DISSERTATION

On the Renormalisability of the Sine–Gordon and Thirring Models

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Die Dissertation widmet sich der Renormormierbarkeit von 1+1 dimensionalen Quantenfeldtheorien wie das Sine-Gordon–Modell (SG–Modell) und das masselose Thirring Modell.

Die Analyse der Renormierbarkeit des SG–Modells führen wir anhand der kausalen Zweipunktgreenfunktion durch. Hierzu betrachten wir die Renormierbarkeit der kausalen Zweipunktgreenfunktion relativ zu Quantenfluktuationen um das triviale Vakuum (den trivialen Sektor) und um das solitonische Vakuum (Solitonsektor).

Im trivialen Sektor berechnen wir die Quantenfluktuationen bis zur zweiten Ordnung in der dimensionsbehafteten Kopplungskonstante α_0 und für alle Ordnungen der dimensionslosen Kopplungskonstante β. Die Summation über alle relevanten Terme zeigt, dass die Theorie unabhängig von der gewählten Renormierungsskala M ist. Die effektive dimensionale Kopplungskonstante stimmt mit der Masse des SG Feldes überein und kann daher mit der physikalischen Kopplungskonstante α_r identifiziert werden.

Die Callan-Symanzik Gleichung für die kausale Zweipunktgreenfunktion wird aufgstellt und gelöst. Wir zeigen hierbei in allen Ordnungen der Störungstheorie in den Kopplungskonstanten α_r und β, dass die kausale Zweipunktgreenfunktion nur von der renormierten Kopplungskonstante α_r und β abhängt.

Bei der Analyse der Renormierbarkeit des SG–Modells für Quantenfluktuationen im Soliton Sektor zeigt sich, dass das Modell analog zum trivialen Sektor in erster Ordung in β^2 renormierbar ist.

Die Analyse der Renormierbarkeit des SG Modells in allen Ordnungen der dimensionslosen Kopplung β ermöglicht, die SG–Quanten für β nahe des kritischen Punktes $\beta^2 = 8\pi$, des sogenannten Kosterlitz-Thouless Phasenüberganges zu betrachten.

Eine wichtige Folgerung unserer Rechnungen ist, dass kein endlicher Korrekturterm zur Masse des SG–Solitons auftritt und wir damit die Vermutung von Zamalodchikov et al. bestätigen, dass ein endlicher Term, welcher bei Autoren wie Dashen et al. und Faddeev et al. auftritt, nur von der Regularisierungs– und Renormierungsprozedur abhängt. Auch wird die Möglichkeit das SG–Modell für Quantumfluktuationen um beliebige klassische Lösungen nicht stö–rungstheoretisch zu renormieren betrachtet.

Die Renormierbarkeit des masselosen Thirring Modells führen wir anhand der fermionischen Zweipunktgreenfunktion und Zweipunktkorrelationsfunktion in allen Ordnungen der Störungstheorie in der Thirring Kopplungskonstante g durch. Es zeigt sich, dass die dynamischen Dimensionen der Thirring Fermionfelder, aus den Greenfunktionen und Korrelationsfelder durch zwei beliebige Paramater parametrisiert werden können, anstatt einem, wie in der Arbeit vom Klaiber aus dem Jahr 1960. Durch eine entsprechede Wahl eines der Parameter haben beide Zweipunktfunktionen die äquivalente dynamische Dimension mit einem freien Parameter. Das gleiche dynamische Verhalten der

fermionischen Thirringfelder erlaubt uns dann die beiden kausalen Zweipunktfunktionen durch Renormierung divergenzenfrei zu wählen.

Die wichtige Konsequenz der Renormierbarkeit des masselosen Thirring Modell ist die Bosonisierung des massiven Thirring Modells zum SG Modell bei der Kopplung $\beta^2 \sim 8\pi$, nahe dem kritischen Punkt des Kosterlitz–Thouless Phasenüberganges.

Abstract

This thesis is devoted to the analysis of the renormalisability of 1+1–dimensional quantum field theories such as the sine–Gordon (SG) model and the massless Thirring model.

The analysis of the renormalisation of the SG model is carried out for the example of the causal two–point Green function. The renormalisation of the causal two–point Green function is investigated with respect to quantum fluctuations of the SG field relative to the trivial vacuum and to the soliton solution. The contributions of quantum fluctuations of the SG field relative to the trivial vacuum are calculated to first and second order in the dimensional coupling constant α_0 and to all orders in the dimensionless coupling constant β. It is shown that after the summation of all contributions calculated to first order in α_0 and to all orders in β^2 and the removal of the ultra–violet cut–off at the normalisation scale M the causal two–point Green function has the shape of a causal two–point Green function of the free SG field with mass independent of the normalisation scale M. The effective dimensional coupling constant obtained in such an approximation coincides with the mass of the SG field and can be identified with the physical coupling constant α_r.

Within the Callan–Symanzik equation approach to the analysis of the renormalisability of quantum field theories, the dependence of the causal two–point Green function only on the coupling constant α_r is proved to all orders of perturbation theory with respect to dimensional and dimensionless coupling constants.

The analysis of the renormalisability of the SG model to all orders in the dimensionless coupling β makes possible a consideration of the behaviour of the SG quanta for coupling constants β in the vicinity of the critical point $\beta^2 = 8\pi$ of the Kosterlitz–Thouless phase transition. Using the Callan–Symanzik equation an agreement of our renormalisation procedure with others, used for the analysis of the SG model for $\beta^2 \approx 8\pi$, is proved to first order in the dimensional and to all order in the dimensionless coupling constants.

An extension of our analysis to the calculation of the contributions of the Gaussian fluctuations around the soliton solution, describing a non–trivial vacuum in the SG model, has shown that the renormalisation of the SG model due to quantum fluctuations around a soliton solution runs parallel the renormalisation of the SG model relative to quantum fluctuations with respect to the trivial vacuum.

The important consequence of this result is the absence of a finite correction to the soliton mass. This confirms the assertion of Zamalodchikov *et al.* on the dependence of a finite correction to the soliton mass, obtained by Dashen et al. and Faddeev et al., on the regularisation and renormalisation procedure.

The renormalisability of the massless Thirring model is investigated to all orders of perturbation theory in the Thirring coupling constant g by the example of the two–point causal Green function of Thirring fermion fields and the two–

point correlation function of left–right fermion densities. It is shown that the dynamical dimensions of the Thirring fermion fields, calculated from two–point causal Green function and two–point correlation function of left–right fermion densities, can be parameterised by two arbitrary parameters instead of one arbitrary parameter that was pointed out by Klaiber in 1960's.

The dependence of dynamical dimensions on an additional arbitrary parameter admits the non–perturbative renormalisability of the massless Thirring model in the sense of the removal of all divergences of correlation functions, calculated to all order of the coupling constant g, by means of the renormalisation of the wave functions of the Thirring fermion fields. It is shown that that such a removal is possible if the dynamical dimensions of the Thirring fermion fields, calculated for different correlation functions, are equal. Having equated the dynamical dimensions obtained from different correlation functions, one arrives at a solution of the massless Thirring model, where all correlation functions of the massless Thirring model are independent of the ultra–violet cut–off and parameterised by one arbitrary parameter.

The main consequence of such a renormalisability of the massless Thirring model is the bosonization of the massive Thirring model to the SG model with coupling constants $\beta^2 \sim 8\pi$ in the vicinity of the critical point of the Kosterlitz–Thouless phase transition.

Moa'e Pi'ye mire

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Contents

Part I

Renormalisability of the sine–Gordon Model

Chapter 1

sine–Gordon model. Introductory comments

The Lagrangian of the sine–Gordon model in 1+1–dimensional space–time is given by [1, 2, 3]

$$\mathcal{L}(x,t) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha}{\beta^2}\,(\cos\beta\vartheta(x) - 1), \tag{1.1}$$

where α and β are a dimensional and dimensionless coupling constants, respectively. In the region of weak coupling $\beta^2 \ll 1$, the Lagrangian $\mathcal{L}(x)$, expanded in powers of β, takes the form

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}\,\alpha\,\vartheta^2(x) + \frac{1}{4!}\,\alpha\,\beta^2\,\vartheta^4(x) - \frac{1}{6!}\,\alpha\,\beta^4\,\vartheta^6(x) + \cdots, \tag{1.2}$$

where we have simplified the notation $\vartheta(x^1,x^0)$ as $\vartheta(x^1,x^0) = \vartheta(x)$ and $x = (x^0,x^1)$ is a 1+1–dimensional vector. The action of the SG model is

$$S[\vartheta] \;=\; \int d^2x\,\mathcal{L}[\vartheta(x)]. \tag{1.3}$$

The Lagrangian (1.1) is invariant under the discrete symmetry operations

$$\begin{aligned}\vartheta(x) &\;\to\; \vartheta'(x) \;=\; -\,\vartheta(x)\\ \vartheta(x) &\;\to\; \vartheta'(x) \;=\; \vartheta(x) + \frac{2\pi N}{\beta},\end{aligned} \tag{1.4}$$

with N being integer numbers $N = 0, \pm1, \pm2, \ldots$ The topological current is given by [1]

$$J^\mu(x) \;=\; \frac{1}{2\pi}\,\varepsilon^{\mu\nu}\,\partial_\nu\vartheta(x). \tag{1.5}$$

The topological current is always conserved and defines the topological charge Q [1, 3]

$$\begin{aligned} \mathrm{Q} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx^1\, J^0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx^1\, \frac{\partial\vartheta(x)}{\partial x^1} = \\ &= \frac{1}{2\pi}\left\{\vartheta(x^1=\infty) - \vartheta(x^1=-\infty)\right\} = N_1 - N_2, \end{aligned} \tag{1.6}$$

where N_1 and N_2 are integers. The energy of the SG field is determined by

$$E[\vartheta] = \int_{-\infty}^{\infty} dx^1 \left(\frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\vartheta(x)}{\partial x^1}\right)^2 + \frac{\alpha}{\beta^2}\left(1-\cos\beta\vartheta(x)\right)\right). \tag{1.7}$$

According to the Principle of Least Action, the equation of motion of the SG field is

$$-\frac{\delta}{\delta\vartheta(x)}S[\vartheta] = \Box\vartheta(x) + \frac{\alpha}{\beta}\sin\beta\vartheta(x) = 0. \tag{1.8}$$

The simplest solutions of (1.8) are constant solutions of the form

$$\vartheta_{\mathrm{N}} = 2N\pi = \mathrm{const.}, \tag{1.9}$$

where N is an integer, defining the Nth vacuum of the theory. They minimize the energy functional (1.7). The SG model is therefore bounded from below. Among the solutions of this equation of motion there are the soliton solutions, which have finite energy and do not change the shape during a motion. We find a static solution of Eq. (1.8) by multiplying it with $\partial\vartheta(x^1)/\partial x^1$ and get by integration

$$\frac{\partial\vartheta(x^1)}{\partial x^1} = \pm\sqrt{2\,\frac{\alpha}{\beta^2}\left(1-\cos\beta\vartheta(x^1)\right)} = \pm 2\sqrt{\frac{\alpha}{\beta^2}}\,\sin\beta\frac{\vartheta(x^1)}{2}. \tag{1.10}$$

Another integration gives

$$x^1 - x^1_0 = \pm\int \frac{d\vartheta}{\sqrt{2\,\frac{\alpha}{\beta^2}\left(1-\cos\beta\vartheta\right)}} = \pm\frac{1}{\sqrt{\alpha}}\,\ln\tan\,\beta\,\frac{\vartheta}{4}. \tag{1.11}$$

Inverting this equation we obtain the static nontrivial solutions

$$\vartheta_{\mathrm{s}}(x^1) = \frac{4}{\beta}\,\arctan\,\exp\left\{\pm\sqrt{\alpha}\,(x^1 - x^1_0)\right\}. \tag{1.12}$$

These are static soliton solutions, where the sign $\pm$ defines a soliton and an antisoliton solution, respectively, and x^1_0 can be interpreted as a center of a soliton and an antisoliton. Choosing $x^1_0 = 0$ we get

$$\vartheta_{\mathrm{s}}(x^1) = \frac{4}{\beta}\,\arctan\,\exp\left\{\pm\sqrt{\alpha}\,x^1\right\}. \tag{1.13}$$

The Lorentz boosted solutions take the form

$$\vartheta_{\rm s}(x^1,x^0) \;=\; \frac{4}{\beta}\,\arctan\,\exp\left\{\pm\frac{\sqrt{\alpha}}{\sqrt{1-u^2}}\,(x^1-u\,x^0)\right\}, \tag{1.14}$$

where u is a velocity. Some more complicated solutions, for example, the soliton–antisoliton $\vartheta_{\rm sa}(x^1,x^0)$ and soliton–soliton $\vartheta_{\rm ss}(x^1,x^0)$ are

$$\begin{aligned}\vartheta_{\rm s\,a}(x^1,x^0) &= \frac{4}{\beta}\,\arctan\frac{\sinh\sqrt{\alpha}\,u\,x^0/\sqrt{1-u^2}}{u\cosh\sqrt{\alpha}\,x^1/\sqrt{1-u^2}},\\ \vartheta_{\rm s\,s}(x^1,x^0) &= \frac{4}{\beta}\,\arctan\frac{u\sinh\sqrt{\alpha}\,x^1/\sqrt{1-u^2}}{\cosh\sqrt{\alpha}\,u\,x^0/\sqrt{1-u^2}}.\end{aligned} \tag{1.15}$$

For $x^0 \to \pm\infty$, i.e. for infinite past and infinite future, the soliton–antisoliton and soliton–soliton solutions behave as follows

$$\begin{aligned}\lim_{x^0\,\to\pm\infty}\vartheta_{\rm s\,a}(x^1,x^0) &= \vartheta_{\rm s}(x^1,x^0)+\vartheta_{\rm a}(x^1,x^0),\\ \lim_{x^0\,\to\pm\infty}\vartheta_{\rm s\,s}(x^1,x^0) &= \vartheta_{\rm s}(x^1,x^0)+\vartheta_{\rm s}(x^1,x^0).\end{aligned} \tag{1.16}$$

This implies that they can describe soliton–antisoliton and soliton–soliton scattering and that the solution (1.14) indeed describes a soliton [1].

1.1 Semiclassical Quantization of sine–Gordon Solitons

In this chapter we give a cursory outline of a semiclassical quantization by using the Wentzel–Kramer–Brillouin (WKB) method or saddle point expansion. This is a well known semiclassical procedure which manifestly relates classical solutions to their quantum levels. The underlying principle of all semiclassical quantization procedures is the so–called Correspondence Principle between classical and quantum mechanical theories [1].

In this approach a quantum field $\vartheta(x)$ introduces to the classical theory a shift of classical solutions $\vartheta_{\rm cl}(x)$

$$\vartheta_{\rm cl}(x) \;\to\; \vartheta_{\rm cl}(x)+\vartheta(x). \tag{1.17}$$

The Lagrangian (1.1) of such a system reads

$$\begin{aligned}\mathcal{L}(x) &= \frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x)+\frac{\alpha}{\beta^2}\,(\cos\beta\vartheta_{\rm cl}(x)-1)\\ &+ \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x)-\frac{\alpha}{\beta^2}\,\cos\beta\vartheta_{\rm cl}(x)\,(1-\cos\beta\vartheta(x))\\ &+ \frac{\alpha}{\beta^2}\,\sin\beta\vartheta_{\rm cl}(x)\,(\beta\vartheta(x)-\sin\beta\vartheta(x)),\end{aligned} \tag{1.18}$$

where we have used the equation of motion (1.8) for $\vartheta_{\rm cl}(x)$.

Expanding in powers of the $\vartheta(x)$–field and keeping only quadratic terms or alternatively Gaussian fluctuations we get

$$\begin{aligned}\mathcal{L}(x) &= \frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x) + \frac{\alpha}{\beta^2}\,(\cos\beta\vartheta_{\rm cl}(x) - 1)\\ &+ \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{\alpha}{2}\,\cos\beta\vartheta_{\rm cl}(x)\,\vartheta^2(x).\end{aligned}\tag{1.19}$$

The action, determined by the Lagrangian (1.19), is

$$S[\vartheta] \;=\; S[\vartheta_{\rm cl}] - \frac{1}{2}\int d^2x\,\vartheta(x)\Big[\partial_\mu\partial^\mu + \alpha\,\cos\beta\vartheta_{\rm cl}(x)\Big]\vartheta(x),\tag{1.20}$$

where we have integrated by parts and dropped the surface term.

For the analysis of the Gaussian fluctuations around a classical solution it is important to know the spectrum of the operator $\partial_\mu\partial^\mu + \alpha\,\cos\beta\vartheta_{\rm cl}(x)$. Let $\vartheta_{\omega,k}(x,t)$ be eigenfunctions of the operator $\partial_\mu\partial^\mu + \alpha\,\cos\beta\vartheta_{\rm cl}(x)$ with eigenvalues $\lambda(\omega,k)$

$$-\frac{\delta S[\vartheta]}{\delta\vartheta(x)} = (\,\Box\,+\,\alpha\cos\beta\vartheta_{\rm cl}(x^1)\,)\vartheta_{\omega,k}(x) \;=\lambda(\omega,k)\,\vartheta_{\omega,k}(x).\tag{1.21}$$

In this case the action (1.20) takes the form

$$\begin{aligned}S[\vartheta] &= \int d^2x\,\mathcal{L}[\vartheta_{\rm cl}]\\ &= S[\vartheta_{\rm cl}] - \frac{1}{2}\int d^2x\,\sum_{\omega,k}\vartheta_{\omega,k}(x)\,\lambda(\omega,k)\,\vartheta_{\omega,k}(x).\end{aligned}\tag{1.22}$$

The contributions of Gaussian fluctuations to different physical quantities are defined in terms of the functional determinant, which is equal to the product of eigenvalues.

Since the contribution of any fluctuation relative to the classical solution or alternatively a non–trivial vacuum are compared with the contribution of the trivial vacuum, in the next section we analyse the spectra of the differential operator $\partial_\mu\partial^\mu + \alpha\,\cos\beta\vartheta_{\rm cl}(x)$ for the solutions $\vartheta_{\rm cl}(x) = \vartheta_0 = 0$ and $\vartheta_{\rm cl}(x) = \vartheta_{\rm s}(x)$.

1.1.1 Spectra of the Operator $\partial_\mu\partial^\mu + \alpha\,\cos\beta\vartheta_{\rm cl}(x)$ for $\vartheta_{\rm cl}(x) = \vartheta_0 = 0$, and $\vartheta_{\rm cl}(x) = \vartheta_{\rm s}(x)$

For a trivial vacuum $\vartheta_{\rm cl}(x) = \vartheta_0 = 0$, when $\cos\beta\vartheta_{\rm cl}(x) = 1$, Eq. (1.21) reduces to the Klein–Gordon equation

$$(\Box + \alpha\,)\;\vartheta_{\omega',k'}(x) \;=\; \lambda(\omega',k')\,\vartheta_{\omega',k'}(x).\tag{1.23}$$

The solutions reads

$$\vartheta_{\omega',k'}(x) \;=\; \frac{1}{2\pi}\,{\rm e}^{-\,i\omega'\,x^0\,+\,ik'\,x^1},\tag{1.24}$$

where the set $\{\omega', k'\}$ of continuous quantum numbers obey the dispersion relation

$$\lambda(\omega', k') \;=\; -\omega'^2 + k'^2 + \alpha, \tag{1.25}$$

where ω' and k' are energy and spatial momentum.

For the static one–soliton solution $\vartheta_{\rm cl}(x) = \vartheta_{\rm s}(x)$ (1.12) the equation (1.21) reads

$$\left(\Box + \alpha - \frac{2\alpha}{\cosh^2\sqrt{\alpha}x^1}\right)\vartheta_{\omega,k}(x) \;=\; \lambda(\omega,k)\,\vartheta_{\omega,k}(x). \tag{1.26}$$

The solutions of this equation are adduced in Appendix A.1, Eq. (A.23), they are

$$\begin{aligned}\vartheta_{\omega,b}(x) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sqrt{\alpha}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha}x^1)}\,{\rm e}^{-\,i\omega\,x^0}\\ \vartheta_{\omega,k}(x) &= \frac{1}{2\pi}\,\frac{k + i\sqrt{\alpha}\,\tanh\sqrt{\alpha}x^1}{\sqrt{k^2+\alpha}}\,{\rm e}^{-\,i\omega\,x^0\,+\,ik\,x^1},\end{aligned} \tag{1.27}$$

where $\vartheta_{\omega,b}(x)$ and $\vartheta_{\omega,k}(x)$ describe a bound state ($k = i\sqrt{\alpha}$) and a continuous set of scattering states, respectively. The eigenvalues $\lambda(\omega,k)$ are defined by

$$\lambda(\omega, k) \;=\; -\omega^2 + k^2 + \alpha. \tag{1.28}$$

The normalised spatial solutions read

$$\begin{aligned}\vartheta_b(x^1) &= \sqrt{\frac{\sqrt{\alpha}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha}x^1)}\\ \vartheta_k(x^1) &= \frac{1}{\sqrt{2\pi}}\,\frac{k + i\sqrt{\alpha}\,\tanh\sqrt{\alpha}x^1}{\sqrt{k^2+\alpha}}\,{\rm e}^{ik\,x^1}.\end{aligned} \tag{1.29}$$

The completeness and orthogonality of the solutions are proved in Appendix A.1.1.

Phase Shift, Levinson's Theorem and Number of Bound States

The asymptotic behavior of the quantum fluctuations for $x^1 \to \pm\infty$ can be defined as follows

$$\vartheta_k(x^1) \to \frac{1}{\sqrt{2\pi}}\,\exp\{ikx^1 \pm i\,\frac{1}{2}\,\delta(k)\}, \tag{1.30}$$

where $\delta(k)$ is a phase shift

$$\delta(k) \;=\; 2\arctan\frac{\sqrt{\alpha}}{k}. \tag{1.31}$$

According to the Levinson theorem, the number of bound states n, caused by the interaction inducing the phase shift $\delta(k)$, is equal to

$$\delta(0) - \delta(\infty) \;=\; n\pi. \tag{1.32}$$

For the phase shift $\delta(k)$ (1.30), induced by the soliton (1.12), we get

$$\delta(0) - \delta(\infty) \ = \ \pi. \tag{1.33}$$

This gives $n = 1$ and agrees well with the existence of the one discrete state $\vartheta_b(x)$, Eq. (1.29) in the soliton sector.

Chapter 2

Renormalisation of the sine–Gordon Model

In this chapter we discuss the renormalisability of the sine–Gordon model. The analysis of the renormalisability of the SG model will be performed perturbatively with respect to causal two–point Green functions. The quantum corrections are calculated relative to the trivial vacuum (1.9) to second order in the coupling constant α and to all orders in β^2.

We show that the SG model is well–defined not only for $\beta^2 < 8\pi$ but for $0 \le \beta^2 < \infty$. An important application of this result is the Fractional Quantum Hall Effect (the FQHE) [4, 5] . Indeed, as has been pointed out in [4, 5] the FQHE is defined by the edge tunnelling of quasi–particles and electrons. In the bosonised form the Hamiltonian of the interaction of quasi–particles and electrons has the form of the sine–Gordon interaction [4]

$$\mathcal{H}_{\rm int}(x) \;=\; -\,\frac{\alpha}{\beta^2}\,\cos\beta\vartheta(x). \tag{2.1}$$

The parameter β^2 is defined by [4]

$$\beta^2 = \begin{cases} 4\pi\nu & \text{for tunneling of quasi} - \text{particles} \\ 4\pi/\nu & \text{for tunneling of electrons} \end{cases} \tag{2.2}$$

where ν is the filling factor [4]. If the coupling constant β^2 obeys the constraint $\beta^2 < 8\pi$, only quasi–particles can be responsible for the FQHE. The participance of electrons in the FQHE is prohibited. However, if there is a possibility for the coupling constant β^2 to be greater than 8π, i.e. $\beta^2 > 8\pi$, the participance of electrons in the FQHE cannot be suppressed. This opens new possibilities for the dynamics of the FQHE.

2.1 Introduction

In this introduction we briefly give the basic definitions that will be used in our study of the renormalisability of the SG model.

The bare Lagrangian of the SG model is given by [6]

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0(\Lambda^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1), \tag{2.3}$$

where $\alpha_0(\Lambda^2)$ is the bare coupling constant at an ultra–violet cut–off Λ. The renormalised coupling constant $\alpha_r(M^2)$ at the renormalisation scale M reads [6, 7, 8]

$$\alpha_r(M^2) \;=\; Z_1^{-1}(\Lambda^2;\beta^2, M^2)\alpha_0(\Lambda^2), \tag{2.4}$$

$Z_1(\Lambda^2;\beta^2, M^2)$ is the renormalisation constant [6]. The renormalised Lagrangian reads

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + Z_1\,\frac{\alpha_r(M^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1). \tag{2.5}$$

The scalar field $\vartheta(x)$ and the dimensionless constant β^2 are non–renormalisable. Expanding the cosine in the Lagrangian (2.5) in polynomials of $\vartheta^2(x)$, we identify the renormalised Lagrangian as

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}\,\alpha_r(M^2)\,\vartheta^2(x) \;+\; \mathcal{L}_{\rm int}(x). \tag{2.6}$$

$\mathcal{L}_{\rm int}(x)$ denotes the interaction part of the renormalised Lagrangian and is defined by

$$\begin{aligned}\mathcal{L}_{\rm int}(x) \;&=\; \alpha_r(M^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x)\\ &+\; (Z_1 - 1)\,\alpha_r(M^2)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x).\end{aligned} \tag{2.7}$$

The renormalised (and normalised) generating functional $Z[J]$ for two–point Green functions is

$$\begin{aligned}Z[J] \;&=\; \int\mathcal{D}\vartheta\,\exp\Big\{i\int d^2x\,\Big(\mathcal{L}(x) + \vartheta(x)J(x)\Big)\Big\}\\ &=\; \int\mathcal{D}\vartheta\,\exp\Big\{i\int d^2x\,\frac{1}{2}\,\Big(\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \alpha_r(M^2)\,\vartheta^2(x)\Big)\Big\}\\ &\quad\times\;\exp\Big\{i\int d^2x\Big(\mathcal{L}_{\rm int}(x) + \vartheta(x)J(x)\Big)\Big\},\end{aligned} \tag{2.8}$$

where $J(x)$ is an external source for the free quanta $\vartheta(x)$ of the SG model. The causal two–point Green function $-i\Delta(x,y;\alpha_r(M^2))$ of the SG field is defined by

$$-i\Delta(x,y;\alpha_r(M^2)) \;\;=\;\; \frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}Z[J]_{J=0}. \tag{2.9}$$

The causal two–point Green function of free SG quanta with mass $\alpha_r(M^2)$ is given by [6]

$$-i\Delta_F(x,y;\alpha_r(M^2)) = \langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(y)\Big)|0\rangle = \int \frac{d^2k}{(2\pi)^2}\,\frac{-\,i\;\mathrm{e}^{-ik(x\,-\,y)}}{\alpha_r(M^2) - k^2 - i0}. \quad (2.10)$$

At $x = y$ the Green function $-i\Delta_F(0;\alpha_r(M^2))$ is equal to

$$-i\Delta_F(0;\alpha_r(M^2)) = \frac{1}{4\pi}\,\ln\Big[\frac{\Lambda^2}{\alpha_r(M^2)}\Big], \quad (2.11)$$

where Λ is a cut–off in Euclidean 2–dimensional momentum space [6].

2.2 Renormalisability of the sine–Gordon model

We perform the analysis of the renormalisability of the two–dimensional SG model by following the standard procedure [7]–[13]. For a Feynman diagram G with L independent loops, I internal boson lines and V_{2n} vertices with $2n$ lines ($n > 0$) the superficial degree of divergence of momentum integrals $\omega(G)$ based on dimensional considerations is

$$\omega(G) \;=\; 2L - 2I. \quad (2.12)$$

The number of independent loops L is defined by

$$L \;=\; I + 1 - \sum_{\{n\}} V_{2n}. \quad (2.13)$$

Substituting (2.13) into (2.12) gives for the superficial degree

$$\omega(G) \;=\; 2 - 2\sum_{\{n\}} V_{2n}. \quad (2.14)$$

This testifies the complete renormalisability of the SG model.

Feynman diagrams with one vertex diverge logarithmically. All other diagrams are convergent [2]. The divergences can be removed by the renormalisation of the dimensional coupling constant α [6].

2.3 Renormalisation of the causal two–point Green function of the sine–Gordon model

The causal two–point Green function of the SG field (2.9) reads explicitly

$$\begin{aligned}
-i\Delta(x,y;\alpha_r(M^2)) \;&=\; \frac{1}{i}\frac{\delta}{\delta J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}Z[J]_{J=0}\\
&=\; \int \mathcal{D}\vartheta\,\vartheta(x)\vartheta(y)\,\exp\Big\{i\int d^2z\,\mathcal{L}_{\rm int}(z)\Big\}\\
&\times\;\; \exp\Big\{\frac{i}{2}\int d^2z\,\Big(\partial_\mu\vartheta(z)\partial^\mu\vartheta(z) - \alpha_r(M^2)\,\vartheta^2(z)\Big)\Big\}. \quad (2.15)
\end{aligned}$$

It is equal to the vacuum expectation value of the time ordered field product of $\vartheta(x)\vartheta(y)\exp\{i\int\mathcal{L}_{\rm int}\}$ [10]

$$-i\Delta(x,y;\alpha_r(M^2)) \;=\; \langle 0|{\rm T}\Big(\vartheta(x)\vartheta(y)\;\exp\Big\{i\int d^2z\,\mathcal{L}_{\rm int}(z)\Big\}\Big)|0\rangle_{\rm c}, \qquad (2.16)$$

where T denotes the time–ordering operator, while the subscript c expresses the fact that only connected graphs are taken into account.

As usual we treat the expression (2.16) perturbativley and expand $\exp\{i\int\mathcal{L}_{\rm int}\}$ in a power series of $\alpha_r(M^2)$ or equivalently due to Eq. (2.7) in powers of $\mathcal{L}_{\rm int}$. It reads, for $y = 0$

$$\begin{aligned} -i\Delta(x,0;\alpha_r(M^2)) &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int\prod_{k=1}^{n}d^2z_k\langle 0|{\rm T}(\vartheta(x)\vartheta(0)\mathcal{L}^n_{\rm int}(z_k))|0\rangle_{\rm c} \\ &= \sum_{n=0}^{\infty}-i\Delta^{(n)}_F(x,0;\alpha_r(M^2)). \end{aligned} \qquad (2.17)$$

To second–order in $\alpha_r(M^2)$ the renormalised causal two–point Green function reads

$$\begin{aligned} -i\Delta(x,0;\alpha_r(M^2)) &= -i\Delta^{(0)}_F(x,y;\alpha_r(M^2))\;+ \\ &\quad -i\Delta^{(1)}_F(x,0;\alpha_r(M^2)) - i\Delta^{(2)}_F(x,0;\alpha_r(M^2)) - \cdots. \end{aligned} \qquad (2.18)$$

The term $-i\Delta^{(0)}_F(x,0;\alpha_r(M^2))$ in (2.18) corresponds to the two–point Green function of free SG quanta, Eq. (2.10). The term $-i\Delta^{(1)}_F(x,0;\alpha_r(M^2))$ proportional to $\alpha_r(M^2)$ contribute, according to the ϕ^4 theory, to the self–energy of the SG quanta.

In momentum space the terms in the expansion (2.17) reads

$$\begin{aligned} -i\tilde{\Delta}^{(n)}_F(p;\alpha_r(M^2)) &= \int d^2x\,{\rm e}^{ipx}\,(-i)\Delta^{(n)}_F(x,0;\alpha_r(M^2)) \\ &= \frac{i^n}{n!}\int\prod_{k=1}^{n}d^2z_k\int d^2x\,{\rm e}^{ipx}\langle 0|{\rm T}(\vartheta(x)\vartheta(0)\mathcal{L}^n_{\rm int}(z_k))|0\rangle_{\rm c}. \end{aligned} \qquad (2.19)$$

2.3.1 Two–point Green function to first–order in $\alpha_r(M^2)$ and to all Orders in β^2

The explicit expression for the first–order correction $-i\Delta^{(1)}(x,0;\alpha_r(M^2))$ to the causal two–point Green function (2.15) is given by

$$\begin{aligned} -i\Delta^{(1)}_F(x,0;\alpha_r(M^2)) &= i\int d^2z\langle 0|{\rm T}\Big(\vartheta(x)\vartheta(0)\mathcal{L}_{\rm int}(z)\Big)|0\rangle_{\rm c} \\ &= i\,\alpha_r(M^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\,\beta^{2(n-1)}\int d^2z\,\langle 0|{\rm T}\Big(\vartheta(x)\vartheta(0)\vartheta^{2n}(z)\Big)|0\rangle_{\rm c} \end{aligned}$$

$$+ i\,\alpha_r(M^2)\,(Z_1-1)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\,\beta^{2(n-1)}\int d^2z\,\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^{2n}(z)\Big)|0\rangle_{\rm c}, \tag{2.20}$$

or by using the relation (2.4)

$$\begin{aligned}-i\Delta^{(1)}_F(x,0;\alpha_r(M^2)) &= i\int d^2z\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\mathcal{L}_{\rm int}(z)\Big)|0\rangle_{\rm c}\\ &= \frac{i\,\alpha_r(M^2)}{2}\int d^2z\langle 0|\mathrm{T}\Big(\vartheta(\mathrm{x})\,\vartheta(0)\,\vartheta^2(\mathrm{z})\Big)|0\rangle_{\rm c}\\ &+\frac{i\,\alpha_r(M^2)Z_1}{\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^n\beta^{2n}}{(2n)!}\langle 0|\mathrm{T}\Big(\vartheta(\mathrm{x})\,\vartheta(0)\,\vartheta^{2\mathrm{n}}(\mathrm{z})\Big)|0\rangle_{\rm c}. \end{aligned}\tag{2.21}$$

Now applying carefully Wick's theorem we turn the expectation values of the time–ordered products $\vartheta(x)\vartheta(0)\vartheta^2(z)$ and $\vartheta(x)\vartheta(0)\vartheta^{2n}(z)$ into an expression of two–point Green functions [10] [1]

$$\begin{aligned}-i\Delta^{(1)}_F(x,0;\alpha_r(M^2)) &= \left[1-Z_1\,\exp\left\{-\frac{1}{2}\,\beta^2\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]\\ &\times i\,\alpha_r(M^2)\int d^2z\,\{-i\Delta_F(x,z;\alpha_r(M^2)\}\,\{-i\Delta_F(z,0;\alpha_r(M^2))\}. \end{aligned}\tag{2.22}$$

Using Eq. (2.11) the first correction term reads

$$\begin{aligned}-i\Delta^{(1)}_F(x,0;\alpha_r(M^2)) &= \left[1-Z_1\,\left(\frac{\alpha_r(M^2)}{\Lambda^2}\right)^{\beta^2/8\pi}\right]\\ &\times i\,\alpha_r(M^2)\int d^2z\,\{-i\Delta_F(x,z;\alpha_r(M^2)\}\,\{-i\Delta_F(z,0;\alpha_r(M^2))\}. \end{aligned}\tag{2.23}$$

We get by adjusting the renormalisation constant $Z_1 = Z_1(\Lambda^2;\beta^2,M^2)$ to

$$Z_1(\Lambda^2;\beta^2,M^2) = \left(\frac{\Lambda^2}{M^2}\right)^{\beta^2/8\pi} \tag{2.24}$$

a well–defined result for the first–order correction $-i\Delta^{(1)}(x,0;\alpha_r(M^2))$

$$\begin{aligned}-i\Delta^{(1)}_F(x,0;\alpha_r(M^2)) &= i\,\alpha_r(M^2)\,\left[1-\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]\\ &\times\int d^2z\,\{-i\Delta_F(x,z;\alpha_r(M^2)\}\,\{-i\Delta_F(z,0;\alpha_r(M^2))\}. \end{aligned}\tag{2.25}$$

[1] The symmetry factor is given as: For the contraction of $\vartheta(x)$ and $\vartheta(0)$ with $\vartheta^2(z)$ there are $2n\,(2n-1)$ possibilities to do this. Two fields at the internal point z contracted to a loop give $(2n-2)(2n-3)/2! = (2n-2)!/(2n-4)!2!$ possibilities. The factor $1/2!$ is due to the fact that both participants of the loop are non distinguishable. Hence the total symmetry factor reads [10]

$$\frac{2n!}{2^{n-1}}\,\frac{1}{(n-1)!},$$

where the symmetry factor $1/(n-1)!$ is due to the $n-1$ non–distinguishable loops.

The renormalisation constant (2.24) corroborates [6].

Thus to first–order in the coupling the renormalised causal two–point Green function (2.16) reads

$$\begin{aligned}
-i\Delta(x,0;\alpha_r(M^2)) &= -i\Delta_F(x,0;\alpha_r(M^2)) - i\Delta_F^{(1)}(x,0;\alpha_r(M^2)) \\
&= -i\Delta_F(x,0;\alpha_r(M^2)) + i\alpha_r(M^2)\left[1-\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right] \\
&\times \int d^2z\,\{-i\Delta_F(x,z;\alpha_r(M^2)\}\,\{-i\Delta_F(z,0;\alpha_r(M^2))\}. \qquad (2.26)
\end{aligned}$$

In momentum space it has the form[2]

$$\begin{aligned}
-i\widetilde{\Delta}(p;\alpha_r(M^2)) &= \frac{-i}{\alpha_r(M^2)-p^2} \\
&+ i\,\alpha_r(M^2)\left[1-\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]\frac{-i}{\alpha_r(M^2)-p^2}\,\frac{-i}{\alpha_r(M^2)-p^2}. \qquad (2.27)
\end{aligned}$$

Definition of the physical coupling constant $\alpha_{\rm ph}$

The effective (or complete) two–point Green function containinig all possible insertions of self–energy $\delta\alpha_r(M^2)$ is represented in momentum space by [10]–[13]

$$\widetilde{\Delta}^{\rm eff}(p;\alpha_r(M^2)) = \frac{-i}{\alpha_r(M^2)+\delta\alpha_r(M^2)-p^2}. \qquad (2.28)$$

Its pole defines the physical mass

$$\alpha_{\rm ph} = \alpha_r(M^2)+\delta\alpha_r(M^2), \qquad (2.29)$$

hence

$$\widetilde{\Delta}^{\rm eff}(p;\alpha_r(M^2)) = \frac{-i}{\alpha_{\rm ph}-p^2}. \qquad (2.30)$$

Following the general procedure [10]–[13] we consider the expression in Eq. (2.27) as the approximation to the effective two–point Green function (2.28) to first–order in the self–energy $\delta\alpha_r(M^2)$. Thus [3]

$$\delta\alpha_r(M^2) = -\,\alpha_r(M^2)\left[1-\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]. \qquad (2.31)$$

[2]Feynman rules in momentum space; or simply, by taking the representation Eq. (2.10) and using the definition for the Dirac δ–function, the Fourier transformed (2.19) is obtained straight forward.

