac{1}{y}-\frac{\omega}{4}L^2y\right]K_M(p,S).$$

We complete the full square in the exponential

$$=\int d^4L \int d^4S\int_0^1 dy\frac{\pi^2 g\omega^3}{32}[5L_\mu - S_\mu]\sum_{\eta=-1}^{1}\eta K_M(p,S)\frac{y^2}{(1-y^2)^3} \tag{4.15}$$
$$\exp\left[-\left[\sqrt{\frac{\omega}{4}y}L-\frac{\frac{i\eta}{4}\tilde{S}}{2\sqrt{\frac{\omega}{4}y}}\right]^2-\frac{\eta^2\tilde{S}^2}{16\omega y}-\frac{\omega}{4}S^2\frac{1}{y}\right],$$

where we have used the abbreviation $\Theta_{\mu\nu}S^\nu=\tilde{S}_\mu$. Now we use $\eta^2=1$ and we substitute the expression in the square brackets by L'. Therefore we get a functional determinant of $\frac{16}{\omega^2 y^2}$ (we are in 4D!)

$$= \int d^4L' \int d^4S \int_0^1 dy \frac{\pi^2 g\omega^3}{32} \sum_{\eta=-1}^{1} \eta[5\frac{L'_\mu + \frac{i\eta}{4\sqrt{\omega y}}\tilde{S}_\mu}{\sqrt{\frac{\omega}{4}y}} - S_\mu]\frac{16}{\omega^2 y^2} K_M(p,S) \tag{4.16}$$

$$\frac{y^2}{(1-y^2)^3} \exp\left[-L'^2 - \frac{\tilde{S}^2}{16\omega y} - \frac{\omega}{4}S^2\frac{1}{y}\right].$$

We are now ready to solve the Gauß integral over L' which yields a factor π^2

$$= \int d^4S \int_0^1 dy \, \frac{\pi^4 g\omega}{2} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\tilde{S}_\mu}{2\omega y} - S_\mu] K_M(p,S) \tag{4.17}$$

$$\frac{1}{(1-y^2)^3} \exp\left[-\frac{\tilde{S}^2}{16\omega y} - \frac{\omega}{4}S^2\frac{1}{y}\right].$$

We now need to insert the 2nd Mehler kernel. Since we have no integral over the external momentum p we can't write this Mehler kernel in terms of long and short variables and must instead use the full expression:

$$K_M(p,S) = \frac{\omega^3}{2\pi^2} \int_0^\infty d\alpha \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2} \exp\left[-\frac{\frac{\omega}{2}(p^2+S^2)(1+e^{-2\alpha}) - 2\omega e^{-\alpha} pS}{1-e^{-2\alpha}}\right]. \tag{4.18}$$

To remove double exponential terms we will also substitute this expression:

$$e^{-\alpha} = v \Rightarrow \alpha = -\ln(v) \Rightarrow \frac{d\alpha}{dv} = -\frac{1}{v}. \tag{4.19}$$

Then the Mehler kernel becomes

$$\begin{aligned} K_M(p,S) =& \frac{\omega^3}{2\pi^2} \int_1^0 dv \left(-\frac{1}{v}\right) \frac{v^2}{(1-v^2)^2} \exp\left[-\frac{\frac{\omega}{2}(p^2+S^2)(1+v^2) - 2\omega vpS}{1-v^2}\right] \\ =& \frac{\omega^3}{2\pi^2} \int_0^1 dv \frac{v}{(1-v^2)^2} \exp\left[-\frac{\frac{\omega}{2}(p^2+S^2)(1+v^2) - 2\omega vpS}{1-v^2}\right]. \end{aligned} \tag{4.20}$$

Upon inserting this Mehler kernel into the tadpole we get

$$
\begin{aligned}
&= \int d^4S \int_0^1 dy\, \frac{\pi^4 g\omega}{2} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\tilde{S}_\mu}{2\omega y} - S_\mu] \qquad (4.21)\\
&\qquad \frac{\omega^3}{2\pi^2} \int_0^1 dv \frac{v}{(1-v^2)^2} \exp\left[-\frac{\frac{\omega}{2}(p^2+S^2)(1+v^2) - 2\omega vpS}{1-v^2}\right]\\
&\qquad \frac{1}{(1-y^2)^3} \exp\left[-\frac{\tilde{S}^2}{16\omega y} - \frac{\omega}{4}S^2\frac{1}{y}\right]\\
&= \int d^4S \int_0^1 dy \int_0^1 dv\, \frac{\pi^4 g\omega}{2} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}]S^\nu\\
&\qquad \frac{\omega^3}{2\pi^2} \frac{v}{(1-v^2)^2} \exp\left[-\frac{\frac{\omega}{2}(p^2+S^2)(1+v^2) - 2\omega vpS}{1-v^2}\right]\\
&\qquad \frac{1}{(1-y^2)^3} \exp\left[-\frac{\Theta^2 S^2}{16\omega y} - \frac{\omega}{4}S^2\frac{1}{y}\right],
\end{aligned}
$$

where we have used the fact that thanks to our choice of $\Theta_{\mu\nu}$ we get $\tilde{S}^2 = \Theta^2 S^2$ with $\Theta^2 := \Theta_{\mu\nu}\Theta^{\mu\nu}$. We'll now have to complete the full square again to be able to solve the Gauß integral

$$
\begin{aligned}
&= \int d^4S \int_0^1 dy \int_0^1 dv \frac{\pi^2 g\omega^4}{4} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}]S^\nu \frac{v}{(1-v^2)^2} \frac{1}{(1-y^2)^3} \\
&\qquad \exp\Bigg[- S^2 \left(\frac{\frac{\omega}{2}(1+v^2)}{1-v^2} + \frac{\omega}{4y} + \frac{\Theta^2}{16\omega y} \right) \\
&\qquad\qquad + S\frac{2\omega vp}{1-v^2} - \frac{\omega}{2}p^2 \frac{1+v^2}{1-v^2} \Bigg] \\
&= \int d^4S \int_0^1 dy \int_0^1 dv \frac{\pi^2 g\omega^4}{4} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}]S^\nu \frac{v}{(1-v^2)^2} \frac{1}{(1-y^2)^3} \\
&\qquad \exp\Bigg[- S^2 \underbrace{\left(\frac{8\omega^2 y(1+v^2) + 4\omega^2(1-v^2) + \Theta^2(1-v^2)}{16\omega y(1-v^2)} \right)}_{A^2} \\
&\qquad + S\frac{2\omega vp}{1-v^2} - \frac{\omega}{2}p^2 \frac{1+v^2}{1-v^2} \Bigg] \\
&= \int d^4S \int_0^1 dy \int_0^1 dv \frac{\pi^2 g\omega^4}{4} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}]S^\nu \frac{v}{(1-v^2)^2} \frac{1}{(1-y^2)^3} \\
&\qquad \exp\left[- \left[SA - \frac{\frac{\omega vp}{1-v^2}}{A} \right]^2 + \frac{\frac{\omega^2 v^2 p^2}{(1-v^2)^2}}{A^2} - \frac{\omega}{2}p^2 \frac{1+v^2}{1-v^2} \right].
\end{aligned}
\tag{4.22}
$$

We now again substitute the expression in the square brackets and therefore get a functional determinant of $\frac{1}{A^4}$

$$
\begin{aligned}
&= \int d^4S \int_0^1 dy \int_0^1 dv \frac{\pi^2 g\omega^4}{4} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}] \frac{1}{A^4} \left[\frac{S'_\nu}{A} + \frac{\omega v p_\nu}{(1-v^2)A^2} \right] \\
&\qquad \frac{v}{(1-v^2)^2} \frac{1}{(1-y^2)^3} \exp\left[-S'^2 + \frac{\frac{\omega^2 v^2 p^2}{(1-v^2)^2}}{A^2} - \frac{\omega}{2}p^2 \frac{1+v^2}{1-v^2} \right].
\end{aligned}
\tag{4.23}
$$

Finally we can solve the last momentum integral

$$= \int_0^1 dy \int_0^1 dv \frac{\pi^4 g\omega^4}{4} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}] \frac{1}{A^4} \frac{\omega v p_\nu}{(1-v^2)A^2}$$

$$\frac{v}{(1-v^2)^2} \frac{1}{(1-y^2)^3} \exp\left[\frac{\frac{\omega^2 v^2 p^2}{(1-v^2)^2}}{A^2} - \frac{\omega}{2} p^2 \frac{1+v^2}{1-v^2}\right]. \tag{4.24}$$

We now reinsert A

$$= \int_0^1 dy \int_0^1 dv \frac{\pi^4 g\omega^4}{4} \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}] \frac{\omega v p_\nu}{(1-v^2)}$$
$$\frac{16^3\omega^3 y^3(1-v^2)^3}{[8\omega^2 y(1+v^2) + 4\omega^2(1-v^2) + \Theta^2(1-v^2)]^3} \frac{v}{(1-v^2)^2} \frac{1}{(1-y^2)^3}$$
$$\exp\left[\frac{\omega^2 v^2 p^2}{(1-v^2)^2} \frac{16\omega y(1-v^2)}{8\omega^2 y(1+v^2) + 4\omega^2(1-v^2) + \Theta^2(1-v^2)} - \frac{\omega}{2} p^2 \frac{1+v^2}{1-v^2}\right]$$
$$= \int_0^1 dy \int_0^1 dv \sum_{\eta=-1}^{1} \eta[\frac{5i\eta\Theta_{\mu\nu}}{2\omega y} - \delta_{\mu\nu}] p_\nu$$
$$\frac{2^{10}\omega^8 y^3 \pi^4 g}{[8\omega^2 y(1+v^2) + 4\omega^2(1-v^2) + \Theta^2(1-v^2)]^3} \frac{v^2}{(1-y^2)^3}$$
$$\exp\left[\frac{v^2 p^2}{(1-v^2)} \frac{16\omega^3 y}{8\omega^2 y(1+v^2) + 4\omega^2(1-v^2) + \Theta^2(1-v^2)} - \frac{\omega}{2} p^2 \frac{1+v^2}{1-v^2}\right]. \tag{4.25}$$

Now that we have derived the final formula we can consider different limits:

$$\lim_{v\to 0} \Rightarrow 0 \tag{4.26}$$
$$\lim_{v\to 1} \Rightarrow \text{finite but not zero} \tag{4.27}$$
$$\lim_{y\to 0} \Rightarrow 0 \tag{4.28}$$
$$\lim_{y\to 1} \Rightarrow \text{infinite} \tag{4.29}$$
$$y = v \ \& \ \lim_{v\to 0} \Rightarrow 0 \tag{4.30}$$
$$y = v \ \& \ \lim_{v\to 1} \Rightarrow \text{infinite.} \tag{4.31}$$

Conclusion: Due to those inconsistent statements we can't really say something definite about convergence in this context. We need other ways to show the convergence of the tadpole.

