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Investigations on Intuitionistic Logic and Modal Logic

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unter der Anleitung von
Ao.Univ.Prof. Dr. Uwe Egly

durch
Patrick Seidelmann
Döblinger Hauptstraße 17/4/8
1190 Wien

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Abstract

In this thesis, we investigate three different non-classical propositional logics, which are relevant to computer science.

The first logic under consideration is *intuitionistic logic*. We present a Hilbert style calculus and a calculus of natural deduction, and show completeness and soundness for both of them with respect to intuitionistic Kripke models.

Then we investigate *normal modal logic*, with an emphasis on K, the basic normal modal logic. We also glimpse at stronger normal modal logics. Here we only present one calculus, a Hilbert style calculus, extended to fit the new requirements to show completeness.

In the last part, we direct our attention to a relatively new non-classical logic; namely *intuitionistic modal logic*. We investigate the soundness and completeness of the system $\mathsf{HK}\Box$, which was first presented in [BD84]. We identified some severe gaps in the completeness proof in this paper.

Chapter 1

Introduction

In the broad field of computer science logic plays an important part. Not only in theoretical computer science, but also sometimes hidden from the unwary observer. The usage of logic ranges from conditional statements in programming languages to applications like chip design and the verification of chips. In most of these applications, classical logic is used. This is the meta-logic usually applied in mathematical reasoning.

Since the beginning of the 20th century, when logic itself has become an area of active interest and research, logics other than classical logic have been devised, because of dissatisfaction with classical logic.

C. I. Lewis criticized the usual (material) implication and created a series of logical systems which were supposed to circumvent these problems. His investigations ultimately lead to our modern systems of normal modal logic.

L. E. J. Brouwer wanted to put mathematics on a more solid foundation than classical logic with its indirect methods. His constructive approach to mathematics (and thus also to logic) has been further developed by many mathematicians and logicians to our modern notion of intuitionistic logic. Although rarely used as a meta-logic (which was Brouwer's original intention), it nevertheless has its merits and applications, especially in the field of computer science.

These are two of the non-classical logics with which we will concern ourselves. Intuitionistic logic is a "sub-logic" of classical logic, as there are fewer "true" statements in intuitionistic logic than in classical logic. In contrast, modal logic has additional connectives (so called modalities) compared to classical logic and thus can express more statements than classical logic, while still permitting all classically true statements.

It is an obvious idea to add modalities to intuitionistic logic and just see what we get. This has first been done by R. A. Bull in [Bul66] and Gisèle Fischer Servi in [FS77]. Since then many other intuitionistic modal logics have been devised and proven complete for some semantics.

In the following of this chapter, we will present some conventions and preliminary definitions to facilitate reading of the rest of this thesis.

In Chapter 2, we treat propositional intuitionistic logic in detail, presenting two calculi and proving completeness for intuitionistic Kripke models

In Chapter 3, we concern ourselves with some propositional normal modal logics and present a soundness and completeness proof for a Hilbert style calculus, which is an extension of one of the calculi presented in Chapter 2.

In Chapter 4 we present the intuitionistic modal logic calculus $\mathsf{\mathbf{HK}}\Box$, as a representative of an intuitionistic modal logic. This logic was first presented in [BD84]. Taking this paper under close scrutiny, we identified severe gaps in the completeness proof, which we point out for future investigation.

1.1 Conventions

1.1.1 Notational Conventions

In order to facilitate reading, we try to adhere as strictly as possible to the following conventions regarding notation:

- We use lowercase Greek letters to denote formulae or formula schemata. Most often we use $\varphi, \psi, \theta, \alpha, \beta$, etc., possibly sub-scripted or primed.

- We use uppercase Greek letters to denote sets of formulae, most often Σ, Λ and Γ, possibly primed.

- We use lowercase Latin letters starting from p for logical variables (i.e. atoms): p, q, r, etc., possibly sub-scripted or primed.

- We use lowercase Latin letters starting from a to denote worlds (or nodes) in Kripke models: a, b, c, etc., possibly sub-scripted or primed.

- We use uppercase characters (and numbers) in a sans serif-font like $\mathsf{4}$ or K for axioms. Logical rules will additionally be put in brackets as, e.g., (MP).

- We use a similar font for different calculi or logical systems as we do for axioms and inference rules, but we use boldface characters as e.g. $\mathsf{\mathbf{L}}$ or $\mathsf{\mathbf{C}}\Box$.

Language

As we present different logical systems in this thesis, the language of the logic may also differ, but here is a super-set of all the elements which we use:

- We use the binary *logical connectives* $\wedge$ (conjunction), $\vee$ (disjunction) and $\rightarrow$ (implication) and the unary logical connective $\neg$ (negation).
- We use the *modalities* $\Box$ and $\Diamond$. Modalities are unary.
- We use logical variables, denoted as described above.
- We only use $\perp$ as logical constant for falsity.
- We may use abbreviations for sub-formulae instead of writing out all formulae.

The language of the different logics is defined in the usual way. So all logical variables, logical constants and sub-formula symbols are in the language of the logic, as well as all permissible combinations thereof with logical connectives and modalities (if modalities are in the respective logic).

Metalanguage

The following symbols will be used (in increasing order of precedence):

- $\Rightarrow$ for "implies"
- $\Leftrightarrow$ for "if and only if"
- $\Leftrightarrow$: for "is defined as"

1.1.2 Logical Conventions and Preliminary Definitions

We have a countable infinite set of variables.

As usual, we make no distinction between formulae and schemata for formulae.

For better readability, we sometimes omit parentheses. To make this possible we have the following decreasing order of precedence of our logical connectives:

- $\neg, \Box, \Diamond$: the unary connective and the modalities bind most strongly,
- $\wedge$,

- $\vee$,
- $\to$, whereas implication is right-associative (i.e., $\varphi \to \psi \to \theta$ is the same as $\varphi \to (\psi \to \theta)$).

For chains of disjunctions and conjunctions, we will usually omit parentheses and assume left-associativity, i.e., we will write $\varphi \wedge \psi \wedge \theta$ instead of $((\varphi \wedge \psi) \wedge \theta)$.

The symbol $\vdash$ (possibly with subscript) is usually used for a syntactic proof relation, whereas the symbol $\vDash$ is used for semantic forcing relation.

$\Sigma \vdash \varphi$ denotes a concrete proof of φ with assumptions in Σ, but—by a slight abuse of notation—we will also use it to express the fact that there is such a proof, i.e., φ *can* be proven when assuming all formulas in Σ to be true.

When using induction arguments over formulae, we need a measure for the complexity of a formula. The *logical complexity*, $\mathrm{lcomp}(\varphi)$, of a formula φ is defined next.