[3]For conventional normal ordering $\alpha_r(M^2) = M^2$ the two–point Green function remains unchanged

$$\widetilde{\Delta}^{\rm eff}(p;\alpha_r(M^2)) = \frac{-i}{\alpha_r(M^2)-p^2} = \frac{-i}{\alpha_{\rm ph}-p^2}.$$

The effective two–point Green function in momentum space reads

$$\begin{aligned}\widetilde{\Delta}^{\rm eff}(p;\alpha_r(M^2)) &= \frac{-i}{\alpha_r(M^2)-p^2}\\ &\times \frac{1}{1-\alpha_r(M^2)\left[1-\left(\dfrac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\right]\dfrac{1}{\alpha_r(M^2)-p^2}}\\ &= \frac{-i}{\alpha_r(M^2)\left(\dfrac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}-p^2}. \end{aligned} \tag{2.32}$$

This yields the physical mass (2.29)

$$\alpha_{\rm ph} \;=\; \alpha_r(M^2)\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\frac{\beta^2}{8\pi}}. \tag{2.33}$$

The relation between the renormalised coupling constant $\alpha_r(M^2)$ and the physical coupling constant $\alpha_{\rm ph}$ reads

$$\alpha_r(M^2) \;=\; \alpha_{\rm ph}\Big(\frac{M^2}{\alpha_{\rm ph}}\Big)^{\tilde{\beta}^2/8\pi} \quad , \quad \tilde{\beta}^2 \;=\; \frac{\beta^2}{1+\dfrac{\beta^2}{8\pi}}. \tag{2.34}$$

The effective two–point Green function (2.28, 2.32) is found to first order in $\alpha_{\rm ph}$ by the effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\rm eff}(x) &= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{\alpha_{\rm ph}}{2}\vartheta^2(x) + \frac{\alpha_{\rm ph}}{\beta^2}\sum_{n=2}^{\infty}\frac{(-1)^n\beta^{2n}}{(2n)!}\,\vartheta^{2n}(x)\\ &= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x)-1). \end{aligned} \tag{2.35}$$

For higher terms $\Delta_F^{(n\geq 2)}(x;\alpha_r(M^2))$ (2.12) we assert that they contribute in terms of the physical coupling constant $\alpha_{\rm ph}$ only

$$\begin{aligned}-i\Delta_F^{(n)}(x,0;\alpha_r(M^2)) &= \frac{i^n}{n!}\int\prod_{k=1}^{n}d^2z_k\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\rm int}(z_k))|0\rangle_{\rm c}\\ &= -i\Delta^{(n)}(x,0;\alpha_{\rm ph}) \quad (\text{for} \quad n\geq 2). \end{aligned} \tag{2.36}$$

We proof this assertion to second order in $\alpha_r(M^2)$.

2.3.2 Two–point Green function to second–order in $\alpha_r(M^2)$ and to all orders in β^2

The second–order correction $-i\Delta_F^{(2)}(x,0;\alpha_r(M^2))$ to the causal two–point Green function (2.15) reads

$$-i\Delta_F^{(2)}(x,0;\alpha_r(M^2)) \;=\; \frac{i^2}{2!}\iint d^2z_1d^2z_2\,\langle 0|\mathrm{T}(\vartheta(x)\vartheta(0)\mathcal{L}_{\rm int}(z_1)\mathcal{L}_{\rm int}(z_2))|0\rangle_c.$$

(2.37)

Since we have already taken terms proportional to $\alpha_r^2(M^2)$ into account when we calculate the effective two point Green function in Eq. (2.32) we have to subtract them from the expression (2.37). If we denote in momentum space these terms by $-i\tilde{\Delta}^{\mathrm{eff}(2)}(p;\alpha_r(M^2))$ they read

$$\begin{aligned}
-i\tilde{\Delta}^{\mathrm{eff}(2)}(p;\alpha_r(M^2)) &= \frac{-i}{\alpha_r(M^2)-p^2}\\
&\times\; i^2\alpha_r^2(M^2)\left[1-Z_1\,\exp\left\{-\frac{1}{2}\,\beta^2\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]^2\\
&\times\; \frac{-i}{\alpha_r(M^2)-p^2}\,\frac{-i}{\alpha_r(M^2)-p^2},
\end{aligned}\tag{2.38}$$

while in real space they are identified as ([10]–[13])

$$\begin{aligned}
-i\Delta^{\mathrm{eff}(2)}(x,0;\alpha_r(M^2)) &= \iint d^2z_1 d^2z_2\,\{-i\Delta_F(x,z_2;\alpha_r(M^2))\}\\
&\times\; i^2\alpha_r^2(M^2)\left[1-Z_1\,\exp\left\{-\frac{1}{2}\,\beta^2\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]^2\\
&\times\; \{-i\Delta_F(z_2,z_1;\alpha_r(M^2)\}\,\{-i\Delta_F(z_1,0;\alpha_r(M^2))\}.
\end{aligned}\tag{2.39}$$

Examining all contractions in Eq. (2.37) and subtracting from that the contributions in Eq. (2.39), thus $-i\Delta_F^{(2)}(x,0;\alpha_r(M^2))+i\Delta^{\mathrm{eff}(2)}(x,0;\alpha_r(M^2))$, we get [4]

$$\begin{aligned}
-i\Delta_F^{(2)}(x,0;\alpha_r(M^2)) &= \left[\alpha_r(M^2)Z_1\,\exp\left\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]^2\\
&\times\frac{1}{\beta^2}\iint d^2z_1 d^2z_2\,\{-i\Delta_F(x,z_1;\alpha_r(M^2))\}\,\{-i\Delta_F(z_1,0;\alpha_r(M^2))\}\\
&\times\Big(\cosh\{-\,\beta^2\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}-1-\frac{1}{2}\,\beta^4\,\{-\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}^2\Big)\\
&-\left[\alpha_r(M^2)Z_1\,\exp\left\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]^2\\
&\times\frac{1}{\beta^2}\iint d^2z_1 d^2z_2\,\{-i\Delta_F(x,z_1;\alpha_r(M^2))\}\,\{-i\Delta_F(z_2,0;\alpha_r(M^2))\}\\
&\times\Big(\sinh\{-\,\beta^2\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}-\beta^2\,\{-\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}\Big).
\end{aligned}\tag{2.40}$$

---

[4] We have identified in Eq. (2.40)

$$-i\Delta_F^{(2)}(x,0;\alpha_r(M^2))+i\Delta^{\mathrm{eff}(2)}(x,0;\alpha_r(M^2))\;\to\;-i\Delta_F^{(2)}(x,0;\alpha_r(M^2))$$

.

The calculations are given in Appendix B explicitly. We replace everywhere $\alpha_r(M^2)$ by $\alpha_{\rm ph}$ by means of the renormalisation constant Z_1 (2.4). This proves our assertion to second order in the coupling constant. The proof for higher orders runs equivalently.

2.4 Physical renormalisation of the sine–Gordon model

Thus, using the results obtained above we can formulate a procedure for the renormalisation of the SG model dealing with physical parameters only. Starting with the Lagrangian (2.3) and performing the renormalisation at the normalisation scale $M^2 = \alpha_{\rm ph}$ we deal with physical parameters only

$$\alpha_{\rm ph} \;=\; Z^{-1}_1(\Lambda^2;\beta^2,\alpha_{\rm ph})\,\alpha_0(\Lambda^2), \tag{2.41}$$

with the renormalisation constant $Z_1(\Lambda^2;\beta^2,\alpha_{\rm ph})$ being now equal to

$$Z_1(\Lambda^2;\beta^2,\alpha_{\rm ph}) \;=\; \Big(\frac{\Lambda^2}{\alpha_{\rm ph}}\Big)^{\beta^2/8\pi}. \tag{2.42}$$

The renormalised Lagrangian is defined by

$$\begin{aligned}\mathcal{L}(x) \;&=\; \frac{1}{2}\,(\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x) + \frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x) - 1)\\ &+\;\; (Z_1 - 1)\,\frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta(x) - 1).\end{aligned} \tag{2.43}$$

From the relation (2.34) at $M^2 = \alpha_{\rm ph}$ follows

$$\alpha_r(\alpha_{\rm ph}) \;=\; \alpha_{\rm ph}. \tag{2.44}$$

The first–order correction $-i\Delta^{(1)}(x,0;\alpha_r(\alpha_{\rm ph}))$, Eq. (2.25), vanishes. Non–trivial perturbative corrections appear only to second– and higher–orders in $\alpha_{\rm ph}$.

One can also show that the results obtained within the physical renormalisation of the SG model can be fully reproduced by using the normal–ordered Lagrangian

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,:\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x):+\frac{\alpha_{\rm ph}}{\beta^2}\;:(\cos\beta\vartheta(x) - 1):. \tag{2.45}$$

In this case all corrections to the two–point Green function are expressed in terms of $\alpha_{\rm ph}$ and finite [6, 2, 1].

2.5 Renormalisation Group approach to the sine–Gordon model

In this section we discuss the renormalisation group approach to the renormalisation of the SG model. We apply the Callan–Symanzik equation to the analysis of the Fourier transform of the two–point Green function of the SG field.

The Callan–Symanzik equation for the Fourier transform of the two–point Green function of the SG field $-i\,\tilde{\Delta}(p;\alpha_r(M^2),\beta^2)$ is equal to [14, 9]

$$\Big[-p\,\frac{\partial}{\partial p}+\hat{\beta}(\alpha_r(M^2),\beta^2)\,\frac{\partial}{\partial\alpha_r(M^2)}-2\Big]\,\tilde{\Delta}(p;\alpha_r(M^2),\beta^2)\ =\ =\ F(0,p;\alpha_r(M^2),\beta^2),\qquad(2.46)$$

where $\hat{\beta}(\alpha_r(M^2),\beta^2)$ denotes the Gell–Man–Low function [7, 9, 10]

$$M\,\frac{\partial\alpha_r(M^2)}{\partial M}\ =\ \hat{\beta}(\alpha_r(M^2),\beta^2).\qquad(2.47)$$

The term $\gamma(\alpha_r(M^2),\beta^2)$ [9], describing an anomalous dimension of the SG field, does not appear in the Callan–Symanzik equation due to the non–renormalisability of the SG quanta $\vartheta(x)$. The function $F(0,p;\alpha_r(M^2),\beta^2)$ in momentum representation is given by [14]

$$F(0,p;\alpha_r(M^2),\beta^2)\ =\ =\ \iint d^2x d^2y\,{\rm e}^{ipx}\,\langle 0|{\rm T}\Big(\hat{\Theta}^{\mu}_{\mu}(y)\vartheta(x)\vartheta(0)\,{\rm e}^{i\int d^2y\,{\cal L}_{\rm int}(y)}\Big)|0\rangle_{\rm c},\qquad(2.48)$$

where $\hat{\Theta}^{\mu}_{\mu}(x)$ denotes the trace over the energy–momentum tensor for the SG model, being equal to

$$\hat{\Theta}_{\mu\nu}(x)\ =\ \partial_{\mu}\vartheta(x)\partial_{\nu}\vartheta(x)-g_{\mu\nu}\,\Big[\frac{1}{2}\partial_{\lambda}\vartheta(x)\partial^{\lambda}\vartheta(x)+\frac{\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x)-1)\Big].\qquad(2.49)$$

The trace reads

$$\hat{\Theta}^{\mu}_{\mu}(x)\ =\ -\,\frac{2\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x)-1)\ =\ 2\,V[\vartheta(x)],\qquad(2.50)$$

where $V[\vartheta(x)]$ is the potential of the SG model. The interaction part ${\cal L}_{\rm int}(y)$ of the sine–Gordon Lagrangian in (2.3) reads

$${\cal L}_{\rm int}(y)\ =\ \frac{\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x)-1)\ =\ -\,\frac{1}{2}\,\hat{\Theta}^{\mu}_{\mu}(x).\qquad(2.51)$$

This allows us to represent the r.h.s of the Callan–Symanzik equation (2.46) in the form

$$F(0,p;\alpha_r(M^2),\beta^2)\ =$$

$$
\begin{aligned}
&= \iint d^2x d^2y\, \mathrm{e}^{ipx}\, \langle 0|\mathrm{T}\Big(2\,V[\vartheta(y)]\vartheta(x)\vartheta(0)\,\mathrm{e}^{i\int d^2y\,\mathcal{L}_{\rm int}(y)}\Big)|0\rangle_{\rm c}\\
&= 2\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)}\iint d^2x d^2y\, \mathrm{e}^{ipx}\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\,\mathrm{e}^{i\int d^2y\,\mathcal{L}_{\rm int}(y)}\Big)|0\rangle_{\rm c}\\
&= 2\,\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)}\,\tilde{\Delta}(p;\alpha_r(M^2),\beta^2). \qquad (2.52)
\end{aligned}
$$

Furthers, we take into account that the two–point Green function $\tilde{\Delta}(p;\alpha_r(M^2),\beta^2)$ is a Lorentz scalar hence should depend on p^2. We change the derivative in Eq. (2.46) as

$$
\partial/\partial p \;=\; 2p\,\partial/\partial p^2. \qquad (2.53)
$$

The Callan–Symanzik equation now reads

$$
\begin{aligned}
&\Big[p^2\frac{\partial}{\partial p^2} - \Big(\frac{1}{2}\,\hat{\beta}(\alpha_r(M^2),\beta^2) - \alpha_r(M^2)\Big)\,\frac{\partial}{\partial\alpha_r(M^2)} + 1\Big]\\
&\qquad\qquad\qquad\times\,\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) \;=\; 0. \qquad (2.54)
\end{aligned}
$$

For the solution of this linear differential equation we look first on the Gell–Mann–Low function (2.35) with respect to the relation we found in Eqs. (2.33) and (2.34) between the renormalised and physical coupling constant, they read

$$
\alpha_{\rm ph} \;=\; \alpha_r(M^2)\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\;,\;\; \alpha_r(M^2) \;=\alpha_{\rm ph}\Big(\frac{M^2}{\alpha_{\rm ph}}\Big)^{\tilde{\beta}^2/8\pi}, \qquad (2.55)
$$

where $\tilde{\beta}^2 = \beta^2/(1+\beta^2/8\pi)$.

The coupling constant $\alpha_r(M^2)$, given by (2.55), satisfy the renormgroup condition $\alpha_r(M^2_2) \;=\; Z_1(M^2_2,M^2_1)\alpha_r(M^2_1)$, which defines a relation between coupling constants $\alpha_r(M^2_1)$ and $\alpha_r(M^2_2)$ at the normalisation scales M_1 and M_2, respectively. The renormalisation constant $Z_1(M^2_2,M^2_1)$ is equal to $Z_1(M^2_2,M^2_1) = (M^2_2/M^2_1)^{\tilde{\beta}^2/8\pi}$.

Now applying the Gell–Mann–Low function on the renormalised coupling constant $\alpha_r(M^2)$ in terms of the renormalisation scale M independent physical coupling constant $\alpha_{\rm ph}$ we have

$$
\hat{\beta}(\alpha_r(M^2),\beta^2) \;=\; M\,\frac{\partial\alpha_r(M^2)}{\partial M} \;=\; \frac{\tilde{\beta}^2}{4\pi}\,\alpha_r(M^2), \qquad (2.56)
$$

Insertion of (2.56) into the Callan–Symanzik equation (2.54) yields

$$
\Big[p^2\frac{\partial}{\partial p^2} + \Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)} + 1\Big]\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) \;=\; 0. \qquad (2.57)
$$

We introduce the dimensionless function $D(p^2;\alpha_r(M^2)$ defined by the relation

$$
\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2)\,p^2 \;=\; D(p^2;\alpha_r(M^2),\beta^2), \qquad (2.58)
$$

hence depending on the dimensionless variables only

$$\tilde{p}^2 = p^2/M^2, \;\; \tilde{\alpha} = \alpha_r(M^2)/M^2. \tag{2.59}$$

Furthers, we introduce dimensionless derivatives in (2.54) by

$$p^2\,\partial/\partial p^2 \;=\; (p^2/M^2)\,(M^2\partial/\partial p^2) \;=\; \tilde{p}^2\,\partial/\partial\tilde{p}^2 \tag{2.60}$$

and

$$\alpha_r(M^2)\,\partial/\partial\alpha_r(M^2) \;=\; \tilde{\alpha}_r(M^2)\,\partial/\partial\tilde{\alpha}_r(M^2). \tag{2.61}$$

Hence, the Callan–Symanzik equation becomes

$$\Big[\tilde{p}^2\frac{\partial}{\partial\tilde{p}^2} + \Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\tilde{\alpha}_r(M^2)\,\frac{\partial}{\partial\tilde{\alpha}_r(M^2)}\Big]D(\tilde{p}^2;\tilde{\alpha}_r(M^2),\beta^2) \;=\; 0, \tag{2.62}$$

where we have differentiated with respect to $\tilde{p}^2$ and multiplied afterwards by $\tilde{p}^2$. The differential equation is an Euler differential equation of first order. The argument of the dimensionless function $D(\tilde{p}^2;\tilde{\alpha}_r(M^2),\beta^2)$ has to obey the characteristic differential equation [15]

$$\Big(1 - \frac{\tilde{\beta}^2}{8\pi}\Big)\frac{d\tilde{p}^2}{\tilde{p}^2} \;=\; \frac{d\tilde{\alpha}}{\tilde{\alpha}}, \tag{2.63}$$

and by direct integration we get the argument to be of the form

$$C = \frac{\tilde{\alpha}}{\tilde{p}^2}\,(\tilde{p}^2)^{\tilde{\beta}^2/8\pi}, \tag{2.64}$$

where C is an arbitrary integration constant. Therefore the two–point Green function in momentum space reads

$$\tilde{\Delta}(p^2;\alpha_r(M^2),\beta^2) \;=\; \frac{1}{p^2}\,D\Big[\frac{\alpha_r(M^2)}{p^2}\,\Big(\frac{p^2}{M^2}\Big)^{\tilde{\beta}^2/8\pi}\Big]. \tag{2.65}$$

Introducing the running coupling constant $\alpha_r(p^2)$ and using the relation given in Equation (2.55) the solution of the Callan–Symanzik equation for the two–point Green function in momentum space becomes

$$\tilde{\Delta}(p^2;\alpha_{\rm ph},\beta^2) = \frac{1}{p^2}\,D\Big[\frac{\alpha_r(p^2)}{p^2}\Big]. \tag{2.66}$$

This proves that the total renormalised two–point Green function of the SG field depends on the physical coupling constant $\alpha_{\rm ph}$ only.

Chapter 3

Renormalisation of the massive sine–Gordon Model

In this chapter, we analyse the renormalisation of the two–point Green function for the massive sine–Gordon model (MSG model). We show that the renormalisation procedure of the two–point Green function in the SG model, developed above, can be applied to the renormalisation of the two–point Green function in the MSG model (the MSG model). We show that the mass operator $m_0^2\vartheta^2(x)$ is soft and does not violate the renormalisability. In the infrared limit $m_0 \to 0$ the physical mass $m_{\rm ph}$ of the MSG model quanta reduces to our result in Eq. (2.33).

3.1 Introduction

The bare Lagrangian under consideration is given by [16]

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}\,m_0^2(\Lambda^2)\,\vartheta^2(x) + \frac{\alpha_0(\Lambda^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1), \tag{3.1}$$

where $\alpha_0(\Lambda^2)$ is the bare coupling constant and $m_0^2(\Lambda^2)$ denotes the bare squared mass parameter of the MSG quanta at an ultraviolet cut–off Λ^2. The renormalised Lagrangian at the renormalisation scale M reads

$$\begin{aligned}\mathcal{L}(x) &= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}\,m_r^2(M^2)\vartheta^2(x)\\ &\quad + \frac{\alpha_r(M^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1) - \frac{1}{2}\,m_r^2(M^2)\,(Z_m - 1)\,\vartheta^2(x)\\ &\quad + (Z_1 - 1)\,\frac{\alpha_r(M^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1)\\ &= \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{1}{2}\,Z_m m_r^2(M^2)\,\vartheta^2(x) + Z_1\,\frac{\alpha_r(M^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1),\end{aligned} \tag{3.2}$$

with $Z_1 = Z_1(\Lambda^2; \beta^2, M^2)$ and $Z_m = Z_m(\Lambda^2; \beta^2, M^2)$ being the renormalisation constants. They relate bare and renormalised parameters as

$$\begin{aligned} m_r^2(\Lambda^2) &= Z_m(\Lambda^2; \beta^2, M^2)\, m_0^2(\Lambda^2) \\ \alpha_r(\Lambda^2) &= Z_1(\Lambda^2; \beta^2, M^2)\, \alpha_0(\Lambda^2). \end{aligned} \tag{3.3}$$

Expanding the cosine in the Lagrangian (3.2) in polynomials of $\vartheta^2(x)$, we identify the renormalised Lagrangian as

$$\mathcal{L}(x) = \frac{1}{2}\left[\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \tilde{m}_r^2(M^2)\,\vartheta^2(x)\right] + \mathcal{L}_{\rm int}(x), \tag{3.4}$$

where $\tilde{m}_r^2(M^2) = m_r^2(M^2) + \alpha_r(M^2)$ is the effective mass of the quanta $\vartheta(x)$ of the MSG model. The two–point Green function generated by the free part in (3.4) reads

$$\begin{aligned} -i\Delta_F(x, y; \tilde{m}_r(M^2)) &= \langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(y)\Big)|0\rangle \\ &= \int \frac{d^2k}{(2\pi)^2}\, \frac{-i\, \mathrm{e}^{-ik\,(x-y)}}{\tilde{m}_r^2(M^2) - k^2 - i0}. \end{aligned} \tag{3.5}$$

At $x = y$ we have

$$-i\Delta_F(0; \Lambda^2; \tilde{m}_r(M^2)) = \frac{1}{4\pi}\, \ln\frac{\Lambda^2}{\tilde{m}_r^2(M^2)}. \tag{3.6}$$

$\mathcal{L}_{\rm int}(x)$ denotes the interaction part of the renormalised Lagrangian of the MSG model and is defined by

$$\begin{aligned} \mathcal{L}_{\rm int}(x) &= -\frac{1}{2}\, m_r^2(M^2)(Z_m - 1)\vartheta^2(x) + \alpha_r(M^2)\sum_{n=2}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x) \\ &\quad + (Z_1 - 1)\alpha_r(M^2)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x). \end{aligned} \tag{3.7}$$

The causal two–point Green function of the MSG field is is defined analogously to that of the SG field, Eq. (2.17)

$$\begin{aligned} -i\Delta_F(x, y; \tilde{m}_r(M^2)) &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int\prod_{k=1}^{n} d^2z_k \langle 0|\mathrm{T}(\vartheta(x)\vartheta(y)\mathcal{L}^n_{\rm int}(z_k))|0\rangle_{\rm c} \\ &= \sum_{n=0}^{\infty} -i\Delta_F^{(n)}(x, y; \tilde{m}_r(M^2)). \end{aligned} \tag{3.8}$$

As in the SG model, $-i\Delta_F^{(0)}(x, y; \tilde{m}_r(M^2))$ corresponds to the two–point Green function of free MSG quanta with mass $\tilde{m}_r(M^2)$, Eq. (3.5). The next term in Eq. (3.8), $-i\Delta_F^{(1)}(x, y; \tilde{m}_r(M^2))$ contributes, according to the ϕ^4 theory, to the self–energy of the MSG quanta.

Since the analysis of the renormalisability of the MSG model runs equivalent to that of the SG model, Chapter 2, we adopt the results of Sections 2.3.1 and 2.3.2 and make some replacements.

3.2 Renormalisation of the causal two–point Green function of the massive sine–Gordon model

The causal two–point Green function (3.8) to first order–correction reads in momentum space [1]

$$
\begin{aligned}
-i\widetilde{\Delta}^{(1)}_F(p;\tilde{m}_r(M^2)) &= \frac{-i}{\tilde{m}^2_r(M^2)-p^2} \\
&+ i\left(-\delta m^2_r(M^2)+\alpha_r(M^2)\left[1-\left(\tilde{m}^2_r(M^2)\Big/M^2\right)^{\beta^2/8\pi}\right]\right) \\
&\times \frac{-i}{\tilde{m}^2_r(M^2)-p^2}\,\frac{-i}{\tilde{m}^2_r(M^2)-p^2},
\end{aligned}
\tag{3.9}
$$

where $\delta m^2_r(M^2) = m^2_r(M^2)(Z_m-1)$.

The effective two–point Green function of the MSG in momentum space

$$
-i\widetilde{\Delta}^{\rm eff}(p;\tilde{m}_r(M^2)) = \frac{-i}{\tilde{m}^2_r(M^2)+\delta\tilde{m}^2_r(M^2)-p^2} = \frac{-i}{\tilde{m}^2_{\rm ph}-p^2}
\tag{3.10}
$$

reads

$$
\begin{aligned}
&-i\widetilde{\Delta}^{\rm eff}(p;\tilde{m}_r(M^2)) = \\
&\quad = -i\left[m^2_r(M^2)+\delta m^2_r(M^2)+\alpha_r(M^2)\left(\tilde{m}^2_r(M^2)\Big/M^2\right)^{\beta^2/8\pi}-p^2\right]^{-1}.
\end{aligned}
\tag{3.11}
$$

The pole of this effective two–point Green function defines the physical mass, $\tilde{m}^2_{\rm ph}$ to

$$
\tilde{m}^2_{\rm ph} = m^2_r(M^2)+\delta m^2_r(M^2)+\alpha_r(M^2)\left(\tilde{m}^2_r(M^2)\Big/M^2\right)^{\beta^2/8\pi}.
\tag{3.12}
$$

Since the pole in (3.11) does not contain divergences the counter–term $\delta m^2_r(M^2)$ can be adjusted to zero. Hence, the physical mass for the MSG model becomes

$$
\tilde{m}^2_{\rm ph} = m^2_r(M^2)+\alpha_r(M^2)\left(\frac{m^2_r+\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}.
\tag{3.13}
$$

In the soft–boson limit, when $m^2_r(M^2)\to 0$, the physical mass of the MSG model field coincides with $\alpha_{\rm ph}$ (2.33). This agrees with the assertion that the SG model is not infrared singular [6] and testifies that the operator $m^2_0\,\vartheta^2(x)$ is soft. This is in agreement with the results obtained by [16].

[1] The calculation of this term runs parallel to that in Eq. (2.21) of the SG model. We have only to replace the factor $\alpha_r(M^2)$ in the first term in Eq. (2.21) by $-\delta m^2_r(M^2)+\alpha_r(M^2)$ and to define the Green functions using Eqs. (3.5) and (3.6).

For finite $m_r^2(M^2)$ and in the perturbative regime $m_r^2(M^2) \gg \alpha_r(M^2)$ the physical mass of the MSG model field is equal to

$$m_{\rm ph}^2 \;=\; m_r^2(M^2) + \alpha_r(M^2)\,\Big(\frac{m_r^2(M^2)}{M^2}\Big)^{\beta^2/8\pi}, \tag{3.14}$$

where we have kept only the leading terms in $\alpha_r(M^2)$ expansion.

The second–order correction to the two–point Green function of the MSG model (3.8) runs completely equivalent to that of the SG model, done in Appendix B. Therefore we take the results of the SG model, found in Appendix B for the second–order contributions to the causal two–point Green function of the MSG model. The second–order contributions are summed up in Eq. (C.6), Appendix C.

We have shown that $(Z_m - 1)$ vanishes, by setting the counter–term $\delta m_r^2(M^2) = m_r^2(M^2)(Z_m - 1)$ in Eq. (C.5) zero. This implies that the mass parameter $m_0(\Lambda^2)$ is unrenormalisable, i.e. $m_0(\Lambda^2) = m_0$. In this case the physical mass of the MSG model field takes in the perturbative regime $m_r^2(M^2) \gg \alpha_r(M^2)$ the form

$$m_{\rm ph}^2 \;=\; m_0^2 + \alpha_r(M^2)\,\Big(\frac{m_0^2}{M^2}\Big)^{\beta^2/8\pi}. \tag{3.15}$$

Since the physical mass of the MSG model field cannot depend on the normalisation scale, we have to set

$$\alpha_{\rm ph} \;=\; \alpha_r(M^2)\,\Big(\frac{m_0^2}{M^2}\Big)^{\beta^2/8\pi} \quad\longrightarrow\quad \alpha_r(M^2) \;=\; \alpha_{\rm ph}\,\Big(\frac{M^2}{m_0^2}\Big)^{\beta^2/8\pi}. \tag{3.16}$$

It is seen that setting the normalisation scale $M = m_0$ the renormalised coupling constant $\alpha_r(m_0^2)$ coincides with the physical one, i.e. $\alpha_r(m_0^2) = \alpha_{\rm ph}$.

3.3 Callan–Symanzik equation approach

In this chapter we apply the renormalisation group analysis to the MSG model. The Callan–Symanzik equation in momentum space reads [14]

$$\Big[-p\,\frac{\partial}{\partial p} + \hat{\beta}(\alpha_r(M^2),\beta^2)\,\frac{\partial}{\partial \alpha_r(M^2)} - 2\Big]\widetilde{\Delta}(p;\alpha_r(M^2),\beta^2) \;=\; F(0,p;\alpha_r(M^2),\beta^2). \tag{3.17}$$

The function $F(0,p;\alpha_r(M^2),\beta^2)$ is given by

$$\begin{aligned} &F(0,p;\alpha_r(M^2),\beta^2)\\ &\quad= \iint d^2x d^2y\, {\rm e}^{ipx}\,\langle 0|{\rm T}\Big(\hat{\Theta}^{\mu}_{\mu}(y)\vartheta(x)\vartheta(0)\,{\rm e}^{i\int d^2y\,\mathcal{L}_{\rm int}(y)}\Big)|0\rangle_{\rm c}. \end{aligned} \tag{3.18}$$

The trace over the energy–momentum tensor $\Theta_{\mu\nu}$ for the MSG model becomes

$$\hat{\Theta}^{\mu}_{\mu}(x) \;=\; m_0^2\,\vartheta^2(x) \;-\; \frac{2\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x) - 1) \;=\; 2\,V[\vartheta(x)]. \tag{3.19}$$

The interaction part $\mathcal{L}_{\rm int}(y)$ of the massive sine–Gordon Lagrangian in (3.1) is

$$\mathcal{L}_{\rm int}(y) \;=\; -\,\frac{1}{2}m_0^2\,\vartheta^2(x) + \frac{\alpha_0}{\beta^2}\,(\cos\beta\vartheta(x) - 1) \;=\; -\,\frac{1}{2}\,\hat{\Theta}^{\mu}_{\mu}(x). \tag{3.20}$$

This gives

$$\begin{aligned} F(0,p;\alpha_r(M^2),\beta^2) \;&=\; \iint d^2x d^2y\, {\rm e}^{ipx}\,\langle 0|{\rm T}\Big(2\,V[\vartheta(y)]\vartheta(x)\vartheta(0)\,{\rm e}^{i\int d^2y\,\mathcal{L}_{\rm int}(y)}\Big)|0\rangle_{\rm c} \\ &= \Big(-\,m_0^2\frac{\partial}{\partial m_0^2} \;+\; 2\,\alpha_r(M^2)\,\frac{\partial}{\partial \alpha_r(M^2)}\Big)\,\widetilde{\Delta}(p;\alpha_r(M^2),\beta^2,m_0^2). \end{aligned} \tag{3.21}$$

The Callan–Symanzik equation (3.1) for the two–point Green function of the MSG model becomes

$$\begin{aligned} \Big[p^2\,\frac{\partial}{\partial p^2} - \Big(\hat{\beta}(\alpha_r(M^2),\beta^2) - 1\Big)\,\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)} \\ -\,\frac{m_0^2}{2}\,\frac{\partial}{\partial m_0^2} + 1\Big]\,\widetilde{\Delta}(p^2;\alpha_r(M^2),\beta^2,m_0^2) \;=\; 0, \end{aligned} \tag{3.22}$$

where we have divided by -2. Further, we have taken the Lorentz invariance of $\widetilde{\Delta}(p^2;\alpha_r(M^2),\beta^2,m_0^2)$ into account. The Gell–Mann–Low function $\hat{\beta}(\alpha_r(M^2),\beta^2)$ for the renormalised coupling constant $\alpha_r(M^2)$

$$\alpha_r(M^2) \;=\; \alpha_{\rm ph}\left(\frac{M^2}{m_0^2}\right)^{\beta^2/8\pi} \tag{3.23}$$

is equal to

$$\beta(\alpha_r(M^2),\beta^2) \;=\; M\,\frac{\partial\alpha_r(M^2)}{\partial M} \;=\; \frac{\beta^2}{4\pi}\,\alpha_r(M^2). \tag{3.24}$$

The Callan–Symanzik equation for the two point Green function of the MSG model reads

$$\begin{aligned} \Big[p^2\,\frac{\partial}{\partial p^2} - \delta(\beta^2)\alpha_r(M^2)\,\frac{\partial}{\partial\alpha_r(M^2)} - \frac{m_0^2}{2}\,\frac{\partial}{\partial m_0^2} + 1\Big] \\ \times\,\widetilde{\Delta}(p^2;\alpha_r(M^2),\beta^2,m_0^2) \;=\; 0, \end{aligned} \tag{3.25}$$

where we have denoted [16]

$$\delta(\beta^2) \;=\; (\beta^2 - 8\pi)/8\pi. \tag{3.26}$$

We introduce the dimensionless function $D(p^2;\alpha_r(M^2),\beta^2,m_0^2)$ via

$$\widetilde{\Delta}(p;\alpha_r(M^2);\beta^2,m_0^2) \;=\; D(p^2;\alpha_r(M^2),\beta^2,m_0^2)/p^2. \tag{3.27}$$

Eq. (3.25) can be simplified by introducing the dimensionless variables

$$t \;=\; \frac{p^2 m_0^2}{M^4}, \quad \tilde{\alpha} \;=\; \alpha_r(M^2)/M^2, \quad \tilde{m}_0^2 \;=\; m_0^2/M^2. \tag{3.28}$$

This gives for the two–point function

$$\widetilde{\Delta}(p;\alpha_r(M^2);\beta^2,m_0^2) \;=\; D(t;\tilde{\alpha})\frac{1}{t}\frac{m_0^2}{M^4}. \tag{3.29}$$

The derivatives becomes

$$p^2\,\partial/\partial p^2 \;=\; t\,\partial/\partial t, \quad m_0^2\,\partial/\partial m_0^2 \;=\; t\,\partial/\partial t. \tag{3.30}$$

Eq. (3.25) reads now

$$\Big[t\frac{\partial}{\partial t} - \delta(\beta^2)\tilde{\alpha}_r(M^2)\,\frac{\partial}{\partial\tilde{\alpha}} - \frac{1}{2}\,t\frac{\partial}{\partial t} - \frac{1}{2}\,\frac{\partial}{\partial m_0^2} + 1\Big]\,D(t;\tilde{\alpha}_r)\frac{1}{t}\frac{m_0^2}{M^4} \;=\; 0. \tag{3.31}$$

The final form of the Callan–Symanzik equation is

$$\Big[t\,\frac{\partial}{\partial t} - 2\delta(\beta^2)\,\tilde{\alpha}\frac{\partial}{\partial\tilde{\alpha}}\Big]\,D(t;\widetilde{\alpha}) \;=\; 0. \tag{3.32}$$

This agrees with the renormalisation group equation obtained by [16].

In order to calculate the Gell–Mann–Low function $2\delta(\beta^2)\widetilde{\alpha}$ to first order in $\widetilde{\alpha}$ and to all orders in β^2 we proceed as in the massless case. The characteristic differential equation [15] for the argument in the dimensionless function $D(t;\tilde{\alpha}_r)$ is given by

$$\frac{dt}{t} \;=\; \frac{d\widetilde{\alpha}}{-2\,\delta(\beta^2)\,\widetilde{\alpha}}. \tag{3.33}$$

The solution is obtained by direct integrations

$$C \;=\; \widetilde{\alpha}t^{2\delta(\beta^2)}, \tag{3.34}$$

where C is an arbitrary integration constant. The two–point Green function in momentum space has a momentum dependence of the form

$$\widetilde{\Delta}(p^2;\alpha_r(M^2),\beta^2,m_0^2) \;=\; \frac{1}{p^2}\,D\Big[\frac{\alpha_{\rm ph}}{m_0^2}\,\Big(\frac{M^2}{m_0^2}\Big)^{\delta(\beta^2)}\,\Big(\frac{p^2 m_0^2}{M^4}\Big)^{2\delta(\beta^2)}\Big]. \tag{3.35}$$

Renormalisation at $M^2 = m_0^2$ gives

$$\widetilde{\Delta}(p^2;\alpha_r(M^2),\beta^2,m_0^2) \;=\; \frac{1}{p^2}\,D\Big[\frac{\alpha_r(p^2)}{m_0^2}\Big], \tag{3.36}$$

where we have introduced the running coupling constant

$$\alpha_r(p^2) \;=\; \alpha_{\rm ph}\,\left(\frac{p^2}{m^2_0}\right)^{2\delta(\beta^2)}. \tag{3.37}$$

For $\delta(\beta^2) < 0$, i.e. $\beta^2 < 8\pi$, the MSG model with quantum fluctuations around a trivial vacuum, calculated to first order in $\alpha_r(M^2)$ and to all order in β^2, is an asymptotically free theory for $p^2 \to \infty$. In turn for $\delta(\beta^2) > 0$, i.e. $\beta^2 > 8\pi$, the running coupling constant $\alpha_r(p^2)$ grows with p^2. Of course, due to a perturbative derivation of the Gell–Mann–Low function (3.24) and the Callan–Symanzik equation (3.25), the running coupling constant $\alpha_r(p^2)$ cannot grow to infinity. The allowed region for momenta p^2 is restricted by the inequality $\alpha_r(p^2) \ll m^2_0$. This gives

$$p^2 \ll m^2_0\Big(\frac{m^2_0}{\alpha_{\rm ph}}\Big)^{1/2\delta(\beta^2)}. \tag{3.38}$$

Thus, we have shown that our results on the renormalisation of the MSG model, carried out for the two–point Green function, agree well with those obtained by [16].

Moreover the mass term of the MSG model is defined by real m_0, which is unrenormalisable. A non–perturbative proof of this assertion we give in the section below.

3.4 Non–perturbative renormalisation of the massive sine–Gordon model

In this section we show that the unrenormalisability of the scale m_0 in the MSG model can be proved non–perturbativly to all order of dimensional α_0 and dimensionless β coupling constants within the path–integral approach.

The generating functional $Z_m[J]$ of Green functions in the MSG model is

$$\begin{aligned} Z_m[J] \;=\; \int {\cal D}\vartheta\;\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_{\mu}\vartheta(x)\,\partial^{\mu}\vartheta(x) - \,\frac{m^2_0}{2}\,\vartheta(x)^2 + \,\vartheta(x)J(x)\\ + \,\frac{\alpha_0}{\beta^2}\,\Big[\cos\beta\,\vartheta(x)\,-1\Big]\Big\}, \end{aligned} \tag{3.39}$$

where $J(x)$ is a source for the SG field $\vartheta(x)$ [6]. Then, we propose to transcribe the generating funcitonal (3.39) into the form

$$Z_m[J] \;=\; \exp\Big\{-\,\frac{i}{2}\,m^2_0\int d^2x\,\frac{\delta}{i\delta J(x)}\frac{\delta}{i\delta J(x)}\Big\}\,Z_0[J], \tag{3.40}$$

where $Z_0[J]$ defines the generating functional of Green functions in the SG model [6].

