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DISSERTATION

Valuations and the Dual L_p Brunn-Minkowski Theory

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Kurzfassung

Während der letzten Jahrzehnte entwickelten sich, ausgehend von der klassischen Brunn-Minkowski Theorie, neue Theorien über konvexe Körper und Sternkörper die mittlerweile wesentlicher Bestandteil der Konvexgeometrie sind.

Einerseits wurde die sogenannte L_p Brunn Minkowski Theorie entwickelt. Ausgangspunkt dafür war eine Erweiterung der Minkowski Addition konvexer Körper. Es wurde eine Vielzahl an klassischen Begriffen der Konvexgeometrie verallgemeinert und bemerkenswerte Analogien zwischen den Theorien bewiesen.

Andererseits gibt es duale Konzepte zu diesen Theorien. Diese behandeln Sternkörper an Stelle konvexer Körper und sind essentiell in geometrischer Tomographie. Der wichtigste Operator der dualen Brunn-Minkowski Theorie ist der Schnittkörperoperator. Letzterer war entscheidend für die Lösung des Busemann-Petty Problems und ist Gegenstand aktueller Forschung. Beispielsweise fragt die "Slicing Conjecture", derzeit eines der wichtigsten ungelösten Probleme des Gebiets, nach gewissen Eigenschaften des Schnittkörpers.

In dieser Arbeit beantworten wir die Frage welcher Operator das L_p Analogon des Schnittkörperoperators ist.

Angeregt durch Ludwigs Charakterisierung des Schnittkörperoperators [39], klassifizieren wir radiale L_p Bewertungen auf konvexen Polytopen. Dies ist Bestandteil des zweiten Kapitels. Das Resultat ist [24] entnommen. Die nichttrivialen Beispiele solcher Bewertungen können als L_p Schnittkörperoperatoren angesehen werden. Überraschenderweise muß man zwischen symmetrischen und unsymmetrischen L_p Schnittkörpern unterscheiden. Im symmetrischen Fall sind die Operatoren die bekannten polaren L_{-p} Schwerpunktkörper. Im Allgemeinen erhalten wir allerdings Abbildungen die mit der verallgemeinerten Minkowski-Funk Transformation in Zusammenhang stehen.

Im dritten Kapitel beweisen wir Beziehungen zwischen Schnittkörpern und deren L_p Analoga. Zum Beispiel führen gewisse Ungleichungen über L_p Schnittkörper zu einer äquivalenten Formulierung der Slicing Conjecture. Wir zeigen weiters, daß Schnittkörper eines konvexen Körpers gleichmäßig durch symmetrische sowie unsymmetrische L_p Schnittkörper approximiert werden können. Außerdem beweisen wir Analogien zwischen klassischen Schnittkörpern und deren L_p Gegenstücken, erhalten aber auch interessante Unterschiede im unsymmetrischen Fall. Beispielsweise sind unsymmetrische L_p Schnittkörper injektiv auf Sternkörpern in allen Dimensionen. Dies steht im Gegensatz zum klassischen Schnittkörper: Nur symmetrische Sternkörper sind eindeutig durch diesen bestimmt.

Die im zweiten Kapitel behandelten radialen L_p Bewertungen machen nur für positives p Sinn. L_p Schnittkörper, im Wesentlichen die einzigen Beispiele solcher Bewertungen, sind auch für negatives p definiert. Kann man Letztere auch für negatives p charakterisieren? Daß dies möglich ist, zeigen wir im vierten Kapitel.

Der letzte Teil dieser Arbeit behandelt spezielle Bewertungen der L_p Brunn-Minkowski Theorie. Wir beweisen eine Charakterisierung der L_p mittleren Breite.

Abstract

Over the last decades, starting from the classical Brunn-Minkowski theory, new theories concerning convex bodies and star bodies emerged and became an essential part of convex geometry.

On the one hand, the so called L_p Brunn-Minkowski theory was developed. The heart of this concept is an extension of Minkowski addition for convex bodies. A huge number of notions from classical convex geometry could be extended to the L_p case and striking analogies between the old and new theory were established.

On the other hand, duals of these two theories exist. They treat star bodies instead of convex bodies and are fundamental in geometric tomography. The most important operator within the dual Brunn-Minkowski theory is the intersection body operator. Beside being the key for the solution of the classical Busemann-Petty problem, it is subject of recent research. For example the slicing conjecture, probably the most famous open problem in the field, asks for certain properties of intersection bodies.

In this thesis, we answer the question which operator is the L_p analogue of the intersection body operator. The work is organized as follows.

Motivated by Ludwig's characterization of the intersection body operator [39], we characterize radial L_p valuations on convex polytopes. This is done in Chapter 2. The result is taken from [24]. The nontrivial examples of such valuations can be viewed as the L_p intersection body operators. Surprisingly it turns out that we have to distinguish between symmetric and nonsymmetric L_p intersection bodies. In the symmetric setting, the operators are the well known polar L_{-p} centroid bodies. But in general, we obtain operators related to generalized Minkowski-Funk transforms.

In Chapter 3, we prove relations between intersection bodies and their L_p analogues. For example, certain inequalities yield an equivalent formulation of the slicing conjecture in terms of L_p intersection bodies. We also prove that every intersection body of a convex body can be uniformly approximated by symmetric as well as nonsymmetric L_p intersection bodies. Further, we establish analogies between classical intersection bodies and their L_p counterparts but obtain interesting differences in the nonsymmetric case. For example, the nonsymmetric L_p intersection body operator is injective on star bodies in all dimensions. This is in contrast to the original context: Only symmetric star bodies are uniquely determined by their intersection bodies.

Radial L_p valuations as studied in Chapter 2 make sense only for positive p. But L_p intersection bodies, basically the only examples of such operators, are defined for negative values of p, too. Can we characterize them also for negative p? The answer is affirmative as is shown in Chapter 4. It contains a classification of L_p intersection bodies for all values of $p \neq 0$.

The last part of this work is devoted to special valuations in the L_p Brunn-Minkowski theory. We establish a characterization of the L_p mean width.

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Contents

Chapter 1

Introduction

The celebrated Busemann-Petty problem [9], posed in 1956, asked the following question concerning *convex bodies* (i.e. nonempty, compact, convex subsets of Euclidean n-space $\mathbb{R}^n$): Is it true, that for two origin-symmetric, n-dimensional convex bodies K, L with the property that the $(n-1)$-dimensional volumes of all hyperplane sections through the origin of K are less than the corresponding ones of L, the volume of K is also smaller than the volume of L? (A convex body K is called *origin-symmetric* if $K = -K$.) About fourty years later, the problem was finally solved by Gardner [13] and Zhang [62] who presented a solution in dimensions three and four, respectively. Using Fourier analytic tools, Gardner, Koldobsky and Schlumprecht [16] gave a unified solution for all dimensions. The answer to the Busemann-Petty problem turned out to be affirmative for $n \leq 4$ and negative for $n > 4$. For a detailed description of the history of the solution we refer to the introduction in [33]. The crucial point in the solution was Lutwak's concept of intersection bodies and the corresponding intersection body operator. To define the latter, we need a bit of notation.

Let $L \subset \mathbb{R}^n$ be a nonempty compact set which is star-shaped with respect to the origin. The *radial function* $\rho(L, \cdot)$ is a real valued function on the unit sphere S^{n-1} which is defined by

$$\rho(L, u) = \max\{r \geq 0 : r\,u \in L\}, \quad u \in S^{n-1}.$$

A *star body* is a nonempty compact set which is star-shaped with respect to the origin and has a continuous radial function. We write $\mathcal{S}^n$ for the set of star bodies in $\mathbb{R}^n$. The *intersection body*, $\mathrm{I}\,L$, of a star body $L \in \mathcal{S}^n$ is the star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of L by $u^{\perp}$, the hyperplane orthogonal to u. So, for $u \in S^{n-1}$,

$$\rho(\mathrm{I}\,L, u) = \mathrm{vol}(L \cap u^{\perp}),$$

where vol denotes $(n-1)$-dimensional volume.

Intersection bodies which arise from centrally symmetric convex bodies first appeared in Busemann [7]. They are important in the theory of area in Finsler spaces. Intersection bodies of star bodies were defined and named by Lutwak [42]. The class of intersection bodies is fundamental in geometric tomography (see e.g. [14]), in affine isoperimetric inequalities (see e.g. [34], [58]) and the geometry of Banach spaces (see e.g. [33], [60]). Note that the Busemann-Petty problem can now be rephrased as follows: Is the implication

$$\mathrm{I}\, K \subset \mathrm{I}\, L \Longrightarrow V(K) \leq V(L)$$

true for arbitrary origin symmetric convex bodies? Here, V stands for volume. This is a first hint at a connection between intersection bodies and the Busemann-Petty problem.

Valuations allow us to obtain characterizations of many important functionals and operators on convex sets by their invariance or covariance properties with respect to suitable groups of transformations (see [25], [31], [49], [50] for information on the classical theory and [1]–[4], [29], [30], [35]–[38], [40], [59] for some of the recent results). For example, Ludwig [39] characterized intersection bodies as $\mathrm{GL}(n)$ covariant valuations. To state this result, we need some additional definitions. A function $\mathrm{Z} : \mathcal{L} \to \langle \mathcal{G}, + \rangle$, where $\mathcal{L}$ is a class of subsets of $\mathbb{R}^n$ and $\langle \mathcal{G}, + \rangle$ is an abelian semigroup, is called a *valuation* if

$$\mathrm{Z}(K \cup L) + \mathrm{Z}(K \cap L) = \mathrm{Z}\, K + \mathrm{Z}\, L,$$

whenever $K, L, K \cup L, K \cap L \in \mathcal{L}$.

Let $\mathcal{P}_0^n$ denote the set of convex polytopes in $\mathbb{R}^n$ that contain the origin in their interiors and let

$$P^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for every } y \in P\}$$

denote the polar body of $P \in \mathcal{P}_0^n$. For $p > 0$, the *L_p-radial sum* $K \mathbin{\tilde{+}_p} L$ of $K, L \in \mathcal{S}^n$ is defined by

$$\rho(K \mathbin{\tilde{+}_p} L, \cdot)^p = \rho(K, \cdot)^p + \rho(L, \cdot)^p.$$

An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ is called *trivial*, if it is a linear combination with respect to $\tilde{+}_p$ of the identity and central reflection. An operator Z is called $\mathrm{GL}(n)$ *covariant of weight* q, $q \in \mathbb{R}$, if for all $\phi \in \mathrm{GL}(n)$ and all bodies Q,

$$\mathrm{Z}(\phi Q) = |\det \phi|^q\, \phi\, \mathrm{Z}\, Q,$$

where $\det \phi$ is the determinant of ϕ. An operator Z is called $\mathrm{GL}(n)$ *covariant*, if Z is $\mathrm{GL}(n)$ covariant of weight q for some $q \in \mathbb{R}$.

Theorem ([39]). *An operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_1 \rangle$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\, \mathrm{I}\, P^*$$

for every $P \in \mathcal{P}_0^n$.

With these considerations in mind, it is natural to ask for a classification result for covariant operators $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ with $p \neq 1$. The solution to this problem is presented in Section 2. From a valuation theoretic point of view, the nontrivial examples of such valuations are therefore the intersection body operators within the dual L_p Brunn-Minkowski theory. To make this even more evident, we will have a glance at the environment of this classification result in the next paragraphs.

Schneider [58] describes the classical Brunn-Minkowski theory in the preface of his monograph as follows:

> Aiming at a brief characterization of Brunn-Minkowski theory, one might say that it is the result of merging two elementary notions for point sets in Euclidean space: vector addition and volume.

Vector addition can be desribed for convex bodies in terms of support functions. So let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, the *support function* $h(K, \cdot)$ of K is defined by

$$h(K, u) = \max\{x \cdot u \,|\, x \in K\} \quad \text{for } \ u \in \mathbb{R}^n,$$

where $x \cdot u$ stands for usual inner product in $\mathbb{R}^n$ of the vectors x and u. The vector or Minkowski sum $K + L$ of two convex bodies K, $L \in \mathcal{K}^n$ is the convex body with support function

$$h(K + L, \cdot) = h(K, \cdot) + h(L, \cdot).$$

Firey [11] extended this definition and introduced L_p Minkowski addition on $\overline{\mathcal{K}_0^n}$, i.e. the set of convex bodies containing the origin. For $p \geq 1$, the L_p Minkowski sum $K +_p L$ of two convex bodies K, $L \in \overline{\mathcal{K}_0^n}$ is the convex body with support function

$$h(K +_p L, \cdot)^p = h(K, \cdot)^p + h(L, \cdot)^p.$$

In analogy to the original context, Lutwak [45] explored relations between L_p Minkowski addition and volume. This can be seen as the starting point of the L_p Brunn-Minkowski theory which became an essential part of convex geometry. Over the last decades, a dual of the L_p Brunn-Minkowski theory emerged. In this dual theory, roughly speaking, convex bodies are replaced by star bodies, L_p Minkowski addition corresponds to L_p radial addition, and projections are substituted by sections. So the dual notion of the intersection body of a star body should be defined as the convex body with support function

$$h(\Pi K, u) = \mathrm{vol}(K |\, u^\perp), \quad u \in S^{n-1}, \quad K \in \mathcal{K}^n,$$

where $K|u^\perp$ is the image of the orthogonal projection of K orthogonal to u. This body is called *projection body* of K and is a classical notion with fundamental impact on convex geometry and geometric tomography (see, e.g. [58], [14]). The duality of projection bodies and intersection bodies was discovered by Lutwak [42] within his studies of the Busemann-Petty problem. If we remember Ludwig's characterization of intersection bodies and use the duality explained before, we expect the projection body to be essentially the only example of a valuation Z with values in $\langle \mathcal{K}^n, + \rangle$ and the property $\mathrm{Z}\,\phi P = |\det \phi|^q \, \phi^{-t}\, \mathrm{Z}\, Q$ for all $\phi \in \mathrm{GL}(n)$ and some $q \in \mathbb{R}$. Indeed, this holds true as shown by Ludwig [38]. Moreover, the classification of such valuations which take values in $\langle \mathcal{K}^n, +_p \rangle$, $p > 1$ was established in this paper, too. The more or less only example is the L_p analogue of the projection body operator. This operator was introduced by Lutwak, Yang and Zhang in [46] where they obtained an L_p version of the Petty projection inequality. With this result in mind and since duality fits well with the characterizations in the case $p = 1$, it is reasonable to call the only nontrivial covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ L_p intersection bodies.

Chapter 2

A Characterization of L_p Intersection Bodies

Our main result is the announced characterization theorem for valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$. A complete answer for the planar case is given in Theorem 3 in Section 2.1.5. For $n \geq 3$, we obtain the following result.

Theorem 1. *For $0 < p < 1$, an operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there are constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_p^+ P^* \,\tilde{+}_p\, c_2 \, \mathrm{I}_p^- P^*$$

for every $P \in \mathcal{P}_0^n$. For $p > 1$, all $\mathrm{GL}(n)$ covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ are trivial.

Here, for $P \in \mathcal{P}_0^n$, the star body $\mathrm{I}_p^+ P$ is defined by

$$\rho(\mathrm{I}_p^+ P, u)^p = \frac{1}{\Gamma(1-p)} \int_{P \cap u^+} |u \cdot x|^{-p} dx, \quad u \in S^{n-1}$$

where $u^+ = \{x \in \mathbb{R}^n : u \cdot x \geq 0\}$ and Γ denotes the Gamma function. We define $\mathrm{I}_p^- P = \mathrm{I}_p^+(-P)$. A change into polar coordinates proves

$$\rho(\mathrm{I}_p^+ P, v)^p = \frac{1}{(n-p)\Gamma(1-p)} \int_{S^{n-1} \cap v^+} |u \cdot v|^{-p} \rho(P, u)^{n-p} \, du, \quad \text{for } v \in S^{n-1},$$

where integration is with respect to spherical Lebesgue measure. Therefore $\rho(\mathrm{I}_p^+ P, \cdot)^p$ is just a multiple of the generalized Minkowski Funk transform M_0^{1-p} of the function $\rho(P, \cdot)^{n-p}$ as introduced by Rubin [57]. The latter is defined as

$$(M_t^\alpha) f(v) = \frac{c_{n,\alpha}}{(1-t^2)^{\alpha-1+n/2}} \int_{u \cdot v > t} (v \cdot u - t)^{\alpha-1} f(u) \, du$$

for suitable functions f on the sphere, where $\alpha > 0$, $t \in (-1, 1)$, and $c_{n,\alpha}$ is a constant depending on the dimension n and α. The operator I_p^+ is also closely related to fractional derivatives and the L_p cosine transform. This connection will be discussed in more detail in Chapter 3. We remark that the normalization in the definition of $\rho(\mathrm{I}_p^+ P, \cdot)$ is chosen such that the limit as p tends to one exists. Of course, this has no effect on our classification result.

As a consequence of Theorem 1, we obtain the following characterization of L_p intersection bodies. For $p < 1$, we call the centrally symmetric star body $\mathrm{I}_p P = \mathrm{I}_p^+ P \tilde{+}_p \mathrm{I}_p^- P$ the L_p *intersection body* of $P \in \mathcal{P}_0^n$. So, for $u \in S^{n-1}$,

$$\rho(\mathrm{I}_p P, u)^p = \int_P |u \cdot x|^{-p} dx. \tag{2.1}$$

We denote by $\mathcal{S}_c^n$ the set of centrally symmetric star bodies in $\mathbb{R}^n$ and classify $\mathrm{GL}(n)$ covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_c^n, \tilde{+}_p \rangle$. The planar case is contained in Theorem 4 in Section 2.1.5. For $n \geq 3$, we obtain the following result.

Theorem 2. *For $0 < p < 1$, an operator $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_c^n, \tilde{+}_p \rangle$ is a non-trivial $\mathrm{GL}(n)$ covariant valuation if and only if there is a constant $c \geq 0$ such that*

$$\mathrm{Z}\, P = c\, \mathrm{I}_p\, P^*$$

for every $P \in \mathcal{P}_0^n$. For $p > 1$, all $\mathrm{GL}(n)$ covariant valuations $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_c^n, \tilde{+}_p \rangle$ are trivial.

Up to normalization, $\mathrm{I}_p P$ equals the polar L_{-p} centroid body of P. Centroid bodies were introduced by Petty [53]. Lutwak and Zhang [48] extended this concept to L_q-centroid bodies for $q > 1$. Gardner and Giannopoulos [15] as well as Yaskin and Yaskina [61] investigated extensions of this notion also for $-1 < q < 1$. L_q centroid bodies themselves were investigated by many different authors (see e.g. [10], [28], [38], [43], [46], [48], [51], [61]). They are also extremely useful tools in different situations. Among others, Paouris [52] used them to prove results concerning concentration of mass for isotropic convex bodies, and Lutwak, Yang and Zhang [47] derived information theoretic inequalities from properties of L_q centroid bodies. Moreover, the concept of L_p centroid bodies leads to important affine isoperimetric inequalities (see e.g. [14]).

2.1 Proof of the Classification Result

We write $x = (x_1, x_2, \ldots, x_n)$ for vectors $x \in \mathbb{R}^n$. The standard basis in $\mathbb{R}^n$ will be denoted by $e_1, e_2, \ldots, e_n$. The norm $\|x\|$ is defined as usual by $\|x\| = \sqrt{x \cdot x}$. Given $A, A_1, A_2, \ldots, A_k \subset \mathbb{R}^n$, we write $[A_1, A_2, \ldots, A_k]$ for the convex hull of $A_1, A_2, \ldots, A_k$, we write $\operatorname{lin} A$ for the linear hull of A, and set $A^\perp = \{x \in \mathbb{R}^n : x \cdot y = 0 \text{ for all } y \in A\}$.

For $L \in \mathcal{S}^n$, we extend the radial function to a homogeneous function defined on $\mathbb{R}^n \backslash \{0\}$ by $\rho(L, x) = \|x\|^{-1} \rho(L, x/\|x\|)$. Then it follows immediately from the definition that

$$\rho(\phi L, x) = \rho(L, \phi^{-1} x), \qquad x \in \mathbb{R}^n \backslash \{0\} \tag{2.2}$$

for $\phi \in \mathrm{GL}(n)$.

We call a valuation $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ an *L_p radial valuation.* If $\mathrm{Z}\, P = \{0\}$ for every P having dimension less than n, Z is called *simple*. A valuation is $\mathrm{GL}(n)$ *contravariant of weight q*, $q \in \mathbb{R}$, if

$$\mathrm{Z}\, \phi P = |\det \phi|^q \phi^{-t}\, \mathrm{Z}\, P$$

for every $\phi \in \mathrm{GL}(n)$ and every $P \in \mathcal{P}_0^n$. Here ϕ^{-t} denotes the transpose of the inverse of ϕ. For $0 < p < 1$, the operators $\mathrm{I}_p^{\pm} : \mathcal{P}_0^n \to \mathcal{S}^n$ and $\mathrm{I}_p : \mathcal{P}_0^n \to \mathcal{S}^n$ are L_p radial valuations and $\mathrm{GL}(n)$ contravariant operators of weight $1/p$.

The following lemma guarantees that a classification of all L_p radial valuations which are $\mathrm{GL}(n)$ covariant with negative weight follows from a classification of all L_p radial valuations which are $\mathrm{GL}(n)$ contravariant with positive weight. Moreover, if we know all L_p radial valuations, $\mathrm{GL}(n)$ covariant of arbitrary weight, we know all $\mathrm{GL}(n)$ contravariant L_p radial valuations and vice versa.

Lemma 1. *Let* Z *be an L_p radial valuation and define another L_p radial valuation* Z^* *by* $\mathrm{Z}^*\, P = \mathrm{Z}\, P^*$ *for every* $P \in \mathcal{P}_0^n$*. Then* Z *is* $\mathrm{GL}(n)$ *covariant of weight q if and only if* Z^* *is* $\mathrm{GL}(n)$ *contravariant of weight $-q$.*

Proof. That Z^* satisfies the valuation property is a consequence of

$$(P \cup Q)^* = P^* \cap Q^*, \quad (P \cap Q)^* = P^* \cup Q^*$$

for polytopes $P, Q \in \mathcal{P}_0^n$ having convex union (see, for example, [58]). The statement of the lemma follows from the fact that $(\phi P)^* = \phi^{-t} P^*$ holds for every $P \in \mathcal{P}_0^n$ and every $\phi \in \mathrm{GL}(n)$. □

We first establish a classification of valuations which are $\mathrm{GL}(n)$ contravariant of weight $q > 0$ and then a classification of valuations which are $\mathrm{GL}(n)$ covariant of weight $q \geq 0$. By Lemma 1, combining these results gives a classification of $\mathrm{GL}(n)$ covariant valuations. The classification result for $n \geq 3$ is contained in Theorem 1. The result for $n = 2$ is stated in Section 2.1.5.

2.1.1 Extension

Given an L_p radial valuation Z, we define another valuation $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ by $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$. Here $\mathcal{C}_+(S^{n-1})$ is the set of non-negative continuous functions on the sphere. We want to extend this valuation to the set $\overline{\mathcal{P}}_0^n$ of convex polytopes which are either in $\mathcal{P}_0^n$ or are the intersection of a polytope in $\mathcal{P}_0^n$ and a polyhedral cone with at most n facets having its apex at the origin. The following preparations will show when such extensions exist. For $1 \leq j \leq n$, let $\overline{\mathcal{P}}_j^n$ denote the set of polytopes which are intersections of polytopes in $\mathcal{P}_0^n$ and j halfspaces bounded by hyperplanes $H_1, \dots, H_j$ containing the origin and having linearly independent normals. We need some more notation. For a hyperplane

$H \subset \mathbb{R}^n$, $\mathcal{P}_0^n(H)$ is the set of convex polytopes in H containing the origin in their interiors relative to H. Let $\overline{\mathcal{C}}_+(S^{n-1})$ denote the superset of $\mathcal{C}_+(S^{n-1})$ consisting of all non-negative functions defined almost everywhere (with respect to spherical Lebesgue measure) on S^{n-1} which are continuous almost everywhere. We write H^+, H^- for the closed halfspaces bounded by H.

For $P \in \mathcal{P}_0^n(H)$ and $A \subset S^{n-1}$, we say that $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ *vanishes on* A *at* P if for $u \in H^-\backslash H$, $v \in H^+\backslash H$ and every $w \in A$, there exists a neighbourhood $A(w)$ of w such that

$$\lim_{u,v \to 0} \mathrm{Y}[P,u,v] = 0 \quad \text{uniformly on } A(w)$$

holds. If there exists a constant $c \in \mathbb{R}$ such that $\mathrm{Y}[P,u,v] \le c$ for $\|u\|, \|v\| \le 1$, $u \in H^-\backslash H$, $v \in H^+\backslash H$ and $[P,u,v] = [P,u] \cup [P,v]$, then we say that $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ is *bounded at* P.

Now we are able to formulate the following lemma proved in [39].

Lemma 2. *Let* $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ *be a valuation.*

1. *If* Y *vanishes on* S^{n-1} *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \mathcal{C}_+(S^{n-1})$.