Definition 1.1.1 (Logical complexity)**.** *The logical complexity of a formula φ is inductively defined as follows:*

$$\mathrm{lcomp}(\varphi) = \begin{cases} 0 & \textit{if } \varphi \textit{ is atomic or a logical variable,} \\ \mathrm{lcomp}(\varphi_1) + \mathrm{lcomp}(\varphi_2) & \textit{if } \varphi = \varphi_1 \circ \varphi_2\textit{, for } \circ \in \{\wedge, \vee, \to\}, \\ \mathrm{lcomp}(\varphi_1) + 1 & \textit{if } \varphi = \circ\varphi\textit{, for } \circ \in \{\neg, \Box\}. \end{cases}$$

Chapter 2

Intuitionistic Logic

2.1 History and Motivation

Intuitionistic Logic was first formulated by Luitzen E. J. Brouwer in 1907 (see [Bro07] for details). Brouwer considered logic as a sub-area of mathematics and not as the foundation of mathematics, as it was and is widely viewed. Logical reasoning is a mathematical activity, logical proofs are nothing but mathematical constructs.

One feature of classical logic which was criticized by Brouwer is the *principle of the excluded middle*, i.e., $A \vee \neg A$ is true in classical logic. When regarding A as a proof of a statement which can neither be proved nor falsified (like Fermat's Theorem at that time, or a still undecided statement like Goldbach's Conjecture), $A \vee \neg A$ is not evident.

In 1931 Arend Heyting presented the *proof interpretation* of intuitionistic logic, which is in the spirit of Brouwer's criticism of classical logic, viewing intuitionistic statements as proofs/mental processes of an abstract mathematician.

Already 1933 Kurt Gödel established the connection between the modal logic **S4** (which will be treated in Section 3.2.1) and intuitionistic logic in [Göd33]. Later several so called *topological interpretations* where found for intuitionistic logic, among those the *Kripke structures* developed by Saul Kripke in 1963 is of most interest to us. As Gödel's identification of a modal logic with intuitionistic logic suggests, this semantics is important for modal logics as well, as we will see in Chapter 3.

Intuitionistic logic and its constructive approach have shown to be quite useful for computer science: A constructive proof of a theorem can also be viewed as a λ-expression (or a computer program) creating witnesses for that theorem.[1] More specifically, the *Curry-Howard Isomorphism*, named after Haskell B. Curry and William A. Howard, who among others discovered this correspondence, links proofs of theorems to types of the typed λ-calculus. So a proof of a certain proposition can be read as a λ-program for obtaining its correspondent type and vice versa. These λ-expressions can be translated into conventional computer programs and executed, hence the link to computer science.

A thorough survey of the Curry-Howard Isomorphism with an introduction to the typed λ-calculus can be found in [Tho91].

2.2 Propositional Intuitionistic Logic

The following is a propositional calculus for intuitionistic logic in the spirit of Heyting as found in [BD84].

Definition 2.2.1 (Hilbert style calculus for intuitionistic logic)**.** *These are the axioms of Heyting's propositional calculus (named* **H***):*

H1 $\varphi \to (\psi \to \varphi)$

H2 $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$

H3 $(\theta \to \varphi) \to ((\theta \to \psi) \to (\theta \to \varphi \wedge \psi))$

H4 $(\varphi \wedge \psi) \to \varphi$

H5 $(\varphi \wedge \psi) \to \psi$

H6 $\varphi \to (\varphi \vee \psi)$

H7 $\psi \to (\varphi \vee \psi)$

H8 $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$

H9 $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$

H10 $\neg\varphi \to (\varphi \to \psi)$

The only rule in **H** *is the* Modus Ponens *rule:*

(MP) $\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$

[1] For readers familiar with LISP, λ-expressions are LISP programs in a broader sense.

Remark 2.2.2. *We will use $\bot$ as a symbol for logical falsity. It can be viewed as a variable with a constant truth value, a logical connective with arity 0 or an abbreviation for any false formula (e.g., $\varphi \wedge \neg\varphi$). In an analogous manner, we sometimes use $\top$, which denotes logical truth. We view $\bot$ as a logical connective with arity 0 and with the logical complexity 0, i.e.,* $\mathrm{lcomp}(\bot) = 0$*. By contrast, $\top$ will be viewed as an abbreviation of the formula $\bot \to \bot$, which is provable in all systems presented in this thesis, as we see in Example 2.2.4.*

Often $\neg\varphi$ is defined to be $\varphi \to \bot$. Then $\neg$ is not in the language of the logic, but rather $\bot$. The preceding axioms are translated in a natural way.

If one of the axioms $\varphi \vee \neg\varphi$ or $\neg\neg\varphi \to \varphi$ is added to the axiom system of Definition 2.2.1, we obtain classical logic. Thus the intuitionistically provable formulae are a subset of the classically provable formulae.

Definition 2.2.3 (Heyting proofs)**.** *A proof $\Sigma \vdash_{\mathsf{H}} \varphi$ in the calculus* **H** *is a sequence $(\psi_1, \psi_2, \ldots, \psi_n)$ of formulae, where $\psi_n = \varphi$ and for all ψ_i either:*

- *ψ_i is a substitution instance of one of the axioms H1,…,H10,*
- *$\psi_i \in \Sigma$ (i.e., ψ_i is a premise),*
- *ψ_i is the conclusion of an application of MP, whose premises are ψ_j, ψ_k with $j, k < i$.*

The length of the proof $\Sigma \vdash_{\mathsf{H}} \varphi$ *is the length of the sequence $(\psi_1, \psi_2, \ldots, \psi_n)$, which is n in this case. If Σ, the set of premises, is empty, we may write $\vdash_{\mathsf{H}} \varphi$ instead of $\emptyset \vdash_{\mathsf{H}} \varphi$ for simplicity.*

As stated above, we will use substitution instances of axioms in our proofs. We obtain a *substitution instance* of a formula schema by consistently replacing all occurrences of the sub-formulae φ, ψ and θ by arbitrary formulae. So $\bot \to (\psi \to \bot)$ is a substitution instance of H1, as is $\varphi \to (\varphi \to \varphi)$, but $\varphi \to (\psi \to \bot)$ is not a substitution instance of H1 as the atomic sub-formula φ is replaced by $\bot$ only at its second occurrence, but not at its first.

Example 2.2.4. *The following is a proof of $\vdash_{\mathsf{H}} \varphi \to \varphi$, where we mark axioms and results of (MP):*
[1: H2] $(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$
[2: H1] $\varphi \to ((\varphi \to \varphi) \to \varphi)$
[3: (MP) 2, 1] $(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$
[4: H1] $\varphi \to (\varphi \to \varphi)$
[5: (MP) 4, 3] $\varphi \to \varphi$

The same proof is depicted as a tree in Figure 2.1.

$$\dfrac{\varphi \to \alpha \qquad \dfrac{\varphi \to (\alpha \to \varphi) \quad (\varphi \to (\alpha \to \varphi)) \to ((\varphi \to \alpha) \to \alpha)}{(\varphi \to \alpha) \to \alpha}\ (\mathsf{MP})}{\alpha}\ (\mathsf{MP})$$

Figure 2.1: Proof of $\alpha : \varphi \to \varphi$ from Example 2.2.4.