For the renormalisation of $Z_0[J]$ follow [6] and expand in powers of α_0 keeping the terms of all orders, we get [2]

$$
\begin{aligned}
Z_0[J] \;=\; & \int \mathcal{D}\vartheta \sum_{n=0}^{\infty}\sum_{p=0}^{n} \frac{i^n}{n!}\left(\frac{\alpha_0}{\beta^2}\right)^n \frac{1}{2^n}\,\frac{n!}{(n-p)!\,p!} \\
& \times \left[\prod_{k=0}^{p}\prod_{l=0}^{n-p}\int d^2x_k\, d^2y_l\right] \exp\Big\{i\,\beta\sum_{k=1}^{p}\vartheta(x_k) - i\beta\sum_{l=1}^{n-p}\vartheta(y_l)\Big\} \\
& \times \exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \vartheta(x)J(x)\Big\}, \qquad (3.41)
\end{aligned}
$$

As has been shown in Section 2 perturbatively and in [6] non–perturbatively by means of the exact calculation of the path integral, the r.h.s. of Eq.(3.41) depends on the renormalised coupling $\alpha_r(M^2)$ only. This proves that the scale m_0 is unrenormalisable.

The calculation of the path integral in Eq. (3.41) are performed in Appendix C.2. The renormalised generating funcitonal of Green functions $Z_m[J]$ of the MSG model is then defined by

$$
\begin{aligned}
Z_m[J] \;=\; & \sum_{n=0}^{\infty}\left(\frac{i}{n!}\,\frac{\alpha_r(M^2)}{2\beta^2}\right)^{2n}\left[\prod_{k=0}^{n}\int d^2x_k\, d^2y_k\right] \\
& \times \exp\Big\{\frac{1}{8\pi}\int d^2x\, d^2y\, J(x)\ln\big(-M^2(x-y)^2 + i0\big)\, J(y) \\
& + \frac{\beta}{4\pi}\int d^2y\sum_{k=1}^{n}\ln\left(\frac{(x_k-y)^2+i0}{(y_k-y)^2+i0}\right) J(y) \\
& + \frac{\beta^2}{4\pi}\sum_{j<k}^{n}\Big(\ln\big(-M^2(x_k-x_j)^2+i0\big) + \ln\big(-M^2(y_k-y_j)^2+i0\big)\Big) \\
& - \frac{\beta^2}{4\pi}\sum_{k=1}^{n}\sum_{l=1}^{n}\ln\big(-M^2(x_k-y_k)^2+i0\big)\Big\} \\
& \times \exp\Big\{+\frac{i}{2}\int d^2x d^2y\,\Big[\frac{m_0}{4\pi}\int d^2z_1\ln\big(-M^2(x-z_1)^2+i0\big)\, J(z_1) \\
& + \frac{m_0}{4\pi}\beta\sum_{k}^{p}\ln\left(\frac{(x_k-x)^2+i0}{(y_k-x)^2+i0}\right)\Big] \\
& \times \Big[\delta(x-y) - m_0^2\Delta(x-y;m_0)\Big] \\
& \times \Big[\frac{m_0}{4\pi}\int d^2z_2\ln\big(-M^2(y-z_2)^2+i0\big)\, J(z_2)
\end{aligned}
$$

[2]
$$\prod_{i=0}^{n}\int d^2x_i \;=\; \int d^2x_1\ldots d^2x_n.$$

$$+\frac{m_0}{4\pi}\beta \ln\left(\frac{(x_k-y)^2+i0}{(y_k-y)^2+i0}\right)\Big]. \tag{3.42}$$

One can see that the limit $m_0 \to 0$ provides no divergences. This confirms our results obatined in Section 3.

Chapter 4

Renormalisation of the sine–Gordon model, caused by quantum fluctuations around a soliton

In this chapter we analyse the renormalisability of Gaussian fluctuations around a soliton. We show that Gaussian fluctuations around a soliton solution are renormalised like quantum fluctuations around the trivial vacuum and do not introduce any singularity to the sine-Gordon model at $\beta^2 = 8\pi$. We calculate the correction to the soliton mass, caused by Gaussian fluctuations around a soliton.

4.1 Introduction

Following [17, 18, 19] (see also [20]) we treat quantum fluctuations of the SG field $\vartheta(x)$ around a soliton solution $\vartheta_s(x)$, Eq. (1.12), by expanding the bare Lagrangian

$$\mathcal{L}(x) \;=\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \frac{\alpha_0(\Lambda^2)}{\beta^2}\,(\cos\beta\vartheta(x) - 1). \tag{4.1}$$

around the classical solution (1.13) to order $\vartheta^2(x)$. This gives

$$\begin{aligned}\mathcal{L}[\vartheta_{\rm s} + \vartheta](x) \;&=\; \mathcal{L}[\vartheta_s(x)]\\ &\quad + \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{\alpha_0(\Lambda^2)}{2}\,\vartheta^2(x)\cos\beta\vartheta_s(x).\end{aligned} \tag{4.2}$$

The partition function for quantum fluctuations around a soliton with respect to quantum fuctuations around the vacuum (1.9) reads

$$Z \;=\; \exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta_{\rm s}(x)]\Big\}$$

$$
\begin{aligned}
&\times \int \mathcal{D}\vartheta \; \exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) \\
&\qquad + \frac{\alpha_0}{\beta^2}\,\cos\beta\vartheta_{\rm s}(x)\cos\beta\vartheta(x) - \frac{\alpha_0}{\beta^2}\,\sin\beta\vartheta_{\rm s}(x)\,\sin\beta\vartheta(x)\Big)\Big\}\Big/ \\
&\times \int \mathcal{D}\vartheta' \; \exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\,\partial_\mu\vartheta'(x)\partial^\mu\vartheta'(x) + \frac{\alpha_0}{\beta^2}\,\cos\beta\vartheta'(x)\Big\}. \qquad (4.3)
\end{aligned}
$$

The partition function for Gaussian fluctuations becomes

$$
\begin{aligned}
Z &= \exp\Big\{i\int d^2x\,\mathcal{L}[\vartheta_{\rm s}(x)]\Big\} \\
&\times \int \mathcal{D}\vartheta\;\exp\Big\{-\frac{i}{2}\int d^2x\,\vartheta(x)\,\Big[\Box + \alpha_0 - \frac{2\alpha_0}{\cosh^2\sqrt{\alpha_0}x}\Big]\,\vartheta(x)\Big\} \\
&\times \int \mathcal{D}\vartheta'\;\exp\Big\{+\frac{i}{2}\int d^2x\,\vartheta'(x)\,\Big[\Box\,+\,\alpha_0\Big]\,\vartheta'(x)\Big\}. \qquad (4.4)
\end{aligned}
$$

In order to ensure the convergence of the path integral we understand the parameter α_0 to have an infinitesimal imaginary part $\alpha_0 - i0$. The partition function reads formally

$$
\begin{aligned}
Z &= \exp\Big\{i\int d^2x\;\mathcal{L}[\vartheta_{\rm s}(x)] + \delta\mathcal{L}_{\rm eff}(x)\Big\} \\
&= \exp\Big\{i\int d^2x\,\mathcal{L}_{\rm eff}(x)\Big\}, \qquad (4.5)
\end{aligned}
$$

where we have introduced the effective Lagrangians $\mathcal{L}_{\rm eff}(x)$ and $\delta\mathcal{L}_{\rm eff}(x)$, the effective correction to the classical part $\mathcal{L}[\vartheta_{\rm s}(x)]$, Eq. (4.2).

Introducing the notation

$$
\delta Z \;=\; \exp\Big\{i\int d^2x\,\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm s}(x)]\Big\} \qquad (4.6)
$$

for the part of the partition function describing quantum fluctuations, then

$$
\begin{aligned}
\delta Z &= \sqrt{\frac{{\rm Det}(\Box + \alpha_0)}{{\rm Det}\Big(\Box + \alpha_0 - \dfrac{2\alpha_0}{\cosh^2(\sqrt{\alpha_0}x)}\Big)}} \\
&= \prod_{\omega,k}\frac{1}{\sqrt{\alpha_0 - \omega^2 + k^2}}\;\prod_{\omega',k'}\sqrt{\alpha_0 - \omega'^2 + k'^2}. \qquad (4.7)
\end{aligned}
$$

Taking both operators $\Box + \alpha_0$ and $\Box + \alpha_0 - 2\alpha_0{\rm sech}^2(\sqrt{\alpha_0}x^1)$, discussed in Section 1.1.1 and in the Appendix A.1, in their eigen–representation we transfer the evaluation of the functional determinants to the more convenient form

$$
\begin{aligned}
\delta Z &= \exp\Big\{-\frac{1}{2}\int d^2x\int d\omega dk\,|\vartheta_k(x)|^2\,\ln(\alpha_0 - \omega^2 + k^2) \\
&\quad - \frac{1}{2}\int d^2x\int d\omega\,|\vartheta_b(x)|^2\,\ln(-\omega^2) \\
&\quad + \frac{1}{2}\int d^2x\int d\omega' dk'\,|\vartheta_{k'}(x)|^2\,\ln(\alpha_0 - \omega'^2 + k'^2)\,\Big\}. \qquad (4.8)
\end{aligned}
$$

Inserting the eigenvalue relations $\lambda(\omega',k') = \alpha_0 - \omega'^2 + k'^2$ (1.25) and $\lambda(\omega) = -\,\omega^2$, $\lambda(\omega,k) = \alpha_0 - \omega^2 + k^2$ (1.28) and the corresponding eigenfunctions Eqs. (1.24) and (1.27) we arrive at

$$\begin{aligned}\delta Z \;&=\; \exp\Big\{-\frac{1}{2}\int d^2x\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\,\frac{k^2+\alpha_0\tanh^2(\sqrt{\alpha_0}x)}{k^2+\alpha_0}\,\ln(\alpha_0-\omega^2+k^2)\\ &\quad -\frac{1}{2}\int d^2x\int\frac{d\omega}{2\pi}\,\frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}\,\frac{\alpha_0}{\cosh^2(\sqrt{\alpha_0}x)}\,\ln(-\omega^2)\\ &\quad +\frac{1}{2}\int d^2x\int\frac{d\omega'}{2\pi}\,\frac{dk'}{2\pi}\,\ln(\alpha_0-\omega'^2+k'^2)\Big\},\end{aligned} \tag{4.9}$$

where we have formulated all integrals in a covariant form by inserting

$$1 \;=\; 2\sqrt{\alpha_0}\int\frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}. \tag{4.10}$$

Using further the identities

$$\frac{1}{\cosh^2(\sqrt{\alpha_0}x)} \;=\; \frac{1}{2}\,(1-\cos\beta\vartheta_{\rm s}(x)) = \frac{\beta^2}{2\alpha_0}\,V[\vartheta_{\rm s}(x)] \tag{4.11}$$

and

$$\begin{aligned}\frac{k^2+\alpha_0\tanh^2(\sqrt{\alpha_0}x)}{k^2+\alpha_0} \;&=\; 1-\alpha_0\frac{1-\tanh^2(\sqrt{\alpha_0}x)}{k^2+\alpha_0}\\ &=\; 1-\frac{\alpha_0}{k^2+\alpha_0}\,\frac{1}{\cosh^2(\sqrt{\alpha_0}x)}\end{aligned} \tag{4.12}$$

we transcribe the partition function into the form

$$\begin{aligned}\delta Z \;&=\; \exp\Big\{-\frac{1}{4}\int d^2x\,\beta^2 V[\vartheta_{\rm s}(x)]\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{1}{k^2+\alpha_0}\Big(\ln(-\omega^2)\\ &\quad -\ln(\alpha_0-\omega^2+k^2)\Big)-\frac{1}{2}\int d^2x\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\,\ln(\alpha_0-\omega^2+k^2)\\ &\quad +\frac{1}{2}\int d^2x\int\frac{d\omega'}{2\pi}\,\frac{dk'}{2\pi}\,\ln(\alpha_0-\omega'^2+k'^2)\Big\}.\end{aligned} \tag{4.13}$$

By the exact cancellation of the last two terms the partition function arrives at

$$\begin{aligned}\delta Z \;&=\; \exp\Big\{-\frac{1}{4}\int d^2x\,\beta^2 V[\vartheta_{\rm s}(x)]\int\frac{d\omega}{2\pi}\frac{dk}{2\pi}\,\frac{1}{k^2+\alpha_0}\\ &\qquad\times\,\big(\ln(-\omega^2)-\,\ln(\alpha_0-\omega^2+k^2)\big)\Big\}.\end{aligned} \tag{4.14}$$

Now, integrating by parts over ω yields

$$\delta Z \;=\; \exp\Big\{\frac{1}{2}\int d^2x\beta^2 V[\vartheta_{\rm cl}(x)]\int\frac{dk}{2\pi}\int\frac{d\omega}{2\pi}\frac{1}{\alpha_0-\omega^2+k^2}\Big\}. \tag{4.15}$$

For our further discussion we pass to the effective Lagrangian $\delta\mathcal{L}_{\rm eff}$ by the definition (4.6), it reads

$$\begin{aligned}\delta\mathcal{L}_{\rm eff}(x) &= \frac{1}{2}\,\beta^2 V[\vartheta_{\rm s}(x)]\int\frac{dk}{2\pi}\frac{d\omega}{2\pi i}\frac{1}{\alpha_0-\omega^2+k^2}\\ &= \frac{1}{2}\,\beta^2 V[\vartheta_{\rm s}(x)]\,\{-i\Delta_F(0;\alpha_0)\}.\end{aligned}\tag{4.16}$$

In order to integrate our last expression in a covariant way we pass to Euclidean momentum space by making a Wick rotation $\omega\to i\,\omega$

$$\begin{aligned}\delta\mathcal{L}_{\rm eff}(x) &= \frac{1}{2}\,\beta^2 V[\vartheta_{\rm s}(x)]\int_0^\Lambda\frac{dp}{4\pi}\frac{2\,p}{\alpha_0+p^2}\\ &= \frac{1}{2}\,\beta^2 V[\vartheta_{\rm s}(x)]\,\frac{1}{4\pi}\ln\Big(\frac{\Lambda^2}{\alpha_0}\Big),\end{aligned}\tag{4.17}$$

where Λ is a momentum cut–off in two–dimensional Euclidean space.

The total Lagrangian, accounting for Gaussian fluctuations around the soliton solution amounts to

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta_{\rm s}(x)\partial^\mu\vartheta_{\rm s}(x)+\frac{\alpha_0}{\beta^2}\Big[1-\frac{\beta^2}{8\pi}\;\ln\Big(\frac{\Lambda^2}{\alpha_0}\Big)\Big]\,(\cos\beta\vartheta_{\rm s}(x)-1).\tag{4.18}$$

For the renormalisation of the divergent factor depending on Λ

$$\begin{aligned}\alpha_r(M^2)Z_1\Big[1-\frac{\beta^2}{8\pi}\;\ln\Big(\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\Big)\Big] &= \alpha_r(M^2)Z_1\Big(\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\Big)^{-\beta^2/8\pi}\\ &= \alpha_r(M^2)Z_1^{1+\beta^2/8\pi}\Big(\frac{\alpha_r(M^2)}{\Lambda^2}\Big)^{\beta^2/8\pi}.\end{aligned}\tag{4.19}$$

we identify to leading order $\mathcal{O}(\beta^2)$ the renormalisation constant Z_1 by

$$Z_1 = \Big(\frac{\Lambda^2}{M^2}\Big)^{\beta^2/8\pi}.\tag{4.20}$$

This confirms our result found in Eq. (2.24). This result coincides with that in [6], appearing there to leading order of Gaussian fluctuations. The renormalised effective Lagrangian reads

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta_{\rm s}(x)\partial^\mu\vartheta_{\rm s}(x)+\frac{\alpha_r(M^2)}{\beta^2}\;\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}(\cos\beta\vartheta_{\rm s}(x)-1).\tag{4.21}$$

By introducing the physical coupling constant $\alpha_{\rm ph}$

$$\alpha_{\rm ph} = \alpha_r(M^2)\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/8\pi}\tag{4.22}$$

the effective Lagrangian becomes

$$\mathcal{L}_{\rm eff}(x) = \frac{1}{2}\partial_\mu\vartheta_{\rm s}(x)\partial^\mu\vartheta_{\rm s}(x)+\frac{\alpha_{\rm ph}}{\beta^2}\,(\cos\beta\vartheta_{\rm s}(x)-1).\tag{4.23}$$

This confirms our result given in Eq. (2.35).

4.2 The soliton mass in terms of the physical coupling

In this section we give the renormalised mass of a soliton solution. We will calculate the contributions caused by quantum fluctuations fully in a Lorentz covariant way.

The soliton mass corrected by Gaussian quantum fluctuations reads

$$M_s \;=\; \int dx\,\Big(\mathcal{L}[\vartheta_{\rm s}] - \delta\mathcal{L}_{\rm eff}[\vartheta_{\rm s}]\Big) \;=\; \frac{8\sqrt{\alpha_0}}{\beta^2} + \Delta M_s. \tag{4.24}$$

The mass correction is given by

$$\Delta M_s \;\;=\;\; -\int dx\,\delta\mathcal{L}_{\rm eff}[\vartheta_{\rm s}(x)]. \tag{4.25}$$

Taking Eq. (4.16) and the definition of $V[\vartheta_{\rm s}(x)]$ (4.11) we get

$$\begin{aligned}\Delta M_s \;&=\; -\frac{1}{2}\,\beta^2\int dx\,V[\vartheta_{\rm s}(x)]\int\frac{dk}{2\pi}\frac{d\omega}{2\pi i}\frac{1}{\alpha_0-\omega^2+k^2}\\ &=\; -2\sqrt{\alpha_0}\int\frac{dk}{2\pi}\frac{d\omega}{2\pi i}\frac{1}{\alpha_0-\omega^2+k^2}.\end{aligned} \tag{4.26}$$

Now, an integration over ω yields for the mass correction to the soliton mass due Gaussian fluctuations the result

$$\Delta M_s \;=\; \int_{-\infty}^{+\infty}\frac{dk}{4\pi}\,\sqrt{\alpha_0+k^2}\,\frac{d\delta(k)}{dk}, \tag{4.27}$$

where $\delta(k)$ is the phase–shift (1.31). Hence, the correction to the soliton mass does not contain a surface term, $-\sqrt{\alpha_0}/\pi$ [1, 18, 19, 21].

Calculating the mass correction (4.26) in a Lorentz covariant way we get with Eq. (4.17) for the soliton mass

$$M_s = \frac{8\sqrt{\alpha_0(\Lambda^2)}}{\beta^2} - \frac{\sqrt{\alpha_0(\Lambda^2)}}{2\pi}\,\ln\left(\frac{\Lambda^2}{\alpha_0(\Lambda^2)}\right). \tag{4.28}$$

The renormalised soliton mass is

$$\begin{aligned}M_s \;&=\; \frac{8\sqrt{\alpha_r(M^2)Z_1}}{\beta^2} - \frac{\sqrt{\alpha_r(M^2)\,Z_1}}{2\pi}\,\ln\left(\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\right)\\ &=\; \frac{8\sqrt{\alpha_r(M^2)}}{\beta^2}\,Z_1^{1/2}\,\left(1-\frac{\beta^2}{16\pi}\,\ln\left(\frac{\Lambda^2}{\alpha_r(M^2)Z_1}\right)\right).\end{aligned} \tag{4.29}$$

It reads to leading order $\mathcal{O}(\beta^2)$

$$M_s \;\;=\;\; \frac{8\sqrt{\alpha_r(M^2)}}{\beta^2}\,Z_1^{1/2}\,\left(\frac{\alpha_r(M^2)}{\Lambda^2}\right)^{\beta^2/16\pi}. \tag{4.30}$$

Substituting (4.20) into (4.30), we have

$$M_s \;=\; \frac{8\sqrt{\alpha_r(M^2)}}{\beta^2}\,\left(\frac{\alpha_r(M^2)}{M^2}\right)^{\beta^2/16\pi} \tag{4.31}$$

and finally in terms of the physical coupling constant (4.22) the soliton mass

$$M_s \;=\; \frac{8\sqrt{\alpha_{\rm ph}}}{\beta^2}. \tag{4.32}$$

The mass of a soliton M_s depends on the physical coupling constant $\alpha_{\rm ph}$. Hence, the contribution of Gaussian fluctuations around a soliton solution is absorbed by the renormalised coupling constant $\alpha_{\rm ph}$ and no singularities of the sine–Gordon model appear at $\beta^2 = 8\pi$.

This result confirms the assertion by Zamolodchikov *et al.* [22], that the singularity of the SG model induced by the finite correction $-\sqrt{\alpha_{\rm ph}}/\pi$ to the soliton mass, caused by Gaussian fluctuations around a soliton solution, is completely due to the regularisation and renormalisation procedure. This has been corroborated in [6].

We have obtained that the soliton mass M_s does not depend on the normalisation scale M. This testifies that the soliton mass M_s is an observable quantity.

4.2.1 Dominance of Gaussian quantum fluctuations around a soliton. Is this a strong or a weak coupling interaction?

Contributions of quantum fluctuations around a soliton are calculated under the assumption of the dominant role of Gaussian fluctuations. In this section we analyse the criteria for the validity of the dominance of Gaussian fluctuations. The Lagrangian, describing the fluctuations around a soliton solution $\vartheta_{\rm s}(x)$ (1.12), is

$$\begin{aligned}\mathcal{L}(x) \;&=\; \frac{1}{2}\,\partial_\mu\vartheta_{\rm s}(x)\partial^\mu\vartheta_{\rm s}(x) + \frac{\alpha}{\beta^2}\,(\cos\beta\vartheta_{\rm s}(x) - 1)\\ &+\; \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) - \frac{\alpha}{\beta^2}\,\cos\beta\vartheta_{\rm s}(x)\,(1-\cos\beta\vartheta(x))\\ &+\; \frac{\alpha}{\beta}\,\sin\beta\vartheta_{\rm s}(x)\,(\beta\vartheta(x) - \sin\beta\vartheta(x)),\end{aligned} \tag{4.33}$$

where we have used the equation of motion (1.8) for $\vartheta_{\rm s}(x)$, and $\vartheta(x)$ is a field fluctuating around a soliton $\vartheta_{\rm s}(x)$.

Expanding in powers of φ and keeping the terms of order of $\mathcal{O}(\vartheta^4(x))$ inclusively we get

$$\begin{aligned}\mathcal{L}(x) \;&=\; \mathcal{L}[\vartheta_{\rm s}(x)] - \frac{1}{2}\,\vartheta(x)(\Box + \alpha\,\cos\beta\vartheta_{\rm s}(x))\,\vartheta(x)\\ &+\; \frac{1}{6}\,\alpha\,\beta^2\,\sin\beta\vartheta_{\rm s}(x)\,\vartheta^3(x) + \frac{1}{24}\,\alpha\,\beta^2\,\cos\beta\vartheta_{\rm s}(x)\,\vartheta^4(x) + \ldots,\end{aligned} \tag{4.34}$$

where $\mathcal{L}[\vartheta_{\rm s}(x)]$ is defined by the first two terms in Eq. (4.33). We have also omitted a contribution of a total divergence proportional to $\partial_\mu(\vartheta(x)\partial^\mu\vartheta(x)) \sim \Box\vartheta^2(x)$.

The dominance of Gaussian fluctuations is determined by the inequalities [23]

$$\begin{aligned}|(\Box\vartheta(x) + \alpha\,\cos\beta\vartheta_{\rm s}(x))\vartheta(x)| &\gg \frac{1}{3}\,\alpha\,\beta^2\,|\sin\beta\vartheta_{\rm s}(x)\,\vartheta^2(x)|,\\ |(\Box\vartheta(x) + \alpha\,\cos\beta\vartheta_{\rm s}(x))\vartheta(x)| &\gg \frac{1}{12}\,\alpha\,\beta^2\,|\cos\beta\vartheta_{\rm s}(x)\,\vartheta^3(x)|,\end{aligned} \tag{4.35}$$

where $\vartheta(x)$ are the eigenfunctions of the operator $(\Box + \alpha\,\cos\beta\vartheta_{\rm s}(x))$ given by Eq.(1.27) with eigenvalues $\lambda(\omega,k) = -\omega^2$ and $\lambda(\omega,k) = \alpha - \omega^2 + k^2$ for the bound and scattering states, respectively. These inequalities mean that the contribution of the quadratic terms, calculated for the eigenfunctions of the operator $(\Box + \alpha\,\cos\beta\vartheta_{\rm s}(x))$, should be much greater than the contributions of the terms of the third and fourth powers of the ϑ–field.

Using the eigenvalues of the operator $(\Box + \alpha\,\cos\beta\vartheta_{\rm s}(x))$ the inequalities (4.35) can be transcribed into the form

$$\begin{aligned}&\omega^2 \gg \frac{1}{3}\,\alpha\,\beta^2\,|\sin\beta\vartheta_{\rm s}(x)\,\vartheta(x)|,\\ &|\alpha - \omega^2 + k^2| \gg \frac{1}{3}\,\alpha\,\beta^2\,|\sin\beta\vartheta_{\rm s}(x)\,\vartheta(x)|,\\ &\omega^2 \gg \frac{1}{12}\,\alpha\,\beta^2\,|\cos\beta\vartheta_{\rm s}(x)\,\vartheta^2(x)|,\\ &|\alpha - \omega^2 + k^2| \gg \frac{1}{12}\,\alpha\,\beta^2\,|\cos\beta\vartheta_{\rm s}(x)\,\vartheta^2(x)|.\end{aligned} \tag{4.36}$$

It is obvious that in the whole region of variation of the eigenvalues, these inequalities can be fulfilled for $\beta = 0$ only.

This means that the calculation of the contribution of Gaussian fluctuations to the soliton mass is to full extent perturbative and related to the weak coupling limit. One can conclude that the Gaussian approximation for the calculation of quantum fluctuations around a soliton is not valid in the non–perturbative regions of the parameter β. Therefore it is not possible to extend the results, obtained by means of Gaussian fluctuations, for the coupling constant $\beta \sim 1$ and $\beta \gg 1$. All results, obtained in the Gaussian approximation must be understood perturbatively for $\beta \ll 1$ only.

Chapter 5

Comparison to the Korepin–Faddeev approach

5.1 Introduction

We calculate in this chapter the quantum contribution to Gaussian order by following the approach given in the manuscript of Faddeev and Korepin [19]. By using this alternative approach we rederive our former result (4.27). We show that the finite term $-m/\pi$ appears due to a different regularisation procedure. The procedure in [19] for the calculation of quantum contributions to Gaussian order

$$\begin{aligned} 1\Big/\sqrt{\det HH_0^{-1}} &= \exp\Big\{-1/2\,\mathrm{tr}\ln HH_0^{-1}\Big\} \\ &= \exp\Big\{-\,i\,\Delta M(t''-t')\Big\} \end{aligned} \tag{5.1}$$

is based on the determination of the derivative

$$\frac{d}{d\varphi}\mathrm{tr}\ln HH_0^{-1} = \int d^2x\,\Big\{H^{-1}\frac{d}{d\varphi}H \,-\, H_0^{-1}\frac{d}{d\varphi}H_0\Big\}, \tag{5.2}$$

acting on a parameter φ connecting two arbitrary Lorentz frames. This derivative simplifies the logarithmic expression to expressions depending on the Green functions $R = H^{-1}$ and $R_0 = H_0^{-1}$ of the underlying operators $H_0 = \Box + m^2$ and $H = \Box + m^2 - 2m^2\mathrm{sech}^2 mx$. Hence,

$$\frac{d}{d\varphi}\mathrm{tr}\ln HH_0^{-1} = \int d^2x\Big\{R\,\frac{d}{d\varphi}H \,-\, R_0\,\frac{d}{d\varphi}H_0\Big\}. \tag{5.3}$$

We formulate the Green functions R_0 and R for the homogeneous differential equations $H_0\psi = 0$ and $H\psi = 0$ for Feynman boundary conditions. The contribution ΔM to the soliton mass generated by quantum fluctuations can be obtained finally, after some calculations by integrating over φ and we get

$$2i\Delta M(t''-t') = \mathrm{tr}\ln HH_0^{-1}. \tag{5.4}$$

5.1.1 Solutions and Green functions to the Gaussian operators H_0 and H

The differential operators under considerations are

$$H_0 \;=\; \frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + m^2 \tag{5.5}$$

and

$$H \;=\; \frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + m^2 - \frac{2\,m^2}{\cosh^2 m x^1}. \tag{5.6}$$

The eigenvalue equation for H_0 is given in Eq. (1.23) with $m^2 = \alpha$ and the corresponding off–shell dispersion relation in Eq. (1.25), $\lambda(\omega, k) = -\omega^2 + k^2 + m^2$. The eigenvalue equation for H is given in Eq. (1.26) with the off–shell dispersion relation in (1.28).

Solutions for $H_0\psi = 0$: The solutions to the equation $\mathbf{H_0}\psi_{\mathbf{0k}} = \mathbf{0}$ for quantum fluctuations in the trivial sector we obtain by setting the eigenvalue equation (1.23)

$$\left[\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + m^2\right]\psi_{0k}(x) \;=\; \lambda(\omega, k)\,\psi_{0k}(x), \tag{5.7}$$

on–shell

$$\lambda(\omega, k) \;=\; -\,\omega^2 + k^2 + m^2 \;=\; 0. \tag{5.8}$$

with the solutions for ω

$$\omega \;=\; \pm\,\omega_k, \tag{5.9}$$

where

$$\omega_k \;=\; \sqrt{k^2 + m^2}. \tag{5.10}$$

We use Feynman boundary conditions assuming that the mass m contains an infinitesimal imaginary part $-i\epsilon$

$$m^2 \to m^2 - i\epsilon \tag{5.11}$$

and

$$\omega \;=\; \pm\sqrt{k^2 + m^2} \mp i\epsilon \;=\; \pm\,\omega_k \mp i\epsilon. \tag{5.12}$$

The solutions $\psi_{0k}(x)$ for positive frequencies we denote by $\psi^+_{0k}(x)$ and those for negative frequencies by $\psi^-_{0k}(x)$. They behave as

$$\begin{aligned}
\psi^-_{0k}(x) \;&=\; f_{0k}(x^1)\,\mathrm{e}^{-i\omega_k x^0}\,\mathrm{e}^{-\,\epsilon\,x^0}, \qquad \lim_{x^0 \to +\infty} \psi^-_{0k}(x) \;=\; 0 \\
\psi^+_{0k}(x) \;&=\; g_{0k}(x^1)\,\mathrm{e}^{+i\omega_k x^0}\,\mathrm{e}^{+\,\epsilon\,x^0}, \qquad \lim_{x^0 \to -\infty} \psi^+_{0k}(x) \;=\; 0.
\end{aligned} \tag{5.13}$$

The Jost functions $f_{0k}(x^1)$ and $g_{0k}(x^1)$ belong to the eigenvalue k satisfying the equations [19]

$$-\frac{\partial^2}{\partial {x^1}^2}\, f_{0k}(x^1) \;=\; k^2\, f_{0k}(x^1), \quad -\frac{\partial^2}{\partial {x^1}^2}\, g_{0k}(x^1) \;=\; k^2\, g_{0k}(x^1). \tag{5.14}$$

They read

$$f_{0k}(x^1) \;=\; \mathrm{e}^{ikx^1}, \qquad g_{0k}(x^1) \;=\; \mathrm{e}^{-ikx^1}. \tag{5.15}$$

In our further calculations we drop the explicit notation (5.11) and understand the mass m to contain implicitly the additional infinitesimal imaginary contribution $-i\epsilon$.

5.1.2 Green function R_0 of the operator H_0:

Using the solutions (5.13) we construct the Green function R_0 of H_0 satisfying Feynman boundary conditions as

$$R_0(x_2|x_1) \;=\; \begin{cases} \displaystyle\int \frac{dk}{2\pi}\, \psi^-_{0k}(x_2)\, \psi^+_{0k}(x_1)/W_{0k}, & x^0_2 > x^0_1 \\[2ex] \displaystyle\int \frac{dk}{2\pi}\, \psi^+_{0k}(x_2)\, \psi^-_{0k}(x_1)/W_{0k}, & x^0_1 > x^0_2, \end{cases} \tag{5.16}$$

with the Wronskian

$$W_{0k} \;=\; -2i\omega_k. \tag{5.17}$$

The prove that R_0 is indeed the Green function of H_0, is given in Appendix D.1.

5.1.3 Solutions for $H\psi = 0$:

The eigenvalue equation for quantum fluctuations $\psi_k(x,t)$ around the SG soliton, Eq. (1.21), reads

$$\left[\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + m^2 - \frac{2\,m^2}{\cosh^2 mx}\right] \psi_k(x) \;=\; \lambda(\omega,k)\, \psi_k(x), \tag{5.18}$$

where $\lambda(\omega,k) = -\omega^2 + k^2 + m^2$. We set this eigenvalue equation on–shell $\lambda(\omega_k,k) = 0$. Using Feynman boundary conditions the solutions $\psi_k(x)$ of $H\psi_k = 0$ read for negative frequencies

$$\begin{aligned} \psi^-_k(x) \;&=\; f_k(x^1)\, \mathrm{e}^{-i\omega_k x^0} \\ &=\; \frac{k + im\tanh mx^1}{k - im}\, \mathrm{e}^{-i\omega_k x^0 + ikx^1}, \end{aligned} \tag{5.19}$$

where the Jost functions $f_k(x^1)$ behave as [19]

$$\lim_{x^1\to-\infty} f_k(x^1) \;=\; \mathrm{e}^{ikx^1}, \qquad \lim_{x^1\to+\infty} f_k(x^1) \;=\; a(k)\,\mathrm{e}^{ikx^1}. \tag{5.20}$$

The solutions $\psi_k(x)$ for positive frequencies read

$$\begin{aligned} \psi_k^+(x) \;&=\; g_k(x^1)\,\mathrm{e}^{+i\omega_k x^0} \\ &=\; \frac{k - im\tanh mx^1}{k-im}\,\mathrm{e}^{+i\omega_k x^0 \,-\, ikx^1}, \end{aligned} \tag{5.21}$$

with the asymptotic behavior of the Jost funtions $g_k(x^1)$

$$\lim_{x^1\to-\infty} g_k(x^1) \;=\; a(k)\,\mathrm{e}^{-i\,k\,x^1}, \qquad \lim_{x^1\to\,\infty} g_k(x^1) \;=\; \mathrm{e}^{-i\,k\,x^1}. \tag{5.22}$$

The (asymptotic) amplitude $a(k)$ is given by

$$a(k) \;=\; \frac{k+im}{k-im}. \tag{5.23}$$

The wave function for the *zero* mode $\omega_k = \lim_{\epsilon\to0}\,\epsilon = 0$ (see Eq. 1.22) reads

$$\psi_0^{\pm}(x) \;=\; \frac{1}{\sqrt{2\pi}}\frac{\sqrt{m\,\pi}}{\cosh mx^1}\;\mathrm{e}^{\pm\,\epsilon\,x^0}. \tag{5.24}$$

The spatial part of Eq. (5.18) fulfills the eigenvalue equation

$$\Big[-\frac{\partial^2}{\partial {x^1}^2} - \frac{2m^2}{\cosh^2 mx^1}\Big]\,\psi_k^{\pm}(x) \;=\; k^2\,\psi_k^{\pm}(x). \tag{5.25}$$

5.1.4 Green function R of the operator H :

Expressing the Green function R as a combination of negative and positive–frequency solutions ψ_k^+ and ψ_k^-, Eqs. (5.19) and (5.21), we have

$$R(x_2|x_1) \;=\; \begin{cases} \displaystyle\int \frac{dk}{2\pi}\psi_k^-(x_2)\;\psi_k^+(x_1)/W_k \;+\; \psi_0^-(x_2)\;\psi_0^+(x_1)/W_0, & x_2^0 > x_1^0 \\[2ex] \displaystyle\int \frac{dk}{2\pi}\psi_k^+(x_2)\;\psi_k^-(x_1)/W_k \;+\; \psi_0^+(x_2)\;\psi_0^-(x_1)/W_0, & x_1^0 > x_2^0, \end{cases} \tag{5.26}$$

with Wronskians

$$W_k \;=\; -2i\omega_k\,a(k), \qquad W_0 \;=\; -2\epsilon. \tag{5.27}$$

The prove that R is indeed the Green function to H is given in Appendix D.1.

5.2 The one–loop mass correction ΔM

In this section we derive the quantum correction ΔM to the soliton mass caused by Gaussian fluctuations (see Eq. (4.7)) by following the procedure used in [19].