2. *If* Y *is bounded and vanishes on* $S^{n-1}\backslash H$ *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0^n$ *bounded by hyperplanes* $H_1, H_2, \dots, H_n$ *containing the origin,* $\overline{\mathrm{Y}}\, P$ *is continuous and bounded on* $S^{n-1}\backslash(H_1 \cup \dots \cup H_n)$.

3. *If* Y *is bounded and vanishes on* $S^{n-1}\backslash H^\perp$ *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0^n$ *bounded by hyperplanes* $H_1, H_2, \dots, H_n$ *containing the origin,* $\overline{\mathrm{Y}}\, P$ *is continuous and bounded on* $S^{n-1}\backslash(H_1^\perp \cup \dots \cup H_n^\perp)$.

4. *If* Y *vanishes on* $S^{n-1}\backslash H^\perp$ *at* P *for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0^n(H)$*, then* Y *can be extended to a simple valuation* $\overline{\mathrm{Y}} : \overline{\mathcal{P}}_0^n \to \overline{\mathcal{C}}_+(S^{n-1})$ *and for* $P \in \overline{\mathcal{P}}_0^n$ *bounded by hyperplanes* $H_1, H_2, \dots, H_n$ *containing the origin,* $\overline{\mathrm{Y}}\, P$ *is continuous on* $S^{n-1}\backslash(H_1^\perp \cup \dots \cup H_n^\perp)$.

The extension is defined inductively for $j = 1, \dots, n$, and convex polytopes $P = P_0 \cap H_1^+ \cap \dots \cap H_j^+$ with $P_0 \in \mathcal{P}_0^n$ and hyperplanes having linearly independent normals: For $u \in H_1 \cap \dots \cap H_{j-1}$, $u \in H_j^-\backslash H$, set

$$\overline{\mathrm{Y}}\, P = \lim_{u \to 0} \overline{\mathrm{Y}}[P,u]$$

on S^{n-1}, $S^{n-1}\backslash(H_1 \cup \cdots \cup H_j)$ or $S^{n-1}\backslash(H_1^\perp \cup \cdots \cup H_j^\perp)$ if Y vanishes on S^{n-1}, $S^{n-1}\backslash H$ or $S^{n-1}\backslash H^\perp$, respectively.

The proof of the following lemma is omitted since it is nearly the same as the proof of Lemma 5 and Lemma 8 in [39].

Lemma 3. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \mathcal{S}^n$ *be an* L_p *radial valuation and define* $\mathrm{Y} : \mathcal{P}_0^n \to \mathcal{C}_+(S^{n-1})$ *by* $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$.

1. *If* Z *is* $\mathrm{GL}(n)$ *covariant of weight* q*, then for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0(H)$*, the following holds: If* $q = 0$*, then* Y *vanishes on* $S^{n-1}\backslash H$ *at* P *and if* $q > 0$*, then* Y *vanishes on* S^{n-1} *at* P*. In both cases,* Y *is bounded at* P*.*

2. *If* Z *is* $\mathrm{GL}(n)$ *contravariant of weight* q*, then for every hyperplane* H *containing the origin and every* $P \in \mathcal{P}_0(H)$*, the following holds: If* $q > 0$*, then* Y *vanishes on* $S^{n-1}\backslash H^\perp$ *at* P *and if* $q > 1$*, then* Y *vanishes on* S^{n-1} *at* P*. For* $q \geq 1$*,* Y *is bounded at* P*.*

Let Z be an L_p radial valuation which is $\mathrm{GL}(n)$ contravariant of weight q. For $q > 0$, Lemma 2 and Lemma 3 guarantee the existence of an extension of $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$ to $\overline{\mathcal{P}}_0^n$ for which we write $\overline{\mathrm{Y}}$. We extend these functions from S^{n-1} to $\mathbb{R}^n\backslash\{0\}$ by making them homogeneous of degree $-p$. From the definition of this extension it follows for $\phi \in \mathrm{GL}(n)$ and $P \in \overline{\mathcal{P}}_0^n$ bounded by hyperplanes $H_1, H_2, \ldots, H_j$ that

$$\overline{\mathrm{Y}}\, \phi P(x) = |\det \phi|^{pq}\, \overline{\mathrm{Y}}\, P(\phi^t x) \tag{2.3}$$

on $S^{n-1}\backslash \phi^{-t}(H_1^\perp \cup \cdots \cup H_j^\perp)$ for $0 < q \leq 1$ and on S^{n-1} for $q > 1$.

If Z is an L_p radial valuation which is $\mathrm{GL}(n)$ covariant of weight q, we proceed as above. For $q \geq 0$, Lemma 2 and Lemma 3 guarantee the existence of an extension of $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$ to $\overline{\mathcal{P}}_0^n$ for which we write $\overline{\mathrm{Y}}$ and which we extend from S^{n-1} to $\mathbb{R}^n\backslash\{0\}$ by making them homogeneous of degree $-p$. From the definition of this extension it follows for $\phi \in \mathrm{GL}(n)$ and $P \in \overline{\mathcal{P}}_0^n$ bounded by hyperplanes $H_1, H_2, \ldots, H_j$ that

$$\overline{\mathrm{Y}}\, \phi P(x) = |\det \phi|^{pq}\, \overline{\mathrm{Y}}\, P(\phi^{-1} x) \tag{2.4}$$

on $S^{n-1}\backslash \phi(H_1 \cup \cdots \cup H_j)$ for $q = 0$ and on S^{n-1} for $q > 0$.

Let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q > 0$ and let $\overline{\mathrm{Z}}$ denote its extension to $\overline{\mathcal{P}}_0^n$. Let $T^n = [0, e_1, \ldots, e_n]$ be the standard simplex in $\mathbb{R}^n$. First, we show that $\overline{\mathrm{Z}}$ is determined on $\overline{\mathcal{P}}_0^n$ by its value on T^n.

Since $\overline{\mathrm{Z}}$ is a simple valuation on $\overline{\mathcal{P}}_0^n$, it suffices to show the statement for a polytope $P \in \overline{\mathcal{P}}_0^n$ contained in a simplicial cone C bounded by n hyperplanes containing the origin and with linearly independent normal vectors. We dissect

$P = T_1 \cup \cdots \cup T_k$, where $T_i \in \overline{\mathcal{P}}_0^n$ are n-dimensional simplices with pairwise disjoint interiors. Let H be a suitable affine hyperplane such that $D = C \cap H$ and $S_i = T_i \cap H$ are $(n-1)$-dimensional simplices. We need the following notions (see [41]). A finite set of $(n-1)$-dimensional simplices αD is called a *triangulation* of D if the simplices have pairwise disjoint interiors and their union equals D. An *elementary move* applied to αD is one of the two following operations: a simplex $S \in \alpha D$ is dissected into two $(n-1)$-dimensional simplices S_1, S_2 by an $(n-2)$-dimensional plane containing an $(n-3)$-dimensional face of S; or the reverse, that is, two simplices $S_1, S_2 \in \alpha D$ are replaced by $S = S_1 \cup S_2$ if S is again a simplex. It is shown in [41] that for every triangulation αD there are finitely many elementary moves that transform αD into the trivial triangulation $\{D\}$. Note that to each $(n-1)$-dimensional simplex $S \in \alpha D$, there corresponds a polytope $Q \in \overline{\mathcal{P}}_0^n$ such that $Q \cap H = S$. If S is dissected by an $(n-2)$-dimensional plane $E \subset H$ into S_1, S_2, then Q is dissected by the cone generated by E into $Q_1, Q_2 \in \overline{\mathcal{P}}_0^n$. Since $\overline{\mathrm{Z}}$ is a simple valuation on $\overline{\mathcal{P}}_0^n$, we obtain $\overline{\mathrm{Z}}\, Q = \overline{\mathrm{Z}}\, Q_1 \tilde{+}_p \overline{\mathrm{Z}}\, Q_2$. The same argument applies for the reverse move. Thus, after finitely many steps, we obtain that $\overline{\mathrm{Z}}\, P = \overline{\mathrm{Z}}\, T_1 \tilde{+}_p \cdots \tilde{+}_p \overline{\mathrm{Z}}\, T_k$. Since $\overline{\mathrm{Z}}$ is $\mathrm{GL}(n)$ contravariant, this proves that $\overline{\mathrm{Z}}$ is determined on $\overline{\mathcal{P}}_0^n$ by $\overline{\mathrm{Z}}\, T^n$.

2.1.2 Some functional equations

We set

$$f(x) = \rho(\overline{\mathrm{Z}}\, T^n, x)^p$$

almost everywhere on $\mathbb{R}^n$. Since Z is $\mathrm{GL}(n)$ contravariant and T^n does not change when the coordinates are permutated, we obtain

$$f(x_1, \ldots, x_n) = f(x_{k_1}, \ldots, x_{k_n}) \tag{2.5}$$

for every permutation $(k_1, \ldots, k_n)$ of $(1, \ldots, n)$. We derive a family of functional equations for f.

For $0 < \lambda_j < 1$ and $j = 2, 3, \ldots, n$, we define two families of linear maps by

$$\begin{aligned} \phi_j e_j &= \lambda_j e_j + (1-\lambda_j) e_1, \qquad & \phi_j e_k &= e_k \text{ for } k \neq j, \\ \psi_j e_1 &= \lambda_j e_j + (1-\lambda_j) e_1, \qquad & \psi_j e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Note that

$$\begin{aligned} \phi_j^{-1} e_j &= \frac{1}{\lambda_j} e_j - \frac{1-\lambda_j}{\lambda_j} e_1, \qquad & \phi_j^{-1} e_k &= e_k \text{ for } k \neq j, \\ \psi_j^{-1} e_1 &= -\frac{\lambda_j}{1-\lambda_j} e_j + \frac{1}{1-\lambda_j} e_1, \qquad & \psi_j^{-1} e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Let H_j be the hyperplane through 0 with normal vector $\lambda_j e_1 - (1-\lambda_j) e_j$. Then we have $T^n \cap H_j^+ = \phi_j T^n$ and $T^n \cap H_j^- = \psi_j T^n$. Since $\overline{\mathrm{Z}}$ is a simple valuation, it follows that

$$\overline{\mathrm{Z}}\, T^n = \overline{\mathrm{Z}}(\phi_j T^n) \tilde{+}_p \overline{\mathrm{Z}}(\psi_j T^n).$$

Since $\overline{\mathrm{Z}}$ is $\mathrm{GL}(n)$ contravariant, this and (2.3) imply

$$f(x) = \lambda_j^{pq} f(\phi_j^t x) + (1-\lambda_j)^{pq} f(\psi_j^t x) \tag{2.6}$$

almost everywhere on $\mathbb{R}^n$ where the set of exception depends on the value of q.

Similar observations can be made if the valuation Z is $\mathrm{GL}(n)$ covariant of weight $q \geq 0$. Then we have by (2.4)

$$f(x) = \lambda_j^{pq} f(\phi_j^{-1} x) + (1-\lambda_j)^{pq} f(\psi_j^{-1} x) \tag{2.7}$$

almost everywhere. Note that (2.5) holds in the covariant case, too.

2.1.3 The 2-dimensional Contravariant Case

Lemma 4. *Let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* $q = 1$. *Then there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = c\, \psi_{\frac{\pi}{2}}(P \;\tilde{+}_p\; (-P))$$

for every $P \in \mathcal{P}_0^2$, *where* $\psi_{\frac{\pi}{2}}$ *denotes the rotation by an angle* $\frac{\pi}{2}$.

Proof. Since

$$\rho(P \cup Q, \cdot) = \max\{\rho(P,\cdot), \rho(Q,\cdot)\}, \quad \rho(P \cap Q, \cdot) = \min\{\rho(P,\cdot), \rho(Q,\cdot)\},$$

formula (2.2) implies that the function $P \mapsto c\psi_{\frac{\pi}{2}}(P \;\tilde{+}_p\; (-P))$ is in fact an L_p radial valuation. Since

$$\psi_{\frac{\pi}{2}} \phi \psi_{\frac{\pi}{2}}^{-1} = (\det \phi) \phi^{-t} \tag{2.8}$$

holds for every $\phi \in \mathrm{GL}(2)$, we obtain by using (2.2)

$$\begin{aligned}
&\rho(c\psi_{\frac{\pi}{2}}(\phi P \;\tilde{+}_p\; (-\phi P)), x)^p \\
&= c^p \rho((\det\phi)P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p + c^p \rho(-(\det\phi)P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p \\
&= c^p \rho(|\det\phi|P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p + c^p \rho(-|\det\phi|P, \psi_{\frac{\pi}{2}}^{-1}\phi^t x)^p \\
&= \rho(|\det\phi|\phi^{-t} c\psi_{\frac{\pi}{2}}(P \;\tilde{+}_p\; (-P)), x)^p.
\end{aligned}$$

This proves the contravariance of weight 1.

From Lemma 2, Lemma 3, and (2.6) we know that

$$f(x) = \lambda_2^p f(\phi_2^t x) + (1-\lambda_2)^p f(\psi_2^t x) \tag{2.9}$$

holds for every $x \in \mathbb{R}^2$ which does not lie in the linear hull of e_1, e_2 or $\lambda_2 e_1 - (1-\lambda_2)e_2$. Thus it follows by induction that for $k = 1, 2, \ldots$,

$$f((\psi_2^{-t})^k x) = \lambda_2^p \sum_{i=1}^{k} (1-\lambda_2)^{p(k-i)} f(\phi_2^t (\psi_2^{-t})^i x) + (1-\lambda_2)^{kp} f(x) \tag{2.10}$$

holds on $\mathbb{R}^2$ except on a set consisting of countably many lines. For suitable $\varepsilon > 0$, we can evaluate (2.10) at $x = e_1 - \varepsilon e_2$. From this we obtain, using the homogeneity and the non-negativity of f, that

$$f(e_1 - (1-\lambda_2)^k \varepsilon (\psi_2^{-t})^k e_2) \geq \lambda_2^p \sum_{i=1}^{k} f(\phi_2^t(e_1 - (1-\lambda_2)^i \varepsilon (\psi_2^{-t})^i e_2)). \qquad (2.11)$$

Note that $(\psi_2^{-t})^k e_2 = -\lambda_2 \sum_{i=0}^{k-1} (1-\lambda_2)^{i-k} e_1 + e_2$. Thus $\|e_1 - (1-\lambda_2)^k \varepsilon (\psi_2^{-t})^k e_2\| \geq 1$. Let $k \to \infty$ in (2.11). By Lemma 2, f is uniformly bounded on $S^1 \backslash \{\pm e_1, \pm e_2\}$. So $f(\phi_2^t(e_1 - (1-\lambda_2)^i \varepsilon (\psi_2^{-t})^i e_2)) \to 0$ as $i \to \infty$. It follows from the continuity properties of f, that $f((1+\varepsilon)(e_1 + (1-\lambda_2)e_2)) = 0$. Taking the limit $\varepsilon \to 0$, we obtain

$$f(1, x_2) = 0, \qquad \text{for } 0 < x_2 < 1. \qquad (2.12)$$

By (2.5), this implies

$$f(x_1, 1) = 0, \qquad \text{for } 0 < x_1 < 1. \qquad (2.13)$$

Relations (2.12), (2.13), and the homogeneity of f imply

$$f(x_1, x_2) = 0, \qquad \text{for } x_1, x_2 > 0. \qquad (2.14)$$

By evaluating (2.10) at $-e_1 - \varepsilon e_2$ we get in a similar way

$$f(-x_1, -x_2) = 0, \qquad \text{for } x_1, x_2 > 0. \qquad (2.15)$$

Formula (2.9) gives

$$f(-1, 1) = \lambda_2^p f(-1, -1 + 2\lambda_2) + (1-\lambda_2)^p f(-1 + 2\lambda_2, 1).$$

In combination with (2.14) and (2.15) we obtain

$$\begin{aligned} f(-1, 1) &= \lambda_2^p f(-1, -1 + 2\lambda_2) \quad \text{for } \tfrac{1}{2} < \lambda_2 < 1, \\ f(-1, 1) &= (1-\lambda_2)^p f(-1 + 2\lambda_2, 1) \quad \text{for } 0 < \lambda_2 < \tfrac{1}{2}. \end{aligned}$$

Hence

$$\begin{aligned} f(-1, x_2) &= \frac{c^p}{(1+x_2)^p} \quad \text{for } 0 < x_2 < 1, \\ f(-x_1, 1) &= \frac{c^p}{(1+x_1)^p} \quad \text{for } 0 < x_1 < 1, \end{aligned}$$

with $c^p = 2^p f(-1, 1)$. Since f is homogeneous of degree $-p$, we get

$$f(-x_1, x_2) = \frac{c^p}{(x_1 + x_2)^p} \qquad \text{for } x_1, x_2 > 0,$$

and by (2.5)

$$f(x_1,-x_2)=\frac{c^p}{(x_1+x_2)^p} \qquad \text{for } \ x_1,x_2>0.$$

Combining these results finally yields

$$f(x)=c^p\rho(\psi_{\frac{\pi}{2}}T^2,x)^p+c^p\rho(\psi_{\frac{\pi}{2}}(-T^2),x)^p$$

almost everywhere on $\mathbb{R}^2$. $\square$

For given $p,q\in\mathbb{R}$, we define the function $g_{p,q}$ on $\mathbb{R}^2$ by

$$g_{p,q}(x_1,x_2)=\begin{cases} (x_1^{pq-p}-x_2^{pq-p})/(x_1-x_2)^{pq} & \text{for } 0\le x_2<x_1,\\ x_1^{pq-p}/(x_1-x_2)^{pq} & \text{for } x_1>0, x_2<0,\\ 0 & \text{otherwise.}\end{cases}$$

Define the linear transformations γ_i, $i=0,1,2$, by

$$\gamma_0(x_1,x_2)=(-x_1,-x_2),\ \ \gamma_1(x_1,x_2)=(x_2,x_1),\ \ \gamma_2(x_1,x_2)=(-x_2,-x_1),$$

that is, γ_0, γ_1, and γ_2 are the reflections with respect to the origin, the first median, and the second median, respectively.

Lemma 5. *Let $f:\mathbb{R}^2\backslash\{0\}\to\mathbb{R}$ be a function positively homogeneous of degree $-p$ such that*

$$f(x_1,x_2)=\lambda^{pq}f(x_1,(1-\lambda)x_1+\lambda x_2)+(1-\lambda)^{pq}f((1-\lambda)x_1+\lambda x_2,x_2) \tag{2.16}$$

holds on $\mathbb{R}^2\backslash\{0\}$ for every $0<\lambda<1$. Then

$$f=f(1,0)\,g_{p,q}+f(-1,0)\,g_{p,q}\circ\gamma_0+f(0,1)\,g_{p,q}\circ\gamma_1+f(0,-1)\,g_{p,q}\circ\gamma_2 \tag{2.17}$$

on $\mathbb{R}^2\backslash\{(x_1,x_2):x_1=x_2\}$.

Proof. Equation (2.16) evaluated at the points $\pm(1,0)$, $\pm(0,1)$, $\pm(-\lambda,1-\lambda)$ and the homogeneity of f yield

$$\begin{aligned}
f(1,1-\lambda) &= \frac{1-(1-\lambda)^{pq-p}}{\lambda^{pq}}f(1,0), && (2.18)\\
f(-1,\lambda-1) &= \frac{1-(1-\lambda)^{pq-p}}{\lambda^{pq}}f(-1,0), && (2.19)\\
f(\lambda,1) &= \frac{1-\lambda^{pq-p}}{(1-\lambda)^{pq}}f(0,1), && (2.20)\\
f(-\lambda,-1) &= \frac{1-\lambda^{pq-p}}{(1-\lambda)^{pq}}f(0,-1), && (2.21)\\
f(-\lambda,1-\lambda) &= \lambda^{pq-p}f(-1,0)+(1-\lambda)^{pq-p}f(0,1), && (2.22)\\
f(\lambda,\lambda-1) &= \lambda^{pq-p}f(1,0)+(1-\lambda)^{pq-p}f(0,-1). && (2.23)
\end{aligned}$$

First, suppose that $x_1 > x_2 \geq 0$. If $x_2 = 0$, it follows from the homogeneity of f that $f(x_1, 0) = x_1^{-p} f(1, 0) = f(1, 0)\, g_{p,q}(x_1, 0)$. For $x_1 > x_2 > 0$ we obtain by (2.18)

$$\begin{aligned} f(x_1, x_2) &= x_1^{-p} f(1, 1 - (1 - x_2/x_1)) = x_1^{-p} \frac{1 - (x_2/x_1)^{pq-p}}{(1 - (x_2/x_1))^{pq}} f(1, 0) \\ &= \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1 - x_2)^{pq}} f(1, 0) = f(1, 0)\, g_{p,q}(x_1, x_2). \end{aligned}$$

Since $g_{p,q} \circ \gamma_0$, $g_{p,q} \circ \gamma_1$, and $g_{p,q} \circ \gamma_2$ are zero for $x_1 > x_2 \geq 0$, (2.17) holds in this part of the plane. (2.19) gives

$$\begin{aligned} f(-x_1, -x_2) &= x_1^{-p} f(-1, (1 - x_2/x_1) - 1) = x_1^{-p} \frac{1 - (x_2/x_1)^{pq-p}}{(1 - (x_2/x_1))^{pq}} f(-1, 0) \\ &= \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1 - x_2)^{pq}} f(-1, 0) = f(-1, 0)(g_{p,q} \circ \gamma_0)(-x_1, -x_2). \end{aligned}$$

But $g_{p,q}$, $g_{p,q} \circ \gamma_1$ as well as $g_{p,q} \circ \gamma_2$ vanish for $x_1 < x_2 < 0$ and therefore (2.17) is true if $x_1 < x_2 < 0$. Using the homogeneity we obtain that (2.17) is correct for $x_1 < 0$, $x_2 = 0$.

Now, assume $x_2 > x_1 \geq 0$. If $x_1 = 0$, then we have

$$\begin{aligned} f(0, x_2) &= x_2^{-p} f(0, 1) = f(0, 1)(g_{p,q} \circ \gamma_1)(0, x_2), \\ f(0, -x_2) &= x_2^{-p} f(0, -1) = f(0, -1)(g_{p,q} \circ \gamma_2)(0, -x_2). \end{aligned}$$

Formulae (2.20) and (2.21) for $x_2 > x_1 > 0$ yield

$$\begin{aligned} f(x_1, x_2) &= x_2^{-p} f(x_1/x_2, 1) = x_2^{-p} \frac{1 - (x_1/x_2)^{pq-p}}{(1 - (x_1/x_2))^{pq}} f(0, 1) \\ &= \frac{x_2^{pq-p} - x_1^{pq-p}}{(x_2 - x_1)^{pq}} f(0, 1) = f(0, 1)(g_{p,q} \circ \gamma_1)(x_1, x_2), \\ f(-x_1, -x_2) &= x_2^{-p} f(-x_1/x_2, -1) = x_2^{-p} \frac{1 - (x_1/x_2)^{pq-p}}{(1 - (x_1/x_2))^{pq}} f(0, -1) \\ &= \frac{x_2^{pq-p} - x_1^{pq-p}}{(x_2 - x_1)^{pq}} f(0, -1) = f(0, -1)(g_{p,q} \circ \gamma_2)(-x_1, -x_2). \end{aligned}$$

Since $g_{p,q}$, $g_{p,q} \circ \gamma_0$, $g_{p,q} \circ \gamma_2$ are zero for $x_2 > x_1 \geq 0$ and $g_{p,q}$, $g_{p,q} \circ \gamma_0$, $g_{p,q} \circ \gamma_1$ vanish for $x_2 < x_1 \leq 0$, it remains to prove identity (2.17) if the coordinates have different signs.

Finally, let x_1 and x_2 be greater than zero. By (2.22) and (2.23) we have

$$\begin{aligned}
f(-x_1,x_2) &= (x_1+x_2)^{-p} f(-x_1/(x_1+x_2), 1-x_1/(x_1+x_2)) \\
&= \frac{x_2^{pq-p}}{(x_1+x_2)^{pq}} f(0,1) + \frac{x_1^{pq-p}}{(x_1+x_2)^{pq}} f(-1,0) \\
&= f(0,1)(g_{p,q}\circ\gamma_1)(-x_1,x_2) + f(-1,0)(g_{p,q}\circ\gamma_0)(-x_1,x_2), \\
f(x_1,-x_2) &= (x_1+x_2)^{-p} f(x_1/(x_1+x_2), x_1/(x_1+x_2)-1) \\
&= \frac{x_2^{pq-p}}{(x_1+x_2)^{pq}} f(0,-1) + \frac{x_1^{pq-p}}{(x_1+x_2)^{pq}} f(1,0) \\
&= f(0,-1)(g_{p,q}\circ\gamma_2)(x_1,-x_2) + f(1,0) g_{p,q}(x_1,-x_2).
\end{aligned}$$

The fact that $g_{p,q}$ and $g_{p,q}\circ\gamma_2$ are zero in the second quadrant and $g_{p,q}\circ\gamma_0$, $g_{p,q}\circ\gamma_1$ are zero in the fourth quadrant completes the proof. □

In the following, we have $q>1$. Therefore Lemma 2 and Lemma 3 imply that f is continuous on S^{n-1}. Thus (2.6) holds on $\mathbb{R}\backslash\{0\}$ and f satisfies the conditions of Lemma 5. Combined with (2.5) this implies that

$$f = f(1,0)(g_{p,q} + g_{p,q}\circ\gamma_1) + f(-1,0)(g_{p,q}\circ\gamma_0 + g_{p,q}\circ\gamma_2) \tag{2.24}$$

on $\mathbb{R}^2\backslash\{0\}$.

