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Population structure and capital accumulation: the role of rising longevity on savings

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Abstract

In recent years the population in most industrialized countries is experiencing a change in the age composition of its members. Most notably, the share of old people will increase as a consequence of low fertility rates and increasing life expectancies. By now, it is not clear how a changing age composition of a society will affect the overall level of savings of an economy. However, this influence has to be considered as crucial, since the saving rate is an important variable in an economy.

This master thesis focuses on the individual level. It deals with the question of how an increasing life expectancy influences consumption and savings at the individual level. By reviewing four alternative models of the literature, the thesis works out the different results on the relation between longevity and individual savings as it will depend on the model specification chosen. All four models are based on the life cycle theory with consumption smoothing. The effects of different pension systems on savings are not examined in these models, though of course they will have an important impact on savings.

While the different models introduced in the master thesis produce different outcomes, we regard the model that assumes an initially low mortality rate and a fixed retirement age to be the most realistic one. Under these assumptions an increasing life expectancy will imply falling saving rates. Assuming alternatively an initially high mortality rate and a fixed retirement age, would result in a positive relation between life expectancy and savings. For a model that assumes no fixed retirement age, the most probable result of an increasing life expectancy is again a decreasing saving rate but additionally also the effect that the population will work longer.

The models used could be improved in many ways. The three most interesting amendments would be the allowance of agents who do not behave fully rational (i.e. either irrational or bounded rational), to move from an analysis at the individual level to an analysis at the household level and to allow for institutionalized transfer systems.

Contents

1 Introduction

Nowadays society is facing a transformation in the age composition of its members. The median age will increase because of a larger proportion of old people as a result of low fertility and increasing life expectancy. This new population structure is supposed to have a large influence on savings. It will change the savings levels of different age groups and therefore can substantially change the overall level of savings of an economy. In addition to the compositional changes, behavioral effects (i.e. changes in the saving rates at different ages) will take place. Savings have a significant influence on the economy as a whole; hence the saving rate is an important economic variable.

The facts of an increasing life expectancy, a larger proportion of old people and changes in the aggregated saving rates can be measured easily on a macroeconomic level. Nevertheless, the question of how an increasing life expectancy influences an individual's choices of consumption and savings over the whole life cycle is not perfectly understood yet. This is crucial because a sound aggregation of the individual's choices of savings in the future would yield an improved forecast of the aggregated saving rate of an economy.

This work gives an overview of various models that aim to link changes in the population structure to savings behavior. Note that savings forecasts are always based on models and therefore the exact specification of the underlying model plays an important role.

The focus of this thesis is on the interaction between individual life length and individual savings with regard to increasing life expectations. It is based on life cycle theories with consumption smoothing and with a focus on the individual decision rather than on macroeconomic examinations.

There is no examination of the implications of different pension systems (pay-as-you-go, pre-funded or mixed systems) on savings, but it is important to note that the actual behavior of course is influenced by the prevailing pension system. The retirement age in pension systems can either be exogenously or endogenously implemented, like in Bloom et al. (2004) where endogenous retirement caused by worsening health in old age is examined.

Chapter 2 is an introduction to life-cycle models. At first savings as an economic variable and reasons for people to save are discussed. Afterwards the Solow model as an example of a model with exogenous saving rates and the Ramsey model as an example of a model with endogenous saving rates is presented. Then inherent differences between micro and macro data for savings are analyzed and explained with an intuitive model. Finally the importance of uncertainty in future income for savings is examined.

1.1 Overview of the papers included in the thesis

In this section I will outline the main characteristics of the 4 chosen papers for which a detailed analysis will be presented. The models chosen are two overlapping generations models and two continuous time models. The two overlapping generations models give different outcomes which are due to different model specifications. The fact of different outcomes is also achieved with the two continuous time models. By comparing 4 models on the same topic the influences of the main model specifications become evident.

A common assumption for all the chosen models is that both the interest rate and the wage rate are exogenously given.

1.1.1 Overlapping generations models

The two chosen overlapping generations models mainly differ in 3 specifications, as is shown in table 1.1: if bequests are allowed or not, if educational investment is taken by parents to their children or is the same for all children (as it is with public schooling), and if there is an actuarially fair annuity market.

The model described in chapter 3 is an overlapping generations model designed by Zhang and Zhang (2001), with the characteristics as shown in table 1.1. Three closely related papers are Ehrlich and Lui (1991), Hu (1995), and de la Croix and Licandro (1999). In Ehrlich and Lui (1991) the model has no annuity market, in Hu (1995) the model has non-altruistic agents, and in de la Croix and Licandro (1999) the difference to the presented model here is that it has non-altruistic agents and an education loan market. The model described in chapter 3 processes opposing effects (saving and investment in human capital) of rising longevity on growth. The new features here are the integration of an altruistic attitude of the parents, an actuarially fair annuity market and no education loan market. Without an education loan market, human capital investment is made to a large extent by the parents, which represents a very realistic important intergenerational link in most industrial nations. As Hu (1995) and de la Croix and Licandro (1999) both assume non-altruistic agents, this link is not incorporated in their models.

The model described in chapter 4 is an overlapping generations model designed by Zhang et al. (2003), with the characteristics as shown in table 1.1. It tries to capture the fact that empirical studies indicate a relation of economic growth to increasing longevity, which depends on the initial level of longevity. To produce this behavior, the presented model has accidental bequests and no actuarially fair annuity market. Lee (1994) found out that with an initial high mortality and low life expectancy, a mortality decline affects primarily children death rates and therefore makes the population in average younger and not older. Meltzer (1992) found out that in this situation also the rates of return to investment in human capital increase. Once child mortality rates are sufficiently small, further mortality declines affect largely the older ages, which has little effect on the rates of return to investment in human capital (Lee, 1994). In Ehrlich and Lui (1991) there is a focus on this early stage of mortality, whereas the model presented in chapter 4 focuses on the later stage. That's why there is sure survival from the schooling period

Table 1.1: Overlapping generations models

Characteristics	model from chapter 3	model from chapter 4
Type of model	Agents are living for 3 periods (schooling time, working time and retirement); infinite number of periods	
Life expectation	sure survival from 1^{st} to 2^{nd} period; probabilistic survival from 2^{nd} to 3^{rd} period	
Intergenerational educational link	investment (goods and time) of a parent in children's human capital	public schooling, financed by a flat-rate income tax
Bequests	no bequests	accidental bequests
Retirement age	exogenous	
Number of children	the same for each agent	1 child per agent
Actuarially fair annuity market	yes	no
Education loan market	no	
Free leisure time to generate a trade-off between working and leisure	no	
Equilibrium result for an increasing life expectancy	individual saving rate increases	individual saving rate increases if initial mortality is high; individual saving rate falls if initial mortality is low

to the working period. As we also have accidental bequests, the approach can be seen as similar to Abel (1985, 1986), Feldstein (1990) and Huggett (1996).

1.1.2 Continuous time models

The two selected continuous time models mainly differ in 4 specifications, as shown in table 1.2: if life expectation is probabilistic or deterministic, how retirement is chosen, the existence of an actuarially fair annuity market, and the possibility of free leisure time to generate a trade-off between working and leisure.

The model described in chapter 5 is a continuous time model designed by Bloom et al. (2004), with the characteristics as shown in table 1.2. Two closely related papers are Chang (1991) and Kalemli-Ozcan and Weil (2002), but both assume an imperfect annuity market, which generates uncertainty about the return to savings and therefore can lower in the case of a high death rate the desired age of retirement with an increasing life expectancy. The new features of the model presented in chapter 5 are an actuarially fair annuity market, the incorporation of health and its effect on the retirement age, and the assumption that a rising life expectancy also generates an improved health status at each age.

Table 1.2: Continuous time models

Characteristics	model from chapter 5	model from chapter 6
Type of model	Agents are living for 2 periods (working time and retirement)	
Life expectation	probabilistic with a constant death rate	deterministic
Intergenerational educational link	no intergenerational links	
Bequests	no	
Retirement age	endogenous as a function of the health status	endogenous as a function of the marginal utility of leisure
Number of children	irrelevant	
Actuarially fair annuity market	yes	no
Education loan market	irrelevant	
Free leisure time to generate a trade-off between working and leisure	no	yes
Equilibrium result for an increasing life expectancy	individual saving rate falls, but the agent will work longer	individual saving rate increases

In chapter 6 a continuous time model designed by Bloom et al. (2003) is described, with the characteristics as shown in table 1.2. Three closely related papers are Lee et al. (2000), Bils and Klenow (2000) and Meltzer (1992). In Lee et al. (2000) the obtained results are very similar and empirical evidence of these results can be found in Tsai et al. (2000) and Hurd et al. (1998). In Bils and Klenow (2000) and Meltzer (1992) they also assume a deterministic life expectation. This assumption is done for simplicity. Furthermore the model introduced in chapter 6 has no intergenerational links, as opposed to Mason (1981, 1988) who accounts for the intergenerational link by using a model on the household level, or Skinner (1985) who accounts for an intergenerational link by assuming the existence of bequests.

2 Basics in life-cycle modeling

In this section I will outline some basic facts and fallacies regarding models of life-cycle behavior.

2.1 Integration of saving in economic models

In theoretical literature saving is a theory of consumption, because it is simply the difference between income and current consumption. On the other hand, most empirical studies are descriptive in terms of saving and not much theoretical. Therefore there still exist two ways of approaching savings in the literature, which are not perfectly unified by now.

At first I outline an overview of the reasons for people to save. It is to note that most of these motives are not considered in the models. Afterwards the Solow model which integrates the saving rate exogenously and the Ramsey model which integrates the saving rate endogenously are presented. These two models give us an idea of different possibilities of integrating savings in economic models.

2.1.1 Reasons for people to save

For a better understanding of the motivation for people to save, I will reproduce here the 9 motives to save by Keynes (1936, page 107-108) and Browning & Lusardi (1996, page 1797):

1. 'To build up a reserve against unforeseen contingencies' *(the precautionary motive)*
2. 'To provide for an anticipated future relationship between the income and the needs of the individual...' *(the life-cycle motive)*
3. 'To enjoy interest and appreciation...' *(the intertemporal substitution motive)*
4. 'To enjoy a gradually increasing expenditure...' *(the improvement motive)*
5. 'To enjoy a sense of independence and the power to do things, though without a clear idea or definite intention of specific action' *(the independence motive)*
6. 'To secure a masse de manoeuvre to carry out speculative or business projects' *(the enterprise motive)*
7. 'To bequeath a fortune' *(the bequest motive)*

8. 'To satisfy pure miserliness, i.e., unreasonable but insistent inhibitions against acts of expenditure as such' *(the avarice motive)*

9. 'To accumulate deposits to buy houses, cars, and other durables' *(the downpayment motive)*

Additionally, Browning & Lusardi (1996, page 1797) remark that 'there is recognition here of considerable heterogeneity in the motives for saving. It is unlikely that a single explanation will suffice for all members of a population at any given time or even for the same person over a long stretch of time. In particular, there is a widespread feeling that the wealthy have different motives to save from the less wealthy.'

2.1.2 The Solow model

The following introduction to the Solow model is based on the description in Barro & Sala-i-Martin, see [BiM95]. This model is a very good example of how to integrate exogenous saving rates in an economic model with a neoclassical production function. We will derive a differential equation for the evolution of the capital-labor ratio and have a look at the conclusions obtained from the system in a steady state where the main one is that the per capita quantities are constant. Finally we will explain the golden rule of capital accumulation.

Model assumptions

We assume a closed economy. This implies that output equals income, and investment equals savings.
There will be no markets and firms, so we will have in mind an agent who acts at the same time as a consuming household and as a producer. Furthermore we will neglect technological progress, which means that the production function does not depend explicitly on t. The two inputs of the production function are physical capital $K(t)$ and labor $L(t)$, so the function can be written with $Y(t)$ as the flow of output produced at time t as

$$Y(t) = F[K(t), L(t)] \tag{2.1}$$

We assume that output can either be consumed, $C(t)$, or invested, $I(t)$. If it is invested, it will create new units of physical capital $K(t)$.
The saving rate is defined as the fraction of output that is saved and denoted by $s(\cdot)$. Therefore $(1-s(\cdot))$ denotes the fraction of output that is consumed. In the Solow model $s(\cdot)$ is assumed to be an exogenously given constant: $s(\cdot) = s$ with $0 \leq s \leq 1$.
Furthermore, capital depreciation is assumed to be constant and given by a constant rate $\delta > 0$. With δ we can express the net increase in the stock of physical capital $\dot{K}$ as inflow (gross investment I) less depreciation δK:

$$\dot{K} = I - \delta K = s \cdot F(K, L, t) - \delta K \tag{2.2}$$

As s and δ are exogenously given, we are able to derive with (2.2) the behavior of K over time if we fix the production function and labor.

Population growth is also simplified by assuming a constant and exogenously given population growth rate n with $\dot{L}/L = n \geq 0$. Furthermore it is convenient to fade out labor force participation problems, so it is assumed that every member of the population participates in the labor market at the same given intensity.

The neoclassical production function

A production function from (2.1) with the following 3 properties is said to be *neoclassical*:

- For all $K > 0$ and $L > 0$, $F(\cdot,\cdot)$ satisfies:

$$\frac{\partial F}{\partial K} > 0, \qquad \frac{\partial^2 F}{\partial K^2} < 0$$

$$\frac{\partial F}{\partial L} > 0, \qquad \frac{\partial^2 F}{\partial L^2} < 0$$

- $F(\cdot,\cdot)$ has constant returns to scale:

$$F(\lambda K, \lambda L) = \lambda \cdot F(K, L) \text{ for all } \lambda > 0$$

- (Inada conditions) The marginal products of capital and labor have the following properties:

$$\lim_{K \to 0}\left(\frac{\partial F}{\partial K}\right) = \lim_{L \to 0}\left(\frac{\partial F}{\partial L}\right) = \infty$$

$$\lim_{K \to \infty}\left(\frac{\partial F}{\partial K}\right) = \lim_{L \to \infty}\left(\frac{\partial F}{\partial L}\right) = 0$$

With constant returns to scale we can reformulate output as

$$Y = F(K, L) = L \cdot F\left(\frac{K}{L}, 1\right) = L \cdot f(k) \tag{2.3}$$

where $k \equiv K/L$ denotes the capital-labor ratio and $f(k) \equiv F(k, 1)$. With per capita output $y \equiv Y/L$ the production function can be reformulated as

$$y = f(k) \tag{2.4}$$

Here we choose for the production function a Cobb-Douglas type function:

$$Y = AK^{\alpha}L^{1-\alpha}, \text{ or } y = Ak^{\alpha} \tag{2.5}$$

where $A > 0$ denotes the technology level and α denotes the constant share parameter with $0 < \alpha < 1$. This function fulfills all 3 neoclassical production function properties.

The fundamental dynamic equation for the capital stock

Let's now have a look at the dynamic behavior of the economy with regard to a neoclassical production function. With (2.2) and dividing both sides by L, we arrive by using (2.3) at

$$\frac{\dot{K}}{L} = s \cdot f(k) - \delta k$$

As the right-hand side consists of per capita dimensions, we can express the left hand side by using the following link between $\dot{K}/L$ and k:

$$\dot{k} \equiv \frac{d(\frac{K}{L})}{dt} = \frac{\dot{K}}{L} - nk, \text{ with } n = \frac{\dot{L}}{L}$$

Reformulating gives

$$\dot{k} = s \cdot f(k) - (n + \delta) \cdot k \tag{2.6}$$

This expression is the fundamental differential equation of the Solow model and consists of only one unknown variable, k. So (2.6) gives us the behavior of k over time. At one hand k increases over time if the saving rate s is positive, and on the other hand k decreases because of depreciation. The term $n + \delta$ of (2.6) can be interpreted as an effective depreciation rate for the capital/labor ratio $k \equiv K/L$. Setting $s = 0$ reveals that k declines because capital K depreciates at the rate δ and because labor L grows at the rate n.

The steady state

In Barro & Sala-i-Martin (1995, page 19), a steady state is defined as 'a situation in which the various quantities grow at constant rates'. Here, the steady state is represented by $\dot{k} = 0$ (for a detailed derivation see Barro & Sala-i-Martin, 1995, page 19, footnote 1) in (2.6), and the corresponding value of k is denoted as k^*. So we get that k^* fulfills

$$s \cdot f(k^*) = (n + \delta) \cdot k^* \tag{2.7}$$

In the steady state k is constant, which gives that y with $y^* = f(k^*)$ and c with $c^* = (1 - s) \cdot f(k^*)$ are also constant. **This means that the per capita quantities k, y and c are all constant in the steady state**. As the levels of the variables K, Y and C are all connected to the per capita quantities k, y and c by the population L which grows at the population growth rate n, we get that K, Y and C all grow in the steady state at the population growth rate n.
Finally we note that the steady-state growth rates of y^*, k^* and c^* are independent of s, n and δ. More precisely, the steady-state growth rates of these per capita quantities are all equal to 0.
This is the reason why the model as presently specified cannot explain long-run per capita growth.

The golden rule of capital accumulation

With given n, δ and $F(\cdot,\cdot)$, there is for every exogenous s a unique steady-state value $k^* > 0$ given by (2.7). That gives a relation between the saving rate and the capital/labor ratio which is expressed by $k^*(s)$, with $dk^*(s)/ds > 0$. Furthermore the steady-state level of per capita consumption is given by $c^* = (1-s) \cdot f[k^*(s)] = f[k^*(s)] - s \cdot f[k^*(s)]$. With (2.7) we get

$$c^*(s) = f[k^*(s)] - (n+\delta) \cdot k^*(s)$$

Steady-state per capita consumption c^* increases for low levels of s and decreases for high levels of s. To get the maximum, we calculate the derivative and set it equal to 0:

$$\frac{dc^*(s)}{ds} = [f'(k^*) - (n+\delta)] \cdot \frac{dk^*}{ds} = 0$$

As $dk^*/ds > 0$, we get that $[f'(k^*) - (n+\delta)] = 0$. By denoting k_{gold} as the value of k^* that corresponds to the maximum of c^*, we can calculate k_{gold} by

$$f'(k_{gold}) = n + \delta$$

This condition is called the golden rule of capital accumulation. It can be interpreted as (see Barro & Sala-i-Martin, 1995, page 20) 'if we provide the same amount of consumption to members of each current and future generation - that is, if we do not provide less to future generations than to ourselves - then the maximum amount of per capita consumption is $c_{gold} = f(k_{gold}) - (n+\delta) \cdot k_{gold}$'. The saving rate corresponding to k_{gold} is denoted by s_{gold}.