4.2.3 The Tadpole with amputated external legs

In this section we amputate the external legs from the tadpole (figure 4.4) by multiplying the inverse of the Mehler kernel from the right onto the expression. We then get $\delta(p-q)$ instead of the Mehler kernel. Therefore, one

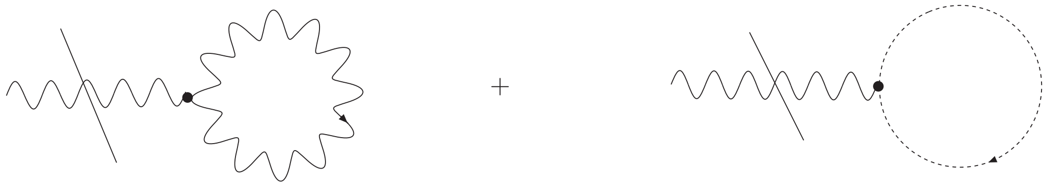


Figure 4.4: Sum of tadpole graphs without external legs

Mehler kernel together with the integral over q goes away and formula (4.6) becomes

$$= \int d^4L \int d^4S\,(2ig)\pi^4\delta^4(p-S)[\frac{5}{2}L_\mu - \frac{1}{2}S_\mu]\sin\left(\frac{L\times p}{4}\right)K_M(L,S). \tag{4.32}$$

We solve the δ-function and get

$$= \int d^4L\,(2ig)\pi^4[\frac{5}{2}L_\mu - \frac{1}{2}S_\mu]\sin\left(\frac{L\times p}{4}\right)K_M(L,p). \tag{4.33}$$

We now write the sine in terms of exponentials and insert the Mehler kernel

$$K_M(L,p) = \frac{\omega^3}{8\pi^2}\int_0^\infty d\alpha\frac{1}{\sinh^2\alpha}\exp\left[-\frac{\omega}{4}p^2\coth(\frac{\alpha}{2}) - \frac{\omega}{4}L^2\tanh(\frac{\alpha}{2})\right] \tag{4.34}$$

to get

$$\begin{aligned} &= \int d^4L(2ig)\pi^4[\frac{5}{2}L_\mu - \frac{1}{2}p_\mu]\sum_\eta \frac{\eta}{2i}e^{\frac{i\eta}{4}L\tilde{p}} \\ &\qquad \frac{\omega^3}{8\pi^2}\int_0^\infty d\alpha\frac{1}{\sinh^2\alpha}\exp\left[-\frac{\omega}{4}p^2\coth(\frac{\alpha}{2}) - \frac{\omega}{4}L^2\tanh(\frac{\alpha}{2})\right]. \end{aligned} \tag{4.35}$$

The next step is to convert the expression into a Gauss integral by completing the square

$$
\begin{aligned}
= \int d^4L \int_0^\infty d\alpha \frac{\omega^3 ig\pi^2}{4}[\frac{5}{2}L_\mu - \frac{1}{2}p_\mu]\frac{1}{\sinh^2\alpha}\sum_\eta \frac{\eta}{2i} \\
\exp\Bigg[-\frac{\omega}{4}p^2\coth(\frac{\alpha}{2}) - \left(\frac{\tilde{p}}{8\sqrt{\frac{\omega}{4}\tanh(\frac{\alpha}{2})}}\right)^2 \\
- \left(L\sqrt{\frac{\omega}{4}\tanh(\frac{\alpha}{2})} - \frac{i\eta\tilde{p}}{8\sqrt{\frac{\omega}{4}\tanh(\frac{\alpha}{2})}}\right)^2\Bigg].
\end{aligned}
\tag{4.36}
$$

We solve the Gauss integral

$$
\begin{aligned}
&= \int_0^\infty d\alpha \frac{\omega^3 ig\pi^4}{4}\sum_\eta \frac{\eta}{2i}[\frac{5}{2}\left(\frac{i\tilde{p}_\mu\eta}{2\omega\tanh(\frac{\alpha}{2})}\right) - \frac{1}{2}p_\mu]\frac{1}{\sinh^2\alpha}\left(\frac{4}{\omega\tanh(\frac{\alpha}{2})}\right)^2 \\
&\qquad \exp\left[-\frac{\omega}{4}p^2\coth(\frac{\alpha}{2}) - \left(\frac{\tilde{p}}{4\sqrt{\omega\tanh(\frac{\alpha}{2})}}\right)^2\right] \\
&= \int_0^\infty d\alpha\, 2\omega^3 ig\pi^4 \sum_\eta \frac{\eta}{2i}\tilde{p}_\mu \left[5\left(\frac{i\eta}{2\omega\tanh(\frac{\alpha}{2})}\right) - 1\right]\frac{1}{\sinh^2\alpha}\left(\frac{1}{\omega\tanh(\frac{\alpha}{2})}\right)^2 \\
&\qquad \exp\left[-\frac{\omega}{4}p^2\coth(\frac{\alpha}{2}) - \frac{\tilde{p}^2}{16\omega\tanh(\frac{\alpha}{2})}\right] \\
&= \int_0^\infty d\alpha\, 5\omega^3 ig\pi^4 \sum_\eta \frac{\eta}{2i}\tilde{p}_\mu \frac{i\eta}{\omega\tanh(\frac{\alpha}{2})}\frac{1}{\sinh^2\alpha}\left(\frac{1}{\omega\tanh(\frac{\alpha}{2})}\right)^2 \\
&\qquad \exp\left[-\frac{\omega}{4}p^2\coth(\frac{\alpha}{2}) - \frac{\tilde{p}^2}{16\omega\tanh(\frac{\alpha}{2})}\right],
\end{aligned}
\tag{4.37}
$$

where the term "1" in the bracket has disappeared because the sum over η gives this term twice apart from a different sign. The aim is now to solve the remaining integral over α. Since we have only terms involving $\frac{\alpha}{2}$ in the exponent, we want to have this dependence in the prefactor as well. Some trigonometric gymnastics yields

$$
\sinh^2\alpha = 4\sinh^2(\frac{\alpha}{2})\cosh^2(\frac{\alpha}{2}). \tag{4.38}
$$

Therefore we get

$$
\begin{aligned}
&= -\int_0^\infty d\alpha\, \frac{5}{i} g\pi^4 \tilde{p}_\mu \frac{\cosh(\frac{\alpha}{2})}{\sinh^5(\frac{\alpha}{2})} \exp\left[-\frac{\omega}{4} p^2 \coth(\frac{\alpha}{2}) - \frac{\tilde{p}^2}{16\omega \tanh(\frac{\alpha}{2})} \right] \\
&= -\int_0^\infty d\alpha\, \frac{5}{i} g\pi^4 \tilde{p}_\mu \frac{\cosh(\frac{\alpha}{2})}{\sinh^5(\frac{\alpha}{2})} \exp\left[-\frac{1}{4} p^2 \coth(\frac{\alpha}{2})(\omega + \frac{\Theta^2}{4\omega}) \right]. \qquad (4.39)
\end{aligned}
$$

We substitute

$$
\begin{aligned}
\coth(\frac{\alpha}{2}) &= \gamma \\
\Rightarrow \frac{d\alpha}{d\gamma} &= \frac{2}{1-\gamma^2} \qquad (4.40)
\end{aligned}
$$

and use the relations

$$
\begin{aligned}
\cosh\left(\operatorname{arcoth}(\gamma)\right) &= \frac{1}{\sqrt{1-\frac{1}{\gamma^2}}} \\
\sinh\left(\operatorname{arcoth}(\gamma)\right) &= \frac{1}{\sqrt{1-\frac{1}{\gamma^2}}\gamma}. \qquad (4.41)
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&= -\int_\infty^1 d\gamma\, \frac{5}{i} g\pi^4 \tilde{p}_\mu \frac{2}{1-\gamma^2} \gamma^5 (1 - \frac{1}{\gamma^2})^4 \exp\left[-\frac{1}{4} p^2 \gamma (\omega + \frac{\Theta^2}{4\omega}) \right] \\
&= \int_1^\infty d\gamma\, \frac{5}{i} g\pi^4 \tilde{p}_\mu \frac{2(1-\gamma^2)^3}{\gamma^3} \exp\left[-\frac{1}{4} p^2 \gamma (\omega + \frac{\Theta^2}{4\omega}) \right]. \qquad (4.42)
\end{aligned}
$$

The remaining integral can be easily solved by complex integration to avoid the problematic pole on the real axis. The result is

$$
\frac{5i\tilde{p}_\mu}{2\pi^2}(3 + 3A + A^2)\frac{e^{-A}}{A^4} \qquad (4.43)
$$

with

$$
A = \frac{p^2}{4}[\omega + \frac{\Theta^2}{4\omega}]. \qquad (4.44)
$$

The result is IR-divergent!

4.2.4 The tadpole coupled with an external field

We can renormalize the divergent result of the previous subsection by coupling an external field to the tadpole.
We start with expression (4.39):

$$\Pi_\mu(p) = \int_0^\infty d\alpha\, 5ig\pi^4 \tilde{p}_\mu \frac{\cosh(\frac{\alpha}{2})}{\sinh^5(\frac{\alpha}{2})} \exp\left[-\frac{1}{4} p^2 \coth(\frac{\alpha}{2})(\omega + \frac{\Theta^2}{4\omega}) \right]. \tag{4.45}$$

Now consider the following expansion

$$\int d^4p \Pi_\mu(p) \Big[A_\mu(0) + p_\nu \left(\partial^p_\nu A_\mu(p)\big|_{p=0} \right) + p_\nu p_\rho \left(\partial^p_\nu \partial^p_\rho A_\mu(p)\big|_{p=0} \right) + \\ + p_\nu p_\rho p_\sigma \left(\partial^p_\nu \partial^p_\rho \partial^p_\sigma A_\mu(p)\big|_{p=0} \right) + \ldots \Big]. \tag{4.46}$$

All terms of even order (i.e. of order 0,2,4...) are zero for symmetry reasons, because the integrand in $\Pi_\mu(p)$ is proportional to p_μ, and therefore odd. A symmetric integral over an odd integrand is always zero.
Of the other terms, we will now show that only the first two, namely orders 1 and 3, diverge:

- *order 1:*

$$\int d^4p\, p_\nu \Pi_\mu(p) = 5ig\pi^4 \int d^4p \int_0^\infty d\alpha\, p_\nu p_\mu e^{-\frac{1}{4}\coth\left(\frac{\alpha}{2}\right) p^2 \left[\omega + \frac{\Theta^2}{4\omega}\right]} \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)}.$$