Lemma 6. *Let* $\mathrm{Z}:\mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* q*. Let* $p>1$*,* $q>1$ *or* $0<p<1$*,* $q>1/p$*. Then*

$$\mathrm{Z}\,P = \{0\}$$

for every $P\in\mathcal{P}_0^2$*.*

Proof. For $x_2>0$ fixed, we obtain by (2.24) that

$$\lim_{x_1\to x_2+} f(x_1,x_2) = \lim_{x_1\to x_2+} \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1-x_2)^{pq}} f(1,0)$$

has to be finite. This implies that $f(1,0)$ has to be zero.

Considering $\lim_{x_1\to x_2+} f(-x_1,-x_2)$ proves $f(-1,0)=0$. So by (2.24), f vanishes on $\mathbb{R}^2\backslash\{0\}$. □

Lemma 7. *For* $p<1$*, let* $\mathrm{Z}:\mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p\rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *contravariant of weight* $q=1/p$*. Then there exist constants* $c_1,c_2\geq 0$ *such that*

$$\mathrm{Z}\,P = c_1\,\mathrm{I}_p^+\,P \,\tilde{+}_p\, c_2\,\mathrm{I}_p^-\,P$$

for every $P\in\mathcal{P}_0^2$*.*

Proof. A simple calculation shows

$$\rho(\mathrm{I}_p^+ T^2, \cdot)^p = (p^2 - 3p + 2)^{-1}(g_{p,1/p} + g_{p,1/p} \circ \gamma_1) \tag{2.25}$$

almost everywhere. Therefore

$$\rho(\mathrm{I}_p^- T^2, \cdot)^p = \rho(\mathrm{I}_p^+ T^2, \gamma_0(\cdot))^p = (p^2 - 3p + 2)^{-1}(g_{p,1/p} \circ \gamma_0 + g_{p,1/p} \circ \gamma_2). \tag{2.26}$$

Combined with (2.24), these equations complete the proof. □

Finally, we consider the case $p < 1$ and $q \in (1, 1/p)$. We define

$$\rho(\mathrm{I}_{p,q}^+ T^2, x)^p = (g_{p,q} + g_{p,q} \circ \gamma_1)(x).$$

The restrictions on q show that $\rho(\mathrm{I}_{p,q}^+ T^2, \cdot)$ is continuous and non-negative on $\mathbb{R}^2 \backslash \{0\}$. By definition, $\rho(\mathrm{I}_{p,q}^+ T^2, \cdot)$ is positively homogeneous of degree -1 and thus the radial function of a star body.

We extend this definition to all simplices in $\mathbb{R}^2$ having one vertex at the origin (we denote this set by $\mathcal{T}_0^2$):

$$\mathrm{I}_{p,q}^+ S = \begin{cases} |\det \phi|^q \phi^{-t} \, \mathrm{I}_{p,q}^+ T^2 & \text{if } S \text{ is 2-dimensional and } S = \phi T^2, \\ \{0\} & \text{otherwise.} \end{cases}$$

Note that $\mathrm{I}_{p,q}^+$ is well defined on $\mathcal{T}_0^2$ since $\rho(\mathrm{I}_{p,q}^+ T^2, \cdot)$ does not change if the coordinates are interchanged. We claim that $\mathrm{I}_{p,q}^+$ is a valuation on $\mathcal{T}_0^2$. To prove this, it suffices to check the valuation property if the two involved simplices coincide in an edge. Since by definition $\mathrm{I}_{p,q}^+$ is GL(2) contravariant, it suffices to check the valuation property for the standard simplex. Thus it suffices to show that

$$\mathrm{I}_{p,q}^+ T^2 = \mathrm{I}_{p,q}^+(T^2 \cap H^+) \mathbin{\tilde{+}_p} \mathrm{I}_{p,q}^+(T^2 \cap H^-)$$

where H is the line with normal vector $\lambda e_1 - (1-\lambda)e_2$, $0 < \lambda < 1$. Therefore we have to prove

$$\begin{aligned} \rho(\mathrm{I}_{p,q}^+ T^2, (x_1, x_2))^p &= \lambda^{pq} \rho(\mathrm{I}_{p,q}^+ T^2, (x_1, (1-\lambda)x_1 + \lambda x_2))^p \\ &\quad + (1-\lambda)^{pq} \rho(\mathrm{I}_{p,q}^+ T^2, ((1-\lambda)x_1 + \lambda x_2, x_2))^p. \end{aligned} \tag{2.27}$$

The case $x_1, x_2 < 0$ is trivial. So assume $x_1 > x_2 \geq 0$. Then $x_1 > (1-\lambda)x_1 + \lambda x_2 \geq 0$, $(1-\lambda)x_1 + \lambda x_2 > x_2 \geq 0$, and the right hand side of (2.27) equals

$$\lambda^{pq} \frac{x_1^{pq-p} - ((1-\lambda)x_1 + \lambda x_2)^{pq-p}}{(x_1 - (1-\lambda)x_1 - \lambda x_2)^{pq}} + (1-\lambda)^{pq} \frac{((1-\lambda)x_1 + \lambda x_2)^{pq-p} - x_2^{pq-p}}{((1-\lambda)x_1 + \lambda x_2 - x_2)^{pq}}$$

which is nothing else than $\rho(\mathrm{I}_{p,q}^+ T^2, (x_1, x_2))^p$. Similar, we obtain (2.27) for points $x_2 > x_1 \geq 0$. To check (2.27) for $(x_1, -x_2)$, $x_1, x_2 > 0$, we first assume that

$(1-\lambda)x_1 - \lambda x_2 > 0$. Then $0 < (1-\lambda)x_1 - \lambda x_2 < x_1$ and the sum appearing in (2.27) equals

$$\lambda^{pq}\frac{x_1^{pq-p} - ((1-\lambda)x_1 - \lambda x_2)^{pq-p}}{(x_1 - (1-\lambda)x_1 + \lambda x_2)^{pq}} + (1-\lambda)^{pq}\frac{((1-\lambda)x_1 - \lambda x_2)^{pq-p}}{((1-\lambda)x_1 - \lambda x_2 + x_2)^{pq}}.$$

If $(1-\lambda)x_1 - \lambda x_2 < 0$, the right hand side of (2.27) is

$$\lambda^{pq}\frac{x_1^{pq-p}}{(x_1 - (1-\lambda)x_1 + \lambda x_2)^{pq}}.$$

These two expressions are equal to $\rho(\mathrm{I}_{p,q}^+ T^2, (x_1, -x_2))^p$. The case $(1-\lambda)x_1 - \lambda x_2 = 0$ is simple and the remaining part can be treated in an analogous way.

Now, we extend the valuation $\mathrm{I}_{p,q}^+$ to $\overline{\mathcal{P}}_0^2$ by setting

$$\rho(\mathrm{I}_{p,q}^+ P, x)^p = \sum_{i \in I} \rho(\mathrm{I}_{p,q}^+ S_i, x)^p,$$

where $\{S_i : i \in I,\ \dim S_i = 2\} \subset \mathcal{T}_0^2$ is a dissection of P, that is, I is finite, $P = \bigcup_{i \in I} S_i$ and no pair of simplices intersects in a set of dimension 2.

Given two different dissections, it is always possible to obtain one from the other by a finite number of the following operations: a simplex is dissected into two 2-dimensional simplices by a line through the origin, or the converse, that is, two simplices whose union is again a simplex are replaced by their union (We remark that the corresponding result holds true for $n \geq 3$, see [41]). Since $\mathrm{I}_{p,q}^+$ is a valuation on $\mathcal{T}_0^2$, this shows that $\mathrm{I}_{p,q}^+$ is well defined on $\overline{\mathcal{P}}_0^2$.

We have to prove that $\mathrm{I}_{p,q}^+$ is a valuation. To do so, let $P, Q \in \overline{\mathcal{P}}_0^2$ be two 2-dimensional convex polytopes such that their union is again convex. We dissect $\mathbb{R}^2$ into 2-dimensional convex cones with apex 0 in such a way that each vertex of $P, Q, P \cap Q, P \cup Q$ lies on the boundary of some cone in this dissection. The intersection of such a cone with the boundary of P and Q are line segments which are either identical, do not intersect, or intersect in their endpoints only. Therefore $\mathrm{I}_{p,q}^+$ is a valuation and obviously it is $\mathrm{GL}(2)$ contravariant of weight q.

We define the L_p radial valuation $\mathrm{I}_{p,q}^-$ by setting $\mathrm{I}_{p,q}^- P = \mathrm{I}_{p,q}^+(-P)$. Now, (2.24) implies the following result.

Lemma 8. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(2)$ contravariant of weight $1 < q < 1/p$. Then there exist constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_{p,q}^+ P \, \tilde{+}_p \, c_2 \, \mathrm{I}_{p,q}^- P$$

for every $P \in \mathcal{P}_0^2$.

2.1.4 The 2-dimensional Covariant Case

Let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ be a valuation which is GL(2) covariant of weight q. Let $q > 0$. As before, let $\mathrm{Y}\, P(\cdot) = \rho(\mathrm{Z}\, P, \cdot)^p$ and denote the extension of Y to $\overline{\mathcal{P}}_0^2$ by $\overline{\mathrm{Y}}$. Note that Lemma 2 and Lemma 3 imply that $\overline{\mathrm{Y}}$ is continuous on S^1. We define the valuation $\hat{\mathrm{Y}}$ by $\hat{\mathrm{Y}}\, P(\cdot) = \overline{\mathrm{Y}}\, P(\psi_{\frac{\pi}{2}}^{-1}(\cdot))$ for every $P \in \overline{\mathcal{P}}_0^2$. From (2.4) and (2.8) it follows that for $\phi \in \mathrm{GL}(2)$ with $\det \phi > 0$

$$\hat{\mathrm{Y}}\, \phi P(x) = |\det \phi|^{pq}\, \overline{\mathrm{Y}}\, P(\phi^{-1} \psi_{\frac{\pi}{2}}^{-1} x) = |\det \phi|^{pq+p}\, \hat{\mathrm{Y}}\, P(\phi^t x)$$

for every $P \in \overline{\mathcal{P}}_0^2$. So $\hat{\mathrm{Y}}\, T^2$ satisfies (2.16) with $q+1$ instead of q. From the GL(2) covariance it follows that $\hat{\mathrm{Y}}\, T^2(x_1, x_2) = \hat{\mathrm{Y}}\, T^2(-x_2, -x_1)$. Thus Lemma 5 shows that

$$\hat{\mathrm{Y}}\, T^2 = \hat{\mathrm{Y}}\, T^2(1,0)(g_{p,q+1} + g_{p,q+1} \circ \gamma_2) + \hat{\mathrm{Y}}\, T^2(0,1)(g_{p,q+1} \circ \gamma_0 + g_{p,q+1} \circ \gamma_1). \quad (2.28)$$

Considering the limit

$$\lim_{x_1 \to x_2 +} \frac{x_1^{pq} - x_2^{pq}}{(x_1 - x_2)^{pq+p}}$$

for fixed $x_2 > 0$, we derive for $p > 1$ that $\hat{\mathrm{Y}}\, T^2(1,0) = \hat{\mathrm{Y}}\, T^2(0,1) = 0$ since $\hat{\mathrm{Y}}\, T^2$ is continuous on $\mathbb{R}^2 \backslash \{0\}$ and has to be finite on the first median. This limit also proves that $\hat{\mathrm{Y}}\, T^2(1,0) = \hat{\mathrm{Y}}\, T^2(0,1) = 0$ for $p < 1$ and $q > 1/p - 1$. Now, (2.28) implies the following result.

Lemma 9. *Let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ be a valuation which is* GL(2) *covariant of weight $q > 0$. Let $p > 1$, $q > 0$ or $0 < p < 1$, $q > 1/p - 1$. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^2$.

For $p < 1$ and $q \in (0, 1/p - 1)$, we define $\mathrm{J}_{p,q}^+$ by

$$\rho(\mathrm{J}_{p,q}^+ T^2, x)^p = (g_{p,q+1} + g_{p,q+1} \circ \gamma_2)(\psi_{\frac{\pi}{2}} x).$$

Similar to the contravariant case, $\mathrm{J}_{p,q}^+$ can be extended to a covariant valuation on $\mathcal{P}_0^2$. We define $\mathrm{J}_{p,q}^-$ by $\mathrm{J}_{p,q}^-\, P = \mathrm{J}_{p,q}^+(-P)$. Now, (2.28) implies the following result.

Lemma 10. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ be a valuation which is* GL(2) *covariant of weight $0 < q < 1/p - 1$. Then there exist constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1\, \mathrm{J}_{p,q}^+\, P\; \tilde{+}_p\; c_2\, \mathrm{J}_{p,q}^-\, P$$

for every $P \in \mathcal{P}_0^2$.

For $q = 1/p - 1$, the continuity of $\hat{\mathrm{Y}}\, T^2$ at the first median and (2.28) yield that $\hat{\mathrm{Y}}\, T^2(1,0) = \hat{\mathrm{Y}}\, T^2(0,1)$. Therefore we obtain the following lemma by using (2.25), (2.26) and the identity $\psi_{\frac{\pi}{2}}\, \mathrm{I}_p\, P = \psi_{\frac{\pi}{2}}^{-1}\, \mathrm{I}_p\, P$.

Lemma 11. *Let* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ *be a valuation which is* $\mathrm{GL}(2)$ *covariant of weight* $q = 1/p - 1$. *Then there exists a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = c\, \psi_{\frac{\pi}{2}}\, \mathrm{I}_p\, P$$

for every $P \in \mathcal{P}_0^2$.

2.1.5 The 2-dimensional Classification Theorems

Using the lemmas of the preceding sections and the planar case of Lemma 12 and Lemma 17, we obtain the following result.

Theorem 3. *For* $0 < p < 1$, *an operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ *is a non-trivial valuation which is* $\mathrm{GL}(2)$ *covariant of weight* q *if and only if there are constants* $c_1, c_2 \geq 0$ *such that*

$$\mathrm{Z}\, P = \begin{cases} c_1\, \mathrm{I}_p^+\, P^* \,\tilde{+}_p\, c_2\, \mathrm{I}_p^-\, P^* & \text{for } q = -1/p \\ c_1\, \mathrm{I}_{p,q}^+\, P^* \,\tilde{+}_p\, c_2\, \mathrm{I}_{p,q}^-\, P^* & \text{for } -1/p < q < -1 \\ c_1 \psi_{\frac{\pi}{2}} (P^* \,\tilde{+}_p\, (-P^*)) & \text{for } q = -1 \\ c_1\, \mathrm{J}_{p,q}^+\, P \,\tilde{+}_p\, c_2\, \mathrm{J}_{p,q}^-\, P & \text{for } 0 < q < 1/p - 1 \\ c_1 \psi_{\frac{\pi}{2}}\, \mathrm{I}_p\, P & \text{for } q = 1/p - 1 \end{cases}$$

for every $P \in \mathcal{P}_0^2$. *For* $p > 1$, *an operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}^2, \tilde{+}_p \rangle$ *is a non-trivial* $\mathrm{GL}(2)$ *covariant valuation if and only if there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\, P = c\, \psi_{\frac{\pi}{2}} (P^* \,\tilde{+}_p\, (-P^*))$$

for every $P \in \mathcal{P}_0^2$.

Next, we consider an operator Z with centrally symmetric images. Note that in this case also the extended operator $\overline{\mathrm{Z}}$ has centrally symmetric images. Using again the lemmas of the preceding sections and the planar case of Lemma 12 and Lemma 17, we obtain the following result. Here $\mathrm{I}_{p,q}\, P = \mathrm{I}_{p,q}^+\, P \,\tilde{+}_p\, \mathrm{I}_{p,q}^-\, P$.

Theorem 4. *For* $0 < p < 1$, *an operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}_c^2, \tilde{+}_p \rangle$ *is a non-trivial valuation which is* $\mathrm{GL}(2)$ *covariant of weight* q *if and only if there is a constant*

$c \geq 0$ *such that*

$$\mathrm{Z}\,P = \begin{cases} c\,\mathrm{I}_p\,P^* & \text{for } q = -1/p \\ c\,\mathrm{I}_{p,q}\,P^* & \text{for } -1/p < q < -1 \\ c\,\psi_{\frac{\pi}{2}}(P^* \tilde{+}_p (-P^*)) & \text{for } q = -1 \\ c\,\psi_{\frac{\pi}{2}}\,\mathrm{I}_{p,q}\,P & \text{for } 0 < q < 1/p - 1 \\ c\,\psi_{\frac{\pi}{2}}\,\mathrm{I}_p\,P & \text{for } q = 1/p - 1 \end{cases}$$

for every $P \in \mathcal{P}_0^2$. *For* $p > 1$, *an operator* $\mathrm{Z} : \mathcal{P}_0^2 \to \langle \mathcal{S}_c^2, \tilde{+}_p \rangle$ *is a non-trivial* $\mathrm{GL}(2)$ *covariant valuation if and only if there is a constant* $c \geq 0$ *such that*

$$\mathrm{Z}\,P = c\,\psi_{\frac{\pi}{2}}(P^* \tilde{+}_p (-P^*))$$

for every $P \in \mathcal{P}_0^2$.

2.1.6 The Contravariant Case for $n \geq 3$

Lemma 12. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 2$, *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $0 < q < 1$. *Then*

$$\mathrm{Z}\,P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. From (2.6) we deduce that for $x \notin \operatorname{lin} e_1 \cup \cdots \cup \operatorname{lin} e_n \cup \operatorname{lin}(\lambda_j e_1 - (1-\lambda_j) e_j)$

$$f(x) = \lambda_j^{pq} f(\phi_j^t x) + (1-\lambda_j)^{pq} f(\psi_j^t x) \tag{2.29}$$

holds. First, we want to show that f is uniformly bounded on the set $S^{n-1} \backslash \{\pm e_1, \dots, \pm e_n\}$. To do so, note that since f is positive, equation (2.29) for $j = 2$ at $(x_1, 1-\lambda_2, x_3, \dots, x_n)$ and $(x_1, -(1-\lambda_2), x_3, \dots, x_n)$ gives

$$\begin{aligned} f(x_1, (1-\lambda_2)(x_1+\lambda_2), x_3, \dots, x_n) &\leq \lambda_2^{-pq} f(x_1, 1-\lambda_2, x_3, \dots, x_n), &(2.30) \\ f(x_1, (1-\lambda_2)(x_1-\lambda_2), x_3, \dots, x_n) &\leq \lambda_2^{-pq} f(x_1, -(1-\lambda_2), x_3, \dots, x_n) &(2.31) \end{aligned}$$

Regarding the limit $x_1 \to -\lambda_2$ one can deduce by (2.30) the boundedness of f in a neighbourhood of $-\lambda_2 e_1$ from the boundedness of f in a neighbourhood of $-\lambda_2 e_1 + (1-\lambda_2) e_2$ where the latter is a consequence of the homogeneity and continuity of f on $S^{n-1} \backslash \{\pm e_1, \dots, \pm e_n\}$. From (2.5) we conclude that such neighbourhoods exists for every $-e_i, i = 1 \dots, n$. Proceeding in an analogous way but taking the limit $x_1 \to \lambda_2$ and taking (2.31) into account on obtains the boundedness in suitable neighbourhoods at $e_i, i = 1 \dots, n$.

From (2.29) we know that

$$f(\phi_2^{-t}x) = \lambda_2^{pq} f(x) + (1-\lambda_2)^{pq} f(\psi_2^t \phi_2^{-t} x)$$

for $x \notin \operatorname{lin} e_1 \cup \cdots \cup \operatorname{lin} e_n \cup \operatorname{lin}(e_1 + (1-\lambda_2)e_2)$. Thus we obtain for $(-1, 1, x_3, \ldots, x_n)$ by using the homogeneity and the non-negativity of f that

$$\lambda_2^{pq-p} f(-1, 1, x_3, \ldots, x_n) \leq f(-\lambda_2, 2-\lambda_2, \lambda_2 x_3, \ldots, \lambda_2 x_n).$$

Since $pq - p < 0$ and f is bounded, this yields

$$f(-1, 1, x_3, \ldots, x_n) = 0, \quad x_3, \ldots, x_n \in \mathbb{R}.$$

Evaluating (2.29) at $(-1, 1, x_3, \ldots, x_n)$ proves

$$0 = \lambda_2^{pq} f(-1, 2\lambda_2 - 1, x_3, \ldots, x_n) + (1-\lambda_2)^{pq} f(2\lambda_2 - 1, 1, x_3, \ldots, x_n).$$

for $\lambda_2 \neq 1/2$. Since f is non-negative,

$$\begin{aligned} f(-1, x_2, x_3, \ldots, x_n) &= 0, \qquad -1 < x_2 < 1, \quad x_2 \neq 0, \quad x_3, \ldots, x_n \in \mathbb{R}, \\ f(x_1, 1, x_3, \ldots, x_n) &= 0, \qquad -1 < x_1 < 1, \quad x_1 \neq 0, \quad x_3, \ldots, x_n \in \mathbb{R}. \end{aligned}$$

Because of (2.3) we also have for $-1 < x_1 < 1$, $-1 < x_2 < 1$, $x_1, x_2 \neq 0$ and arbitrary $x_3, \ldots, x_n$

$$\begin{aligned} f(x_1, -1, x_3, \ldots, x_n) &= 0, \\ f(1, x_2, x_3, \ldots, x_n) &= 0. \end{aligned}$$

These last four equations prove that f is equal to zero almost everywhere on $\mathbb{R}^n$. $\square$

Lemma 13. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 3$, *be a valuation which is* $\mathrm{GL}(n)$ *contravariant of weight* $q = 1$. *Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. By Lemma 2 and Lemma 3, f is continuous and uniformly bounded on S^{n-1} except on $\operatorname{lin} e_1 \cup \cdots \cup \operatorname{lin} e_n$. By (2.6), we have for $2 \leq j \leq n$

$$f(x) = \lambda_j^p f(\phi_j^t x) + (1-\lambda_j)^p f(\psi_j^t x) \tag{2.32}$$

on $\mathbb{R}^n$ except on a finite union of lines. Using this repeatedly, we get

$$\begin{aligned} f(x) &= \lambda_2^p \cdots \lambda_n^p f(\phi_n^t \cdots \phi_2^t x) + \sum_{j=3}^{n} \lambda_2^p \cdots \lambda_{j-1}^p (1-\lambda_j)^p f(\psi_j^t \phi_{j-1}^t \cdots \phi_2^t x) \\ &\quad + (1-\lambda_2)^p f(\psi_2^t x) \\ &\geq \lambda_2^p \cdots \lambda_n^p f(\phi_n^t \cdots \phi_2^t x) + (1-\lambda_2)^p f(\psi_2^t x). \end{aligned}$$

This implies for $k = 1, 2, \ldots,$

$$f((\psi_2^{-t})^k x) \geq \lambda_2^p \cdots \lambda_n^p \sum_{i=1}^{k} (1-\lambda_2)^{p(k-i)} f(\phi_n^t \cdots \phi_2^t (\psi_2^{-t})^i x) \tag{2.33}$$

except on countably many lines. Define $x' = x_3 e_3 + \cdots + x_n e_n$. Evaluating (2.33) at suitable $e_1 + x'$ and multiplying by $(1-\lambda_2)^{-pk}$ shows that

$$f(e_1 + (1-\lambda_2)^k x') \geq \lambda_2^p \cdots \lambda_n^p \sum_{i=1}^{k} f(\phi_n^t \cdots \phi_2^t (e_1 + (1-\lambda_2)^i x')).$$

Let $k \to \infty$. Since f is uniformly bounded and continuous at $\phi_n^t \cdots \phi_2^t e_1 = e_1 + (1-\lambda_2)e_2 + \cdots + (1-\lambda_n)e_n$, it follows that $f(\phi_n^t \cdots \phi_2^t e_1) = 0$. So we get

$$f(1, x_2, \ldots, x_n) = 0, \quad 0 < x_2, \ldots, x_n < 1.$$

From (2.5) we obtain (using the homogeneity of f) that

$$f(1, x_2, \ldots, x_n) = 0, \quad x_2, \ldots, x_k > 0, \quad 0 < x_{k+1}, \ldots, x_n < 1.$$

So $f(x_1, \ldots, x_n) = 0$ for $x_1, \ldots, x_n > 0$. Considering $-e_1 + x'$ and (2.33) like before shows $f(-x_1, \ldots, -x_n) = 0$ for $x_1, \ldots, x_n > 0$.