A saving rate that is bigger as s_{gold} at all times must be inefficient because a higher per capita consumption c could be generated at all times simply by reducing the saving rate to s_{gold}. An economy that oversaves like as described is called dynamically inefficient. If the saving rate is below s_{gold}, then a rise in the saving rate would reduce c in current times and during part of the transition phase to the new steady state. The overall outcome will therefore be seen as positive or negative depending on the households' time preferences, precisely on the exchange factor of today's consumption against future consumption.

2.1.3 The Ramsey model

The following introduction to the Ramsey model is based on the description in Barro & Sala-i-Martin, see [BiM95]. This model is a very good example of a growth model with consumption optimization, and hence endogenous savings. The main conclusions are that the per capita quantities are growing in the steady state and that inefficient oversaving is impossible. Finally we will note that the Solow model is a special case of the Ramsey model.

Model assumptions

All households receive wages in exchange for labor services, earn interest income on assets, consume goods and save. Each household consists of at least one adult. An adult

is everybody in the population who works.
We consider an immortal extended family although individuals die in finite time. This approach is reasonable if we consider altruistic parents.
The growth rate of the population is n, exogenously given and constant. With normalizing the number of adults in one household at time 0 to 1, the adult population in one household at time t is e^{nt}.
Let $c(t)$ be the consumption per adult person. Each household wants to maximize its overall utility U:

$$U = \int_0^\infty u[c(t)] \cdot e^{nt} \cdot e^{-\rho t} dt \tag{2.8}$$

$u[c(t)]$ denotes the utility of consumption per person at time t, e^{nt} denotes the number of adults in the household, and $e^{-\rho t}$ denotes the discount factor for time t with $\rho > 0$ being the rate of time preference.
We assume $u(c)$ to be increasing and concave: $u'(c) > 0$, $u''(c) < 0$. Furthermore it should satisfy the Inada conditions: $u'(c) \to \infty$ as $c \to 0$ and $u'(c) \to 0$ as $c \to \infty$.
We assume $\rho > n$ which implies in (2.8) a finite integral for a constant $c(t) = \overline{c}$.
Finally it is to note that (2.8) assumes that ρ stays unchanged for the individuals and even across generations.
We assume a closed economy where households can only lend to and borrow from each other, with a real rate of return $r(t)$. The household's net assets per person, $a(t)$, can be positive (net lender) or negative (net borrower).
The wage rate paid for 1 unit of labor services per time unit is $w(t)$. We assume for every adult to deliver in 1 time unit exactly 1 unit of labor services. In an equilibrium state the labor market will clear, so there is no involuntary unemployment in the model.
The flow budget constraint for the household (without time subscripts) is given by

$$\dot{a} = w + ra - c - na \tag{2.9}$$

That means that the change in assets equals the wage income, plus the interest income (lender) or payments (borrower) of the existing assets, less consumption and less the expansion of the population.
Finally, to rule out ever-growing negative assets for a household, we assume the following restriction:

$$\lim_{t\to\infty} \left\{ a(t) \cdot \exp\left[-\int_0^t [r(\mu) - n] d\mu \right] \right\} \geq 0 \tag{2.10}$$

We will see later that (2.10) is derived from the market equilibrium.

First-order conditions

The present value Hamiltonian of the problem is given by

$$J = u(c)e^{nt} \cdot e^{-\rho t} + \nu \cdot \dot{a} = u(c)e^{-(\rho-n)t} + \nu \cdot [w + (r-n)a - c] \tag{2.11}$$

where ν represents the present-value shadow price of income. The first-order conditions for a maximum of U yield:

$$\frac{\partial J}{\partial c} = 0 \Rightarrow \nu = u'(c)e^{-(\rho-n)t} \tag{2.12}$$

$$\dot{\nu} = -\frac{\partial J}{\partial a} \Rightarrow \dot{\nu} = -(r-n)\nu \tag{2.13}$$

The last equation is called Euler equation or Ramsey rule of optimal saving.
Differentiation of (2.12) with respect to t and using (2.12) and (2.13) leads to

$$r = \rho - \left(\frac{du'/dt}{u'}\right) = \rho - \left[\frac{u''(c)\cdot c}{u'(c)}\right]\cdot\frac{\dot{c}}{c} \tag{2.14}$$

As $\left(-\left[\frac{u''(c)\cdot c}{u'(c)}\right]\right) > 0$ we see that the return to saving r and the discount rate ρ are connected by a positive term times $\dot{c}/c$ which represents the way consumption is smoothed over time. We see that $\left[\frac{u''(c)\cdot c}{u'(c)}\right]$ needs to be asymptotically constant if we want to find a steady state with constant r and $\dot{c}/c$.
For further analysis we need an explicit form of $u(c)$, and we assume it of the following style (constant intertemporal elasticity of substitution utility function):

$$u(c) = \frac{c^{1-\theta}-1}{1-\theta} \text{ with } \theta > 0 \tag{2.15}$$

With (2.15), $\left[\frac{u''(c)\cdot c}{u'(c)}\right]$ equals $-\theta$. This simplifies (2.14) to

$$\frac{\dot{c}}{c} = \frac{r-\rho}{\theta} \tag{2.16}$$

So the per capita consumption path over time is dependent on r, ρ and θ. It is rising if $r > \rho$ and falling if $r < \rho$.

Integration of (2.13) results in

$$\nu(t) = \nu(0)\cdot\exp\left\{-\int_0^t [r(\mu)-n]d\mu\right\} \text{ with } \nu(0) = u'[c(0)]$$

The transversality condition is given by $\lim_{t\to\infty}[\nu(t)\cdot a(t)] = 0$. Together with the result of $\nu(t)$ and a constant $u'[c(0)]$ we get

$$\lim_{t\to\infty}\left\{a(t)\cdot\exp\left[-\int_0^t [r(\mu)-n]d\mu\right]\right\} = 0 \tag{2.17}$$

Optimizing households will satisfy (2.17) instead of only satisfying (2.10).

The consumption function

First we define the average interest rate as

$$\bar{r}(t) = \frac{1}{t}\cdot\int_0^t r(\mu)d\mu$$

If we solve (2.9) as a first-order linear differential equation in a we get for any $T \geq 0$:

$$a(T) \cdot e^{-[\overline{r}(T)-n]T} + \int_0^T c(t)e^{-[\overline{r}(t)-n]t}dt = a(0) + \int_0^T w(t)e^{-[\overline{r}(t)-n]t}dt$$

Letting T approach infinity, $T \to \infty$, $a(T) \cdot e^{-[\overline{r}(T)-n]T}$ vanishes because of (2.17) and we get the intertemporal budget constraint

$$\int_0^\infty c(t)e^{-[\overline{r}(t)-n]t}dt = a(0) + \int_0^\infty w(t)e^{-[\overline{r}(t)-n]t}dt = a(0) + \tilde{w}(0) \tag{2.18}$$

This means that the present value of consumption equals initial assets $a(0)$ plus the present value of wage income $\tilde{w}(0)$.
Integration from 0 to t of (2.16) leads to

$$c(t) = c(0) \cdot e^{(1/\theta)[\overline{r}(t)-\rho]t}$$

This expression gives us consumption at time t as a function of $c(0)$ and some constants. Together with (2.18) we can calculate the consumption function at time 0:

$$c(0) = \frac{a(0) + \tilde{w}(0)}{\int_0^\infty e^{[\overline{r}\cdot(1-\theta)/\theta-\rho/\theta+n]t}dt} \tag{2.19}$$

With an expression of $c(0)$, and $c(t)$ as a function of $c(0)$ we now have the whole consumption path.

Firms

Firms produce an output Y and pay for the inputs labor L and capital K:

$$Y = F(K, L)$$

The function $F(\cdot, \cdot)$ has the properties of a neoclassical production function as defined in section 2.1.2. We furthermore assume labor-augmenting technological progress at a constant rate:

$$Y = F(K, \hat{L}) \text{ with effective labor } \hat{L} \equiv L \cdot A(t)$$

$A(t)$ denotes the level of technology with a constant growth rate $x \geq 0$. With assuming $A(0) = 1$ we get $A(t) = e^{xt}$. We define quantities per unit of effective labor as:

$$\hat{y} \equiv Y/\hat{L} \qquad , \qquad \hat{k} \equiv K/\hat{L}$$

This results in

$$\hat{y} = \frac{Y}{\hat{L}} = \frac{F(K, L)}{\hat{L}} = F(\frac{K}{\hat{L}}, 1) = f(\hat{k})$$

The net rate of return for a household for 1 unit of capital equals the rental price for capital services R less capital depreciation $\delta \geq 0$. On the other hand, households can lend funds with an interest rate r. As these two possibilities are supposed to be perfect substitutes, we get $R - \delta = r$, or $R = r + \delta$.

For a firm the profit equals sales of output less factor payments for capital and labor. If we normalize the price of one unit of K in units of C to 1, we arrive at:

$$\text{Profit } = F(K, \hat{L}) - (r+\delta) \cdot K - wL = \hat{L} \cdot [f(\hat{k}) - (r+\delta) \cdot \hat{k} - we^{-xt}] \tag{2.20}$$

The firm will take r, w and $\hat{L}$ as given and maximize its profit which results in

$$f'(\hat{k}) = r + \delta \tag{2.21}$$

In a perfect competition equilibrium, the profit of the firm in (2.20) must equal 0. With (2.21) we get

$$w = e^{xt}[f(\hat{k}) - (r+\delta) \cdot \hat{k}] = e^{xt}[f(\hat{k}) - f'(\hat{k}) \cdot \hat{k}] \tag{2.22}$$

Equilibrium

As households only lend from other households, the average household has 0 net debts. Together with the facts that the whole capital stock in a closed economy stays in the economy, and each adult provides 1 unit of labor, we get that the assets per adult person a have to equal the capital per worker k. With $a = k$, $\hat{k} \equiv K/\hat{L} = ke^{-xt}$ and $\hat{c} \equiv C/\hat{L} = ce^{-xt}$ we can rewrite (2.9) and use (2.21) and (2.22) to get

$$\dot{\hat{k}} = f(\hat{k}) - \hat{c} - (x + n + \delta) \cdot \hat{k} \tag{2.23}$$

This is the resource constraint for the economy. The change in the capital per worker equals output less consumption and less depreciation. (2.23) determines the evolution of $\hat{k}$ over time and therefore also of $\hat{y} = f(\hat{k})$ over time, with the only inconvenience of being dependent on $\hat{c}$. If we would take constant savings, the link would be $\hat{c} = (1-s)f(\hat{k})$ as in the Solow model. Here we use (2.16) and (2.21) to arrive at

$$\frac{\dot{\hat{c}}}{\hat{c}} = \frac{\dot{c}}{c} - x = \frac{1}{\theta} \cdot [f'(\hat{k}) - \delta - \rho - \theta x] \tag{2.24}$$

(2.23) and (2.24) give a differential equations system in $\hat{c}$ and $\hat{k}$ which determine their evolution over time for a given $\hat{k}(0)$ and the transversality condition. We can rewrite this condition (2.17) by using $\hat{k} = ke^{-xt}$ and $a = k$ to get

$$\lim_{t \to \infty} \left\{ \hat{k} \cdot \exp\left[- \int_0^t [f'(\hat{k}) - \delta - x - n] d\mu \right] \right\} = 0 \tag{2.25}$$

For the limes to be 0 we need that $f'(\hat{k}) - \delta - x - n > 0$, or $f'(\hat{k}) - \delta > x + n$, which means that the steady state rate of return has to be bigger than the steady state growth rate of K.

The steady state

We now analyze if the differential equation system (2.23) and (2.24) allows a steady state, which means (Barro & Sala-i-Martin, 1995, page 72) 'a situation in which the various quantities grow at constant rates'. In the Ramsey model, the steady state corresponds to $\dot{\hat{k}} = \dot{\hat{c}} = 0$, which means that $\hat{k}$**,** $\hat{c}$ **and** $\hat{y}$ **are constant**. This implies that k, c and y grow at the rate x. Furthermore it implies that K, C and Y grow at the rate $n + x$. These results are similar to those obtained in the Solow model in 2.1.2.

The steady state values $\hat{k}^*$ and $\hat{c}^*$ of $\hat{k}$ and $\hat{c}$ we get by setting the differential equation system of (2.23) and (2.24) to 0. If we do this for (2.23) we get

$$\hat{c} = f(\hat{k}) - (x + n + \delta) \cdot \hat{k} \tag{2.26}$$

Taking the first derivative of this expression with regard to $\hat{k}$ gives us the peak of the curve at $f'(\hat{k}) = x + n + \delta$. With (2.21) we see that the interest rate r equals $x + n$ at the point in the steady state which gives the maximum of $\hat{c}$. This point also defines the golden-rule level of $\hat{k}$.

Setting (2.24) to 0 gives

$$f'(\hat{k}^*) = \delta + \rho + \theta x \tag{2.27}$$

So the interest rate in the steady state $r^* = f'(\hat{k}^*) - \delta$ must equal the overall discount rate $\rho + \theta x$ in which θx captures the fact that c is growing at the rate x.

Finally we now can determine the steady state values ($\hat{k}^*$, $\hat{c}^*$): with (2.27) we get $\hat{k}^*$ and with (2.26) we get $\hat{c}^*$.
From (2.25) we got that $f'(\hat{k}) - \delta > x + n$ must hold true, and with (2.27) we arrive at

$$\rho > n + (1 - \theta)x \tag{2.28}$$

(2.28) implies that $f'(\hat{k}^*) = \delta + \rho + \theta x > \delta + x + n = f'(\hat{k}_{gold})$. With $f''(\hat{k}) < 0$ we get $\hat{k}^* < \hat{k}_{gold}$. As $\hat{k}_{gold}$ corresponds to the golden-rule level of $\hat{k}$ which leads to a maximum of $\hat{c}$ in the steady state, we see that **in the Ramsey model there is no inefficient oversaving possible** like in the Solow model. This is because in the Solow model the saving rate has no restrictions except being between 0 and 1.

The saving rate in the Ramsey model equals $s = [\hat{y} - \hat{c}]/\hat{y} = 1 - [\hat{c}/\hat{y}] = 1 - [\hat{c}/f(\hat{k})]$. In the case of a Cobb-Douglas production function we get for the steady state saving rate s^*:

$$s^* = \alpha \cdot \frac{x + n + \delta}{\delta + \rho + \theta x} \tag{2.29}$$

With (2.28) we see that $s^* < \alpha$ where α is the gross capital share.

To sum up, the Solow model with an exogenous saving rate cannot explain per capita growth of capital, consumption or income in the steady state because in this state these quantities are all constant. So population growth or changes in the saving rate don't affect these quantities. On the other hand, the Ramsey model with an endogenous saving rate is able to explain steady state per capita growth of capital, consumption and income. In both models the overall level of capital, consumption and income is influenced by the saving rate and the population growth rate, but only the Ramsey model can additionally explain steady state per capita growth.

2.2 Differences between micro and macro data for savings

This short introduction on aggregation issues is based on the work of Weil (1994), see [Wei94]. Using data at the household level (micro data), most studies have found little or no dissaving of the elderly. Contrarily, when using data at the aggregate level (macro data), studies frequently found that the pure fact of a large elderly population decreases a population's saving rate. As the micro data does not take into account intergenerational relations, it is straightforward that if aggregated micro data does not produce the observed macro data which includes all interactions naturally, there must be interactions among households which have to be added for consistency.

2.2.1 Empirical verifications

In the work of Weil (1994, page 59-60), an empirical regression with macro data of different countries is performed. The derived result is that, depending on the exact model specifications for the regression, a 1% shift of the population from the working age group to the elderly group reduces the saving rate between 0.52% and 1.78%. So, there is indeed a macroeconomic influence of the share of old people on the saving rate.

Weil (1994, page 61-62) also performs an empirical regression using micro data of the USA. The results are shown in table 2.1, where we can see that both consumption and income have roughly similar hump-shaped characteristics, but we cannot observe any sign of dissaving by people aged 65 years or older (the target age group for retirees). So, on an individual level, there is no observation of dissaving among the elderly. It is to remark that relations between people cannot be observed by this kind of data.

2.2.2 A demonstrative model

For a better understanding of these differences, let's assume a two-generation economy with intergenerational relations where the first generation is represented by children and the second generation by parents. Every child has one parent, but every parent can have several children. Children and parents don't live in the same household, which corresponds to relations between adult children and their parents.

The saving of an individual is assumed to be connected to its age and the intergenerational relations. The saving of a child is given by

$$s_1 = \beta_1 + \pi_{2,1} \cdot \left(\frac{1}{1 + \#siblings} \right)$$

Table 2.1: Consumption, income and saving in micro data (Weil, 1994, page 62)

	Consumption		Income		Saving	
Age group	Coefficient	S.E.	Coefficient	S.E.	Coefficient	S.E.
0-4	-135	280	-190	336	-55	364
5-9	-347	297	-1357	356	-1010	386
10-14	149	301	-1177	360	-1325	390
15-19	2345	270	759	323	-1586	350
20-24	5974	248	4397	297	-1577	322
25-29	8952	246	9998	295	1046	320
30-34	11204	271	12304	325	1100	352
35-39	12068	296	14156	355	2087	384
40-44	11562	332	13948	398	2385	431
45-49	10746	337	12744	403	1998	437
50-54	7956	350	10377	420	2421	455
55-59	7978	338	10990	405	3012	439
60-64	7517	340	8049	408	532	442
65-69	7776	363	7549	435	-227	471
70-74	6296	409	6772	490	476	530
75-79	6271	522	6245	625	-26	677
80+	5400	509	5645	610	246	661
N	12019		12019		12019	
R^2	0.114		0.146		0.010	

where β_1 is the saving of a child if it had no intergenerational link and $\pi_{2,1}$ represents the effect of a parent on the saving of a child. The first term reflects the behavior without intergenerational influence and the second term should represent the influence of parents. As every child has exactly one parent, the number of parents cannot be of any influence. Therefore the influence is expressed by the number of siblings a child has.