With $p_\nu p_\mu = \frac{\partial}{\partial z_\nu}\frac{\partial}{\partial z_\rho}\Theta_{\mu\rho} e^{-zp}\Big|_{z=0}$ we can solve the Gauß integral by completing the square

$$= 5ig\pi^4 \int d^4p \int_0^\infty d\alpha\, \frac{\partial}{\partial z_\nu}\frac{\partial}{\partial z_\rho}\Theta_{\mu\rho} e^{-zp} e^{-\frac{1}{4}\coth\left(\frac{\alpha}{2}\right) p^2 \left[\omega + \frac{\Theta^2}{4\omega}\right]} \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)}\Bigg|_{z=0}. \tag{4.47}$$

With $B_\alpha = \coth\left(\frac{\alpha}{2}\right)\left[\omega + \frac{\Theta^2}{4\omega}\right]$ we get

$$\begin{aligned} &= 5ig\pi^4 \int d^4p \int_0^\infty d\alpha\, \frac{\partial}{\partial z_\nu}\frac{\partial}{\partial z_\rho}\Theta_{\mu\rho} e^{-\frac{1}{4}B_\alpha\left(p+\frac{2z}{B_\alpha}\right)^2 + \frac{z^2}{B_\alpha}} \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)}\Bigg|_{z=0} \\ &= 5ig\pi^4 \int_0^\infty d\alpha\, \frac{\partial}{\partial z_\nu}\frac{\partial}{\partial z_\rho}\Theta_{\mu\rho} \frac{16\pi^2}{B_\alpha^2} e^{\frac{z^2}{B_\alpha}} \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)}\Bigg|_{z=0}. \end{aligned} \tag{4.48}$$

Auxiliary calculation:

$$\frac{\partial}{\partial z_\nu}\frac{\partial}{\partial z_\rho}\left(e^{\frac{z^2}{B_\alpha}}\right)\Big|_{z=0}=\frac{\partial}{\partial z_\nu}\left(e^{\frac{z^2}{B_\alpha}}\frac{2z_\rho}{B_\alpha}\right)\Big|_{z=0}=\frac{2\delta_{\nu\rho}}{B_\alpha}. \tag{4.49}$$

Thus we get

$$\begin{aligned}
=&5ig\pi^4\int_0^\infty d\alpha\,\Theta_{\mu\nu}\frac{32\pi^2}{B_\alpha^3}\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)}\\
=&160ig\pi^6\int_0^\infty d\alpha\,\Theta_{\mu\nu}\frac{1}{\left(\coth\left(\frac{\alpha}{2}\right)\left[\omega+\frac{\Theta^2}{4\omega}\right]\right)^3}\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh^5\left(\frac{\alpha}{2}\right)}\\
=&160ig\pi^6\int_0^\infty d\alpha\,\frac{\Theta_{\mu\nu}}{\left(\omega+\frac{\Theta^2}{4\omega}\right)^3}\frac{1}{\cosh^2\left(\frac{\alpha}{2}\right)\sinh^2\left(\frac{\alpha}{2}\right)}\\
=&640ig\pi^6\frac{\Theta_{\mu\nu}}{\left(\omega+\frac{\Theta^2}{4\omega}\right)^3}\int_0^\infty d\alpha\,\frac{1}{\sinh^2\alpha}.
\end{aligned} \tag{4.50}$$

An integral over $\frac{1}{\sinh^2\alpha}$ gives $\coth\alpha$ which we have to evaluate at the boundaries of the integral. The upper boundary ∞ is no problem since it gives one, while the lower boundary is a problem since it gives $+\infty$,

$$=-640ig\pi^6\frac{\Theta_{\mu\nu}}{\left(\omega+\frac{\Theta^2}{4\omega}\right)^3}\lim_{\epsilon\to 0}\left(1-\coth\epsilon\right), \tag{4.51}$$

but we can approximate this function

$$1-\coth\epsilon=1-\frac{1}{\epsilon}+\mathcal{O}(\epsilon) \tag{4.52}$$

and get

$$=-640ig\pi^6\frac{\Theta_{\mu\nu}}{\left(\omega+\frac{\Theta^2}{4\omega}\right)^3}\lim_{\epsilon\to 0}\left(1-\frac{1}{\epsilon}+\mathcal{O}(\epsilon)\right), \tag{4.53}$$

which is renormalizable.

• *order 3:*

From the previous calculation it is obvious, that we will have a parameter integral which is proportional to

$$\int_0^\infty d\alpha\frac{1}{\sinh(\frac{\alpha}{2})\cosh^3\left(\frac{\alpha}{2}\right)}\approx\lim_{\epsilon\to 0}\left[K-\ln(\epsilon)+\mathcal{O}(\epsilon^2)\right],\quad K\in\mathbb{R}, \tag{4.54}$$

because looking at the expansion (4.46) we see that in the 3^{rd} order we have two additional differentiations with respect to the momentum. Therefore we get in eq. (4.49) another factor B_α down. This kills a factor $\sinh\frac{\alpha}{2}$ and yields a factor $\cosh\frac{\alpha}{2}$ in the denominator.

• *order 5 and higher:*
These orders are finite.

The two divergent terms can be removed by renormalization, i.e. considering appropriate counter terms in the action. However, the remaining (finite) expressions are non-zero. The fact that these graphs do not vanish means we need to find the correct vacuum by solving the equations of motion. This is a task of work in progress [29].

4.3 The 2-point Tadpole (self-energy graph)

4.3.1 General considerations

We consider the following loop graph (figure 4.5)

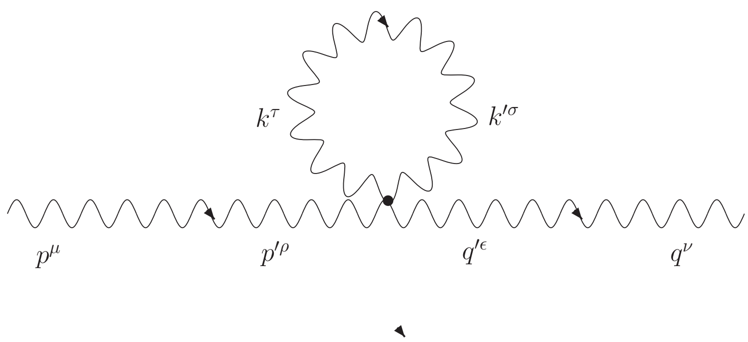


Figure 4.5: 2-point-tadpole

which is a correction to the self energy at the 1 loop level. If we amputate the external legs we are left with the following analytical expression

$$
\begin{aligned}
&\int d^4p' \int d^4q' \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \delta^{\mu\rho}\delta^{\tau\sigma}\delta^{\nu\epsilon}\delta^4(p-p')\delta^4(q-q')M_K(k,k') \\
&\qquad\qquad\qquad\qquad\qquad\qquad V^{4A}(p',-q',-k,k') \\
&= \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} \delta^{\mu\rho}\delta^{\tau\sigma}\delta^{\nu\epsilon} M_K(k,k')(-4g^2)(2\pi)^4\delta^4(p-q-k+k') \\
&\qquad \Big[(g_{\rho\tau}g_{\sigma\epsilon} - g_{\rho\epsilon}g_{\sigma\tau})\sin\left(\frac{p\times -k}{2}\right)\sin\left(\frac{k'\times -q}{2}\right) \\
&\qquad + (g_{\rho\sigma}g_{\tau\epsilon} - g_{\rho\epsilon}g_{\sigma\tau})\sin\left(\frac{p\times k'}{2}\right)\sin\left(\frac{-k\times -q}{2}\right) \\
&\qquad + (g_{\rho\sigma}g_{\tau\epsilon} - g_{\rho\tau}g_{\sigma\epsilon})\sin\left(\frac{-k\times k'}{2}\right)\sin\left(\frac{p\times -q}{2}\right)\Big].
\end{aligned}
\tag{4.55}
$$

Eliminating the $\delta^{\tau\epsilon}$ function we realize that the 3^{rd} term vanishes and the first two get an additional factor -3. We also eliminate the other δ-functions and get

$$
\begin{aligned}
&= (-4g^2)(2\pi)^4(-3g_{\mu\nu}) \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} M_K(k,k')\delta^4(p-q-k+k') \\
&\qquad \Big[\sin\left(\frac{p\times k'}{2}\right)\sin\left(\frac{k\times q}{2}\right) + \sin\left(\frac{p\times k}{2}\right)\sin\left(\frac{k'\times q}{2}\right)\Big].
\end{aligned}
\tag{4.56}
$$

We now substitute the expression in terms of long and short variables

$$
k - k' = S \quad \& \quad k + k' = L. \tag{4.57}
$$

This substitution yields a Jacobian $\frac{1}{16}$ and we get

$$
\begin{aligned}
=&-12g^2\frac{1}{16}(2\pi)^4 g_{\mu\nu}\int\frac{d^4L}{(2\pi)^4}\int\frac{d^4S}{(2\pi)^4}M_K(L,S)\delta^4(p-q-S)\\
&\left[\sin\left(\frac{p\times\frac{L-S}{2}}{2}\right)\sin\left(\frac{\frac{L+S}{2}\times q}{2}\right)+\sin\left(\frac{p\times\frac{L+S}{2}}{2}\right)\sin\left(\frac{\frac{L-S}{2}\times q}{2}\right)\right]\\
=&-12g^2\frac{1}{16}g_{\mu\nu}\int d^4L\ M_K(L,p-q)\\
&\left[\sin\left(\frac{p\times\frac{L-p+q}{2}}{2}\right)\sin\left(\frac{\frac{L+p-q}{2}\times q}{2}\right)+\sin\left(\frac{p\times\frac{L+p-q}{2}}{2}\right)\sin\left(\frac{\frac{L-p+q}{2}\times q}{2}\right)\right]\\
=&-12g^2\frac{1}{16}g_{\mu\nu}\int d^4L\ M_K(L,p-q)\\
&\left[\sin\left(\frac{p\times(L+q)}{4}\right)\sin\left(\frac{(L+p)\times q}{4}\right)+\sin\left(\frac{p\times(L-q)}{4}\right)\sin\left(\frac{(L-p)\times q}{4}\right)\right],
\end{aligned}
\tag{4.58}
$$

where we have eliminated the δ-function with the integral over S.
The task is now to rewrite the sine in terms of exponentials

$$
\begin{aligned}
&=-12g^2\frac{1}{16}\left(-\frac{1}{4}\right)g_{\mu\nu}\int d^4L\ M_K(L,p-q)\\
&\left[e^{\frac{i}{4}(p\times q+p\times L+p\times q+L\times q)}-e^{\frac{i}{4}(p\times q+p\times L-p\times q-L\times q)}\right.\\
&-e^{\frac{i}{4}(-p\times q-p\times L+p\times q+L\times q)}+e^{\frac{i}{4}(-p\times q-p\times L-p\times q-L\times q)}\\
&+e^{\frac{i}{4}(-p\times q+p\times L+L\times q-p\times q)}-e^{\frac{i}{4}(-p\times q+p\times L-L\times q+p\times q)}\\
&\left.-e^{\frac{i}{4}(p\times q-p\times L+L\times q-p\times q)}+e^{\frac{i}{4}(p\times q-p\times L-L\times q+p\times q)}\right]\\
&=-12g^2\frac{1}{16}\left(-\frac{1}{4}\right)g_{\mu\nu}\int d^4L\ M_K(L,p-q)\\
&\left[e^{\frac{i}{4}(2p\times q+p\times L+L\times q)}-e^{\frac{i}{4}(p\times L-L\times q)}\right.\\
&-e^{\frac{i}{4}(-p\times L+L\times q)}+e^{\frac{i}{4}(-2p\times q-p\times L-L\times q)}\\
&+e^{\frac{i}{4}(-2p\times q+p\times L+L\times q)}-e^{\frac{i}{4}(p\times L-L\times q)}\\
&\left.-e^{\frac{i}{4}(-p\times L+L\times q)}+e^{\frac{i}{4}(2p\times q-p\times L-L\times q)}\right].
\end{aligned}
\tag{4.59}
$$