Note that (2.32) for $j = 2$ and arbitrary $c \geq 1$ at $(c, c, -1, c, \ldots, c)$ proves (since $p \neq 1$) that $f(c, c, -1, c, \ldots, c) = 0$. Let $x_1 < 0$, $x_2, \ldots, x_n > 0$, and $(1-\lambda_j)x_1 + \lambda_j x_j > 0$. By (2.32) and the fact that f vanishes at points having all coordinates greater than zero we get

$$f(\phi_n^t \cdots \phi_2^t x) = \frac{1}{\lambda_2^p \cdots \lambda_n^p} f(x)$$

except on finitely many lines. Thus we obtain

$$\begin{aligned} &\lambda_2^{-p} \cdots \lambda_n^{-p} f(-1, c-\varepsilon, c, \ldots, c) = f(\phi_n^t \cdots \phi_2^t (-1, c-\varepsilon, c, \ldots, c)) \\ &= f(-1, -1 + \lambda_2(1+c-\varepsilon), -1 + \lambda_3(1+c), \ldots, -1 + \lambda_n(1+c)) \end{aligned}$$

for suitable $\varepsilon > 0$ and $\lambda_2, \ldots, \lambda_n > 1/(1+c-\varepsilon)$. The continuity of f shows

$$f(-1, x_2, \ldots, x_n) = 0, \quad 0 < x_2, \ldots, x_n < c.$$

But $c \geq 1$ was arbitrary, so $f(-1, x_2, \ldots, x_n) = 0$ for $x_2, \ldots, x_n > 0$. The homogeneity yields $f(-x_1, x_2, \ldots, x_n) = 0$ for $x_1, x_2, \ldots, x_n > 0$. In conclusion, $f(x_1, \ldots, x_n) = 0$ if at most one coordinate is negative. Suppose $f(x_1, \ldots, x_n) = 0$ where at most $1 \leq k < n-1$ coordinates are negative. Let x be chosen such that

$x_1, \dots, x_{k+1} < 0$ and $x_{k+2}, \dots, x_n > 0$. Suppose $x_2 < x_1 < 0$. Choose λ_2 with $0 < x_1/x_2 < \lambda_2 < 1$. Then

$$\begin{aligned} (\psi_2^{-t} x)_1 = (\phi_2^t \psi_2^{-t} x)_1 &= \frac{x_1}{1-\lambda_2} - \frac{\lambda_2}{1-\lambda_2} x_2 > 0, \\ (\psi_2^{-t} x)_i = (\phi_2^t \psi_2^{-t} x)_i > 0, & \qquad i = k+2, \dots, n. \end{aligned}$$

Since $f(\psi_2^{-t} x) = \lambda_2^{pq} f(\phi_2^t \psi_2^{-t} x) + (1-\lambda_2)^{pq} f(x)$ we obtain $f(x) = 0$. By (2.5) we conclude $f(x) = 0$ for the case $x_1 < x_2 < 0$. □

In the following, we have $q > 1$. Therefore Lemma 2 and Lemma 3 imply that f is continuous on $\mathbb{R}^n \backslash \{0\}$. In the proof of Lemmas 14 to 16, we use the following remark. Suppose we have two functions f_1, f_2 which are continuous on $\mathbb{R}^n \backslash \{0\}$ satisfying (2.5) and such that for $0 < \lambda_j < 1$, $j = 2, \dots, n$,

$$f_i(x) = \lambda_j^{pq} f_i(\phi_j^t x) + (1-\lambda_j)^{pq} f_i(\psi_j^t x)$$

holds on $\mathbb{R}^n$. Further assume that these functions are equal for all points where at most two coordinates do not vanish. Then an argument similar to that at the end of the last proof shows that these functions have to be equal.

Lemma 14. *For $p > 1$, let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q > 1$. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. Define $\tilde{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$. Then $\tilde{f}$ is continuous and satisfies the conditions of Lemma 5. The proof of Lemma 6 shows $\tilde{f} = 0$. By (2.5) this implies that $f(x_i e_i + x_j e_j) = 0$ for arbitrary $1 \leq i, j \leq n$. Thus f vanishes on $\mathbb{R}^n \backslash \{0\}$. □

Lemma 15. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q = 1/p$. Then there exist constants $c_1, c_2 \geq 0$ such that*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_p^+ \, P \;\tilde{+}_p\; c_2 \, \mathrm{I}_p^- \, P$$

for every $P \in \mathcal{P}_0^n$.

Proof. For $x = x_1 e_1 + x_2 e_2$, note that $\rho(\mathrm{I}_p^{\pm} \, T^n, x)$ is a multiple of $\rho(\mathrm{I}_p^{\pm} \, T^2, (x_1, x_2))$. This and an analogous argument as before proves the lemma. □

Lemma 16. *For $p < 1$, let $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$ be a valuation which is $\mathrm{GL}(n)$ contravariant of weight $q > 1$, $q \neq 1/p$. Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. By (2.6) we have

$$f(x) = \lambda^{pq} f(\phi_2^t x) + (1-\lambda)^{pq} f(\psi_2^t x)$$

on $\mathbb{R}^n \backslash \{0\}$. Since e_3 is an eigenvector of ϕ_2^t and ψ_2^t with eigenvalue 1, we get $f(\pm e_i) = 0$ for $i = 1, \ldots, n$. For $\tilde{f}(x_1, x_2) = f(x_1 e_1 + x_2 e_2)$ this implies $\tilde{f}(1,0) = \tilde{f}(-1,0) = 0$. Lemma 5 proves $\tilde{f} = 0$. □

2.1.7 The Covariant Case for $n \geq 3$

Lemma 17. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 2$, *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* $q = 0$. *Then there exist constants* $c_1, c_2 \geq 0$ *such that*

$$\mathrm{Z}\, P = c_1\, P \,\tilde{+}_p\, c_2(-P)$$

for every $P \in \mathcal{P}_0^n$.

Proof. By Lemma 2 and Lemma 3, f is continuous and uniformly bounded on $S^{n-1} \backslash (e_1^\perp \cup \cdots \cup e_n^\perp)$. By (2.7), the equation

$$f(x) = f(\phi_j^{-1} x) + f(\psi_j^{-1} x) \tag{2.34}$$

holds for $x \notin e_1^\perp \cup \cdots \cup e_n^\perp \cup (\lambda_j e_1 - (1-\lambda_j) e_j)^\perp$. Using this, we get by induction for $k = 1, 2, \ldots$,

$$f(\phi_2^k x) = \sum_{i=1}^{k} f(\psi_2^{-1} \phi_2^i\, x) + f(x) \tag{2.35}$$

for $x \notin e_2^\perp \cup \cdots \cup e_n^\perp \bigcup_{i=1}^\infty (e_1 + a_i e_2)^\perp$ and a suitable sequence (a_i). Define $x' = x_1 e_1 + x_3 e_3 + x_4 e_4 + \cdots + x_n e_n$ where $x_1, x_3, x_4, \ldots, x_n \neq 0$ and $x_1 \neq 1 - a_i$ for every i. Then (2.35) at $e_2 - e_1 + x'$ and the non-negativity of f show

$$f(\lambda_2^k (e_2 - e_1) + x') \geq \sum_{i=1}^{k} f(\psi_2^{-1} (\lambda_2^i (e_2 - e_1) + x')).$$

Let $k \to \infty$. Since f is uniformly bounded, $\lim_{i\to\infty} f(\psi_2^{-1}(\lambda_2^i (e_2 - e_1) + x')) = 0$. The continuity properties of f yield

$$f\left(\frac{x_1}{1-\lambda_2}, \frac{-\lambda_2 x_1}{1-\lambda_2}, x_3, \ldots, x_n\right) = 0, \quad \text{for } x_1, x_3, \ldots, x_n \neq 0. \tag{2.36}$$

From (2.36) we obtain that

$$f(x_1, x_2, \ldots, x_n) = 0, \quad \text{for } x_1, x_2, \ldots, x_n \neq 0 \text{ and not all } x_i \text{ have the same sign.}$$

For $j = 2, 3, \dots, n$ and $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ where all the x_i have the same sign, it follows that at least two coordinates of $\psi_j^{-1}\phi_j x$ have different signs. Thus (2.34) gives

$$f(\phi_n \cdots \phi_2 x) = f(x), \quad \text{for } x \notin e_1^\perp \cup \cdots \cup e_n^\perp \cup \bigcup_{k=0}^{n-2} (e_1 + \sum_{i=0}^{k} (1-\lambda_{2+i}) e_{2+i})^\perp. \tag{2.37}$$

Evaluating (2.37) at $(1, \dots, 1)$ gives

$$f(1 + (1-\lambda_2) + \cdots + (1-\lambda_n), \lambda_2, \dots, \lambda_n) = f(1, \dots, 1),$$

from which we conclude

$$f(x_1, \dots, x_n) = f(1, \dots, 1), \quad \text{for } 0 < x_2, \dots, x_n < 1,\ x_1 = n - x_2 - \cdots - x_n. \tag{2.38}$$

But (2.37) for positive $x_2, \dots, x_n$ is nothing else than

$$f(x_1, \dots, x_n) = f(x_1 + (1-\lambda_2)x_2 + \cdots + (1-\lambda_n)x_n, \lambda_2 x_2, \dots, \lambda_n x_n).$$

Choosing sufficiently small $\lambda_2, \dots, \lambda_n$, we obtain by (2.38)

$$f(x_1, \dots, x_n) = f(1, \dots, 1), \quad \text{for } x_1, \dots, x_n > 0,\ x_1 = n - x_2 - \cdots - x_n.$$

Similarly, we derive

$$f(-x_1, \dots, -x_n) = f(-1, \dots, -1), \quad \text{for } x_1, \dots, x_n > 0,\ x_1 = n - x_2 - \cdots - x_n.$$

This shows that $f(x) = c_1\, \rho(T^n, x)^p + c_2\, \rho(-T^n, x)^p$. □

Lemma 18. *Let* $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}^n, \tilde{+}_p \rangle$, $n \geq 3$, *be a valuation which is* $\mathrm{GL}(n)$ *covariant of weight* $q > 0$. *Then*

$$\mathrm{Z}\, P = \{0\}$$

for every $P \in \mathcal{P}_0^n$.

Proof. Since $q > 0$, Lemma 2 and Lemma 3 imply that f is continuous on S^{n-1}. By (2.7) we have

$$f(x) = \lambda_j^{pq} f(\phi_j^{-1} x) + (1-\lambda_j)^{pq} f(\psi_j^{-1} x) \tag{2.39}$$

on $\mathbb{R}^n \backslash \{0\}$. The vector e_3 is an eigenvector with eigenvalue 1 of ϕ_2^{-1} and ψ_2^{-1}. So for $pq \neq 1$, (2.39) and (2.5) imply $f(\pm e_k) = 0$ for $k = 1, 2, \dots, n$. For $pq = 1$, (2.39) evaluated at e_j for $j > 1$ yields

$$f(e_j)\lambda_j^{-p} = f(e_j - (1-\lambda_j)e_1).$$

Since $f(e_j - e_1)$ has to be finite and f is continuous, $f(e_j)$ has to be zero. Thus also in this case $f(\pm e_k) = 0$ for $k = 1, 2, \ldots, n$.

Hence (2.39) gives

$$f((1-\lambda_j)e_1 + \lambda_j e_j) = \lambda^{pq} f(e_j) + (1-\lambda_j)^{pq} f(e_1) = 0.$$

Therefore $f(x_1 e_1 + e_j) = 0$ for positive x_1. Using (2.39) again shows

$$f(-e_1) = \lambda_j^{pq} f(-e_1) + (1-\lambda_j)^{pq+p} f(-e_1 + \lambda_j e_j),$$

and so $f(x_1 e_1 + e_j) = 0$ for $x_1 \leq -1$. But

$$f(e_j) = \lambda_j^{pq+p} f(-(1-\lambda_j)e_1 + e_j) + (1-\lambda_j)^{pq} f(e_j),$$

which proves, together with the observations made before, that $f(x_1 e_1 + e_j) = 0$ for all x_1. By (2.5) this implies that $f(e_1 + x_j e_j) = 0$ for all x_j. The homogeneity of f shows $f(x_1 e_1 + x_j e_j) = 0$ for all x_1, x_j. Thus f vanishes on all points with at most two coordinates not equal to zero.

We use induction on the number of non-vanishing coordinates. We assume that f equals zero on points with $(j-1)$ non-vanishing coordinates. Set $x' = x_2 e_2 + \cdots + x_{j-1} e_{j-1}$. By (2.39),

$$f((1-\lambda_j)e_1 + \lambda_j e_j + x') = \lambda_j^{pq} f(e_j + x') + (1-\lambda_j)^{pq} f(e_1 + x') = 0,$$

which gives $f(x_1 e_1 + e_j + x'/\lambda_j) = 0$ for $x_1 > 0$. Therefore $f(x_1 e_1 + e_j + x') = 0$ for all $x_1 > 0$ and $x' = x_2 e_2 + \cdots + x_{j-1} e_{j-1}$. But by (2.39)

$$\begin{aligned} f(-e_1 + x') &= \lambda_j^{pq} f(-e_1 + x') + (1-\lambda_j)^{pq+p} f(-e_1 + \lambda_j\, e_j + (1-\lambda_j)x'), \\ f(e_j + x') &= \lambda_j^{pq+p} f(-(1-\lambda_j)e_1 + e_j + x') + (1-\lambda_j)^{pq} f(e_j + \lambda_j\, x'). \end{aligned}$$

So $f(x_1 e_1 + e_j + x') = 0$ for all x_1 and $x' = x_2 e_2 + \cdots + x_{j-1} e_{j-1}$. By (2.5), $f(e_1 + x_j e_j + x') = 0$ for all x_j and $x' = x_2 e_2 + \cdots + x_{j-1} e_{j-1}$. The homogeneity of f finally shows that $f(x_1 e_1 + \cdots + x_j e_j) = 0$ for all $x_1, \ldots, x_j$. $\square$

Chapter 3

Properties of L_p Intersection Bodies

In the preceeding section we proved that there is essentially one way how to define intersection bodies within the dual L_p Brunn-Minkowski theory. Now, we will have a closer look at these objects.

In Section 3.1 we start by establishing an approximation result for intersection bodies. A motivation for doing so is the somehow surprising result of Theorem 1. One would have expected a one-parametric set of solutions as in the corresponding result on intersection bodies. So it is reasonable to explore what happens to L_p intersection bodies and their nonsymmetric analogues for p close to one. The answer is given in Theorem 5. We prove that every intersection body of a convex body is the limit of (nonsymmetric) L_p intersection bodies. This enables us to obtain results on intersection bodies from corresponding ones on their L_p analogues.

In the remaining sections of this chapter, we derive results which further indicate that the operator I_p can be viewed as the L_p analogue of the intersection body operator. Results of this type have already been given. An extremely nice example which perfectly shows the analogy of I and I_p is the following. Goodey and Weil proved in [18] that the intersection body of a star body is the limit of finite radial sums of ellipsoids. Here, the limit is taken with respect to the metric on $\mathcal{S}^n$ induced from uniform convergence of radial functions. (In fact, they proved even more, namely that such limits are precisely a slightly more general class of intersection bodies.) Thus the radial function $\rho(\mathrm{I}\,K, \cdot)$ of a star body K is the uniform limit of L_1 radial sums of ellipsoids, i.e.

$$E_1 \tilde{+} E_2 \tilde{+} \cdots \tilde{+} E_k.$$

What one would expect is that $\rho(\mathrm{I}_p\,K, \cdot)$ can be uniformly approximated by L_p radial sums of the form

$$E_1 \tilde{+}_p E_2 \tilde{+}_p \cdots \tilde{+}_p E_k.$$

In fact this is true, as was shown by Kalton, Koldobsky, Yaskin and Yaskina [28]. An other example where I_p behaves like I can be found in Yaskin's and Yaskina's solution of the L_p Busemann-Petty problem [61]. We discuss this in detail in Section 3.5.

We will not only deal with the operator I_p but also with its companion I_p^+. What we can conclude from results of Sections 3.3 and 3.5 is that properties of I and I_p which are true for origin symmetric sets become true also for non symmetric ones. One just has to work with I_p^+ instead of I or I_p!

Before we continue, we want to point out a connection between L_p intersection bodies and cosine transforms. For a function $f \in C(S^{n-1})$ and $p < 1$, the L_{-p} cosine transform is defined by

$$\mathrm{C}_{-p} f(v) = \int_{S^{n-1}} |u \cdot v|^{-p} f(u)\, du, \quad \text{for } v \in S^{n-1}.$$

We further introduce the nonsymmetric L_p cosine transform

$$\mathrm{C}_{-p}^+ f(v) = \int_{S^{n-1} \cap v^+} |u \cdot v|^{-p} f(u)\, du, \quad \text{for } v \in S^{n-1}.$$

A change into polar coordinates proves

$$\rho(\mathrm{I}_p K, v)^p = ((n-p)\Gamma(1-p))^{-1} (\mathrm{C}_{-p}\, \rho(K,\cdot)^{n-p})(v), \tag{3.1}$$
$$\rho(\mathrm{I}_p^+ K, v)^p = ((n-p)\Gamma(1-p))^{-1} (\mathrm{C}_{-p}^+\, \rho(K,\cdot)^{n-p})(v), \tag{3.2}$$

for a star body K and $v \in S^{n-1}$. This enables us to prove that I_p and I_p^+ map the unit ball $B^n \subset \mathbb{R}^n$ to balls of radii r_{I_p} and $r_{\mathrm{I}_p^+}$, respectively. Indeed, relation (3.2) yields

$$\begin{aligned} \rho(\mathrm{I}_p^+ B^n, v)^p &= \frac{\omega_{n-1}}{(n-p)\Gamma(1-p)} \int_0^1 t^{-p}(1-t^2)^{(n-3)/2}\, dt \\ &= \frac{\omega_{n-1}\Gamma((1-p)/2)\Gamma((n-1)/2)}{2(n-p)\Gamma(1-p)\Gamma((n-p)/2)}. \end{aligned}$$

Note that the volume κ_n and the surface area ω_n of B^n are given by

$$\kappa_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}, \quad \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{3.3}$$

So by (3.3) and the formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\Gamma\left(x + \frac{1}{2}\right), \tag{3.4}$$

which holds for complex numbers x and $x + \frac{1}{2}$ that do not belong to $-\mathbb{N} \cup \{0\}$, we obtain

$$r_{\mathrm{I}_p^+}^p = \frac{2^p \pi^{n/2}}{(n-p)\Gamma((n-p)/2)\Gamma(1-p/2)} \tag{3.5}$$

for $p < 1$ which are not integers. Obviously, $r_{\mathrm{I}_p}^p = 2 r_{\mathrm{I}_p^+}^p$.

3.1 Approximation of Intersection Bodies

The next theorem clarifies the behavior of the L_p intersection body of a convex body as p tends to one. Before we go into detail, we collect some topological notions. Let $\mathcal{K}^n$ be topologized as usual by the topology induced from the Hausdorff distance

$$\delta(K,L) = \sup_{u\in S^{n-1}} |h(K,u) - h(L,u)| =: \|h(K,\cdot) - h(L,\cdot)\|_\infty, \quad \text{for } K,L \in \mathcal{K}^n.$$

The natural metric on $\mathcal{S}^n$ is

$$\tilde{\delta}(K,L) = \|\rho(K,\cdot) - \rho(L,\cdot)\|_\infty, \quad \text{for } K,L \in \mathcal{S}^n.$$

Denote by $\mathcal{K}_0^n$ the set of convex bodies containing the origin in their interiors. The announced approximation result is as follows.

Theorem 5. *For every $K \in \mathcal{K}_0^n$, we have*

$$\tilde{\delta}(\mathrm{I}_p^\pm K, \mathrm{I}\, K) \to 0 \quad \textit{and} \quad \tilde{\delta}(\mathrm{I}_p K, 2\,\mathrm{I}\, K) \to 0,$$

for $p \nearrow 1$.

This makes it plausible that there exists only a one-parametric set of L_p radial valuations for $p = 1$ whereas we have a two-parametric one for $p < 1$.

Before we start to prove the theorem, we remark that the radial function of I_p^+ can be given in terms of fractional derivatives. Suppose h is a continuous, integrable function on $\mathbb{R}$ that is m-times continuously differentiable in some neighborhood of zero. For $-1 < q < m$, $q \neq 0, 1, \ldots, m-1$, the fractional derivative of order q of the function h at zero is defined as

$$\begin{aligned} h^{(q)}(0) \;&=\; \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left(h(t) - h(0) - \cdots - h^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!} \right) dt \\ &\quad + \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} h(t)\, dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!(k-q)}. \end{aligned}$$

For a non-negative integer $k < m$ we have

$$\lim_{q\to k} h^{(q)}(0) = (-1)^k \frac{d^k}{dt^k} h(t)|_{t=0}. \tag{3.6}$$

If $0 < p < 1$ and $K \in \mathcal{K}_0^n$, we get by Fubini's theorem

$$\rho(\mathrm{I}_p^+ K, v)^p = \frac{1}{\Gamma(1-p)} \int_0^\infty t^{-p} A_{K,v}(t)\, dt = A_{K,v}^{(p-1)}(0) \tag{3.7}$$

where $A_{K,v}(t) := \mathrm{vol}(K \cap \{x \in \mathbb{R}^n : x \cdot v = t\})$ denotes the parallel section function of K in direction $v \in S^{n-1}$. For details on fractional derivatives we refer to [33, Section 2.6].

Proof. Suppose $0 < p < 1$. First, we prove the pointwise convergence

$$\rho(\mathrm{I}_p^+ K, u) \to \rho(\mathrm{I}\, K, u), \quad u \in S^{n-1}, \tag{3.8}$$

as p tends to one (cf. [33, page 9]). We can approximate $K \in \mathcal{K}_0^n$ with respect to the Hausdorff metric by bodies belonging to $\mathcal{K}_0^n$ which have infinitely smooth support functions (cf. [58, Theorem 3.3.1]). Note that

$$\rho(K^*, \cdot) = \frac{1}{h(K, \cdot)} \quad \text{for every } K \in \mathcal{K}_0^n. \tag{3.9}$$

This yields an approximation of K with respect to the radial metric by convex bodies with infinitely smooth radial functions. Note that by (3.5) and the representation of the radial function of I as spherical Radon transform (see [42, formula 8.5]) we obtain for $K_1, K_2 \in \mathcal{K}^n(r, R)$ with $R > 1$, $p > 1/2$

$$\begin{aligned} |\rho(\mathrm{I}_p^+ K_1, u) - \rho(\mathrm{I}_p^+ K_2, u)| &\leq 4R^n \pi^{n/2} \gamma_0^{-3} \tilde{\delta}(K_1, K_2), \\ |\rho(\mathrm{I}\, K_1, u) - \rho(\mathrm{I}\, K_2, u)| &\leq (n-1)\omega_{n-1} R^{n-2} \tilde{\delta}(K_1, K_2), \end{aligned}$$

where $\gamma_0 > 0$ denotes the minimum of the Gamma function on $\mathbb{R}^+$. So in order to derive (3.8), we can restrict ourselves to bodies $K \in \mathcal{K}_0^n$ with sufficiently smooth radial functions. For such bodies, $A_{K,u}$ is continuously differentiable in a neighbourhood of 0 (cf. [33, Lemma 2.4]). So (3.6) and (3.7) prove (3.8).
For $k \in \mathbb{N}$, let $0 < p_k < 1$ be an increasing sequence which converges to one. Define functions

$$\begin{aligned} f_k^1(u) &:= \rho(\mathrm{I}_{p_k}^+ K, u)^{-1} \left(\frac{\Gamma(1+n) V(K \cap u^+)}{\Gamma(1 - p_k + n)} \right)^{1/p_k}, \\ f_k^2(u) &:= \left(\frac{\Gamma(1 - p_k + n)}{\Gamma(1+n)} \right)^{1/p_k}, \\ f_k^3(u) &:= V(K \cap u^+)^{-1/p_k}, \end{aligned}$$

on S^{n-1}. We need the following result of Borell [5] which was strengthened in [17]: For a compact convex set K with nonempty interior and an integrable, concave function $f : K \to \mathbb{R}^+$, the function

$$F(q) := \left(\frac{1}{nB(q+1, n) V(K)} \int_K f(x)^q \, dx \right)^{\frac{1}{q}},$$

where B denotes the beta function, is decreasing on $(-1, 0)$. Thus the sequence f_k^1 is increasing.
Since o is an interior point of K, there exists a constant $c > 0$ such that $cV(K \cap u^+) \geq 1$ for every $u \in S^{n-1}$. Thus $c^{-1/p_k} f_k^3$ is increasing, too. So f_k^1

and $c^{-1/p_k} f_k^3$ are monotone sequences of continuous functions converging pointwise to continuous functions on a compact set. Therefore they converge uniformly by Dini's theorem. Thus

$$\rho(\mathrm{I}^+_{p_k} K, u)^{-1} = f_k^1 f_k^2 f_k^3(u) \to \rho(\mathrm{I}\, K, u)^{-1}$$

uniformly for $k \to \infty$.
The other assertion of the theorem immediately follows from the definition $\mathrm{I}^-_p K = \mathrm{I}^+_p(-K)$ and the relation

$$\mathrm{I}_p K = \mathrm{I}^+_p K \,\tilde{+}_p\, \mathrm{I}^-_p K.$$

$\square$

3.2 An L_p Ellipsoid Formula

In [8] Busemann showed that the volume of a centered ellipsoid $E \subset \mathbb{R}^n$ can essentially be obtained by averaging over certain powers of $(n-1)$-dimensional volumes of its hyperplane sections. To be precise,

$$V(E)^{n-1} = \frac{\kappa_n^{n-2}}{n\kappa_{n-1}^n} \int_{S^{n-1}} \mathrm{vol}(E \cap u^\perp)^n \, du. \tag{3.10}$$

This formula is the hyperplane case of a more general version due to Furstenberg and Tzkoni [12]. They proved a similar formula for i-dimensional sections, $0 < i < n$, where the average is taken with respect to the rotation invariant probability measure on the i-dimensional Grassmann manifold. Guggenheimer [23] established a companion of (3.10) which involves the surface area of E, $S(E)$:

$$V(E)^{n-1} S(E) = \frac{\kappa_n^{n-1}}{\kappa_{n-1}^{n+1}} \int_{S^{n-1}} \mathrm{vol}(E \cap u^\perp)^{n+1} \, du. \tag{3.11}$$

Lutwak [44] obtained a more general ellipsoid formula which contains (3.10) and (3.11) as special cases:

$$\frac{\kappa_n^{n-2}}{\kappa_{n-1}^n} \int_{S^{n-1}} \mathrm{vol}(E \cap u^\perp)^{n+1} \,\mathrm{vol}(F \cap u^\perp)^{-1} \, du = V(E)^{n-1} V(F)^{-1} \int_{\partial E} h(F,u) \, d\nu(u).$$

Here, E, F are centered ellipsoids and $d\nu(u)$ denotes the area element of ∂E whose outer unit normal is u. Moreover, this result establishes a formula similar to (3.11) involving the mean width of E.