The saving of a parent is given by

$$s_2 = \beta_2 + \pi_{1,2} \cdot (\#children)$$

where β_2 is the saving of a parent if it had no intergenerational link and $\pi_{1,2}$ represents the effect of a child on the saving of a parent. In the second term we see that the number of children is the influential parameter.

It is important to understand that the β coefficients are not the average savings but the savings without any intergenerational links. This means that any saving not caused by the number of siblings for a child, or the number of children for a parent is incorporated in the β coefficients. We see that $s_1 = \beta_1$ if a child has an infinite number of siblings, and $s_2 = \beta_2$ if a parent has no children.

Let's now analyze the results taken from micro data. If we think of a set of household data where each household consists only of members of a single generation, we don't get the number of siblings or children of each person. If n_i represents the number of parents with i children, and w_1 and w_2 are the total population in each age group, we get the mean number of children per parent by

$$\frac{\#children}{\#parents} = \frac{\sum_{i\geq 0} n_i i}{\sum_{i\geq 0} n_i} = \frac{w_1}{w_2}$$

Similarly, we get the mean of the quantity $\frac{1}{1+\#siblings}$ among children by

$$\frac{\sum_{i\geq 1} \frac{1}{i} n_i i}{\sum_{i\geq 0} n_i i} = \frac{w_2 - n_0}{w_1}$$

We assume for simplicity $n_0 = 0$, which means that every parent has at least 1 child. This is not unrealistic because even if an old person has no child, he or she can still affect the saving of the young. Weil (1994, page 66) mentions correctly that 'In the case of accidental bequests, for example, the wealth of the childless elderly will be passed to *someone*, and so may have the same effect on saving of the young as the wealth of elderly people with children.'

With this assumption we get for the mean savings

$$\bar{s}_1 = \beta_1 + \pi_{2,1} \cdot \frac{w_2}{w_1}$$

$$\bar{s}_2 = \beta_2 + \pi_{1,2} \cdot \frac{w_1}{w_2}$$

These $\bar{s}$ parameters are exactly what is measured by the micro data, and not the β parameters! So, micro data reflects mean intergenerational links rather than no intergenerational link at all.

If we now aggregate our micro behavior to the level of the whole population, we get for saving per capita:

$$S = w_1 \cdot \overline{s}_1 + w_2 \cdot \overline{s}_2 = w_1 \left(\beta_1 + \pi_{2,1} \cdot \frac{w_2}{w_1} \right) + w_2 \left(\beta_2 + \pi_{1,2} \cdot \frac{w_1}{w_2} \right) = w_1(\beta_1 + \pi_{1,2}) + w_2(\beta_2 + \pi_{2,1})$$

So, this result would now reflect the information in macro data. If we now analyze age specific savings by regressing S on the fraction of the population in each age group, we would get

$$\widetilde{s}_1 = \beta_1 + \pi_{1,2}$$

$$\widetilde{s}_2 = \beta_2 + \pi_{2,1}$$

Due to this the resulting $\widetilde{s}$ parameters are neither equal to the β parameters, which reflect pure savings, nor to the $\overline{s}$ parameters, which reflect mean savings. These $\widetilde{s}$ parameters still incorporate intergenerational effects by the π parameters.

So, it is followed that with intergenerational relations it is impossible to forecast aggregate savings for changes of the age structure correctly by using micro data which does not incorporate intergenerational relations. To get an idea of the quantity of the error, we just need to take a look at the differences:

$$\overline{s}_1 - \widetilde{s}_1 = \pi_{2,1} \cdot \frac{w_2}{w_1} - \pi_{1,2}$$

$$\overline{s}_2 - \widetilde{s}_2 = \pi_{1,2} \cdot \frac{w_1}{w_2} - \pi_{2,1}$$

As w_1 and w_2 is greater than 0 and given, we see that the age-specific saving estimates for micro and macro data will differ if the π parameters are not equal to 0. Though this would mean that intergenerational links are present.

At this point we can see that as in our framework the elderly have higher savings in micro than in macro data, $\overline{s}_2 > \widetilde{s}_2$, either $\pi_{1,2} > 0$ or $\pi_{2,1} < 0$ must hold. This means that there is a positive intergenerational link of the young on the saving of the elderly or a negative intergenerational link of the elderly on the saving of the young.

Finally, we know that one of the main sources of intergenerational relations is bequests. An expectation about receiving a future bequest already increases household consumption and therefore lowers savings. In this sense, it is not even necessary to restrict to bequests, because inter vivos gifts from the elderly have clearly the same effect. So, even if the elderly do not dissave themselves, they still lower average savings by lowering the savings of the young.

2.3 The importance of uncertainty

For a better understanding of the importance of uncertainty, let's consider a very simple two-period model. This example is taken from [BL96]. An agent maximizes expected utility U given by

$$U(c_1, c_2) = \ln c_1 + \ln c_2$$

Table 2.2: Shortened outcome of the calculations from Browning and Lusardi (1996, page 1802)

	$\epsilon = 0$ $y_1 = 1,\, y_2 = 2$	$\epsilon = 0.01$ $y_1 = 2,\, y_2 = 1$	$\epsilon = 0.01$ $y_1 = 1,\, y_2 = 2$
First period consumption	1.5	1.49	0.98
Saving rate	-0.5	0.255	0.02

where c_1 and c_2 represent certain consumption in period 1 and 2. y_1 denotes first period earnings plus any starting wealth and earnings in period 2 are supposed to be stochastic: 0 earnings with probability ϵ and $\frac{y_2}{1-\epsilon}$ with probability $(1-\epsilon)$. Therefore an increase in ϵ gives a mean preserving spread in future earnings risk, because the expectation of second period income is y_2 and independent of ϵ:

$$\mathbf{E}y_2 = \epsilon \cdot 0 + (1-\epsilon)\frac{y_2}{1-\epsilon} = y_2$$

Finally, the real rate of interest is supposed to be 0.

Browning and Lusardi (1996, page 1802) calculate 3 scenarios: perfect certainty ($\epsilon = 0$) with a low y_1, and small uncertainty ($\epsilon = 0.01$) with either a high or low y_1. The results are shortened and presented in table 2.2.

The results reveal that with an initial high y_1, small uncertainty about the future income only has a small influence, as first period consumption with uncertainty is 1.49, as opposed to 1.50 with certainty. However, if y_1 is initially low, small uncertainty already hugely influences first period consumption, as it reaches only 0.98, but certainty gives 1.5. So, a model which deals with certainty about future income or the length of life can highly differ in the outcome to a model with even very small uncertainty. Additionally, in table 2.2 is presented the saving rate. Introducing small uncertainty with an initial low y_1 would change the saving rate from -0.5 to 0.02, so it is very advisable to pay attention to modeling variables with certainty.

The differences between micro and macro data show us the importance of accounting for interactions between individuals. The empirical evidence suggests that an increased share of old people should decrease the overall saving rate of a population. Furthermore variables with certainty should be used with caution because the change from certainty to even very small uncertainty could be fundamental for the outcome.

3 Longevity and economic growth in a dynastic family model with an annuity market (Zhang and Zhang 2001)

In this section I will present the model published by Zhang and Zhang, see [ZZ01]. It is a dynastic family model with an annuity market and examines the impact of a rising life expectancy on economic growth. It belongs to the class of overlapping generations models without bequests.

We will see that it can explain rising life-cycle savings if longevity rises, and also shows that even with endogenous fertility (which falls with increasing longevity), the overall labor supply to working age population still rises with a rising longevity.

3.1 Motivation

In industrial countries, life expectancy is rising and also the fraction of retirees in relation to the whole population has been rising. Furthermore this fraction is expected to rise even more in the next few decades, so in relation to the working age population there will be a considerable higher fraction of retirees. In most industrial countries, birth rates are nowadays around the replacement level. All that raises the question about the impact of population aging on the economic growth rate.

3.2 The model

The model will use an overlapping generations approach with an infinite number of periods where every agent lives for 3 periods (child in young age, worker in middle age and retiree in old age).

Assumptions

- Overlapping generations with every agent living for 3 periods
- Two-sector growth model (production and education)
- Economy with a single good

- Perfect competition in the goods market
- Physical capital fully depreciates in one period
- Infinite number of periods
- Physical capital investment and old-age consumption come from life-cycle savings (no support from children to parents)
- Only link between generations: investment of parents in children's human capital
- No bequests
- Actuarially fair annuity market
- No education loan market
- Sure survival from young to middle age
- Probability p of survival from middle to old age
- Each middle-aged agent has n_t identical children, born between young and middle age
- A middle-aged agent has no free leisure time, all time is devoted to rearing children, educating children and working
- The interest rate r_t and the wage rate w_t are exogenous

In the model the human capital investments of children are made to a large extent by its' parents, and represents the only intergenerational transfer. For Zhang and Zhang (2001, page 270), this is 'perhaps the main form of intergenerational linkage in most industrial nations where the average schooling years of workers are about 10 years or higher'.

Model formulation

The utility V_t of a representative middle-aged agent with t denoting a period in time is assumed to be logarithmic:

$$V_t = ln(c_t^t) + p \cdot ln(c_{t+1}^t) + \eta \cdot ln(n_t) + \alpha V_{t+1}, \qquad 0 < \alpha < 1, \quad \eta > 0 \tag{3.1}$$

where the subscript t is the current period in time and the superscript t represents the generation born in $(t-1)$. c_t^t is middle-age consumption, c_{t+1}^t is old-age consumption, η represents the taste for the number of children, α describes the taste for the welfare of children and V_{t+1} is the utility per child.

As we assume perfect competition in the goods market, we can use the following Cobb-Douglas-type aggregate production function

$$Y_t = DK_t^{\theta}(L_t l_t H_t)^{1-\theta}, \qquad D > 0, \quad 0 < \theta < 1 \tag{3.2}$$

where Y_t is aggregate output, D the productivity parameter, K_t aggregate physical capital, θ the share parameter associated with physical capital, L_t the number of agents in middle age living in period t, l_t the input of labor per middle-aged agent and H_t the human capital per middle-aged agent. $(L_t l_t H_t)$ represents effective labor.

As all companies want to maximize their profits, we have $profits = output - input = Y_t - [w \cdot (L_t l_t H_t) + (1 + r_t) \cdot K_t]$ with w as the wage rate per effective unit of labor, r the interest rate and $(1 + r_t)$ being the cost of capital because capital fully depreciates in one period. With the first order conditions of $\frac{\partial(profits)}{\partial(L_t l_t H_t)} = 0$, $\frac{\partial(profits)}{\partial K_t} = 0$ we get

$$w_t = \frac{\partial Y_t}{\partial(L_t l_t H_t)} = (1 - \theta) D e_t^{\theta} \tag{3.3}$$

$$1 + r_t = \frac{\partial Y_t}{\partial K_t} = \theta D e_t^{\theta - 1}$$

with $e_t = K_t/(L_t l_t H_t)$ being the ratio of physical capital to effective labor.

The human capital of a child H_{t+1} is assumed to accumulate Cobb-Douglas like:

$$H_{t+1} = A q_t^{\beta} (h_t H_t)^{1-\beta}, \qquad A > 0, \qquad 0 < \beta < 1 \tag{3.4}$$

where A is a scaling parameter[1], q_t the investment of goods from the parent for this child, β the share parameter[2] associated with q_t, and h_t the investment of time from the parent for this child.

With v being the fixed units of time to rearing one child $(0 < v < 1)$, a middle-aged agent with n_t children uses vn_t units of time to rearing children. Furthermore an agent provides for the children's education $h_t n_t$ units of time. If we assume that there is no additional free leisure time in middle age, an agent uses the remaining l_t units of time for earning a wage income $w_t l_t H_t$:

$$l_t = 1 - vn_t - h_t n_t \tag{3.5}$$

A middle-aged agent spends the income on own consumption c_t^t, savings $s_t w_t l_t H_t$ with s_t being the saving rate, and the investment of goods in the education of children, $q_t n_t$.

The budget constraints for a middle-aged agent are:

$$c_t^t = w_t l_t H_t (1 - s_t) - q_t n_t \tag{3.6}$$

That means that middle-age consumption equals earnings less savings and investment in children.

$$c_{t+1}^t = \frac{1 + r_{t+1}}{p} s_t w_t l_t H_t \tag{3.7}$$

Here we see that old-age consumption equals savings $s_t w_t l_t H_t$ times the expected rate of return on savings which is composed of two terms: $(1 + r_{t+1})$ being the return rate on capital and $1/p$ representing the investment in an actuarially fair annuity market.

[1] In fact, A is not defined in [ZZ01].

[2] In fact, β is not defined in [ZZ01].

Here we see that an increasing life expectancy, represented by an increasing p, influences not only directly the utility function (3.1) but also the budget constraint (3.7).

Markets clear when physical capital accumulation equals savings:

$$K_{t+1} = L_t s_t w_t l_t H_t \tag{3.8}$$

3.3 The optimization problem

If we insert (3.6), (3.7) and (3.5) into (3.1), we arrive at the following optimization problem where the agent chooses the number of children n_t, the investment in goods q_t and time h_t for every child and the saving rate s_t:

$$\begin{aligned} V_t(H_t) = \max_{h_t,n_t,q_t,s_t} \{ & ln[w_t(1 - vn_t - h_t n_t)H_t(1 - s_t) - q_t n_t] + \\ & + p \cdot ln \left[\frac{1 + r_{t+1}}{p} s_t w_t (1 - vn_t - h_t n_t) H_t \right] + \eta \cdot ln(n_t) + \alpha V_{t+1}(H_{t+1}) \} \end{aligned} \tag{3.9}$$

subject to the human capital evolution of a child:

$$H_{t+1} = A q_t^\beta (h_t H_t)^{1-\beta}$$

First order conditions

If we differentiate (3.9) with respect to s_t, we get

$$\frac{\partial V_t(H_t)}{\partial s_t} = \frac{1}{c_t^t}[-w_t(1 - vn_t - h_t n_t)H_t] + \frac{p}{c_{t+1}^t}\left[\frac{1 + r_{t+1}}{p} w_t(1 - vn_t - h_t n_t)H_t\right] = 0$$

and therefore

$$\frac{1}{c_t^t} = \frac{1 + r_{t+1}}{c_{t+1}^t} \tag{3.10}$$

or, equally,

$$\frac{1}{c_t^t} = p\frac{1 + r_{t+1}}{p \cdot c_{t+1}^t}$$

The interpretation is clearly stated by Zhang and Zhang (2001, page 272) that 'the utility forgone from saving one more unit in middle age is equal to the utility obtained from receiving $(1 + r_{t+1})/p$ units more in old age at a chance p'.

Differentiating (3.9) with respect to q_t gives

$$\frac{\partial V_t(H_t)}{\partial q_t} = \frac{1}{c_t^t}[-n_t] + \alpha\frac{\partial V_{t+1}(H_{t+1})}{\partial q_t} = 0$$

With

$$\frac{\partial V_{t+1}(H_{t+1})}{\partial q_t} = \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial q_t} = \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{\beta H_{t+1}}{q_t}$$

we get

$$\frac{n_t}{c_t^t} = \alpha \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{\beta H_{t+1}}{q_t} \tag{3.11}$$

If we differentiate (3.9) with respect to h_t, we arrive at

$$\frac{\partial V_t(H_t)}{\partial h_t} = \frac{1}{c_t^t}[w_t(-n_t)H_t(1-s_t)] + \frac{p}{c_{t+1}^t}\left[\frac{1+r_{t+1}}{p}s_t w_t(-n_t)H_t\right] + \alpha \frac{\partial V_{t+1}(H_{t+1})}{\partial h_t} = 0$$

With

$$\frac{\partial V_{t+1}(H_{t+1})}{\partial h_t} = \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial h_t} = \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{(1-\beta)H_{t+1}}{h_t}$$

and (3.10) we arrive at

$$\frac{w_t n_t H_t}{c_t^t} = \alpha \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{(1-\beta)H_{t+1}}{h_t} \tag{3.12}$$

The envelope condition is

$$\frac{\partial V_t(H_t)}{\partial H_t} = \frac{1}{c_t^t}[w_t(1-vn_t-h_t n_t)(1-s_t)] + \frac{p}{c_{t+1}^t}\left[\frac{1+r_{t+1}}{p}s_t w_t(1-vn_t-h_t n_t)\right] + \alpha \frac{\partial V_{t+1}(H_{t+1})}{\partial H_t}$$

With using

$$\frac{\partial V_{t+1}(H_{t+1})}{\partial H_t} = \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{\partial H_{t+1}}{\partial H_t} = \frac{\partial V_{t+1}(H_{t+1})}{\partial H_{t+1}} \cdot \frac{(1-\beta)H_{t+1}}{H_t} = \frac{w_t n_t h_t}{\alpha c_t^t}$$

where the last equality uses (3.12), and using (3.10) results in

$$\frac{\partial V_t(H_t)}{\partial H_t} = \frac{w_t(1-vn_t)}{c_t^t} \tag{3.13}$$

As we already used (3.10) to eliminate c_{t+1}^t, we now use (3.13), with replacing t by $(t+1)$, to simplify (3.11) and (3.12):

$$\frac{n_t}{c_t^t} = \alpha \frac{w_{t+1}(1-vn_{t+1})}{c_{t+1}^{t+1}} \cdot \frac{\beta H_{t+1}}{q_t} \tag{3.14}$$

$$\frac{w_t n_t H_t}{c_t^t} = \alpha \frac{w_{t+1}(1-vn_{t+1})}{c_{t+1}^{t+1}} \cdot \frac{(1-\beta)H_{t+1}}{h_t} \tag{3.15}$$

If we rewrite (3.14) and (3.15) by

$$\frac{q_t n_t}{c_t^t} = \frac{\alpha w_{t+1}(1-vn_{t+1})\beta H_{t+1}}{c_{t+1}^{t+1}}$$

$$\frac{w_t n_t H_t h_t}{c_t^t} = \frac{\alpha w_{t+1}(1-vn_{t+1})(1-\beta)H_{t+1}}{c_{t+1}^{t+1}}$$

where the interpretation is stated clearly by Zhang and Zhang (2001, page 273) that 'the losses in utility from investing an additional unit of goods and time in children's education equal the gains in children's welfare by these investments, respectively.'