As we have seen in the last example, a proof in a "linear notation" can be transformed to a proof in "tree notation" and vice versa. In general, the latter can be longer than the former.

Example 2.2.5. *When falsity $\bot$ is in the language, then we can show that, for all φ, it holds that $\vdash_{\mathsf{H}} \bot \to \varphi$. This means that, whenever we can deduce $\bot$, arbitrary formulae can be deduced (*ex falso quod libet*).*
Since H10 is an abbreviation for $(\psi \to \bot) \to (\psi \to \varphi)$, with the substitution instance $(\bot \to \bot) \to (\bot \to \varphi)$ and Example 2.2.4, we get the following tree proof:

$$\dfrac{\overset{\textit{by Example 2.2.4}}{\overset{\triangledown}{\bot \to \bot}} \qquad \overset{\mathsf{\textit{H10}}}{(\bot \to \bot) \to (\bot \to \varphi)}}{(\bot \to \varphi)}\ (\mathsf{\textit{MP}})$$

In the above proof tree, the symbol $\triangledown$ denotes a known sub-proof, which we omit for better readability. We will use this symbol whenever we already have a proof for the formula which is written directly below $\triangledown$. In this case, the formula $\bot \to \bot$ is a substitution instance of $\varphi \to \varphi$, which is the formula proven in Example 2.2.4.

We will now formulate the *Deduction Theorem* for intuitionistic propositional logic with Heyting proofs.

Theorem 2.2.6 (Deduction Theorem for intuitionistic Heyting proofs)**.**

$$\Sigma, \varphi \vdash_{\mathsf{H}} \psi \Rightarrow \Sigma \vdash_{\mathsf{H}} \varphi \to \psi$$

Proof. Suppose there is a proof $\Sigma, \varphi \vdash_{\mathsf{H}} \psi$, which is a sequence of formulae of the form $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_n = \psi$. We will transform this proof into a proof $\Sigma \vdash_{\mathsf{H}} \varphi \to \psi$ of the form $(\beta_1, \ldots, \beta_m)$ with $\beta_m = \varphi \to \psi$.

We proceed by induction on n, the length of the proof $(\alpha_1, \ldots, \alpha_n)$ and show that, for every such proof, there is a proof $\Sigma \vdash_{\mathsf{H}} \varphi \to \psi$.

Induction Base. $n = 1$.
If α_n is an *axiom*, then we trivially have $\Sigma, \varphi \vdash_{\mathsf{H}} \alpha_n$, because, via the axiom

$\mathsf{H1}$ and modus ponens, we get

$$\frac{\alpha_n \qquad \alpha_n \to (\varphi \to \alpha_n)}{\varphi \to \alpha_n}(\mathsf{MP})$$

and therefore we have $\Sigma \vdash_{\mathsf{H}} \varphi \to \alpha_n$.
If $\alpha_n \in \Sigma$, then we proceed analogously.
If $\alpha_n = \varphi$, then we have $\vdash_{\mathsf{H}} \varphi \to \varphi$ by Example 2.2.4 and thus $\Sigma \vdash_{\mathsf{H}} \varphi \to \varphi$.

Induction Hypothesis. Suppose $n \geq 1$ and assume that, for all proofs $\Sigma, \varphi \vdash_{\mathsf{H}} \psi$ of length $\leq n$, we have corresponding proofs $\Sigma \vdash_{\mathsf{H}} \varphi \to \psi$.

Induction Step. For the induction step, let us consider a proof $\Sigma, \varphi \vdash_{\mathsf{H}} \psi$ of length $n+1$, which is of the form $(\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ with $\alpha_{n+1} = \psi$.
If α_{n+1} is an axiom, an element of Σ or $\alpha_{n+1} = \varphi$, this corresponds essentially to the base case above and we would just have to add some formulae to the proof $\Sigma \vdash_{\mathsf{H}} \varphi \to \alpha_n$ (which exists by the induction hypothesis).

On the other hand, if α_{n+1} is the result of an application of (MP) of the form

$$\frac{\alpha_i \qquad \alpha_j}{\alpha_{n+1}}\ (\mathsf{MP}),$$

where $i, j < n+1$ and without loss of generality $\alpha_j = \alpha_i \to \alpha_{n+1}$. Then we proceed as follows: By the induction hypothesis, there are two proofs $\Sigma \vdash_{\mathsf{H}} \varphi \to \alpha_i$ and $\Sigma \vdash_{\mathsf{H}} \varphi \to \alpha_j$. So with the substitution instance of axiom $\mathsf{H2}$: $\varphi \to (\alpha_i \to \alpha_{n+1}) \to ((\varphi \to \alpha_i) \to (\varphi \to \alpha_{n+1}))$ and two applications of modus ponens, we get $\Sigma \vdash_{\mathsf{H}} \varphi \to \alpha_{n+1}$.

This concludes the proof of the theorem. $\square$

We will now introduce a semantics for intuitionistic logic: the *possible world semantics* first conceived by Saul Kripke in [Kri65].

The idea behind this semantics is that we have a certain state of knowledge any given moment in a certain world. This knowledge follows—more or less—classical rules.
So when we know that φ is true, we also know that, for any given ψ, the formula $\psi \to \varphi$ is also true, just as it would be in a classical setting. When we have φ and ψ, we also have $\varphi \wedge \psi$ and so forth. But it is possible for us to enlarge our knowledge and obtain new facts by going to a new world accessible from our current one.

The whole "universe" of all accessible worlds is our Kripke model. Only formulae which are universally true for the model (i.e., they are true in all worlds of the model) are true in the model.

In the following definition, we create a formal framework for this kind of model.

Definition 2.2.7 (Kripke structures, Kripke models for intuitionistic logic)**.** *A model*

$$\mathscr{M} = (N, \leq, D)$$

is a tuple, where N is a partially ordered, non-empty set of nodes N (or worlds), $\leq$ is a reflexive and transitive relation over N and D is the domain function

$$D : N \to \mathscr{P}(\mathbb{VAR}),^{2}$$

satisfying the constraint

$$\forall a, b \in N : \ a \leq b \Rightarrow D(a) \subseteq D(b).$$

When we do not need to consider the domain function D of a Kripke model, the pair $(N, \leq)$ is called a Kripke structure*.*

Remark 2.2.8. *"For all $a \in \mathscr{M}$" denotes all elements of N in a model $\mathscr{M} = (N, \leq, D)$.*

By the means of a forcing relation $\vDash$, we first introduce the concept of truth in a world of a Kripke model and then expand this notion to the whole model and classes of models (i.e., all models satisfying all formulas in some set Σ).