We extend this formula using L_p intersection bodies. From our equation one can obtain the formulas of Busemann, Guggenheimer, and Lutwak by taking the limit $p \to 1$.

Theorem 6. *For $0 < p < 1$ and two centered ellipsoids E and F we have*

$$\widetilde{V}_{p-2}(\mathrm{I}_p^+ E, \mathrm{I}_p^+ F) = r^n_{\mathrm{I}_p^+}\kappa_n^{2-n/p} V(E)^{(n-3p+2)/p} V(F)^{(p-2)/p} V_{2-p}(E, F). \tag{3.12}$$

The terms V_{2-p} and $\widetilde{V}_{p-2}$ stand for certain L_p mixed and dual L_p mixed volumes. As we mentioned in the introduction, Lutwak [45] extended the classical Brunn-Minkowski theory to the L_p Brunn Minkowski theory in the Ninteen Ninetees. The starting point of his studies was the mixed L_p-Quermassintegral. For $p \geq 1$, Lutwak defined

$$V_p(K, L) = \frac{p}{n} \lim_{\varepsilon \downarrow 0} \frac{V(K +_p \varepsilon^{1/p} L) - V(K)}{\varepsilon} \tag{3.13}$$

and proved the formula

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} \, dS(K, u), \tag{3.14}$$

where $K, L \in \mathcal{K}_0^n$ and $S(K, \cdot)$ denotes the surface area measure of K. The corresponding notion within the dual L_p Brunn-Minkowski follows from merging volume with radial L_p addition. By the polar formula for volume we have

$$\widetilde{V}_p(K, L) := \frac{p}{n} \lim_{\varepsilon \downarrow 0} \frac{V(K \tilde{+}_p \varepsilon^{1/p} L) - V(K)}{\varepsilon} = \frac{1}{n} \int_{S^{n-1}} \rho(L, u)^p \rho(K, u)^{n-p} \, du \tag{3.15}$$

for two star bodies $K, L \in \mathcal{S}^n$ and $0 < p < n$. If K and L contain the origin in their interiors, we can define $\widetilde{V}_p(K, L)$ for arbitrary p.

Proof. We denote by $\bar{E}$, $\bar{F}$ the ellipsoids which are dilates of E, F with volume κ_n. So

$$\bar{E} = \lambda E \quad \text{and} \quad \bar{F} = \mu F$$

where

$$\lambda := (\kappa_n / V(E))^{1/n} \quad \text{and} \quad \mu := (\kappa_n / V(F))^{1/n}.$$

We write $\phi_{\bar{E}}$ for the linear transformation which maps the unit ball B_n to $\bar{E}$. So $\phi_{\bar{E}}$ has determinant ± 1.
The main tool in the proof will be the equation

$$\widetilde{V}_{p-2}(\bar{E}^*, \bar{F}^*) = V_{2-p}(\bar{E}, \bar{F}). \tag{3.16}$$

From (3.13) and (3.15) we get for $\phi \in \mathrm{SL}(n)$

$$\widetilde{V}_{p-2}(\phi K, \phi L) = \widetilde{V}_{p-2}(K, L), \quad V_{p-2}(\phi K, \phi L) = V_{p-2}(K, L).$$

Identity (3.14)shows

$$V_{2-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^{2-p} h(K, u)^{p-1} \, dS(K, u).$$

Hence
$$V_{2-p}(B_n, L) = \widetilde{V}_{p-2}(B_n, L^*).$$
These preparations enable us to derive (3.16) by
$$\begin{aligned}
\widetilde{V}_{p-2}(\bar{E}^*, \bar{F}^*) &= \widetilde{V}_{p-2}((\phi_{\bar{E}} B_n)^*, \bar{F}^*) = \widetilde{V}_{p-2}(\phi_{\bar{E}}^{-t} B_n, \bar{F}^*) = \widetilde{V}_{p-2}(B_n, \phi_{\bar{E}}^t \bar{F}^*) \\
&= V_{2-p}(B_n, (\phi_{\bar{E}}^t \bar{F}^*)^*) = V_{2-p}(B_n, \phi_{\bar{E}}^{-1} \bar{F}) = V_{2-p}(\phi_{\bar{E}} B_n, \bar{F}) \\
&= V_{2-p}(\bar{E}, \bar{F}).
\end{aligned}$$
We use obvious homogeneity properties of $\widetilde{V}_{p-2}$ and V_{p-2}, which follow from their integral representations, for extending (3.16) to our ellipsoids E and F. Indeed,
$$\begin{aligned}
\widetilde{V}_{p-2}(E^*, F^*) &= \widetilde{V}_{p-2}((\lambda^{-1}\bar{E})^*, (\mu^{-1}\bar{F})^*) = \widetilde{V}_{p-2}(\lambda \bar{E}^*, \mu \bar{F}^*) \\
&= \lambda^{n+2-p}\mu^{p-2}\widetilde{V}_{p-2}(\bar{E}^*, \bar{F}^*) = \lambda^{n+2-p}\mu^{p-2} V_{2-p}(\bar{E}, \bar{F}) \\
&= \lambda^{2n} V_{2-p}(E, F). \qquad (3.17)
\end{aligned}$$
As was shown at the beginning of this chapter, I_p^+ maps the unit ball B_n to the ball $r_{\mathrm{I}_p^+} B_n$, so by the $\mathrm{GL}(n)$ contravariance of I_p^+ we have
$$\begin{aligned}
\mathrm{I}_p^+ E &= \mathrm{I}_p^+ \lambda^{-1}\bar{E} = \lambda^{1-n/p}\, \mathrm{I}_p^+ \bar{E} = \lambda^{1-n/p}\, \mathrm{I}_p^+ \phi_{\bar{E}} B_n \\
&= \lambda^{1-n/p} r_{\mathrm{I}_p^+} \phi_{\bar{E}}^{-t} B_n = \lambda^{1-n/p} r_{\mathrm{I}_p^+} \bar{E}^* \\
&= \lambda^{-n/p} r_{\mathrm{I}_p^+} E^*.
\end{aligned}$$
We obtain
$$\begin{aligned}
\widetilde{V}_{p-2}(\mathrm{I}_p^+ E, \mathrm{I}_p^+ F) &= r_{\mathrm{I}_p^+}^n \lambda^{-n/p(n+2-p)} \mu^{-n/p(p-2)} \widetilde{V}_{p-2}(E^*, F^*) \\
&= r_{\mathrm{I}_p^+}^n \lambda^{-n/p(n+2-p)+2n} \mu^{-n/p(p-2)} V_{2-p}(E, F).
\end{aligned}$$
Substituting the values of λ and μ finishes the proof. □

An application of Theorem 5 in (3.12) for the special choice $E = F$ proves Busemann's formula (3.10). Guggenheimer's relation (3.11) is the limiting case $p \nearrow 1$ for $F = B_n$ of (3.12). Taking the limit $p \nearrow 1$ in (3.12) without further assumptions on the ellipsoids yields the formula of Lutwak for intersection bodies.

3.3 Injectivity Results

We start by collecting some basic facts about spherical harmonics. All of them can be found, for example, in [21].

Let $\{Y_{kj} : j = 1, \ldots, N(n,k)\}$ be an orthonormal basis of the real vector space of spherical harmonics of order $k \in \mathbb{N}_0$ and dimension n. We write

$$f \sim \sum_{k=0}^{\infty} Y_k \tag{3.18}$$

for the condensed harmonic expansion of a function $f \in L_2(S^{n-1})$ where

$$Y_k = \sum_{j=1}^{N(n,k)} (f, Y_{kj}) Y_{kj}.$$

Here, (f,g) stands for the usual scalar product $\int_{S^{n-1}} fg\,du$ on $L_2(S^{n-1})$. The norm induced by this scalar product is denoted by $\|.\|_2$. For a bounded integrable function $\Phi : [-1,1] \to \mathbb{R}$ we define a transformation T_Φ on $C(S^{n-1})$ by

$$(\mathrm{T}_\Phi\, f)(v) := \int_{S^{n-1}} \Phi(u \cdot v) f(u)\, du, \quad v \in S^{n-1}.$$

If Y_k is a spherical harmonic of degree k, then the Funk-Hecke Theorem states that

$$\mathrm{T}_\Phi\, Y_k = a_{n,k}(\mathrm{T}_\Phi) Y_k \tag{3.19}$$

with

$$a_{n,k}(\mathrm{T}_\Phi) = \omega_{n-1} \int_{-1}^{1} \Phi(t) P_k^n(t)(1-t^2)^{(n-3)/2}\, dt \tag{3.20}$$

where P_k^n is the Legendre polynomial of dimension n and degree k. If (3.18) holds, then

$$\mathrm{T}_\Phi f \sim \sum_{k=0}^{\infty} a_{n,k}(\mathrm{T}_\Phi) Y_k. \tag{3.21}$$

This remains true for arbitrary Φ provided the induced transformation T_Φ maps continuous functions to continuous functions, satisfies $(\mathrm{T}_\Phi\, f, g) = (f, \mathrm{T}_\Phi\, g)$ for all $f, g \in C(S^{n-1})$ as well as (3.19). So (3.21) and Parseval's equality show that such transformations T_Φ are injective on $C(S^{n-1})$ if all multipliers $a_{n,k}(\mathrm{T}_\Phi)$ are not equal to zero.
If $m \geq 0$, Δ_o^m stands for the m-times iterated Beltrami operator. For a function $f : S^{n-1} \to \mathbb{R}$ for which (3.18) holds and $\Delta_o^m f$ exists and is continuous, we have

$$\Delta_o^m f \sim (-1)^m \sum_{k=0}^{\infty} k^m (k+n-2)^m Y_k. \tag{3.22}$$

We will deal with smooth functions on the sphere and their development into series of spherical harmonics. For this purpose, we need information on the behavior of derivatives of spherical harmonics. For an n-dimensional spherical harmonic Y_k of order k and all $u \in S^{n-1}$

$$|(D^\alpha Y_k(x/\|x\|))_{x=u}| \leq c_{n,|\alpha|} k^{n/2+|\alpha|-1} \|Y_k\|_2, \tag{3.23}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha = \partial^{|\alpha|}/(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Define

$$c_{n,k,p} = \frac{\pi^{n/2-1}\Gamma(1-p)\Gamma((k+p)/2)}{2^{-p}\Gamma((n+k-p)/2)}. \tag{3.24}$$

Lemma 19. *Assume $p < 1$ and that p is not an integer. Then the multipliers of C^+_{-p} and C_{-p} are*

$$a_{n,k}(\mathrm{C}^+_{-p}) = \begin{cases} (-1)^{k/2+1} c_{n,k,p} \cos\left(\pi \frac{1+p}{2}\right) & k \text{ even}, \\ (-1)^{(k-1)/2} c_{n,k,p} \sin\left(\pi \frac{1+p}{2}\right) & k \text{ odd}, \end{cases}$$

and

$$a_{n,k}(\mathrm{C}_{-p}) = \begin{cases} (-1)^{k/2+1} 2 c_{n,k,p} \cos\left(\pi \frac{1+p}{2}\right) & k \text{ even}, \\ 0 & k \text{ odd}. \end{cases}$$

The multipliers $a_{n,k}(\mathrm{C}_{-p})$ appeared in their full generality already in [32] and [56]. In our situation they are an obvious consequence of the formula for $a_{n,k}(\mathrm{C}^+_{-p})$. In dimensions three and higher, Rubin [57] calculated $a_{n,k}(\mathrm{C}^+_{-p})$ using associated Legendre functions. We present a more elementary proof and establish the representation of the multipliers also in dimension two.

Proof. First, we assume that $n = 2$. Then the relation

$$P_k^2(t) = \cos(k \arccos t), \qquad k \in \mathbb{N}_0.$$

holds for $t \in [-1, 1]$. Therefore we obtain

$$\begin{aligned} a_{2,k}(\mathrm{C}^+_{-p}) &= 2\int_0^1 t^{-p}(1-t^2)^{-1/2} \cos(k \arccos t)\, dt \\ &= 2\int_0^{\pi/2} \cos^{-p} t \cos kt \, dt \\ &= \frac{\pi\Gamma(1-p)}{2^{-p}\Gamma((2-p+k)/2)\Gamma((2-p-k)/2)}, \end{aligned}$$

where the last equality follows from [54, 2.5.11, formula 22]. If $x \in \mathbb{C}$ is not a real integer, then Euler's reflection formula states

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Thus

$$\frac{\pi}{\Gamma((2-p-k)/2)} = \Gamma((p+k)/2)\sin(\pi(p+k)/2),$$

which finally gives

$$a_{2,k}(\mathrm{C}^+_{-p}) = \frac{\Gamma(1-p)\sin(\pi(k+p)/2)\Gamma((k+p)/2)}{2^{-p}\Gamma((2+k-p)/2)}.$$

An application of a standard addition theorem to the involved sine proves the first part of the lemma in dimension two.
Now, let $n \geq 3$. Then we can use the following connection between Legendre polynomials P_k^n and Gegenbauer polynomials $C_k^{(n-2)/2}$:

$$P_k^n(t) = \binom{k+n-3}{n-3}^{-1} C_k^{(n-2)/2}(t). \tag{3.25}$$

Assume further that $k = 2m+1$, $m \in \mathbb{N}_0$. Combining (3.25) and (3.20) we obtain

$$a_{n,k}(\mathrm{C}^+_{-p}) = \omega_{n-1}\binom{k+n-3}{n-3}^{-1}\int_0^1 t^{-p}(1-t^2)^{(n-3)/2}C_k^{(n-2)/2}(t)\,dt.$$

The odd part of [55, 2.21.2, formula 5] yields the following expression for $a_{n,k}(\mathrm{C}^+_{-p})$:

$$\omega_{n-1}\binom{k+n-3}{n-3}^{-1}\frac{(-1)^m 2^{2m}}{(2m+1)!}\left(\frac{n-2}{2}\right)_{m+1}\left(\frac{1+p}{2}\right)_m B\left(\frac{n-1}{2}+m, \frac{2-p}{2}\right),$$

where $(a)_l$ denotes the Pochhammer symbol. Rewriting this in terms of Gamma functions gives

$$\begin{aligned} a_{n,k}(\mathrm{C}^+_{-p}) \quad = \quad & \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)}\binom{k+n-3}{n-3}^{-1}\frac{(-1)^{(k-1)/2}2^{k-1}}{k!} \\ & \frac{\Gamma((n+k-1)/2)}{\Gamma((n-2)/2)}\frac{\Gamma((p+k)/2)}{\Gamma((1+p)/2)}\frac{\Gamma((n-2+k)/2)\Gamma((2-p)/2)}{\Gamma((n+k-p)/2)}. \end{aligned} \tag{3.26}$$

Formula (3.4) yields

$$\begin{aligned} \Gamma\left(\frac{n-2+k}{2}\right)\Gamma\left(\frac{n-1+k}{2}\right) &= \frac{\Gamma(n-2+k)\sqrt{\pi}}{2^{n-3+k}}, \\ \Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n-1}{2}\right) &= \frac{\Gamma(n-2)\sqrt{\pi}}{2^{n-3}}. \end{aligned}$$

Substituting this in relation (3.26) and using an representation of the binomial coefficient occuring in (3.26) in terms of gamma functions one obtains

$$a_{n,k}(\mathrm{C}^+_{-p}) = \frac{\pi^{(n-1)/2}(-1)^{(k-1)/2}\Gamma((k+p)/2)\Gamma((2-p)/2)}{\Gamma((1+p)/2)\Gamma((n+k-p)/2)}.$$

Since

$$\begin{aligned}
\Gamma\left(\frac{1+p}{2}\right) &= \frac{\pi}{\Gamma((1-p)/2)\sin(\pi(1+p)/2)}, \\
\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) &= \frac{\sqrt{\pi}\Gamma(1-p)}{2^{-p}},
\end{aligned}$$

we obtain the desired representation of $a_{n,k}(\mathrm{C}^+_{-p})$ in the odd case.
If k is even, one can proceed in a similar way by using the even case of [55, formula 2.21.2, 5]. The computation of the multipliers of C_{-p} is an easy consequence of the results above since Legendre polynomials of even degree are even and of odd degree are odd. □

An immediate consequence of Lemma 19 and the remarks before it is

Theorem 7. *If $p < 1$ is not an integer, then the transformations $\mathrm{C}^+_{-p} : C(S^{n-1}) \to C(S^{n-1})$ and $\mathrm{C}_{-p} : C_e(S^{n-1}) \to C_e(S^{n-1})$ are injective.*

($C_e(S^{n-1})$ stands for continuous, even functions on the sphere.) The representations of the multipliers $a_{n,k}(\mathrm{C}^+_{-p})$ and $a_{n,k}(\mathrm{C}_{-p})$ obtained in Lemma 19 allow us to extend them to all $p \in \mathbb{R}\backslash\mathbb{Z}$. Moreover, they give us the possibility of examining their growth as k becomes large. In fact, by Stirling's formula

$$|a_{n,k}(\mathrm{C}^+_{-p})| \;=\; O(k^{|p|}), \quad k \to \infty, \tag{3.27}$$

for $p \in \mathbb{R}\backslash\mathbb{Z}$. Moreover, for $0 < p < 1$ exist constants c_1, c_2 which depend only on n such that for sufficiently large k

$$|a_{n,k}(\mathrm{C}^+_{-p})^{-1}| \leq \begin{cases} c_1 \left|\cos\left(\pi\frac{1+p}{2}\right)\right|^{-1} 2^{-p/2}\Gamma(1-p)^{-1}k^{\beta} & k \text{ even,} \\ c_2 \left|\sin\left(\pi\frac{1+p}{2}\right)\right|^{-1} 2^{-p/2}\Gamma(1-p)^{-1}k^{\beta} & k \text{ odd,} \end{cases} \tag{3.28}$$

where $\beta = n/2 - p$.
For $f \in C^\infty(S^{n-1})$ which satisfies (3.18) we set for arbitrary $p \in \mathbb{R}\backslash\mathbb{Z}$

$$\mathrm{C}^+_{-p} f(u) := \sum_{k=0}^{\infty} a_{n,k}(\mathrm{C}^+_{-p})Y_k(u), \quad \text{for } u \in S^{n-1}. \tag{3.29}$$

In order to show that this is well defined, note the following. An application of Parseval's equality to (3.22) proves $\|Y_k\|_2 = O(k^{-2m})$, $k \to \infty$. Combining this with (3.27) and (3.23) for $\alpha = 0$, it follows that the series occuring in (3.29) is uniformly convergent on S^{n-1} since f is infinitely smooth. In fact, $\mathrm{C}^+_{-p}(f)$ is infinitely smooth by analogous arguments and well-known facts on convergence of infinite series.
Let $C^\infty_e(S^{n-1})$ and $C^\infty_o(S^{n-1})$ denote the subspaces of even and odd infinitely smooth functions on the sphere, respectively. Denote by π_e, π_o the projections

which assign to each $f \in C^\infty(S^{n-1})$ its even part $(f(u)+f(-u))/2$ and odd part $(f(u)-f(-u))/2$, respectively. Define

$$\begin{aligned} c_e^{-1} &:= 2^n\pi^{n-2}\Gamma(1-p)\Gamma(1-n+p)\cos(\pi(1+p)/2)\cos(\pi(1+n-p)/2),\\ c_o^{-1} &:= 2^n\pi^{n-2}\Gamma(1-p)\Gamma(1-n+p)\sin(\pi(1+p)/2)\sin(\pi(1+n-p)/2). \end{aligned}$$

The terms which involve gamma functions with a dependence on k and p in the representations of the multipliers $a_{n,k}(\mathrm{C}^+_{-p})$ reverse if one replaces p by $n-p$. Therefore we obtain the following

Theorem 8. *If p is not an integer, the transformation C^+_p is a bijection of $C^\infty(S^{n-1})$. Moreover, the inversion formula*

$$(\mathrm{C}^+_{-p})^{-1} = \mathrm{C}^+_{p-n} \circ (c_e\pi_e + c_o\pi_o)$$

holds.

For $n \geq 3$ this was shown in [57] and the inversion formula for C_{-p} can be found in [56].
Now, we return to geometry. The geometric reformulation of Theorem 7 is as follows.

Theorem 9. *For $p < 1$, $p \notin \mathbb{Z}$, the operators $\mathrm{I}^\pm_p : \mathcal{S}^n \to \mathcal{S}^n$ and $\mathrm{I}_p : \mathcal{S}^n_e \to \mathcal{S}^n_e$ are injective.*

($\mathcal{S}^n_e$ denotes the set of symmetric star bodies in $\mathbb{R}^n$.) We point out that the nonsymmetric L_p intersection body operator I^+_p determines also nonsymmetric star bodies uniquely. This is in contrast to its classical analogue which is injective only on centrally symmetric sets. Note that there exists work of Groemer [22] and Goodey and Weil [19] which ensures that certain sections determine also a nonsymmetric body uniquely. But in the L_p theory, the nonsymmetric L_p intersection body operator is itself injective on all star bodies.

A stability version of Theorem 9 is as follows.

Theorem 10. *Suppose $0 < p < 1$. For $\gamma \in (0, 1/(1+\beta))$ and $K, L \in \mathcal{K}^n(r,R)$ there is a constant c_1 depending only on r, R, p, n, γ such that*

$$\delta(K,L) \leq c_1\tilde{\delta}(\mathrm{I}^+_p K, \mathrm{I}^+_p L)^{2\gamma/(n+1)}.$$

If in addition K and L are symmetric, then

$$\delta(K,L) \leq c_2\tilde{\delta}(\mathrm{I}_p K, \mathrm{I}_p L)^{2\gamma/(n+1)},$$

where c_2 is again a constant depending just on r, R, p, n, γ.

The proof of this result follows the approach suggested by Bourgain and Lindenstrauss [6] which was also used in [27] to establish stability results involving transformations T_Φ for bounded Φ.