We can now rewrite the expression in terms of cosines

$$12g^2\frac{1}{16}g_{\mu\nu}\int d^4L\ M_K(L,p-q)$$
$$\left[-\cos\left(\frac{(p+q)\times L}{4}\right)+\cos\left(\frac{(p-q)\times L}{4}\right)\cos\left(\frac{p+q}{2}\right)\right]. \tag{4.60}$$

4.3.2 The 2-point Tadpole with amputated external legs

We now insert the Mehler kernel (2.100) into (4.60) and arrive at

$$\begin{aligned}&12g^2\frac{1}{16}g_{\mu\nu}\frac{\omega^3}{8\pi^2}\int d^4L\int\limits_0^\infty d\alpha\\ &\frac{1}{(\sinh(\alpha))^2}\exp\left[-\frac{\omega}{4}\left(\coth\left(\frac{\alpha}{2}\right)(p-q)^2+\tanh\left(\frac{\alpha}{2}\right)L^2\right)\right]\\ &\left[-\cos\left(\frac{(p+q)\times L}{4}\right)+\cos\left(\frac{(p-q)\times L}{4}\right)\cos\left(\frac{p+q}{2}\right)\right].\end{aligned} \tag{4.61}$$

We now have to rewrite the cosines in terms of exponentials

$$\begin{aligned}&3g^2\frac{1}{16}g_{\mu\nu}\frac{\omega^3}{8\pi^2}\int d^4L\int\limits_0^\infty d\alpha\\ &\frac{1}{(\sinh(\alpha))^2}\exp\left[-\frac{\omega}{4}\left(\coth\left(\frac{\alpha}{2}\right)(p-q)^2+\tanh\left(\frac{\alpha}{2}\right)L^2\right)\right]\\ &\sum_\xi\sum_\eta\left[2e^{\frac{i\xi}{4}L(\tilde{p}+\tilde{q})}-e^{\frac{i\xi}{4}(2p\times q+\eta L(\tilde{p}-\tilde{q}))}\right],\end{aligned} \tag{4.62}$$

with $\tilde{p}^\mu=\Theta^{\mu\nu}p^\nu$.
The task is now to complete the square to be able to solve the Gauss integral

over L

$$3g^2\frac{1}{16}g_{\mu\nu}\frac{\omega^3}{8\pi^2}\int d^4L\int\limits_0^\infty d\alpha\frac{1}{(\sinh(\alpha))^2}\sum_\xi\sum_\eta e^{-\frac{\omega}{4}\coth\left(\frac{\alpha}{2}\right)(p-q)^2}$$
$$\left[2e^{-\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)L^2+\frac{i\xi}{4}L(\tilde{p}+\tilde{q})}-e^{-\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)L^2+\frac{i\xi}{4}(2p\times q+\eta L(\tilde{p}-\tilde{q}))}\right]$$
$$=3g^2\frac{1}{16}g_{\mu\nu}\frac{\omega^3}{8\pi^2}\int d^4L\int\limits_0^\infty d\alpha\frac{1}{(\sinh(\alpha))^2}\sum_\xi\sum_\eta e^{-\frac{\omega}{4}\coth\left(\frac{\alpha}{2}\right)(p-q)^2}$$
$$\left[2e^{-\left(\sqrt{\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)}L-\frac{\frac{i\xi}{4}(\tilde{p}+\tilde{q})}{2\sqrt{\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)}}\right)^2-\frac{\xi^2(\tilde{p}+\tilde{q})^2}{16\omega\tanh\left(\frac{\alpha}{2}\right)}}\right.$$
$$\left.-e^{-\left(\sqrt{\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)}L-\frac{\frac{i\xi\eta}{4}(\tilde{p}-\tilde{q})}{2\sqrt{\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)}}\right)^2-\frac{\xi^2\eta^2(\tilde{p}-\tilde{q})^2}{16\omega\tanh\left(\frac{\alpha}{2}\right)}}\right]. \tag{4.63}$$

We are now ready to solve $\int d^4L$

$$=3g^2\frac{1}{16}g_{\mu\nu}\frac{\omega^3}{8\pi^2}\int\limits_0^\infty d\alpha\frac{1}{(\sinh(\alpha))^2}\sum_\xi\sum_\eta e^{-\frac{\omega}{4}\coth\left(\frac{\alpha}{2}\right)(p-q)^2}$$
$$\left[2\frac{\pi^2}{\left(\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)\right)^2}e^{-\frac{(\tilde{p}+\tilde{q})^2}{16\omega\tanh\left(\frac{\alpha}{2}\right)}}-\frac{\pi^2}{\left(\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)\right)^2}e^{-\frac{(\tilde{p}-\tilde{q})^2}{16\omega\tanh\left(\frac{\alpha}{2}\right)}}\right], \tag{4.64}$$

with $\xi^2=1$ and $\eta^2=1$. The expression is now independent of ξ and η and therefore the sums give just a factor $2*2=4$. Some numerical factors cancel and we are left with

$$=12g^2g_{\mu\nu}\frac{\omega}{8}\int\limits_0^\infty d\alpha\frac{1}{(\sinh(\alpha))^2}\frac{1}{\left(\tanh\left(\frac{\alpha}{2}\right)\right)^2}e^{-\frac{\omega}{4}\coth\left(\frac{\alpha}{2}\right)(p-q)^2}$$
$$\left[2e^{-\frac{(\tilde{p}+\tilde{q})^2}{16\omega}\coth\left(\frac{\alpha}{2}\right)}-e^{-\frac{(\tilde{p}-\tilde{q})^2}{16\omega}\coth\left(\frac{\alpha}{2}\right)}\right]. \tag{4.65}$$

Mathematica knows this integral:

$$\int\limits_0^\infty\frac{1}{\sinh^2(\alpha)}\frac{1}{\tanh^2\left(\frac{\alpha}{2}\right)}e^{-A\coth\left(\frac{\alpha}{2}\right)}d\alpha=\frac{1+A\,e^{-A}}{A^3}. \tag{4.66}$$

Therefore we get

$$12g^2 g_{\mu\nu}\frac{\omega}{8}\sum_{j=1}^{2}\left[(-1)^{j+1}\frac{j}{A_j^3}\left(1+A_j e^{-A_j}\right)\right] \tag{4.67}$$

with

$$\begin{aligned} A_1 &= \frac{\omega}{4}(p-q)^2+(\tilde{p}+\tilde{q})^2\frac{1}{16\omega} \\ A_2 &= \frac{\omega}{4}(p-q)^2+(\tilde{p}-\tilde{q})^2\frac{1}{16\omega}, \end{aligned} \tag{4.68}$$

where A_1 and A_2 have the following limits

$$\begin{aligned} A_1 &\xrightarrow{p=q} \tilde{p}^2\frac{1}{4\omega} \\ A_2 &\xrightarrow{p=q} 0 \end{aligned} \tag{4.69}$$

We analyze:

In the nonplanar part $(\tilde{p}+\tilde{q})$ the expression becomes finite for $p=q$. In the planar part $(\tilde{p}-\tilde{q})$ we discover an evil divergence of degree 6 for $p=q$.

Such a huge divergence is indeed fatal and couldn't be renormalized. But we expect a better behaviour for the graph if we add two external fields to the graph. The reason why this will probably help is the following: Looking at ordinary noncommutative ϕ**4 theory we have the quadratic divergent part $\Pi_{\mu\nu}(p)$ multiplied by a delta function $\delta(p-p')$ which makes the divergence even worse (the delta function is infinite at $p=p'$). One then integrates over the expression and is left with the quadratic divergence of $\Pi_{\mu\nu}(p)$ which then has to be taken care of by renormalization.
Here, we don't have a delta-function explicitly but some sort of, namely we understand the Mehler kernel as kind of a smeared delta function. To extract the degree of divergence we want to know, we therefore need to integrate over the whole expression. We'll do this by coupling an external field to the expression in the next chapter.

4.3.3 The 2-point Tadpole with external fields

We start at expression (4.60). In order to extract the divergence and to compute the relevant counter terms we expand the integrand depending on q around $q=p$, except for the Mehler kernel (because the Mehler kernel still depends on the parameter α). We will also leave out constant factors and therefore get

$$
\begin{aligned}
\Sigma \propto & -\int d^4p \int d^4q \tilde{A}_\rho(p) \int d^4L\, \mathcal{B}_\rho(q) M_K(p-q, L) \\
= & -\int d^4p \int d^4q \tilde{A}_\rho(p) \int d^4L \Big(\mathcal{B}_\rho(p) + (p-q)_\alpha (\partial^q_\alpha \mathcal{B}_\rho)(p) \\
& + \frac{1}{2}(p-q)_\alpha (p-q)_\beta (\partial^q_\alpha \partial^q_\beta \mathcal{B}_\rho)(p) + \dots \Big) K_M(p-q, L)
\end{aligned} \tag{4.70}
$$

with

$$
\mathcal{B}_\rho = \tilde{A}_\rho(q) \left[-\cos\left(\frac{(p+q)\times L}{4}\right) + \cos\left(\frac{(p-q)\times L}{4}\right)\cos\left(\frac{p+q}{2}\right)\right], \tag{4.71}
$$

where $\mathcal{B}_\rho$ not only depends on q, but also on p and L (and of course θ).