Finally, if we differentiate (3.9) with respect to n_t, we get

$$\frac{\partial V_t(H_t)}{\partial n_t} = \frac{1}{c_t^t}[w_t(-v-h_t)H_t(1-s_t)-q_t] + \frac{p}{c_{t+1}^t}\left[\frac{1+r_{t+1}}{p}s_t w_t(-v-h_t)H_t\right] + \frac{\eta}{n_t} = 0$$

and therefore by using (3.10) we have

$$\frac{w_t(v+h_t)H_t + q_t}{c_t^t} = \frac{\eta}{n_t} \tag{3.16}$$

The interpretation is clearly stated by Zhang and Zhang (2001, page 273) that 'the loss in utility from adding one child, which arises from the forgone earning and the increased education investment for this child, is compensated by the gain in utility from enjoying the additional child.'

Equilibrium

The equilibrium is characterized by setting the number of children $n_t = n$, the investment of time for each child $h_t = h$, and the saving rate $s_t = s$ as constant. Note that with n_t and h_t being constant, also $l_t = l$ will be constant. Furthermore the ratio of investment of goods for each child to labor income, $\gamma_q = \frac{q_t}{w_t l_t H_t}$ is supposed to be constant.

Dividing (3.6) by $w_t l_t H_t$ results in

$$\gamma_{c1} \equiv \frac{c_t^t}{w_t l_t H_t} = 1 - s - n\frac{q_t}{w_t l H_t} = 1 - s - n\gamma_q \tag{3.17}$$

So, we see that the ratio γ_{c1} of middle-age consumption to labor income is constant too.

With (3.15) we get

$$\frac{w_t n H_t}{\gamma_{c1}\cdot w_t l H_t} = \alpha\frac{w_{t+1}(1-vn)}{\gamma_{c1}\cdot w_{t+1} l H_{t+1}}\cdot\frac{(1-\beta)H_{t+1}}{h}$$

where the simplification gives us

$$h = \frac{\alpha(1-\beta)(1-vn)}{n} \tag{3.18}$$

This expression gives us the first equation for the equilibrium.

Using (3.14) gives us

$$\frac{n}{\gamma_{c1}\cdot w_t l H_t} = \alpha\frac{w_{t+1}(1-vn)}{\gamma_{c1}\cdot w_{t+1} l H_{t+1}}\cdot\frac{\beta H_{t+1}}{\gamma_q\cdot w_t l H_t}$$

and simplifying leads to

$$\gamma_q = \frac{\alpha\beta(1-vn)}{n(1-vn-hn)}$$

If we use (3.18) we finally get

$$\gamma_q = \frac{\alpha\beta(1-vn)}{n(1-vn-\alpha(1-\beta)(1-vn))} = \frac{\alpha\beta(1-vn)}{n(1-vn)[1-\alpha(1-\beta)]} = \frac{\alpha\beta}{n[1-\alpha(1-\beta)]} \tag{3.19}$$

Here we obtain the second equation for the equilibrium.

Taking (3.10), using (3.7), and multiplying by $w_t l H_t$ leads to

$$\frac{w_t l H_t}{c_t^t} = \frac{(1+r_{t+1})p \cdot w_t l H_t}{(1+r_{t+1}) s w_t l H_t} = \frac{p}{s}$$

which results in

$$\gamma_{c1} = \frac{s}{p} \tag{3.20}$$

If we use (3.20) and (3.17) we get

$$s = \left(1+\frac{1}{p}\right)^{-1}(1-n\gamma_q) = \frac{p}{p+1}\left(1-\frac{\alpha\beta}{1-\alpha(1-\beta)}\right) = \frac{p(1-\alpha)}{(p+1)[1-\alpha(1-\beta)]} \tag{3.21}$$

This expression is the third equation for the equilibrium.

Using (3.20) and (3.21) gives us instantly

$$\gamma_{c1} = \frac{1-\alpha}{(p+1)[1-\alpha(1-\beta)]}$$

Here the fourth equation for the equilibrium is obtained.

Taking (3.16) and using (3.6),(3.18) (3.19), (3.5) and (3.21) results in

$$\begin{aligned} \frac{\eta}{n} &= \frac{w_t(v+h)H_t + \gamma_q w_t l H_t}{w_t l H_t(1-s) - w_t l H_t \gamma_q n} = \\ &= \frac{(v+\frac{\alpha(1-\beta)(1-vn)}{n}) + \frac{\alpha\beta}{n[1-\alpha(1-\beta)]}(1-vn-\alpha(1-\beta)(1-vn))}{(1-vn-\alpha(1-\beta)(1-vn))(1-\frac{p(1-\alpha)}{(p+1)[1-\alpha(1-\beta)]}-\frac{\alpha\beta}{n[1-\alpha(1-\beta)]}n)} \end{aligned}$$

This equation only involves the variable n. Reformulation yields

$$n = \frac{\eta(1-\alpha)-\alpha(p+1)}{v(1-\alpha)(p+1+\eta)}$$

This expression is the last equation for the equilibrium.

To sum up, the equations describing the equilibrium solution of h, γ_q, s, γ_{c1} and n are:

$$h = \frac{\alpha(1-\beta)(1-vn)}{n} \tag{3.22}$$

$$\gamma_q = \frac{\alpha\beta}{[1-\alpha(1-\beta)]n} \tag{3.23}$$

$$s = \frac{p(1-\alpha)}{(1+p)[1-\alpha(1-\beta)]} \tag{3.24}$$

$$\gamma_{c1} = \frac{1-\alpha}{(1+p)[1-\alpha(1-\beta)]}$$

$$n = \frac{\eta(1-\alpha)-\alpha(1+p)}{v(1-\alpha)(1+\eta+p)} \tag{3.25}$$

With (3.25) we derive a condition for positive fertility ($n > 0$) for η: $\eta > \eta_{min} = \alpha(1+p)/(1-\alpha)$. This condition shows us that we have a positive fertility ($n > 0$) if the taste for the number of children η is bigger than η_{min} which is a function of the taste for the welfare of children α and the probability of survival p.

(3.4) can be rewritten with the growth rate of human capital per worker g_{ht}, and by using (3.3) in the last equality, as

$$1+g_{ht} = \frac{H_{t+1}}{H_t} = \frac{Aq_t^{\beta}(hH_t)^{1-\beta}}{H_t} = \frac{A[\gamma_q w_t l H_t]^{\beta}(hH_t)^{1-\beta}}{H_t} = A[\gamma_q(1-\theta)De_t^{\theta}l]^{\beta}h^{1-\beta} \tag{3.26}$$

Here we see that the only parameter not fixed on the right-hand side is e_t.

If we denote $k_t = K_t/L_t$ as physical capital per worker, then we can derive an expression for the growth rate of physical capital per worker g_{kt} by using (3.8), (3.3) and the definition of e_t:

$$1+g_{kt} = \frac{k_{t+1}}{k_t} = \frac{K_{t+1}L_t}{L_{t+1}K_t} = \frac{(L_t s w_t l H_t)L_t}{(L_t n)K_t} = \frac{s(1-\theta)De_t^{\theta-1}}{n} \tag{3.27}$$

With these two equations we express the evolution of e_t:

$$\begin{aligned} e_{t+1} \equiv \frac{K_{t+1}}{L_{t+1}lH_{t+1}} &= \frac{K_t n(1+g_{kt})}{(nL_t)l(H_t(1+g_{ht}))} = e_t \cdot \frac{n\frac{s(1-\theta)De_t^{\theta-1}}{n}}{n\cdot A[\gamma_q(1-\theta)De_t^{\theta}l]^{\beta}h^{1-\beta}} = \\ &= \left\{\frac{s[(1-\theta)D]^{1-\beta}}{nA(\gamma_q l)^{\beta}h^{1-\beta}}\right\} e_t^{\theta(1-\beta)} \end{aligned} \tag{3.28}$$

As the exponent of e_t is between 0 and 1, $0 < \theta(1-\beta) < 1$, we get a global convergence of the ratio of physical capital to effective labor to

$$e = \left\{\frac{s[(1-\theta)D]^{1-\beta}}{nA(\gamma_q l)^{\beta}h^{1-\beta}}\right\}^{\frac{1}{1-\theta(1-\beta)}} \tag{3.29}$$

(3.5) and (3.22) imply

$$l = (1-vn)[1-\alpha(1-\beta)] \tag{3.30}$$

If we insert this expression into (3.29) and the result into (3.26) we get

$$1+g=$$

$$A\left[\gamma_q(1-\theta)D\left\{\frac{s[(1-\theta)D]^{1-\beta}}{nA(\gamma_q(1-vn)[1-\alpha(1-\beta)])^{\beta}h^{1-\beta}}\right\}^{\frac{\theta}{1-\theta(1-\beta)}}(1-vn)[1-\alpha(1-\beta)]\right]^{\beta}h^{1-\beta}$$

$$=\left\{(Ah^{1-\beta})^{1-\theta}\cdot[1-\alpha(1-\beta)]^{\beta(1-\theta)}\cdot[\gamma_q(1-vn)]^{\beta(1-\theta)}\cdot[(1-\theta)D]^{\beta}\cdot\left(\frac{s}{n}\right)^{\beta\theta}\right\}^{\frac{1}{1-\theta(1-\beta)}} \tag{3.31}$$

We denote g_{ht} with g because we get the same result by using (3.26), which means that the growth rate of human and physical capital is the same. Here we see that a higher growth rate g depends positively on the investment of goods $q_t = \gamma_q \cdot w_t l H_t$ and the investment of time h_t from the parents for their children, and negatively on fertility n.

3.4 Results

Now we will have a closer look at the implications that we get in the equilibrium.

Proposition 3.1 *A rise in life expectancy increases the saving rate,* $\frac{\partial s}{\partial p} > 0$*, and the ratio of physical capital investment to income,* $\frac{\partial (K_{t+1}/Y_t)}{\partial p} > 0$.

Proof Taking the derivative of (3.24) with respect to p leads instantly to $\frac{\partial s}{\partial p} > 0$. With (3.2) and (3.8) we get

$$\frac{K_{t+1}}{Y_t} = \frac{L_t s w_t l H_t}{DK_t^{\theta}(L_t l H_t)^{1-\theta}} = s w_t \frac{(L_t l H_t)^{\theta}}{DK_t^{\theta}} = s(1-\theta)$$

where the last equality comes from using (3.3). So, we see that

$$\frac{\partial (K_{t+1}/Y_t)}{\partial p} = (1-\theta)\cdot\frac{\partial s}{\partial p}$$

and therefore, as $(1-\theta) > 0$, that $\frac{\partial (K_{t+1}/Y_t)}{\partial p} > 0$. □

So, we see that in the equilibrium, a longer life expectancy drives the agents to save a higher fraction of income and therefore invest a higher fraction of income in physical capital. Furthermore we see with (3.24) that $\frac{\partial^2 s}{\partial p^2} < 0$, which means that the saving rate and the ratio of physical capital investment are both concave in p. So, the rise of s diminishes with an ongoing rise of p.

Proposition 3.2 *A rise in life expectancy reduces fertility,* $\frac{\partial n}{\partial p} < 0$*, but raises labor supply,* $\frac{\partial l}{\partial p} > 0$.

Proof Taking the derivative of (3.25) with respect to p leads instantly to $\frac{\partial n}{\partial p} < 0$. With (3.30) we get

$$\frac{\partial l}{\partial p} = [1 - \alpha(1-\beta)]v \cdot \left(-\frac{\partial n}{\partial p}\right)$$

As $[1 - \alpha(1-\beta)]v > 0$, we finally get that $\frac{\partial l}{\partial p} > 0$. □

This result is very important: a rising life expectancy increases the number of agents in old age, and a reduction of fertility reduces the number of workers. So, the fraction of old-age people to the whole population must rise if life expectancy increases, which is consistent with statistics in many industrialized countries. On the other hand, middle-aged agents use their time differently: they shift time used for children to time used for working in average, a fact that can be explained by an increasing participation of women in the labor market in many industrialized countries in the past, as they represent approximately 50% of the population.

Proposition 3.3 *A rise in life expectancy increases human capital investments:* $\frac{\partial \gamma_q}{\partial p} > 0$ *and* $\frac{\partial h}{\partial p} > 0$.

Proof Taking the derivative of (3.23) with respect to p leads to

$$\frac{\partial \gamma_q}{\partial p} = \frac{\alpha\beta}{[1 - \alpha(1-\beta)]n^2} \cdot \left(-\frac{\partial n}{\partial p}\right) \tag{3.32}$$

As $\frac{\alpha\beta}{[1-\alpha(1-\beta)]n^2} > 0$, and using proposition 3.2 gives us $\frac{\partial \gamma_q}{\partial p} > 0$. Taking the derivative of (3.22) leads to

$$\frac{\partial h}{\partial p} = \frac{\alpha(1-\beta)}{n^2} \cdot \left(-\frac{\partial n}{\partial p}\right) \tag{3.33}$$

As $\frac{\alpha(1-\beta)}{n^2} > 0$, and using proposition 3.2 gives us $\frac{\partial h}{\partial p} > 0$. □

Here we see that a rising life expectancy increases human capital investments through decreasing fertility, so there is an adjustment between the quality and the quantity of children.

Proposition 3.4 *A rise in life expectancy increases per capita growth:* $\frac{\partial g}{\partial p} > 0$.

Proof Taking the derivative of g with respect to p leads to

$$\frac{\partial g}{\partial p} = \frac{\partial g}{\partial h} \cdot \frac{\partial h}{\partial p} + \frac{\partial g}{\partial \gamma_q} \cdot \frac{\partial \gamma_q}{\partial p} + \frac{\partial g}{\partial n} \cdot \frac{\partial n}{\partial p} + \frac{\partial g}{\partial s} \cdot \frac{\partial s}{\partial p} \tag{3.34}$$

With (3.31) we get that

$$\frac{\partial g}{\partial h} > 0, \quad \frac{\partial g}{\partial \gamma_q} > 0, \quad \frac{\partial g}{\partial n} < 0, \quad \frac{\partial g}{\partial s} > 0$$

With the results from proposition 3.1-3.3, $\frac{\partial h}{\partial p} > 0$, $\frac{\partial \gamma_q}{\partial p} > 0$, $\frac{\partial n}{\partial p} < 0$ and $\frac{\partial s}{\partial p} > 0$, we get that $\frac{\partial g}{\partial p} > 0$. □

Here we see that a rising life expectancy increases per capita growth.

Proposition 3.5 *When fertility n is fixed, a rise in life expectancy increases the saving rate,* $\frac{\partial s}{\partial p} > 0$*, and the ratio of investment in physical capital to income,* $\frac{\partial (K_{t+1}/Y_t)}{\partial p} > 0$*. Therefore it also increases per capita growth,* $\frac{\partial g}{\partial p} > 0$*.*

Proof With n being fixed, we have $\frac{\partial n}{\partial p} = 0$. As n does not interfere in the proof of proposition 3.1, the result remains the same for constant n. So, we get $\frac{\partial s}{\partial p} > 0$ and $\frac{\partial (K_{t+1}/Y_t)}{\partial p} > 0$. With (3.32) we instantly get $\frac{\partial \gamma_q}{\partial p} = 0$, and with (3.33) we get $\frac{\partial h}{\partial p} = 0$. With these results, (3.34) is reduced to

$$\frac{\partial g}{\partial p} = \frac{\partial g}{\partial s} \cdot \frac{\partial s}{\partial p}$$

As $\frac{\partial g}{\partial s} > 0$ is still true, we get that $\frac{\partial g}{\partial p} > 0$ for constant n. □

Here we see that if we fix fertility and have a rising life expectancy p, the only force responsible for the rise of the growth rate g is the saving rate s or the ratio of physical capital investment to income. The result is the same as with an endogenous fertility, but in the latter case there are four driving forces of an increasing growth rate g at an increasing life expectancy p, as seen in proposition 3.4: the investment of time from a parent for a child h, the ratio of goods for each child to labor income γ_q, fertility n and the saving rate s.

3.5 Conclusion

Importance of the parameter β

The model can rule out the influence of savings or physical capital investment to long-run growth by setting β equal to 0 in (3.31), which would mean that any investment of goods for a child has no influence on its human capital. But this would not be in line with most empirical evidence, because typically one of the growth determinants is saving or investment in physical capital.

Opposing effects on the saving rate s

With a rising life expectancy p, there are different opposing effects:

- If a worker expects a longer lifetime in retirement, he or she will save more and reduce both consumption while working and the investment in the education of his or her children to save for retirement.
- On the other hand, rising longevity may lower the total rate of return of annuity investments, which leads the worker to save less for retirement.
- If an altruistic parent expects the children to live longer, he or she may invest more in their education.

Interpretation of the results

In this section we saw an example of a dynastic family model with an annuity market. The model shows us that the effect of rising life expectancy on equilibrium growth, which represents long-run growth, is positive. To reach this conclusion, we have considered the influence on several growth determinants: fertility, savings, ratio of physical capital investment to income, labor supply and human capital investments.

In general a rising life expectancy increases savings and human capital investments, reduces fertility and all that results in increasing long-run growth. If we take fertility as constant, we still have increasing long-run growth because of a positive effect on savings and investment in the ratio of physical capital investment to income.

The assumptions of the model seem to be quite relevant to the current situation in industrial countries, so the outcome of the model could give hope to a positive effect of an aging population.

4 Rising longevity, education, savings, and growth (Zhang et al. 2003)

In this section I will present the paper of Zhang et al., see [ZZL03]. An overlapping generations model without annuity markets constitutes the modeling framework. Since agents won't reach retirement for certain, accidental bequests occur which generate heterogeneity among the population. The aim of the paper is to investigate the influence of an exogenously rising longevity on economic growth and on the saving rate. In addition we assume public education which is financed by a tax, and the tax rate is chosen endogenously by majority voting of the population.

4.1 Motivation

Nowadays in most industrialized countries people are worried about an unfavorable impact of an aging population on aggregated capital and economic growth. People get in average older because of an increasing life expectancy and low birth rates. A longer life for an individual means that bequests are received in average later in life, and are diminished by an in average longer consumption period.