Proof. In the proof we denote by $d_1, d_2, \dots$ constants which depend on r, R, p, γ and n. We write $c_1, c_2, \dots$ for constants depending on r, R, n only. Define

$$\tilde{\delta}_2(K, L) = \|\rho(K, \cdot) - \rho(L, \cdot)\|_2.$$

The ball $B(0, r)$ is contained in K, L, hence

$$\tilde{\delta}_2(K, L) \leq ((n-p)r^{n-p-1})^{-1} \|\rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}\|_2.$$

Groemer [20] proved that

$$\delta(K, L) \leq 2 \left(\frac{8\kappa_{n-1}}{n(n+1)} \right)^{-1/(n+1)} R^2 r^{-(n+3)/(n+1)} \tilde{\delta}_2(K, L)^{2/(n+1)}.$$

Therefore

$$\delta(K, L) \leq c_1((n-p)r^{n-p-1})^{-2/(n+1)} \|\rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}\|_2^{2/(n+1)}. \tag{3.30}$$

The operator I_p^+ maps balls to balls by (3.5). Since $\mathrm{I}_p^+ B(0, r) \subset \mathrm{I}_p^+ K, \mathrm{I}_p^+ L$, we get

$$\|\rho(\mathrm{I}_p^+ K, \cdot)^p - \rho(\mathrm{I}_p^+ L, \cdot)^p\|_2 \leq p(r^{n/p-1} r_{\mathrm{I}_p^+})^{p-1} \tilde{\delta}_2(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L).$$

Together with the trivial estimate $\tilde{\delta}_2(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L) \leq \sqrt{\omega_n} \tilde{\delta}(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L)$ we deduce that

$$\|\rho(\mathrm{I}_p^+ K, \cdot)^p - \rho(\mathrm{I}_p^+ L, \cdot)^p\|_2 \leq c_2(r^{n/p-1} r_{\mathrm{I}_p^+})^{p-1} \tilde{\delta}(\mathrm{I}_p^+ K, \mathrm{I}_p^+ L). \tag{3.31}$$

So by (3.30) and (3.31) it is enough to prove

$$\|\rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}\|_2 \leq d_7 \|\rho(\mathrm{I}_p^+ K, \cdot)^p - \rho(\mathrm{I}_p^+ L, \cdot)^p\|_2^\gamma,$$

for some constant d_7. For simplicity we write $f := \rho(K, \cdot)^{n-p} - \rho(L, \cdot)^{n-p}$ and $\bar{f} := 1/\Gamma(1-p) f$.
Relation (3.9) and the estimate

$$|h(K_1, u) - h(K_2, v)| \leq \hat{R}\|u - v\| + \max\{|u|, |v|\} \delta(K, L)$$

for arbitrary vectors u, v and convex bodies K_1, K_2 contained in $B(0, \hat{R})$ (cf. [58, Lemma 1.8.10]) proves that f is a Lipschitz function on S^{n-1} with a Lipschitz constant $\Lambda(f)$ which is at most $2(n-p)R^{n-p+1} r^{-1}$.
Assume (3.18) holds for f. Since $f \in C(S^{n-1})$, the Poisson transform f_τ satisfies

$$f_\tau(u) := \frac{1}{\omega_n} \int_{S^{n-1}} \frac{1 - \tau^2}{(1 + \tau^2 - 2\tau(u \cdot v))^{n/2}} f(v)\, dv = \sum_{k=0}^{\infty} \tau^k Y_k(u), \quad \text{for } u \in S^{n-1}.$$

for $0 < \tau < 1$ (cf. [21, Theorem 3.4.16]).
Since $(-\beta/(e\ln\tau))^\beta$ is the maximal value of the function $x \to x^\beta \tau^x, x > 0$, we have

$$k^\beta \tau^k (1-\tau)^\beta \le \left(\frac{\beta}{-e\ln\tau}\right)^\beta (1-\tau)^\beta \le \left(\frac{\beta}{e}\right)^\beta \left(\frac{1-\tau}{-\ln\tau}\right)^\beta \le \left(\frac{\beta}{e}\right)^\beta, \quad \text{for } k \in \mathbb{N}_0. \tag{3.32}$$

By (3.28) we derive the existence of a constant c_3 and a positive integer N such that

$$\begin{aligned} k^{-\beta} &\le c_3 \max\left\{\left|\cos\left(\pi\frac{1+p}{2}\right)\right|^{-1}, \left|\sin\left(\pi\frac{1+p}{2}\right)\right|^{-1}\right\}\cdot \\ &\quad \cdot 2^{-p/2}\Gamma(1-p)^{-1}|a_{n,k}(\mathrm{C}^+_{-p})| \\ &=: c_3\alpha(p)|a_{n,k}(\mathrm{C}^+_{-p})| \end{aligned}$$

for $k \ge N$. Define

$$d_1 = \max\left\{\max_{1\le k<N}\{\tau^k(\beta/e)^{-\beta}\alpha(p)^{-1}(1-\tau)^\beta|a_{n,k}(\mathrm{C}^+_{-p})|^{-1}\}, c_3\right\}.$$

Thus by (3.32)

$$\tau^k \le d_1(\beta/e)^\beta\alpha(p)(1-\tau)^{-\beta}|a_{n,k}(\mathrm{C}^+_{-p})| =: d_2(1-\tau)^{-\beta}|a_{n,k}(\mathrm{C}^+_{-p})|, \quad \text{for } k \in \mathbb{N}_0.$$

Combining this with Parseval's equation and (3.21) gives

$$\begin{aligned} \|f_\tau\|_2^2 &= \sum_{k=0}^\infty \tau^{2k}\|Y_k\|_2^2 \le d_2^2(1-\tau)^{-2\beta}\sum_{k=0}^\infty |a_{n,k}(\mathrm{C}^+_{-p})|^2\|Y_k\|_2^2 \\ &= d_2^2(1-\tau)^{-2\beta}\|\,\mathrm{C}^+_{-p} f\|_2^2 = d_3^2(1-\tau)^{-2\beta}\|\,\mathrm{C}^+_{-p}\bar{f}\|_2^2 \end{aligned} \tag{3.33}$$

where $d_3 := \Gamma(1-p)d_2$. The Cauchy-Schwarz inequality, the estimate $\|f - f_\tau\|_\infty \le c_4\Lambda(f)(1-\tau)\ln(2/(1-\tau))$ for $\tau \in [1/4, 1)$ (cf. [21, Lemma 5.5.8]) and (3.33) yield

$$\begin{aligned} \|f\|_2^2 &\le |(f, f - f_\tau)| + |(f, f_\tau)| \le \int_{S^{n-1}} |f(u)|\,du\|f - f_\tau\|_\infty + \|f\|_2\|f_\tau\|_2 \\ &\le (\sqrt{\omega_n}\|f - f_\tau\|_\infty + \|f_\tau|_2)\|f\|_2 \\ &\le \left(c_5 r^{-1}(n-p)R^{n-p+1}(1-\tau)\ln\frac{2}{1-\tau} + d_3(1-\tau)^{-\beta}\|\,\mathrm{C}^+_{-p}\bar{f}\|_2\right)\|f\|_2. \end{aligned} \tag{3.34}$$

By (3.5), the quotient $\|\,\mathrm{C}^+_p\bar{f}\|_2/R^{n-p}$ can be bounded from above by $c_6 r^p_{\mathrm{I}^+_p}$. If we set

$$d_4 := c_6\frac{(4/3)^{1+\beta}}{\ln(8/3)} r^p_{\mathrm{I}^+_p},$$

then

$$d_4(1-\tau)^{1+\beta}\ln\frac{2}{1-\tau}=\frac{\|\,\mathrm{C}^+_{-p}\bar{f}\|_2}{R^{n-p}}.$$

for a certain value $\tau\in[1/4,1)$. So finally for this τ and every $\gamma\in(0,1/(1+\beta))$ we have by (3.34)

$$\begin{aligned}
\|f\|_2 &\leq \left(c_5 r^{-1}(n-p)R^{n-p+1}d_4^{-1}R^{p-n}+d_3\right)\|\,\mathrm{C}^+_{-p}\bar{f}\|_2(1-\tau)^{-\beta}\\
&=: d_5\|\,\mathrm{C}^+_{-p}\bar{f}\|_2(1-\tau)^{-\beta}\\
&= R^{(n-p)(1-\gamma)}d_5 d_4^{1-\gamma}(1-\tau)^{1-\gamma(1+\beta)}\left(\log\frac{2}{1-\tau}\right)^{1-\gamma}\|\,\mathrm{C}^+_{-p}\bar{f}\|_2^{\gamma}\\
&\leq R^{(n-p)(1-\gamma)}d_5 d_4^{1-\gamma}\max\{(3/4)^{1-\gamma(1+\beta)}(\ln(8/3))^{1-\gamma},\\
&\quad 2^{1-\gamma(1+\beta)}((1-\gamma)/(e(1-\gamma(1+\beta))))^{1-\gamma}\}\|\,\mathrm{C}^+_{-p}\bar{f}\|_2^{\gamma}\\
&\leq d_5 d_4^{1-\gamma}d_6\|\,\mathrm{C}^+_{-p}\bar{f}\|_2^{\gamma}.
\end{aligned}$$

In conclusion we obtain

$$\begin{aligned}
\delta(K,L) &\leq c_7((n-p)r^{n-p-1})^{-2/(n+1)}(d_5 d_4^{1-\gamma}d_6)^{2/(n+1)}\cdot\\
&\quad\cdot\left(c_2 p(r^{n/p-1}r_{\mathrm{I}_p^+})^{p-1}\right)^{\gamma/(n+1)}\tilde{\delta}(\mathrm{I}_p^+K,\mathrm{I}_p^+L)^{2\gamma/(n+1)}.
\end{aligned}$$

This settles the first part of the theorem. The proof of the second part follows the same lines noting that f is now an even function and therefore the odd coefficients in the condensed harmonic expansion of f vanish. □

An other application of Theorem 5 is the proof of a stability theorem for intersection bodies (cf. [20]).

Corollary. *For $\gamma\in(0,2/n)$ and centrally symmetric $K,L\in\mathcal{K}^n(r,R)$ there is a constant c depending only on r,R,n,γ such that for*

$$\delta(K,L)\leq c\tilde{\delta}(\mathrm{I}\,K,\mathrm{I}\,L)^{2\gamma/(n+1)}.$$

Proof. Choose $\gamma_p=2/(n-2p+2)+\gamma-2/n$. Then the second part of Theorem 10 gives

$$\delta(K,L)\leq c_2(\tilde{\delta}(\mathrm{I}_p K,2\,\mathrm{I}\,K)+\tilde{\delta}(2\,\mathrm{I}\,K,2\,\mathrm{I}\,L)+\tilde{\delta}(2\,\mathrm{I}\,L,\mathrm{I}_p L))^{2\gamma_p/(n+1)}.$$

The sine-term in the definition of $\alpha(p)$ is not involved within the centrally symmetric case. Therefore the constant c_2 converges as p tends to one as one can see from the definitions of constants d_i. □

The next two results particularly show the announced analogy between intersection bodies and their L_p analogues. A star body is called L_p intersection body if it is contained in $\mathrm{I}_p\,\mathcal{S}^n$.

Theorem 11. *Suppose $0 < p < 1$ and let $S \in \mathcal{S}^n$ be an L_p intersection body. Then there exists a unique centered star body S_c with $\mathrm{I}_p\, S_c = S$. Moreover, this star body is characterized by having smaller volume than any other star body in the preimage $\mathrm{I}_p^{-1}\, S$.*

For intersection bodies, the corresponding result was proved by Lutwak [42]. To construct the desired body of the last theorem we need the following definition. For each star body $K \in \mathcal{S}^n$ we define a symmetric star body by

$$\widetilde{\nabla}_p K := \frac{1}{2} \cdot K \mathbin{\tilde{+}}_{n-p} \frac{1}{2} \cdot (-K).$$

The expression $1/2 \cdot K$ stands for $2^{1/(p-n)}K$. We will make use of the following inequalities for $0 < p < 1$. From Hölder's inequality and the polar formula for volume we obtain the dual L_{n-p} Minkowski and the dual L_p Minkowski inequality

$$\begin{aligned} \widetilde{V}_{n-p}(K, L)^n &\leq V(K)^p V(L)^{n-p}, && (3.35)\\ \widetilde{V}_p(K, L)^n &\leq V(K)^{n-p} V(L)^p, && (3.36) \end{aligned}$$

for arbitrary star bodies K and L. If $K, L \neq \{0\}$, equality holds in (3.35) or (3.36) if and only if K and L are dilates. The polar formula for volume of star bodies together with the linearity properties of dual mixed volumes give

$$V(K\tilde{+}_{n-p}L) = \widetilde{V}_{n-p}(K\tilde{+}_{n-p}L, K\tilde{+}_{n-p}L) = \widetilde{V}_{n-p}(K\tilde{+}_{n-p}L, K)+\widetilde{V}_{n-p}(K\tilde{+}_{n-p}L, L).$$

Thus (3.35) yields the dual L_p Kneser-Süss inequality

$$V(K \mathbin{\tilde{+}}_{n-p} L)^{(n-p)/n} \leq V(K)^{(n-p)/n} + V(L)^{(n-p)/n}. \tag{3.37}$$

Equality holds for star bodies $K, L \in \mathcal{S}^n$, $K, L \neq \{0\}$, if and only if they are dilates.

Proof. Let $\overline{S} \in \mathcal{S}^n$ be chosen such that $\mathrm{I}_p\, \overline{S} = S$. The star body

$$S_c := \widetilde{\nabla}_p \overline{S}$$

is centrally symmetric. Representation (3.1) immediately shows that $\mathrm{I}_p\, \widetilde{\nabla}_p S_c = S$. But I_p is injective on centrally symmetric sets which proves the first part of the theorem.
Since $(1/2) \cdot K = (1/2)^{1/(n-p)}K$, we obtain from (3.37) that

$$V(\widetilde{\nabla}_p K) \leq V(K) \tag{3.38}$$

with equality if and only if K is centered. If K is an arbitrary star body which is mapped to S by I_p, then $\widetilde{\nabla}_p K = \widetilde{\nabla}_p \overline{S}$. So

$$V(\widetilde{\nabla}_p \overline{S}) = V(\widetilde{\nabla}_p K) \leq V(K)$$

with equality if and only if K is centered by (3.38). This establishes the second part of the theorem. $\square$

Theorem 12. *For given star bodies $K, L \in \mathcal{S}^n$ and $0 < p < 1$, the following statements are equivalent:*

$$\begin{aligned} \mathrm{I}_p K &= \mathrm{I}_p L, && (3.39)\\ \widetilde{\nabla}_p K &= \widetilde{\nabla}_p L, && (3.40)\\ \widetilde{V}_p(K,M) &= \widetilde{V}_p(L,M), \quad \textit{for each centered star body } M \in \mathcal{S}^n. && (3.41) \end{aligned}$$

Formally setting $p = 1$ and $\mathrm{I}_1 = \mathrm{I}$, the corresponding equivalence (3.39) $\Leftrightarrow$ (3.41) was established in [42] and (3.39) $\Leftrightarrow$ (3.40) can be found in [14].

Proof. First, since $\mathrm{I}_p K = \mathrm{I}_p \widetilde{\nabla}_p K$ as well as $\mathrm{I}_p L = \mathrm{I}_p \widetilde{\nabla}_p L$ and I_p is injective on centrally symmetric star bodies, (3.39) implies (3.40). Conversely, the identity $\widetilde{\nabla}_p K = \widetilde{\nabla}_p L$ means

$$\frac{1}{2}\rho(K,v)^{n-p} + \frac{1}{2}\rho(-K,v)^{n-p} = \frac{1}{2}\rho(L,v)^{n-p} + \frac{1}{2}\rho(-L,v)^{n-p}$$

for every $v \in S^{n-1}$. Therefore

$$\begin{gathered} \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,v)^{n-p}\,dv + \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,-v)^{n-p}\,dv = \\ \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,v)^{n-p}\,dv + \frac{1}{2}\int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,-v)^{n-p}\,dv \end{gathered}$$

The invariance properties of the spherical Lebesgue measure show that (3.39) holds.
Second, suppose that (3.39) holds. Thus

$$\int_{S^{n-1}} |u\cdot v|^{-p}\rho(K,v)^{n-p}\,dv = \int_{S^{n-1}} |u\cdot v|^{-p}\rho(L,v)^{n-p}\,dv, \quad \forall\, u \in S^{n-1}.$$

By Fubini's theorem we conclude

$$\int_{S^{n-1}} \rho(K,v)^{n-p}\int_{S^{n-1}} |u{\cdot}v|^{-p} f(u)\,du dv = \int_{S^{n-1}} \rho(L,v)^{n-p}\int_{S^{n-1}} |u{\cdot}v|^{-p} f(u)\,du dv,$$

for suitable f. The remarks after Theorem 8 show that

$$\int_{S^{n-1}} \rho(K,v)^{n-p}F(v)\,dv = \int_{S^{n-1}} \rho(L,v)^{n-p}F(v)\,dv, \quad \text{for } F \in C_e^\infty(S^{n-1}).$$

An approximation argument proves that $\widetilde{V}_p(K,M) = \widetilde{V}_p(L,M)$ for each centered star body M.
Finally, assume that (3.41) holds. Define a centered star body M by

$$\rho(M,u)^p := \int_{S^{n-1}} |u\cdot v|^{-p} f(v)\,dv,$$

where f is now a continuous, nonnegative function on the sphere. Applying (3.41) for this special M, we get

$$\int_{S^{n-1}} f(v)(\rho(\mathrm{I}_p K, v)^p - \rho(\mathrm{I}_p L, v)^p)\, dv = 0. \tag{3.42}$$

For arbitrary continuous functions f we can deduce (3.42) by writing f as the difference of its positive and negative part. Thus $\rho(\mathrm{I}_p K, \cdot)^p = \rho(\mathrm{I}_p L, \cdot)^p$. □

3.4 L_p Version of Hensley's Result on Intersection Bodies and the Slicing Conjecture

A compact set $K \subset \mathbb{R}^n$ with volume 1 is said to be in isotropic position if for each unit vector u

$$\int_K (x \cdot u)^2 = L_K^2,$$

where L_K denotes the isotropic constant of K. Hensley [26] proved the existence of absolute (not depending on K and n) constants c_1, c_2 with

$$c_1 \leq \frac{\rho(\mathrm{I}\, K, u)}{\rho(\mathrm{I}\, K, v)} \leq c_2, \quad \forall\, u, v \in S^{n-1}, \tag{3.43}$$

for symmetric convex bodies K in isotropic position. In fact, even more is true, namely

$$\frac{\tilde{c}_1}{L_K} \leq \rho(\mathrm{I}\, K, u) \leq \frac{\tilde{c}_2}{L_K} \tag{3.44}$$

for all unit vectors u and universal constants $\tilde{c}_1, \tilde{c}_2$.

One of the major open problems in the field of convexity is the so called slicing conjecture. It asks whether the isotropic constant for centrally symmetric bodies can be bounded from above by a universal constant. Relation (3.44) shows that this is equivalent to bound $\|\rho(\mathrm{I}\, K, \cdot)^{-1}\|_\infty$ by a constant independent of the dimension and the symmetric body K which is supposed to be in isotropic position.

We prove that (3.43) is also true for L_p intersection bodies and establish an equivalent formulation of the slicing conjecture in terms of L_p intersection bodies. To do so, we prove inequalitites between radial functions of I and I_p first.

Theorem 13. *Suppose $0 < p < 1$. For all symmetric $K \in \mathcal{K}_0^n$ with volume one there exist constants c_1, c_2 independent of the dimension, the body K and p, such that*

$$c_1 \rho(\mathrm{I}\, K, u) \leq \rho(\mathrm{I}_p K, u) \leq c_2 \rho(\mathrm{I}\, K, u)$$

holds for every direction $u \in S^{n-1}$.

Proof. We use the following two facts which can be found in [51]. For a measurable function $f : \mathbb{R}^n \to \mathbb{R}^+$ which has values less than or equal 1 and a symmetric convex body $Q \in \mathcal{K}_0^n$ the function

$$F_1(q) := \left(\frac{\int_{\mathbb{R}^n} \rho(Q,x)^{-q} f(x)\, dx}{\int_Q \rho(Q,x)^{-q}\, dx} \right)^{1/(n+q)}$$

is an increasing function of q on $(-n, \infty)$.
Suppose $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\psi(0) = 0$, ψ and $\psi(x)/x$ are increasing on an interval $(0, \nu]$, and $\psi(x) = \psi(\nu)$ for $x \geq \nu$. Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing, continuous function which vanishes at $\psi(\nu)$. Then

$$F_2(q) := \left(\frac{\int_0^\infty h(\psi(x)) x^q\, dx}{\int_0^\infty h(x) x^q\, dx} \right)^{1/(1+q)}$$

is a decreasing function of q on $(-1, \infty)$ (provided that the integrals make sense). To prove the second inequality take $f(x) := A_{K,u}(x)/A_{K,u}(0)$ and $Q := [-1, 1] \subset \mathbb{R}$. Brunn's theorem shows that this f satisfies the above assumptions to ensure that $F_1(-p) \leq F_1(0)$, that is

$$\left(\frac{(1-p) \int_{\mathbb{R}} |x|^{-p} A_{K,u}(x)\, dx}{2 \operatorname{vol}(K \cap u^\perp)} \right)^{1/(1-p)} \leq \frac{1}{2 \operatorname{vol}(K \cap u^\perp)}.$$

Thus by (3.7)

$$\rho(\mathrm{I}_p K, u) \leq \frac{2}{(\Gamma(2-p))^{1/p}} \rho(\mathrm{I}\, K, u).$$

We have $\lim_{p \to 0} (\Gamma(2-p))^{1/p} = \exp(\gamma - 1) > 0$ where γ denotes the Euler-Mascheroni constant. For all other values of $p \in (0, 1]$ we trivially have that $\Gamma(2-p))^{1/p} > 0$. This shows that $\Gamma(2-p))^{1/p}$ can be bounded from below on $(0, 1)$ by a constant.

To establish the first inequality take $h(x) = (1-x)^{n-1} \mathbb{I}_{[0,1]}(x)$, $x \geq 0$ and $\psi(x) = 1 - (A_{K,u}(x)/A_{K,u}(0))^{1/(n-1)}$ for arbitrary $u \in S^{n-1}$. ($\mathbb{I}$ stands for the indicator function.) Brunn's theorem shows that ψ is a convex function on $[0, h(K,u)]$. Therefore these two functions satisfy the above conditions to guarantee the monotonicity of F_2. Hence $F_2(-p) \geq F_2(0)$, which can be rewritten as

$$\left(\frac{\int_0^\infty A_{K,u}(x) x^{-p}\, dx}{\operatorname{vol}(K \cap u^\perp) B(1-p, n)} \right)^{1/(1-p)} \geq \frac{n}{2 \operatorname{vol}(K \cap u^\perp)}.$$

Using (3.7), we obtain

$$\rho(\mathrm{I}_p K, u) \geq 2 \left(\frac{\Gamma(n) n^{1-p}}{\Gamma(1+n-p)} \right)^{1/p} \rho(\mathrm{I}\, K, u).$$

We want to show that

$$\frac{\Gamma(n)n^{1-p}}{\Gamma(1+n-p)} \geq 1$$

for every $n \in \mathbb{N}$ and $p \in (0,1)$. So we have to prove that

$$\ln \Gamma(n+1-p) + p \ln n \leq \ln \Gamma(n+1). \tag{3.45}$$

Since the Gamma function is logarithmic convex we get

$$\begin{aligned} \ln \Gamma(n+1-p) &= \ln \Gamma((1-p)(n+1)+pn) \\ &\leq (1-p)\ln \Gamma(n+1) + p \ln \Gamma(n) \\ &= (1-p)\ln n + \ln \Gamma(n). \end{aligned}$$

This immediately implies (3.45).

□

Hensley's original relation combined with Theorem 13 gives the L_p analogue of Hensley's result.

Theorem 14. *Assume $0 < p < 1$. There exist constants c_1, c_2 independent of the dimension, the body K and p, such that for symmetric bodies $K \in \mathcal{K}_0^n$ in isotropic position*

$$c_1 \leq \frac{\rho(\mathrm{I}_p K, u)}{\rho(\mathrm{I}_p K, v)} \leq c_2$$

for all $u, v \in S^{n-1}$.

By Theorem 13 the slicing conjecture is equivalent to

Question 1. *Does there exist a constant c independent of the dimension and the body K such that*

$$\|\rho(\mathrm{I}_p K, \cdot)^{-1}\|_\infty \leq c$$

for all symmetric $K \subset \mathcal{K}^n$ in isotropic position and some $p \in (0,1)$?

3.5 L_p Busemann-Petty Problems

As we already remarked in the introduction, the Busemann-Petty problem asks whether the implication

$$\mathrm{I}\, K \subset \mathrm{I}\, L \Longrightarrow V(K) \leq V(L)$$

holds for arbitrary origin symmetric $K, L \in \mathcal{K}^n$. The obvious analogue of this question for other values $0 < p < 1$ is to ask the following: does $\mathrm{I}_p K \subset \mathrm{I}_p L$ for origin symmetric $K, L \in \mathcal{K}^n$ imply $V(K) \leq V(L)$? We refer to this question as the symmetric L_p Busemann-Petty problem. This was stated and solved in terms of polar L_p centroid bodies by Yaskin and Yaskina [61]. Their result shows that the answer is positive if and only if $n \leq 3$. Since $\mathrm{I}_p K \subset \mathrm{I}_p L$ is equivalent to $\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L$ for origin symmetric bodies K, L, the symmetric L_p Busemann-Petty problem asks whether

$$\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L \Longrightarrow V(K) \leq V(L) \tag{3.46}$$

holds for arbitrary origin symmetric $K, L \in \mathcal{K}^n$. If we allow the bodies in (3.46) to be arbitrary elements of $\mathcal{K}_0^n$, we call this question the nonsymmetric L_p Busemann-Petty problem.

To each body K which is not centered, one can construct bodies L such that the desired implications for the original as well as the symmetric L_p Busemann-Petty problem fail. Our goal is to show that Lutwak's connections on intersection bodies (which will be described in detail below) also hold in the nonsymmetric L_p case. This proves in particular that there are nonsymmetric bodies K for which (3.46) holds! Therefore we obtain a sufficient condition to compare volumes of two nonsymmetric bodies.

That (3.46) is true for centered ellipsoids can be seen from (3.12) for $E = F$. Then

$$V(\mathrm{I}_p^+ E) = r_{\mathrm{I}_p^+}^n \kappa_n^{2-n/p} V(E)^{n/p-1},$$

which immediately implies that (3.46) holds for ellipsoids.

Lutwak's first connection, as proved in [42], states that the answer to the Busemann-Petty problem is affirmative if the body with smaller sections is an intersection body. The assumption of convexity of the involved bodies can be omitted in this case; it suffices to deal with star bodies. The L_p analogue of this result is the following theorem.

Theorem 15. *Let $0 < p < 1$ and $K, L \in \mathcal{S}_0^n$. If K is a nonsymmetric L_p intersection body, i.e. contained in $\mathrm{I}_p^+ \mathcal{S}^n$, then*

$$\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L,$$

implies

$$V(K) \leq V(L),$$

with equality only if $K = L$.