The quantum correction by Gaussian fluctuations read

$$\begin{aligned}
\mathrm{Det}\sqrt{\frac{H_0}{H}} &= \sqrt{\frac{\mathrm{Det}(\Box + m^2)}{\mathrm{Det}\Big(\Box + m^2 - \dfrac{2m^2}{\cosh^2 mx^1}\Big)}} \\
&= \exp\Big\{ - 1/2\,\mathrm{tr}\ln HH_0^{-1}\Big\} \\
&= \exp\Big\{ -\ i\,\Delta M(t'' - t')\Big\}.
\end{aligned} \tag{5.28}$$

Hence the mass contribution is given by

$$\Delta M \;=\; -i/2 \int dx^1 \ln HH_0^{-1}. \tag{5.29}$$

We follow [19] and differentiate the exponent with respect to the Lorentz parameter φ connecting two arbitrary Lorentz frames (x^0, x^1) and (y^0, y^1) (see the definition in Eq. (F.4))

$$\begin{aligned}
\frac{d}{d\varphi}\,\mathrm{tr}\ln H\,H_0^{-1} \;=& \int d^2x \Bigg\{ \lim_{x'\to x} \frac{d}{d\varphi}(\Box_x + v(x))\,H^{-1}(x'|x) \\
& \qquad - \lim_{x'\to x} \frac{d}{d\varphi}(\Box_x + v_0(x))\,H_0^{-1}(x'|x)\Bigg\},
\end{aligned} \tag{5.30}$$

where the potentials are given by $v_0 = m^2$ and $v = m^2 - 2m^2\mathrm{sech}^2 mx^1$. Eq. (5.30) is derived in Appendix D.2. In terms of the Green functions $R(x|x) = H^{-1}(x|x)$ and $R_0(x,|x) = H_0^{-1}(x,|x)$, Eqs. (5.16) and (5.26), we have

$$\begin{aligned}
d_\varphi \ln\det HH_0^{-1} \;=& \int_{t'}^{t''}\!\!\int d^2x\ \lim_{x'\to x} d_\varphi H(x)\,R(x'|x) \\
&- \int_{t'}^{t''}\!\!\int d^2x\ \lim_{x'\to x} d_\varphi H_0(x)\,R_0(x'|x) \\
=& \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,\psi_k^-(x)\,d_\varphi H\psi_k^+(x)/W_k \\
&+ \int_{t'}^{t''}\!\!\int d^2x\,\psi_0^-(x)\,d_\varphi H\psi_0^+(x)/W_0 \\
&- \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,\psi_{0k}^-(x)\,d_\varphi H_0\psi_{0k}^+(x)/W_{0k},
\end{aligned} \tag{5.31}$$

where we have abbreviated the derivative $d/d\varphi\cdot$ with $d_\varphi\cdot$ Further we simplify the notation of $\psi(x)$ as $\psi(x) = \psi$. We rewrite (5.31) with $H\psi_k = 0,\ H\psi_0 = 0$ and

$H_0\psi_{0k} = 0$ by using the differentials with respect to φ

$$
\begin{aligned}
H\psi_k^+ &= \left(\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + v\right)\psi_k^+, \quad d_\varphi H\psi_k^+ = -\, H d_\varphi \psi_k^+ \\
H\psi_0^+ &= \left(\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + v\right)\psi_0^+, \quad d_\varphi H\psi_0^+ = -\, H d_\varphi \psi_0^+
\end{aligned}
\tag{5.32}
$$

for the soliton sector and

$$
H_0\psi_{0k}^+ = \left(\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + v_0\right)\psi_{0k}^+, \quad d_\varphi H_0\,\psi_{0k}^+ = -\, H_0\, d_\varphi \psi_{0k}^+, \tag{5.33}
$$

for the trivial sector. This gives

$$
\begin{aligned}
d_\varphi \ln\det HH_0^{-1} &= -\int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\,\psi_k^-\; H d_\varphi \psi_k^+ \\
&\quad - \int_{t'}^{t''}\!\!\int d^2x\; W_0^{-1}\,\psi_0^-\; H d_\varphi\psi_0^+ \;+\; \int_{t'}^{t''}\!\!\int dt\,dx\,\frac{dk}{2\pi}\,W_{0k}^{-1}\,\psi_{0k}^-\,H_0 d_\varphi \psi_{0k}^+ \\
&= -\int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\,\psi_k^-\,\Big(\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + v\Big) d_\varphi\psi_k^+ \\
&\quad - \int_{t'}^{t''}\!\!\int d^2x\; W_0^{-1}\,\psi_0^-\,\Big(\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + v\Big) d_\varphi\psi_0^+ \\
&\quad + \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\,\psi_{0k}^-\,\Big(\frac{\partial^2}{\partial x^{0\,2}} - \frac{\partial^2}{\partial x^{1\,2}} + v_0\Big) d_\varphi\psi_{0k}^+ .
\end{aligned}
\tag{5.34}
$$

Integration by parts over x^0 and x^0 gives in Appendix D.2.2

$$
\begin{aligned}
d_\varphi \ln\det HH_0^{-1} &= -\int dx^1\,\frac{dk}{2\pi}\,\Big\{W_k^{-1}\psi_k^-\,\overleftrightarrow{\partial}_{x^0}\,d_\varphi\psi_k^+ \;-\; W_{0k}^{-1}\psi_{0k}^-\,\overleftrightarrow{\partial}_{x^0}\,d_\varphi\psi_{0k}^+\Big\}\Big|_{t'}^{t''} \\
&\quad - \int dx^1\,\Big\{W_0^{-1}\psi_0^-\,\overleftrightarrow{\partial}_{x^0}\,d_\varphi\psi_0^+\Big\}\Big|_{t'}^{t''},
\end{aligned}
\tag{5.35}
$$

where $(a\,\overleftrightarrow{\partial}_{x^0} b) = a\,(\partial_{x^0} b) - (\partial_{x^0} a) b$. In Appendix D.2.3 we evaluate the d_φ and get in Eq. (D.34)

$$
\ln\det HH_0^{-1} = \frac{1}{2\pi}\frac{(t''-t')}{\cosh\varphi}\int dx^1\,dk\; i\omega_k\left\{\frac{1}{a(k)}\psi_k^-\,\psi_k^+ \;-\; \psi_{0k}^-\,\psi_{0k}^+\right\}. \tag{5.36}
$$

If we introduce the spectral density $\rho(k)$, Eq. (5.36) reads

$$
\ln\det HH_0^{-1} = \frac{(t''-t')}{\cosh\varphi}\int dk\; i\sqrt{k^2+m^2}\,\rho(k). \tag{5.37}
$$

In β parametrisation [19], with $k = m\sinh\beta$ and $dk = m\cosh\beta\,d\beta = \omega_k\,d\beta = \sqrt{k^2+m^2}\,d\beta$, it becomes

$$
\ln\det HH_0^{-1} = \frac{(t''-t')}{\cosh\varphi}\int d\beta\; im\cosh\beta\,\rho(\beta), \tag{5.38}
$$

where the spectral density $\rho(\beta)$ fulfills

$$\int d\beta\,\rho(\beta) \;=\; -\frac{1}{\pi}\int d\beta\frac{1}{\cosh\beta} \;=\; -1. \tag{5.39}$$

Eq. (5.38) reads

$$\ln\det HH_0^{-1} \;=\; -\frac{im}{\pi}\frac{(t''-t')}{\cosh\varphi}\int d\beta. \tag{5.40}$$

Therefore, using Eq. (5.29)

$$\Delta M(t''-t') = 1/2i\;\ln\det HH_0^{-1} \tag{5.41}$$

the quantum contribution to the soliton mass reads in β parametrisation

$$\Delta M \;=\; -\frac{m}{2\pi}\int d\beta. \tag{5.42}$$

In k parametrisation we get for the spectral density

$$\begin{aligned}
\rho(k) \;&=\; \frac{1}{2\pi}\int dx^1\,\left\{\frac{1}{a(k)}\,\psi_k^-\,\psi_k^+ \;-\;\psi_{0k}^-\,\psi_{0k}^+\right\} \;= \\
&=\; \frac{1}{2\pi}\int dx^1\,\left\{\frac{k-im}{k+im}\frac{k+im\tanh mx^1}{k-im}\frac{k-im\tanh mx^1}{k-im}\;-\;1\right\} \\
&=\; \frac{1}{2\pi}\int dx^1\,\left\{\frac{k^2+m^2\tanh^2 mx^1}{k^2+m^2}\;-\;1\right\} \\
&=\; \frac{1}{2\pi}\int dx^1\,\frac{m^2}{k^2+m^2}\,\frac{-1}{\cosh^2 mx^1}\;=\;-\frac{1}{2\pi}\frac{m^2}{k^2+m^2}\,\frac{2}{m} \\
&=\; -\frac{1}{\pi}\frac{m}{k^2+m^2}.
\end{aligned} \tag{5.43}$$

Integration over the spectral parameter k gives

$$\begin{aligned}
\int dk\,\rho(k) \;&=\; -\frac{1}{\pi}\int_{-\infty}^{+\infty} dk\,\frac{m}{k^2+m^2} \\
&=\; -\frac{1}{\pi}\;\mathrm{arctan}\frac{k}{m}\,\Big|_{-\infty}^{+\infty} \;=\; -1.
\end{aligned} \tag{5.44}$$

This confirms relation (1.33). Hence, the continuum spectrum in the soliton sector is shifted down by one mode with respect to the continuum spectrum of the vacuum sector.

In k parametrisation we obtain the mass correction

$$\Delta M \;=\; -\frac{m}{2\pi}\int dk\,\frac{1}{\sqrt{k^2+m^2}}, \tag{5.45}$$

where we have used Eqs. (5.37) and (5.41). This can be rewritten to

$$\begin{aligned}\Delta M &= \int \frac{dk}{2\pi} \sqrt{k^2+m^2} \, \frac{-m}{k^2+m^2} \\ &= \int \frac{dk}{2\pi} \sqrt{k^2+m^2} \, \frac{d}{dk} \arctan \frac{m}{k}, \end{aligned} \tag{5.46}$$

and with the definition of the phase shift (1.31), the quantum correction ΔM (5.46) reads

$$\Delta M = \frac{1}{2} \int \frac{dk}{2\pi} \sqrt{k^2+m^2} \, \frac{d\delta(k)}{dk}. \tag{5.47}$$

This result for the mass correction of the SG soliton due to Gaussian fluctuations confirms our result found in Eq. (4.27). In [19] Eq. (5.47) is modified by changing from the energy momentum cut–off to mode number cut–off. This change gives the additional finite term $-\sqrt{\alpha}/\pi$ to (5.47).

In Chapter 4 we have performed our calculation in a Lorentz covariant form in continuous space–time within the mode number cut–off regularisation. In [24] we confirm within the mode number cut–off regularisation our continuous space–time approach.

Chapter 6

On the non–renormalisability of the sine–Gordon model with respect to quantum fluctuations around non–trivial classical solutions

In this chapter we discuss the renormalisability of the SG model, caused by quantum fluctuations around non–trivial solutions $\vartheta_{\rm cl}(x)$. We perform the discussion to all orders of dimensional and dimensionless coupling constants within the path–integral approach. We show that the generating functional of Green function of quantum fluctuations around non–trivial solutions $\vartheta_{\rm cl}(x)$ depends on the ultra–violet cut–off. This implies a non–renormalisability of the quantum field theory of quantum fluctuations around non–trivial solutions $\vartheta_{\rm cl}(x)$.

Below we perform the discussion on the renormalisabilty of the SG model caused by quantum fluctuations around soliton–antisoliton and soliton–soliton solutions $\vartheta_{\rm cl}(x)$ (1.15).

The generating functional of fluctuations $\vartheta(x)$ around soliton–antisoliton and soliton–soliton solutions $\vartheta_{\rm cl}(x)$ reads

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\vartheta\,\exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\,\partial_{\mu}\vartheta_{\rm cl}(x)\partial^{\mu}\vartheta_{\rm cl}(x)\\
&+ \frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x) + \frac{\alpha_0}{\beta^2}\,(\cos\beta(\vartheta_{\rm cl}(x)+\vartheta(x)) - 1)\\
&+ \alpha_0\,\sin\beta\vartheta_{\rm cl}(x)\,\vartheta(x) + \vartheta(x)J(x)\Big]\Big\}, \qquad (6.1)
\end{aligned}
$$

where $J(x)$ is an external source of a fluctuating field $\vartheta(x)$. The external source obeys the constraint [6]

$$\int d^2x\, J(x) \;=\; 0. \tag{6.2}$$

The term proportional to $\vartheta(x)$ one can also unify with the external source defining a new external source $\tilde{J}(x)$

$$\widetilde{J}(x) = J(x) + \alpha_0\,\sin\beta\vartheta_{\rm cl}(x), \tag{6.3}$$

which does not violate the constraint (6.2) due to the anti–symmetry of the soliton–antisoliton and soliton–soliton solutions with respect to the transformations $x^1 \to -\,x^1$ and $x^0 \to -\,x^0$. Indeed, the soliton–antisoliton solution $\vartheta_{sa}(x^1,x^0)$ (see Eq.(1.15)) changes sign under time revearsal $x^0 \to -\,x^0$, whereas the soliton–soliton solution $\vartheta_{ss}(x^1,x^0)$ (see Eq.(1.15)) is anti–symmetric with respect to the parity transformation $x^1 \to -\,x^1$.

For the regularisation and renormalisation of the generating functional of Green functions we expand the r.h.s. of Eq.(6.1) in powers of $\cos\beta\,(\vartheta_{\rm cl}(x) + \vartheta(x))$. This yields

$$\begin{aligned}
Z[\tilde{J}] \;=\;& \exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_{\mu}\vartheta_{\rm cl}(x)\partial^{\mu}\vartheta_{\rm cl}(x)\Big\}\\
&\int\mathcal{D}\vartheta\,\sum^{\infty}_{n=0}\frac{i^n}{n!}\,\Big(\frac{\alpha_0}{\beta^2}\Big)^n\prod^{n}_{i=0}\int d^2x_i\,\cos\beta\,(\vartheta_{\rm cl}(x_i) + \vartheta(x_i))\\
&\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x) - \frac{1}{2}\,\mu^2\,\vartheta^2(x) + \vartheta(x)\widetilde{J}(x)\Big\},
\end{aligned} \tag{6.4}$$

where μ is an infrared cut–off, which should be taken in the limit $\mu \to 0$. The product $\prod^{n}_{i=0}\int d^2x_i\,\cos\beta\,(\vartheta_{\rm cl}(x_i) + \vartheta(x_i))$ means

$$\begin{aligned}
\prod^{n}_{i=0}\int d^2x_i\,\cos\beta\,\Big(\vartheta_{\rm cl}(x_i) + \vartheta(x_i)\Big) \;=\;& \int d^2x_1\ldots d^2x_n\\
\times\,\cos\beta\,(\vartheta_{\rm cl}(x_1) + \vartheta(x_1))&\ldots\cos\beta\,(\vartheta_{\rm cl}(x_n) + \vartheta(x_n)).
\end{aligned} \tag{6.5}$$

Using the exponential representation for the cosine–function, the r.h.s. of Eq.(6.4) can be transcribed into the form

$$\begin{aligned}
Z[\tilde{J}] \;=\;& \exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_{\mu}\vartheta_{\rm cl}(x)\partial^{\mu}\vartheta_{\rm cl}(x)\Big\}\\
\times&\int\mathcal{D}\vartheta\,\sum^{\infty}_{n=0}\sum^{n}_{p=0}\frac{i^n}{n!}\,\Big(\frac{\alpha_0}{\beta^2}\Big)^n\,\frac{1}{2^n}\,\frac{n!}{(n-p)!\,p!}\,\Bigg[\prod^{p}_{k=0}\prod^{n-p}_{l=0}\int d^2x_k\,d^2y_l\Bigg]\\
\times&\exp\Big\{i\,\beta\sum^{p}_{k=1}(\vartheta_{\rm cl}(x_k) + \vartheta(x_k)) - \sum^{n-p}_{l=1}\beta(\vartheta_{\rm cl}(y_l) + \vartheta(y_l))\Big\}\\
\times&\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x) - \frac{1}{2}\,\mu^2\,\vartheta^2(x) + \vartheta(x)\widetilde{J}(x)\Big\}.
\end{aligned} \tag{6.6}$$

The terms depending on the classical solution $\vartheta_{\rm cl}(x)$ can be written in front of the path integral. This gives

$$
\begin{aligned}
Z[\tilde J] \;&=\; \exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x)\Big\}\\
&\times\sum_{n=0}^{\infty}\sum_{p=0}^{n}\Big(\frac{\alpha_0}{\beta^2}\Big)^n\,\frac{1}{2^n}\,\frac{i^n}{(n-p)!\,p!}\,\Big[\prod_{k=0}^{p}\prod_{l=0}^{n-p}\int d^2x_k\,d^2y_l\Big]\\
&\exp\Big\{i\,\beta\Big(\sum_{k=1}^{p}\vartheta_{\rm cl}(x_k)\;-\;\sum_{l=1}^{n-p}\vartheta_{\rm cl}(y_l)\Big)\Big\}\,\int\mathcal{D}\vartheta\,\exp\Big\{i\,\beta\sum_{k=1}^{p}\vartheta(x_k)-i\beta\sum_{l=1}^{n-p}\vartheta(y_l)\Big\}\\
&\qquad\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x)-\frac{1}{2}\,\mu^2\,\vartheta^2(x)+\vartheta(x)\widetilde J(x)\Big\}.
\end{aligned}
\tag{6.7}
$$

Integrating over the fluctuating field $\vartheta(x)$ we obtain

$$
\begin{aligned}
Z[\tilde J] \;&=\; \exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x)\Big\}\\
&\times\sum_{n=0}^{\infty}\sum_{p=0}^{n}\Big(\frac{\alpha_0}{\beta^2}\Big)^n\,\frac{1}{2^n}\,\frac{i^n}{(n-p)!\,p!}\,\Big[\prod_{k=0}^{p}\prod_{l=0}^{n-p}\int d^2x_k\,d^2y_l\Big]\\
&\exp\Big\{i\,\beta\Big(\sum_{k=1}^{p}\vartheta_{\rm cl}(x_k)\;-\;\sum_{l=1}^{n-p}\vartheta_{\rm cl}(x_l)\Big)\Big\}\;\exp\Big\{\frac{i}{2}\int d^2x\,d^2y\,\widetilde J(x)\Delta(x-y;\mu)\widetilde J(y)\\
&+i\beta\int d^2y\Big(\sum_{k=1}^{p}\Delta(x_k-y;\mu)-\sum_{l=1}^{n-p}\Delta(y_l-y;\mu)\Big)\widetilde J(y)-\,i\beta^2\sum_{k=1}^{p}\sum_{l=1}^{n-p}\Delta(x_k-y_l;\mu)\\
&+i\frac{\beta^2}{2}\Big(\sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\Delta(x_{k_1}-x_{k_2};\mu)+\sum_{l_1=1}^{n-p}\sum_{l_2=1}^{n-p}\Delta(x_{l_1}-x_{l_2};\mu)\Big)\Big\}.
\end{aligned}
\tag{6.8}
$$

The causal two–point function is

$$
\Delta(x-y;\mu)\;=\;-\frac{i}{4\pi}\,\ln\big(-\mu^2(x-y)^2+i0\big)\,,\quad \Delta(0;\mu)\;=\;\frac{i}{4\pi}\,\ln\Big(\frac{\Lambda^2}{\mu^2}\Big)\,. \tag{6.9}
$$

The term in the exponent, proportional to $\tilde J$, can be rewritten as

$$
\begin{aligned}
&\exp\Big\{i\beta\int d^2y\Big(\sum_{k=1}^{p}\Delta(x_k-y;\mu)-\sum_{l=1}^{n-p}\Delta(y_l-y;\mu)\Big)\widetilde J(y)\Big\}\\
&=\;\exp\Big\{-\,\frac{i}{4\pi}\,i\beta\int d^2y\ln\Big((-\mu^2)^{2p-n}\prod_{k=1}^{p}\prod_{l=1}^{n-p}\frac{(x_k-y;\mu)^2+i0}{(y_l-y)^2+i0}\Big)\widetilde J(y)\Big\}\\
&=\;\exp\Big\{\frac{1}{4\pi}\,(2p-n)\,\ln(-\mu^2)\int d^2y\widetilde J(y)\\
&+\frac{1}{4\pi}\int d^2y\,\ln\Big(\prod_{k=1}^{p}\prod_{l=1}^{n-p}\frac{(x_k-y)^2+i0}{(y_l-y)^2+i0}\Big)\widetilde J(y)\Big\}.
\end{aligned}
\tag{6.10}
$$

It is seen that the dependence on the infrared cut–off vanishes due to the constraint (6.2). In the terms, which do not contain the external sources

$$\exp\Big\{-i\beta^2\sum_{k=1}^{p}\sum_{l=1}^{n-p}\Delta(x_k-y_l;\mu)$$
$$+\frac{1}{2}i\beta^2\Big(\sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\Delta(x_{k_1}-x_{k_2};\mu)+\sum_{l_1=1}^{n-p}\sum_{l_2=1}^{n-p}\Delta(x_{l_1}-x_{l_2};\mu)\Big)\Big\}, \tag{6.11}$$

the dependence on the infrared cut–off vanishes due to the relation

$$\begin{aligned}
&\frac{i}{4\pi}\,i\beta^2p(p-n)\,\ln(-\mu^2)-\frac{1}{2}\frac{i}{4\pi}\,i\beta^2p^2\,\ln(-\mu^2)-\frac{1}{2}\frac{i}{4\pi}\,i\beta^2(n-p)^2\,\ln(-\mu^2)\\
&=\;-\frac{\beta^2}{8\pi}\,\big(2p^2-2pn-p^2-n^2+2pn-p^2\big)\ln(-\mu^2)\\
&=\;-\frac{\beta^2}{8\pi}\,\big(-n^2+2pn\big)\ln(-\mu^2),
\end{aligned}\tag{6.12}$$

only for $n=2p$. This gives

$$\begin{aligned}
Z[\widetilde{J}]\;&=\;\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x)\Big\}\\
&\times\sum_{n=0}^{\infty}\Big(\frac{\alpha_0}{2\beta^2}\Big)^{2n}\,\Big(\frac{i}{n!}\Big)^2\,\Big[\prod_{k=0}^{n}\prod_{l=0}^{n}\int d^2x_k\,d^2y_l\Big]\\
&\times\;\exp\Big\{i\,\beta\Big(\sum_{k=1}^{n}\vartheta_{\rm cl}(x_k)\;-\;\sum_{l=1}^{n}\vartheta_{\rm cl}(y_l)\Big)\Big\}\\
&\times\;\exp\Big\{\frac{i}{2}\,\int d^2x\,d^2y\,\widetilde{J}(x)\Delta(x-y;\mu)\widetilde{J}(y)\\
&+i\beta\int d^2y\Big(\sum_{k=1}^{n}\Delta(x_k-y;\mu)-\sum_{l=1}^{n}\Delta(y_l-y;\mu)\Big)\widetilde{J}(y)\\
&+i\beta^2\Big(\sum_{l<k}^{n}\Delta(x_l-x_k;\mu)+\sum_{l<k}^{n}\Delta(y_l-y_k;\mu)\Big)\\
&+\frac{1}{2}\,i\,2n\,\beta^2\Delta(0;\mu)-\;i\beta^2\,\sum_{k=1}^{n}\sum_{l=1}^{n}\Delta(x_l-y_k;\mu)\Big\}.
\end{aligned}\tag{6.13}$$

Replacing the bare coupling $\alpha_0(\Lambda^2)$ as

$$\alpha_0(\Lambda^2)\;=\;Z_1(\Lambda^2;\beta^2,M^2)\,\alpha_r(M^2), \tag{6.14}$$

where the renormalisation constant is equal to

$$Z_1(\Lambda^2;\beta^2,M^2)\;=\;\Big(\frac{\Lambda^2}{M^2}\Big)^{\beta^2/8\pi} \tag{6.15}$$

and M is the renormalisation scale, we obtain the following expression for the generating functional of Green functions

$$
\begin{aligned}
&Z[\widetilde{J}] = \exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x)\Big\}\\
&\times\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\,\Big(\frac{\alpha_r(M^2)}{2\beta^2}\Big)^{2n}\Big[\prod_{k=0}^{n}\int d^2x_k\,d^2y_k\Big]\\
&\times\,\Big[\Big(\frac{\Lambda^2}{M^2}\Big)^{\beta^2/8\pi}\Big(\frac{\mu^2}{\Lambda^2}\Big)^{\beta^2/8\pi}\Big]^{2n}\exp\Big\{i\,\beta\Big(\sum_{k=1}^{n}\phi(x_k)\,-\,\sum_{k=1}^{n}\phi(y_k)\Big)\Big\}\\
&\times\,\exp\Big\{\frac{1}{8\pi}\int d^2x\,d^2y\,\widetilde{J}(x)\ln\big(-\mu^2(x-y)^2+i0\big)\,\widetilde{J}(y)\\
&+\frac{\beta}{4\pi}\int d^2y\sum_{k=1}^{n}\ln\Big(\frac{(x_k-y)^2+i0}{(y_k-y)^2+i0}\Big)\,\widetilde{J}(y)\\
&+\frac{\beta^2}{4\pi}\sum_{j<k}^{n}\Big(\ln\big(-\mu^2(x_k-x_j)^2+i0\big)+\ln\big(-\mu^2(y_k-y_j)^2+i0\big)\,\Big)\\
&-\frac{\beta^2}{4\pi}\,\sum_{k=1}^{n}\sum_{l=1}^{n}\ln\big(-\mu^2(x_k-y_k)^2+i0\big)\,\Big\}.
\end{aligned}
\tag{6.16}
$$

The dependence on the infrared cut–off μ is cancelled due to the relation

$$
\frac{\beta}{8\pi}\,2n\ln(-\mu^2)+\frac{\beta}{4\pi}\,n(n-1)\ln(-\mu^2)-\frac{\beta}{4\pi}\,n^2\ln(-\mu^2)\;=\;0. \tag{6.17}
$$

Thus, we arrive at the following expression for the generating functional of Green functions

$$
\begin{aligned}
&Z[\widetilde{J}]\;=\;\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta_{\rm cl}(x)\partial^\mu\vartheta_{\rm cl}(x)\Big\}\\
&\times\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\,\Big(\frac{\alpha_r(M^2)}{2\beta^2}\Big)^{2n}\Big[\prod_{k=0}^{n}\int d^2x_k\,d^2y_k\Big]\\
&\times\,\exp\Big\{i\,\beta\Big(\sum_{k=1}^{n}\vartheta_{\rm cl}(x_k)\,-\,\sum_{k=1}^{n}\vartheta_{\rm cl}(y_k)\Big)\Big\}\\
&\times\,\exp\Big\{\frac{1}{8\pi}\int d^2x\,d^2y\,\widetilde{J}(x){\rm ln}(-M^2(x-y)^2+i0)\widetilde{J}(y)\\
&-\frac{1}{8\pi}\int d^2x\,d^2y\,\widetilde{J}(x){\rm ln}\Big(\frac{M^2}{\mu^2}\Big)\widetilde{J}(y)\\
&+\frac{\beta}{4\pi}\int d^2y\sum_{k=1}^{n}{\rm ln}(\frac{(x_k-y)^2+i0}{(y_k-y)^2+i0})\widetilde{J}(y)\\
&+\frac{\beta^2}{4\pi}\sum_{j<k}^{n}\Big(\ln\big(-M^2(x_k-x_j)^2+i0\big)\,+\,{\rm ln}(-M^2(y_k-y_j)^2+i0)\Big)
\end{aligned}
$$

$$- \frac{\beta^2}{4\pi}\sum^{n}_{k=1}\sum^{n}_{l=1}\ln(-M^2(x_k - y_k)^2 + i0)\Big\}, \qquad (6.18)$$

where the term, proportional to $\ln(M^2/\mu^2)$, vanishes due to the constraint (6.2). This gives

$$\begin{aligned}
Z[\widetilde{J}] \;&=\; \exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\,\partial_{\mu}\vartheta_{\rm cl}(x)\partial^{\mu}\vartheta_{\rm cl}(x)\Big]\Big\}\\
&\times\sum^{\infty}_{n=0}\frac{(-1)^n}{(n!)^2}\,\Big(\frac{\alpha_r(M^2)}{2\beta^2}\Big)^{2n}\Big[\prod^{n}_{k=0}\int d^2x_k\,d^2y_k\Big]\\
&\times\;\exp\Big\{i\,\beta\Big(\sum^{n}_{k=1}\vartheta_{\rm cl}(x_k)\;-\;\sum^{n}_{k=1}\vartheta_{\rm cl}(y_k)\Big)\Big\}\\
&\times\;\exp\Big\{\frac{1}{8\pi}\;\int d^2x\,d^2y\,\widetilde{J}(x)\ln(-M^2(x-y)^2+i0)\widetilde{J}(y)\\
&+\frac{\beta}{4\pi}\int d^2y\sum^{n}_{k=1}\ln(\frac{(x_k-y)^2+i0}{(y_k-y)^2+i0})\widetilde{J}(y)\\
&+\frac{\beta^2}{4\pi}\sum^{n}_{j<k}\Big(\ln\big(-M^2(x_k-x_j)^2+i0\big)\;+\;\ln(-M^2(y_k-y_j)^2+i0)\Big)\\
&-\frac{\beta^2}{4\pi}\sum^{n}_{k=1}\sum^{n}_{l=1}\ln(-M^2(x_k-y_k)^2+i0)\Big\}.
\end{aligned} \qquad (6.19)$$

The r.h.s. of Eq. (6.19) depends on the ultra–violet cut–off Λ in the form of

$$\frac{\alpha_0(\Lambda^2)}{\beta^2}\sin\vartheta_{\rm cl}(x) \qquad (6.20)$$

which is included in the source $\widetilde{J}(x)$ (6.3) and appears in the definition of correlation functions of the fields of quantum flcutuations around soliton–antisoliton and soliton–soliton solutions. This means that quantum fluctuations around soliton–antisoliton and soliton–soliton solutions in the SG model are defined by a non–renormalisable theory.

Since the result can be valid for any classical solution $\vartheta_{\rm cl}(x)$, the results obtained above can make a hint to quantum field theory of quantum fluctuations around classical solutions of sine–Gordon model equation of motion are non–renormalisable. This can be possibly illustrated by using the mathematical representation

$$Z[\tilde{J}] \;=\; \exp\Big\{\frac{\alpha_0(\Lambda^2)}{\beta^2}\int d^2x\,\sin\vartheta_{\rm cl}(x)\,\frac{\delta}{\delta J(x)}\Big\}\;Z[J], \qquad (6.21)$$

where $Z[J]$ is given by

$$Z[J] \;=\int \mathcal{D}\vartheta\;\exp\Big\{i\int d^2x\,\Big[\frac{1}{2}\partial_\mu\vartheta_{\rm cl}(x)\vartheta_{\rm cl}(x)+\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x)$$
$$+\frac{\alpha_0}{\beta^2}\,(\cos\beta((\vartheta_{\rm cl}(x)+\vartheta(x))-1)+\vartheta(x)J(x)\Big]\Big\}. \tag{6.22}$$

which corresponds to the generating functional for Green functions of quantum fluctuations around the trivial vacuum.

Chapter 7

Conclusion for the sine–Gordon model

We have investigated the renormalisability of the sine–Gordon model. We have analysed the renormalisability of the two–point Green function to second order in α and to all orders in β^2. We have shown that the divergences appearing in the sine–Gordon model can be removed by the renormalisation of the dimensional coupling constant $\alpha_0(\Lambda^2)$. We remind that the coupling constant β^2 has not to be renormalised. This agrees well with a possible interpretation of the coupling constant β^2 as $\hbar$ [2, 25]. The perturbation theory is developed with respect to the renormalised dimensional coupling constant $\alpha_r(M^2)$ depending on the normalisation scale M and the dimensionless coupling constant β^2. Quantum fluctuations relative to the trivial vacuum calculated to first order in $\alpha_r(M^2)$ and to arbitrary order in β^2 form a physical coupling constant $\alpha_{\rm ph}$ after the removal of divergences. The physical coupling constant $\alpha_{\rm ph}$ is finite and does not depend on the normalisation scale M. We have argued that the total renormalised two–point Green function depends on the physical coupling constant $\alpha_{\rm ph}$ only. In order to illustrate this assertion (i) we have calculated the correction to the two–point Green function to second order in $\alpha_r(M^2)$ and to all orders in β^2 and (ii) we have solved the Callan–Symanzik equation for the two–point Green function with the Gell–Mann–Low function, defined to all orders in $\alpha_r(M^2)$ and β^2. We have found that the two–point Green function of the sine–Gordon field depends on the running coupling constant $\alpha_r(p^2) = \alpha_{\rm ph}(p^2/\alpha_{\rm ph})^{\tilde{\beta}^2/8\pi}$, where $\tilde{\beta}^2 = \beta^2/(1+\beta^2/8\pi) < 1$ for any β^2.

We have analysed the renormalisation of the two–point Green function of the massive sine–Gordon model. We have shown that the mass operator $m_0^2\vartheta^2(x)$ is soft. In the infrared limit $m_0 \to 0$ the physical mass of the MSG model quanta reduces to our result (2.33). For $m_0^2 \gg \alpha_r(M^2)$ we have shown that the mass parameter m_0 is unrenormalisable. The physical coupling constant $\alpha_{\rm ph}$ has been calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2. This has allowed to calculate the Gell–Mann–Low function and to derive the Callan–Symanzik equation for the two–point Green function. We have shown that the Callan–Symanzik equation reduces to the form used by [16] to the same order in perturbation theory. Solving this equation we have calculated the running coupling constant and found that for $\beta^2 < 8\pi$ the massive sine–Gordon model with quantum fluctuations around a trivial vacuum, calculated to first order in $\alpha_r(M^2)$ and to all orders in β^2, is asymptotically free for infinitely large momenta. In turn, for $\beta^2 > 8\pi$ the running coupling constant $\alpha_r(p^2)$ grows with p^2. But since $\alpha_r(p^2)$ has been calculated perturbatively for $m_0^2 \gg \alpha_r(M^2)$, the running coupling constant should obey the constraint $m_0^2 \gg \alpha_r(p^2)$. This restricts the region of the allowed momenta $p^2 \ll m_0^2(m_0^2/\alpha_{\rm ph})^{1/2\delta(\beta^2)}$ with $\delta(\beta^2) = (\beta^2 - 8\pi)/8\pi$ [16]. All these results do not contradict those obtained by [16]. We discuss the renormazability of the massive sine–Gordon model non–perturbativly for quantum fluctuations around a vacuum. We confirm result that the mass parameter m_0 does not introduce any divergences.

In addition to the analysis of the renormalisability of the sine–Gordon model with respect to quantum fluctuations relative to the trivial vacuum, we have analysed the renormalisability of the sine–Gordon model with respect to quantum fluctuations around a soliton. Following [17, 18] and [19] we have taken into account only Gaussian fluctuations.

For the calculation of the effective Lagrangian, induced by Gaussian fluctuations, we have used the path–integral approach and integrated over the field $\vartheta(x)$, fluctuating around a soliton. This has allowed to express the effective Lagrangian in terms of a functional determinant. For the calculation of the contribution of the functional determinant we have used the eigenfunctions and eigenvalues of the differential operator, describing the evolution of the field $\vartheta(x)$. We have shown that the renormalised effective Lagrangian, induced by Gaussian fluctuations around a soliton, coincides to leading order in β^2 with the renormalised Lagrangian of the SG model, caused by quantum fluctuations around the trivial vacuum to first order in α_0 and to second order in β^2. After the removal of divergences the soliton mass is equal to the mass of a soliton, calculated without quantum corrections, up to the replacement $\alpha_0 \to \alpha_{\rm ph}$. This implies that Gaussian fluctuations around a soliton do not produce any quantum corrections to the soliton mass. Hence, no non–perturbative singularities of the sine–Gordon model at $\beta^2 = 8\pi$ can be induced by Gaussian fluctuations around a soliton.

We compare our continuous space–time approach to Gaussian fluctuations around soliton with that of Faddeev *et al.* [19]. We derive our result for the mass correction to the soliton induced by Gaussian fluctuations within the formalism of [19]. The finite term appearing in [19] is due to a different regularisation scheme. In [19] and also in [18] the mass correction were obtained by summation over on–shell energy momenta, while in our approach we stay manifestly Lorentz covariant through our calculations from the beginning. The summation over the quantum numbers is calculated in a completely covariant way. We confirm our continuous space–time regularisation scheme within the discretised regularisation approach [24].

The renormalisation of the sine–Gordon model, which was carried out before 1979 in [26]–[29], has been discussed well by Amit *et al.*. After 1980, as has been pointed out by Nándori *et al.* [30], the main results on the renormalisation of the sine–Gordon model in two dimensions have been obtained in [31]–[38]. In these papers the sine–Gordon model has been investigated at finite temperature in connection with the XY model and the existence of phase transitions.

Unlike [16, 26]–[38] our results can be applied to the analysis of the FQHE (the Fractional Quantum Hall Effect) [4, 5]. As has been shown in [25], the massive Thirring model, which can describe one–dimensional edge fermions [4, 5], bosonises to the sine–Gordon model for $\beta^2 > 8\pi$. According to [25], for $\beta^2 > 8\pi$ the sine–Gordon system produces mainly solitons, which can play an important role in the FQHE [39].

Finally, we consider the renormalisability of the sine–Gordon model with respect to quantum fluctuations around any nontrivial classical solution. We show, that quantum fluctuations around any classical solution in the sG model are defined by a nonrenormalisable theory.

Part II

Renormalisibility of the Massless Thirring Model

Chapter 8

On the renormalisation of the Thirring model

8.1 Introduction

The massless Thirring model [40] is an exactly solvable quantum field theoretic model of fermions with a non–trivial four-fermion interaction in $1+1$–dimensional space–time defined by the Lagrangian invariant under the chiral group $U_{\rm V}(1) \times U_{\rm A}(1)$

$$\mathcal{L}_{\rm Th}(x) \;=\; \bar{\psi}(x) i\gamma^{\mu}\partial_{\mu}\psi(x) - \frac{1}{2}\, g\, \bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(x)\gamma_{\mu}\psi(x), \tag{8.1}$$

where $\psi(x)$ is a massless Dirac fermion field and g is a dimensionless coupling constant that can be both positive and negative.

A solution of the Thirring model assumes the development of a procedure for the calculation of any correlation function [41]–[52]. As has been shown by Hagen [43] and Klaiber [44], the correlation functions of massless Thirring fermion fields can be parameterised by one arbitrary parameter. In Hagen's notation this parameter is ξ. Below we show that the correlation functions in the massless Thirring model can be parameterised by two parameters . This confirms the results obtained by Harada *et al.* [53] (see also [54, 58]) for the chiral Schwinger model. In our notation these parameters are $\bar{\xi}$ and $\bar{\eta}$. The region of variation of these parameters is restricted by the condition for the norms of the wave functions of the states related to the components of the fermion vector current to be positive. For $\bar{\eta} = 1$ the parameter $\bar{\xi}$ is equal to Hagen's parameter $\bar{\xi} = \xi$. The parameters $\bar{\xi}$ and $\bar{\eta}$ we use for the analysis of the non–perturbative renormalisability of the massless Thirring model in the sense that a dependence of any correlation function on the ultra–violet cut–off Λ can be removed by the renormalisation of the wave function of Thirring fermion fields only. We show that the independence of any correlation function of an ultra–violet cut–off exists only if the dynamical dimensions of Thirring fermion fields, calculated from different correlation functions, are equal. The dynamical dimensions of the known solutions for causal Green functions and left–right correlation functions of the massless Thirring model are different [41]–[52]. The existence of different dynamical dimensions of Thirring fermion fields obtained from different correlation functions has been regarded by Jackiw as a problem of 1+1–dimensional quantum field theories [47]. We show that the determinant $\mathrm{Det}(i\hat{\partial} + \hat{A})$, where A_μ is an external vector field, can be parameterised by two parameters. For this aim we calculate the vacuum expectation value of the vector current and show that the ambiguous parameterisation of the determinant $\mathrm{Det}(i\hat{\partial} + \hat{A})$ is fully caused by the regularisation procedure [53]–[58]. We analyse the constraints on the parameters $\bar{\xi}$ and $\bar{\eta}$ imposed by the positive definiteness of the norms of the wave functions of the states related to the components of the vector fermion current. We show that the positive definiteness of these norms does not prohibit the possibility for the dynamical dimensions of massless Thirring fermion fields to be equal. According to the equivalence of the massive Thirring model to the sine–Gordon model [2], the constraints on the parameters $\bar{\eta}$ and $\bar{\xi}$ together with the requirement of the non–perturbative renormalisability of the massless Thirring model lead to the strongly coupled sine–Gordon field with the coupling constant $\beta^2 \sim 8\pi$. The behaviour and renormalisability of the sine–Gordon model for the coupling constants $\beta^2 \sim 8\pi$ has been investigated in [16, 24, 32]-[36, 59].