The empirical analysis shows us that the relation of economic growth to mortality decline could depend on the initial level of mortality. Table 4.1 reports the average growth rates of the gross domestic product per capita (1960-1985) and the ratios of private investment to the gross domestic product (1970-1985) for countries with different initial life expectancies in 1960. As the initial life expectancy rises from below 60 to 69 years, the private investment ratio and the growth rate rise, but with a life expectancy of 70 years or more the private investment ratio and the growth rate fall. Indeed, a declining mortality can have different demographic effects depending on the initial level of mortality. If we start from a high mortality level, a decline in mortality mainly diminishes death rates in childhood, which strongly raises the population growth rate thus making the population younger on average. In this case the declining mortality

Table 4.1: Investment ratios, growth rates and initial life expectancy (Zhang et al. 2003, page 85)

Life expectancy at birth in 1960 (years)	<60	60-64	65-69	≥ 70
Number of countries	41	8	14	12
Private investment/GDP (1970-1985 (%))	14	20	23	22
Average growth rates (1960-1985 (%))	1.88	3.18	3.36	2.50

raises the rates of return to investment in human capital. Once life expectancy has reached the level of around 65, a further decline in mortality strongly influences the older population. Therefore it does not much influence the population growth rate and the rates of return to investment in human capital because death rates in childhood are already at a very low level.

Here we will focus on initial low mortalities, which means that further declines strongly affect the older population.

4.2 The model

The following life-cycle model constitutes an overlapping generations model of three-period-lived agents with accidental bequests. As the amounts of the received bequests can differ, the agents are inhomogeneous. The received bequests depend on the entire mortality history of their ancestors.

First, every agent has a schooling period, then a working period with 1 unit of labor time, and finally a retirement period. In young adulthood, agents save for consumption in retirement and pay taxes to finance public education for the current generation of children. Furthermore they vote for the level of this flat-rate income tax. If an agent dies before retirement, his or her wealth is considered as an accidental bequest to the own child. There is no trade-off between working and leisure time because every agent works the same amount of time in life.

Assumptions and notations

- Every generation lives for 3 periods: schooling, working and retirement period
- Every agent survives for sure to childhood and young adulthood
- With a probability $p \in (0, 1)$ the agent dies after working and before starting retirement
- Every agent has exactly 1 child which is born between the schooling and the working period of the parent
- Market of only 1 good
- Infinite number of periods
- Imperfect capital markets
- Accidental bequests are considered
- Physical capital depreciates completely in 1 period
- Only public education is available for children
- Every agent works the same amount of time in life

- Growth is endogenous

Since every agent has exactly 1 child, the population size of children equals the population size of young adults and is normalized to 1. This means that aggregate capital stock equals average capital stock per worker.

As bequests are accidental, the received bequest for a child depends on the entire mortality history of its family. We will denote with subscript t the time period and superscript j the type-j-agents in a generation. A type-j-agent is defined as an agent whose exactly j consecutive previous family generations have died before entering into retirement. That gives the number of type-j-agents as $(p)^j(1-p)$ where $(p)^j$ represents the j consecutive died generations before retirement, and $(1-p)$ the generation living before which had to survive to retirement. As our model has an infinite number of periods, there are also infinitely many types of agents, which means $j \in \mathbf{N}$.

We will indicate variables without parentheses, $.^j$, for a type-j-agent, and with parentheses, $(.)^j$, if j represents the exponent.

Human capital accumulation equation

The human capital of a type-j-agent at time t, h_t^j, accumulates according to

$$h_t^j = A \cdot q_{t-1}^{\alpha} \cdot h_{t-1}^{1-\alpha}, \quad 0 < \alpha < 1 \tag{4.1}$$

where A is a scaling parameter[1], q_{t-1} the average amount of goods invested by the preceding generation for public schooling, α the share parameter and h_{t-1} the average human capital of the preceding generation.

This formula indicates that agents of the same generation have the same human capital stock: $h_t^j = h_t$.

Utility function

The utility function U_t^j is defined as

$$U_t^j = \ln[c_{1,t}^j] + (1-p)\delta \cdot \ln[c_{2,t+1}^j] + \phi \cdot \ln[q_t], \quad 0 < \delta < 1, \quad 0 < \phi < 1 \tag{4.2}$$

where $c_{1,t}^j$ denotes consumption in young adulthood, p the probability of dying before retirement, δ the subjective discount factor on the expected utility from consumption in retirement, $c_{2,t+1}^j$ consumption in retirement, ϕ the subjective discount factor on the expected utility from the quality of schools for children and q_t the quality of schools for children.

Here we see that $\delta < 1$ indicates that certain consumption in young adulthood is preferred to uncertain consumption in retirement, and $\phi < 1$ means that own young adulthood consumption is of more utility than the quality of schools for children.

[1] In fact, A is not defined in [ZZL03].

Consumption functions

The consumption in young adulthood of a type-j-agent is

$$c_{1,t}^j = b_t^j \cdot (1 + r_t) + w_t^j \cdot (1 - \tau_t) - s_t^j \tag{4.3}$$

where b_t^j is the bequest at the beginning of t, r_t the interest rate, w_t^j the wage rate, τ_t the wage tax rate and s_t^j the amount of savings.

This formula simply states the budget equation: consumption equals total income (bequests with interests plus after tax wage) less savings.

The consumption in retirement of a type-j-agent is given by

$$c_{2,t+1}^j = (1 + r_{t+1}) \cdot s_t^j \tag{4.4}$$

The formula states that if the agent survives to retirement, he or she consumes the whole stock of savings.

If we denote consumption in young adulthood as a fraction of after-tax young-age income $[b_t^j \cdot (1 + r_t) + w_t^j \cdot (1 - \tau_t)]$ by γ_c with $0 \leq \gamma_c \leq 1$, we can rewrite (4.3) as

$$c_{1,t}^j = \gamma_c \cdot [b_t^j \cdot (1 + r_t) + w_t^j \cdot (1 - \tau_t)] \tag{4.5}$$

With (4.3) and (4.5) we instantly get

$$s_t^j = (1 - \gamma_c) \cdot [b_t^j \cdot (1 + r_t) + w_t^j \cdot (1 - \tau_t)] \tag{4.6}$$

where we already see that $(1 - \gamma_c)$ represents the saving rate. Using (4.6) and (4.4) gives us

$$\frac{c_{2,t+1}^j}{1 + r_{t+1}} = (1 - \gamma_c) \cdot [b_t^j \cdot (1 + r_t) + w_t^j \cdot (1 - \tau_t)] \tag{4.7}$$

Bequests

Agents who die before retirement leave their savings as accidental bequests to their children. Agents who survive to retirement leave nothing:

$$b_t^j = s_{t-1}^{j-1} \quad \text{for} \quad j \geq 1 \qquad \text{and} \qquad b_t^j = 0 \quad \text{for} \quad j = 0 \tag{4.8}$$

Production function

The production function is given by

$$y_t = D \cdot k_t^\theta \left[\sum_{j=0}^{\infty} h_t^j \cdot l_t \cdot (p)^j (1 - p) \right]^{1-\theta} = D \cdot k_t^\theta \cdot h_t^{1-\theta}, \quad D > 0, \quad 0 < \theta < 1 \tag{4.9}$$

where D is a scaling parameter[2], k_t the aggregate or average physical capital stock, θ the share parameter, h_t the human capital, l_t the labor input per worker which is assumed

[2] In fact, D is not defined in [ZZL03].

to equal 1, and $(p)^j(1-p)$ the number of type-j-agents. The last equality uses $h_t^j = h_t$, $l_t = 1$ and $\sum_{j=0}^{\infty}(p)^j = \frac{1}{1-p}$.

As physical capital depreciates completely in 1 period and the production factors obtain their marginal products, we arrive at:

$$w_t^j = (1-\theta) \cdot D \left(\frac{k_t}{h_t}\right)^{\theta} h_t^j = (1-\theta) \cdot D \cdot e_t^{\theta} \cdot h_t^j \tag{4.10}$$

$$1 + r_t = \theta \cdot D \cdot \left(\frac{h_t}{k_t}\right)^{1-\theta} = \theta \cdot D \cdot e_t^{\theta-1} \tag{4.11}$$

where $e_t = \frac{k_t}{h_t}$ is the physical-human capital ratio. As $h_t^j = h_t$, we have $w_t^j = w_t$, which means that the wage rate is the same for all workers in 1 generation.

Market clearing

Markets clear when physical capital equals savings:

$$k_{t+1} = \sum_{j=0}^{\infty} s_t^j \cdot (p)^j(1-p) = s_t \tag{4.12}$$

s_t represents weighted average or aggregate savings.

Government budget constraint

The government is assumed to have a balanced budget in each period. Therefore the budget constraint is given by

$$q_t = \tau_t \sum_{j=0}^{\infty} w_t^j \cdot (p)^j(1-p) = \tau_t \cdot w_t \tag{4.13}$$

4.3 Equilibrium and results

The optimization problem is solved in two steps. In young adulthood, agents first choose savings with regard to (4.1), (4.3), (4.4) and (4.10), taking as given τ_t, b_t^j, q_{t-1}, h_{t-1} and e_t (thus with using (4.11) also r_t). Afterwards, agents vote for their preferred tax rate, and the applied tax rate is chosen in a political equilibrium.

In the first step of the problem, an agent wants to choose savings s_t^j optimally. Therefore we take the derivative of (4.2) with regard to s_t^j, using (4.3) and (4.4):

$$\frac{\partial U_t^j}{\partial s_t^j} = \frac{1}{c_{1,t}^j}(-1) + (1-p)\delta\frac{1}{c_{2,t+1}^j}(1+r_{t+1}) = 0$$

$$\frac{1}{c_{1,t}^j} = \frac{(1-p)\delta}{s_t^j} = \frac{(1-p)\delta}{b_t^j \cdot (1+r_t) + w_t^j \cdot (1-\tau_t) - c_{1,t}^j}$$

$$c_{1,t}^j = \frac{1}{1+(1-p)\delta}[b_t^j \cdot (1+r_t) + w_t^j \cdot (1-\tau_t)]$$

$$\gamma_c = \frac{1}{1+(1-p)\delta} \tag{4.14}$$

So, in the optimum the fraction of income consumed in young adulthood γ_c increases with a lower subjective preference parameter δ and a higher mortality rate p. It is independent of the tax rate τ_t.

As we already know that the saving rate is $(1-\gamma_c)$, we get

$$1-\gamma_c = \frac{(1-p)\delta}{1+(1-p)\delta} \tag{4.15}$$

This result yields that in the optimum the saving rate $(1-\gamma_c)$ rises with a higher δ and a lower mortality rate p.

If we take (4.6) and use the fact that $b_t^0 = 0$ we get

$$s_t^0 = (1-\gamma_c)w_t(1-\tau_t) \tag{4.16}$$

This equation determines the savings of a type-0-agent.

Taking again (4.6) and using (4.16) gives us

$$s_t^j = (1-\gamma_c) \cdot b_t^j \cdot (1+r_t) + s_t^0 \tag{4.17}$$

This equation separates the savings of a type-j-agent into a fraction dependent of the received bequest and a fraction dependent on type-0-agent savings.

Taking (4.12) and using (4.17) leads us to

$$\begin{aligned} k_{t+1} &= \sum_{j=0}^{\infty} \left\{[(1-\gamma_c) \cdot b_t^j \cdot (1+r_t) + s_t^0] \cdot (p)^j(1-p)\right\} \\ &= (1-\gamma_c)(1+r_t)\sum_{j=0}^{\infty}\left\{b_t^j \cdot (p)^j(1-p)\right\} + s_t^0 = (1-\gamma_c)(1+r_t) \cdot b_t + s_t^0 \end{aligned} \tag{4.18}$$

where b_t represents the weighted average or aggregate bequest. This equation connects the next period's physical capital stock with the current period average or aggregate bequest and the current savings of type-0-agents.

If we look at aggregate or average savings in period $t-1$, s_{t-1}, we see that as a fraction p of each type of agents die before retirement, this fraction $p \cdot s_{t-1}$ is the next period's accidental aggregate or average bequest b_t:

$$b_t = p \cdot s_{t-1} = p \cdot k_t \tag{4.19}$$

where the last equality uses (4.12).

If we denote the growth rate of physical capital as $1 + g_{k,t} = \frac{k_{t+1}}{k_t}$, we get by using (4.18) and (4.19):

$$k_{t+1} = \frac{k_{t+1}}{s_t^0} s_t^0 = \frac{\frac{k_{t+1}}{k_t}}{\frac{s_t^0}{k_t}} s_t^0 = \frac{1 + g_{k,t}}{1 + g_{k,t} - \frac{(1-\gamma_c)(1+r_t)b_t}{k_t}} s_t^0 = \frac{1 + g_{k,t}}{1 + g_{k,t} - (1-\gamma_c)(1+r_t)p} s_t^0 \tag{4.20}$$

Here the aggregate physical capital stock of the next period is linked to the savings of this period's type-0-agents. It increases with higher savings s_t^0, a lower growth rate of physical capital $(1 + g_{k,t})$, a higher saving rate $(1 - \gamma_c)$, a higher interest rate r_t and a higher mortality rate p.

If we assume $p = 0$, which means that it is certain to survive until retirement, we see from (4.20) that $k_{t+1} = s_t = s_t^0$. So, aggregate savings s_t equal aggregate savings of type-0-agents s_t^0, which can only be if there are no accidental bequests. If $p > 0$, these accidental bequests increase the physical capital accumulation, which results in $k_{t+1} = s_t > s_t^0$.

Using (4.20) gives us

$$1 + g_{k,t} = \frac{k_{t+1}(1-\gamma_c)(1+r_t)p}{k_{t+1} - s_t^0}$$

With (4.16) and (4.18) we get

$$\begin{aligned} 1 + g_{k,t} &= \frac{[(1-\gamma_c)(1+r_t)b_t + (1-\gamma_c)w_t(1-\tau_t)](1-\gamma_c)(1+r_t)p}{[(1-\gamma_c)(1+r_t)b_t + s_t^0] - s_t^0} \\ &= \frac{[(1+r_t)b_t + w_t(1-\tau_t)](1-\gamma_c)p}{b_t} \end{aligned}$$

With (4.19), (4.10) and (4.11) we arrive at

$$\begin{aligned} 1 + g_{k,t} &= \frac{[\theta \cdot D \cdot e_t^{\theta-1} \cdot p \cdot k_t + (1-\theta)D \cdot e_t^{\theta} \cdot h_t(1-\tau_t)](1-\gamma_c)p}{p \cdot k_t} \\ &= \frac{(1-\gamma_c)De_t^{\theta-1}}{k_t} [\theta \cdot p \cdot k_t + (1-\theta)e_t \cdot h_t(1-\tau_t)] \end{aligned}$$

Using the fact that $e_t = \frac{k_t}{h_t}$ gives us

$$1 + g_{k,t} = D(1-\gamma_c)\left[\theta \cdot p + (1-\theta)(1-\tau_t)\right] e_t^{\theta-1} \tag{4.21}$$

Here we see that the growth rate of physical capital decreases if p decreases, which means that the people have a higher probability of survival until retirement.

Defining the growth rate of human capital by $1 + g_{h,t} = \frac{h_{t+1}}{h_t}$ and using (4.1), (4.10) and (4.13) gives us

$$1+g_{h,t} \equiv \frac{h_{t+1}}{h_t} = A \cdot q_t^{\alpha} \cdot h_t^{-\alpha} = A(\tau_t w_t)^{\alpha} \cdot w_t^{-\alpha}(1-\theta)^{\alpha} D^{\alpha} \cdot e_t^{\alpha\theta} = A \cdot \tau_t^{\alpha}(1-\theta)^{\alpha} D^{\alpha} \cdot e_t^{\alpha\theta} \tag{4.22}$$

Here we see that the growth rate of human capital is positively influenced by the tax rate τ_t, which is clear because the tax rate defines the amount of goods the population invests in the next generation's human capital.

With (4.21) and (4.22) we get the evolution of the physical-human capital ratio:

$$\begin{aligned} e_{t+1} &\equiv \frac{k_{t+1}}{h_{t+1}} = \frac{(1+g_{k,t})k_t}{(1+g_{h,t})h_t} = \frac{D(1-\gamma_c)\left[\theta \cdot p + (1-\theta)(1-\tau_t)\right] e_t^{\theta-1}}{A \cdot \tau_t^{\alpha}(1-\theta)^{\alpha} D^{\alpha} \cdot e_t^{\alpha\theta}} e_t \\ &= \frac{D^{1-\alpha}(1-\gamma_c)\left[\theta \cdot p + (1-\theta)(1-\tau_t)\right]}{A \cdot \tau_t^{\alpha}(1-\theta)^{\alpha}} e_t^{(1-\alpha)\theta} \end{aligned}$$

Solving this equation for the steady state ratio $e^* = e_{t+1} = e_t$ with a constant tax rate τ results in

$$e^* = \left[\frac{D^{1-\alpha}(1-\gamma_c)\left[\theta \cdot p + (1-\theta)(1-\tau)\right]}{A \cdot \tau^{\alpha}(1-\theta)^{\alpha}}\right]^{\frac{1}{1-(1-\alpha)\theta}}$$

Inserting this result into either (4.21) or (4.22) results in

$$1+g = \left[\frac{\{D(1-\gamma_c)\left[\theta \cdot p + (1-\theta)(1-\tau)\right]\}^{\alpha\theta}}{\{A \cdot \tau^{\alpha}(1-\theta)^{\alpha} D^{\alpha}\}^{\theta-1}}\right]^{\frac{1}{1-(1-\alpha)\theta}} \tag{4.23}$$

Here we see that the growth rate is affected by the mortality rate p directly through one term, and indirectly through the saving rate $(1-\gamma_c)$ and the tax rate τ.

The behavior of the growth rate for changes of the saving rate and the tax rate are given by

Proposition 4.1

$$\frac{\partial g}{\partial(1-\gamma_c)} > 0; \qquad \frac{\partial g}{\partial \tau} > 0 \text{ if } \tau < 1-\theta+p\theta$$

Proof Taking the derivative of (4.23) with respect to $(1-\gamma_c)$ and to τ immediately produces the results. □

This means that a rise in the saving rate or tax rate has a positive effect on the growth rate, as long as the tax rate is not too high.