We remark that the approach of the following proof can be used to derive a similar statement also for other values of p. The only difference is that the inequality for the volumes reverses. This method also shows that the assumption of convexity is not necessary in the corresponding result for polar L_p-centroid bodies in [61].

Proof. For a star body $\overline{K}$ with $\mathrm{I}_p^+ \overline{K} = K$, the definition of generalized dual mixed volumes and Fubini's theorem prove

$$V(K) = \widetilde{V}_p(K, K) = \widetilde{V}_p(\overline{K}, \mathrm{I}_p^+ K), \quad \widetilde{V}_p(L, K) = \widetilde{V}_p(\overline{K}, \mathrm{I}_p^+ L).$$

Since

$$\begin{aligned}\widetilde{V}_p(\overline{K}, \mathrm{I}_p^+ K) &= \frac{1}{n}\int_{S^{n-1}} \rho(\overline{K}, u)^{n-p}\left(\frac{\rho(\mathrm{I}_p^+ K, u)}{\rho(\mathrm{I}_p^+ L, u)}\right)^p \rho(\mathrm{I}_p^+ L, u)^p\, du \\ &\le \max_{u\in S^{n-1}}\left(\frac{\rho(\mathrm{I}_p^+ K, u)}{\rho(\mathrm{I}_p^+ L, u)}\right)^p \widetilde{V}_p(\overline{K}, \mathrm{I}_p^+ L),\end{aligned}$$

we have

$$\frac{V(K)}{\widetilde{V}_p(L, K)} \le \max_{u\in S^{n-1}}\left(\frac{\rho(\mathrm{I}_p^+ K, u)}{\rho(\mathrm{I}_p^+ L, u)}\right)^p. \tag{3.47}$$

Since $\mathrm{I}_p^+ K \subset \mathrm{I}_p^+ L$, the claimed inequality for the volumes is an immediate consequence of (3.47) and (3.36). The equality case of the theorem follows from the equality case of the dual L_p Minkowski inequality. □

The next result is a negative counterpart of Theorem 15.

Theorem 16. *Suppose we have an infinitely smooth star body $L \in \mathcal{S}_0^n$ which is not a nonsymmetric L_p intersection body. Then there exists a star body K such that*

$$\mathrm{I}_p^+ K \subsetneq \mathrm{I}_p^+ L,$$

but

$$V(L) < V(K).$$

This is the analogue of Lutwak's second connection on intersection bodies. The latter is the same statement as Theorem 16 but for intersection bodies instead of nonsymmetric L_p intersection bodies.

Proof. By Theorem 8 there exists a function $f \in C^\infty(S^{n-1})$ such that

$$\rho(L, \cdot)^p = \mathrm{C}_{-p}^+ f.$$

Since L is not a nonsymmetric L_p intersection body, f must assume negative values. Therefore we are able to choose a nonconstant function $\bar{f} \in C^\infty(S^{n-1})$ such that

$$\bar{f}(u) \ge 0, \quad \text{when } f(u) < 0,$$

and

$$\bar{f}(u) = 0, \quad \text{when } f(u) \ge 0.$$

Choose another function $\bar{\bar{f}} \in C^\infty(S^{n-1})$ such that $\mathrm{C}_{-p}^+ \bar{\bar{f}} = \bar{f}$. Now, since the origin is an interior point of L, we can find a constant $\lambda > 0$ with

$$\rho(L, \cdot)^{n-p} - \lambda\bar{\bar{f}} > 0.$$

Define a star body Q by $\rho(Q,\cdot)^{n-p} := \rho(L,\cdot)^{n-p} - \lambda\bar{\bar{f}}(\cdot)$. Then

$$\rho(\mathrm{I}_p^+ Q,\cdot)^p = \rho(\mathrm{I}_p^+ L,\cdot)^p - \lambda((n-p)\Gamma(1-p))^{-1}\bar{f}.$$

Hence

$$\rho(\mathrm{I}_p^+ Q,\cdot)^p \leq \rho(\mathrm{I}_p^+ L,\cdot)^p, \quad \text{when } f(u) < 0, \tag{3.48}$$

and

$$\rho(\mathrm{I}_p^+ Q,\cdot)^p = \rho(\mathrm{I}_p^+ L,\cdot)^p, \quad \text{when } f(u) \geq 0. \tag{3.49}$$

By linearity properties of generalized dual mixed volumes and the self adjointness of C_{-p}^+ we therefore have

$$\begin{aligned}
V(L) - \widetilde{V}_p(Q,L) &= \frac{1}{n}\int_{S^{n-1}} \left(\rho(L,u)^{n-p} - \rho(Q,u)^{n-p}\right)\rho(L,u)^p\,du \\
&= \frac{1}{n}\int_{S^{n-1}} \left(\rho(L,u)^{n-p} - \rho(Q,u)^{n-p}\right)\mathrm{C}_{-p}^+ f(u)\,du \\
&= \frac{(n-p)\Gamma(1-p)}{n}\int_{S^{n-1}} \left(\rho(\mathrm{I}_p^+ L,u)^p - \rho(\mathrm{I}_p^+ Q,u)^p\right) f(u)\,du \\
&< 0.
\end{aligned}$$

So from (3.36) we get

$$V(L) < V(Q).$$

Relations (3.48) and (3.49) show that $\mathrm{I}_p^+ Q \subset \mathrm{I}_p^+ L$. Set

$$\varepsilon := \left(\frac{1}{2}\left(1 + \frac{V(L)}{V(Q)}\right)\right)^{1/n}.$$

Then $\varepsilon < 1$ and the body $K := \varepsilon Q$ has the desired properties. $\square$

Chapter 4

Star Body Valued Valuations

In Chapter 2 we obtained a classification of valuations Z on convex polytopes taking values in $\langle \mathcal{S}^n, \tilde{+}_p \rangle$ for positive p. Is it possible to characterize L_p intersection bodies for negative values of p also? One has to be careful, since L_p radial addition makes no sense for negative p. (The origin could be contained in the boundary of a star body.) But L_p radial addition is well defined for all $p \neq 0$ if we restrict it to star bodies containing the origin in their interiors. Obviously, we can assume $\langle \mathcal{S}_0^n, \tilde{+}_p \rangle$, $p \neq 0$ to be a monoid by adjoining an identity element e and adopt addition properly. A geometric intuition behind these new elements would be the following. For $p > 0$ the identity element can be seen as the origin and for $p < 0$ as the whole $\mathbb{R}^n$.

Of course we will try to classify operators by their co- or contravariance properties. Since we will deal with operators which can have the abstract identity element as value, we have to adopt the above affine notions properly. We can regard covariance of operators in an algebraic setting by using group actions. Let G denote a subgroup of the general linear group $\mathrm{GL}(n)$ and let $\mathcal{L}$ be a subset of $\mathcal{K}^n$. Define actions by

$$\begin{array}{rcl} a_1 : G \times \mathcal{L} & \to & \mathcal{L} \\ (\phi, L) & \to & \phi L, \end{array} \qquad \begin{array}{rcl} a_2 : G \times \mathcal{S}^n & \to & \mathcal{S}^n \\ (\phi, K) & \to & |\det \phi|^q \phi K, \end{array} \tag{4.1}$$

where $q \in \mathbb{R}$. An operator $\mathrm{Z} : \mathcal{L} \to \mathcal{S}^n$ is called G *covariant of weight* q if

$$\mathrm{Z}\, a_1(\phi, L) = a_2(\phi, \mathrm{Z}\, L), \quad \text{for } \phi \in G,\ L \in \mathcal{L}. \tag{4.2}$$

If G equals the whole $\mathrm{GL}(n)$, this definition is equivalent to the definition given in the introduction. Let us come back to our monoid induced from $\mathcal{S}_0^n$. Of course, the action a_2 is well defined on $G \times \mathcal{S}_0^n$ and there is exactly one possible choice how to extend it to the monoid $\langle \mathcal{S}_0^n \cup \{e\}, \tilde{+}_p \rangle$: We have to define $a_2(\phi, e) = e$ for all $\phi \in G$. So if we speak of G covariant operators $\mathrm{Z} : \mathcal{L} \to \mathcal{S}_0^n \cup \{e\}$ in the sequel, we mean that they satisfy (4.2) with respect to the extended action a_2. Covariant operators are those which are covariant of weight q for some $q \in \mathbb{R}$. Contravariant operators are defined in an analogous way.

Denote by $\overline{\mathcal{P}}_0^n$ convex polytopes that contain the origin. Define $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n \cup \{e\}, \tilde{+}_p \rangle$ by

$$\mathrm{Z}\, P = \begin{cases} e & \dim P < n \\ c_1 \,\mathrm{I}_p^+ P \,\tilde{+}_p\, c_2 \,\mathrm{I}_p^- P & \dim P = n \end{cases}$$

for positive constants c_1, c_2. This is a $\mathrm{GL}^+(n)$ contravariant valuation and we write $c_1 \,\mathrm{I}_p^+ \tilde{+}_p c_2 \,\mathrm{I}_p^-$ for it. Here, $\mathrm{GL}^+(n)$ stands for linear maps with positive determinant. As we will see, this is the only example of such valuations. To state the exact result, we call a valuation $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n \cup \{e\}, \tilde{+}_p \rangle$ trivial if it is constant with value e. For simplicity, we write $\langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ instead of $\langle \mathcal{S}_0^n \cup \{e\}, \tilde{+}_p \rangle$ below.

Theorem 17. *Suppose $p \neq 0$. Let $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ be a $\mathrm{GL}^+(n)$ covariant valuation. If $n = 2$ and $p < 1$, Z is nontrivial if and only if there exist positive constants c_1, c_2 with*

$$\mathrm{Z}\, P = \psi_{\pi/2}^{-1}(c_1 \,\mathrm{I}_p^+ P \,\tilde{+}_p\, c_2 \,\mathrm{I}_p^- P), \quad \forall P \in \mathcal{P}_0^2.$$

For all other values of p, Z is trivial. For $n \geq 3$, Z is always trivial.
If $p < 1$ and $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ is a nontrivial $\mathrm{GL}^+(n)$ contravariant valuation, then exist positive constants c_1, c_2 with

$$\mathrm{Z}\, P = c_1 \,\mathrm{I}_p^+ P \,\tilde{+}_p\, c_2 \,\mathrm{I}_p^- P, \quad \forall P \in \overline{\mathcal{P}}_0^n.$$

For $p \geq 1$ there exist only the trivial examples.

Recall that $\psi_{\pi/2}$ denotes the rotation about the angle $\pi/2$. By $\psi_{\pi/2}^{-1}(c_1 \,\mathrm{I}_p^+ P \,\tilde{+}_p\, c_2 \,\mathrm{I}_p^- P)$ we understand $a_2(\psi_{\pi/2}^{-1}, c_1 \,\mathrm{I}_p^+ P \,\tilde{+}_p\, c_2 \,\mathrm{I}_p^- P)$. For n-dimensional P this is, as indicated by the notation, just a rotation.

4.1 Proof

A finite set αP of n-dimensional simplices is a triangulation of an n-dimensional polytope $P \subset \mathbb{R}^n$ if the union of all simplices in αP equals P and no pair of simplices intersects in a set of dimension n. Especially, a starring at $x \in P$ is a triangulation αP where every simplex in αP has a vertex at x.

Let $\langle M, +, e \rangle$ be a monoid with identity element e. We call a valuation $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle M, +, e \rangle$ simple, if polytopes of dimension less than n are mapped to e. Denote by $\mathcal{T}_0^n$ n-dimensional simplices with one vertex at the origin.

Lemma 20. *Every simple valuation $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle M, +, e \rangle$ with values in an abelian monoid with cancellation law is uniquely determined by its values on n-dimensional simplices having one vertex at the origin.*

Proof. Let $P \in \overline{\mathcal{P}}_0^n$ be an n-dimensional polytope. First, we prove that for arbitrary $x \in P$ there exists a starring of P at x. This can be seen by induction on the dimension. For $n = 1$ it is trivial. Suppose that the assertion is true for $(n-1)$-dimensional polytopes and denote by $F_j, j = 1, \dots, k$ the facets of an n-dimensional polytope P. We choose starrings $\alpha_j F_j$ of F_j for those facets which do not contain the given point x. Thus the convex hulls of x and the $(n-1)$-dimensional simplices in $\alpha_j F_j$ define the desired starring.

The proof of the lemma is finished if we can show that

$$P = P_1 \cup P_2 \cup \ldots \cup P_k, \quad P, P_1, \dots, P_k \in \overline{\mathcal{P}}_0^n, \quad \dim(P_i \cap P_j) < n \quad \text{for } i \neq j$$

implies

$$\mathrm{Z}\, P = \sum_{i=1}^{k} \mathrm{Z}\, P_i$$

for an n dimensional polytope $P \in \overline{\mathcal{P}}_0^n$. We proceed by induction on k. For $k = 1, 2$ this is trivial. It holds also true if $P_1 = P$ since then $\dim P_i < n$ for $i \neq 1$ and Z is assumed to be simple. Suppose that our desired conclusion holds true for at most $k-1$ involved polytopes. Without loss of generality assume $\dim P_1 = n$ and that P_1 is a proper subpolytope of P. Then P_1 has a facet F containing the origin such that P has points contained in $\operatorname{int}(\operatorname{lin} F)^+$ and as well as $\operatorname{int}(\operatorname{lin} F)^-$. Write $H := \operatorname{lin} F$ for simplicity and assume that $P_1 \subset H^-$. Define

$$P^- := P \cap H^-, \quad P^+ := P \cap H^+, \quad P_i^- := P_i \cap H^-, \quad P_i^+ := P_i \cap H^+,$$

for $i = 1, \dots, k$. From the fact that $P^+ = P_2^+ \cup \ldots \cup P_k^+$, the induction hypothesis and the simplicity of Z we obtain

$$\mathrm{Z}\, P^+ = \sum_{i=2}^{k} \mathrm{Z}\, P_i^+ = \sum_{i=1}^{k} \mathrm{Z}\, P_i^+,$$

and therefore

$$\begin{aligned} \mathrm{Z}\, P + \sum_{i=1}^{k} \mathrm{Z}\, P_i^- &= \mathrm{Z}\, P^+ + \mathrm{Z}\, P^- + \sum_{i=1}^{k} \mathrm{Z}\, P_i^- \\ &= \sum_{i=1}^{k} (\mathrm{Z}\, P_i^- + \mathrm{Z}\, P_i^+) + \mathrm{Z}\, P^- \\ &= \sum_{i=1}^{k} \mathrm{Z}\, P_i + \mathrm{Z}\, P^- \qquad (4.3) \end{aligned}$$

If $P_1^- = P^-$ we have $\sum_{i=1}^{k} \mathrm{Z}\, P_i^- = \mathrm{Z}\, P^-$ and we are done by the cancellation law. Otherwise, we can proceed as above but now for the polytope P^-. So cutting with a suitable hyperplane H_2 gives

$$P^{-,2} := P^- \cap H_2^-, \quad P^{+,2} := P^- \cap H_2^+, \quad P_i^{-,2} := P_i^- \cap H_2^-, \quad P_i^{+,2} := P_i^- \cap H_2^+,$$

and

$$\operatorname{Z} P^- + \sum_{i=1}^k \operatorname{Z} P_i^{-,2} = \sum_{i=1}^k \operatorname{Z} P_i^- + \operatorname{Z} P^{-,2}.$$

By (4.3) we therefore get

$$\begin{aligned}
\operatorname{Z} P + \sum_{i=1}^k \operatorname{Z} P_i^- + \sum_{i=1}^k \operatorname{Z} P_i^{-,2} &= \sum_{i=1}^k \operatorname{Z} P_i + \operatorname{Z} P^- + \sum_{i=1}^k \operatorname{Z} P_i^{-,2} \\
&= \sum_{i=1}^k \operatorname{Z} P_i + \sum_{i=1}^k \operatorname{Z} P_i^- + \operatorname{Z} P^{-,2}.
\end{aligned}$$

The cancellation law again proves

$$\operatorname{Z} P + \sum_{i=1}^k \operatorname{Z} P_i^{-,2} = \sum_{i=1}^k \operatorname{Z} P_i + \operatorname{Z} P^{-,2}.$$

Repeating this procedure finitely many times (depending on the number of supporting hyperplanes of P_1 which contain the origin), we are in the situation that $P_1^{-,m} = P^{-,m}$. □

For $K \in \mathcal{S}_0^n$ let the *Minkowski functional* $\|\cdot\|_K$ be defined as

$$\|x\|_K = \min\{\lambda \geq 0 : \, x \in \lambda K\}, \quad \text{for } \, x \in \mathbb{R}^n.$$

Note that the Minkowski functional of K is just the reciprocal of the radial function.

Lemma 21. *For $p \neq 0$, let $\operatorname{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ be an $\operatorname{GL}^+(n)$ co- or contravariant operator of arbitrary weight. Then Z is simple.*

Proof. Because of the assumed $\operatorname{GL}^+(n)$ co- or contravariance it is enough to prove that a polytope $P \in \overline{\mathcal{P}}_0^n$ which is contained in $e_n^\perp$ is mapped to the identity element of $\langle \mathcal{S}_0^n, \tilde{+}_p \rangle$.
For $s > 0$ define a linear map $\phi \in \operatorname{GL}^+(n)$ by

$$\phi e_i = e_i, \quad i = 1, \ldots, n-1, \quad \text{and} \quad \phi e_n = s e_n.$$

First, assume that Z is $\operatorname{GL}^+(n)$ contravariant of weight q and suppose $\operatorname{Z} P \in \mathcal{S}_0^n$. Then

$$\operatorname{Z} P = \operatorname{Z} \phi P = (\det \phi)^q \phi^{-t} \operatorname{Z} P,$$

and therefore

$$\|x\|_{\operatorname{Z} P} = \|(\det \phi)^{-q} \phi^t x\|_{\operatorname{Z} P} = (\det \phi)^{-q} \|\phi x\|_{\operatorname{Z} P} \tag{4.4}$$

for every $x \in \mathbb{R}^n$. $\mathrm{Z}\, P$ contains an Euclidean ball with center at the origin and is contained in such a ball. Thus there exist positive constants c_1, c_2 with

$$c_1 \|x\| \le \|x\|_{\mathrm{Z}\, P} \le c_2 \|x\|, \quad \text{for every } \; x \in \mathbb{R}^n. \tag{4.5}$$

This together with (4.4) implies

$$c_1 \sqrt{\sum_{i=1}^{n-1} x_i^2 + (s x_n)^2} \le c_2 s^q \sqrt{\sum_{i=1}^{n} x_i^2}, \tag{4.6}$$

for all $x \in \mathbb{R}^n$. Note that $s > 0$ was arbitrary. So taking the limit $s \to 0^+$ in (4.6) evaluated at e_1 yields a contradiction for positive q. If $q = 0$, then the limit $s \to \infty$ in relation (4.6) at e_n gives a contradiction. Finally, for negative q regard (4.6) at e_1 and let $s \to \infty$. We obtain again a contradiction. Thus we proved that for all weights q the image of $\mathrm{Z}\, P$ is not contained in $\mathcal{S}_0^n$ and therefore has to be e.
If Z is $\mathrm{GL}(n)$ covariant, then, on the assumption that $\mathrm{Z}\, P \in \mathcal{S}_0^n$, one derives from

$$\|x\|_{\mathrm{Z}\, P} = (\det \phi)^{-q} \|\phi^{-1} x\|_{\mathrm{Z}\, P}$$

that

$$c_1 s^q \sqrt{\sum_{i=1}^{n} x_i^2} \le c_2 \sqrt{\sum_{i=1}^{n-1} x_i^2 + \left(\frac{x_n}{s}\right)^2}$$

holds on $\mathbb{R}^n$. For $q > 0$, $q = 0$, $q < 0$ regard this inequality at points e_1, e_n, e_1 and take limits $s \to \infty$, $s \to \infty$ and $s \to 0$, respectively. As above we obtain that $\mathrm{Z}\, P$ has to be the identity element. □

From Lemma 20 and Lemma 21 we conclude that a $\mathrm{GL}^+(n)$ co- or contravariant valuation $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ is uniquely determined by its value on the standard simplex $T^n := \operatorname{conv}\{0, e_1, \dots, e_n\}$.

For $0 < \lambda < 1$, we define two families of linear maps by

$$\begin{aligned} \phi e_2 &= \lambda e_2 + (1-\lambda) e_1, \qquad & \phi e_k &= e_k \text{ for } k \neq 2, \\ \psi e_1 &= \lambda e_2 + (1-\lambda) e_1, \qquad & \psi e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Note that

$$\begin{aligned} \phi^{-1} e_2 &= \frac{1}{\lambda} e_2 - \frac{1-\lambda}{\lambda} e_1, \qquad & \phi_j^{-1} e_k &= e_k \text{ for } k \neq 2, \\ \psi^{-1} e_1 &= -\frac{\lambda}{1-\lambda} e_2 + \frac{1}{1-\lambda} e_1, \qquad & \psi_j^{-1} e_k &= e_k \text{ for } k \neq 1. \end{aligned}$$

Let H be the hyperplane through 0 with normal vector $\lambda e_1 - (1-\lambda) e_2$. Then we have $T^n \cap H^+ = \phi T^n$ and $T^n \cap H^- = \psi T^n$. So for a simple valuation Z we obtain

$$\mathrm{Z}\, T^n = \mathrm{Z}(\phi T^n) \,\tilde{+}_p\, \mathrm{Z}(\psi T^n). \tag{4.7}$$

Lemma 22. *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^2 \to \langle \mathcal{S}_0^2, \tilde{+}_p \rangle$ *be a valuation which is* $\mathrm{GL}^+(2)$ *contravariant of weight* q*. If* $p < 1$*,* Z *is nontrivial and* $q = 1/p$*, then exist positive constants* c_1*,* c_2 *with*

$$\mathrm{Z}\, P = c_1 \, \mathrm{I}_p^+ \, P \, \tilde{+}_p \, c_2 \, \mathrm{I}_p^- \, P, \quad \forall P \in \overline{\mathcal{P}}_0^2.$$

In all other cases Z *is trivial.*

Proof. Assume $\mathrm{Z}\, T^2 \in \mathcal{S}_0^2$ and set $f(x) := \rho(\mathrm{Z}\, T^2, x)^p \in C(\mathbb{R}^2 \backslash \{0\})_+$, i.e. positive continuous functions on $\mathbb{R}^2 \backslash \{0\}$. Thus f is positively homogeneous of degree $-p$ and (4.7) implies

$$f(x_1, x_2) = \lambda^{pq} f(x_1, (1-\lambda)x_1 + \lambda x_2) + (1-\lambda)^{pq} f((1-\lambda)x_1 + \lambda x_2, x_2) \quad (4.8)$$

for arbitrary $x \in \mathbb{R}^2 \backslash \{0\}$. In Lemma 5 it has been shown that such a function f has to be of the form

$$f = f(1,0)\, g_{p,q} + f(-1,0)\, g_{p,q} \circ \gamma_0 + f(0,1)\, g_{p,q} \circ \gamma_1 + f(0,-1)\, g_{p,q} \circ \gamma_2 \quad (4.9)$$

on $\mathbb{R}^2 \backslash \{(x_1, x_2) : x_1 = x_2\}$, where the function $g_{p,q}$ on $\mathbb{R}^2$ is defined by

$$g_{p,q}(x_1, x_2) = \begin{cases} (x_1^{pq-p} - x_2^{pq-p})/(x_1 - x_2)^{pq} & \text{for } 0 \le x_2 < x_1, \\ x_1^{pq-p}/(x_1 - x_2)^{pq} & \text{for } x_1 > 0, x_2 < 0, \\ 0 & \text{otherwise.} \end{cases}$$

and the linear transformations γ_i, $i = 0, 1, 2$ are

$$\gamma_0(x_1, x_2) = (-x_1, -x_2), \;\; \gamma_1(x_1, x_2) = (x_2, x_1), \;\; \gamma_2(x_1, x_2) = (-x_2, -x_1).$$

Let $p < 1$ and $q = 1/p$. Since f is continuous, (4.9) yields

$$f(x_1, x_1) = \lim_{x_2 \to x_1^-} \frac{x_1^{1-p} - x_2^{1-p}}{x_1 - x_2} f(1,0) = \lim_{x_2 \to x_1^+} \frac{x_2^{1-p} - x_1^{1-p}}{x_2 - x_1} f(0,1),$$

for positive x_1. Thus $f(1,0) = f(0,1)$ and an analogous observation for negative values x_1 proves $f(-1,0) = f(0,-1)$. So (4.9) simplifies to

$$f = f(1,0)\,(g_{p,q} + g_{p,q} \circ \gamma_1) + f(-1,0)(\, g_{p,q} \circ \gamma_0 + g_{p,q} \circ \gamma_2).$$

An elementary calculation shows

$$\rho(\mathrm{I}_p^+ \, T^2, \cdot)^p = (p^2 - 3p + 2)^{-1} (g_{p,1/p} + g_{p,1/p} \circ \gamma_1)$$

almost everywhere. Therefore

$$\rho(\mathrm{I}_p^- \, T^2, \cdot)^p = \rho(\mathrm{I}_p^+ \, T^2, \gamma_0(\cdot))^p = (p^2 - 3p + 2)^{-1} (g_{p,1/p} \circ \gamma_0 + g_{p,1/p} \circ \gamma_2).$$