Definition 2.2.9 (Forcing relation)**.** *The forcing relation $\vDash$ establishes truth in a model $\mathscr{M} = (N, \leq, D)$ and is defined inductively as follows:*

1. $a \vDash p \Leftrightarrow: p \in D(a)$;
2. $a \vDash \psi \wedge \varphi \Leftrightarrow: a \vDash \psi$ *and* $a \vDash \varphi$;
3. $a \vDash \psi \vee \varphi \Leftrightarrow: a \vDash \psi$ *or* $a \vDash \varphi$;
4. $a \vDash \psi \to \varphi \Leftrightarrow:$ *for all b with $a \leq b : b \vDash \psi \Rightarrow b \vDash \varphi$;*
5. $a \vDash \neg\varphi \Leftrightarrow:$ *for all b with $a \leq b : b \nvDash \varphi$.*

We say φ holds in a model $\mathscr{M}$, $\mathscr{M}$ forces φ or $\mathscr{M} \vDash \varphi$ holds, if, for all worlds a of $\mathscr{M}$, $a \vDash \varphi$ holds. We say $\mathscr{M} \vDash \Sigma$ holds, if $\mathscr{M} \vDash \sigma$ holds for all $\sigma \in \Sigma$. Furthermore, we say $\Sigma \vDash \varphi$ holds, if, for all models with $\mathscr{M} \vDash \Sigma$, also $\mathscr{M} \vDash \varphi$ holds.

Remark 2.2.10 (Alternative definition of $\neg\varphi$)**.** *When $\neg\varphi$ is defined as $\varphi \to \bot$, item number 5 of the previous definition is a consequence of the fact that $\bot$ is never forced at any world (i.e., for all $a \in \mathscr{M}$, $a \nvDash \bot$ holds).*

[2] $\mathscr{P}(\mathbb{VAR})$ is the power-set of the infinite, countable set of propositional atoms.

Lemma 2.2.11 (Intuitionistic Heredity)**.**

$$a \leq b \Rightarrow (a \vDash \varphi \Rightarrow b \vDash \varphi)$$

Proof. The above statement expresses the fact that every succeeding world forces all formulae which have been forced in its $\leq$-predecessor. Roughly speaking, true statements stay true over the $\leq$-relation.

We have to prove that, for every world b, for which $a \leq b$ holds, all statements φ, which are forced in a, are also forced in b. As always, when considering implications as $a \vDash \varphi \Rightarrow b \vDash \varphi$, we only have to regard the case that the left hand side of the implication is true, because otherwise the statement as a whole is trivially true.

We prove the lemma by induction on $\mathrm{lcomp}(\varphi)$, the logical complexity of φ. Assume that $a \leq b$ holds.

Induction Base. Consider φ with $\mathrm{lcomp}(\varphi) = 0$, i.e., φ is of the form p, where p is a variable, or φ is $\perp$. As $\perp$ is not forced in any world, $a \vDash \perp$ is never true and so in this case the statement is trivially true, as we have stated above.
Now suppose $a \vDash p$ holds. By Definition 2.2.9, this is equivalent to $p \in D(a)$. By assumption, we have $a \leq b$, which implies $D(a) \subseteq D(b)$ by Definition 2.2.7. Thus, we know that $p \in D(b)$ and obtain $b \vDash \varphi$.

Induction Hypothesis. Suppose $n \geq 0$ and assume, that, for all formulas φ with $\mathrm{lcomp}(\varphi) \leq n$ and all worlds a, b with $a \leq b$, $b \vDash \varphi$ holds, if $a \vDash \varphi$ holds.

Induction Step. Consider φ with $\mathrm{lcomp}(\varphi) = n + 1$. Since $n \geq 0$, φ contains at least one connective. We perform a case distinction according to the outermost logical connective, as, for the remaining immediate subformulae, the desired property holds by the induction hypothesis.

φ is of the form $\psi \wedge \sigma$. Now suppose $\psi \wedge \sigma$ is forced in a. By item 2 of Definition 2.2.9, $a \vDash \psi$ and $a \vDash \sigma$ also holds. We apply the induction hypothesis and obtain $b \vDash \psi$ and $b \vDash \sigma$ for all worlds b with $a \leq b$. Having both $b \vDash \psi$ and $b \vDash \sigma$, once again by item 2 of Definition 2.2.9, we obtain $b \vDash \psi \wedge \sigma$ for all b with $a \leq b$.

φ is of the form $\psi \vee \sigma$. This case of the induction step is analogous to the case above with the slight difference that we quote item 3 of Definition 2.2.9 and we only know that ψ *or* σ is forced in a (and b respectively).

φ is of the form $\psi \rightarrow \sigma$. Suppose $a \vDash \psi \rightarrow \sigma$ holds. We use an indirect approach and suppose that $b \nvDash \varphi$. By Definition 2.2.9, item 4, there is a c with $b \leq c$ for which $c \vDash \psi$ holds and $c \vDash \sigma$ does not hold. As—by the transitivity of $\leq$—it holds that $a \leq c$, by Definition 2.2.9, item 4, $a \nvDash \psi \rightarrow \sigma$ holds. This is a contradiction to our assumption, so $b \vDash \varphi$ holds.

φ is of the form $\neg\psi$. Suppose $\neg\psi$ is forced in a. Then $b \nvDash \psi$ holds for all b with $a \leq b$ by Definition 2.2.9, item 5. Now, for all c with $b \leq c$, we also have $c \nvDash \psi$, because the successors of a are included in the successors of b, since $a \leq b$ and transitivity of $\leq$ hold. Thus we obtain $b \vDash \neg\psi$ by item 5 of Definition 2.2.9.

This concludes the proof of the lemma. □

Lemma 2.2.12 (Soundness of **H** with respect to Kripke models for intuitionistic logic)**.** *If* $\Sigma \vdash_{\mathsf{H}} \varphi$ *holds, then* $\Sigma \vDash \varphi$ *holds.*

Proof. To prove the soundness of Heyting proofs with respect to Kripke models, we have to show that, whenever we can construct a proof for φ by iteratively applying (MP) on elements of Σ and axioms of **H** (i.e., $\Sigma \vdash_{\mathsf{H}} \varphi$ holds), the following is also true:
All Kripke models which force all elements of Σ also force φ, or in symbols: $\forall \mathscr{M} : \mathscr{M} \vDash \Sigma \Rightarrow \mathscr{M} \vDash \varphi$ (cf. Definition 2.2.9).

To obtain this result, we will somehow follow our proof semantically; we will show that all axioms of **H** are forced in any Kripke model and that, whenever two formulae φ and $\varphi \rightarrow \psi$ are forced in any Kripke model, so is ψ, thus obtaining the correctness for the rule (MP).