To get a better understanding of the implications on growth with a rising longevity, we focus on an exogenous tax rate to keep it simple:

Proposition 4.2 *For a given tax rate τ, $\frac{\partial g}{\partial p} < 0$ unless $\tau \geq \frac{1-\theta(1-p)[2+\delta(1-p)]}{1-\theta}$. In particular, if p is sufficiently high, then $\frac{\partial g}{\partial p} < 0$. If p is low, then the sign of $\frac{\partial g}{\partial p}$ is ambiguous in general and positive for sufficiently large δ, θ and τ.*

Proof For a constant τ, the mortality rate p influences g as in (4.23) only through $(1-\gamma_c)[p\theta + (1-\tau)(1-\theta)]$, with γ_c as given in (4.14). Therefore, $\frac{\partial g}{\partial p}$ is signed by $f(p) \equiv \theta(1-p)[1+\delta(1-p)] - p\theta - (1-\tau)(1-\theta)$ and $f'(p) < 0$. The restriction on τ for signing $\frac{\partial g}{\partial p}$ negatively follows immediately. Evidently, $f(p) < 0$ and therefore $\frac{\partial g}{\partial p} < 0$ if p is sufficiently high. □

This proposition states that for an exogenous tax rate, rising longevity has a positive effect on the growth rate if initial mortality is sufficiently high. If initial mortality is low, the effect is ambiguous.

We should also note that the saving rate as stated in (4.15) increases if p falls, because $\frac{\partial(1-\gamma_c)}{\partial p} < 0$. At the same time the saving rate is concave in p because $\frac{\partial^2(1-\gamma_c)}{\partial p^2} < 0$. So, a higher longevity (a lower p) increases the saving rate, but this increase gets smaller with an increasing longevity.

Determination of the tax rate

The tax rate is determined endogenously through majority voting in two steps. First we calculate the preferred tax rate τ_t^j for each type of agents, and then we will determine the actual tax τ_t. As agents in retirement don't pay any taxes because we only consider an income tax, their welfare is not affected by the tax. Furthermore their grandchildren are not part of the utility function, so they are indifferent to the level of the tax applied. That's why in our model they don't have a vote.

Taking (4.2) and using (4.5), (4.7) and (4.5) leads to

$$\begin{aligned} U_t^j &= \ln[c_{1,t}^j] + (1-p)\delta \cdot \ln[c_{2,t+1}^j] + \phi \cdot \ln[q_t] \\ &= \ln[\gamma_c[b_t^j(1+r_t) + w_t(1-\tau_t^j)]] + \\ &\quad + (1-p)\delta \cdot \ln[(1+r_{t+1})(1-\gamma_c)[b_t^j(1+r_t) + w_t(1-\tau_t^j)]] + \phi \cdot \ln[\tau_t^j \cdot w_t] \end{aligned}$$

where we finally get to the problem formulation in the second stage with the utility function V_t^j which is maximized with regard to τ_t^j:

$$\begin{aligned} V_t^j =& \ln\gamma_c + [1+(1-p)\delta]\ln[b_t^j(1+r_t) + w_t(1-\tau_t^j)] \\ &+ (1-p)\delta\ln[(1+r_{t+1})(1-\gamma_c)] + \phi\ln\tau_t^j + \phi\ln w_t \end{aligned} \tag{4.24}$$

Here we take as given b_t^j, e_t, r_t, r_{t+1} and w_t. Note that γ_c is independent of τ_t^j by (4.14).

With this expression we derive the preferred tax rate for each type of agent as follows:

Proposition 4.3 *The preferred tax rate of type-j-agents is*

$$\tau_t^j = \frac{\phi}{1+\phi+(1-p)\delta}\left[1 + \frac{\theta p}{1-\theta}\frac{b_t^j}{b_t}\right]$$

In addition, $\frac{d\tau_j^0}{dp} > 0$.

Proof Taking the derivative of (4.24) with respect to τ_t^j results in

$$\frac{\partial V_t^j}{\partial \tau_t^j} = [1+(1-p)\delta]\frac{-w_t}{b_t^j(1+r_t) + w_t(1-\tau_t^j)} + \frac{\phi}{\tau_t^j} = 0$$

and therefore

$$\phi[b_t^j(1+r_t) + w_t(1-\tau_t^j)] = w_t[1+(1-p)\delta]\tau_t^j$$

$$\tau_t^j = \frac{\phi}{1+\phi+(1-p)\delta}\left[1+\frac{b_t^j(1+r_t)}{w_t}\right]$$

Using (4.10), (4.11), (4.19) and $e_t = \frac{k_t}{h_t}$ gives us

$$\frac{1+r_t}{w_t} = \frac{\theta D e_t^{\theta-1}}{(1-\theta)De_t^{\theta}h_t} = \frac{\theta}{(1-\theta)k_t} = \frac{\theta p}{(1-\theta)b_t}$$

which leads us to

$$\tau_t^j = \frac{\phi}{1+\phi+(1-p)\delta}\left[1+\frac{\theta p}{1-\theta}\frac{b_t^j}{b_t}\right]$$

Taking the derivative of (4.24) with respect to τ_t^j two times results in

$$\frac{\partial^2 V_t^j}{\partial(\tau_t^j)^2} = -w_t[1+(1-p)\delta]\frac{w_t}{[b_t^j(1+r_t)+w_t(1-\tau_t^j)]^2} - \frac{\phi}{(\tau_t^j)^2} < 0$$

So, the preferred tax rate is a local maximum.

As $b_t^0 = 0$ we have $\tau_t^0 = \frac{\phi}{1+\phi+(1-p)\delta}$ and we get

$$\frac{\partial \tau_t^0}{\partial p} = \frac{-\phi}{[1+\phi+(1-p)\delta]^2}(-\delta) > 0 \square$$

Here we observe 3 important facts:

- With given bequests b_t^j, the preferred tax rate of all types of agents falls if longevity increases. This shows us that resources are shifted from investments in children's human capital to own consumption in retirement.

- The preferred tax rate rises with the type-j bequest b_t^j. This fact is clear because the quality of schools for children is a normal good for the parent, so a higher received bequest raises demand for a higher quality of schools.

- The preferred tax rate of type-0-agents is independent of time t: $\tau_t^0 = \tau^0$.

Let's now have a look at the equilibrium tax rate. Only agents in young adulthood have the right to vote, and each type of them votes for its preferred tax rate. As every agent has one vote, each τ_t^j is voted by a fraction of $(p)^j(1-p)$ type-j-agents. If $p < \frac{1}{2}$ then type-0-agents form the majority, otherwise no single type forms a majority. The preferred tax rate in the political equilibrium is the median tax rate, which is given by the preferred tax rate of type-j^*-agents such that $\sum_{i\geq j^*}(p)^i(1-p) \geq \frac{1}{2}$ and $\sum_{i\leq j^*}(p)^i(1-p) \geq \frac{1}{2}$. So the median tax rate is the preferred tax rate of the median type-j-agent population, which are called type-j^*-agents. By reformulation we get

$$\sum_{i\geq j^*}(p)^i(1-p) = (p)^{j^*}(1-p)\sum_{k\geq 0}(p)^k = (p)^{j^*}(1-p)\frac{1}{1-p} = (p)^{j^*} \geq \frac{1}{2}$$

$$j^* \leq \frac{\ln[\frac{1}{2}]}{\ln p} \tag{4.25}$$

Due to the fact that the expression on the right-hand side of this inequation is a real value, j^* is the rounded down integer of this expression.

Using the balanced growth condition $\frac{s_t^j}{s_{t-1}^j} = \frac{s_t}{s_{t-1}}$ and defining $a_t \equiv \frac{(1-\gamma_c)(1+r_t)}{s_t/s_{t-1}} = \frac{(1-\gamma_c)(1+r_t)}{1+g_{k,t}}$ by using (4.12), $\frac{s_t}{s_{t-1}} = \frac{k_{t+1}}{k_t} = 1 + g_{k,t}$. With (4.11) and (4.21) we get

$$a_t = \frac{(1-\gamma_c)\theta \cdot D \cdot e_t^{\theta-1}}{D(1-\gamma_c)\left[\theta \cdot p + (1-\theta)(1-\tau_t)\right] e_t^{\theta-1}} = \frac{\theta}{\theta \cdot p + (1-\theta)(1-\tau_t)}$$

Using (4.17) and (4.8) gives us $s_t^j = (1-\gamma_c)(1+r_t)s_{t-1}^{j-1} + s_t^0 = a_t \frac{s_t}{s_{t-1}} s_{t-1}^{j-1} + s_t^0 = a_t \cdot \frac{s_t^{j-1}}{s_{t-1}^{j-1}} \cdot s_{t-1}^{j-1} + s_t^0 = a_t \cdot s_t^{j-1} + s_t^0$. Therefore we get

$$s_t^j = \sum_{k=0}^{j} a_t^k \cdot s_t^0 = \frac{1-(a_t)^{j+1}}{1-a_t} s_t^0 \tag{4.26}$$

Dividing (4.20) by $1 + g_{k,t}$ results in $s_t = \frac{1}{1-a_t \cdot p} s_t^0$. With this result, (4.8), (4.19) and (4.26) we get

$$\frac{b_t^j}{b_t} = \frac{s_{t-1}^{j-1}}{p \cdot s_{t-1}} = \frac{\frac{1-(a_t)^j}{1-a_t} s_t^0}{p \cdot \frac{1}{1-a_t \cdot p} s_t^0} = \frac{(1-(a_t)^j)(1-a_t \cdot p)}{p(1-a_t)} \tag{4.27}$$

With this expression and proposition 4.3 we finally get the equilibrium tax rate $\tau_t^{j^*}$:

$$\tau_t^{j^*} = \frac{\phi}{1+\phi+(1-p)\delta}\left[1 + \frac{\theta}{1-\theta}\frac{(1-(a_t)^j)(1-a_t \cdot p)}{1-a_t}\right] \tag{4.28}$$

and

$$a_t = \frac{\theta}{\theta \cdot p + (1-\theta)(1-\tau_t^{j^*})} \tag{4.29}$$

These two equations determine the solution for a_t and $\tau_t^{j^*}$. With a given value p we calculate j^* by using (4.25). With this result we get a_t and $\tau_t^{j^*}$, which are actually time invariant, $a_t = a$ and $\tau_t^{j^*} = \tau^{j^*}$.

As a next step we investigate the change of steady-state growth with a rising longevity:

Proposition 4.4 *With an endogenously determined tax rate for public education by a majority of voters, $\frac{dg}{dp} < 0$ if p is large enough.*

For a small enough p, we have $\frac{dg}{dp} > 0$ when δ and θ are sufficiently large.

Proof From (4.23) we see that the sign of $\frac{dg}{dp}$ is determined by the first-order derivative with respect to p of $\Phi(p) \equiv \tau^{1-\theta}(1-\gamma_c)^\theta[\theta \cdot p + (1-\tau)(1-\theta)]^\theta$. Let $\tau = \tau^{j^*}$. If $p \to 1$ then $j^* \to \infty$ by (4.25), $a < 1$ by (4.29) and $\frac{b_t^{j^*}}{b_t} \to 1$ by (4.27). Note that as $p \to 1$,

Table 4.2: Growth effects of rising longevity with labor income taxes (Zhang et al. 2003, page 94)

$\alpha = 0.15$, $\delta = 0.995$, $\theta = 0.36$, $\phi = 0.06$ and $A = D = 3.0$			
Adult mortality (p)	Saving/income (s/y (%))	Public education/income (q/y (%))	Annual growth ($g^{1/25}$ (%))
0.98	1.83	5.58	2.18
0.90	8.25	5.29	2.63
0.80	14.59	4.87	2.77
0.72	18.62	4.47	2.80
0.60	23.31	3.72	2.77
0.50	26.07	3.53	2.77
0.40	28.44	2.32	2.56
0.30	29.81	2.19	2.54
0.20	30.64	2.07	2.52
0.04	31.04	1.91	2.48
0.02	30.99	1.89	2.47

$\phi < 1$ implies $\tau^{j^*} < 1$. This and (4.28) leads us to the fact that $\frac{d\tau^{j^*}}{dp}$ is finite. As $p \to 1$, $\Phi'(p) \to -\infty$.

If $p \leq \frac{1}{2}$, type-0-agents form a majority. So, the preferred tax rate is $\tau^{j^*} = \tau^0 = \frac{\phi}{1+\phi+(1-p)\delta}$. With $p \to 0$, $\Phi'(p)$ is signed by $\delta(1-\theta)(1+\delta)[(1-\theta)(1+\delta) - \phi\theta] + \theta(1+\delta)(1+\phi+\delta)[\theta(2+\phi+\delta) - 1]$ which is positive for δ and θ large enough. □

Here we see that the net impact of rising longevity on steady-state growth is ambiguous: if initial mortality p is high, rising longevity has a positive effect on steady-state growth g, whereas with an initial low p, rising longevity has a negative effect on g, given a large enough δ and θ.

The authors of [ZZL03] did some numerical analysis with plausible values for the parameters. The results are given in table 4.2. The ratio of aggregate saving to aggregate income is derived by using successively (4.12), (4.18), (4.16), (4.11), (4.10), (4.9) and (4.19):

$$\frac{s_t}{y_t} = \frac{k_{t+1}}{y_t} = \frac{(1-\gamma_c)(1+r_t)b_t + (1-\gamma_c)w_t(1-\tau^{j^*})}{y_t} =$$
$$= (1-\gamma_c)\frac{\theta \cdot D \cdot e_t^{\theta-1} \cdot b_t + (1-\theta)D \cdot e_t^{\theta} \cdot h_t(1-\tau^{j^*})}{D \cdot k_t^{\theta} \cdot h_t^{1-\theta}} = (1-\gamma_c)[\theta \cdot p + (1-\tau^{j^*})(1-\theta)$$

Here we see that with an adult mortality p going from 0.98 to 0.02, the ratio of aggregate saving to aggregate income $\frac{s_t}{y_t}$ rises first and finally declines a bit. On the other hand the ratio of public education investment to aggregate income $\frac{q_t}{y_t} = \tau^{j^*}(1-\theta)$ (derived by using (4.13), (4.10) and (4.9)) only falls. With that fact we finally get that initially the net effect of lower adult mortality on growth is positive, and later negative.

The authors furthermore investigated a more rational voting process which considers not only the direct impact of the tax rate, but also an indirect impact. There is the possibility of considering the influence of the tax rate on investment decisions of other individuals, which influences the aggregate physical-human capital ratio in the next period, e_{t+1}. This fact influences the return to savings, r_{t+1}. With these rational reflections the model becomes non-analytic. Therefore the authors did a numerical simulation. It results in the same behavior of the ratio of savings to income, which is our focus here. Therefore this approach is not discussed here in more detail.

4.4 Conclusion

In this section we took a closer look at the growth rate with exogenously rising longevity, in the framework of a two-sector endogenous growth model without annuity markets. As agents don't reach retirement for certain, we have accidental bequests which generate heterogeneity among the population. Public education is financed by a tax, where the tax rate is chosen endogenously by majority voting.

Driving forces

This model gives for increasing longevity an ambiguous result on growth because a declining mortality affects growth in 3 ways:

- It raises the saving rate and therefore raises the rate of physical capital accumulation;
- It reduces accidental bequests which lowers investment and therefore lowers the rate of physical capital accumulation;
- If initial mortality is low, it may lead the median voter to decrease the tax rate and therefore lowers the human capital accumulation. If initial mortality is high, there is a positive effect on growth.

In the numerical analysis, with the results in table 4.2, we see that if initial mortality is high, a declining mortality has a positive effect on the saving rate. However, if mortality is low, as in most industrialized countries nowadays, a furthermore declining mortality has a negative effect on the saving rate.

Shortcomings of the model

The presented model has several shortcomings, which should be taken into account. The main simplifications are:

- A very simple tax system: in childhood and retirement, no direct or indirect taxes are paid;

- The voting process: in reality tax rates are fixed by the government which is mostly elected by public voting, but does not necessarily reflect exactly the public opinion on preferred tax rates. Furthermore old agents should have a vote, too.
- No trade-off between working and leisure time because every agent works the same amount of time in life;

5 The effect of improvements in health and longevity on optimal retirement and saving (Bloom et al. 2004)

In the following section I will present the model published by Bloom et al., see [BCM04]. It is a continuous time model of agents living for two periods, a working and a retirement period. In each period, agents can choose a consumption level. In the first period they also decide on their retirement and whether to save for the second period, i.e. for retirement. If the agent survives until retirement, the saved assets will be the only funding of consumption in retirement, as there are no transfers either from the state or directly from the working population to the retirees.

This model can be seen as a benchmark for social security systems because the optimal retirement age derived from the model should be close to the mandatory retirement age of a well designed social security system. Otherwise people would be obliged to retire from work before it would be necessary for them personally.

5.1 Motivation

In the last 100 years there have been significant improvements in the overall health level and an increase in the life expectancy. According to Lee (2003, page 168) the average life expectancy at birth for the whole world was 30 years in 1900, 65 years in 2000 and is projected to reach 81 years in 2100.

This significant change in people's time horizons is supposed to have an important influence on their economic life-cycle behavior. Let's give one short example to clarify this: if we think of choosing between an education at university and starting to work after high school, then education at university would be an investment in one's human capital. It will be chosen because it is expected to pay off afterwards by higher salaries. So, doing this investment reflects the expectation of living a sufficient time after finishing this education. Given 2 exemplary life expectations, 30 years and 65 years, then clearly in the first case it would be unattractive to invest in an education at university, whilst in the second case it could be an attractive investment. This example shows that changes in the planning time horizons could have significant influences in people's choices over their whole life.

5.2 The model

Motivation of saving

In the literature there are several different reasons for people to save, as already mentioned in section 2.1.1. In the model presented here we will only consider the motivation of saving for retirement, and ignore all the other reasonings.

There is one important difference between developing countries and industrial countries: in developing countries, the elderly often benefit from intra-family transfers, whilst in industrial countries they often benefit from well-working social security systems.

Given these facts, it is clear that the transfer systems of a state affect the incentives to save and to retire for the people.