This proves the first part of the lemma. For other weights we investigate the relation

$$f(x_1, x_2) = \frac{x_1^{pq-p} - x_2^{pq-p}}{(x_1 - x_2)^{pq}} f(1, 0) \tag{4.10}$$

for $x_1 > x_2 > 0$. For $pq \leq 0$ or $pq > 0$ and $pq-1 < 0$, the right hand side converges to zero when $x_2 \to x_1^-$. If $pq > 0$ and $pq - 1 > 0$ it assumes arbitrary large values as $x_2 \to x_1^-$. This is a contradiction to the assumption that $f \in C(\mathbb{R}^2 \backslash \{0\})_+$. If $q = 1/p$ and $p \geq 1$, then the right hand side of (4.10) is less or equal than zero which is again a contradiction. $\square$

Lemma 23. *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^2 \to \langle \mathcal{S}_0^2, \tilde{+}_p \rangle$ *be a* $\mathrm{GL}^+(2)$ *covariant valuation. If* $p < 1$, Z *is nontrivial and contravariant of weight* $q = 1/p - 1$, *then exist positive constants* c_1, c_2 *with*

$$\mathrm{Z}\, P = \psi_{\pi/2}^{-1}(c_1\, \mathrm{I}_p^+\, P \,\tilde{+}_p\, c_2\, \mathrm{I}_p^-\, P), \quad \forall P \in \overline{\mathcal{P}}_0^2,$$

In all other cases, Z *is trivial.*

Proof. Define an operator $\overline{\mathrm{Z}}$ by

$$\overline{\mathrm{Z}} P := a_2(\psi_{\pi/2}, \mathrm{Z}\, P), \qquad \text{for every } \ P \in \overline{\mathcal{P}}_0^2.$$

Since

$$a_2(\psi_{\pi/2}, S_1 \,\tilde{+}_p\, S_2) = a_2(\psi_{\pi/2}, S_1) \,\tilde{+}_p\, a_2(\psi_{\pi/2}, S_2)$$

for all $S_1, S_2 \in \mathcal{S}_0^n \cup \{e\}$, $\overline{\mathrm{Z}}$ is a valuation. For every $\phi \in \mathrm{GL}^+(2)$ we have

$$\psi_{\pi/2} \phi \psi_{\pi/2}^{-1} = (\det \phi) \phi^{-t}.$$

If Z is $\mathrm{GL}^+(2)$ covariant of weight q, then $\overline{\mathrm{Z}}$ is contravariant of weight $q + 1$. This and the already established characterization result of Lemma 22 finish the proof. $\square$

Lemma 24. *Let* $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ *be a* $\mathrm{GL}^+(n)$ *contravariant valuation for* $n \geq 3$. *If* $p < 1$, Z *is nontrivial and contravariant of weight* $q = 1/p$, *then exist positive constants* c_1, c_2 *with*

$$\mathrm{Z}\, P = c_1\, \mathrm{I}_p^+\, P \,\tilde{+}_p\, c_2\, \mathrm{I}_p^-\, P, \quad \forall P \in \overline{\mathcal{P}}_0^n.$$

In all other cases Z *is trivial.*

Proof. Assume that $\mathrm{Z}\, T^n \in \mathcal{S}_0^n$ and set

$$f(x) := \rho(\mathrm{Z}\, T^n, x)^p.$$

Further, we define a function $\tilde{f} : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ by

$$\tilde{f}(x_1, x_2) := f(x_1 e_1 + x_2 e_2).$$

From (4.7) we obtain

$$f(x) = \lambda^{pq} f(\phi^t x) + (1-\lambda)^{pq} f(\psi^t x) \tag{4.11}$$

for arbitrary $x \in \mathbb{R}^n\backslash\{0\}$. But $e_3, \ldots, e_n$ are fixpoints of the linear maps ϕ^t and ψ^t, so $\tilde{f}$ satisfies (4.8). So the proof of Lemma 22 shows that a nontrivial valuation can only exist for $p < 1$ and $q = 1/p$. Moreover,

$$\tilde{f}(x_1, x_2) = \tilde{c_1}\rho(\mathrm{I}_p^+ T^2, (x_1, x_2))^p + \tilde{c_2}\rho(\mathrm{I}_p^- T^2, (x_1, x_2))^p$$

for $p < 1$ and $q = 1/p$. Since $\rho(\mathrm{I}_p^\pm T^n, x_1e_1 + x_2e_2)^p$ are positive multiples of $\rho(\mathrm{I}_p^\pm T^2, (x_1, x_2))^p$, there exist positive constants c_1, c_2 such that

$$f(x_1e_1 + x_2e_2) = c_1\rho(\mathrm{I}_p^+ T^n, x_1e_1 + x_2e_2)^p + c_2\rho(\mathrm{I}_p^- T^n, x_1e_1 + x_2e_2)^p.$$

For simplicity we will write $f_1(x) = c_1\rho(\mathrm{I}_p^+ T^n, x)^p + c_2\rho(\mathrm{I}_p^- T^n, x)^p$.
We arrived at the following situation: $f, f_1 : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ are two continuous functions satisfying (4.11). They are invariant under even permutations of indices and are equal on $\mathrm{lin}\{e_1, e_2\}\backslash\{0\}$.
We will show that f coincides with f_1 on $\mathbb{R}^n\backslash\{0\}$. Because of the invariance properties it is enough to prove

$$f_1(x) = f_2(x),\ \forall\ x \in \mathrm{lin}\{e_{i_1}, \ldots, e_{i_k}\} \Longrightarrow f_1(x) = f_2(x),\ \forall\ x \in \mathrm{lin}\{e_1, \ldots, e_{k+1}\} \tag{4.12}$$

for $2 \leq k \leq n-1$. Let x be contained in $\mathrm{lin}\{e_1, \ldots, e_{k+1}\}$. Suppose $0 < x_1/x_2 < 1$ and let $\lambda := x_1/x_2$. Then

$$\begin{aligned}(\psi^{-t}x)_1 = (\phi^t\psi^{-t}x)_1 &= \frac{x_1}{1-\lambda} - \frac{\lambda}{1-\lambda}x_2 = 0,\\ (\psi^{-t}x)_i = (\phi^t\psi^{-t}x)_i &= 0, \quad i = k+2, \ldots, n.\end{aligned}$$

By (4.11) follows $f(\psi^{-t}x) = \lambda^{pq} f(\phi^t\psi^{-t}x) + (1-\lambda)^{pq} f(x)$ and the analogous relation is true for f_1. So (4.12) holds for $0 < x_1 < x_2$ and $x_2 < x_1 < 0$.
For $0 < \lambda := (x_1 - x_2)/x_1 < 1$ we obtain

$$\begin{aligned}(\phi^{-t}x)_2 = (\psi^t\phi^{-t}x)_2 &= -\frac{1-\lambda}{\lambda}x_1 + \frac{1}{\lambda}x_2 = 0,\\ (\psi^{-t}x)_i = (\phi^t\psi^{-t}x)_i &= 0, \quad i = k+2, \ldots, n.\end{aligned}$$

Since $f(\phi^{-t}x) = \lambda^{pq} f(x) + (1-\lambda)^{pq} f(\psi^t\phi^{-t}x)$ and f_1 satisfies the same identity, (4.12) holds for $0 < x_2 < x_1$ and $x_1 < x_2 < 0$.
For $x_1, x_2 \neq 0$ and $\mathrm{sgn}(x_1) \neq \mathrm{sgn}(x_2)$ define $0 < \lambda := x_1/(x_1 - x_2) < 1$. Then

$$(\phi^t x)_2 = (\phi^t x)_i = (\psi^t x)_1 = (\psi^t x)_i = 0, \quad i = k+2, \ldots, n.$$

As before we conclude that (4.11) implies (4.12) for $x_1 < 0, x_2 > 0$ and $x_1 > 0, x_2 < 0$. The continuity of f and f_1 concludes the proof of (4.12). $\square$

Lemma 25. *Every* $\mathrm{GL}^+(n)$ *covariant valuation* $\mathrm{Z} : \overline{\mathcal{P}}_0^n \to \langle \mathcal{S}_0^n, \tilde{+}_p \rangle$ *for* $n \geq 3$ *is trivial.*

Proof. Assume $\mathrm{Z}\,T^n \in \mathcal{S}_0^n$ and set $f(x) := \rho(\mathrm{Z}\,T^n, x)^p \in C(\mathbb{R}^n \backslash \{0\})_+$. Thus (4.7) implies

$$f(x) = \lambda^{pq} f(\phi^{-1} x) + (1-\lambda)^{pq} f(\psi^{-1} x) \tag{4.13}$$

on $\mathbb{R}^n \backslash \{0\}$. Since e_3 is an eigenvector with eigenvalue 1 of ϕ^{-1} and ψ^{-1}, we get

$$1 = \lambda^{pq} + (1-\lambda)^{pq}, \quad \forall\ 0 < \lambda < 1.$$

For $q \neq 1/p$ this is not possible. If $q = 1/p$ evaluate (4.13) at e_1 to obtain

$$f(e_1) = (1-\lambda)^p f(e_1 - \lambda e_2), \quad \forall\ 0 < \lambda < 1.$$

Taking the limit $\lambda \to 1$ yields a contradiction. $\square$

Chapter 5

L_p Minkowski Addition and L_p Mean Width

Recently, Paouris [52] proved a sharp concentration of mass inequality for isotropic convex bodies. Besides L_p intersection bodies, the L_p mean width was an essential tool in his proof. This notion is part of L_p Brunn-Minkowski theory. For $p \geq 1$ and a convex body $K \in \mathcal{K}_0^n$ it is defined as

$$w_p(K) := \left(\int_{S^{n-1}} h(K,u)^p \, du \right)^{\frac{1}{p}}.$$

If p equals one, this is (up to normalization) the classical notion of the mean width of a convex body. Note that the definition of mean width makes sense also for convex bodies which do not contain the origin.

To establish a classification of w_p, we need the concept of L_p Minkowski additive functionals. A function $\mathrm{Z} : \mathcal{K}_0^n \to \mathbb{R}$ is called L_p *Minkowski additive*, if

$$\mathrm{Z}(K +_p L) = \mathrm{Z}\, K + \mathrm{Z}\, L, \quad \text{for every } \ K, L \in \mathcal{K}_0^n.$$

We remark that every L_p Minkowski additive function is a valuation. This is an immediate consequence of the equation

$$(K \cup L) +_p (K \cap L) = K +_p L, \quad \text{for } \ K, L, K \cup L \in \mathcal{K}_0^n. \tag{5.1}$$

Formula (5.1) itself follows from the identities

$$h(K \cup L, \cdot) = \max\{h(K, \cdot), h(L, \cdot)\}, \ h(K \cap L, \cdot) = \min\{h(K, \cdot), h(L, \cdot)\},$$

provided that $K \cup L$ is convex.

Constant multiples of the mean width are the only examples of L_1 Minkowski additive functions which are invariant under proper rotations and continuous (with respect to Hausdorff distance) at the unit ball (see, e.g. Hadwiger's book [25]). Hadwiger's approach can be used to prove also the L_p analogue of this classification.

Theorem 18. *If* $\mathrm{Z} : \mathcal{K}_0^n \to \mathbb{R}$ *is* L_p *Minkowski additive, invariant under proper rotations, and continuous at the unit ball, then* $\mathrm{Z}(K)$ *is a constant multiple of* $w_p(K)^p$ *for all* $K \in \mathcal{K}_0^n$.

Proof. We set

$$h(\lambda \cdot K, u)^p := \lambda h(K, u)^p, \quad u \in S^{n-1},$$

for nonnegative λ. An L_p *rotation mean* K' of a convex body $K \in \mathcal{K}_0^n$ is defined as

$$K' := \frac{1}{m} \cdot (\rho_1 K +_p \cdots +_p \rho_m K),$$

where $\rho_1, \ldots, \rho_m$ are rotations of $\mathbb{R}^n$. We start by proving that for any $K \in \mathcal{K}_0^n$ there exists a sequence of L_p rotation means which converges to a ball with center at the origin and positive radius.

One can find positive numbers r_1, r_2 such that $B(0, r_1) \subset K \subset B(0, r_2)$. Therefore $B(0, r_1) \subset K' \subset B(0, r_2)$ for every L_p rotation mean K' of K. In particular, the set of L_p rotation means is bounded.

Let $r(L)$, $L \in \mathcal{K}_0^n$, denote the radius of the smallest ball with center at the origin which contains L and define $\overline{r} = \inf\{r(K') |\, K' \text{ rotation mean of } K\}$. Note that $\overline{r} \geq r_1 > 0$. By Blaschke's selection principle we can find a sequence $(K'_j)_{j \in \mathbb{N}}$ of L_p rotation means such that $r(K'_j) \to \overline{r}$ and $(K'_j)_{j \in \mathbb{N}}$ itself converges to a $\overline{K} \in \mathcal{K}_0^n$. Since $r(\cdot)$ is continuous, we obtain $r(\overline{K}) = \overline{r}$.

We prove $\overline{K} = B(0, \overline{r})$. Assume that it is false. Then we can find a neighbourhood $U \subset S^{n-1}$ with $h(\overline{K}, u) < \overline{r}$ for every $u \in U$. By compactness of S^{n-1}, there are finitely many rotations $\rho_1, \ldots, \rho_m$ with $\bigcup_{i=1}^m \rho_i U = S^{n-1}$. Define

$$Q := \frac{1}{m} \cdot (\rho_1 \overline{K} +_p \cdots +_p \rho_m \overline{K}).$$

For every $u \in S^{n-1}$ we have

$$h(Q, u)^p = \frac{1}{m} \sum_{i=1}^m h(\rho_i \overline{K}, u)^p = \frac{1}{m} \sum_{i=1}^m h(\overline{K}, \rho_i^{-1} u)^p < \overline{r}^p,$$

since u is contained in $\rho_j U$ for some j. Since support functions are continuous, $h(Q, u) \leq \overline{r} - \varepsilon$ for an $\varepsilon > 0$.

Let L_n and M_n be two sequences in $\mathcal{K}_0^n$ which converge to L, $M \in \mathcal{K}_0^n$. Then

$$L_n +_p M_n \to L +_p M$$

Therefore we obtain

$$\frac{1}{m} \cdot (\rho_1 K'_j +_p \cdots +_p \rho_m K'_j) \to Q. \tag{5.2}$$

We have $\rho_k(\lambda \cdot L +_p \mu \cdot M) = \lambda \cdot \rho_k L +_p \mu \cdot \rho_k M$, $\lambda \cdot (\lambda \cdot L) = \lambda^2 \cdot L$ for arbitrary $L, M \in \mathcal{K}_0^n$ and $\lambda, \mu > 0$. This together with (5.2) contradicts the choice of $\overline{r}$.

We have shown that we can find a sequence $(K'_j)_{j\in\mathbb{N}}$ of L_p rotation means which converges to a ball with positive radius $\overline{r}$ and center at the origin for any body $K \in \mathcal{K}_0^n$. This will prove our classification result. Indeed, from the fact that $2 \cdot K = K +_p K$ we deduce that $\mathrm{Z}(2 \cdot K) = 2\,\mathrm{Z}\,K$. By induction we obtain $\mathrm{Z}(n \cdot K) = n\,\mathrm{Z}\,K$ for every $n \in \mathbb{N}$. For $p, q \in \mathbb{N}$ we therefore get $p\,\mathrm{Z}\,K = \mathrm{Z}(p \cdot K) = \mathrm{Z}(q(p/q) \cdot K) = q\,\mathrm{Z}((p/q) \cdot K)$. Thus $\mathrm{Z}(r \cdot K) = r\,\mathrm{Z}\,K$ for all rational numbers r. Choose a sequence $(q_j)_{j\in\mathbb{N}}$ of rational numbers which converges to $\overline{r}$. Then $q_j^{-1}K'_j \to B^n$. Thus

$$\mathrm{Z}\,B^n = \lim_{j\to\infty} \mathrm{Z}\,q_j^{-1}K'_j = \lim_{j\to\infty} q_j^{-1}\,\mathrm{Z}\,K'_j = \lim_{j\to\infty} q_j^{-1}\,\mathrm{Z}\,K = \overline{r}^{-1}\,\mathrm{Z}\,K.$$

The same calculation can be done for w_p^p and so

$$\mathrm{Z}\,K = \overline{r}\,\mathrm{Z}\,B^n = (w_p(B^n)^{-p}\,\mathrm{Z}\,B^n)w_p(K)^p.$$

$\square$

Bibliography

[1] S. Alesker. Continuous rotation invariant valuations on convex sets. *Ann. of Math. (2)* **149** (1999), 977–1005.

[2] S. Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.* **11** (2001), 244–272.

[3] S. Alesker. Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. *J. Differential Geom.* **63** (2003), 63–95.

[4] S. Alesker. The multiplicative structure on continuous polynomial valuations. *Geom. Funct. Anal.* **14** (2004), 1–26.

[5] C. Borell. Integral inequalities for generalized convex bodies. *J. Math. Anal. Appl.* **43** (1973), 419–440.

[6] J. Bourgain and J. Lindenstrauss. Projection bodies. In *Geometric aspects of functional analysis (1986/87)*, (edited by J. Lindenstrauss and V. D. Milman), volume 1317 of *Lecture Notes in Math.* Springer, Berlin, 1988, (250–270).

[7] H. Busemann. A theorem on convex bodies of the Brunn-Minkowski type. *Proc. Nat. Acad. Sci. U. S. A.* **35** (1949), 27–31.

[8] H. Busemann. Volume in terms of concurrent cross-sections. *Pacific J. Math.* **3** (1953), 1–12.

[9] H. Busemann and C. M. Petty. Problems on convex bodies. *Math. Scand.* **4** (1956), 88–94.

[10] S. Campi and P. Gronchi. The L^p-Busemann-Petty centroid inequality. *Adv. Math.* **167** (2002), 128–141.

[11] W. J. Firey. p-means of convex bodies. *Math. Scand.* **10** (1962), 17–24.

[12] H. Furstenberg and I. Tzkoni. Spherical functions and integral geometry. *Israel J. Math.* **10** (1971), 327–338.

[13] R. Gardner. A positive answer to the Busemann-Petty problem in three dimensions. *Ann. of Math. (2)* **140** (1994), 435–447.

[14] R. Gardner. *Geometric tomography.* Cambridge University Press, Cambridge, 1995.

[15] R. Gardner and A. Giannopoulos. p-cross-section bodies. *Indiana Univ. Math. J.* **48** (1999), 593–613.

[16] R. Gardner, A. Koldobsky and T. Schlumprecht. An analytic solution to the Busemann-Petty problem on sections of convex bodies. *Ann. of Math. (2)* **149** (1999), 691–703.

[17] R. Gardner and G. Zhang. Affine inequalities and radial mean bodies. *Amer. J. Math.* **120** (1998), 505–528.

[18] P. Goodey and W. Weil. Intersection bodies and ellipsoids. *Mathematika* **42** (1995)(2), 295–304.

[19] P. Goodey and W. Weil. Average section functions for star-shaped sets. *Adv. in Appl. Math.* **36** (2006), 70–84.

[20] H. Groemer. Stability results for convex bodies and related spherical integral transformations. *Adv. in Math.* **109** (1994), 45–74.

[21] H. Groemer. *Geometric Applications of Fourier Series and Spherical Harmonics.* Cambridge Univ. Press, Cambridge, 1996.

[22] H. Groemer. On a spherical integral transform and sections of star bodies. *Monatsh. Math.* **126** (1998), 117–124.

[23] H. Guggenheimer. A formula of Furstenberg-Tzkoni type. *Israel J. Math.* **14** (1973), 281–282.

[24] C. Haberl and M. Ludwig. A characterization of L_p intersection bodies. *Int. Math. Res. Not.* **2006** (2006), Article ID 10548, 29 pages.

[25] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.* Springer, Berlin, 1957.

[26] D. Hensley. Slicing convex bodies-bounds for slice area in terms of the body's covariance. *Proc. Amer. Math. Soc.* **79** (1980), 619–625.

[27] D. Hug and R. Schneider. Stability results involving surface area measures of convex bodies. *Rend. Circ. Mat. Palermo (2) Suppl.* **70** (2002)(II), 21–51.

[28] N. J. Kalton, A. Koldobsky, V. Yaskin and M. Yaskina. The geometry of L_0. *Canadian J. Math.* , to appear.

[29] D. A. Klain. Star valuations and dual mixed volumes. *Adv. Math.* **121** (1996), 80–101.

[30] D. A. Klain. Invariant valuations on star-shaped sets. *Adv. Math.* **125** (1997), 95–113.

[31] D. A. Klain and G.-C. Rota. *Introduction to geometric probability.* Cambridge University Press, Cambridge, 1997.

[32] A. Koldobsky. The Busemann-Petty problem via spherical harmonics. *Adv. Math.* **177** (2003), 105–114.

[33] A. Koldobsky. *Fourier analysis in convex geometry*, volume 116 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005.

[34] K. Leichtweiß. *Affine geometry of convex bodies.* Johann Ambrosius Barth, Heidelberg, 1998.

[35] M. Ludwig. Projection bodies and valuations. *Adv. Math.* **172** (2002), 158–168.

[36] M. Ludwig. Valuations on polytopes containing the origin in their interiors. *Adv. Math.* **170** (2002), 239–256.

[37] M. Ludwig. Ellipsoids and matrix valued valuations. *Duke Math. J.* **119** (2003), 159–188.

[38] M. Ludwig. Minkowski valuations. *Trans. Amer. Math. Soc.* **357** (2005), 4191–4213.

[39] M. Ludwig. Intersection bodies and valuations. *Amer. J. Math.* **128** (2006), 1409–1428.

[40] M. Ludwig and M. Reitzner. A characterization of affine surface area. *Adv. Math.* **147** (1999), 138–172.

[41] M. Ludwig and M. Reitzner. Elementary moves on triangulations. *Discrete Comput. Geom.* **35** (2006), 527–536.

[42] E. Lutwak. Intersection bodies and dual mixed volumes. *Adv. Math.* **71** (1988), 232–261.

[43] E. Lutwak. Centroid bodies and dual mixed volumes. *Proc. Lond. Math. Soc.* **60** (1990), 365–391.

[44] E. Lutwak. On some ellipsoid formulas of Busemann, Furstenberg and Tzkoni, Guggenheimer, and Petty. *J. Math. Anal. Appl.* **159** (1991), 18–26.

[45] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.* **38** (1993), 131–150.

[46] E. Lutwak, D. Yang and G. Zhang. L_p affine isoperimetric inequalities. *J. Differential Geom.* **56** (2000), 111–132.

[47] E. Lutwak, D. Yang and G. Zhang. Moment-entropy inequalities. *Ann. Probab.* **32** (2004), 757–774.

[48] E. Lutwak and G. Zhang. Blaschke-Santaló inequalities. *J. Differential Geom.* **47** (1997), 1–16.

[49] P. McMullen. Valuations and dissections. In *Handbook of Convex Geometry, Vol. B*, (edited by P. Gruber and J. Wills). North-Holland, Amsterdam, 1993, (933–990).

[50] P. McMullen and R. Schneider. Valuations on convex bodies. In *Convexity and its applications*, (edited by P. Gruber and J. Wills). Birkhäuser, 1983, (170–247).

[51] V. D. Milman and A. Pajor. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space. In *Geometric aspects of functional analysis (1987–88)*, volume 1376 of *Lecture Notes in Math.* Springer, Berlin, 1989, (64–104).

[52] G. Paouris. Concentration of mass on isotropic convex bodies. *Geom. Funct. Anal.* **16** (2006), 1021–1049.

[53] C. M. Petty. Centroid surfaces. *Pacific J. Math.* **11** (1961), 1535–1547.

[54] A. Prudnikov, Y. Brychov and O. Marichev. *Integrals and series, volume 1.* Gordon and Breach Science Publishers, New York, 1986.

[55] A. Prudnikov, Y. Brychov and O. Marichev. *Integrals and series, volume 2.* Gordon and Breach Science Publishers, New York, 1986.

[56] B. Rubin. Fractional integrals and wavelet transforms associated with Blaschke-Levy representations on the sphere. *Israel J. Math.* **114** (1999), 1–27.

[57] B. Rubin. Generalized Minkowski-Funk transforms and small denominators on the sphere. *Fract. Calc. Appl. Anal.* **3** (2000)(2), 177–203.

[58] R. Schneider. *Convex bodies: the Brunn–Minkowski theory.* Cambridge Univ. Press, Cambridge, 1993.

[59] R. Schneider and F. E. Schuster. Rotation equivariant Minkowski valuations. *Int. Math. Res. Not.* (2006), Article ID 72894.

[60] A. C. Thompson. *Minkowski geometry.* Cambridge University Press, Cambridge, 1996.

[61] V. Yaskin and M. Yaskina. Centroid bodies and comparison of volumes. *Indiana Univ. Math. J.* **55** (2006), 1175–1194.

[62] G. Zhang. A positive solution to the Busemann-Petty problem in $\mathbf{R}^4$. *Ann. of Math. (2)* **149** (1999), 535–543.

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