First we show for every axiom of **H** that it holds in any world a of any Kripke model $\mathscr{M}$ and thus *a fortiori* in the Kripke models which force all elements of Σ. To obtain this result, we consider an arbitrary world a and its successor(s). Since $\leq$ is reflexive, every world has at least one successor. So when we consider a world a, we may also have to regard its successors b, i.e., the worlds b such that $a \leq b$ holds. By the properties of $\leq$, we need not consider successors of successors, i.e., the worlds c such that $a \leq b \leq c$ holds, as these are included in the set $\{b \mid a \leq b \text{ holds}\}$.

H1: $\varphi \rightarrow (\psi \rightarrow \varphi)$. Suppose $a \vDash \varphi \rightarrow (\psi \rightarrow \varphi)$ does not hold. By Definition 2.2.9, item 4, this is only the case, if there are worlds b with $a \leq b$ which force φ but which do not force $\psi \rightarrow \varphi$.
If some b forces φ, all successors of those b also force φ by Intuitionistic Heredity (Lemma 2.2.11). So, by the definition mentioned above, all b with $a \leq b$ trivially force $\psi \rightarrow \varphi$. But then $a \vDash \varphi \rightarrow (\psi \rightarrow \varphi)$ holds

which is a contradiction to our assumption. Therefore H1 must be forced in all worlds a.

H2: $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$. In order to show that H2 holds in every world a, we consider all b with $a \leq b$. If there is no b for which $b \vDash \varphi \to (\psi \to \theta)$ and $b \vDash \varphi \to \psi$ holds, H2 is trivially forced in a.
On the other hand, if there is a b with $a \leq b$ and $b \nvDash \varphi$ holds, the following implications hold in b: $\varphi \to (\psi \to \theta)$, $\varphi \to \psi$ and $\varphi \to \theta$. So we get $b \vDash (\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$ by Definition 2.2.9 item 4 and Intuitionistic Heredity (Lemma 2.2.11). Now consider the case that $b \vDash \varphi \to (\psi \to \theta)$, $b \vDash \varphi \to \psi$ and $b \vDash \varphi$ holds. By the latter two it follows that $b \vDash \psi$ also holds, by the first and third, we have $b \vDash \psi \to \theta$. $b \vDash \theta$ holds by Definition 2.2.9, item 4 So $\varphi \to \theta$ is forced by those b and so is H2.
Now we have established that b forces H2 for all b with $a \leq b$, thus by reflexivity of $\leq$, we know that a forces H2.

H3: $(\theta \to \varphi) \to ((\theta \to \psi) \to (\theta \to \varphi \land \psi))$. Using a similar argument as the one above for H2, we focus on successors b of a which force $\theta \to \varphi$ as well as $\theta \to \psi$, since $a \vDash (\theta \to \varphi) \to ((\theta \to \psi) \to (\theta \to \varphi \land \psi))$ is trivially true, if there is no such world. As all implications hold trivially, when $b \nvDash \theta$ holds, we only consider those worlds b with $b \vDash \theta$. By Definition 2.2.9, item 4 and reflexivity of $\leq$, we then have $b \vDash \varphi$ and $b \vDash \psi$. Thus by item 2 of the same definition, we have $b \vDash \varphi \land \psi$ as well as $b \vDash \theta \to \varphi \land \psi$. So all successors of a which force $\theta \to \varphi$ and $\theta \to \psi$ also force $\theta \to \varphi \land \psi$, which means that all successors of a—a included—force H3.

H4: $(\varphi \land \psi) \to \varphi$. Suppose for all b with $a \leq b$ that $b \nvDash (\varphi \land \psi)$ holds. Then $b \vDash (\varphi \land \psi) \to \varphi$ is trivially true and so is $a \vDash (\varphi \land \psi) \to \varphi$.
Now suppose there are some b with $b \vDash \varphi \land \psi$. For those b, the relation $b \vDash \varphi$ is true by Definition 2.2.9, item 2. So we get $a \vDash (\varphi \land \psi) \to \varphi$ by item 4 of the same definition.

H5: $(\varphi \land \psi) \to \psi$. The proof is similar to the one for H4.

H6: $\varphi \to (\varphi \lor \psi)$. To show that $\varphi \to (\varphi \lor \psi)$ is forced in every world a of every Kripke model, we suppose that φ is forced in some worlds b, which are successors of a. (Otherwise we can apply the same argument as for axiom H4 and conclude that $a \vDash \varphi \to (\varphi \lor \psi)$ is trivially true.)

Each world b with $b \vDash \varphi$ also forces $\varphi \vee \psi$ by Definition 2.2.9 item 3, so those b and a also force $\varphi \rightarrow (\varphi \vee \psi)$.

H7: $\psi \rightarrow (\varphi \vee \psi)$. The proof is similar to the one for H6.

H8: $(\varphi \rightarrow \theta) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \vee \psi \rightarrow \theta))$. We will proceed in a fashion similar to the cases H2 and H3 to prove that H8 is forced by every world a of every Kripke model. We focus on the successors b of a which force $\varphi \rightarrow \theta$ and $\psi \rightarrow \theta$. When $b \vDash \varphi$ holds, it follows that b forces θ. Analogously when $b \vDash \psi$ holds, b also forces θ. So when $b \vDash \varphi$ or $b \vDash \psi$ holds, $b \vDash \theta$ holds. By Definition 2.2.9 item 3 and 4, this is equivalent to the fact that $b \vDash \varphi \vee \psi \rightarrow \theta$ holds. So once again, for all successors of a, H8 either holds trivially (if $\varphi \rightarrow \theta$ or $\psi \rightarrow \theta$ is not forced) or by the above argument.

H9: $(\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$. We consider all b, for which $a \leq b$ holds and which force $\varphi \rightarrow \neg\psi$. By Definition 2.2.9 item 5, this means that all b must not force ψ, when $b \vDash \varphi$ holds. So when $b \vDash \psi$ holds, b must not force φ, which means that we have $b \vDash \psi \rightarrow \neg\varphi$. By reflexivity of $\leq$, we have $a \vDash \psi \rightarrow \neg\varphi$, whenever $a \vDash \varphi \rightarrow \neg\psi$ holds. Therefore, by Definition 2.2.9 item 4, any a forces H9.

H10: $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$. Here we only have to consider the case that no b with $a \leq b$ forces φ, which, by Definition 2.2.9 item 5, means that $a \vDash \neg\varphi$. Otherwise the outermost implication of H10 would be trivially true. When no b forces φ, all b force $\varphi \rightarrow \psi$ by Definition 2.2.9 item 4. So, by this definition, $b \vDash \neg\varphi \rightarrow (\varphi \rightarrow \psi)$ also holds. Thus a forces H10.

We have shown that arbitrary worlds of arbitrary Kripke models force all of the axioms presented in Definition 2.2.1. It remains to show that the rule (MP) is correct for this semantics. So, whenever we have $a \vDash \varphi$ and $a \vDash \varphi \rightarrow \psi$, $a \vDash \psi$ must also be true. By Definition 2.2.9 item 4, $a \vDash \varphi \rightarrow \psi$ holds, if and only if, for all b with $a \leq b$, $b \vDash \varphi$ implies $b \vDash \psi$. By reflexivity, we get $a \vDash \psi$ and thus have shown the correctness of (MP).