Exogenous variables

In the model, there are assumed to be several exogenously given variables:

- Interest rate r
- Disutility of labor ν: it describes the negative value of investing time to work and will depend on life expectancy Z and age t
- Rate of time preference δ: the discount factor of an agent will be $e^{-\delta t}$
- Real wage rate w: an agent will earn a wage $w(t)$ at age t if working, so it will depend on age t
- Rate of wage growth σ: $\dot{w} = \sigma w$
- Constant death rate λ (with $1/\lambda$ = life expectancy Z)

Environment

For the model there are several general assumptions:

- Complete capital markets: existence of perfect annuity markets
- Impossibility of using consumption c and health services to have a longer life, and no link from labor supply to health status and life expectancy: Z will be independent of c and on the argument if the agent has retired or not at age t
- Working life starts at the beginning of the life cycle (no schooling period)
- No bequests

Health

Health during the lifetime of an agent is represented by Z, which will have an influence on the decision when to retire. So, the disutility of work will depend on a worker's health status.

The decision to retire is only caused by a poor health status, which means that as long as an agent is healthy enough, he or she will not retire.

A rising life expectancy is linked to improved health at each age t, so an agent will live both a longer and a healthier life span (as opposed to the possibility of living longer, but at very poor health in the additional years in life).

Wealth effect

If we think of an increase in life expectancy, it would be tempting to think of a result where the proportion of lifetime spent working to total life remains constant, and consumption & the saving rate remain unchanged, e.g. if an agent who expects to live 60 years will work 40 years (proportion $40/60 = 2/3$), then an agent who expects to live 90 years will choose to work 60 years (same proportion $60/90 = 2/3$).

Evidently this is not the case because of the wealth effect: a longer lifetime spent working permits more accumulation of compound interests on savings, which generates a higher potential wealth at retirement compared to the proportionality result.

Lifetime expected utility

The lifetime expected utility U is given by:

$$U = \int_0^\infty e^{-(\delta+\lambda)t}\,[u(c(t)) - \chi_t \cdot \nu(Z,t)]\,dt \quad \text{with} \quad \frac{\partial \nu(Z,t)}{\partial t} > 0, \frac{\partial \nu(Z,t)}{\partial Z} < 0$$

$e^{-\delta t}$ is the discount factor, with δ being the subjective rate of time preference. $e^{-\lambda t}$ is the probability of being alive at age t, with λ being the constant death rate. $[u(c(t)) - \chi_t \cdot \nu(Z,t)]$ gives the instantaneous utility at age t, with $u(c(t))$ being the utility function, $\nu(Z,t)$ being the disutility of working, χ_t being an indicator function with 1 for working and 0 for retired, and Z being the life expectancy.

The disutility of work $\nu(Z,t)$ should grow in t in a way that once it is more interesting to be retired than to work, it should stay this way. So, the indicator function χ_t gives the retirement age which is the age t where χ_t changes its value from 1 to 0. $\frac{\partial \nu(Z,t)}{\partial Z} < 0$ represents the assumption of a both longer and healthier life span.

This formula specifies for given variables δ, λ, Z, a given disutility of work $\nu(Z,t)$, and a given utility function $u(c(t))$ for every chosen consumption path $c(t)$ in t and retirement age χ_t the lifetime expected utility U of that agent.

The budget constraint

The budget constraint is given by:

$$\frac{dW(t)}{dt} = \chi_t \cdot w(t) + (\lambda + r)W(t) - c(t)$$

$W(t)$ denotes the state variable and describes the overall wealth at age t. $w(t)$ denotes the wage at age t which the agent only receives if he is working at age t. The effective interest rate $(\lambda + r)$ is bigger than r because of the mortality risk the agent faces. As we have no bequests, this term describes a perfect annuity market. $c(t)$ is the consumption at age t and decreases the overall wealth.

The transversality condition is given by $\lim_{t\to\infty} W(t) \geq 0$. It is important to notice that there is not necessarily equality. Agents may want to hold positive wealth at every age t because they don't know when they will die.

5.3 Optimal solution

The Hamiltonian

The lifetime expected utility is maximized with respect to the budget constraint. The control variables for this optimization problem are $c(t)$ and χ_t, so the agent chooses the consumption path and the retirement age.

The Hamiltonian for this problem is:

$$H = e^{-(\delta+\lambda)t}\left[u(c(t)) - \chi_t \cdot \nu(Z,t)\right] + \phi\left[\chi_t \cdot w(t) + (\lambda + r)W(t) - c(t)\right]$$

The costate equation for this problem yields

$$\dot{\phi} = -\frac{\partial H}{\partial W(t)} = -\phi(\lambda + r) \tag{5.1}$$

The first order conditions of this problem are given by

$$\frac{\partial H}{\partial c(t)} = e^{-(\delta+\lambda)t} \cdot u'(c(t)) - \phi = 0 \tag{5.2}$$

$$\begin{aligned} \frac{\partial H}{\partial \chi_t} &= -e^{-(\delta+\lambda)t} \cdot \nu(Z,t) + \phi \cdot w(t) \geq 0 \text{ when } \chi_t = 1 \\ \frac{\partial H}{\partial \chi_t} &= -e^{-(\delta+\lambda)t} \cdot \nu(Z,t) + \phi \cdot w(t) \leq 0 \text{ when } \chi_t = 0 \end{aligned} \tag{5.3}$$

With (5.2) we get

$$\phi = e^{-(\delta+\lambda)t} \cdot u'(c) \tag{5.4}$$

Taking the first derivative with respect to t yields

$$\dot{\phi} = -(\delta + \lambda) \cdot e^{-(\delta+\lambda)t} \cdot u'(c) + e^{-(\delta+\lambda)t} \cdot u''(c) \cdot \dot{c} \tag{5.5}$$

If we insert (5.4) and (5.5) into (5.1), we arrive at

$$\dot{c}(t) = (r - \delta)\frac{u'(c(t))}{-u''(c(t))} \tag{5.6}$$

Obviously a rising consumption level over time can be observed for a concave utility function ($u'(x) > 0$, $u''(x) < 0$) if r is bigger than δ, even without an explicit expression of $u(c(t))$.

Inserting (5.4) into (5.3) results in

$$\chi_t = 1 \Leftrightarrow u'(c(t)) \cdot w(t) \geq \nu(Z,t) \tag{5.7}$$

This equivalence shows that an agent works at age t, if the marginal utility of consumption times the wage exceeds the disutility of work. If we assume that the utility function is concave and $\nu(Z,t)/w(t)$ is non-decreasing then we get a distinctive age t where we have equality for a concave utility function.

5.4 Results

Additional assumptions

For a detailed analysis we assume the utility function and the disutility of work to have an explicit form.

For the utility function we assume the form of constant relative risk aversion:

$$\begin{aligned} u(c) &= \frac{c^{1-\beta}}{1-\beta} && \text{for } \beta \geq 0 \text{ and } \beta \neq 1 \\ u(c) &= log(c) && \text{for } \beta = 1 \end{aligned}$$

Here, β is both the coefficient of relative risk aversion and the inverse of the intertemporal elasticity of substitution.

For the disutility of work we assume an explicit form as given by:

$$\nu(Z,t) = d \cdot e^{t/Z} = d \cdot e^{\lambda t}$$

d describes the intensity of the disutility of work. Here we see that both conditions, $\frac{\partial \nu(Z,t)}{\partial t} > 0$ and $\frac{\partial \nu(Z,t)}{\partial Z} < 0$ are satisfied, which means that the disutility of work rises in λ ($\lambda = 1/Z$) and t.

Optimal consumption path $c(t)$

With these additional assumptions, we now derive some further explicit results. Inserting them into (5.6) gives the optimal growth rate of consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{r-\delta}{\beta}$$

where the consumption path results in

$$c(t) = c_0 \cdot e^{\frac{r-\delta}{\beta} t} \tag{5.8}$$

with c_0 as initial level of consumption. c_0 can be derived by the budget constraint over the whole life:

$$\int_0^\infty e^{-(\lambda+r)t} c(t)dt = \int_0^R e^{-(\lambda+r)t} w(t)dt$$

The left side represents the discounted present value of consumption over the whole life cycle, and the right side the discounted present value of earnings over the whole life cycle. R is the optimal retirement age. $e^{-(\lambda+r)t}$ gives the market discount rate of capital. With (5.8) and $w(t) = w_0 e^{\sigma t}$, where w_0 is the initial wage rate, we arrive at

$$\int_0^\infty e^{-(\lambda+r)t} c_0 \cdot e^{\frac{r-\delta}{\beta}t} dt = \int_0^R e^{-(\lambda+r)t} w_0 e^{\sigma t} dt$$

Integration and assuming $\sigma < \lambda + r$ yields

$$\left[c_0 \frac{e^{-\left(\frac{r(\beta-1)+\lambda\beta+\delta}{\beta}\right)t}}{-\left(\frac{r(\beta-1)+\lambda\beta+\delta}{\beta}\right)} \right]_0^\infty = w_0 \left[\frac{e^{(\sigma-(\lambda+r))t}}{\sigma-(\lambda+r)} \right]_0^R$$

With the assumption of $\beta \geq 1$ we finally get an expression of the initial consumption:

$$c_0 = w_0 \frac{r(\beta-1)+\lambda\beta+\delta}{\beta(\lambda+r-\sigma)} \left(1 - e^{(\sigma-\lambda-r)R}\right) \text{ with } \beta \geq 1 \text{ and } \sigma < \lambda + r \tag{5.9}$$

To have $\sigma < \lambda + r$ for any λ, we choose $\sigma < r$. Equation (5.9) is the first explicit link between R and c_0 which shows us that a rise in the retirement age R results in a rise in the initial consumption c_0.

Optimal retirement age R

With (5.7) we get that R is given by the marginal condition

$$\nu(Z,R) = w(R) \cdot u'(c(R))$$

which can be written as

$$d \cdot e^{\lambda R} = w_0 \cdot e^{\sigma R} \left[c_0 \cdot e^{\frac{r-\delta}{\beta}R} \right]^{-\beta} \tag{5.10}$$

This equation is the second link between R and c_0 which shows us that a rise in the retirement age R results here also in a rise in the initial consumption c_0.

Determination of the time path of consumption

With (5.9) and (5.10) we can determine c_0 and R, and together with (5.8) we finally get the optimal time path of consumption.

Solving (5.9) and (5.10) gives us

$$d \cdot e^{\lambda \boldsymbol{R}} = w_0^{1-\beta} \left[\frac{\beta(\lambda+r-\sigma)}{r(\beta-1)+\lambda\beta+\delta} \right]^\beta \frac{e^{(\sigma+\delta-r)\boldsymbol{R}}}{(1-e^{(\sigma-\lambda-r)\boldsymbol{R}})^\beta} \tag{5.11}$$

$$\boldsymbol{c_0} = w_0 \frac{r(\beta-1)+\lambda\beta+\delta}{\beta(\lambda+r-\sigma)} \left[1 - \left(\frac{w_0 \boldsymbol{c_0}^{-\beta}}{d}\right)^{\frac{\sigma-\lambda-r}{\lambda-\delta-\sigma+r}}\right] \tag{5.12}$$

Unfortunately, these are 2 implicit functions of R and c_0 which we cannot solve explicitly for an arbitrary β. This is why we need to look for another solution to derive some results.

5.5 A special case

Special case of $\delta = r = \sigma = 0$

With $\delta = r = \sigma = 0$, (5.11) and (5.12) simplify to:

$$d \cdot e^{\lambda R} = w_0^{1-\beta} \left[1 - e^{-\lambda R}\right]^{-\beta} \text{ or } \lambda R = \frac{R}{Z} = f_1(w_0, d, \beta)$$

This equation shows us that the optimal retirement age is a fixed proportion of life expectancy because w_0, d and β are fixed exogenously, which gives $\frac{R}{Z} = const.$

$$c_0 = w_0 \left[1 - \frac{d}{w_0 c_0^{-\beta}}\right] \text{ or } c_0 = f_2(w_0, d, \beta)$$

This equation reveals that the optimal consumption level is independent of the life expectancy because w_0, d and β are fixed exogenously, which gives $c_0 = const.$ Additionally, $r = \delta = 0$ results in constant consumption over time.

Very special case of $\delta = r = \sigma = 0$ and $\beta = 2$

For $\delta = r = \sigma = 0$ and $\beta = 2$ there is an explicit solution of (5.11) and (5.12):

$$R = log \left[\frac{1 + 2dw_0 + \sqrt{1 + 4dw_0}}{2dw_0}\right] Z$$

$$c_0 = \frac{\sqrt{1 + 4dw_0} - 1}{2d}$$

As there is now an explicit solution for a very special case, the idea for δ, r and $\sigma \neq 0$ is to derive an approximation by linearizing around the solution for R and c_0 around the point $\delta = r = \sigma = 0$.

Approximation for $\beta = 2$ around the point $\delta = r = \sigma = 0$

If r, δ and σ are small and $\beta = 2$, we can derive the following approximation by using the implicit function theorem and linearizing around the solution for R and c_0 around the point $\delta = r = \sigma = 0$:

$$
\begin{aligned}
R =& log\left[\frac{1+2dw_0+\sqrt{1+4dw_0}}{2dw_0}\right] Z + \left(log\left[\frac{1+2dw_0+\sqrt{1+4dw_0}}{2dw_0}\right] - \frac{2}{\sqrt{1+4dw_0}}\right) Z^2\sigma + \\
&+ \left(\frac{1}{\sqrt{1+4dw_0}} + log(2) - log\left[\frac{1+2dw_0+\sqrt{1+4dw_0}}{dw_0}\right]\right) Z^2 r + \\
&+ \frac{log\left[\frac{1+2dw_0+\sqrt{1+4dw_0}}{2dw_0}\right] - 1}{\sqrt{1+4dw_0}} Z^2\delta
\end{aligned}
$$

$$
\begin{aligned}
c_0 =& \frac{\sqrt{1+4dw_0}-1}{2d} + \frac{\sqrt{1+4dw_0}-1}{4d\sqrt{1+4dw_0}} Z(2\sigma - r) + \\
&+ \frac{\sqrt{1+4dw_0} - 1 + (\sqrt{1+4dw_0}-1)^2 log\left[\frac{4dw_0}{(\sqrt{1+4dw_0}-1)^2}\right]}{4d\sqrt{1+4dw_0}} Z\delta
\end{aligned}
$$

With this approximation for the problem, we are now able to derive some results.

Results for $\beta = 2$ and the approximation

The results derived from the approximations for parameter changes are captured in table 5.1.

Table 5.1: Results for $\beta = 2$ and the approximation (Bloom et al., 2004, page 18)

Parameter increased	Retirement age	Initial consumption
Interest rate r	Falls	Falls
Rate of time preference δ	Falls if $R < Z$ when $r = \delta = \sigma = 0$	Rises
Wage growth rate σ	Rises	Rises
Disutility of labor d	Falls	Falls
Initial wage level w_0	Falls	Rises but less than proportionately
Life expectancy Z	Rises but less than proportionately if $r = \delta$ and if $R < Z$ when $r = \delta = \sigma = 0$	Rises if $r = \delta$

Especially for life expectancy Z we see that, under certain assumptions, the retirement age rises less than proportionately according to Z, which means a more than proportional increase in time spent in retirement. For example, if $Z_{initial} = 60$ $years$, $R_{initial} = 50$ $years$ and $Z_{new} = 90$ $years$, the proportionality result would be $R^{prop}_{new} = (50/60)\cdot 90 = 75$ $years$. But if the optimal retirement age results in $R^{opt}_{new} = 70$ $years$, we get that the agent doubles the time spent in retirement (20 years instead of 10 years) if the life expectancy increases by 50% (proportionately it should be $10 \cdot 1.5 = 15$ years in retirement).

Furthermore there is an increase in initial consumption, which means a higher consumption at each age t. The reason which makes both more consumption and a longer time spent in retirement possible is the wealth effect, which results in an increase in lifetime expected utility.

Effect on the saving rate

If life expectancy Z rises (or $\lambda = 1/Z$ falls), we have a higher initial consumption c_0 which results in a higher consumption at each age t. With a lower λ the effective interest rate $(\lambda + r)$ on annuities fall, so we have a lower total return on savings which makes saving less attractive.

As consumption rises and total income (including the return on wealth) falls, we get a reduction of the observed saving rate of an agent at each age t conditional on working at age t.

So, we conclude that a longer life expectancy generates a lower saving rate at each age t (but the agent will work a longer lifetime).

Main effects of increased life expectancy

If we compare an agent 1 with a given life expectancy Z and an agent 2 with a higher life expectancy who works the same proportion of life, agent 2 would have a higher than proportional level of accumulated assets because of the wealth effect. To maximize his or her lifetime expected utility, agent 2 would smooth consumption and adjust his retirement age, which means that agent 2 would choose a higher level of initial consumption and an earlier retirement than the proportionality result.

So, the proportion of lifetime spent in employment falls, but the main effect of increased health and longevity is an increase in the lifetime spent working.

5.6 Conclusion

In the presented model there are 3 major influences for the retirement age and the saving rate:

- Higher levels of lifetime income result in an earlier retirement and a higher saving rate
- Lower disutility of working results in longer working lives and a lower saving rate
- Longer life expectancies and healthier lives lead to longer working lives and a lower saving rate

Depending on the strength of these influences, the working life and saving rate either rises or falls.

Shortcomings

Due to an easier handling, some simplifications were implemented in the model. A more complete and realistic model should respect the points listed below.

- Lack of foresight to save for retirement (assuming a non-rational agent)
- Time inconsistency in preferences (assuming a non-rational agent)
- No perfect capital market
- Possible negative externalities if people suffer an impoverished old age (a possible redistributional element)
- Welfare gains from intergenerational (or institutionalized) transfers
- An accurate health measurement to treat health and longevity as two separate variables.

6 Longevity and life-cycle savings (Bloom et al. 2003)

In this section I will present the model published by Bloom et al., see [BCG03]. It is based on a life-cycle model which accounts for health and longevity, but abstracts from bequests and family structures. In this model an increase in longevity alone tends to increase the relative length of retirement, which generates higher saving rates in working life. A higher health level alone can postpone retirement and therefore results in a longer working life. The overall effect of increasing longevity and a higher health level is ambiguous.

Increases in life expectancy may increase the saving rate at each age, but the effect on aggregate savings is only temporary. An increase in longevity in a stationary population (stable age distribution and no population growth) gives an age structure with a higher proportion of old-age population, so that in the long run higher age-specific saving rates are accompanied by a greater number of retirees who are dissaving. However, the long run can be even more than 50 years.