▷ To zeroth order we obtain

$$
\begin{aligned}
\Sigma^{(0)} = & -\int d^4p \tilde{A}_\rho(p) \int d^4L \int d^4q \mathcal{B}_\rho(p) K_M(p-q, L) \\
= & -\frac{\omega^3}{8\pi^2} \int d^4p \tilde{A}^2(p) \int d^4L \int d^4q \int_0^\infty d\alpha \left(1 - \cos\left(\frac{p \times L}{2}\right)\right) \\
& \frac{e^{-\frac{\omega}{4}[(p-q)^2 \coth\frac{\alpha}{2} + L^2 \tanh\frac{\alpha}{2}]}}{\sinh^2\alpha}.
\end{aligned} \tag{4.72}
$$

We can now integrate out $\int d^4q$ by substituting $q' = p - q$. We get a Jacobian of -1 which exactly compensates with reexchanging the integration bounds $-\infty$ and $+\infty$ which have exchanged due to our substitution. The result is

$$
= -\frac{\omega^3}{8\pi^2} \int d^4p \tilde{A}^2(p) \int d^4L \int_0^\infty d\alpha \left(1 - \cos\left(\frac{p\times L}{2}\right)\right) \frac{\pi^2}{\left(\frac{\omega}{4}\coth\frac{\alpha}{2}\right)^2} \frac{e^{-\frac{\omega}{4}L^2\tanh\frac{\alpha}{2}}}{\sinh^2\alpha}. \tag{4.73}
$$

We can write the cosine as

$$
\cos\left(\frac{p\times L}{2}\right) = \sum_{\xi=-1}^{1} e^{\frac{i\xi}{2}p\times L} = \sum_{\xi=-1}^{1} e^{\frac{i\xi}{2}L\times p} = \sum_{\xi=-1}^{1} e^{\frac{i\xi}{2}L\tilde{p}}. \tag{4.74}
$$

Therefore the exponent becomes

$$
\frac{i\xi}{2}L\tilde{p} - \frac{\omega}{4}L^2\tanh\frac{\alpha}{2} = -\left(L\sqrt{\frac{\omega}{4}\tanh\frac{\alpha}{2}} + \frac{\frac{i\xi}{2}\tilde{p}}{\cancel{2}\sqrt{\frac{\omega}{\cancel{4}}\tanh\frac{\alpha}{2}}}\right)^2 - \frac{\tilde{p}^2}{4\omega\tanh\frac{\alpha}{2}}, \tag{4.75}
$$

where we have completed the square. Now we are able to integrate out L:

$$= -\frac{\omega^3}{8\pi^2}\int d^4p\tilde{A}^2(p)\int_0^\infty d\alpha\left(1-e^{-\frac{\tilde{p}^2}{4\omega}\coth\frac{\alpha}{2}}\right)\frac{\pi^2}{\left(\frac{\omega}{4}\cancel{\coth\frac{\alpha}{2}}\right)^2}\frac{\pi^2}{\left(\frac{\omega}{4}\cancel{\tanh\frac{\alpha}{2}}\right)^2}\frac{1}{\sinh^2\alpha}$$

$$= -\frac{\omega^3}{8\pi^2}\left(\frac{4\pi}{\omega}\right)^4\int d^4p\tilde{A}^2(p)\int_0^\infty d\alpha\frac{\left(1-e^{-\frac{\tilde{p}^2}{4\omega}\coth\frac{\alpha}{2}}\right)}{\sinh^2\alpha}. \tag{4.76}$$

The first term, $\int\frac{d\alpha}{\alpha^2}$, just produces the usual Λ^2 divergence. The second term is crucial. It is what remains of the UV/IR mixing. For $\tilde{p}^2\neq 0$ integration gives

$$\int_0^\infty d\alpha\frac{e^{-\frac{\tilde{p}^2}{4\omega}\coth\frac{\alpha}{2}}}{\sinh^2\alpha} < \int_0^\infty d\alpha\frac{e^{-\frac{\tilde{p}^2}{4\omega}\coth\frac{\alpha}{2}}}{\sinh^2\frac{\alpha}{2}} = \frac{8\omega}{\tilde{p}^2}e^{-\frac{\tilde{p}^2}{4\omega}}. \tag{4.77}$$

This is finite for non-zero momentum, but diverges for $p=0$. Regulating the integral yields

$$\Sigma^{(0)}_{\text{reg}} = \int_\epsilon^\infty d\alpha\frac{1-e^{-\frac{\tilde{p}^2}{4\omega}\coth\frac{\alpha}{2}}}{\sinh^2\frac{\alpha}{2}} = \frac{4}{e^\epsilon-1}-\frac{8\omega e^{-\frac{\tilde{p}^2}{4\omega}}}{\tilde{p}^2}+\frac{8\omega e^{-\frac{\tilde{p}^2(1+e^\epsilon)}{4\omega(e^\epsilon-1)}}}{\tilde{p}^2}, \tag{4.78}$$

but this expression has a limit for $\tilde{p}^2\to 0$, the divergences cancel.

▷ All odd orders of the expansion are zero, which we will prove for the first order in appendix (A.2.2).

▷ For the second order we need the second derivative of $\mathcal{B}_\rho$:

$$\begin{aligned}\partial^q_\alpha\partial^q_\beta\mathcal{B}_\rho =&(\partial^q_\alpha\partial^q_\beta\tilde{A}_\rho)(p)\left(-\cos\frac{(p+q)\times L}{4}+\cos\frac{(p-q)\times L}{4}\cos\frac{p+q}{2}\right)\\ &+\tilde{A}_\rho\Bigg[\frac{\tilde{L}_\alpha\tilde{L}_\beta}{16}\left(\cos\frac{(p+q)\times L}{4}-\cos\frac{(p-q)\times L}{4}\cos\frac{p\times q}{2}\right)\\ &\qquad+\frac{\tilde{p}_\alpha\tilde{p}_\beta}{4}\cos\frac{(p-q)\times L}{4}\cos\frac{p\times q}{2}\\ &\qquad+\left(\frac{\tilde{L}_\alpha\tilde{p}_\beta}{8}\sin\left(\frac{(p-q)\times L}{4}\right)\sin\left(\frac{p\times q}{2}\right)+(\alpha\leftrightarrow\beta)\right)\Bigg]\\ &+\partial^q_\beta\tilde{A}_\rho()_\alpha+\partial^q_\alpha\tilde{A}_\rho()_\beta,\end{aligned} \tag{4.79}$$

where the expression in the brackets is the first derivative of $\mathcal{B}_\rho$ with respect to q which we already calculated in appendix (A.2.2). We now need to evaluate $\partial^q_\alpha \partial^q_\beta \mathcal{B}_\rho$ at $q = p$. Therefore,

$$(\partial^q_\alpha \partial^q_\beta \mathcal{B}_\rho)(p) = (\partial^q_\alpha \partial^q_\beta \tilde{A}_\rho)(p)\left(1 - \cos\left(\frac{p \times L}{2}\right)\right) \tag{4.80}$$

$$- \tilde{A}_\rho(p)\left(\frac{\tilde{L}_\alpha \tilde{L}_\beta}{16}\left(1 - \cos\frac{p \times L}{2}\right) + \frac{\tilde{p}_\alpha \tilde{p}_\beta}{4}\right) \tag{4.81}$$

$$+ (\partial_\alpha \tilde{A}_\rho)(p)\frac{\tilde{L}_\beta}{4}\sin\left(\frac{p \times L}{2}\right) + (\partial_\beta \tilde{A}_\rho)(p)\frac{\tilde{L}_\alpha}{4}\sin\left(\frac{p \times L}{2}\right). \tag{4.82}$$

For convenience, we split the second order contribution:

$$\Sigma^{(2)} = \Sigma^{(2,1)} + \Sigma^{(2,2)} + \Sigma^{(2,3)} \tag{4.83}$$

corresponding to the lines (4.80), (4.81) and (4.82). Starting with $\Sigma^{(2,1)}$ we have

$$\begin{aligned}\Sigma^{(2,1)} &= -\int d^4p \int d^4q \tilde{A}_\rho(p)(\partial^q_\alpha \partial^q_\beta \mathcal{B}_\rho)(p) \int d^4L\, \frac{1}{2}(p-q)_\alpha (p-q)_\beta K_M(p-q, L) \\ &= -\frac{64\pi^2}{\omega^2} \int d^4p \tilde{A}_\rho(p)(\partial^q_\alpha \partial^q_\beta \mathcal{B}_\rho)(p) \int_\epsilon^\infty d\alpha \frac{1 - e^{-\frac{\tilde{p}^2 \coth\frac{\alpha}{2}}{4\omega}}}{\sinh\frac{\alpha}{2}\cosh^3\frac{\alpha}{2}},\end{aligned} \tag{4.84}$$

where we have already solved the Gauss integral over L and over q, which works like described in appendix (A.2.2). $\Sigma^{(2,1)}$ is logarithmically divergent. Again the term (4.78) appears, curing the UV/IR mixing problem.

The next part is

$$
\begin{aligned}
\Sigma^{(2,2)} &= \int d^4p\tilde{A}^2(p)\int d^4q\int d^4L\,(p-q)_\alpha(p-q)_\beta K_M(p-q,L) \\
&\quad\left(\frac{\tilde{L}_\alpha\tilde{L}_\beta}{16}\left(1-\cos\frac{p\times L}{2}\right)+\frac{\tilde{p}_\alpha\tilde{p}_\beta}{4}\right) \\
&= \int d^4p\,\tilde{A}^2(p)\int d^4q\int d^4L\,\left(\frac{\tilde{L}_\alpha\tilde{L}_\beta}{16}\left(1-\cos\frac{p\times L}{2}\right)+\frac{\tilde{p}_\alpha\tilde{p}_\beta}{4}\right) \\
&\quad(p-q)_\alpha(p-q)_\beta\int\limits_\epsilon^\infty\frac{d\alpha}{\sinh^2\alpha}e^{-\frac{\omega}{4}\left[(p-q)^2\coth\frac{\alpha}{2}+L^2\tanh\frac{\alpha}{2}\right]} \\
&= \int d^4p\tilde{A}^2(p)\left[\frac{16\pi^2}{\omega^2}\tilde{p}^2\int\limits_\epsilon^\infty\frac{d\alpha}{\sinh\frac{\alpha}{2}\cosh^3\frac{\alpha}{2}}\right. && (4.85)\\
&\qquad+\frac{4\pi^2\theta^2}{\omega^4}\tilde{p}^2\int\limits_\epsilon^\infty\frac{d\alpha\,e^{-\frac{\tilde{p}^2\coth\frac{\alpha}{2}}{4\omega}}}{\sinh^3\frac{\alpha}{2}\cosh\frac{\alpha}{2}} && (4.86)\\
&\qquad\left.+\frac{32\pi^2\theta^2}{\omega^3}\int\limits_\epsilon^\infty\frac{d\alpha}{\sinh^2\frac{\alpha}{2}\cosh^2\frac{\alpha}{2}}\left(1-e^{-\frac{\tilde{p}^2\coth\frac{\alpha}{2}}{4\omega}}\right)\right]. && (4.87)
\end{aligned}
$$

The contribution (4.85) is logarithmically divergent, (4.86) is finite, whereas **(4.87) is quadratically divergent**, for $p^2\neq 0$. This sounds like bad news because this quadratic divergence will also occur in higher order contributions, but in the end it turns out that all quadratically divergent parts of the expansion can be summed up and give just one quadratically divergent term multiplied with a phase. This can of course be compensated by one counter term which corresponds to a renormalization of the wave function (because the "evil" quadratic divergence comes from the 4-photon vertex which is included in the kinetic term $F_{\mu\nu}F_{\mu\nu}$ as proved in section (3.5)).
What's left to mention is that

$$\Sigma^{(2,3)} = 0. \tag{4.88}$$

Chapter 5

Outlook

Considering the Mehler kernel section in this diploma thesis one is immediately forced to realize that the missing link in this picture is the Mehler kernel in Minkowski space, corresponding to the nature of our universe, which is Minkowskian. The reason that this is not an easy task is that in non-commutative Minkowskian geometry time and space do not commute. A straightforward Wick rotation of the 4-dimensional Mehler kernel in chapter (2.7) back to Minkowski space is certainly not the right way to go, since the t or τ in formula (2.90) is only a parameter, the real time is hidden in the space time variable x^μ. Although serious effort has been taken, nobody has yet been able to come up with the right answer.