8.2 Generating functional of correlation functions

The generating functional of vacuum expectation values of products of massless Thirring fermion fields, i.e. correlation functions, is defined by

$$\begin{aligned} Z_{\rm Th}[J,\bar{J}] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp i\int d^2x\,\Big[\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x) - \frac{1}{2}\,g\,\bar{\psi}(x)\gamma^\mu\psi(x)\bar{\psi}(x) \\ &\quad + \bar{\psi}(x)J(x) + \bar{J}(x)\psi(x)\Big]. \end{aligned} \tag{8.2}$$

It can be represented also as follows

$$Z_{\rm Th}[J,\bar{J}] = \exp\Big\{\frac{i}{2}\,g\int d^2x\,\frac{\delta}{\delta A_\mu(x)}\frac{\delta}{\delta A^\mu(x)}\Big\}Z^{(0)}_{\rm Th}[A;J,\bar{J}]\Big|_{A=0}, \tag{8.3}$$

where we have denoted

$$\begin{aligned} Z^{(0)}_{\rm Th}[A;J,\bar{J}] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp i\int d^2x\,\Big[\bar{\psi}(x)i\gamma^\mu\partial_\mu\psi(x) + \bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \\ &\quad + \bar{\psi}(x)J(x) + \bar{J}(x)\psi(x)\Big]. \end{aligned} \tag{8.4}$$

The functional $Z^{(0)}_{\rm Th}[A;J,\bar{J}]$ is a generating functional of vacuum expectation values of products of massless fermion fields of the massless Schwinger model coupled to an external vector field $A_\mu(x)$ [60]. The integration over fermion fields can be carried out explicitly and we get

$$Z^{(0)}_{\rm Th}[A;J,\bar{J}] \;=\; {\rm Det}(i\hat{\partial} + \hat{A})\,\exp\Big\{i\iint d^2x\,d^2y\,\bar{J}(x)\,G(x,y)_A\,J(y)\Big\}, \tag{8.5}$$

where $G(x,y)_A$ is the two–point causal fermion Green function obeying the equation

$$i\gamma^{\mu}\Big(\frac{\partial}{\partial x^{\mu}} - i\,A_{\mu}(x)\Big)G(x,y)_A \;=\; -\,\delta^{(2)}(x-y). \tag{8.6}$$

As has been shown in the Appendix E.1, the functional determinant ${\rm Det}(i\hat{\partial}+\hat{A})$ can be parameterised by two parameters

$${\rm Det}(i\hat{\partial} + \hat{A}) \;=\; \exp\Big\{\frac{i}{2}\iint d^2xd^2y\,A_{\mu}(x)\,D^{\mu\nu}(x-y)\,A_{\nu}(y)\Big\}, \tag{8.7}$$

where we have denoted

$$D^{\mu\nu}(x-y) \;=\; \frac{\bar{\xi}}{\pi}\,g^{\mu\nu}\,\delta^{(2)}(x-y) - \frac{\bar{\eta}}{\pi}\,\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x_{\nu}}\Delta(x-y;\mu). \tag{8.8}$$

Here $\bar{\xi}$ and $\bar{\eta}$ are two parameters, $g^{\mu\nu}$ is the metric tensor and $\Delta(x-y;\mu)$ is the causal two–point Green function of a free massless (pseudo)scalar field

$$i\Delta(x-y;\mu) = \frac{1}{4\pi}\ln[-\mu^2(x-y)^2 + i\,0]. \tag{8.9}$$

It obeys the equation $\Box_x\Delta(x-y;\mu) = \delta^{(2)}(x-y)$, where μ is an infrared cut–off.

The appearance of two parameters is caused by the dependence of the determinant ${\rm Det}(i\hat{\partial}+\hat{A})$ on the regularisation procedure [53]–[58]. In Appendix E.4 we find a constraint for these parameters imposed by the positive definiteness of the norms of the wave functions of the states related to the components of the fermion vector current. The parameters $\bar{\xi}$ and $\bar{\eta}$ are related to Hagen's parameter ξ as $\bar{\xi} = \xi$ and $\bar{\eta} = 1$.

The solution of the equation (8.6) is equal to

$$\begin{aligned} G(x,y)_A \;&=\; G_0(x-y) \\ \times\,\exp\Big\{&-i\,(g^{\alpha\beta} - \varepsilon^{\alpha\beta}\,\gamma^5)\int d^2z\,\frac{\partial}{\partial z^{\alpha}}[\Delta(x-z;\mu) - \Delta(y-z;\mu)]\,A_{\beta}(z)\Big\}, \end{aligned} \tag{8.10}$$

where $\varepsilon^{\alpha\beta}$ is the antisymmetric tensor defined by $\varepsilon^{01} = 1$ and $G_0(x-y)$ is the Green function of a free massless fermion field

$$G_0(x-y) \;=\; i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}}\Delta(x-y;\mu) = \frac{1}{2\pi}\,\frac{\gamma^{\mu}(x-y)_{\mu}}{(x-y)^2 - i\,0} \tag{8.11}$$

satisfying the equation $i\gamma^{\mu}\partial_{\mu}G_0(x-y) = -\,\delta^{(2)}(x-y)$.

Any correlation function of the massless Thirring fermion fields can be defined by functional derivatives of the generating functional (8.3) and calculated in terms of the two–point Green functions $G(x,y)_A$ and $\Delta(x-y;\mu)$. Below we calculate the causal two–point Green function $G(x,y)$ and the correlation function $C(x,y)$ of the left–right fermion densities defined by

$$\begin{aligned}
G(x,y) &= i\,\langle 0|\mathrm{T}(\psi(x)\bar{\psi}(y))|0\rangle = \frac{1}{i}\,\frac{\delta}{\delta \bar{J}(x)}\,\frac{\delta}{\delta J(y)} Z_{\mathrm{Th}}[J,\bar{J}]\Big|_{J=\bar{J}=0},\\
C(x,y) &= \Big\langle 0\Big|\mathrm{T}\Big(\bar{\psi}(x)\Big(\frac{1-\gamma^5}{2}\Big)\psi(x)\,\bar{\psi}(y)\Big(\frac{1+\gamma^5}{2}\Big)\psi(y)\Big)\Big|0\Big\rangle\\
&= \frac{1}{i}\frac{\delta}{\delta J(x)}\Big(\frac{1-\gamma^5}{2}\Big)\frac{1}{i}\frac{\delta}{\delta \bar{J}(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}\Big(\frac{1+\gamma^5}{2}\Big)\frac{1}{i}\frac{\delta}{\delta \bar{J}(y)} Z_{\mathrm{Th}}[J,\bar{J}]\Big|_{J=\bar{J}=0},
\end{aligned} \tag{8.12}$$

where T is the time–ordering operator. The main aim of the investigation of these correlation functions is the calculation of the dynamical dimensions of the massless Thirring fermion fields and the analysis of the possibility to make them equal [47].

8.3 Two–point causal Green function $G(x,y)$

In terms of the generating functional (8.2) the two–point Green function $G(x,y)$ is defined by

$$\begin{aligned}
G(x,y) &= \frac{1}{i}\,\frac{\delta}{\delta \bar{J}(x)}\,\frac{\delta}{\delta J(y)} Z_{\mathrm{Th}}[J,\bar{J}]\Big|_{J=\bar{J}=0}\\
&= \exp\Big\{\frac{i}{2}\,g\int d^2z\,\frac{\delta}{\delta A_{\mu}(z)}\frac{\delta}{\delta A^{\mu}(z)}\Big\}\\
&\times \exp\Big\{\frac{i}{2}\iint d^2z_1 d^2z_2\,A_{\lambda}(z_1)\,D^{\lambda\varphi}(z_1-z_2)\,A_{\varphi}(z_2)\Big\}\,G(x,y)_A\Big|_{A=0}.
\end{aligned} \tag{8.13}$$

The calculation of (8.13) reduces to the calculation of the path integral

$$\begin{aligned}
G(x,y) &= \frac{1}{2\pi}\,\frac{\gamma^{\mu}(x-y)_{\mu}}{(x-y)^2 - i\,0}\int \mathcal{D}^2u\,\exp\Big\{-\frac{i}{2}\int d^2z\,u_{\mu}(z)u^{\mu}(z)\\
&- \frac{i}{2}\,g\iint d^2z_1 d^2z_2\,u_{\mu}(z_1)\,D^{\mu\nu}(z_1-z_2)\,u_{\nu}(z_2) + \sqrt{g}\,(g^{\alpha\beta} - \varepsilon^{\alpha\beta}\,\gamma^5)\\
&\times \int d^2z\,\frac{\partial}{\partial z^{\alpha}}[\Delta(x-z;\mu) - \Delta(y-z;\mu)]\,u_{\beta}(z)\Big\}.
\end{aligned} \tag{8.14}$$

Symbolically (8.14) can be written as

$$\begin{aligned}
G(x,y) &= \frac{1}{2\pi}\,\frac{\gamma^{\mu}(x-y)_{\mu}}{(x-y)^2 - i\,0}\int \mathcal{D}^2u\,\exp\Big\{-\frac{i}{2}\,u_{\mu}(1+gD)^{\mu\nu}u_{\nu} +\\
&+ \sqrt{g}\,\partial^{\mu}(\Delta_x - \Delta_y)\,u_{\mu} - \sqrt{g}\,\gamma^5\,\partial_{\mu}(\Delta_x - \Delta_y)\,\varepsilon^{\mu\nu}\,u_{\nu}\Big\}.
\end{aligned} \tag{8.15}$$

The integration over u can be carried out by quadratic extension. This yields

$$\begin{aligned}G(x,y) &= \frac{1}{2\pi}\,\frac{\gamma^{\mu}(x-y)_{\mu}}{(x-y)^2 - i\,0}\\ &\times\,\exp\Big\{-\,\frac{i}{2}\,g\,(\partial^x_{\mu}\Delta_x - \partial^y_{\mu}\Delta_y)\Big(\frac{1}{1+gD}\Big)^{\mu\nu}(\partial^x_{\nu}\Delta_x - \partial^y_{\nu}\Delta_y)\\ &+\frac{i}{2}\,g\,(\partial^x_{\mu}\Delta_x - \partial^y_{\mu}\Delta_y)\,\varepsilon^{\mu\alpha}\Big(\frac{1}{1+gD}\Big)_{\alpha\beta}\varepsilon^{\beta\nu}\,(\partial^x_{\nu}\Delta_x - \partial^y_{\nu}\Delta_y)\Big\}.\end{aligned}\tag{8.16}$$

For the subsequent calculation we have to construct the matrix $(1+gD)^{-1}$. The matrix $(1+gD)$ has the following elements

$$(1+gD)^{\mu\alpha}(x,z) = \Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\,g^{\mu\alpha}\,\delta^{(2)}(x-z) - \bar{\eta}\,\frac{g}{\pi}\,\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x_{\alpha}}\Delta(x-z;\mu).\tag{8.17}$$

The elements of the matrix $(1+gD)^{-1}$ we define as (see Appendix E.2)

$$((1+gD)^{-1})_{\alpha\nu}(z,y) = A\,g_{\alpha\nu}\,\delta^{(2)}(z-y) + B\,\frac{\partial}{\partial z^{\alpha}}\frac{\partial}{\partial z^{\nu}}\Delta(z-y;\mu).\tag{8.18}$$

The matrices $(1+gD)$ and $(1+gD)^{-1}$ should obey the condition

$$\int d^2z\,(1+gD)^{\mu\alpha}(x,z)((1+gD)^{-1})_{\alpha\nu}(z,y) = g^{\mu}_{\nu}\,\delta^{(2)}(x-y).\tag{8.19}$$

This gives

$$\begin{aligned}((1+gD)^{-1})_{\alpha\nu}(z,y) &= \frac{g_{\alpha\nu}}{1+\bar{\xi}\,\dfrac{g}{\pi}}\,\delta^{(2)}(z-y)\\ &+\frac{g}{\pi}\,\frac{\bar{\eta}}{\Big(1+\bar{\xi}\,\dfrac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\dfrac{g}{\pi}\Big)}\,\frac{\partial}{\partial z^{\alpha}}\frac{\partial}{\partial z^{\nu}}\Delta(z-y;\mu).\end{aligned}\tag{8.20}$$

Using (8.20) we obtain (see Appendix E.3)

$$\begin{aligned}&-\frac{i}{2}\,g\,(\partial^x_{\mu}\Delta_x - \partial^y_{\mu}\Delta_y)\Big(\frac{1}{1+gD}\Big)^{\mu\nu}(\partial^x_{\nu}\Delta_x - \partial^y_{\nu}\Delta_y)\\ &\qquad= \frac{g}{1+(\bar{\xi}-\bar{\eta})\,\dfrac{g}{\pi}}\,[i\Delta(0;\mu) - i\Delta(x-y;\mu)],\\ &\frac{i}{2}\,g\,(\partial^x_{\mu}\Delta_x - \partial^y_{\mu}\Delta_y)\varepsilon^{\mu\alpha}\Big(\frac{1}{1+gD}\Big)_{\alpha\beta}\varepsilon^{\beta\nu}(\partial^x_{\nu}\Delta_x - \partial^y_{\nu}\Delta_y)\\ &\qquad= -\,\frac{g}{1+\bar{\xi}\,\dfrac{g}{\pi}}[i\Delta(0;\mu) - i\Delta(x-y;\mu)],\end{aligned}\tag{8.21}$$

where $i\Delta(0;\mu)$ is equal to

$$i\Delta(0;\mu) = -\,\frac{1}{4\pi}\ln\Big(\frac{\Lambda^2}{\mu^2}\Big).\tag{8.22}$$

Thus, the two–point Green function reads

$$
\begin{aligned}
G(x,y) &= \frac{1}{2\pi}\,\frac{\gamma^\mu(x-y)_\mu}{(x-y)^2 - i\,0}\,e^{\,4\pi\, d_{\bar\psi\psi}(g)\,[i\Delta(0;\mu) - i\Delta(x-y;\mu)]}\\
&= -\,\frac{\Lambda^2}{2\pi}\,\frac{\gamma^\mu(x-y)_\mu}{-\Lambda^2(x-y)^2 + i\,0}\,[-\Lambda^2(x-y)^2 + i\,0\,]^{-\,d_{(\bar\psi\psi)}(g)}\\
&= \Lambda\, G(d_{\bar\psi\psi}(g);\Lambda x,\Lambda y), \qquad (8.23)
\end{aligned}
$$

where $d_{\bar\psi\psi}(g)$ is the dynamical dimension of the Thirring fermion field defined by [47]

$$
d_{\bar\psi\psi}(g) = \frac{g^2}{4\pi^2}\,\frac{\bar\eta}{\Big(1+\bar\xi\,\dfrac{g}{\pi}\Big)\Big(1+(\bar\xi-\bar\eta)\,\dfrac{g}{\pi}\Big)}. \qquad (8.24)
$$

Now we are proceeding to the calculation of the correlation function $C(x,y)$.

8.4 Two–point correlation function $C(x,y)$

According to Eq.(8.12), the two–point correlation function $C(x,y)$ of the left–right fermion densities is defined by

$$
\begin{aligned}
C(x,y) &= \frac{1}{i}\frac{\delta}{\delta J(x)}\Big(\frac{1-\gamma^5}{2}\Big)\frac{1}{i}\frac{\delta}{\delta \bar J(x)}\frac{1}{i}\frac{\delta}{\delta J(y)}\Big(\frac{1+\gamma^5}{2}\Big)\frac{1}{i}\frac{\delta}{\delta \bar J(y)} Z_{\rm Th}[J,\bar J]\Big|_{J=\bar J=0}\\
&= -\,\exp\Big\{\frac{i}{2}\,g\int d^2z\,\frac{\delta}{\delta A_\mu(z)}\frac{\delta}{\delta A^\mu(z)}\Big\}\\
&\times\ \exp\Big\{\frac{i}{2}\iint d^2z_1 d^2z_2\, A_\lambda(z_1)\,D^{\lambda\varphi}(z_1-z_2)\,A_\varphi(z_2)\Big\}\\
&\times\ {\rm tr}\Big\{G(y,x)_A\Big(\frac{1-\gamma^5}{2}\Big)G(x,y)_A\Big(\frac{1+\gamma^5}{2}\Big)\Big\}\Big|_{A=0}. \qquad (8.25)
\end{aligned}
$$

This reduces to the calculation of the path integral

$$
\begin{aligned}
&C(x,y) = \frac{1}{4\pi^2}\,\frac{1}{(x-y)^2 - i\,0}\\
&\times\int \mathcal{D}^2u\,\exp\Big\{-\frac{i}{2}\,u_\mu(1+gD)^{\mu\nu}u_\nu - 2\sqrt{g}\,\partial_\mu(\Delta_x-\Delta_y)\,\varepsilon^{\mu\nu}\,u_\nu\Big\}\\
&= \frac{1}{4\pi^2}\,\frac{1}{(x-y)^2 - i\,0}\\
&\times\,\exp\Big\{2\,i\,g\,(\partial^x_\mu\Delta_x - \partial^y_\mu\Delta_y)\,\varepsilon^{\mu\alpha}\Big(\frac{1}{1+gD}\Big)_{\alpha\beta}\varepsilon^{\beta\nu}\,(\partial^x_\nu\Delta_x - \partial^y_\nu\Delta_y)\Big\}. \qquad (8.26)
\end{aligned}
$$

The result is

$$
C(x,y) = \frac{1}{4\pi^2}\,\frac{1}{(x-y)^2 - i\,0}\,e^{\,8\,\pi\, d_{(\bar\psi\psi)^2}\,[i\Delta(0;\mu) - i\Delta(x-y;\mu)]}
$$

$$
\begin{aligned}
&= -\frac{\Lambda^2}{4\pi^2}\,\frac{1}{-\Lambda^2(x-y)^2 + i\,0}\,[-\Lambda^2(x-y)^2 + i\,0]^{-\,2d_{(\bar{\psi}\psi)^2}} \\
&= \Lambda^2\, C(d_{(\bar{\psi}\psi)^2}(g); \Lambda x, \Lambda y).
\end{aligned}
\tag{8.27}
$$

The dynamical dimension $d_{(\bar{\psi}\psi)^2}$ is equal to

$$
d_{(\bar{\psi}\psi)^2}(g) \;=\; -\frac{g}{2\pi}\,\frac{1}{1 + \bar{\xi}\,\dfrac{g}{\pi}}. \tag{8.28}
$$

For $\bar{\xi}$ and $\bar{\eta}$, restricted only by the constraint caused by the positive definiteness of the norms of the wave functions of the states related to the components of the fermion vector current, the dynamical dimensions of the massless Thirring model, calculated for the two–point causal Green function (8.24) and the correlation function of the left–right fermion densities (8.28), are not equal. According to Jackiw [47], this is a problem of quantum field theories in 1+1–dimensional space–time. However, equating $d_{(\bar{\psi}\psi)^2}(g)$ and $d_{\bar{\psi}\psi}(g)$ we get the constraint on the parameter $\bar{\eta}$

$$
\bar{\eta} \;=\; \frac{2\pi}{g}\Big(1 + \bar{\xi}\,\frac{g}{\pi}\Big). \tag{8.29}
$$

As has been shown in Appendix E.4, the constraint on the region of variation of parameters $\bar{\xi}$ and $\bar{\eta}$, imposed by the positive definiteness of the norms of the wave functions of the states related to the components of the vector current, does not prevent from the equality of dynamical dimensions $d_{(\bar{\psi}\psi)^2}(g) = d_{\bar{\psi}\psi}(g)$.

This indicates that the massless Thirring model is renormalisable. The dependence on the ultra–violet cut–off Λ can be removed by the renormalisation of the wave functions of Thirring fermion fields for both the $2n$–point Green functions $G(x_1, ..., x_n; y_1, ..., y_n)$ and the $2n$–point correlation functions $C(x_1, ..., x_n; y_1, ..., y_n)$ of the left–right fermion densities, the cut–off dependent parts of which are proportional to $(\Lambda^2)^{-\,nd_{\bar{\psi}\psi}(g)}$ and $(\Lambda^2)^{-\,2nd_{(\bar{\psi}\psi)^2}(g)}$, respectively. Such a dependence can be proved by direct calculations. The dynamical dimension of the Thirring fermion fields is equal to $d_{\psi}(g) = d_{(\bar{\psi}\psi)^2}(g)$ defined by Eq.(8.28). We have to emphasize that, according to Jackiw [47], the dynamical dimension of the operator $:\psi(x_1)\ldots\psi(x_n)\bar{\psi}(y_1)\ldots\bar{\psi}(y_n):$, where $:\ldots:$ stands for normal ordering, should differ from $2nd_{\psi}(g)$. This means that the dynamical dimensions of the Thirring fermion fields are *nondistributive* [47]. However, the *nondistributive* property of the dynamical dimensions of the Thirring fermion fields does not influence the renormalisability of the massless Thirring model.

8.5 Non–perturbative renormalisation

According to the standard procedure of renormalisation in quantum field theory [8] the renormalisability of the massless Thirring model should be understood as a possibility to remove all ultra–violet divergences by renormalisation of the wave function of the massless Thirring fermion field $\psi(x)$ and the coupling constant g.

Let us rewrite the Lagrangian (8.1) in terms of *bare* quantities

$$\mathcal{L}_{\rm Th}(x) \;=\; \bar{\psi}_0(x) i\gamma^{\mu}\partial_{\mu}\psi_0(x) - \frac{1}{2}\, g_0\, \bar{\psi}_0(x)\gamma^{\mu}\psi_0(x)\bar{\psi}_0(x)\gamma_{\mu}\psi_0(x), \qquad (8.30)$$

where $\psi_0(x)$, $\bar{\psi}_0(x)$ are *bare* fermionic field operators and g_0 is the *bare* coupling constant.

The renormalised Lagrangian $\mathcal{L}(x)$ of the massless Thirring model should then read [8]

$$\begin{aligned}\mathcal{L}_{\rm Th}(x) &= \bar{\psi}(x) i\gamma^{\mu}\partial_{\mu}\psi(x) - \frac{1}{2}\, g\, \bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(x)\gamma_{\mu}\psi(x)\\ &\quad +(Z_2 - 1)\,\bar{\psi}(x) i\gamma^{\mu}\partial_{\mu}\psi(x) - \frac{1}{2}\, g\,(Z_1 - 1)\,\bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(x)\gamma_{\mu}\psi(x)\\ &= Z_2\,\bar{\psi}(x) i\gamma^{\mu}\partial_{\mu}\psi(x) - \frac{1}{2}\, g\, Z_1\,\bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(x)\gamma_{\mu}\psi(x), \qquad (8.31)\end{aligned}$$

where Z_1 and Z_2 are the renormalisation constants of the coupling constant and the wave function of the fermion field.

The renormalised fermionic field operator $\psi(x)$ and the coupling constant g are related to *bare* quantities by the relations [8]

$$\begin{aligned}\psi_0(x) &= Z_2^{1/2}\,\psi(x),\\ g_0 &= Z_1 Z_2^{-2}\, g. \qquad (8.32)\end{aligned}$$

For the correlation functions of massless Thirring fermions the renormalisability of the massless Thirring model means the possibility to replace the ultra–violet cut–off Λ by a finite scale M by means of the renormalisation constants Z_1 and Z_2.

According to the general theory of renormalisation [8], the renormalisation constants Z_1 and Z_2 depend on the renormalised quantities g, the infrared scale μ, the ultra–violet scale Λ and the finite scale M. As has been shown above the Green functions and left–right fermion density correlation functions do not depend on the infrared cut–off. Therefore, we can omit it. This defines the renormalisation constants as follows

$$\begin{aligned}Z_1 &= Z_1(g, M; \Lambda),\\ Z_2 &= Z_2(g, M; \Lambda). \qquad (8.33)\end{aligned}$$

For the analysis of the feasibility of the replacement $\Lambda \to M$ it is convenient to introduce the following notations

$$\begin{aligned}G^{(0)}(x_1,\ldots,x_n;y_1,\ldots,y_n) &= \Lambda^n\, G^{(0)}(d_{(\bar{\psi}\psi)}(g_0);\Lambda x_1,\ldots,\Lambda x_n;\Lambda y_1,\ldots,\Lambda y_n),\\ C^{(0)}(x_1,\ldots,x_n;y_1,\ldots,y_n) &= \Lambda^{2n}\, C^{(0)}(d_{(\bar{\psi}\psi)^2}(g_0);\Lambda x_1,\ldots,\Lambda x_n;\Lambda y_1,\ldots,\Lambda y_n). \qquad (8.34)\end{aligned}$$

The transition to a finite scale M changes the functions (8.34) as follows

$$
\begin{aligned}
&G^{(0)}(x_1,\ldots,x_n;y_1,\ldots,y_n) = \\
&\quad = \left(\frac{\Lambda}{M}\right)^{-2nd_{(\bar\psi\psi)}(g)} M^n\, G^{(0)}(d_{(\bar\psi\psi)}(g_0);Mx_1,\ldots,Mx_n;My_1,\ldots,My_n),\\
&C^{(0)}(x_1,\ldots,x_n;y_1,\ldots,y_n)\\
&\quad = \left(\frac{\Lambda}{M}\right)^{-4nd_{(\bar\psi\psi)^2}(g)} M^{2n}\, C^{(0)}(d_{(\bar\psi\psi)^2}(g_0);Mx_1,\ldots,Mx_n;My_1,\ldots,My_n).
\end{aligned}
\tag{8.35}
$$

The renormalised correlation functions are related to the *bare* ones by the relations [8]:

$$
\begin{aligned}
&G^{(r)}(x_1,\ldots,x_n;y_1,\ldots,y_n) \;=\; Z_2^{-n}\, G^{(0)}(x_1,\ldots,x_n;y_1,\ldots,y_n)\\
&= Z_2^{-n}\left(\frac{\Lambda}{M}\right)^{-2nd_{(\bar\psi\psi)}(g)} M^n\, G^{(0)}(d_{(\bar\psi\psi)}(Z_1Z_2^{-2}g);Mx_1,\ldots,Mx_n;My_1,\ldots,My_n),\\
&C^{(r)}(x_1,\ldots,x_n;y_1,\ldots,y_n) = Z_2^{-2n}C^{(0)}(x_1,\ldots,x_n;y_1,\ldots,y_n)\\
&= Z_2^{-2n}\left(\frac{\Lambda}{M}\right)^{-4nd_{(\bar\psi\psi)^2}(g)} M^{2n}\, C^{(0)}(d_{(\bar\psi\psi)^2}(Z_1Z_2^{-2}g);Mx_1,\ldots,Mx_n;My_1,\ldots,My_n).
\end{aligned}
\tag{8.36}
$$

Renormalisability demands the relations

$$
\begin{aligned}
G^{(r)}(x_1,\ldots,x_n;y_1,\ldots,y_n) &= M^n\, G^{(r)}(d_{(\bar\psi\psi)}(g);Mx_1,\ldots,Mx_n;My_1,\ldots,My_n),\\
C^{(r)}(x_1,\ldots,x_n;y_1,\ldots,y_n) &= M^{2n}\, C^{(r)}(d_{(\bar\psi\psi)^2}(g);Mx_1,\ldots,Mx_n;My_1,\ldots,My_n),
\end{aligned}
\tag{8.37}
$$

which impose constraints on the dynamical dimensions and renormalisation constants

$$
\begin{aligned}
d_{(\bar\psi\psi)}(g) &= d_{(\bar\psi\psi)}(Z_1Z_2^{-2}g),\\
d_{(\bar\psi\psi)^2}(g) &= d_{(\bar\psi\psi)^2}(Z_1Z_2^{-2}g)
\end{aligned}
\tag{8.38}
$$

and

$$
Z_2^{-1}\left(\frac{\Lambda}{M}\right)^{-2d_{(\bar\psi\psi)}(g)} \;=\; Z_2^{-1}\left(\frac{\Lambda}{M}\right)^{-2d_{(\bar\psi\psi)^2}(g)} \;=\; 1. \tag{8.39}
$$

The constraints (8.38) on the dynamical dimensions are fulfilled only if the renormalisation constants are related by

$$
Z_1 \;=\; Z_2^2. \tag{8.40}
$$

The important consequence of this relation is that the coupling constant g of the massless Thirring model is unrenormalised, i.e.

$$g_0 = g. \tag{8.41}$$

The unrenormalisability of the coupling constant, $g_0 = g$, is not a new result and it has been obtained in [61, 62] for the massive Thirring model.

The unrenormalisability of the coupling constant, $g_0 = g$, implies also that the Gell–Mann–Low β–function, defined by [8]

$$M\frac{dg}{dM} = \beta(g, M), \tag{8.42}$$

should vanish, since g is equal to g_0, which does not depend on M, i.e. $\beta(g, M) = 0$.

The constraint (8.39) is fulfilled for $d_{(\bar{\psi}\psi)}(g) = d_{(\bar{\psi}\psi)^2}(g)$ only. In this case the dependence of the $2n$–point causal Green functions and the $2n$–point correlation functions of left–right fermion densities on the ultra–violet cut–off Λ can be simultaneously removed by renormalisation of the wave function of the massless Thirring fermion fields. This means the massless Thirring model is non–perturbative renormalisable.

8.6 Conclusion

We have found the most general expressions for the causal two–point Green function and the two–point correlation function of left–right fermion densities with dynamical dimensions parameterised by two parameters. The variation of these parameters is restricted by the positive definiteness of the norms of the wave functions of the states related to the components of the fermion vector current (see E.4).

Our expressions incorporate those obtained by Hagen, Klaiber and within the path–integral approach [43]–[52]. Indeed, for Hagen's parameterisation of the functional determinant with the parameters $\bar{\xi} = \xi$ and $\bar{\eta} = 1$ the dynamical dimensions $d_{\bar{\psi}\psi}(g)$ and $d_{(\bar{\psi}\psi)^2}(g)$ take the form

$$d_{\bar{\psi}\psi}(g) = \frac{g^2}{2\pi^2}\frac{1}{\Big(1 + \xi\,\dfrac{g}{\pi}\Big)\Big(1 - \eta\,\dfrac{g}{\pi}\Big)}, \qquad d_{(\bar{\psi}\psi)^2}(g) = -\frac{g}{2\pi}\frac{1}{1 + \xi\,\dfrac{g}{\pi}}. \tag{8.43}$$

For $\xi = 1$ we get

$$d_{\bar{\psi}\psi}(g) = \frac{g^2}{2\pi^2}\frac{1}{1 + \dfrac{g}{\pi}}, \qquad d_{(\bar{\psi}\psi)^2}(g) = -\frac{g}{2\pi}\frac{1}{1 + \dfrac{g}{\pi}}. \tag{8.44}$$

These are the dynamical dimensions of the Green functions and correlation functions of left–right fermion densities obtained by Klaiber [44] and within the path–integral approach [45]–[52].

We have shown that the dynamical dimensions $d_{\bar{\psi}\psi}(g)$ and $d_{(\bar{\psi}\psi)^2}(g)$ can be made equal. This fixes the parameter $\bar{\eta}$ in terms of the parameter $\bar{\xi}$ and gives the dynamical dimension of the massless Thirring fermion fields to

$$d_{\bar{\psi}\psi}(g) \;=\; d_{(\bar{\psi}\psi)^2}(g) \;=\; d_{\psi}(g) \;=\; -\frac{g}{2\pi}\,\frac{1}{1+\bar{\xi}\,\dfrac{g}{\pi}}. \tag{8.45}$$

As has been pointed out by Jackiw [47], the inequality of dynamical dimensions of fermion fields obtained from different correlation functions is a problem of 1+1–dimensional quantum field theories. The equality of the dynamical dimensions $d_{\bar{\psi}\psi}(g)$ and $d_{(\bar{\psi}\psi)^2}(g)$ is not suppressed by the positive definiteness of the norms of the wave functions of the states related to the components of the vector currents. The positive definiteness of the norms of the wave functions of these states imposes some constraints on the variation of the parameters $\bar{\eta}$ and $\bar{\xi}$, demanding the parameter $1+\bar{\xi}\,g/\pi$ to be negative, i.e. $1+\bar{\xi}\,g/\pi < 0$.

This makes the massless Thirring model renormalisable in the sense that the dependence of correlation functions of Thirring fermion fields on the ultra-violet cut–off can be removed by renormalisation of the wave function of Thirring fermion fields only. We have corroborated this assertion within the standard renormalisation procedure.

The removal of divergences of the massless Thirring model by the renormalisation of the wave function of the Thirring fermion fields has been analysed by Marino and Swieca [68] within the Mandelstam representation of massless Thirring fermion fields [63]. The divergences of the correlation functions were mapped into electrostatic (self–interaction) divergences of an associated system of point–like charges and removed by the renormalisation of the wave function.

¿From the constraint $-g\,(1+\bar{\xi}\,g/\pi) > 0$ there follows that the coupling constant β^2 of the sine–Gordon model is of order $\beta^2 \sim 8\pi$. The behaviour and renormalisability of the sine–Gordon model for the coupling constants $\beta^2 \sim 8\pi$ has been investigated in [16, 24, 32]-[36, 59].

We would like to accentuate that the dynamical dimensions of the massless Thirring fermion fields are *nondistributive* [47], but this property does not influence the renormalisability of the massless Thirring model discussed above.

Appendix A

Calculations to Chapter 1

A.1 Gaussian fluctuations

To solve the eigenvalue problem given by

$$\left(\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + \alpha - \frac{2\alpha}{\cosh^2 \sqrt{\alpha} x^1} \right) \vartheta_{\omega,k}(x) \;=\; \lambda(\omega,k)\, \vartheta_{\omega,k}(x), \tag{A.1}$$

where

$$\lambda(\omega,k) \;=\; -\omega^2 + k^2 + \alpha \tag{A.2}$$

we make the ansatz [69]

$$\vartheta_{\omega,k}(x) \;=\; \mathrm{e}^{-i\omega x^0}\, \vartheta_k(x^1). \tag{A.3}$$

This gives

$$\left(\omega^2 + \frac{\partial^2}{\partial {x^1}^2} - \alpha + 2\alpha\, \mathrm{sech}^2 \sqrt{\alpha} x^1 \right) \vartheta_k(x^1) \;=\; -\, \lambda(\omega,k)\, \vartheta_k(x^1). \tag{A.4}$$

By a change of variables

$$\xi \;=\; \tanh \sqrt{\alpha} x^1, \quad -1 < \xi < 1. \tag{A.5}$$

we arrive at

$$\left\{ \alpha(1-\xi^2) \frac{d}{d\xi} \left[(1-\xi^2) \frac{d}{d\xi} \right] + \lambda(\omega,k) + \omega^2 - \alpha + 2\alpha(1-\xi^2) \right\} \vartheta_k(\xi) \;=\; 0. \tag{A.6}$$

A division by $\alpha(1-\xi^2)$ yields

$$\left\{ \frac{d}{d\xi} \left[(1-\xi^2) \frac{d}{d\xi} \right] + 2 + \frac{\lambda(\omega,k) + \omega^2 - \alpha}{\alpha(1-\xi^2)} \right\} \vartheta_k(\xi) \;=\; 0. \tag{A.7}$$

This is a special case of the Legendre differential equation [70, 71]

$$\left\{ \frac{d}{d\xi} \left[(1-\xi^2) \frac{d}{d\xi} \right] + \lambda' - \frac{m^2}{(1-\xi^2)} \right\} \vartheta_k(\xi) \ = \ 0, \tag{A.8}$$

with

$$\lambda' \ = \ l(l+1) \ = \ 2, \quad m^2 \ = \ 1 - \frac{\lambda(\omega, k) + \omega^2}{\alpha} \ = \ -\frac{k^2}{\alpha} \tag{A.9}$$

Since this is a Sturm–Liouville problem we get an orthogonal set of eigenfunctions with real eigenvalues. In order to get square integrable functions corresponding to bound states, the hypergeometric functions have to become polynomials. These are the associated Legendre polynomials

$$P_l^m(\xi) \ = \ \frac{(1-\xi^2)^{m/2}}{2^l l!} \frac{d^{l+m}}{d\xi^{l+m}} (\xi^2 - 1)^l \tag{A.10}$$

with $l = 1$ the condition for m reads

$$0 \leq m \leq l \ \Rightarrow \ m \ = \ 0, 1. \tag{A.11}$$

This gives the polynomials

$$P_0^0(\xi) \ = \ 1, \quad P_1^0(\xi) \ = \ \xi, \quad P_1^1(\xi) \ = \ (1-\xi^2). \tag{A.12}$$

The constant solution $P_0^0(\xi)$ can be dropped. For the normalisable solutions the eigenvalues read

$$m^2 \ = \ -k_{0,1}^2/\alpha \ = \ 0, \ 1. \tag{A.13}$$

For $k_0 = 0$ we have the eigenfunction

$$P_1^0(x^1) \ = \ N_1^0 \tanh \sqrt{\alpha} x^1 \tag{A.14}$$

and for $k_1 = i\sqrt{\alpha}$ the corresponding eigenfunctions depending on the variable x^1 reads

$$P_1^{i\sqrt{\alpha}}(x^1) \ = \ N_1^{i\sqrt{\alpha}} \mathrm{sech} \sqrt{\alpha} x^1, \tag{A.15}$$

N_1^0 and $N_1^{i\sqrt{\alpha}}$ denote the corresponding norms. The $P_1^{i\sqrt{\alpha}}(x)$ is the so called zero mode which is the partial derivative of the soliton solution (1.12) with respect to x_0

$$\begin{aligned} \frac{d}{dx_0^1} \vartheta_{\mathrm{s}}(x^1) \ &= \ \frac{4}{\beta} \frac{d}{dx_0^1} \arctan \exp \sqrt{\alpha}(x^1 - x_0^1) \\ &= \ \frac{4}{\beta} \mathrm{sech} \sqrt{\alpha}(x^1 - x_0^1) \ \propto \ P_1^{i\sqrt{\alpha}}(x^1 - x_0^1). \end{aligned} \tag{A.16}$$

Therefore, it shifts $\vartheta_{\mathrm{s}}(x^1)$ and is not a real bound state. It corresponds to the translation invariance of the theory. We will denote k_1 in our further calculations as $k_1 \to k_0 = i\sqrt{\alpha}$. We denote the normalised solution (A.15) as

$$\vartheta_b(x^1) \ = \ \sqrt{\frac{\sqrt{\alpha}}{2}} \frac{1}{\cosh(\sqrt{\alpha} x^1)}. \tag{A.17}$$

For the scattering states the solutions read

$$P_l^m(\xi) \;=\; \frac{1}{\Gamma[1-m]}\left[\frac{\xi+1}{\xi-1}\right]^{m/2} F\Big(-l,\, l+1,\, 1-m;\, \frac{1-\xi}{2}\Big), \tag{A.18}$$

(see [71]). This gives with the relation for the eigenvalue k

$$m^2 \;=\; -k^2/\alpha \;\;\Rightarrow\;\; m \;=\; ik/\sqrt{\alpha} \tag{A.19}$$

and with $l=1$ ($l=-2$ gives the same solutions)

$$P_1^{ik/\sqrt{\alpha}}(\xi) \;=\; N_1^{ik/\sqrt{\alpha}}\left[\frac{\xi+1}{\xi-1}\right]^{\frac{ik}{\sqrt{2\alpha}}} F\Big(-1,\, 2,\, 1-ik/\sqrt{\alpha};\, \frac{1-\xi}{2}\Big), \tag{A.20}$$

for the scattering solutions depending on x

$$P_1^{ik/\sqrt{\alpha}}(x^1) \;=\; N_1^{ik/\sqrt{\alpha}}\,\frac{k+i\sqrt{\alpha}\tanh\sqrt{\alpha}x^1}{k+i\sqrt{\alpha}}\,\mathrm{e}^{ikx^1}. \tag{A.21}$$

For the normalised scattering solutions we write

$$\vartheta_k(x^1) \;=\; \frac{1}{\sqrt{2\pi}}\,\frac{k+i\sqrt{\alpha}\tanh\sqrt{\alpha}x^1}{\sqrt{k^2+\alpha}}\,\mathrm{e}^{ikx^1}. \tag{A.22}$$

Since the solution in (A.14) is the limit $k=0$ of (A.21) [70] we have only one discrete mode (A.17).