6.1 Motivation

If we regard savings in the life-cycle model, we see that agents save in young age to finance consumption in old age. So, in theory and without bequests, the amount of dissaving of the old should be equal to the amount of savings of the young, which means that in a stationary population we have no change in the aggregate capital stock. If this is not the case, there can be aggregate savings or dissavings, as with population growth or youth dependency (the presence of children increases the consumption requirements of young families).

In this section we will have a closer look at the influence of changes in longevity on aggregate savings.

An increase in longevity is assumed to be accompanied by reduced morbidity. However it would be interesting to look at the effect of increased longevity and reduced morbidity separately. Unfortunately we don't have comprehensive data on ill health, so we use life expectancy as a general health measure which accounts for both. The effect of an increase in longevity without changing morbidity on the saving rate clearly is positive because the agent will, in general, be unable to work much longer (unchanged morbidity) and therefore needs to finance a longer retirement (increased longevity). If an increase in longevity is accompanied by a reduction in morbidity, the overall effect on the saving rate is ambiguous.

6.2 The model

Assumptions

The assumptions of the model are:

- Every agent maximizes his or her lifetime utility
- The family structure has no effect on consumption and savings
- No bequests
- Lifetime T is known: it is independent of consumption, spending on health care and the timing of death (which could be important for the savings behavior)
- Every agent can choose his or her retirement age
- Exogenously fixed variables: the time path of health h_t, the time path of wages w_t, the interest rate r and the initial stock of wealth W_0.

Equations

Every agent maximizes lifetime utility given by

$$\int_0^T e^{-\delta t} \cdot U(c_t, l_t, h_t) dt \tag{6.1}$$

where δ denotes the subjective rate of time preference, t denotes age, U denotes the instantaneous utility function, c denotes consumption, l denotes leisure and h denotes health. $e^{-\delta t}$ represents the discount factor on future utility. The control variables to choose are the consumption path, c_t, and leisure time, l_t.

Every agent is faced by the constraints

$$c_t \geq 0, \quad 1 \geq l_t \geq 0, \quad W_T \geq 0 \tag{6.2}$$

which mean that consumption is never negative, leisure time is somewhere between 0 and 100% of the available time for working which is normalized to 1, and the stock of wealth W at the end of life is never negative.

The stock of wealth at age t, W_t, evolves according to

$$\frac{dW_t}{dt} = rW_t + (1 - l_t)w_t - c_t \tag{6.3}$$

Here we see that the change of wealth (the savings) is given by the interest r on existing wealth W_t, plus the working time $(1 - l_t)$ times the wage rate w_t less consumption c_t.

6.3 Optimal path of consumption and leisure

Let's assume the instantaneous utility function U as increasing and strictly concave in each argument. The Hamiltonian of the optimization problem is

$$H = e^{-\delta t} \cdot U(c_t, l_t, h_t) + \phi[rW_t + (1 - l_t)w_t - c_t]$$

The costate equation yields

$$\dot{\phi} = -\frac{\partial H}{\partial W_t} = -r \cdot \phi \tag{6.4}$$

The optimality conditions yield

$$\frac{\partial H}{\partial c_t} = e^{-\delta t} \cdot \frac{\partial U}{\partial c_t} - \phi = 0 \tag{6.5}$$

$$\frac{\partial H}{\partial l_t} = e^{-\delta t} \cdot \frac{\partial U}{\partial l_t} - w_t \cdot \phi = 0 \quad \text{for } 0 \leq l_t \leq 1 \tag{6.6}$$

(6.5) and (6.6) immediately give

$$w_t \frac{\partial U}{\partial c_t} = \frac{\partial U}{\partial l_t} \quad \text{for } 0 \leq l_t \leq 1$$

Therefore, the optimal path of consumption and leisure satisfies (without the subscripts t):

$$\frac{\partial U}{\partial l} = w_t \frac{\partial U}{\partial c} \text{ if } 0 < l < 1, \quad \frac{\partial U}{\partial l} \geq w_t \frac{\partial U}{\partial c} \text{ if } l = 1, \quad \frac{\partial U}{\partial l} \leq w_t \frac{\partial U}{\partial c} \text{ if } l = 0$$

Here we see that agents work until their marginal utility of extra leisure $\frac{\partial U}{\partial l}$ equals the marginal utility of extra consumption they can afford to buy when they work, $w_t \frac{\partial U}{\partial c}$. Retirement ($l = 1$) happens when the marginal utility of extra leisure $\frac{\partial U}{\partial l}$ always exceeds the marginal utility of extra consumption $w_t \frac{\partial U}{\partial c}$, even if the agent doesn't work at all. We will denote the optimal plan by (c_t^*, l_t^*), which also determines wealth holdings W_t^* over time.

6.4 Results

We will make two further assumptions:

Assumption 1

Initial wealth $W_0 = 0$ and the optimal path $W_t^* \geq 0$ for all $0 \leq t \leq T$ for any T. This means that consumption cannot be financed through debts. Agents have to save to be able to dissave later on.

Assumption 2

Consumption and leisure are both normal goods:

$$\frac{\partial c_t^*}{\partial W_0} \geq 0, \quad \frac{\partial l_t^*}{\partial W_0} \geq 0 \qquad \text{for all } 0 \leq t \leq T$$

This assumption implies that increasing initial wealth, W_0, increases both c_t^* and l_t^* at all times.

If we define the saving rate s_t as

$$s_t = \frac{y_t - c_t}{y_t} \quad \text{where } y_t = rW_t + (1 - l_t)w_t \tag{6.7}$$

then we can state the following proposition:

Proposition 6.1 *Let $W_0 = 0$. With assumptions 1 and 2, an increase in longevity increases the saving rate at every age, if we keep everything else the same.*

Proof We shall denote (c_t^*, l_t^*) as the initial optimal plan for life expectancy T, and (c_t^{**}, l_t^{**}) as the new optimal plan at a rising life expectancy. With the use of assumption 1, we can conclude that wealth at time T in the new optimal plan is bigger or equal to 0, $W_T^{**} \geq 0$. If we retain the new optimal plan after T and focus on the time interval [0, T], we see that the new optimal plan cannot be improved in this interval as it maximizes

$$\int_0^T e^{-\delta t} \cdot U(\widetilde{c}_t, \widetilde{l}_t, h_t)dt$$

subject to

$$\widetilde{c}_t \geq 0, \quad 1 \geq \widetilde{l}_t \geq 0, \quad \widetilde{W}_T \geq W_T^{**}$$

With stating the latter optimization problem, it is clear that in the optimum the wealth at the end of life, $\widetilde{W}_T$, will be as little as possible. Any residual wealth has no direct utility, but could be used for extra consumption which would increase total utility. Therefore the optimal plan maximizes the latter optimization problem with the constraint $\widetilde{W}_T \geq W_T^{**}$.

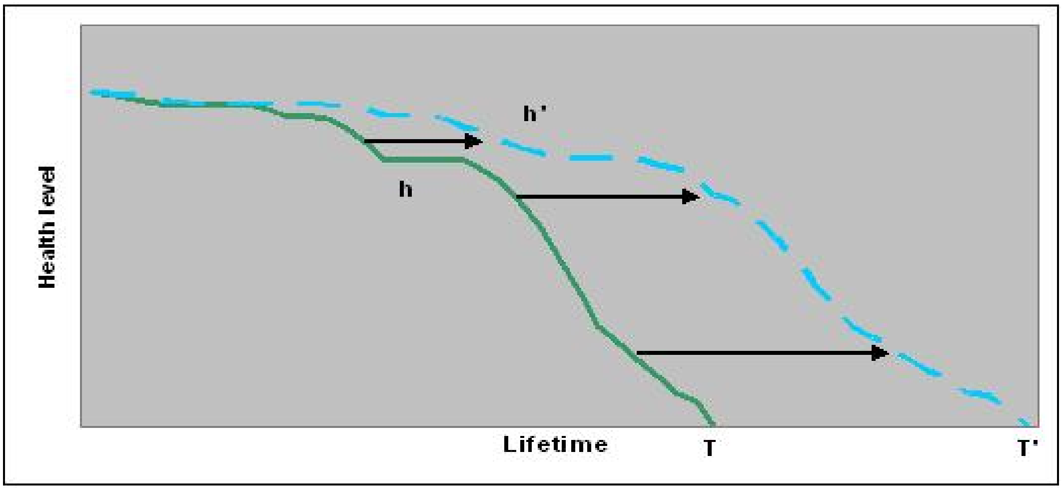
The latter optimization problem can be transformed to an equivalent one by maximizing over [0, T] subject to

$$c_t \geq 0, \quad 1 \geq l_t \geq 0, \quad W_T \geq 0 \text{ but with } W_0 = -W_T^{**} \cdot e^{-rT}$$

Here, final minimum wealth W_T^{**} is discounted to initial negative wealth W_0 which retains the equivalency. So, we have the original problem of maximizing over [0, T], with the only difference of lower initial wealth. By assumption 2 it follows that on [0, T] we have $c_t^{**} \leq c_t^*$ and $l_t^{**} \leq l_t^*$. With (6.7) we see that $s_0^{**} \geq s_0^*$. At each point in time, consumption is lower. With (6.3) we see that wealth rises more quickly under the new optimal plan and interest payments on the higher wealth boost the difference. So we see that at every time t income under the new plan is higher and consumption is lower. Therefore we have under the new plan a higher saving rate. □

The proposition only assumes an increase in longevity, and does not say anything about the health status. In practice, an increase in longevity will be linked closely to a better overall health status, and empirical regressions will count both effects as one, if there is no control for morbidity separately.

Let's now suppose a proportional increase in longevity from T to $T' = \lambda T$ with λ being constant, and also a proportionately improving health status which implies a delay of the effects of aging at work, which means productivity, by supposing $h'_{\lambda t} = h_t$ and $w'_{\lambda t} = w_t$. The effect of a λ bigger than 1 for a given h_t is shown in the following diagram:



With these assumptions, an agent maximizes

$$\int_0^{T'} e^{-\delta t} \cdot U(c_t, l_t, h'_t) dt$$

subject to

$$c_t \geq 0, \quad 1 \geq l_t \geq 0, \quad W_{T'} \geq 0, \quad \frac{dW_t}{dt} = rW_t + (1 - l_t) w'_t - c_t$$

With a variable substitution, $z = \frac{t}{\lambda}$, and defining $K_z = \frac{W_t}{\lambda}$, we instantly get

$$\int_0^{T'} e^{-\delta t} \cdot U(c_t, l_t, h'_t) dt = \int_0^{T} e^{-\delta \lambda z} \cdot U(c_{\lambda z}, l_{\lambda z}, h_z) \lambda dz = \lambda \int_0^{T} e^{-\delta \lambda z} \cdot U(\widetilde{c}_z, \widetilde{l}_z, h_z) dz$$

subject to

$$\widetilde{c}_z \geq 0, \quad 1 \geq \widetilde{l}_z \geq 0, \quad K_T \geq 0, \quad \frac{dK_z}{dz} = \frac{\frac{1}{\lambda} dW_t}{\frac{1}{\lambda} dt} = r\lambda K_z + (1 - \widetilde{l}_z) w_z - \widetilde{c}_z$$

where $\widetilde{c}_z$ and $\widetilde{l}_z$ are the new control variables in this optimization problem.

If $r = \delta = 0$, then the new maximization problem yields the same consumption and leisure path, because the problem is the same as in (6.1) to (6.3), apart from the multiplier λ which makes no difference for the optimization. This means that the optimal decision at time t in the original problem is the same in the new problem at time λt. This result is a proportionality result, which does not take into account the compound interest effect, because of $r = 0$, and the subjective preference rate, because of $\delta = 0$.

In general, a longer life can be associated with a greater or less than proportional increase in health at each age. The theory normally suggests for a longer life a higher saving rate, but it can get ambiguous when taking into account that it can be associated with a more than proportional increase in health and therefore with higher productivity and lower disutility of work.

6.5 Conclusion

Increasing longevity and a higher health level are likely to have a big influence on life-cycle behavior as people will have both a longer and healthier life. To obtain results on the influence of these demographic changes on savings, we need to understand whether an increasing longevity and a higher health level increases or decreases the length of the working life. An increasing longevity alone seems to increase the relative length of retirement, which generates higher saving rates in working life. With a proportional increase in longevity and health we see that in the case of $r = \delta = 0$ the saving rate stays unchanged. So a higher health level alone has the effect of decreasing the saving rate and therefore postponing retirement to smooth consumption. The higher productivity and lower disutility of work here result in a longer working life.

The overall effect of increasing longevity and a higher health level is ambiguous.

Driving forces

Different effects appear if longevity increases:

- As longevity rises, saving rates rise at every age to finance increased old-age consumption needs.
- An increased longevity is assumed to be accompanied with general health improvements at each age that may increase old-age productivity and wages, which represents an incentive to retire later (depending on the interest rate and the rate of time preference).
- Health improvements that increase longevity and also reduce morbidity may allow a large enough increase in the length of working life so that saving rates will fall.

Shortcomings of the model

The decision of how much to save is in practice not only due to the need of consumption smoothing. There can be other factors too:

- Credit constraints can have an impact on borrowing and aggregate savings
- Habit formation can increase savings during periods of rapid income growth

- Institutional arrangements: i.e. a mandatory retirement age will increase the saving rate if longevity increases; pension schemes can have a huge impact on savings behavior
- Availability of appropriate financial markets to invest all savings
- Likelihood of inflation because inflation can decrease the value of financial assets

The biggest point however is that the savings decision is maybe made on the household level, rather than from every individual itself. Therefore changes of longevity and in the demographic structure influence the decision through the family structure and not directly at the individual consumption smoothing level. This is especially important if we think of developing countries with large families, because our model does not take into account any intra-family or intra-household transfers, nor consumption needs of a potentially large number of children.

7 Conclusion

In this work the effects of a transformation in the age composition of a society on the saving rate is analyzed. Therefore the motives for people to save are reviewed and the motive to save for retirement is given priority.

In the first part of the thesis two standard aggregate saving models - the Solow model which incorporates savings exogenously and the Ramsey model which integrates savings endogenously - as well as selected empirical estimates on the relation between age structure and saving rates, are reviewed. As opposed to the Solow model, in the Ramsey model, endogenous savings eliminate the possibility of inefficient oversaving in the economy and the model is additionally able to explain steady state per capita growth. Empirical evidence indicates differences in the relation between age structure and savings when micro and macro level data is compared. These findings hint towards the existence of intergenerational relations among households that are not properly taken care of in micro level data.

Since the aggregate saving rate of an economy is based on individual decisions on savings, the focus of the master thesis is on individual life cycle models. Four different models from the literature on individual life cycle savings and longevity are reviewed. The aim is to better understand which model specification is the most adequate to replicate observed empirical patterns of the relation between life cycle savings and longevity.

7.1 Results derived from the models

The 4 chosen papers which are analyzed describe two overlapping generations models and two continuous time models. The approach of comparing 4 models on the same topic emphasizes the influence of the main model specifications. An assumption for all these models is that both the interest rate and the wage rate are exogenously given.

The two chosen overlapping generations models mainly differ in 3 specifications: if bequests are allowed or not, if educational investment into children is taken by parents individually or is the same for all children, and if there is an actuarially fair annuity market. In the case of an initial high mortality rate, both overlapping generations models from chapter 3 and 4 give the same result of an increasing saving rate if life expectancy increases. If the initial mortality rate is low, the effect of an increasing life expectancy on the saving rate is ambiguous: the model from chapter 3 still has as result an increasing saving rate, but the model from chapter 4 gives a decreasing saving rate. Since these two models differ in their behavior at an initial low mortality if life expectancy increases, it is to note that the model from chapter 4 seems to be more adapted to reality. It includes the fact that the vast majority of children are in public schools and furthermore considers bequests.

The two selected continuous time models mainly differ in 4 specifications: if life expectation is probabilistic or deterministic, how retirement is chosen, the existence of an actuarially fair annuity market, and the possibility of free leisure time to generate a trade-off between working and leisure. In the case of the continuous time models, the assumption of an endogenous retirement age is unrealistic, because in most countries there is a fixed retirement age. Nevertheless it is important to consider a freely chosen personal retirement age because it is interesting to compare it with a mandatory retirement age of a given economy. The model from chapter 5 gives at an increasing life expectancy a decreasing saving rate which is explained by the fact that the length of lifetime spent with working increases. This is opposed to the outcome of the model from chapter 6 where an increasing life expectancy increases the saving rate. Since these two models have a different behavior at an increasing life expectancy, it is to note that the more realistic model would be the model from chapter 5. It assumes a probabilistic life expectancy and the fact of a deterministic life expectancy can influence the results dramatically, as already explained in section 2.3.

In most industrial countries which nowadays face a change in the composition of its age structure because of an increasing life expectancy, the most probable change in the saving rate will be negative. This result is based on the assumptions of a fixed retirement age and an initial low mortality rate. On the other hand, if a country has a fixed retirement age and an initial high mortality rate, the result with highest probability will be an increasing saving rate. In the case of a freely chosen retirement age, the most probable result will be a falling saving rate and people who will work a longer lifetime.

7.2 Possible improvements

There are several opportunities to improve the existing framework to get a more realistic model:

- The most important improvement would be to allow agents to behave not completely rational. This approach would introduce behavioral economics with agents who are short-sighted and are not able to process economic information perfectly. Additionally, agents would use heuristics and rules of thumb to forecast the future and simply lack the necessary willpower to save the necessary amount in each point in time.
  So an agent could be modeled as forecasting at each time point t the future only with a rule of thumb and only until $t + h$, with h being the forecast horizon of the short-sighted agent.

- Given the fact that actual savings decisions are rather made by households than by individuals, it would be appealing to do the whole analysis on the household level instead of the individual level and to take into account intra-family and intra-household transfers.
  So the agent would represent a whole household in the analysis. The size and the

composition of the members of the households and inter-household links need to be considered.

- Another important point would be the inclusion of an institutionalized transfer system, for example a tax system, a pension system or an incentive system. Since in most industrialized countries the government pays annuities to people in old age, an integrated institutionalized transfer system would be appropriate.
  As the main reason for saving in the master thesis is to save for retirement, an institutionalized transfer system which pays annuities in retirement could largely influence the saving rate.

- The introduction of an accurate health measurement would allow treating health and longevity as two separate variables and therefore would give more insights about the role of the health level in life-cycle decisions.

List of Tables

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