Regarding the loop calculations of the model [20], only the Tadpole graphs are calculated in this diploma thesis. Higher loop calculations are in progress, e.g. we have strong indications that the self-energy graph with two insertions (2-loop-niveau) will be renormalizable too.
Due to the huge success of the Grosse-Wulkenhaar model [16] and our first loop calculations with this model (which is basically an extension of the G.W. model to gauge theories) we expect to be able to renormalize this model to all orders.

Appendix A

Detailed Calculations

A.1 The Mehler kernel

A.1.1 The Mehler kernel in terms of trigonometric functions

We start at formula (2.75)

$$K_{osc}(x, y, t) = \frac{1}{\sqrt{\pi}} \frac{e^{-it/2}}{(1 - e^{-2it})^{1/2}} \exp\Big(- \frac{(e^{-it}x - y)^2}{1 - e^{-2it}} - \frac{x^2}{2} + \frac{y^2}{2}\Big). \quad \text{(A.1)}$$

Upon bringing the exponent to the same denominator we get

$$\begin{aligned}
=&\frac{1}{\sqrt{\pi}} \left(\frac{e^{-it}}{1 - e^{-2it}}\right)^{\frac{1}{2}} \exp\Big(- \frac{2(e^{-it}x - y)^2 - x^2(1 - e^{-2it}) + y^2(1 - e^{-2it})}{2(1 - e^{-2it})}\Big)\\
=&\frac{1}{\sqrt{\pi}} \left(\frac{e^{-it}}{1 - e^{-2it}}\right)^{\frac{1}{2}} \exp\Big(- \frac{2(e^{-2it}x^2 - 2xye^{-it} + y^2) + x^2(1 - e^{-2it}) - y^2(1 - e^{-2it})}{2(1 - e^{-2it})}\Big)\\
=&\frac{1}{\sqrt{\pi}} \left(\frac{e^{-it}}{1 - e^{-2it}}\right)^{\frac{1}{2}} \exp\Big(- \frac{e^{-2it}x^2 - 4xye^{-it} + y^2 + x^2 + y^2e^{-2it}}{2(1 - e^{-2it})}\Big)\\
=&\frac{1}{\sqrt{\pi}} \left(\frac{e^{-it}}{1 - e^{-2it}}\right)^{\frac{1}{2}} \exp\Big(- \frac{x^2(e^{-2it} + 1) + y^2(1 + e^{-2it}) - 4xye^{-it}}{2(1 - e^{-2it})}\Big)\\
=&\frac{1}{\sqrt{\pi}} \left(\frac{1}{e^{it} - e^{-it}}\right)^{\frac{1}{2}} \exp\Big(- \frac{x^2(e^{it} + e^{-it}) + y^2(e^{it} + e^{-it}) - 4xy}{2(e^{it} - e^{-it})}\Big).
\end{aligned} \quad \text{(A.2)}$$

By using

$$\begin{aligned}
e^{it} + e^{-it} =& 2i \sin t\\
e^{it} - e^{-it} =& 2 \cos t
\end{aligned} \quad \text{(A.3)}$$

we get

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi}} \left(\frac{1}{2i \sin t} \right)^{\frac{1}{2}} \exp \Big(- \frac{x^2(2\cos t) + y^2(2\cos t) - 4xy}{2(2i\sin t)} \Big) \\
&= \left(\frac{1}{2\pi i \sin t} \right)^{\frac{1}{2}} \exp \Big(- \frac{x^2 \cos t + y^2 \cos t - 2xy}{2i \sin t} \Big) \\
&= \frac{1}{\sqrt{2\pi i \ \sin t}} \exp \Big(i \frac{x^2 + y^2}{2} \cot t - i \frac{xy}{\sin t} \Big). \qquad \text{(A.4)}
\end{aligned}
$$

A.1.2 The Fourier transformation of the Mehler kernel

We want to Fourier transform $\Delta(x, y)$ =(2.90). For a shorter calculation we will directly start with the Mehler kernel in terms of long and short variables (2.93) and show its Fourier transform is equivalent to (2.100).
The Fourier transform of $\Delta(x, y)$ is given by

$$
\tilde{\Delta}(p, q) = \frac{1}{(2\pi)^4} \int d^4x \int d^4y e^{-ipx} e^{-iqy} \Delta(x, y). \qquad \text{(A.5)}
$$

We subsitute in terms of long and short variables

$$
S = k - k', \quad L = k + k' \quad \Rightarrow \quad k = \frac{L + S}{2}, \quad k' = \frac{L - S}{2}. \qquad \text{(A.6)}
$$

This is a two-dimensional variable substitution, and therefore we get a Jacobian:

$$
\begin{aligned}
J &= \begin{pmatrix} \frac{\partial k}{\partial L} & \frac{\partial k}{\partial S} \\ \frac{\partial k'}{\partial L} & \frac{\partial k'}{\partial S} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\
\Rightarrow \det J &= -\frac{1}{2} \stackrel{4-dim}{\longrightarrow} \left(-\frac{1}{2} \right)^4 = \frac{1}{16}. \qquad \text{(A.7)}
\end{aligned}
$$

(A.5) therefore becomes

$$
\tilde{\Delta}(p, q) = \frac{1}{(2\pi)^4} \int d^4L \int d^4S \frac{1}{16} e^{-ip\frac{L+S}{2}} e^{-iq\frac{L-S}{2}} \Delta(L, S), \qquad \text{(A.8)}
$$

with $\Delta(L, S)$ given by (2.93).
We rearrange the terms in the exponential

$$
\begin{aligned}
&\frac{1}{16(2\pi)^4} \frac{1}{\pi^2 \omega} \int d^4L \int d^4S \int_0^\infty d\tau \frac{1}{\left(2 \sinh \frac{\tau}{2}\right)^2} \\
&\exp \left[-S^2 \frac{\coth \frac{\tau}{2}}{4\omega} - \frac{i}{2} S(p - q) - L^2 \frac{\tanh \frac{\tau}{2}}{4\omega} - \frac{i}{2} L(p + q) \right], \qquad \text{(A.9)}
\end{aligned}
$$

complete the square

$$\frac{1}{16(2\pi)^4}\frac{1}{\pi^2\omega}\int d^4L\int d^4S\int_0^\infty d\tau\frac{1}{\left(2\sinh\frac{\tau}{2}\right)^2}$$
$$\exp\left[-\left(S\sqrt{\frac{\coth\frac{\tau}{2}}{4\omega}}+\frac{i(p-q)}{4\sqrt{\frac{\coth\frac{\tau}{2}}{4\omega}}}\right)^2+\left(\frac{i(p-q)}{4\sqrt{\frac{\coth\frac{\tau}{2}}{4\omega}}}\right)^2\right.$$
$$\left.-\left(L\sqrt{\frac{\tanh\frac{\tau}{2}}{4\omega}}+\frac{i(p+q)}{4\sqrt{\frac{\tanh\frac{\tau}{2}}{4\omega}}}\right)^2+\left(\frac{i(p+q)}{4\sqrt{\frac{\tanh\frac{\tau}{2}}{4\omega}}}\right)^2\right] \tag{A.10}$$

and solve the Gauß integral

$$\frac{1}{16(2\pi)^4}\frac{1}{\pi^2\omega}\int_0^\infty d\tau\frac{1}{\left(2\sinh\frac{\tau}{2}\right)^2}\frac{\pi^2}{\left(\frac{\coth\frac{\tau}{2}}{4\omega}\right)^2}\frac{\pi^2}{\left(\frac{\tanh\frac{\tau}{2}}{4\omega}\right)^2}$$
$$\exp\left[-\frac{\omega(p-q)^2}{4\coth\frac{\tau}{2}}-\frac{\omega(p+q)^2}{4\tanh\frac{\tau}{2}}\right].$$

Fortunately the prefactors $\tanh\frac{\tau}{2}$ and $\coth\frac{\tau}{2}$ cancel one another. So, finally we end up at

$$\frac{\omega^3}{\pi^2}\int_0^\infty\frac{1}{\left(2\sinh\frac{\tau}{2}\right)^2}\exp\left[-\frac{\omega}{4}\left((p-q)^2\tanh\frac{\tau}{2}+(p+q)^2\coth\frac{\tau}{2}\right)\right]. \tag{A.11}$$

A.2 Loop calculations

A.2.1 4-photon vertex - explicit calculation

Applying (3.46) to (3.45) gives

$$\begin{aligned}V^{4A}=&-(2\pi)^{16}\frac{\delta}{\delta\tilde{A}_\lambda(-k_4)}\frac{\delta}{\delta\tilde{A}_\tau(-k_3)}\frac{\delta}{\delta\tilde{A}_\sigma(-k_2)}\\&\frac{-2ig^2}{2}\int\frac{d^4q_1...q_4}{(2\pi)^{16}}\delta(\sum_i q_i)(2\pi)^4e^{-\frac{i}{2}(q_1\times q_2)}\sin\left(\frac{q_3\times q_4}{2}\right)\\&\Big[\delta^\rho_\mu\delta(k_1+q_1)\tilde{A}_\nu(q_2)\tilde{A}_\mu(q_3)\tilde{A}_\nu(q_4)+\delta^\rho_\nu\delta(k_1+q_2)\tilde{A}_\mu(q_1)\tilde{A}_\mu(q_3)\tilde{A}_\nu(q_4)\\&+\delta^\rho_\mu\delta(k_1+q_3)\tilde{A}_\mu(q_1)\tilde{A}_\nu(q_2)\tilde{A}_\nu(q_4)+\delta^\rho_\nu\delta(k_1+q_4)\tilde{A}_\mu(q_1)\tilde{A}_\nu(q_2)\tilde{A}_\mu(q_3)\Big],\end{aligned} \tag{A.12}$$

where we have used

$$\frac{\delta A_\mu(x)}{\delta A_\nu(y)} = \delta_\mu^\nu \delta(x-y). \tag{A.13}$$