With these results it is obvious that, once we can construct a proof $\Sigma \vdash_{\mathsf{H}} \varphi$, $\Sigma \vDash \varphi$ holds (i.e., for all models $\mathscr{M}$: $\mathscr{M} \vDash \Sigma \Rightarrow \mathscr{M} \vDash \varphi$): $\mathscr{M}$ forces all elements of Σ by precondition and we have shown that any Kripke model forces all axioms of **H**. Furthermore any formula, which is the result of an application of (MP) of two formulae forced by $\mathscr{M}$, is also forced by $\mathscr{M}$. $\square$

The completeness of Kripke models with respect to the Heyting calculus is more difficult to show. First we take a detour to prove the completeness property $\Sigma \vDash \varphi \Rightarrow \Sigma \vdash_{\mathsf{H}} \varphi$. We introduce a different notion of proof based on the *Natural Deduction* calculus (originally introduced by Gerhard Gentzen in [Gen35]). The basic idea behind this calculus is the availability of introduction and elimination rules for all logical connectives. The most elementary proof in this system is the proof of φ from the premise φ. By applications of the rules, the formula which is proven can more or less complex, depending on whether we use an introduction or an elimination rule and assumptions can be eliminated (canceled) or added.

We denote by $\Sigma \vdash_{\mathsf{ND}} \varphi$ the provability of φ from a set Σ of formulae (the assumptions) by rules of the Natural Deduction calculus **ND**. Then we will show the completeness of Kripke models for that calculus via a *Model Existence Lemma* and prove that all formulae which are provable by Natural Deduction are also provable by our Heyting calculus, i.e., we will show that $\Sigma \vdash_{\mathsf{ND}} \varphi \Rightarrow \Sigma \vdash_{\mathsf{H}} \varphi$ holds.

Together with the correctness result for the Heyting calculus, we will eventually obtain

$$\Sigma \vdash_{\mathsf{H}} \varphi \Rightarrow \Sigma \models \varphi \Rightarrow \Sigma \vdash_{\mathsf{ND}} \varphi \Rightarrow \Sigma \vdash_{\mathsf{H}} \varphi$$

and thus the equivalence of all three notions.

Definition 2.2.13 (Natural Deduction calculus for intuitionistic propositional logic)**.** *The Natural Deduction calculus for intuitionistic logic (named* **ND***) for propositional logic has the following introduction and elimination rules for the logical connectives:*

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi}\ (\wedge I) \qquad\qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge E) \quad \frac{\varphi \wedge \psi}{\psi}\ (\wedge E)$$

$$\frac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}\ (\to I) \qquad\qquad \frac{\varphi \quad \varphi \to \psi}{\psi}\ (\to E)$$

$$\frac{\varphi}{\varphi \vee \psi}\ (\vee I) \quad \frac{\psi}{\varphi \vee \psi}\ (\vee I) \qquad\qquad \frac{\psi \vee \varphi \quad \begin{array}{c}[\varphi]\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}[\psi]\\ \vdots\\ \theta\end{array}}{\theta}\ (\vee E)$$

$$\dfrac{\begin{matrix}[\varphi]\\ \vdots\\ \bot\end{matrix}}{\neg\varphi}\ (\neg I) \qquad\qquad \dfrac{\varphi \quad \neg\varphi}{\bot}\ (\neg E)$$

$$\dfrac{\bot}{\varphi}\ (\bot)$$

The formulae above the line are called premises, the formula below the line is called conclusion. A formula in square brackets is called a canceled hypothesis.

In this notation $(\circ \mathsf{I})$ is the introduction rule for the logical connective $\circ$, where $(\circ \mathsf{E})$ is the corresponding elimination rule.
$\bot$ has no introduction rule, as we do not want to be able to deduce logical falsity in our calculus. The rule $(\to \mathsf{E})$ coincides with modus ponens (MP) from the Heyting calculus.

Remark 2.2.14. *In the above calculus, if we interpret $\neg\varphi$ as $\varphi \to \bot$, the rules $(\neg I)$ and $(\neg E)$ are just instances of $(\to I)$ and $(\to E)$ with $\psi = \bot$.*

Analogously to Remark 2.2.2, if the rule reductio ad absurdum

$$\dfrac{\begin{matrix}[\neg\varphi]\\ \vdots\\ \bot\end{matrix}}{\varphi}\ (RAA)$$

is added to the Natural Deduction calculus, we obtain classical logic.

Definition 2.2.15 (Natural Deduction proofs)**.** *A proof $\Sigma \vdash_{\mathsf{ND}} \varphi$ is a tree, whose root is φ and whose leaves are either canceled assumptions or formulae from Σ. The inner nodes of the tree are conclusions of the rule applications of* **ND***, whose premises occur above the conclusion in the proof tree.*

The length of the proof $\Sigma \vdash_{\mathsf{ND}} \varphi$ *is the number of formulas in the proof, which corresponds to the number of nodes in the proof tree.*

Proofs in **ND** are usually drawn as trees, whose roots are at the bottom. When all leaf nodes are canceled, we have a proof $\vdash_{\mathsf{ND}} \varphi$, where φ is the formula at the root of the proof tree.

When applying one of the rules ($\rightarrow$I), ($\neg$I) or ($\vee$E), we may cancel assumptions. In Definition 2.2.13, the assumptions which we may cancel are depicted in square brackets. It is possible to cancel none, some or all occurrences of the assumption in the proof tree above the rule application. To denote that the respective assumption is canceled, we put it in square brackets and—for better readability—add subscripts to the applications of rules which cancel assumptions and the assumptions canceled by them, as can be observed in the following examples.

The calculus **ND** is very convenient and thus we will use it to prove formulae, we need in the later chapters as well. This is possible because intuitionistic logic is the weakest logic we will treat, and all the other systems are stronger in the sense that every intuitionistically provable formula is also provable in the stronger systems.

Example 2.2.16 ($\vdash_{\mathsf{ND}} \varphi \wedge (\varphi \rightarrow \psi) \rightarrow \psi$)**.** *This is a proof tree for the formula $\varphi \wedge (\varphi \rightarrow \psi) \rightarrow \psi$ in the calculus of Natural Deduction:*

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge (\varphi \rightarrow \psi)]_1}{\varphi}\,(\wedge E) \qquad \dfrac{[\varphi \wedge (\varphi \rightarrow \psi)]_1}{\varphi \rightarrow \psi}\,(\wedge E)}{\psi}\,(\rightarrow E)}{\varphi \wedge (\varphi \rightarrow \psi) \rightarrow \psi}\,(\rightarrow I)_1$$

The derived formula is a "paraphrase" of (MP).