The set of eigensolutions to the stability operator of the soliton ϑ_{s}

The normalised complete set of solutions (A.3) to the eigenvalue equation (A.1) read

$$\begin{aligned}
\vartheta_{\omega,b}(x) \;&=\; \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\sqrt{\alpha}}{2}}\,\frac{1}{\cosh(\sqrt{\alpha}x^1)}\,\mathrm{e}^{-\,i\omega\, x^0}\\
\vartheta_{\omega,k}(x) \;&=\; \frac{1}{2\pi}\,\frac{k+i\sqrt{\alpha}\,\tanh\sqrt{\alpha}x^1}{\sqrt{k^2+\alpha}}\,\mathrm{e}^{-\,i\omega\, x^0\,+\,ik\, x^1}.
\end{aligned} \tag{A.23}$$

A.1.1 Completeness and orthogonality of the eigensolutions of the stability operator

Since, the underlying differential operator is hermitian the eigensolutions form a complete orthogonal set of solutions normalised to the Dirac delta function. The solutions (A.23) form a complete set

$$\begin{aligned}
&\int_{-\infty}^{+\infty} dk\, d\omega\, \vartheta^*_{\omega,k}(x')\vartheta_{\omega,k}(x)\\
&\qquad +\vartheta^*_{b,\,\omega}(x')\vartheta_{b,\,\omega}(x) \;=\; \delta(x'^1-x^1)\,\delta(x'^0-x^0).
\end{aligned} \tag{A.24}$$

The proof of completeness is easily performed. The integration over ω gives immediately the Dirac delta function. For the k integration we have

$$\begin{aligned}
\delta(x'^1 - x^1) \;=\; & \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \mathrm{e}^{ik(x^1 - x'^1)} \\
& + \sqrt{\alpha}\, \left(\tanh\sqrt{\alpha}x^1 - \tanh\sqrt{\alpha}x'^1\right) \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, \frac{ik}{k^2+\alpha}\, \mathrm{e}^{ik(x^1 - x'^1)} \\
& + \alpha\, \left(\tanh\sqrt{\alpha}x'^1\, \tanh\sqrt{\alpha}x^1 \, - 1\right) \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{1}{k^2+\alpha}\, \mathrm{e}^{ik(x^1 - x'^1)} \\
& + \frac{\sqrt{\alpha}}{2}\, \frac{1}{\cosh(\sqrt{\alpha}x'^1\,)}\, \frac{1}{\cosh\sqrt{\alpha}x^1},
\end{aligned} \tag{A.25}$$

we see, that the last three lines have to vanish. After integration over k and using elementary trigonometric properties we have

$$\begin{aligned}
& \delta(x'^1 - x^1) - \frac{\sqrt{\alpha}}{2} \Big(\tanh\sqrt{\alpha}x^1 - \tanh\sqrt{\alpha}x'^1 - \tanh\sqrt{\alpha}x'^1\, \tanh\sqrt{\alpha}x^1 \, + 1 \Big) \\
& \times \exp\{-\sqrt{\alpha}k\,(x^1 - x'^1)\} \;+\; \frac{\sqrt{\alpha}}{2}\, \frac{1}{\cosh(\sqrt{\alpha}x'^1\,)}\, \frac{1}{\cosh\sqrt{\alpha}x^1} \\
& = \; \delta(x'^1 - x^1).
\end{aligned} \tag{A.26}$$

The scattering solutions are normalised to the Dirac delta function

$$\int_{-\infty}^{+\infty} d^2x\, \vartheta^*_{k',\omega'}(x)\, \vartheta_{k,\omega}(x) \;\;=\;\; \delta(k\,' - k)\, \delta(\omega' - \omega). \tag{A.27}$$

Again the ω integration gives immediately the delta function. For the k integration we obtain

$$\begin{aligned}
\int_{-\infty}^{+\infty} dx^1\, \vartheta^*_{k'}(x^1)\, \vartheta_k(x^1) \;\;&=\;\; \frac{1}{k'^2 - k^2} \left(\vartheta^*_{k'}(x^1) \frac{d}{dx^1} \vartheta_k(x^1) - \vartheta_k(x^1) \frac{d}{dx^1} \vartheta^*_{k'}(x^1) \right)\Big|_{-\infty}^{\infty} \\
&=\;\; \delta(k' - k).
\end{aligned} \tag{A.28}$$

The discrete eigenfunction $\vartheta_{b,\,\omega}(x)$ is normalised to unity

$$\int_{-\infty}^{+\infty} dx^1\, \vartheta^*_b(x^1)\, \vartheta_b(x^1) \;=\; 1. \tag{A.29}$$

Appendix B

Calculations to Chapter 2

B.1 Calculations to Equation (2.38)

The second–order quantum corrections $-i\Delta^{(2)}_F(x;\alpha_r(M^2))$, Eq. (2.38), to the causal two–point Green function $-i\Delta(x;\alpha_r(M^2))$ (2.16) reads

$$
\begin{aligned}
-i\Delta^{(2)}_F(x,0;\alpha_r(M^2)) \;&=\; -\frac{1}{2}\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\mathcal{L}_{\rm int}(z_1)\,\mathcal{L}_{\rm int}(z_2)\Big)|0\rangle_{\rm c}\\
&= -\frac{1}{2}\frac{\alpha_r^2}{4}\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^2(z_2)\vartheta^2(z_1)\Big)|0\rangle_{\rm c}\\
&\quad -\frac{1}{2}\frac{\alpha_r^2(M^2)Z_1}{\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^n\beta^{2n}}{(2n)!}\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^{2n}(z_2)\vartheta^2(z_1)\Big)|0\rangle_{\rm c}\\
&\quad -\frac{1}{2}\frac{\alpha_r^2(M^2)Z_1^2}{\beta^4}\sum_{n_2=1}^{\infty}\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1+n_2}\,\beta^{2(n_1+n_2)}}{(2n_2)!\,(2n_1)!}\\
&\quad \times\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^{2n_2}(z_2)\vartheta^{2n_1}(z_2)\Big)|0\rangle_{\rm c}.
\end{aligned}
\qquad \text{(B.1)}
$$

For the calculations below we introduce the notation $-i\Delta^{(2)}_{F\,j}(x,0;\alpha_r(M^2))$ denoting the three terms in Eq. (B.1), respectively. Hence,

$$
\begin{aligned}
-i\Delta^{(2)}_F(x,0;\alpha_r(M^2)) \;&=\; -i\Delta^{(2)}_{F\,1}(x,0;\alpha_r(M^2))\\
&\quad -i\Delta^{(2)}_{F\,2}(x,0;\alpha_r(M^2)) - i\Delta^{(2)}_{F\,3}(x,0;\alpha_r(M^2)).
\end{aligned}
\qquad \text{(B.2)}
$$

The diagrams generated by the expression (B.1) are characterized by the number of their internal lines between the two vertices at z_1 and z_2. Hence they define two classes. We call them briefly odd–class and even–class.

Before we pass to the evaluation of all possible contractions in (B.1) we should discuss briefly the symmetry factors.

B.1.1 Symmetry factors

The symmetry factor Σ for a diagram with two external legs l_1 and l_2 at x and 0 and two vertices V_{2n_1} and V_{2n_2} with $2n_1$ and $2n_2$ legs λ_1 and λ_2 at z_1 and z_2 we obtain as follows:

There exists an overall symmetry factor $2!$ since interchanging the two vertices does not affect the topology of the diagram, hence $\Sigma \propto 2!$. The contraction of l_1 with λ_1 yields a factor $2n_1$ and the contraction of l_2 with λ_1 gives the factor $(2n_1-1)$, therefore $\Sigma \propto 2!(2n_1)(2n_1-1)$. Now, a diagram with $2k$ contractions between λ_1 and λ_2 gives a factor $(2n_1-2)!/(2n_1-2-2k)!$ at V_{2n_1} and a factor $(2n_2)!/(2n_2-2k)!$ at V_{2n_2}. Since $2k$ contractions cannot be distinguished a factor $1/2k!$ has be taken into account. This gives

$$\Sigma \propto 2!\,(2n_1)(2n_1-1)\,\frac{(2n_1-2)!}{(2n_1-2-2k)!}\,\frac{2n_2!}{(2n_2-2k)!}\,\frac{1}{2k!}. \tag{B.3}$$

Generally, two legs at a vertex with $2n$ free legs yields a factor $2n!/(2n-2)!2!$ if they are contracted to a loop. A factor $1/m!$ has to be taken into account, if $2m$ legs are contracted to form loops.

Applying this consideration to the remaining $(2n_1-2-2k)$ legs at V_{2n_1} and to the remaining $(2n_2-2k)$ legs at V_{2n_2} we obtain finally the symmetry factor

$$\begin{aligned}\Sigma &= 2!\,\frac{2n_1!}{(2n_1-2-2k)!}\,\frac{2n_2!}{(2n_2-2k)!}\,\frac{1}{2k!}\\ &\times \frac{(2n_1-2-2k)!}{2^{n_1-1-k}}\,\frac{1}{(n_1-1-k)!}\,\frac{(2n_2-2k)!}{2^{n_2-k}}\,\frac{1}{(n_2-k)!}\\ &= 2!\,2n_1!\,2n_2!\,\frac{1}{2k!}\,\frac{1}{(n_1-1-k)!\,2^{n_1-1-k}}\,\frac{1}{(n_2-k)!\,2^{n_2-k}}.\end{aligned} \tag{B.4}$$

This symmetry factor corresponds to diagrams belonging to the even–class with $2k$ internal lines.

For odd–class diagrams with $2k+1$ internal lines we proceed equivalently and obtain the symmetry factor

$$\begin{aligned}\Sigma &= 2!\,\frac{2n_1!}{(2n_1-1-(2k+1))!}\,\frac{2n_2!}{(2n_2-1-(2k+1))!}\,\frac{1}{(2k+1)!}\\ &\times \frac{(2n_1-1-(2k+1))!}{2^{n_1-1-k}}\,\frac{1}{(n_1-1-k)!}\,\frac{(2n_2-1-(2k+1))!}{2^{n_2-1-k}}\,\frac{1}{(n_2-1-k)!}\\ &= 2!\,2n_1!\,2n_2!\,\frac{1}{(2k+1)!}\,\frac{1}{(n_1-1-k)!\,2^{n_1-1-k}}\,\frac{1}{(n_2-1-k)!\,2^{n_2-1-k}}.\end{aligned} \tag{B.5}$$

B.1.2 Calculations to Equation (B.1)

Performing the contractions in the first term $-i\Delta^{(2)}_{F\,1}(x,0;\alpha_r(M^2))$ in Eq. (B.1) we get the contribution

$$-\alpha_r^2(M^2)\iint d^2z_1 d^2z_2\,\{-i\Delta_F(x,z_2;\alpha_r(M^2))\}$$

$$\times \quad \{-i\Delta_F(z_2, z_1; \alpha_r(M^2)\} \, \{-i\Delta_F(z_1, 0; \alpha_r(M^2))\}. \qquad \text{(B.6)}$$

This diagrams belongs to the odd–class. It has to be neglected in the summation (B.13) due to Eq. (2.39).

For the second term $-i\Delta^{(2)}_{F\,2}(x, 0; \alpha_r(M^2))$ in Eq. (B.1)) we have two topological different diagrams. The first one, belonging to the odd–class gives the contraction

$$\begin{aligned}
&2\,\alpha_r^2(M^2)\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\frac{\beta^2}{8\pi}} \iint d^2z_1 d^2z_2\,\{-i\Delta_F(x, z_2; \alpha_r(M^2))\} \\
&\qquad \times \{-i\Delta_F(z_2, z_1; \alpha_r(M^2))\}\,\{-i\Delta_F(z_1, 0; \alpha_r(M^2))\}. \qquad \text{(B.7)}
\end{aligned}$$

This contribution we drop in the summation (B.13) due to Eq. (2.39) too. The next diagram generated by the second term of Eq. (B.1)) contributes

$$\begin{aligned}
&-\frac{1}{2}\frac{\alpha_r^2(M^2)Z_1}{\beta^2}\sum_{n=2}^{\infty}\frac{(-1)^n\beta^{2n}}{2^{n-2}(n-2)!}\frac{2!}{2!}\,\{-i\Delta_F(0; \alpha_r(M^2))\}^{n-2} \\
&\quad \times \iint d^2z_1 d^2z_2\,\{i\Delta_F(x, z_2; \alpha_r(M^2))\} \\
&\quad \times \{-i^2\Delta_F(z_2, z_1; \alpha_r(M^2))\}^2\,\{-i\Delta_F(z_1, 0; \alpha_r(M^2))\} \\
&\quad = -\frac{1}{2}\,\alpha_r^2(M^2)\beta^2\,Z_1\,\exp\Big\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0; \alpha_r(M^2))\}\Big\} \\
&\quad \times \iint d^2z_1 d^2z_2\,\{-i\Delta_F(x, z_2; \alpha_r(M^2))\} \\
&\quad \times \{-i\Delta_F(z_2, z_1; \alpha_r(M^2))\}^2\,\{-i\Delta_F(z_1, 0; \alpha_r(M^2))\} \\
&\quad = -\frac{1}{2}\,\alpha_r^2(M^2)\beta^2\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\frac{\beta^2}{8\pi}} \iint d^2z_1 d^2z_2\,\{-i\Delta_F(x, z_2; \alpha_r(M^2))\} \\
&\quad \times \{-i\Delta_F(z_2, z_1; \alpha_r(M^2))\}^2\,\{-i\Delta_F(z_1, 0; \alpha_r(M^2))\}. \qquad \text{(B.8)}
\end{aligned}$$

This term we have to take into account. For the proof of our assertion in Section 2.3.2 we expand the factor $(\alpha_r/M^2)^{\beta^2/8\pi}$ to order $\mathcal{O}((\alpha_r/M^2)^{\beta^2/4\pi})$, it reads $(\alpha_r/M^2)^{\beta^2/8\pi} = (\alpha_r/M^2)^{\beta^2/4\pi} - \cdots$. Thus

$$\begin{aligned}
&-\frac{1}{2}\,\alpha_r^2(M^2)\beta^2\Big(\frac{\alpha_r}{M^2}\Big)^{\frac{\beta^2}{4\pi}} \iint d^2z_1 d^2z_2\,\{-i\Delta_F(x, z_2; \alpha_r(M^2))\} \\
&\qquad \times \{-i\Delta_F(z_2, z_1; \alpha_r(M^2))\}^2\,\{-i\Delta_F(z_1, 0; \alpha_r(M^2))\} - \cdots. \qquad \text{(B.9)}
\end{aligned}$$

Now, we arrive at the more involved third term $-i\Delta^{(2)}_{F\,3}(x, 0; \alpha_r(M^2))$ of Eq. (B.1). The first representation of the odd–class that with one internal line gives the contraction

$$\begin{aligned}
&-\alpha_r^2(M^2)\Big(\frac{\alpha_r(M^2)}{M^2}\Big)^{\frac{\beta^2}{4\pi}} \iint d^2z_1 d^2z_2\,\{-i\Delta_F(x, z_2; \alpha_r(M^2))\} \\
&\qquad \times \quad \{-i\Delta_F(z_2, z_1; \alpha_r(M^2))\}\,\{-i\Delta_F(z_1, 0; \alpha_r(M^2))\}. \qquad \text{(B.10)}
\end{aligned}$$

This contribution is already taken into account in Eq. (2.39) and is therefore neglected in the summation (B.13). The contractions of any odd–class diagram with $2k+1$ internal lines reads

$$\begin{aligned}
&-\frac{\alpha_r^2(M^2)}{\beta^4}\,Z_1^2\,\exp\left\{-\frac{\beta^2}{2}\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}^2\\
&\qquad\times\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_2;\alpha_r(M^2))\}\\
&\qquad\times\frac{\{-i\beta^2\Delta_F(z_2,z_1;\alpha_r(M^2))\}^{2k+1}}{(2k+1)!}\,\{-i\Delta_F(z_1,0;\alpha_r(M^2))\},
\end{aligned}\tag{B.11}$$

while the contractions of any even–class diagram with $2k>0$ internal lines becomes

$$\begin{aligned}
&+\frac{\alpha_r^2(M^2)}{\beta^4}\,Z_1^2\left(\frac{\alpha_r^2(M^2)}{\Lambda^2}\right)^{\frac{\beta^2}{4\pi}}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_2;\alpha_r(M^2))\}\\
&\qquad\times\frac{\{-i\beta^2\Delta_F(z_2,z_1;\alpha_r(M^2))\}^{2k}}{(2k)!}\,\{-i\Delta_F(z_1,0;\alpha_r(M^2))\}.
\end{aligned}\tag{B.12}$$

The summation of all odd– and even–class diagrams given in Eqs. (B.9), (B.11) (for $k>1$) and (B.12) gives

$$\begin{aligned}
-i\Delta^{(2)}_F(x,0;\alpha_r(M^2)) \;=\;& \left[\alpha_r(M^2)Z_1\,\exp\left\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]^2\\
&\times\frac{1}{\beta^2}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_1;\alpha_r(M^2))\}\,\{-i\Delta_F(z_1,0;\alpha_r(M^2))\}\\
&\times\Big(\cosh\{-\beta^2\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}-1-\frac{1}{2}\,\beta^4\,\{-\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}^2\Big)\\
&-\left[\alpha_r(M^2)Z_1\,\exp\left\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0;\alpha_r(M^2))\}\right\}\right]^2\\
&\times\frac{1}{\beta^2}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_1;\alpha_r(M^2))\}\,\{-i\Delta_F(z_2,0;\alpha_r(M^2))\}\\
&\times\Big(\sinh\{-\,\beta^2\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}-\beta^2\,\{-\,i\Delta_F(z_1,z_2;\alpha_r(M^2))\}\Big).
\end{aligned}\tag{B.13}$$

Appendix C

Calculations to Chapter 3

C.1 Calculations to Equation (3.8)

The second–order quantum corrections $-i\Delta^{(2)}_F(x;\tilde{m}_r(M^2))$ to the causal two–point Green function $-i\Delta(x;\alpha_r(M^2))$, Eq. (3.8) of the MSG model reads

$$-i\Delta^{(2)}_F(x,0;\alpha_r(M^2)) \;=\; -\frac{1}{2}\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\mathcal{L}_{\rm int}(z_1)\,\mathcal{L}_{\rm int}(z_2)\Big)|0\rangle_{\rm c}, \tag{C.1}$$

where the interaction Lagrangian is given in Eq. (3.2). However, rewriting the interaction Lagrangian to the form

$$\mathcal{L}_{\rm int}(x) \;=\; \frac{1}{2}\,\delta M\vartheta^2(x) + Z_1\alpha_r(M^2)\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\beta^{2(n-1)}\vartheta^{2n}(x), \tag{C.2}$$

with the definition $\delta M = -\delta m_r^2(M^2) + \alpha_r(M^2)$, Eq. (C.1) becomes

$$\begin{aligned}
-i\Delta^{(2)}_F(x,0;\alpha_r(M^2)) \;=&\\
=&\; -\frac{1}{2}\frac{\delta M^2}{4}\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^2(z_2)\vartheta^2(z_1)\Big)|0\rangle_{\rm c}\\
&-\frac{1}{2}\frac{\delta M\alpha_r(M^2)Z_1}{\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^n\beta^{2n}}{(2n)!}\\
&\times\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^{2n}(z_2)\vartheta^2(z_1)\Big)|0\rangle_{\rm c}\\
&-\frac{1}{2}\frac{\alpha_r^2(M^2)Z_1^2}{\beta^4}\sum_{n_2=1}^{\infty}\sum_{n_1=1}^{\infty}\frac{(-1)^{n_1+n_2}\,\beta^{2(n_1+n_2)}}{(2n_2)!\,(2n_1)!}\\
&\times\iint d^2z_1 d^2z_2\;\langle 0|\mathrm{T}\Big(\vartheta(x)\vartheta(0)\vartheta^{2n_2}(z_2)\vartheta^{2n_1}(z_2)\Big)|0\rangle_{\rm c}.
\end{aligned} \tag{C.3}$$

The first term in Eq. (C.3) corresponding to (B.7) has to be dropped due to Eq. (3.11). Because this term was already taken into account when we derive the effective two–point function (3.11).

The relevant contribution of the second term in Eq. (C.1) corresponding to the (B.8) reads

$$\begin{aligned}
&-\frac{1}{2}\,\delta M\alpha_r(M^2)\beta^2\Big(\frac{\tilde{m}_r^2(M^2)}{M^2}\Big)^{\frac{\beta^2}{8\pi}}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_2;\tilde{m}_r^2(M^2))\}\\
&\qquad\times\{-i\Delta_F(z_2,z_1;\tilde{m}_r^2(M^2))\}^2\,\{-i\Delta_F(z_1,0;\tilde{m}_r^2(M^2))\}\\
&\qquad= -\frac{1}{2}\Big\{-\delta m_r^2(M^2)+\alpha_r(M^2)\Big\}\,\alpha_r(M^2)\beta^2\Big(\frac{\tilde{m}_r^2(M^2)}{M^2}\Big)^{\frac{\beta^2}{8\pi}}\\
&\qquad\times\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_2;\tilde{m}_r^2(M^2))\}\\
&\qquad\times\{-i\Delta_F(z_2,z_1;\alpha_r(M^2))\}^2\,\{-i\Delta_F(z_1,0;\tilde{m}_r^2(M^2))\}.
\end{aligned}\tag{C.4}$$

Since this expression is a well–defined one the counter–term can be set zero, as done in Eq. (3.13). Hence, this contribution is of order $\mathcal{O}((\tilde{m}_r^2/M^2)^{\beta^2/4\pi})$

$$\begin{aligned}
&-\frac{1}{2}\,\alpha_r^2(M^2)\beta^2\Big(\frac{\tilde{m}_r^2(M^2)}{M^2}\Big)^{\frac{\beta^2}{4\pi}}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_2;\tilde{m}_r^2(M^2))\}\\
&\qquad\times\{-i\Delta_F(z_2,z_1;\tilde{m}_r^2(M^2))\}^2\,\{-i\Delta_F(z_1,0;\tilde{m}_r^2(M^2))\}-\cdots.
\end{aligned}\tag{C.5}$$

Continuing for the third term in Eq. (C.1) completely in the way as for the third term of Eq. (B.1) the summation of all odd– and even–class diagrams yields

$$\begin{aligned}
-i\Delta_F^{(2)}(x,0;\tilde{m}_r(M^2)) \;=\; &\Bigg[\alpha_r(M^2)Z_1\,\exp\Big\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0;\tilde{m}_r(M^2))\}\Big\}\Bigg]^2\\
&\times\frac{1}{\beta^2}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_1;\tilde{m}_r(M^2))\}\,\{-i\Delta_F(z_1,0;\tilde{m}_r(M^2))\}\\
&\times\Big(\cosh\{-\beta^2\,i\Delta_F(z_1,z_2;\tilde{m}_r(M^2))\}-1-\frac{1}{2}\,\beta^4\,\{-\,i\Delta_F(z_1,z_2;\tilde{m}_r(M^2))\}^2\Big)\\
&-\Bigg[\alpha_r(M^2)Z_1\,\exp\Big\{-\frac{\beta^2}{2}\,\{-i\Delta_F(0;\tilde{m}_r(M^2))\}\Big\}\Bigg]^2\\
&\times\frac{1}{\beta^2}\iint d^2z_1d^2z_2\,\{-i\Delta_F(x,z_1;\tilde{m}_r(M^2))\}\,\{-i\Delta_F(z_2,0;\tilde{m}_r(M^2))\}\\
&\times\Big(\sinh\{-\beta^2\,i\Delta_F(z_1,z_2;\tilde{m}_r(M^2))\}-\beta^2\,\{-\,i\Delta_F(z_1,z_2;\tilde{m}_r(M^2))\}\Big).
\end{aligned}\tag{C.6}$$

C.2 Calculations to Section 3.4

In this section we derive the partition function Z_m (3.41) for fluctuations of the MSG field.

$$Z_m[J] \;=\; \exp\Big\{-\,\frac{i}{2}\,m_0^2\int d^2x\,\frac{\delta}{i\delta J(x)}\,\frac{\delta}{i\delta J(x)}\Big\}\,Z[J], \tag{C.7}$$

we derive first $Z[J]$, the partition function of the SG model by following [6]. It reads

$$\begin{aligned} Z[J] \;&=\; \mathrm{e}^{-i\alpha_0/\beta^2}\int\mathcal{D}\vartheta\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{i^n}{n!}\left(\frac{\alpha_0}{\beta^2}\right)^n\frac{1}{2^n}\,\frac{n!}{(n-p)!\,p!}\\ &\times\left[\prod_{k=0}^{p}\prod_{l=0}^{n-p}\int d^2x_k\,d^2y_l\right]\exp\Big\{i\,\beta\sum_{k=1}^{p}\vartheta(x_k)-i\beta\sum_{l=1}^{n-p}\vartheta(y_l)\Big\}\\ &\times\,\exp\Big\{i\int d^2x\,\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x)-\frac{1}{2}\,\mu^2\,\vartheta^2(x)+\vartheta(x)J(x)\Big\}, \end{aligned} \tag{C.8}$$

where μ is an infrared regulator. The constant $\exp\{-i\alpha_0/beta^2\}$ we put into the measure. The source $J(x)$ obeys the constraint [6]

$$\int d^2x\,J(x) \;=\; 0. \tag{C.9}$$

We perform the path integration over the field ϑ by making a quadratic extension, which can be represented in the following symbolic form

$$\begin{aligned} &-\mathbb{D}^2\vartheta^2+2\vartheta J+2\beta\sum_{k=1}^{p}\vartheta_k-2\beta\sum_{l=1}^{n-p}\vartheta_l\\ &=\;\Big(i\mathbb{D}\vartheta+(i\mathbb{D})^{-1}[J+\beta\sum_{k=1}^{p}1_k-\beta\sum_{l=1}^{n-p}1_l)]\Big)^2\\ &\quad-(-\Delta_F)\Big[J+\beta\sum_{k=1}^{p}1_k-\beta\sum_{l=1}^{n-p}1_l\Big]^2, \end{aligned} \tag{C.10}$$

where the notation 1_k means the Dirac delta function $1_k = \delta(x_k)$. The causal two–point function are [6]

$$\Delta(x-y;\mu) \;=\; -\,\frac{i}{4\pi}\,\ln\left(-\mu^2(x-y)^2+i0\right), \quad \Delta(0;\mu) \;=\; \frac{i}{4\pi}\,\ln\left(\frac{\Lambda^2}{\mu^2}\right), \tag{C.11}$$

where Λ is an ultra–violet cut–off in two–dimensional Euclidean space. Path integration over ϑ gives

$$Z[J] \;=\; \sum_{n=0}^{\infty}\sum_{p=0}^{n}\left(\frac{\alpha_0}{\beta^2}\right)^n\frac{1}{2^n}\,\frac{i^n}{(n-p)!\,p!}\left[\prod_{k=0}^{p}\prod_{l=0}^{n-p}\int d^2x_k\,d^2y_l\right]$$

$$
\begin{aligned}
&\times\ \exp\Big\{\frac{i}{2}\int d^2x\,d^2y\,J(x)\Delta(x-y;\mu)J(y)\\
&+i\beta\int d^2y\Big(\sum_{k=1}^{p}\Delta(x_k-y;\mu)-\sum_{l=1}^{n-p}\Delta(y_l-y;\mu)\Big)J(y)\\
&-i\beta^2\sum_{k=1}^{p}\sum_{l=1}^{n-p}\Delta(x_k-y_l;\mu)\\
&+i\frac{\beta^2}{2}\Big(\sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\Delta(x_{k_1}-x_{k_2};\mu)+\sum_{l_1=1}^{n-p}\sum_{l_2=1}^{n-p}\Delta(y_{l_1}-y_{l_2};\mu)\Big)\Big\}.
\end{aligned}
\tag{C.12}
$$

It is seen that the dependence on the infrared cut–off vanishes due to the constraint (C.9). In the terms, which do not contain the external sources

$$
\begin{aligned}
&\exp\Big\{-i\beta^2\sum_{k=1}^{p}\sum_{l=1}^{n-p}\Delta(x_k-y_l;\mu)\\
&+\frac{1}{2}i\beta^2\Big(\sum_{k_1=1}^{p}\sum_{k_2=1}^{p}\Delta(x_{k_1}-x_{k_2};\mu)+\sum_{l_1=1}^{n-p}\sum_{l_2=1}^{n-p}\Delta(y_{l_1}-y_{l_2};\mu)\Big)\Big\},
\end{aligned}
\tag{C.13}
$$

the dependence on the infrared cut–off vanishes due to the relation

$$
\begin{aligned}
&\frac{i}{4\pi}\,i\beta^2p(p-n)\,\ln(-\mu^2)-\frac{1}{2}\frac{i}{4\pi}\,i\beta^2p^2\,\ln(-\mu^2)-\frac{1}{2}\frac{i}{4\pi}\,i\beta^2(n-p)^2\,\ln(-\mu^2)\\
&=-\frac{\beta^2}{8\pi}\,\big(2p^2-2pn-p^2-n^2+2pn-p^2\big)\ln(-\mu^2)\\
&=-\frac{\beta^2}{8\pi}\,\big(-n^2+2pn\big)\ln(-\mu^2),
\end{aligned}
\tag{C.14}
$$

for $n=2p$ only [6]. We rewrite the partition function $Z_m[J]$, Eq. (C.7), to

$$
Z_m[J]\;=\;\int\mathcal{D}u\exp\Big\{-\frac{i}{2}\int d^2x\,u^2(x)\;+\;im_0\int d^2xu(x)\,\frac{\delta}{i\delta J(x)}\Big\}\,Z[J].
\tag{C.15}
$$

This gives explicitly

$$
\begin{aligned}
Z_m[J]\;&=\;\mathrm{e}^{-i\alpha_0/\beta^2}\sum_{n=0}^{\infty}\sum_{p=0}^{n}\Big(\frac{\alpha_0}{\beta^2}\Big)^n\frac{1}{2^n}\,\frac{i^n}{(n-p)!\,p!}\\
&\times\Big[\prod_{k=0}^{p}\prod_{l=0}^{n-p}\int d^2x_k\,d^2y_l\Big]\int\mathcal{D}u\exp\Big\{-\frac{i}{2}\int d^2xu^2(x)\Big\}\\
&\times\ \exp\Big\{\frac{i}{2}\int d^2x\,d^2y\,\left(J(x)+im_0u(x)\right)\Delta(x-y;\mu)\left(J(y)+im_0u(y)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ i\beta \int d^2y \Big(\sum_{k=1}^{p} \Delta(x_k - y;\mu) - \sum_{l=1}^{n-p} \Delta(y_l - y;\mu) \Big) \left(J(y) + im_0 u(y)\right) \\
&- i\beta^2 \sum_{k=1}^{p} \sum_{l=1}^{n-p} \Delta(x_k - y_l;\mu) \\
&+ i\frac{\beta^2}{2} \Big(\sum_{k_1=1}^{p} \sum_{k_2=1}^{p} \Delta(x_{k_1} - x_{k_2};\mu) + \sum_{l_1=1}^{n-p} \sum_{l_2=1}^{n-p} \Delta(x_{l_1} - x_{l_2};\mu) \Big) \Big\}.
\end{aligned}
\tag{C.16}
$$

We make a quadratic extension of the terms depending on u

$$
-\frac{i}{2}\,u^2 + \frac{i}{2}\,(J + im_0 u)\,\Delta\,(J + im_0 u) + i\beta\Big(\sum_k \Delta - \sum_l \Delta\Big)\,(J + im_0 u)\,, \tag{C.17}
$$

in symbolic form

$$
\begin{aligned}
&-\frac{i}{2}\left(\mathbb{D}'u - \mathbb{D}'^{-1}\Big[im_0\Delta J + im_0\beta\Big(\sum_k \Delta - \sum_l \Delta\Big)\Big]\right)^2 \\
&\qquad\qquad + \frac{i}{2}\,\mathbb{D}'^{-2}\Big[im_0\Delta J + im_0\beta\Big(\sum_k \Delta - \sum_l \Delta\Big)\Big]^2,
\end{aligned}
\tag{C.18}
$$

where $\mathbb{D}'^2 = 1 + m_0^2\Delta$. Path integration over u yields

$$
\begin{aligned}
\exp\Big\{ &+ \frac{i}{2} \int d^2x d^2y \Big[im_0 \int d^2z_1 \Delta(x - z_1;\mu) J(z_1) \\
&+ im_0\beta\Big(\sum_{k_1}^{p} \Delta(x_{k_1} - x;\mu) - \sum_{l_1}^{n-p} \Delta(y_{l_1} - x;\mu) \Big)\Big] \\
&\times \Big[\delta(x - y) + m_0^2\Delta(x - y;\mu)\Big]^{-1} \\
&\times \Big[im_0 \int d^2z_2 \Delta(y - z_2;\mu) J(z_2) \\
&+ im_0\beta\Big(\sum_{k_2}^{p} \Delta(x_{k_2} - y;\mu) - \sum_{l_2}^{n-p} \Delta(y_{l_2} - y;\mu) \Big)\Big],
\end{aligned}
\tag{C.19}
$$

where we have written only those factors of Eq. (C.16), which are integrated.

The inverse of $F(x-y)$, where

$$
F(x - y) \;=\; \delta(x - y) + m_0^2\Delta(x - y;\mu) \tag{C.20}
$$

we find as

$$
F^{-1}(y - z) \;=\; \delta(x - y) - m_0^2\Delta(x - y;(m_0^2 + \mu^2)^{1/2}). \tag{C.21}
$$

The relation holds

$$\int d^2y\, F(x-y)\, F^{-1}(y-z) \;=\; \delta(x-z). \tag{C.22}$$

This gives the expression (C.19)

$$\begin{aligned}
\exp\Big\{ + \;\frac{i}{2} \int d^2x d^2y \Big[& im_0 \int d^2z_1 \Delta(x-z_1;\mu) J(z_1) \\
& + im_0\beta\Big(\sum_{k_1}^{p} \Delta(x_{k_1}-x;\mu) - \sum_{l_1}^{n-p} \Delta(y_{l_1}-x;\mu)\Big)\Big] \\
& \times \Big[\delta(x-y) - m_0^2\Delta(x-y;(m_0^2+\mu^2)^{1/2})\Big] \\
& \times \Big[im_0 \int d^2z_2 \Delta(y-z_2;\mu) J(z_2) \\
& + im_0\beta\Big(\sum_{k_2}^{p} \Delta(x_{k_2}-y;\mu) - \sum_{l_2}^{n-p} \Delta(y_{l_2}-y;\mu)\Big)\Big].
\end{aligned} \tag{C.23}$$

Now let us discuss the dependence of this factor on the infrared cut–off μ. Inserting the two–point Green functions (C.11) yields

$$\begin{aligned}
\exp\Big\{ + \;\frac{i}{2} \int d^2x d^2y \Big[& \frac{m_0}{4\pi} \int d^2z_1 \ln\big(-\mu^2(x-z_1)^2 + i0\big)\, J(z_1) \\
& + \frac{m_0}{4\pi}\beta\Big(\sum_{k_1}^{p} \ln\big(-\mu^2(x_{k_1}-x)^2 + i0\big) - \sum_{l_1}^{n-p} \ln\big(-\mu^2(y_{l_1}-x)^2 + i0\big)\Big)\Big] \\
& \times \Big[\delta(x-y) - \frac{im_0^2}{4\pi} \ln\big(-\mu^2(x-y)^2 + i0\big)\Big]^{-1} \\
& \times \Big[\frac{m_0}{4\pi} \int d^2z_2 \ln\big(-\mu^2(y-z_2)^2 + i0\big)\, J(z_2) \\
& + \frac{m_0}{4\pi}\beta\Big(\sum_{k_2}^{p} \ln\big(-\mu^2(x_{k_2}-y)^2 + i0\big) - \sum_{l_2}^{n-p} \ln\big(-\mu^2(y_{l_2}-y)^2 + i0\big)\Big)\Big],
\end{aligned} \tag{C.24}$$

and introducing the renormalisation scale M [6], the infrared cut–off dependence reads explicitly

$$\begin{aligned}
\exp\Big\{ + \;\frac{i}{2} \int d^2x d^2y \Big[& \frac{m_0}{4\pi} \int d^2z_1 \Big\{ \ln\big(-M^2(x-z_1)^2 + i0\big) + \ln\Big(\frac{\mu^2}{M^2}\Big)\Big\} J(z_1) \\
& + \frac{m_0}{4\pi}\beta \,\ln\Big(\big(-M^2\big)^{2p-n} \prod_{k_1}^{p}\prod_{l_1}^{n-p} \frac{(x_{k_1}-x)^2 + i0}{(y_{l_1}-x)^2 + i0}\Big) \\
& + \frac{m_0}{4\pi}\beta(2p-n)\ln\Big(\frac{\mu^2}{M^2}\Big)\Big] \\
\times \Big[\delta(x-y) & - m_0^2\Delta(x-y;(m_0^2+\mu^2)^{1/2})\Big]
\end{aligned}$$

$$
\begin{aligned}
&\times \Big[\frac{m_0}{4\pi} \int d^2 z_2 \Big\{ \ln\big(-M^2 (y - z_2)^2 + i0\big) + \ln\Big(\frac{\mu^2}{M^2}\Big)\Big\} J(z_2) \\
&+ \frac{m_0}{4\pi}\beta \ln\Big(\big(-M^2\big)^{2p-n} \prod_{k_2}^{p} \prod_{l_2}^{n-p} \frac{(x_{k_1} - x)^2 + i0}{(y_{l_1} - x)^2 + i0}\Big) \\
&+ \frac{m_0}{4\pi}\beta (2p - n) \ln\Big(\frac{\mu^2}{M^2}\Big)\Big].
\end{aligned} \tag{C.25}
$$

We see by taking the constraint (C.9) and taking relation $p = n/2$ (C.14) [6] into account that the dependence on the infrared cut–off vanishes. Hence the mass term m_0 does not violate the renormalisability of the MSG model.