Further evaluation gives

$$\begin{aligned}
= -(2\pi)^{16} & \frac{\delta}{\delta \tilde{A}_\lambda(k_4)} \frac{\delta}{\delta \tilde{A}_\tau(k_3)} \\
& \frac{-2ig^2}{2} \int \frac{d^4q_1...q_4}{(2\pi)^{16}} \delta(\sum_i q_i)(2\pi)^4 e^{-\frac{i}{2}(q_1\times q_2)} \sin\left(\frac{q_3\times q_4}{2}\right) \\
& \Big[\delta_\mu^\rho \delta(k_1+q_1)\Big(\delta_\nu^\sigma \delta(k_2+q_2)\tilde{A}_\mu(q_3)\tilde{A}_\nu(q_4) \\
& \quad + \delta_\mu^\sigma \delta(k_2+q_3)\tilde{A}_\nu(q_2)\tilde{A}_\nu(q_4) \\
& \quad + \delta_\nu^\sigma \delta(k_2+q_4)\tilde{A}_\nu(q_2)\tilde{A}_\mu(q_3)\Big) \\
& + \delta_\nu^\rho \delta(k_1+q_2)\Big(\delta_\mu^\sigma \delta(k_2+q_1)\tilde{A}_\mu(q_3)\tilde{A}_\nu(q_4) \\
& \quad + \delta_\mu^\sigma \delta(k_2+q_3)\tilde{A}_\mu(q_1)\tilde{A}_\nu(q_4) \\
& \quad + \delta_\nu^\sigma \delta(k_2+q_4)\tilde{A}_\mu(q_1)\tilde{A}_\mu(q_3)\Big) \\
& + \delta_\mu^\rho \delta(k_1+q_3)\Big(\delta_\mu^\sigma \delta(k_2+q_1)\tilde{A}_\nu(q_2)\tilde{A}_\nu(q_4) \\
& \quad + \delta_\nu^\sigma \delta(k_2+q_2)\tilde{A}_\mu(q_1)\tilde{A}_\nu(q_4) \\
& \quad + \delta_\nu^\sigma \delta(k_2+q_4)\tilde{A}_\mu(q_1\tilde{A}_\nu(q_2)\Big) \\
& + \delta_\nu^\rho \delta(k_1+q_4)\Big(\delta_\mu^\sigma \delta(k_2+q_1)\tilde{A}_\nu(q_2)\tilde{A}_\mu(q_3) \\
& \quad + \delta_\nu^\sigma \delta(k_2+q_2)\tilde{A}_\mu(q_1)\tilde{A}_\mu(q_3) \\
& \quad + \delta_\mu^\sigma \delta(k_2+q_3)\tilde{A}_\mu(q_1)\tilde{A}_\nu(q_2)\Big)\Big].
\end{aligned}$$

Evaluating the last 2 variations leads to

$$
\begin{aligned}
=&-(2\pi)^{16}\frac{\delta}{\delta\tilde{A}_{\lambda}(k_4)}\frac{\delta}{\delta\tilde{A}_{\tau}(k_3)} \qquad \text{(A.14)}\\
&\frac{-2ig^2}{2}\int\frac{d^4q_1...q_4}{(2\pi)^{16}}\delta(\sum_i q_i)(2\pi)^4 e^{-\frac{i}{2}(q_1\times q_2)}\sin\left(\frac{q_3\times q_4}{2}\right)\\
&\Big[\delta^{\rho}_{\mu}\delta(k_1+q_1)\Big[\delta^{\sigma}_{\nu}\delta(k_2+q_2)\left(\delta^{\tau}_{\mu}\delta(k_3+q_3)\delta^{\lambda}_{\nu}\delta(k_4+q_4)+\delta^{\tau}_{\nu}\delta(k_3+q_4)\delta^{\lambda}_{\mu}\delta(k_4+q_3)\right)\\
&\qquad+\delta^{\sigma}_{\mu}\delta(k_2+q_3)\left(\delta^{\tau}_{\nu}\delta(k_3+q_2)\delta^{\lambda}_{\nu}\delta(k_4+q_4)+\delta^{\tau}_{\nu}\delta(k_3+q_4)\delta^{\lambda}_{\nu}\delta(k_4+q_2)\right)\\
&\qquad+\delta^{\sigma}_{\nu}\delta(k_2+q_4)\left(\delta^{\tau}_{\nu}\delta(k_3+q_2)\delta^{\lambda}_{\mu}\delta(k_4+q_3)+\delta^{\tau}_{\mu}\delta(k_3+q_3)\delta^{\lambda}_{\nu}\delta(k_4+q_2)\right)\Big]\\
&+\delta^{\rho}_{\nu}\delta(k_1+q_2)\Big[\delta^{\sigma}_{\mu}\delta(k_2+q_1)\left(\delta^{\tau}_{\mu}\delta(k_3+q_3)\delta^{\lambda}_{\nu}\delta(k_4+q_4)+\delta^{\tau}_{\nu}\delta(k_3+q_4)\delta^{\lambda}_{\mu}\delta(k_4+q_3)\right)\\
&\qquad+\delta^{\sigma}_{\mu}\delta(k_2+q_3)\left(\delta^{\tau}_{\mu}\delta(k_3+q_1)\delta^{\lambda}_{\nu}\delta(k_4+q_4)+\delta^{\tau}_{\nu}\delta(k_3+q_4)\delta^{\lambda}_{\mu}\delta(k_4+q_1)\right)\\
&\qquad+\delta^{\sigma}_{\nu}\delta(k_2+q_4)\left(\delta^{\tau}_{\mu}\delta(k_3+q_1)\delta^{\lambda}_{\mu}\delta(k_3+q_1)+\delta^{\tau}_{\mu}\delta(k_3+q_3)\delta^{\lambda}_{\mu}\delta(k_4+q_1)\right)\Big]\\
&+\delta^{\rho}_{\mu}\delta(k_1+q_3)\Big[\delta^{\sigma}_{\mu}\delta(k_2+q_1)\left(\delta^{\tau}_{\nu}\delta(k_3+q_2)\delta^{\lambda}_{\nu}\delta(k_4+q_4)+\delta^{\tau}_{\nu}\delta(k_3+q_4)\delta^{\lambda}_{\nu}\delta(k_4+q_2)\right)\\
&\qquad+\delta^{\sigma}_{\nu}\delta(k_2+q_2)\left(\delta^{\tau}_{\mu}\delta(k_3+q_1)\delta^{\lambda}_{\nu}\delta(k_4+q_4)+\delta^{\tau}_{\nu}\delta(k_3+q_4)\delta^{\lambda}_{\mu}\delta(k_4+q_1)\right)\\
&\qquad+\delta^{\sigma}_{\nu}\delta(k_2+q_4)\left(\delta^{\tau}_{\mu}\delta(k_3+q_1)\delta^{\lambda}_{\nu}\delta(k_4+q_2)+\delta^{\tau}_{\nu}\delta(k_3+q_2)\delta^{\lambda}_{\mu}\delta(k_4+q_1)\right)\Big]\\
&+\delta^{\rho}_{\nu}\delta(k_1+q_4)\Big[\delta^{\sigma}_{\mu}\delta(k_2+q_1)\left(\delta^{\tau}_{\nu}\delta(k_3+q_2)\delta^{\lambda}_{\mu}\delta(k_4+q_3)+\delta^{\tau}_{\mu}\delta(k_3+q_3)\delta^{\lambda}_{\nu}\delta(k_4+q_2)\right)\\
&\qquad+\delta^{\sigma}_{\nu}\delta(k_2+q_2)\left(\delta^{\tau}_{\mu}\delta(k_3q_1)\delta^{\lambda}_{\mu}\delta(k_4+q_3)+\delta^{\tau}_{\mu}\delta(k_3+q_3)\delta^{\lambda}_{\mu}\delta(k_4+q_1)\right)\\
&\qquad+\delta^{\sigma}_{\mu}\delta(k_2+q_3)\left(\delta^{\tau}_{\mu}\delta(k_3+q_1)\delta^{\lambda}_{\nu}\delta(k_4+q_2)+\delta^{\tau}_{\nu}\delta(k_3+q_2)\delta^{\lambda}_{\mu}\delta(k_4+q_1)\right)\Big]\Big].
\end{aligned}
\tag{A.15}
$$

Summation over the indices gives

$$
\begin{aligned}
=& -(2\pi)^{16} \frac{\delta}{\delta \tilde{A}_\lambda(k_4)} \frac{\delta}{\delta \tilde{A}_\tau(k_3)} \qquad \text{(A.16)} \\
& \frac{-2ig^2}{2} \int \frac{d^4 q_1 ... q_4}{(2\pi)^{16}} \delta(\sum_i q_i)(2\pi)^4 e^{-\frac{i}{2}(q_1 \times q_2)} \sin\left(\frac{q_3 \times q_4}{2}\right) \\
& \Bigg[\delta(k_1+q_1) \Big[\delta(k_2+q_2) \left(g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_3)\delta(k_4+q_4) + g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_4)\delta(k_4+q_3) \right) \\
& \qquad + \delta(k_2+q_3) \left(g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_2)\delta(k_4+q_4) + g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_4)\delta(k_4+q_2) \right) \\
& \qquad + \delta(k_2+q_4) \left(g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_2)\delta(k_4+q_3) + g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_3)\delta(k_4+q_2) \right) \Big] \\
& + \delta(k_1+q_2) \Big[\delta(k_2+q_1) \left(g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_3)\delta(k_4+q_4) + g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_4)\delta(k_4+q_3) \right) \\
& \qquad + \delta(k_2+q_3) \left(g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_1)\delta(k_4+q_4) + g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_4)\delta(k_4+q_1) \right) \\
& \qquad + \delta(k_2+q_4) \left(g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_1)\delta(k_3+q_1) + g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_3)\delta(k_4+q_1) \right) \Big] \\
& + \delta(k_1+q_3) \Big[\delta(k_2+q_1) \left(g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_2)\delta(k_4+q_4) + g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_4)\delta(k_4+q_2) \right) \\
& \qquad + \delta(k_2+q_2) \left(g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_1)\delta(k_4+q_4) + g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_4)\delta(k_4+q_1) \right) \\
& \qquad + \delta(k_2+q_4) \left(g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_1)\delta(k_4+q_2) + g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_2)\delta(k_4+q_1) \right) \Big] \\
& + \delta(k_1+q_4) \Big[\delta(k_2+q_1) \left(g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_2)\delta(k_4+q_3) + g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_3)\delta(k_4+q_2) \right) \\
& \qquad + \delta(k_2+q_2) \left(g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_1)\delta(k_4+q_3) + g^{\rho\sigma} g^{\tau\lambda} \delta(k_3+q_3)\delta(k_4+q_1) \right) \\
& \qquad + \delta(k_2+q_3) \left(g^{\rho\lambda} g^{\sigma\tau} \delta(k_3+q_1)\delta(k_4+q_2) + g^{\rho\tau} g^{\sigma\lambda} \delta(k_3+q_2)\delta(k_4+q_1) \right) \Big] \Bigg].
\end{aligned}
\tag{A.17}
$$