Example 2.2.17 ($\vdash_{\mathsf{ND}} (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))$)**.**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]_1 \quad [\varphi \rightarrow \psi]_3}{\psi}\,(\rightarrow E) \qquad [\psi \rightarrow \theta]_2}{\theta}\,(\rightarrow E)}{\varphi \rightarrow \theta}\,(\rightarrow I)_1}{(\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta)}\,(\rightarrow I)_2}{(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \theta) \rightarrow (\varphi \rightarrow \theta))}\,(\rightarrow I)_3$$

This formula describes the fact that the logical connective $\rightarrow$ behaves as a transitive relation: When $\varphi \rightarrow \psi$ is provable and $\psi \rightarrow \theta$ is provable, so is $\varphi \rightarrow \theta$.

Example 2.2.18 ($\vdash_{\mathsf{ND}} (\varphi \to \psi) \to ((\varphi \wedge \theta) \to (\psi \wedge \theta))$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \theta]_1}{\varphi}\,(\wedge E) \qquad [\varphi \to \psi]_2}{\psi}\,(\to E) \qquad \dfrac{[\varphi \wedge \theta]_1}{\theta}\,(\wedge E)}{\psi \wedge \theta}\,(\wedge I)}{(\varphi \wedge \theta) \to (\psi \wedge \theta)}\,(\to I)_1}{(\varphi \to \psi) \to ((\varphi \wedge \theta) \to (\psi \wedge \theta))}\,(\to I)_2$$

Example 2.2.19 ($\vdash_{\mathsf{ND}} \neg((\varphi \vee \psi) \wedge \neg\varphi \wedge \neg\psi)$). *In the following, we abbreviate $(\varphi \vee \psi) \wedge \neg\varphi \wedge \neg\psi$ by α. And—by a slight abuse of notation—we apply the rule ($\wedge E$) in a more general form to conjunctions with more than two operands.*

$$\dfrac{\dfrac{\dfrac{[\alpha]_2}{\varphi \vee \psi}\,(\wedge E) \qquad \dfrac{[\varphi]_1 \qquad \dfrac{[\alpha]_2}{\neg\varphi}\,(\wedge E)}{\bot}\,(\neg E) \qquad \dfrac{[\psi]_1 \qquad \dfrac{[\alpha]_2}{\neg\psi}\,(\wedge E)}{\bot}\,(\neg E)}{\bot}\,(\vee E)_1}{\neg((\varphi \vee \psi) \wedge \neg\varphi \wedge \neg\psi)}\,(\neg I)_2$$

Example 2.2.20 ($\vdash_{\mathsf{ND}} \neg(\varphi \wedge (\varphi \to \psi) \wedge \neg\psi)$). *In the following, we abbreviate $(\varphi \wedge (\varphi \to \psi) \wedge \neg\psi)$ by α. Furthermore, we apply the rule ($\wedge E$) in a more general form to conjunctions with more than two operands.*

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha]_1}{\varphi}\,(\wedge E) \qquad \dfrac{[\alpha]_1}{\varphi \to \psi}\,(\wedge E)}{\psi}\,(\to E) \qquad \dfrac{[\alpha]_1}{\neg\psi}\,(\wedge E)}{\bot}\,(\neg E)}{\neg(\varphi \wedge (\varphi \to \psi) \wedge \neg\psi)}\,(\neg I)_1$$

The completeness proof (i.e., the proof that everything which holds in all models also can be proven in the calculus under consideration) is much easier and comes more natural, when using Natural Deduction instead of a Hilbert-style calculus. The drawback of proving completeness for the calculus **ND** is that we still have to show that our two notions of proof produce the same set of formulae ($\Sigma \vdash_{\mathsf{H}} \varphi \Leftrightarrow \Sigma \vdash_{\mathsf{ND}} \varphi$), but this is also not too difficult to show.

We will use an indirect approach to show the completeness of the calculus **ND** for Kripke models, by showing that, whenever a formula φ cannot be proven for some assumptions in Σ, there exists a counter model, which forces all formulae in Σ but not φ. In order to create such a counter model for φ with $\Sigma \nvdash_{\mathsf{ND}} \varphi$, we need the notion of prime theories, as our Kripke counter model will have prime theories as its nodes. We will proceed as in [vD04] to create such a counter-model.

Definition 2.2.21 (Prime Theory (wrt. a calculus L))**.** *Σ is called a* prime theory *with respect to a calculus L if*

(i) Σ is closed under $\vdash_{\mathsf{L}}$ (i.e., $\Sigma \vdash_{\mathsf{L}} \varphi \Rightarrow \varphi \in \Sigma$), where $\vdash_{\mathsf{L}}$ is the proof relation of the calculus L, and

(ii) for all $\varphi \vee \psi \in \Sigma$, $\varphi \in \Sigma$ or $\psi \in \Sigma$.

The calculus L might not be explicitly noted when it is obvious from the context.

Lemma 2.2.22. *Suppose Σ does not prove φ ($\Sigma \nvdash_{\mathsf{ND}} \varphi$). Then there is a prime theory $\Sigma' \supseteq \Sigma$ (with respect to ND) with $\Sigma' \nvdash_{\mathsf{ND}} \varphi$.*

Proof. We will construct a prime theory Σ' with $\Sigma' \nvdash_{\mathsf{ND}} \varphi$ by creating a chain of formula sets $\Sigma = \Sigma_0, \Sigma_1, \ldots, \Sigma_n, \ldots$, all of which do not force φ. The prime theory is obtained by taking the union of all these sets.

We start with the construction of the Σ_n and show by induction on n that, $\Sigma_n \nvdash_{\mathsf{ND}} \varphi$ holds for all $n \in \mathbb{N}$:

Induction Base. $\Sigma_0 = \Sigma$. $\Sigma \nvdash_{\mathsf{ND}} \varphi$ holds by the assumption of the lemma.

Induction Hypothesis. Suppose $n \geq 0$ and assume that $\Sigma_n \nvdash_{\mathsf{ND}} \varphi$ holds.

Induction Step. Consider Σ_{n+1}. We have to show that $\Sigma_{n+1} \nvdash_{\mathsf{ND}} \varphi$ holds. We choose any enumeration of all well formed formulae and take the first disjunction $\psi_1 \vee \psi_2$, which can be proven by Σ_n (i.e., $\psi_1 \vee \psi_2$ such that $\Sigma_n \vdash_{\mathsf{ND}} \psi_1 \vee \psi_2$ holds) and which we have not treated in a previous step.