Performing the infrared limit $\mu \to 0$ we get

$$
\begin{aligned}
\exp\Big\{ &+ \frac{i}{2} \int d^2 x d^2 y \Big[\frac{m_0}{4\pi} \int d^2 z_1 \ \ln\big(-M^2 (x - z_1)^2 + i0\big) J(z_1) \\
&+ \frac{m_0}{4\pi}\beta \ln\Big(\prod_{k_1}^{p} \prod_{l_1}^{n-p} \frac{(x_{k_1} - x)^2 + i0}{(y_{l_1} - x)^2 + i0}\Big)\Big] \\
&\times \Big[\delta(x - y) - m_0^2 \Delta(x - y; m_0)\Big] \\
&\times \Big[\frac{m_0}{4\pi} \int d^2 z_2 \ln\big(-M^2 (y - z_2)^2 + i0\big) J(z_2) \\
&+ \frac{m_0}{4\pi}\beta \ln\Big(\prod_{k_2}^{p} \prod_{l_2}^{n-p} \frac{(x_{k_1} - x)^2 + i0}{(y_{l_1} - x)^2 + i0}\Big)\Big].
\end{aligned} \tag{C.26}
$$

Finally, by inserting (C.26) into (C.16) we obtain for the renormalised partition function $Z_m[J]$ of the MSG

$$
\begin{aligned}
Z[J] \ = \ & \sum_{n=0}^{\infty} \Big(\frac{i}{n!} \frac{\alpha_r(M^2)}{2\beta^2}\Big)^{2n} \Big[\prod_{k=1}^{n} \int d^2 x_k \, d^2 y_k\Big] \\
& \times \ \exp\Big\{\frac{1}{8\pi} \int d^2 x \, d^2 y \, J(x) \ln\big(-M^2 (x - y)^2 + i0\big) J(y) \\
& + \frac{\beta}{4\pi} \int d^2 y \sum_{k=1}^{n} \ln\Big(\frac{(x_k - y)^2 + i0}{(y_k - y)^2 + i0}\Big) J(y) \\
& + \frac{\beta^2}{4\pi} \sum_{j<k}^{n} \Big(\ln\big(-M^2 (x_k - x_j)^2 + i0\big) + \ \ln\big(-M^2 (y_k - y_j)^2 + i0\big) \Big) \\
& - \frac{\beta^2}{4\pi} \sum_{k=1}^{n} \sum_{l=1}^{n} \ln\big(-M^2 (x_k - y_k)^2 + i0\big) \Big\} \\
& \times \ \exp\Big\{ + \frac{i}{2} \int d^2 x d^2 y \Big[\frac{m_0}{4\pi} \int d^2 z_1 \ln\big(-M^2 (x - z_1)^2 + i0\big) J(z_1) \\
& + \frac{m_0}{4\pi}\beta \sum_{k}^{p} \ln\Big(\frac{(x_k - x)^2 + i0}{(y_k - x)^2 + i0}\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\times \Big[\delta(x-y) - m_0^2 \Delta(x-y; m_0)\Big] \\
&\times \Big[\frac{m_0}{4\pi}\int d^2 z_2 \ln\big(-M^2 (y-z_2)^2 + i0\big)\, J(z_2) \\
&+ \frac{m_0}{4\pi}\beta \ln\left(\frac{(x_k-y)^2 + i0}{(y_k-y)^2 + i0}\right)\Big].
\end{aligned}
\tag{C.27}
$$

Appendix D

Calculations to Chapter 5

D.1 Calculations to Section 5.1

To prove that R_0 in Eq. (5.16) is indeed the Green function to H_0, i.e. is

$$\left[\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + m^2\right] R_0(x_2, t_2|x_1, t_1) \;=\; \delta(x_1^0 - x_2^0)\,\delta(x_1^1 - x_2^1) \tag{D.1}$$

we integrate Eq. (D.1) over x_2^0 in the interval $[x_1^0 - \epsilon, x_1^0 + \epsilon]$ with $\epsilon \to 0$

$$\partial_{x_2^0} R_0(x_2|x_1)\,\Big|_{x_1^0-\epsilon}^{x_1^0+\epsilon} + \,2\epsilon(k^2 + m^2)\,R_0(x_2|x_1) \;=\; \delta(x_2^1 - x_1^1), \tag{D.2}$$

where we have used

$$-\frac{\partial^2}{\partial {x^1}^2}\,R_0(x_2|x_1) \;=\; k^2\,R_0(x_2|x_1). \tag{D.3}$$

Since the second term on the l.h.s. of Eq. (D.2) goes to zero for $\epsilon \to 0$ we have

$$\begin{aligned}
\partial_{x_2^0} R_0(x_2|x_1)\,\Big|_{x_1^0-\epsilon}^{x_1^0+\epsilon} \;&=\; -\int \frac{dk}{2\pi}\, i\omega_k\, \psi_{0k}^-(x_2^1, x_1^0)\,\psi_{0k}^+(x_1^1, x_1^0)/W_{0k} \\
&\quad -\int \frac{dk}{2\pi}\, i\omega_k\, \psi_{0k}^+(x_1^1, x_1^0)\,\psi_{0k}^-(x_2^1, x_1^0)/W_{0k} \\
&=\; \frac{1}{2}\int \frac{dk}{2\pi}\, \mathrm{e}^{ik(x_2^1 - x_1^1)} + \frac{1}{2}\int \frac{dk}{2\pi}\, \mathrm{e}^{-ik(x_2^1 - x_1^1)} \;=\; \delta(x_2^1 - x_1^1).
\end{aligned} \tag{D.4}$$

This proves Eq. (D.1).

The proof that R in Eq. (5.26) is indeed the Green function to H runs equivalently. Integrating over the basic relation between R and H

$$\left[\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + m^2 - \frac{2\,m^2}{\cosh^2 mx^1}\right] R(x_2|x_1) \;=\; \delta(x_1^0 - x_2^0)\,\delta(x_1^1 - x_2^1), \tag{D.5}$$

over x_2^0 in the infinitesimal interval $[x_1^0-\epsilon, x_1^0+\epsilon]$ with $\epsilon \to 0$. This means

$$\partial_{t_2} R(x_2,t_2|x_1,t_1)\Big|_{t_1-\epsilon}^{t_1+\epsilon} + 2\epsilon\Big(k^2+m^2\Big)\, R(x_2,t_1|x_1,t_1) \;=\; \delta(x_2-x_1), \qquad \text{(D.6)}$$

where we have used Eq. (5.25). Neglecting on the l.h.s. the second term proportional to ϵ and inserting the Wronskians[1] (5.27) we have explicitly

$$\begin{aligned}
&\partial_{x_2^0} R(x_2|x_1)\Big|_{x_1^0-\epsilon}^{x_1^0+\epsilon} = \\
&\quad= -\epsilon\,\psi_0^-(x_2^1,x_1^0)\,\psi_0^+(x_1^1,x_1^0)/W_0 - \epsilon\,\psi_0^+(x_2^1,x_1^0)\,\psi_0^-(x_1^1,x_1^0)/W_0 \\
&\quad+ \int \frac{dk}{2\pi}\Big\{- i\omega_k\,\psi_k^-(x_2^1,x_1^0)\,\psi_k^+(x_1^1,x_1^0)/W_k - i\omega_k\,\psi_k^+(x_2^1,x_1^0)\,\psi_k^-(x_1^1,x_1^0)/W_k\Big\} \\
&\quad= \int \frac{dk}{2\pi}\,\frac{1}{a(k)}\;\psi_k^-(x_2^1,x_1^0)\;\psi_k^+(x_1^1,x_1^0) \;+\; \psi_0^+(x_2^1,x_1^0)\,\psi_0^-(x_1^1,x_1^0) \\
&\quad= \int \frac{dk}{2\pi}\,\frac{k-im}{k+im}\;\frac{k+im\tanh mx_2^1}{k-im}\;\frac{k-im\tanh mx_1^1}{k-im}\;\mathrm{e}^{ik(x_2^1-x_1^1)} \\
&\qquad\qquad + \frac{1}{2\pi}\,\frac{m\,\pi}{\cosh mx_2^1\,\cosh mx_1^1} \\
&\quad= \int \frac{dk}{2\pi}\;\frac{k+im\tanh mx_2^1}{k+im}\;\frac{k-im\tanh mx_1^1}{k-im}\;\mathrm{e}^{ik(x_2^1-x_1^1)} \\
&\qquad\qquad + \frac{1}{2}\,\frac{m}{\cosh mx_2^1\,\cosh mx_1^1} \\
&\quad= \delta(x_2^1-x_1^1).
\end{aligned} \qquad \text{(D.7)}$$

This confirms R to be the Green function of H, satisfying Feynman boundary conditions.

D.2 Calculations to Section 5.2

D.2.1 Calculations to Equation (5.30)

We consider in this section the expression $d_\varphi \mathrm{tr}\ln H \;=\; \mathrm{tr}\;(1/H\, d_\varphi H)$

$$\begin{aligned}
d_\varphi \mathrm{tr}\ln H &= \mathrm{tr}\, d_\varphi \ln H \\
&= \int \frac{dk}{2\pi}\,\langle k|\,\frac{1}{H}\,d_\varphi H\,|k\rangle\,.
\end{aligned} \qquad \text{(D.8)}$$

The operator $1/H$ we can write as

$$\frac{1}{H} \;=\; \int \frac{dk'}{2\pi}\;\big|k'\big\rangle\,\frac{1}{\lambda(k')}\,\big\langle k'\big|$$

[1] The Wronskians are defined at $x_1^0 = x_2^0$ to give a continuous Green function R at this point.

$$
\begin{aligned}
&= \int d^2x' d^2y' \frac{dk'}{2\pi} \left|x'\right\rangle \left\langle x' \mid k'\right\rangle \frac{1}{\lambda(k')} \left\langle k' \mid y'\right\rangle \left\langle y'\right| \\
&= \int d^2x' d^2y' \left|x'\right\rangle G(x'|y') \left\langle y'\right| . \qquad \text{(D.9)}
\end{aligned}
$$

And analog only for the operator $d_\varphi H$

$$
\begin{aligned}
d_\varphi H &= d_\varphi \int \frac{dk''}{2\pi} \left|k''\right\rangle \lambda(k'') \left\langle k''\right| \\
&= d_\varphi \int d^2x'' d^2y'' \frac{dk''}{2\pi} \left|x''\right\rangle \left\langle x'' \mid k''\right\rangle \lambda(k'') \left\langle k'' \mid y''\right\rangle \left\langle y''\right| \\
&= d_\varphi \int d^2x'' d^2y'' \left|x''\right\rangle H(x''|y'') \left\langle y''\right| . \qquad \text{(D.10)}
\end{aligned}
$$

Now we use a picture where changes of Lorentz systems with φ modify the eigenvalues of H, $\lambda(k) \to \lambda(k, \varphi)$ and the wave functions $\langle x|k\rangle$

$$
\begin{aligned}
d_\varphi H &= d_\varphi \int \frac{dk}{2\pi} |k\rangle \lambda(k) \langle k| = \int \frac{dk}{2\pi} |k\rangle d_\varphi \lambda(k) \langle k| \\
&= \int d^2x'' d^2y'' \frac{dk''}{2\pi} \left|x''\right\rangle \left\langle x'' \mid k''\right\rangle d_\varphi \left(\Box_{y''} + v(y'')\right) \left\langle k'' \mid y''\right\rangle \left\langle y''\right| \\
&= \int d^2x'' d^2y'' \frac{dk''}{2\pi} \left|x''\right\rangle \left\langle x'' \mid k''\right\rangle d_\varphi v(y'') \left\langle k'' \mid y''\right\rangle \left\langle y''\right| \\
&= \int d^2x'' d^2y'' \left|x''\right\rangle \delta(x'' - y'') \, d_\varphi v(y'') \left\langle y''\right| \\
&= \int d^2x'' \left|x''\right\rangle d_\varphi v(x'') \left\langle x''\right| . \qquad \text{(D.11)}
\end{aligned}
$$

The differential operator $\left(\Box_{y''} + v(y'')\right)$ is Hermitian. Inserting Eqs. (D.9) and (D.11) into Eq. (D.8) we arrive at

$$
\begin{aligned}
d_\varphi \mathrm{tr} \ln H &= \mathrm{tr} \, d_\varphi \ln H \\
&= \int d^2x' d^2y' d^2x'' \frac{dk}{2\pi} \left\langle k \mid x'\right\rangle G(x'|y') \left\langle y' \mid x''\right\rangle d_\varphi v(x'') \left\langle x'' \mid k\right\rangle \\
&= \int d^2x' d^2x'' \frac{dk}{2\pi} \left\langle k \mid x'\right\rangle G(x'|x'') d_\varphi v(x'') \left\langle x'' \mid k\right\rangle \\
&= \int d^2x' d^2x'' G(x'|x'') \, d_\varphi v(x'') \, \delta(x' - x'') \\
&= \int d^2x' G(x'|x') \, d_\varphi v(x'), \qquad \text{(D.12)}
\end{aligned}
$$

setting $x' = x$ we read finally for the expression (D.8)

$$
\begin{aligned}
d_\varphi \mathrm{tr} \ln H &= \int d^2x \, G(x|x) \, d_\varphi v(x) \\
&= \lim_{x' \to x} \int d^2x \, d_\varphi \left(\Box_x + v(x)\right) \, G(x'|x), \qquad \text{(D.13)}
\end{aligned}
$$

where the differential operator $(\Box_x + v(x)) = \big(\partial^2/\partial^2 x^0 - \partial^2/\partial^2 x^1 + v(x)\big)$ effects only the argument x on the r.h.s of $\lim_{x'\to x} G(x|x') = G(x|x)$. The potential $v(x)$ reads $v(x^1) = m^2 - 2m^2 \mathrm{sech}^2 \mathrm{mx}^1$. Changing to a co–moving system with coordinates (y^0, y^1) the derivative $d_\varphi v(x)$ of the potential with respect to the parameter φ (see Definition in Eq. (F.4)) reads

$$d_\varphi v(x^1) \quad = \quad d_\varphi v(x^1 = y^1 \cosh\varphi + y^0 \sinh\varphi). \tag{D.14}$$

We are allowed to express the derivative $d_\varphi v(x^1) = d_\varphi v(y^1, y^0)$ in terms of (y^1, y^0) only

$$\frac{d}{d\varphi} \quad = \quad \Big(\frac{dy^1}{d\varphi}\frac{d}{dy^1} + \frac{dy^0}{d\varphi}\frac{d}{dy^0}\Big) \;=\; -\Big(y^0\,\frac{d}{dy^1} + y^1\,\frac{d}{dy^0}\Big). \tag{D.15}$$

Hence, the derivative d_φ in Eq. (D.8) has to be understood in the form (D.15) only. Using the relation $H\psi = 0$, Eq. (D.8), reads

$$\begin{aligned} d_\varphi \mathrm{tr}\ln H \quad &= \quad \lim_{x'\to x}\int d^2x\, d_\varphi\,(\Box_x + v(x))\; G(x'|x),\,, \\ &= \;-\int d^2x\,\int \frac{dk}{2\pi}\,\psi_k^-(x)\left(\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + v(x)\right) d_\varphi\psi_k^+ \Big/ W_k, \end{aligned} \tag{D.16}$$

where W_k is the Wronskian given in Eq. (5.27).

D.2.2 Calculations to Equation (5.34)

Integrating in Eq. (5.34) by parts over t and x yields

$$\begin{aligned} d_\varphi \ln\det HH_0^{-1} \;=\; &-\int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\frac{d}{dx^0}\Big(\psi_k^-\,\frac{\partial\, d_\varphi\psi_k^+}{\partial x^0} - \frac{\partial\psi_k^-}{\partial x^0}\,d_\varphi\psi_k^+\Big) \\ &+ \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\frac{d}{dx^1}\Big(\psi_k^-\,\frac{\partial\, d_\varphi\psi_k^+}{\partial x^1} - \frac{\partial\psi_k^-}{\partial x^1}\,d_\varphi\psi_k^+\Big) \\ &- \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\,d_\varphi\psi_k^+\,\Big(\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + v\Big)\psi_k^- \\ &- \int_{t'}^{t''}\!\!\int d^2x\,W_0^{-1}\frac{d}{dx^0}\Big(\psi_0^-\,\frac{\partial\, d_\varphi\psi_0^+}{\partial x^0} - \frac{\partial\psi_0^-}{\partial x^0}\,d_\varphi\psi_0^+\Big) \\ &+ \int_{t'}^{t''}\!\!\int d^2x\,W_0^{-1}\frac{d}{dx^1}\Big(\psi_0^-\,\frac{\partial\, d_\varphi\psi_0^+}{\partial x^1} - \frac{\partial\psi_0^-}{\partial x^1}\,d_\varphi\psi_0^+\Big) \\ &- \int_{t'}^{t''}\!\!\int d^2x\,W_0^{-1} d_\varphi\psi_0^+\,\Big(\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + v\Big)\psi_0^- \\ &+ \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\frac{d}{dx^0}\Big(\psi_{0k}^-\,\frac{\partial\, d_\varphi\psi_{0k}^+}{\partial x^0} - \frac{\partial\psi_{0k}^-}{\partial x^0}\,d_\varphi\psi_{0k}^+\Big) \\ &- \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\frac{d}{dx^1}\Big(\psi_{0k}^-\,\frac{\partial\, d_\varphi\psi_{0k}^+}{\partial x^1} - \frac{\partial\psi_{0k}^-}{\partial x^1}\,d_\varphi\psi_{0k}^+\Big) \\ &+ \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\,d_\varphi\psi_{0k}^+\,\Big(\frac{\partial^2}{\partial {x^0}^2} - \frac{\partial^2}{\partial {x^1}^2} + v_0\Big)\psi_{0k}^- \end{aligned} \tag{D.17}$$

Using the homogeneous differential equation $H\psi = 0$ and due to Eq. (D.21) this expression simplifies

$$S \;=\; \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\frac{d}{dx^1}\left(\psi_k^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_k^+\right) + \int_{t'}^{t''}\!\!\int d^2x\,W_0^{-1}\frac{d}{dx^1}\left(\psi_0^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_0^+\right) - \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\frac{d}{dx^1}\left(\psi_{0k}^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_{0k}^+\right) \;=\; 0, \tag{D.18}$$

where $(a\overleftrightarrow{\partial}_{x^1}b) = a\,(\partial_{x^1}b) - (\partial_{x^1}a)\,b$ to

$$\begin{aligned} d_\varphi \ln\det HH_0^{-1} \;=\; &-\int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\frac{d}{dx^0}\left(\psi_k^-\,\frac{\partial\,d_\varphi\psi_k^+}{\partial x^0} - \frac{\partial\psi_k^-}{\partial x^0}\,d_\varphi\psi_k^+\right) \\ &- \int_{t'}^{t''}\!\!\int d^2x\,W_0^{-1}\frac{d}{dx^0}\left(\psi_0^-\,\frac{\partial\,d_\varphi\psi_0^+}{\partial x^0} - \frac{\partial\psi_0^-}{\partial x^0}\,d_\varphi\psi_0^+\right) \\ &+ \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\frac{d}{dx^0}\left(\psi_{0k}^-\,\frac{\partial\,d_\varphi\psi_{0k}^+}{\partial x^0} - \frac{\partial\psi_{0k}^-}{\partial x^0}\,d_\varphi\psi_{0k}^+\right). \end{aligned} \tag{D.19}$$

With the notation $\overleftrightarrow{\partial}_{x^0}$ defined via $(a\overleftrightarrow{\partial}_{x^0}b) = a\,(\partial_{x^0}b) - (\partial_{x^0}a)b$ we arrive at

$$\begin{aligned} d_\varphi \ln\det HH_0^{-1} \;=\; &-\int dx^1\,\frac{dk}{2\pi}\,\left\{W_k^{-1}\psi_k^-\,\overleftrightarrow{\partial}_{x^0}\,d_\varphi\psi_k^+ \;-\; W_{0k}^{-1}\psi_{0k}^-\,\overleftrightarrow{\partial}_{x^0}\,d_\varphi\psi_{0k}^+\right\}\Big|_{t'}^{t''} \\ &- \int dx^1\left\{W_0^{-1}\psi_0^-\,\overleftrightarrow{\partial}_{x^0}\,d_\varphi\psi_0^+\right\}\Big|_{t'}^{t''}. \end{aligned} \tag{D.20}$$

Calculations to Equation (D.18)

We calculate in this subsection the spatial surface–term S (D.18)

$$S \;=\; \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_k^{-1}\frac{d}{dx^1}\left(\psi_k^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_k^+\right) + \int_{t'}^{t''}\!\!\int d^2x\,W_0^{-1}\frac{d}{dx^1}\left(\psi_0^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_0^+\right) - \int_{t'}^{t''}\!\!\int d^2x\,\frac{dk}{2\pi}\,W_{0k}^{-1}\frac{d}{dx^1}\left(\psi_{0k}^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_{0k}^+\right) \;=\; 0. \tag{D.21}$$

The first term reads

$$\begin{aligned} &\int_{t'}^{t''} dx^0\,\frac{dk}{2\pi}\,W_k^{-1}\left(\psi_k^-\,\overleftrightarrow{\partial}_{x^1}\,d_\varphi\psi_k^+\right)\Big|_{-\infty}^{\infty} \\ &= \int_{t'}^{t''} dx^0\,\frac{dk}{2\pi}\,W_k^{-1}\,a(k)\Big(\mathrm{e}^{-i\omega_k x^0 + ikx^1}\,\frac{\overleftrightarrow{d}}{dx^1}\,d_\varphi\Big\{\mathrm{e}^{i\omega_k x^0 - ikx^1}\Big\}\Big)\Big|_{-\infty}^{\infty} \\ &= \int_{t'}^{t''} dx^0\,\frac{dk}{2\pi}\,W_k^{-1}\,a(k)\Big(\mathrm{e}^{-i\omega_k x^0 + ikx^1}\,\frac{\overleftrightarrow{d}}{dx^1}\,\mathrm{e}^{i\omega_k x^0 - ikx^1}\Big)\,d_\varphi\Big\{i\omega_k x^0 - ikx^0\Big\}\Big|_{-\infty}^{\infty} \\ &= \int_{t'}^{t''} dx^0\,\frac{dk}{2\pi}\,W_k^{-1}\,a(k)\,(-2ik)d_\varphi\Big\{i\omega_k x^0 - ikx^1\Big\}\Big|_{-\infty}^{\infty} \\ &+ \int_{t'}^{t''} dx^0\,\frac{dk}{2\pi}\,W_k^{-1}\,a(k)\,\frac{d}{dx^1}\,d_\varphi\Big\{i\omega_k x^0 - ikx^1\Big\}\Big|_{-\infty}^{\infty}. \end{aligned} \tag{D.22}$$

By using the dependence of the coordinates (x,t) on the parameter φ, see Eq. (F.4) the derivatives with respect to φ read $dx^1/d\varphi = x^0$ and $dx^0/d\varphi = x^1$, yielding

$$\int_{t'}^{t''} dx^0\, \frac{dk}{2\pi}\, W_k^{-1} \Big(\psi_k^- \overleftrightarrow{\partial}_{x^1}\, d\psi_k^+ \Big) \Big|_{-\infty}^{\infty} \;=\; \int_{t'}^{t''} dx^0\, \frac{dk}{2\pi}\, W_k^{-1}\, a(k)\, (-2ik) \Big\{ i\omega_k x^1 \Big\} \Big|_{-\infty}^{\infty}. \tag{D.23}$$

For the third term in Eq. (D.21) we get

$$-\int_{t'}^{t''}\!\!\int d^2x\, \frac{dk}{2\pi}\, W_{0k}^{-1} \frac{d}{dx^1} \Big(\psi_{0k}^- \overleftrightarrow{\partial}_{x^1}\, d\psi_{0k}^+ \Big) \;=\; -\int_{t'}^{t''} dx^0\, \frac{dk}{2\pi}\, W_{0k}^{-1} (-2ik) \Big\{ i\omega_k x^1 \Big\} \Big|_{-\infty}^{\infty}. \tag{D.24}$$

Hence by inserting the Wronskians $W_k = -2i\,\omega_k a(k)$ and $W_{0k} = -2i\omega_k$ we can see that the sum of the first and third term of Eq. (D.21) vanish. For the second term we use $W_0 = -2\epsilon$ and get

$$\int_{t'}^{t''} dx^0\, W_0^{-1} \Big(\psi_0^- \overleftrightarrow{\partial}_{x^1}\, d_\varphi \psi_0^+ \Big) \Big|_{-\infty}^{\infty} \;=\; \int_{t'}^{t''} dx^0\, W_0^{-1} \Big(\psi_0^- \overleftrightarrow{\partial}_{x^1}\, \partial_x \psi_0^+\, x^0 \Big) \Big|_{-\infty}^{\infty} \;=\; 0. \tag{D.25}$$

D.2.3 Calculations to Equation (5.35)

Eq. (5.35) is given by

$$\begin{aligned} d_\varphi \ln\det HH_0^{-1} \;=\; & -\int dx^1\, \frac{dk}{2\pi}\, \Big\{ W_k^{-1} \psi_k^- \overleftrightarrow{\partial}_{x^0}\, d_\varphi \psi_k^+ \;-\; W_{0k}^{-1} \psi_{0k}^- \overleftrightarrow{\partial}_{x^0}\, d_\varphi \psi_{0k}^+ \Big\} \Big|_{t'}^{t''} \\ & -\int dx^1 \Big\{ W_0^{-1} \psi_0^- \overleftrightarrow{\partial}_{x^0}\, d_\varphi \psi_0^+ \Big\} \Big|_{t'}^{t''}. \end{aligned} \tag{D.26}$$

Using for the differential $d_\varphi = d/d\varphi$ Eq. (D.15) we get [19]

$$\begin{aligned} -\psi_k^- \overleftrightarrow{\partial}_{x^0} \Big(y^0 \frac{d}{dy^1} + y^1 \frac{d}{dy^0} \Big) \psi_k^+ \;=\; & -\frac{x^0}{\cosh\varphi}\, \psi_k^- \overleftrightarrow{\partial}_{x^0} \frac{d\psi_k^+}{dy^0} \\ & -\Big\{ \frac{1}{\cosh\varphi}\, \psi_k^-\, \frac{d\psi_k^+}{dy^1} + \psi_k^- \overleftrightarrow{\partial}_{x^0} \Big[y^1 \frac{d}{dy^0}\, \psi_k^+ - \tanh\varphi\, y^1 \frac{d}{dy^1} \psi_k^+ \Big] \Big\}, \end{aligned} \tag{D.27}$$

where we have used

$$\begin{aligned} \psi_k^- \overleftrightarrow{\partial}_{x^0}\, y^0 \frac{d}{dy^1}\, \psi_k^+ \;=\; & \frac{1}{\cosh\varphi}\, \psi_k^- \overleftrightarrow{\partial}_{x^0} \left(y^0 \cosh\varphi + y^1 \sinh\varphi \right) \frac{d}{dy^1}\, \psi_k^+ \\ & - \psi_k^- \overleftrightarrow{\partial}_{x^0} \tanh\varphi\, y^1 \frac{d}{dy^1} \psi_k^+ \\ = \; & \frac{1}{\cosh\varphi}\, \psi_k^- \overleftrightarrow{\partial}_{x^0}\, x^0 \frac{d}{dy^1}\, \psi_k^+ \;-\; \psi_k^- \overleftrightarrow{\partial}_{x^0} \tanh\varphi\, y^1 \frac{d}{dy^1} \psi_k^+ \\ = \; & \frac{x^0}{\cosh\varphi}\, \psi_k^- \overleftrightarrow{\partial}_{x^0} \frac{d}{dy^1}\, \psi_k^+ \;+\; \frac{1}{\cosh\varphi}\, \psi_k^-\, \frac{d}{dy^1}\, \psi_k^+ \;-\; \psi_k^- \overleftrightarrow{\partial}_{x^0} \tanh\varphi\, y^1 \frac{d}{dy^1} \psi_k^+. \end{aligned} \tag{D.28}$$

The expression inside the curly bracket in Eq. (D.27) cancels due to time–independence [19]. For Eq. (D.26) follows

$$
\begin{aligned}
\frac{d}{d\varphi}\,\ln\det HH_0^{-1} \;=& \\
=& \int dx^1\,\frac{dk}{2\pi}\,\frac{x^0}{\cosh\varphi}\,\left\{W_k^{-1}\psi_k^-\overleftrightarrow{\partial}_{x^0}\,\frac{d}{dy^1}\psi_k^+ \,-\, W_{0k}^{-1}\psi_{0k}^-\overleftrightarrow{\partial}_{x^0}\,\frac{d}{dy^1}\psi_{0k}^+\right\}\Big|_{t'}^{t''} \\
&+\int dx^1\,\frac{x^0}{\cosh\varphi}\left\{W_0^{-1}\psi_0^-\overleftrightarrow{\partial}_{x^0}\,\frac{d}{dy^1}\psi_0^+\right\}\Big|_{t'}^{t''}.
\end{aligned}
\tag{D.29}
$$

Changing the derivatives as

$$
\begin{aligned}
\frac{d}{dy^1}\psi_k^+ \;&=\; \frac{1}{\cosh\varphi}\frac{d}{dx^1}\,g_k(x^1)\,\mathrm{e}^{i\omega_k x^0} \\
&=\; i\omega_k\tanh\varphi\,\psi_k^+ + \frac{1}{\cosh\varphi}\frac{d}{dx^1}\,\psi_k^+,
\end{aligned}
\tag{D.30}
$$

the second term in Eq. (D.29) containing the derivative d/dx^1 can be dropped as can be seen by inserting into Eq. (D.29)

$$
\begin{aligned}
&\int dx^1\,\frac{dk}{2\pi}\,\frac{1}{\cosh\varphi}\,W_k^{-1}\,\psi_k^-\overleftrightarrow{\partial}_{x^0}\,\frac{d}{dx^1}\psi_k^+ + \int dx^1\,\frac{1}{\cosh\varphi}\,W_0^{-1}\,\psi_0^-\overleftrightarrow{\partial}_{x^0}\,\frac{d}{dx^1}\psi_0^+ \\
&= \frac{1}{\cosh\varphi}\int dx^1\,2i\bar{\omega}\,\frac{dk}{2\pi}\,\frac{1}{-2i\bar{\omega}a(k)}\,\psi_k^-\,\frac{d}{dx^1}\,\psi_k^+ + \frac{1}{\cosh\varphi}\int dx^1\,2\epsilon\,\frac{1}{-2\epsilon}\,\psi_0^-\,\frac{d}{dx^1}\,\psi_0^+ \\
&= \frac{1}{\cosh\varphi}\int dx^1\,\frac{d}{dx^1}\left\{\delta(0) - \frac{1}{2}\frac{1}{\cosh^2 mx^1}\right\} + \frac{1}{\cosh\varphi}\int dx^1\,\frac{d}{dx^1}\frac{1}{2}\frac{1}{\cosh^2 mx^1} \;=\; 0.
\end{aligned}
\tag{D.31}
$$

Inserting the first term in Eq. (D.30) into Eq. (D.29) gives

$$
\begin{aligned}
\ln\det HH_0^{-1} \;=& \\
=& \int dx^1\,\frac{dk}{2\pi}\,d\varphi\,\frac{t}{\cosh^2\varphi}\,\sinh\varphi\,i\omega_k\left\{W_k^{-1}\psi_k^-\overleftrightarrow{\partial}_{x^0}\,\psi_k^+ - W_{0k}^{-1}\psi_{0k}^-\overleftrightarrow{\partial}_{x^0}\,\psi_{0k}^+\right\}\Big|_{t'}^{t''} \\
&+\int dx^1\,\frac{t}{\cosh^2\varphi}\,\sinh\varphi\,\epsilon\left\{W_0^{-1}\psi_0^-\overleftrightarrow{\partial}_{x^0}\,\psi_0^+\right\}\Big|_{t'}^{t''} \\
=& -\int dx^1\,\frac{dk}{2\pi}\,\frac{(t''-t')}{\cosh\varphi}\,i\omega_k\left\{W_k^{-1}\psi_k^-\overleftrightarrow{\partial}_{x^0}\,\psi_k^+ - W_{0k}^{-1}\psi_{0k}^-\overleftrightarrow{\partial}_{x^0}\,\psi_{0k}^+\right\} \\
&-\int dx^1\,\frac{(t''-t')}{\cosh\varphi}\,\epsilon\left\{W_0^{-1}\psi_0^-\overleftrightarrow{\partial}_{x^0}\,\psi_0^+\right\}.
\end{aligned}
\tag{D.32}
$$

and with the Wronskians (5.17) and (5.27) we arrive at

$$
\begin{aligned}
\ln\det HH_0^{-1} \;=& \\
=& \frac{1}{4\pi}\frac{(t''-t')}{\cosh\varphi}\int dx^1\,dk\left\{\frac{1}{a(k)}\,\psi_k^-\overleftrightarrow{\partial}_{x^0}\,\psi_k^+ - \psi_{0k}^-\overleftrightarrow{\partial}_{x^0}\,\psi_{0k}^+\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\frac{(t''-t')}{\cosh\varphi}\int dx^1\,\left\{\psi_0^-\,\overleftrightarrow{\partial}_{x^0}\,\psi_0^+\right\}\\
&= \frac{1}{2\pi}\frac{(t''-t')}{\cosh\varphi}\int dx^1\,dk\;i\omega_k\left\{\frac{1}{a(k)}\,\psi_k^-\,\psi_k^+ - \psi_{0k}^-\,\psi_{0k}^+\right\}\\
&+ \frac{(t''-t')}{\cosh\varphi}\int dx^1\,\epsilon\left\{\psi_0^-\,\psi_0^+\right\}. \qquad \text{(D.33)}
\end{aligned}
$$

For $\epsilon\to 0$ the discrete mode does not contribute, hence the expression (5.35) becomes

$$
\ln\det HH_0^{-1} \;=\; \frac{1}{2\pi}\frac{(t''-t')}{\cosh\varphi}\int dx^1\,dk\;i\omega_k\left\{\frac{1}{a(k)}\psi_k^-\,\psi_k^+\;-\;\psi_{0k}^-\,\psi_{0k}^+\right\}. \quad \text{(D.34)}
$$

D.3 Comparing the eigensolutions 1.22 for the soliton sector with those in the manuscript [19]

In order to simplify the comparison with [19] we redefine our solutions (1.27) by

$$
\begin{aligned}
\vartheta_{0\,\omega}(x) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{\pi\sqrt{\alpha}}{2\pi}}\,\frac{1}{\cosh\sqrt{\alpha}x^1}\,\mathrm{e}^{-i\omega x^0} \;=\; \frac{1}{2\pi}\,\vartheta_0(x^1)\,\mathrm{e}^{-i\omega x^0},\\
\vartheta_{k,\omega}(x) &= \frac{1}{2\pi}\,\frac{k+i\sqrt{\alpha}\,\tanh\sqrt{\alpha}x^1}{\sqrt{k^2+\alpha}}\,\mathrm{e}^{-i\omega x^0+ikx^1} \;=\; \frac{1}{2\pi}\,\frac{1}{\omega_k}\,\vartheta_k(x^1)\,\mathrm{e}^{-i\omega x^0}.
\end{aligned} \qquad \text{(D.35)}
$$

with $\omega_k=\sqrt{k^2+\alpha}$. The functions $\vartheta_k(x)$ and $\vartheta_0(x)$ are defined as

$$
\begin{aligned}
\vartheta_0(x^1) &= \sqrt{\frac{\pi\sqrt{\alpha}}{2\pi}}\,\frac{1}{\cosh\sqrt{\alpha}x^1}\\
\vartheta_k(x^1) &= (k+i\sqrt{\alpha}\,\tanh\sqrt{\alpha}x^1)\,\mathrm{e}^{ikx^1}.
\end{aligned} \qquad \text{(D.36)}
$$

Using these solutions the Green function reads

$$
\begin{aligned}
G(x|x') &= \int\frac{dk}{2\pi}\,\frac{d\omega}{2\pi}\,\vartheta^*_{k,\omega}(x)\,\frac{1/\omega_k^2}{-\,\omega^2+k^2+\alpha}\,\vartheta_{k,\omega}(x')\\
&\quad+\frac{1}{2\pi}\int\frac{d\omega}{2\pi}\,\vartheta^*_{0\,\omega}(x)\,\frac{1}{-\,\omega^2}\,\vartheta_{0\,\omega}(x')\\
&= \int\frac{dk}{2\pi}\,\frac{d\omega}{2\pi}\,\vartheta^*_k(x^1)\,\frac{1/\omega_k^2}{-\,\omega^2+k^2+\alpha}\,\vartheta_k(x'^1)\,\mathrm{e}^{i\omega(x^0\,-\,x'^0)}\\
&\quad+\frac{1}{2\pi}\int\frac{d\omega}{2\pi}\,\vartheta^*_0(x^1)\,\frac{1}{-\,\omega^2}\,\vartheta_0(x'^1)\,\mathrm{e}^{i\omega(x^0\,-\,x'^0)}.
\end{aligned} \qquad \text{(D.37)}
$$

Integration over ω and taking Feynman boundary conditions into account $\alpha\to\alpha-i\epsilon$ gives

$$
G(x|x') \;=
$$

$$= \begin{cases} \int dk\, \vartheta_k^*(x^1)\, \frac{i/\omega_k^2}{4\pi\omega_k}\, \vartheta_k(x'^1)\, \mathrm{e}^{-i\omega_k(x^0 - x'^0)} & x^0 > x'^0 \\ \int dk\, \vartheta_k^*(x^1)\, \frac{i/\omega_k^2}{4\pi\omega_k}\, \vartheta_k(x'^1)\, \mathrm{e}^{-i\omega_k(x'^0 - x^0)} & x'^0 > x^0 \end{cases}$$
$$+ \begin{cases} \vartheta_0^*(x^1)\, \frac{i}{-4\pi i\epsilon}\, \vartheta_0(x'^1)\, \mathrm{e}^{-\epsilon(x^0 - x'^0)} & x^0 > x'^0 \\ +\vartheta_0^*(x^1)\, \frac{i}{-4\pi i\epsilon}\, \vartheta_0(x'^1)\, \mathrm{e}^{-\epsilon(x'^0 - x^0)} & x'^0 > x^0\,. \end{cases} \tag{D.38}$$

In order to compare with [19] we introduce the parameter β by $k = \sqrt{\alpha}\sinh\beta$, $\omega_k = \sqrt{\alpha}\cosh\beta$ and $dk = \sqrt{\alpha}\cosh d\beta$. This gives for the Green function (D.38)

$$G(x|x') =$$
$$= \int d\beta \begin{cases} \vartheta_\beta^*(x^1)\, \frac{a(\beta)}{\alpha\cosh^2\beta}\, \frac{1}{-4\pi i\, a(\beta)}\, \vartheta_\beta(x'^1)\, \mathrm{e}^{-im\cosh\beta(x^0 - x'^0)} & x^0 > x'^0 \\ \vartheta_\beta^*(x^1)\, \frac{a(\beta)}{\alpha\cosh^2\beta}\, \frac{1}{-4\pi i\, a(\beta)}\, \vartheta_\beta(x'^1)\, \mathrm{e}^{-im\cosh\beta(x'^0 - x^0)} & x'^0 > x^0 \end{cases}$$
$$+ \begin{cases} \vartheta_0^*(x^1)\, \frac{1}{-4\pi\epsilon}\, \vartheta_0(x'^1)\, \mathrm{e}^{-\epsilon(x^0 - x'^0)} & x^0 > x'^0 \\ \vartheta_0^*(x^1)\, \frac{1}{-4\pi\epsilon}\, \vartheta_0(x'^1)\, \mathrm{e}^{-\epsilon(x'^0 - x^0)} & x'^0 > x^0\,, \end{cases} \tag{D.39}$$

where the amplitude $a(\beta)$ is given by $a(\beta) = (\sinh\beta + i)/(\sinh\beta - i)$. Using $\cosh^2\beta = (\sinh\beta + i)(\sinh\beta - i)$ and the definition of the Wronskian $W_\beta = -4\pi i\cosh\beta a(\beta)$ in [19] and $W_{0\beta} = -4\pi\epsilon$

$$\begin{aligned} \frac{a(\beta)}{\alpha\cosh^2\beta}\, \frac{1}{-4\pi i\, a(\beta)} &= \frac{(\sinh\beta + i)/(\sinh\beta - i)}{\alpha(\sinh\beta + i)(\sinh\beta - i)}\, \frac{1}{W_\beta} \\ &= \frac{1}{\alpha(\sinh\beta - i)^2}\, \frac{1}{W_\beta}. \end{aligned} \tag{D.40}$$

This allows us to introduce the negative frequency solutions $\bar{\psi}_\beta^-(x)$ and the positive frequency solutions $\bar{\psi}_\beta^+(x)$. The negative frequency solution is related to $\vartheta_\beta^*(x)$, corresponding to the complex conjugate function in Eq. (D.36) in β parametrisation

$$\begin{aligned} \bar{\psi}_\beta^-(x) &= \frac{\vartheta_\beta^*(x^1)}{\sqrt{\alpha}(\sinh\beta - i)}\, \mathrm{e}^{-i\sqrt{\alpha}\cosh\beta x^0} \\ &= \frac{\sqrt{\alpha}\sinh\beta - i\,\sqrt{\alpha}\, \tanh\sqrt{\alpha}x^1}{\sqrt{\alpha}(\sinh\beta - i)}\, \mathrm{e}^{-i\sqrt{\alpha}\cosh\beta x^0 - i\sqrt{\alpha}\sinh\beta x^1}, \end{aligned} \tag{D.41}$$

while the positive frequency solutions are related to $\vartheta_\beta(x)$, Eq. (D.36) in β parametrisation by

$$\begin{aligned}\bar{\psi}^+_\beta(x) &= \frac{\vartheta_\beta(x^1)}{\sqrt{\alpha}(\sinh\beta - i)}\,\mathrm{e}^{-im\cosh\beta x^0} \\ &= \frac{\sqrt{\alpha}\sinh\beta + i\sqrt{\alpha}\,\tanh\sqrt{\alpha}x^1}{\sqrt{\alpha}(\sinh\beta - i)}\,\mathrm{e}^{i\sqrt{\alpha}\cosh\beta x^0 + i\sqrt{\alpha}\sinh\beta x^1}. \end{aligned} \tag{D.42}$$

The discrete mode reads

$$\psi^\pm_0(x) = \vartheta_0(x^1)\,\mathrm{e}^{\pm\,\epsilon x^0}. \tag{D.43}$$

Now, the solutions $\bar{\psi}^-_\beta(x)$ and $\bar{\psi}^+_\beta(x)$ are related to those given in the manuscript [19] $\psi^-_\beta(x)$ and $\psi^+_\beta(x)$ Eqs. (5.19) and (5.21) by the identification $\sqrt{\alpha} \to m$ and $x \to -x$.