Well, this is a long expression, and it would be nice to sort it somehow. We will sort it with respect to the metric expressions. Furthermore, we will now solve the remaining integrals with the δ functions:

$$
\begin{aligned}
&= (2ig^2)\delta(\sum_i k_i)(2\pi)^4 \\
&\Bigg[g^{\rho\tau} g^{\sigma\lambda} \Big[e^{\frac{i}{2}k_1\times k_2} \sin\left(\frac{k_3\times k_4}{2}\right) + e^{\frac{i}{2}k_1\times k_4} \sin\left(\frac{k_3\times k_2}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_2\times k_1} \sin\left(\frac{k_4\times k_3}{2}\right) + e^{\frac{i}{2}k_4\times k_1} \sin\left(\frac{k_3\times k_2}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_3\times k_2} \sin\left(\frac{k_1\times k_4}{2}\right) + e^{\frac{i}{2}k_3\times k_4} \sin\left(\frac{k_1\times k_2}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_2\times k_3} \sin\left(\frac{k_4\times k_1}{2}\right) + e^{\frac{i}{2}k_4\times k_3} \sin\left(\frac{k_2\times k_1}{2}\right) \Big] \\
&+ g^{\rho\lambda} g^{\sigma\tau} \Big[e^{\frac{i}{2}k_1\times k_2} \sin\left(\frac{k_4\times k_3}{2}\right) + e^{\frac{i}{2}k_1\times k_3} \sin\left(\frac{k_4\times k_2}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_2\times k_1} \sin\left(\frac{k_3\times k_4}{2}\right) + e^{\frac{i}{2}k_3\times k_1} \sin\left(\frac{k_2\times k_4}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_4\times k_2} \sin\left(\frac{k_1\times k_3}{2}\right) + e^{\frac{i}{2}k_4\times k_3} \sin\left(\frac{k_1\times k_2}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_2\times k_4} \sin\left(\frac{k_3\times k_1}{2}\right) + e^{\frac{i}{2}k_3\times k_4} \sin\left(\frac{k_2\times k_1}{2}\right) \Big] \\
&+ g^{\rho\sigma} g^{\tau\lambda} \Big[e^{\frac{i}{2}k_1\times k_3} \sin\left(\frac{k_2\times k_1}{2}\right) + e^{\frac{i}{2}k_1\times k_4} \sin\left(\frac{k_2\times k_3}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_3\times k_4} \sin\left(\frac{k_4\times k_2}{2}\right) + e^{\frac{i}{2}k_4\times k_1} \sin\left(\frac{k_3\times k_2}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_2\times k_3} \sin\left(\frac{k_1\times k_4}{2}\right) + e^{\frac{i}{2}k_2\times k_4} \sin\left(\frac{k_1\times k_3}{2}\right) \\
&\qquad + e^{\frac{i}{2}k_3\times k_2} \sin\left(\frac{k_4\times k_1}{2}\right) + e^{\frac{i}{2}k_4\times k_2} \sin\left(\frac{k_3\times k_1}{2}\right) \Big] \Bigg].
\end{aligned} \tag{A.18}
$$

Now, we can sum up some expressions because they are identical and we can merge some exponentials to a sine:

$$= (-4g^2)\delta(\sum_i k_i)(2\pi)^4$$
$$\Bigg[g^{\rho\tau}g^{\sigma\lambda}\Big[\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)+\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_3\times k_2}{2}\right)\Big]$$
$$+g^{\rho\lambda}g^{\sigma\tau}\Big[\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_4\times k_3}{2}\right)+\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_4\times k_2}{2}\right)\Big]$$
$$+g^{\rho\sigma}g^{\tau\lambda}\Big[\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)+\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_2\times k_3}{2}\right)\Big]\Bigg]. \tag{A.19}$$

This can also be written in the following form, corresponding to the result of D.N. Blaschke et. al. ([19]):

$$\begin{aligned} V^{4A} = {} & -4g^2\delta\left(\sum_i k_i\right)(2\pi)^4 \\ & \Big[\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)(g^{\rho\tau}g^{\sigma\lambda}-g^{\rho\lambda}g^{\sigma\tau}) \\ & +\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right)(g^{\rho\sigma}g^{\tau\lambda}-g^{\rho\lambda}g^{\sigma\tau}) \\ & +\sin\left(\frac{k_1\times k_4}{2}\right)\sin\left(\frac{k_2\times k_3}{2}\right)(g^{\rho\sigma}g^{\tau\lambda}-g^{\rho\tau}g^{\sigma\lambda})\Big]. \end{aligned} \tag{A.20}$$

A.2.2 The first order of the 2-point tadpole

In this section we will show that the first order of the expansion (in the external fields) of the 2 point tadpole vanishes.
For this task we take a closer look at the integral over q in the first order of expression (4.70):

$$\Sigma^{(1)} = -\int d^4p\int d^4q\,\tilde{A}_\rho(p)\int d^4L\,(p-q)_\alpha(\partial^q_\alpha\mathcal{B}_\rho)(p)K_M(p-q,L). \tag{A.21}$$

To evaluate this we need to calculate the derivative of $\mathcal{B}_\rho$:

$$
\begin{aligned}
\partial^q_\alpha \mathcal{B}_\rho =& \partial^q_\alpha \left[\tilde{A}_\rho(q) \left[-\cos\left(\frac{(p+q)\times L}{4}\right) + \cos\left(\frac{(p-q)\times L}{4}\right)\cos\left(\frac{p+q}{2}\right)\right]\right] \\
=& \frac{\partial \tilde{A}_\rho(q)}{\partial q^\alpha}\left[-\cos\left(\frac{(p+q)\times L}{4}\right) + \cos\left(\frac{(p-q)\times L}{4}\right)\cos\left(\frac{p+q}{2}\right)\right] \\
&+ \tilde{A}_\rho(q)\left[-\sin\left(\frac{p\times q}{2}\right)\tilde{p}_\alpha \cos\left(\frac{(p-q)\times L}{4}\right)\right. \\
&\qquad + \cos\left(\frac{p\times q}{2}\sin\left(\frac{(p-q)\times L}{4}\right)\frac{\tilde{L}_\alpha}{4}\right) \\
&\qquad \left. + \sin\left(\frac{(p+q)\times L}{4}\right)\frac{\tilde{L}_\alpha}{4}\right]
\end{aligned}
\tag{A.22}
$$

and evaluate it at $q = p$:

$$
\begin{aligned}
&\partial^q_\alpha \mathcal{B}_\rho \Big|_{q=p} \\
&= - \left.\frac{\partial \tilde{A}_\rho(q)}{\partial q^\alpha}\right|_{q=p} \cos\left(\frac{p\times L}{2}\right) + \tilde{A}_\rho(p)\sin\left(\frac{p\times L}{2}\right)\frac{\tilde{L}_\alpha}{4}.
\end{aligned}
\tag{A.23}
$$

Thus, the full expression for the first order is

$$
\begin{aligned}
\Sigma^{(1)} =& -\int d^4p \int d^4q\, \tilde{A}_\rho(p)(p-q)_\alpha \int d^4L\, K_M(p-q,L) \\
&\left[- \left.\frac{\partial \tilde{A}_\rho(q)}{\partial q^\alpha}\right|_{q=p} \cos\left(\frac{p\times L}{2}\right) + \tilde{A}_\rho(p)\sin\left(\frac{p\times L}{2}\right)\frac{\tilde{L}_\alpha}{4}\right].
\end{aligned}
\tag{A.24}
$$

We now want to solve the integral over q. Mathematically, this is a delicate procedure, because normally shifting a function which is not of Schwarz type is not allowed. Anyway, this problem can be avoided by performing the shift, solving the integral and then reshifting before the integration boundaries are inserted. This feature will be illustrated in the following neat example:

$$
\begin{aligned}
\int_{-\infty}^{\infty} dq(q-2) &= \frac{q^2}{2} - 2q\Big|_{-\infty}^{\infty} = -\infty \\
\int_{-\infty}^{\infty} dq(q-2) \overset{!}{=} \int_{-\infty}^{\infty} dq'\, q' &= \frac{q'^2}{2}\Big|_{-\infty}^{\infty} = 0
\end{aligned}
\tag{A.25}
$$

which gives two different results for the shifted and the non shifted version. Of course the 2nd line is wrong. However, we can as explained above reshift after the integration and get the right result:

$$\int_{-\infty}^{\infty} dq(q-2) \stackrel{!}{=} \int_{-\infty}^{\infty} dq'\, q' = \frac{q'^2}{2}\Big|_{\partial q'} = \frac{(q-2)^2}{2}\Big|_{-\infty}^{\infty} = -\infty. \tag{A.26}$$

We will now use exactly the same trick here for our integral

$$\begin{aligned}
\Sigma^{(1)} =& -\int d^4p \int d^4q'\, \tilde{A}_\rho(p) q'_\alpha \int d^4L\, K_M(q', L) \\
& \left[-\left.\frac{\partial \tilde{A}_\rho(q)}{\partial q^\alpha}\right|_{q=p} \cos\left(\frac{p\times L}{2}\right) + \tilde{A}_\rho(p) \sin\left(\frac{p\times L}{2}\right) \frac{\tilde{L}_\alpha}{4} \right] \\
=& -\int d^4p \int d^4q'\, \tilde{A}_\rho(p) q'_\alpha \int d^4L \frac{\omega^3}{8\pi^2} \int_0^\infty d\beta \frac{1}{\sinh^2\beta} e^{-\frac{\omega}{4}q'^2 \coth\frac{\beta}{2} + L^2 \tanh\frac{\beta}{2}} \\
& \left[-\left.\frac{\partial \tilde{A}_\rho(q)}{\partial q^\alpha}\right|_{q=p} \cos\left(\frac{p\times L}{2}\right) + \tilde{A}_\rho(p) \sin\left(\frac{p\times L}{2}\right) \frac{\tilde{L}_\alpha}{4} \right].
\end{aligned}$$

The integral

$$\int d^4q'\, q' e^{-q'^2 * A} = -\frac{e^{-q'^2 * A}}{2A}\Big|_{\partial q'} = -\frac{e^{-(p-q)^2 * A}}{2A}\Big|_{-\infty}^{\infty} = 0 \tag{A.27}$$

and so we get 0, as promised.

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