Appendix E

Calculations to the Thirring Model

E.1 On the parameterisation of the functional determinant

The result of the calculation of the functional determinant $\mathrm{Det}(i\hat{\partial} + \hat{A})$ is related to the vacuum expectation value $\langle j^\mu(x)\rangle$ of the vector current $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$. Using (8.4), the vacuum expectation value of the vector current can be defined by

$$\begin{aligned}\langle j^\mu(x)\rangle &= \frac{1}{i}\frac{\delta}{\delta A_\mu(x)}\ln Z^{(0)}_{\rm th}[A,J,\bar{J}]\Big|_{\bar{J}=J=0} = \\ &= \frac{1}{i}\frac{\delta}{\delta A_\mu(x)}\ln\mathrm{Det}(i\hat{\partial}+\hat{A}) = \int d^2y\, D^{\mu\nu}(x-y)\,A_\nu(y),\end{aligned} \qquad \text{(E.1)}$$

where $D^{\mu\nu}(x-y)$ is given by (8.7) and parameterised by two parameters $\bar{\xi}$ and $\bar{\eta}$. Hence, the calculation of the vacuum expectation value of the vector current should show how many parameters one can use for the parameterisation of the Green function $D^{\mu\nu}(x-y)$ or the functional determinant $\mathrm{Det}(i\hat{\partial}+\hat{A})$. According to Hagen [43], $\langle j^\mu(x)\rangle$ can be determined by

$$\langle j^\mu(x)\rangle = \lim_{y\to x}\mathrm{tr}\Big\{iG(x,y)_A\gamma^\mu \exp\, i\int_x^y dz^\nu(a\,A_\nu(z) + b\,\gamma^5\,A_{5\nu}(z))\Big\}, \qquad \text{(E.2)}$$

where a and b are parameters and $A^\nu_5(z) = -\,\varepsilon^{\nu\beta}A_\beta(z)$. The fermion Green function $G(y,x)_A$ is given by (8.10). The requirement of covariance relates the parameters a and b. This provides the parameterisation of the functional determinant $\mathrm{Det}(i\hat{\partial}+\hat{A})$ by one parameter. In Hagen's notation this is the parameter ξ.

In order to show that the functional determinant $\mathrm{Det}(i\hat{\partial}+\hat{A})$ can be parameterised by two parameters we propose to define the vacuum expectation value of the vector current (E.2) as follows

$$\langle j^\mu(x)\rangle = \lim_{y\to x}\mathrm{tr}\Big\{iG(x,y)_A\gamma^\mu \exp\, i\int_x^y dz^\nu\Big(a\,A_\nu(z) + b\,\gamma^5\,A_{5\nu}(z)$$
$$+c\int d^2t\,\frac{\partial}{\partial t^\nu}\frac{\partial}{\partial t_\beta}\Delta(z-t;\mu)\,A_\beta(t) + d\,\gamma^5\int d^2t\,\frac{\partial}{\partial t^\nu}\frac{\partial}{\partial t_\beta}\Delta(z-t;\mu)\,A_{5\beta}(t)\Big)\Big\}, \tag{E.3}$$

where c and d are additional parameters and $\Delta(z-t;\mu)$ is determined by (8.9). Under the gauge transformation $A_\nu \to A'_\nu = A_\nu + \partial_\nu\phi$ the third term in Eq. (E.3) behaves like the first one, whereas the fourth one is gauge invariant. The exponent of Eq. (E.3) has the most general form constrained by dimensional considerations and gauge invariance.

The vacuum expectation value of the vector current can be transcribed into the form

$$\langle j^\mu(x)\rangle = \lim_{y\to x}\frac{i}{2\pi}\frac{(x-y)_\rho}{(x-y)^2 - i0}\,\mathrm{tr}\Big\{\gamma^\rho\exp\big(-i(g^{\alpha\beta}-\varepsilon^{\alpha\beta}\gamma^5\int d^2z\,\frac{\partial}{\partial z^\alpha}[\Delta(x-z;\mu))$$
$$\times - \Delta(y-z;\mu)]A_\beta(z)\big)\,\gamma^\mu\exp\, i\int_x^y dz^\nu\,\Big(aA_\nu(z) + b\gamma^5 A_{5\nu}(z)$$
$$+c\int d^2t\,\frac{\partial}{\partial t^\nu}\frac{\partial}{\partial t_\beta}\Delta(z-t;\mu)A_\beta(t) + d\gamma^5\int d^2t\,\frac{\partial}{\partial t^\nu}\frac{\partial}{\partial t_\beta}\Delta(z-t;\mu)A_{5\beta}(t)\Big)\Big\}. \tag{E.4}$$

For the calculation of the r.h.s. of Eq. (E.4) we apply the spatial–point–slitting technique. We set $y^0 = x^0$ and $y^1 = x^1 \pm \epsilon$, taking the limit $\epsilon \to 0$. This gives

$$\langle j^\mu(x)\rangle =$$
$$= \lim_{\epsilon\to 0}\frac{i}{2\pi}\frac{1}{\mp\epsilon}\mathrm{tr}\Big\{\gamma^1\Big[1 \mp i\epsilon(g^{\alpha\beta}-\varepsilon^{\alpha\beta}\gamma^5)\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^\alpha}\int d^2z\,\Delta(x-z;\mu)A_\beta(z)\Big]$$
$$\times\gamma^\mu\Big[1 \pm i\epsilon\Big(aA_1(x) + b\gamma^5 A_{51}(x) + c\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_\beta(t)$$
$$+d\gamma^5\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_{5\beta}(t)\Big)\Big]\Big\} =$$
$$= \mp\lim_{\epsilon\to 0}\frac{ig^{1\mu}}{\pi\epsilon} + \lim_{\epsilon\to 0}\frac{i}{2\pi\epsilon}\Big[i\epsilon(2g^{1\mu}g^{\alpha\beta} + 2\varepsilon^{1\mu}\varepsilon^{\alpha\beta})\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^\alpha}\int d^2z\,\Delta(x-z;\mu)A_\beta(z)$$
$$\mp i\epsilon\Big(2ag^{1\mu}A_1(x) + 2b\varepsilon^{1\mu}A_{51}(x) + 2cg^{1\mu}\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_\beta(t)$$
$$+2d\,\varepsilon^{1\mu}\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_{5\beta}(t)\Big)\Big], \tag{E.5}$$

Taking the symmetric limit we get

$$\langle j^\mu(x)\rangle \;\; = \;\; \frac{1}{\pi}\Big[-(g^{1\mu}g^{\alpha\beta} + \varepsilon^{1\mu}\varepsilon^{\alpha\beta})\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^\alpha}\int d^2z\,\Delta(x-z;\mu)A_\beta(z)$$

$$+\Big(ag^{1\mu}A_1(x)+b\varepsilon^{1\mu}A_{51}(x)+cg^{1\mu}\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_\beta(t)$$
$$+d\varepsilon^{1\mu}\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_{5\beta}(t)\Big)\Big]. \tag{E.6}$$

The components of the current are equal to

$$\begin{aligned}\langle j^0(x)\rangle &= \frac{1}{\pi}\varepsilon^{\alpha\beta}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^\alpha}\int d^2z\,\Delta(x-z;\mu)A_\beta(z)\\ &\quad -\frac{b}{\pi}A_{51}(x)-\frac{d}{\pi}\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_{5\beta}(t),\\ \langle j^1(x)\rangle &= \frac{1}{\pi}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x_\alpha}\int d^2z\,\Delta(x-z;\mu)A_\alpha(z)\\ &\quad -\frac{a}{\pi}\,A_1(x)-\frac{c}{\pi}\int d^2t\,\frac{\partial}{\partial t^1}\frac{\partial}{\partial t_\beta}\Delta(x-t;\mu)A_\beta(t).\end{aligned} \tag{E.7}$$

Using $A_{5\mu}=-\varepsilon_{\mu\nu}A^\nu$ and $\Box\Delta(x-y;\mu)=\delta^{(2)}(x-y)$ the zero component can be transcribed into the form

$$\begin{aligned}\langle j^0(x)\rangle &= \frac{1}{\pi}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^0}\int d^2z\,\Delta(x-z;\mu)A_1(z)-\frac{1}{\pi}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1}\int d^2z\,\Delta(x-z;\mu)A_0(z)\\ &\quad +\frac{b}{\pi}\,A^0(x)+\frac{d}{\pi}\int d^2t\frac{\partial}{\partial t^1}\frac{\partial}{\partial t^0}\Delta(x-t;\mu)A_1(t)\\ &\quad -\frac{d}{\pi}\int d^2t\frac{\partial}{\partial t^1}\frac{\partial}{\partial t^1}\Delta(x-t;\mu)A_0(t)=\\ &= -\frac{1}{\pi}\,\frac{\partial}{\partial x^0}\frac{\partial}{\partial x_\mu}\int d^2z\,\Delta(x-z;\mu)A_\mu(z)+\frac{d}{\pi}\,A^0(x)\\ &\quad +\frac{b+1}{\pi}\,A^0(x)-\frac{d}{\pi}\int d^2t\frac{\partial}{\partial t^0}\frac{\partial}{\partial t_\mu}\Delta(x-t;\mu)A_\mu(t)=\\ &= -\frac{1+d}{\pi}\frac{\partial}{\partial x_0}\frac{\partial}{\partial x_\mu}\int d^2z\,\Delta(x-z;\mu)A_\mu(z)+\frac{b+d+1}{\pi}\,A^0(x).\end{aligned} \tag{E.8}$$

Comparing the time component with the spatial one, given by

$$\langle j^1(x)\rangle = \frac{c-1}{\pi}\,\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_\mu}\int d^2z\,\Delta(x-z;\mu)A_\mu(z)+\frac{a}{\pi}\,A^1(x), \tag{E.9}$$

we obtain that the covariance of the vacuum expectation value of the vector current takes place for $c=-d$ and $b+d+1=a$ only.

Thus, the vacuum expectation value of the vector current is

$$\begin{aligned}\langle j^\mu(x)\rangle &= \frac{\bar{\xi}}{\pi}\,A^\mu(x)-\frac{\bar{\eta}}{\pi}\,\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\int d^2z\,\Delta(x-z;\mu)A_\nu(z)\\ &= \int d^2y\,D^{\mu\nu}(x-y)\,A_\nu(y),\end{aligned} \tag{E.10}$$

where $\bar{\eta}$ and $\bar{\xi}$ are parameters related to the parameters a, b, c and d as $\bar{\xi} = a$ and $\bar{\eta} = 1 - c$. The vacuum expectation value of the vector current, given by (E.10), supports the possibility to parametrise the functional determinant $\mathrm{Det}(i\hat{\partial} + \hat{A})$ as well as the Green function $D^{\mu\nu}(x - y)$ by two parameters (8.8).

E.2 Calculation to Equation (8.18)

The inverse $(1 + gD)^{-1}$ of the matrix

$$(1 + gD)^{\mu\alpha}(x, z) \;=\; \Big(1 + \bar{\xi}\,\frac{g}{\pi}\Big)\, g^{\mu\alpha}\,\delta^{(2)}(x - z) - \bar{\eta}\,\frac{g}{\pi}\,\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\alpha}\Delta(x - z; \mu). \quad \text{(E.11)}$$

has to obey the condition

$$\int d^2z\,(1 + gD)^{\mu\alpha}(x, z)((1 + gD)^{-1})_{\alpha\nu}(z, y) \;=\; g^{\mu}_{\;\nu}\,\delta^{(2)}(x - y). \quad \text{(E.12)}$$

Making the ansatz for the elements of the inverse matrix $(1 + gD)^{-1}$ as

$$((1 + gD)^{-1})_{\alpha\nu}(z, y) \;=\; \widetilde{A}\,g_{\alpha\nu}\,\delta^{(2)}(z - y) + \widetilde{B}\,\frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial z^\nu}\Delta(z - y; \mu). \quad \text{(E.13)}$$

By introducing the notation

$$A \;=\; \Big(1 + \bar{\xi}\,\frac{g}{\pi}\Big), \quad B \;=\; \bar{\eta}\,\frac{g}{\pi}, \quad \text{(E.14)}$$

we have for the product in Eq. (E.12)

$$\begin{aligned}
&\int d^2z\,(1 + gD)^{\mu\alpha}(x, z)((1 + gD)^{-1})_{\alpha\nu}(z, y)\\
&= A\widetilde{A}g^{\mu}_{\nu} + A\widetilde{B}\int d^2z\,\delta^{(2)}(x - z)\,\frac{\partial}{\partial z_\mu}\frac{\partial}{\partial z^\nu}\Delta(z - y; \mu)\\
&-B\widetilde{A}\int d^2z\,\delta^{(2)}(z - y; \mu)\,\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\nu}\Delta(x - z; \mu)\\
&-B\widetilde{B}\int d^2z\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\nu}\Delta(x - z; \mu)\,\frac{\partial}{\partial z_\mu}\frac{\partial}{\partial z^\nu}\Delta(z - y; \mu).\\
&= A\widetilde{A}g^{\mu}_{\nu} + A\widetilde{B}\,\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\nu}\Delta(x - y; \mu) - B\widetilde{A}\,\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\nu}\Delta(x - y; \mu)\\
&-B\widetilde{B}\frac{\partial}{\partial z_\mu}\frac{\partial}{\partial z^\nu}\Delta(z - y; \mu) \;=\; g^{\mu}_{\;\nu}\,\delta^{(2)}(x - y).
\end{aligned} \quad \text{(E.15)}$$

Hence, the coefficients of the inverse matrix are fixed as

$$\widetilde{A} \;=\; A^{-1}, \qquad A\widetilde{B} - B\widetilde{A} - B\widetilde{B} \;=\; 0. \quad \text{(E.16)}$$

Using Eq. (E.14) they read

$$\widetilde{A} \;=\; \frac{1}{1+\bar{\xi}\,\frac{g}{\pi}}, \qquad \widetilde{B} \;=\; \frac{g}{\pi}\frac{\bar{\eta}}{1+\frac{g}{\pi}(\xi-\eta)}\frac{1}{1+\bar{\xi}\,\frac{g}{\pi}}. \tag{E.17}$$

Thus the matrix $(1\,+\,gD)^{-1}$ is

$$\begin{aligned}((1+gD)^{-1})_{\alpha\nu}(z,y) \;&=\; \frac{1}{1+\xi\,\dfrac{g}{\pi}}\,g_{\alpha\nu}\,\delta^{(2)}(z-y;\mu)\\ &\quad+\frac{g}{\pi}\frac{\bar{\eta}}{1+\frac{g}{\pi}(\xi-\eta)}\frac{1}{1+\bar{\xi}\,\frac{g}{\pi}}\,\frac{\partial}{\partial z^{\alpha}}\frac{\partial}{\partial z^{\nu}}\Delta(z-y;\mu).\end{aligned} \tag{E.18}$$

E.3 Calculations to Equation (8.21)

Calculation of the first term

The first term in Eq. (8.21) reads

$$\begin{aligned}&-\frac{i}{2}\,g\,\partial(\Delta_x-\Delta_y)\,\frac{1}{1+gD}\,\partial(\Delta_x-\Delta_y)\\ &=\;-\frac{i}{2}\,g\iint d^2z_1d^2z_2\;\frac{\partial}{\partial z_{1\alpha}}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\Big[\widetilde{A}\,g_{\alpha\nu}\,\delta^{(2)}(z_1-z_2;\mu)\\ &\qquad+\widetilde{B}\,\frac{\partial}{\partial z_1^{\alpha}}\frac{\partial}{\partial z_1^{\nu}}\Delta(z_1-z_2;\mu)\Big]\,\frac{\partial}{\partial z_{2\nu}}\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big).\end{aligned} \tag{E.19}$$

Integrating over z_2 and afterwards partial integration over z_1 yields for the first term in Eq. (E.19) to become

$$\begin{aligned}&-\frac{i}{2}\,g\int d^2z_1\frac{\partial}{\partial z_{1\alpha}}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\;\widetilde{A}\,g_{\alpha\nu}\\ &\qquad\qquad\times\frac{\partial}{\partial z_{1\nu}}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\\ &\quad-\frac{i}{2}\,g\iint d^2z_1d^2z_2\;\frac{\partial}{\partial z_{1\alpha}}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\widetilde{B}\,\frac{\partial}{\partial z_1^{\alpha}}\frac{\partial}{\partial z_1^{\nu}}\Delta(z_1-z_2;\mu)\\ &\qquad\qquad\times\frac{\partial}{\partial z_{2\nu}}\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big)\\ &=\;-\frac{i}{2}\,g\widetilde{A}\int d^2z_1(-1)\frac{\partial}{\partial z_1^{\alpha}}\frac{\partial}{\partial z_{1\alpha}}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\\ &\qquad\qquad\qquad\times\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\\ &\quad-\frac{i}{2}\,g\,\widetilde{B}\iint d^2z_1d^2z_2\,(-1)^3\,\frac{\partial}{\partial z_1^{\alpha}}\frac{\partial}{\partial z_{1\alpha}}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\\ &\qquad\qquad\qquad\times\frac{\partial}{\partial z_2^{\nu}}\frac{\partial}{\partial z_{1\nu}}\Delta(z_1-z_2;\mu)\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big).\end{aligned} \tag{E.20}$$

Using $\Box\Delta(x-y;\mu) = \delta(x-y)$, we have

$$
\begin{aligned}
&-\frac{i}{2}\,g\,\partial(\Delta_x-\Delta_y)\,\frac{1}{1+gD}\,\partial(\Delta_x-\Delta_y) \\
&\quad= \frac{i}{2}\,g\widetilde{A}\,2\,[\Delta(0;\mu)-\Delta(x-y;\mu)] + \frac{i}{2}\,g\,\widetilde{B}\,2\,[\Delta(0;\mu)-\Delta(x-y;\mu)]. \qquad \text{(E.21)}
\end{aligned}
$$

Inserting the coefficients given in Eq. (E.16) we arrive at

$$
\begin{aligned}
&-\frac{i}{2}\,g\,\partial(\Delta_x-\Delta_y)\,\frac{1}{1+gD}\,\partial(\Delta_x-\Delta_y) \\
&\quad= i\,g\,[\Delta(0;\mu)-\Delta(x-y;\mu)]\left(\frac{1}{1+\frac{g}{\pi}\,\bar{\xi}} + \frac{1}{1+\frac{g}{\pi}\,(\bar{\xi}-\bar{\eta})}\right) \\
&\quad= i\,g\,\frac{1}{1+\frac{g}{\pi}\,(\bar{\xi}-\bar{\eta})}\,[\Delta(0;\mu)-\Delta(x-y;\mu)]. \qquad \text{(E.22)}
\end{aligned}
$$

Calculation of the second term

$$
\begin{aligned}
&\frac{i}{2}\,g\,\partial(\Delta_x-\Delta_y)\,\varepsilon\,\frac{1}{1+gD}\,\varepsilon\,\partial(\Delta_x-\Delta_y) \\
&= \frac{i}{2}\,g\iint d^2z_1d^2z_2\,\frac{\partial}{\partial z_1^\gamma}\Big(\Delta(x-z_1;\mu)\Delta(y-z_1)\Big)\varepsilon^{\gamma\alpha}\Big[\widetilde{A}\,g_{\alpha\nu}\,\delta^{(2)}(z_1-z_2) \\
&\quad+\widetilde{B}\,\frac{\partial}{\partial z_1^\alpha}\frac{\partial}{\partial z_1^\nu}\Delta(z_1-z_2;\mu)\Big]\,\varepsilon^{\nu\delta}\,\frac{\partial}{\partial z_2^\delta}\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big). \qquad \text{(E.23)}
\end{aligned}
$$

Using the formulas

$$
\varepsilon^{\gamma\alpha}\,\varepsilon^{\nu\delta} = -\,g^{\alpha\delta}\,g^{\gamma\nu} + g^{\alpha\nu}\,g^{\gamma\delta} \qquad \text{(E.24)}
$$

we have

$$
\begin{aligned}
&\frac{i}{2}\,g\iint d^2z_1d^2z_2\;\widetilde{A}\frac{\partial}{\partial z_1^\gamma}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)g^{\gamma\delta}\,\delta^{(2)}(z_1-z_2) \\
&\qquad\qquad\frac{\partial}{\partial z_2^\delta}\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big) \\
&\quad+\widetilde{B}\,\frac{\partial}{\partial z_1^\gamma}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\frac{\partial}{\partial z_1^\alpha}\frac{\partial}{\partial z_1^\nu}\Delta(z_1-z_2;\mu)\,g^{\alpha\nu}\,g^{\gamma\delta} \\
&\qquad\qquad\frac{\partial}{\partial z_2^\delta}\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big) \\
&\quad+\widetilde{B}\,\frac{\partial}{\partial z_1^\gamma}\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\frac{\partial}{\partial z_1^\alpha}\frac{\partial}{\partial z_1^\nu}\Delta(z_1-z_2;\mu)\,(-g^{\alpha\delta}\,g^{\gamma\nu}) \\
&\qquad\qquad\frac{\partial}{\partial z_2^\delta}\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big), \qquad \text{(E.25)}
\end{aligned}
$$

integrating partially we get

$$
\begin{aligned}
&-\frac{i}{2}\,g\int d^2z_1\;\widetilde{A}\,\Big(\delta(x-z_1)-\delta(y-z_1)\Big)\,\Big(\Delta(x-z_1;\mu)-\Delta(y-z_1;\mu)\Big)\\
&\quad-\widetilde{B}\,\Big(\delta(x-z_1)-\delta(y-z_1)\Big)\,\Big(\Delta(x-z_2;\mu)-\Delta(y-z_2;\mu)\Big)\\
&\quad+\widetilde{B}\,\Big(\delta(x-z_1)-\delta(y-z_1)\Big)\,\Delta(z_1-z_2;\mu)\,\Big(\delta(x-z_2)-\delta(y-z_2)\Big).
\end{aligned}
\tag{E.26}
$$

Since the last two terms cancel we have

$$
\frac{i}{2}\,g\,\partial(\Delta_x-\Delta_y)\,\varepsilon\,\frac{1}{1+gD}\,\varepsilon\,\partial(\Delta_x-\Delta_y)\;=\;-\,i\,g\,\frac{1}{1+\frac{g}{\pi}\bar{\xi}}\,[\Delta(0;\mu)-\Delta(x-y;\mu)].
\tag{E.27}
$$

Summing both terms in Eqs. (E.22) and (E.27) gives for the argument in the exponent of the causal two–point Green function (8.16)

$$
\begin{aligned}
&i\,g\,\left(\frac{1}{1+\frac{g}{\pi}\,(\bar{\xi}-\bar{\eta})}\;-\;\frac{1}{1+\frac{g}{\pi}\bar{\xi}}\right)\,[\Delta(0;\mu)-\Delta(x-y;\mu)]\\
&\quad=\;i\,\frac{g^2}{\pi}\,\frac{\bar{\eta}}{1+\frac{g}{\pi}\,(\bar{\xi}-\bar{\eta})}\,\frac{1}{1+\frac{g}{\pi}\bar{\xi}}\,[\Delta(0;\mu)-\Delta(x-y;\mu)].
\end{aligned}
\tag{E.28}
$$

The argument in the exponent of the correlation function (8.23) reads

$$
2\,i\,g\,\frac{1}{1+\frac{g}{\pi}\bar{\xi}}\,[\Delta(0;\mu)-\Delta(x-y;\mu)].
\tag{E.29}
$$

E.4 Constraints on the parameters $\bar{\xi}$ and $\bar{\eta}$ from the norms of the wave functions of the states related to the components of the vector current

The dependence of the functional determinant $\mathrm{Det}(i\hat{\partial}+\hat{A})$ on two parameters leads to the dependence of the two–point correlation function $\langle 0|\mathrm{T}(j^{\mu}(x)j^{\nu}(y))|0\rangle$ on these parameters. Following Johnson [41] we get for the vacuum expectation value $\langle 0|\mathrm{T}(j^{\mu}(x)j^{\nu}(y))|0\rangle$

$$
\begin{aligned}
i\langle 0|\mathrm{T}(j^{\mu}(x)j^{\nu}(y))|0\rangle\;&=\;-\frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\,\frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x_{\nu}}\Delta(x-y;\mu)\\
&\quad+\;\frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\,g^{\mu 0}g^{\nu 0}\,\delta^{(2)}(x-y).
\end{aligned}
\tag{E.30}
$$

This gives the following vacuum expectation values

$$\begin{aligned}\langle 0|j^0(x)j^0(y)|0\rangle &= -\frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\Big(\frac{\partial}{\partial x^1}\Big)^2 D^{(+)}(x-y)\\ \langle 0|j^1(x)j^1(y)|0\rangle &= -\frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\Big(\frac{\partial}{\partial x^1}\Big)^2 D^{(+)}(x-y),\end{aligned} \tag{E.31}$$

where $D^{(\pm)}(x-y)$ are the Wightman functions given by

$$D^{(\pm)}(x-y) = \int \frac{d^2k}{(2\pi)^2}\, 2\pi\theta(k^0)\delta(k^2)\, e^{\mp ik\cdot(x-y)} \tag{E.32}$$

We have taken into account that

$$\Delta(x-y;\mu) = i\theta(x^0-y^0)\,D^{(+)}(x-y) + i\theta(y^0-x^0)\,D^{(-)}(x-y). \tag{E.33}$$

According to Wightman and Streater [64] and Coleman [65], we can define the wave functions of the states

$$\begin{aligned}|h;j^0\rangle &= \int d^2x\,h(x)\,j^0(x)|0\rangle,\\ |h;j^1\rangle &= \int d^2x\,h(x)\,j^1(x)|0\rangle,\end{aligned} \tag{E.34}$$

where $h(x)$ is the test function from the Schwartz class $h(x)\in\mathcal{S}(\mathbb{R}^2)$ [64].

The norms of the states (E.34) are equal to [64, 65]

$$\begin{aligned}&\langle j^0;h|h;j^0\rangle = \iint d^2x d^2y\, h^*(x)\,\langle 0|j^0(x)j^0(y)|0\rangle\, h(y) =\\ &= \frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\int\frac{d^2k}{(2\pi)^2}\,2\pi\,(k^0)^2\theta(k^0)\delta(k^2)\,|\tilde{h}(k)|^2,\\ &\langle j^1;h|h;j^1\rangle = \frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\\ &\qquad\times \iint d^2x d^2y\, h^*(x)\,\langle 0|j^1(x)j^1(y)|0\rangle\, h(y) =\\ &= \frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big)}\int\frac{d^2k}{(2\pi)^2}\,2\pi\,(k^0)^2\theta(k^0)\delta(k^2)\,|\tilde{h}(k)|^2,\end{aligned} \tag{E.35}$$

where $\tilde{h}(k)$ is the Fourier transform of the test function $h(x)$. Since the norms of the states (E.34) should be positive, we get the constraint

$$\bar{\eta}\,\Big(1+\bar{\xi}\,\frac{g}{\pi}\Big)\Big(1+(\bar{\xi}-\bar{\eta})\,\frac{g}{\pi}\Big) > 0. \tag{E.36}$$

This assumes that $\bar{\eta} \neq 0$. For $\bar{\eta}$, constrained by the requirement of the renormalisability of the massless Thirring model (8.30), the inequality (8.36) reduces to the form

$$- g\left(1 + \bar{\xi}\,\frac{g}{\pi}\right) > 0. \tag{E.37}$$

This inequality is fulfilled for

$$g > 0\;,\; 1 + \bar{\xi}\,\frac{g}{\pi} < 0 \quad , \quad g < 0\;,\; 1 + \bar{\xi}\,\frac{g}{\pi} > 0. \tag{E.38}$$

Using the vacuum expectation value of the two–point correlation function of the vector currents (E.30) we can calculate the Schwinger term, defining the equal–time commutator. For the equal–time commutator $[j^0(x), j^1(y)]_{x^0=y^0}$ we get

$$\begin{aligned}[j^0(x), j^1(y)]_{x^0=y^0} &= -\,c\,i\frac{\partial}{\partial x^1}\delta(x^1 - y^1) = \\ &= -\,\frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1 + \bar{\xi}\,\dfrac{g}{\pi}\Big)\Big(1 + (\bar{\xi} - \bar{\eta})\,\dfrac{g}{\pi}\Big)}\,i\frac{\partial}{\partial x^1}\delta(x^1 - y^1)\end{aligned} \tag{E.39}$$

with the Schwinger term c equal to

$$c = \frac{\bar{\eta}}{\pi}\,\frac{1}{\Big(1 + \bar{\xi}\,\dfrac{g}{\pi}\Big)\Big(1 + (\bar{\xi} - \bar{\eta})\,\dfrac{g}{\pi}\Big)}. \tag{E.40}$$

Due to the constraint (E.37) the Schwinger term is always positive. For $\bar{\eta} = 1$ our expression (E.40) for the equal–time commutator coincides with that obtained by Hagen [43].

Using (E.40) we can analyse the Bjorken–Johnson–Low (BJL) limit for the Fourier transform of the two–point correlation function of the vector currents [66, 67]. Following [67], we consider the Fourier transform

$$T_{\mu\nu}(q) = i\int d^2x\, e^{\,iq\cdot x}\,\langle A|\mathrm{T}(j_{\mu}(x) j_{\nu}(0))|B\rangle, \tag{E.41}$$

where $q = (q^0, q^1)$ and $|A\rangle$ and $|B\rangle$ are quantum states [67]. In our case these are vacuum states $|A\rangle = |B\rangle = |0\rangle$. This gives

$$T_{\mu\nu}(q) = i\int d^2x\, e^{\,iq\cdot x}\,\langle 0|\mathrm{T}(j_{\mu}(x) j_{\nu}(0))|0\rangle. \tag{E.42}$$

According to the BJL theorem [66, 67], in the limit $q^0 \to \infty$ the r.h.s of Eq. (E.41) behaves as follows [67]

$$\begin{aligned}T_{\mu\nu}(q) &= -\,\frac{1}{q^0}\int^{+\infty}_{-\infty} dx^1\, e^{\,-iq^1x^1}\,\langle 0|[j_{\mu}(0,x^1), j_{\nu}(0)]|0\rangle \\ &\quad -\frac{i}{(q^0)^2}\int^{+\infty}_{-\infty} dx^1\, e^{\,-iq^1x^1}\,\langle 0|[\partial_0 j_{\mu}(0,x^1), j_{\nu}(0)]|0\rangle + O\Big(\frac{1}{(q^0)^3}\Big)\end{aligned} \tag{E.43}$$

For the time–space component of the two–point correlation function we get

$$T_{01}(q^0,q^1) \;=\; -\frac{1}{q^0}\int^{+\infty}_{-\infty} dx^1\, e^{\,-iq^1x^1}\langle 0|[j_0(0,x^1),j_1(0)]|0\rangle$$
$$-\frac{i}{(q^0)^2}\int^{+\infty}_{-\infty} dx^1\, e^{\,-iq^1x^1}\langle 0|[\partial_0 j_0(0,x^1),j_1(0)]|0\rangle + O\Big(\frac{1}{(q^0)^3}\Big) \tag{E.44}$$

Using (E.39) for the BJL limit of $T_{01}(q^0,q^1)$ we obtain

$$T_{01}(q^0,q^1) \;=\; \frac{q^1}{q^0}\,\frac{\bar\eta}{\pi}\,\frac{1}{\Big(1+\bar\xi\,\dfrac{g}{\pi}\Big)\Big(1+(\bar\xi-\bar\eta)\,\dfrac{g}{\pi}\Big)} + O\Big(\frac{1}{(q^0)^3}\Big). \tag{E.45}$$

For $\bar\eta = 1$ this reproduces the result which can be obtained using Hagen's solution [43]. One can show that due to conservation of the vector current the term proportional to $1/q_0^2$ vanishes. The asymptotic behaviour of the Fourier transform of the two–point correlation function of the vector currents places no additional constraints on the parameters $\bar\xi$ and $\bar\eta$.

The inequality (E.36) leads to the following interesting consequences. According to Coleman [2], the coupling constant β^2 of the sine–Gordon model is related to the coupling constant g of the Thirring model as

$$\frac{\beta^2}{8\pi} = \frac{1}{2} + d_{(\bar\psi\psi)^2}(g) = \frac{1}{2}\Big(1 - \frac{g}{\pi}\,\frac{1}{1+\bar\xi\,\dfrac{g}{\pi}}\Big). \tag{E.46}$$

Hence, for the constraint (E.38) the coupling constant β^2 is of order $\beta^2 \sim 8\pi$. The behaviour and renormalisability of the sine–Gordon model for the coupling constants $\beta^2 \sim 8\pi$ has been investigated in [24].

Appendix F

Definitions

Metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{F.1}$$

Quabla operator:

$$\Box = \partial_\mu \partial^\mu . \tag{F.2}$$

Notation for integrals

$$\int \equiv \int_{-\infty}^{\infty} \tag{F.3}$$

Lorentz transformation

$$\begin{aligned} y^1 &= x^1 \cosh\varphi - x^0 \sinh\varphi, \quad & y^0 &= x^0 \cosh\varphi - x^1 \sinh\varphi \\ x^1 &= y^1 \cosh\varphi + y^0 \sinh\varphi, \quad & x^0 &= y^0 \cosh\varphi + y^1 \sinh\varphi, \end{aligned} \tag{F.4}$$

with velocity parameter v, $v = \tanh\varphi$.

ϵ–Tensor

$$\varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \varepsilon_{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = g_{\mu\alpha}\varepsilon^{\alpha\beta}g_{\beta\nu} = -\varepsilon^{\mu\nu} \tag{F.5}$$

$$\varepsilon^{\mu\nu}\varepsilon^{\lambda\rho} = -g^{\mu\lambda}g^{\rho\nu} + g^{\mu\rho}g^{\nu\lambda} \qquad \varepsilon_{\mu\alpha}\varepsilon^{\alpha\nu} = g^\mu_\nu \tag{F.6}$$

γ–Matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{F.7}$$

Relations for γ Matrices

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \qquad \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0 \tag{F.8}$$

$$\gamma^\mu\gamma^5 = -\varepsilon^{\mu\nu}\gamma_\nu \qquad \gamma^\mu\gamma^\nu = g^{\mu\nu} + \varepsilon^{\mu\nu}\gamma^5 \tag{F.9}$$

Useful trace formulas in Minkowski space

$$\begin{aligned}
\mathrm{tr}\left\{\gamma^\alpha\gamma^\beta\right\} &= \;\; 2\,g^{\alpha\beta} \\
\mathrm{tr}\left\{\gamma^\alpha\gamma^\beta\gamma^5\right\} &= -2\,\varepsilon^{\alpha\beta} \\
\mathrm{tr}\left\{\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right\} &= \;\; 2\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right) \\
\mathrm{tr}\left\{\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^5\right\} &= -2\left(g^{\alpha\beta}\varepsilon^{\gamma\delta} + \varepsilon^{\alpha\beta}g^{\gamma\delta}\right)
\end{aligned} \tag{F.10}$$

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List of Publications

1) *On the renormalizability of the massless Thirring model*
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3) *Are there Local Minima in the Magnetic Monopole Potential in Compact QED?*
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