We construct Σ_{n+1} as follows:

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\psi_1\} & \text{if } \Sigma_n, \psi_1 \nvdash_{\mathsf{ND}} \varphi, \\ \Sigma_n \cup \{\psi_2\} & \text{otherwise.} \end{cases}$$

It remains to show that Σ_{n+1} does not force φ.
If $\Sigma_{n+1} = \Sigma_n \cup \{\psi_1\}$, then $\Sigma_{n+1} \nvdash_{\mathsf{ND}} \varphi$ by construction.
Now let $\Sigma_{n+1} = \Sigma_n \cup \{\psi_2\}$. Then $\Sigma_n \cup \{\psi_1\} \vdash_{\mathsf{ND}} \varphi$ holds by the construction of Σ_{n+1}. Suppose $\Sigma_{n+1} \vdash_{\mathsf{ND}} \varphi$ holds. Then we have a proof $\Sigma_n \cup \{\psi_1\} \vdash_{\mathsf{ND}} \varphi$ and a proof $\Sigma_n \cup \{\psi_2\} \vdash_{\mathsf{ND}} \varphi$. As we have $\Sigma_n \vdash_{\mathsf{ND}} \psi_1 \vee \psi_2$, we can combine it with these proofs by applying ($\vee\mathsf{E}$) and get a proof $\Sigma_n \vdash_{\mathsf{ND}} \varphi$. This is a contradiction to the induction hypothesis, so when Σ_{n+1} is $\Sigma_n \cup \{\psi_2\}$ we also have $\Sigma_{n+1} \nvdash_{\mathsf{ND}} \varphi$. This concludes the induction proof.

The set $\Sigma' = \bigcup_{n\in\mathbb{N}} \Sigma_n$, the union of all Σ_n, does not prove φ. If it would, there would be a finite **ND**-proof, which only uses formulae contained in some Σ_n. So this Σ_n would prove φ ($\Sigma_n \vdash_{\mathsf{ND}} \varphi$), which is a contradiction to the construction of Σ_n.

Now we still have to show that $\Sigma' = \bigcup_{n\in\mathbb{N}} \Sigma_n$ is a prime theory.

Property (i): If $\Sigma' \vdash_{\mathsf{ND}} \psi$, then there is some n such that $\Sigma_n \vdash_{\mathsf{ND}} \psi$ holds and consequently $\Sigma_n \vdash_{\mathsf{ND}} \psi \vee \psi$ can be proven by an application of ($\vee$I). Sooner or later, we will treat the disjunction $\psi \vee \psi$ in our construction of the Σ_k and thus construct a Σ_k which contains ψ. Therefore, for every ψ with $\Sigma' \vdash_{\mathsf{ND}} \psi$, $\psi \in \Sigma'$ holds.

Property (ii): For every $\psi \vee \theta \in \Sigma'$, there is a Σ_i such that $\psi \vee \theta \in \Sigma_i$. This Σ_i satisfies $\Sigma_i \vdash_{\mathsf{ND}} \psi \vee \theta$. So $\psi \vee \theta$ is one of the disjunctions from which we choose in the induction step of the construction of the Σ_i. As there are only enumerably many well formed formulae, we will treat this disjunction somewhere in our construction and add either ψ or θ to some Σ_j. Now there is either $\psi \in \Sigma'$ or $\theta \in \Sigma'$. □

To prove the completeness of a logical system with respect to some semantics, the construction of a canonical model is an often employed technique. Usually we call a model *canonical*, if it forces all formulae, provable either in some logical system itself or provable by some set of premises, and is a counter model for all other formulae, or—in our case—at least a counter model for a specific formula which is of interest.

Lemma 2.2.23 (Model Existence Lemma for $\vdash_{\mathsf{ND}}$). *If $\Sigma \nvdash_{\mathsf{ND}} \varphi$, then there is a (canonical) Kripke model $\mathscr{M}$ with a node a, such that $\mathscr{M} \vDash \Sigma$ and $a \nvDash \varphi$*

Proof. We will create an infinite, countable Kripke model by associating finite sequences of natural numbers to its nodes. So let us quickly introduce some notation which we will use in this proof: $\langle\rangle$ is the empty sequence. $\overrightarrow{n} = \langle n_1, n_2, \ldots, n_k\rangle$ is a sequence of k numbers and $\langle \overrightarrow{n}, i\rangle$ is the same sequence with i at its end, to spell it out: $\langle n_1, n_2, \ldots, n_k, i\rangle$. The relation $\leq$ is defined as the sub-sequence relation on finite sequences of natural numbers.

For each node $\overrightarrow{n}$, we will construct a prime theory $\Sigma_{\overrightarrow{n}} \supseteq \Sigma$ as in Lemma 2.2.22 and show that $\overrightarrow{n} \vDash \Sigma_{\overrightarrow{n}}$ holds. The model will be a rooted tree.

For the first node of our canonical model, we construct a prime theory $\Sigma_{\langle\rangle} \supseteq \Sigma$, and at node $\langle\rangle$, we force all atoms occurring as elements in $\Sigma_{\langle\rangle}$ (i.e., we define $D(\langle\rangle)$ to be the intersection of the set of all atoms and $\Sigma_{\langle\rangle}$).

Now, for any node $\overrightarrow{n}$, we enumerate all pairs of formulae

$$\langle\psi_0, \theta_0\rangle, \langle\psi_1, \theta_1\rangle, \langle\psi_2, \theta_2\rangle, \ldots$$

such that, for all i, $\Sigma_{\overrightarrow{n}}, \psi_i \nvdash_{\mathsf{ND}} \theta_i$ holds.

The direct successor $\langle \overrightarrow{n}, i \rangle$ of $\overrightarrow{n}$ will be created as follows:
We construct a prime theory $\Sigma_{\langle \overrightarrow{n}, i \rangle}$ out of $\Sigma_{\overrightarrow{n}}, \psi_i$ which still does not force φ (via Lemma 2.2.22) and to obtain $D(\langle \overrightarrow{n}, i \rangle)$, we force all atoms in $\Sigma_{\langle \overrightarrow{n}, i \rangle}$ at $\langle \overrightarrow{n}, i \rangle$.

Now that we have established the form and the method of construction for our canonical Kripke model $\mathscr{M}$, we will prove that, for all worlds $\overrightarrow{n} \in \mathscr{M}$, $\overrightarrow{n} \vDash \Sigma$ holds and that $\langle\rangle \nvDash \varphi$ holds. In fact we will prove the following—even stronger—statement: for all $\overrightarrow{n} \in \mathscr{M}$:

$$\overrightarrow{n} \vDash \psi \Leftrightarrow \Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi$$

As all $\Sigma_{\overrightarrow{n}}$ are super-sets of Σ by their construction, for any formula $\sigma \in \Sigma$, $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \sigma$ holds since $\sigma \in \Sigma_{\overrightarrow{n}}$. So, once we prove the above equivalence statement, for all $\sigma \in \Sigma$ the relation $\overrightarrow{n} \vDash \sigma$ holds. On the other hand, by the construction of $\Sigma_{\langle\rangle}$, we have $\Sigma_{\langle\rangle} \nvdash_{\mathsf{ND}} \varphi$, which we then show to be equivalent to $\langle\rangle \nvDash \varphi$.

We will proceed by induction on $\mathrm{lcomp}(\psi)$, the logical complexity of an arbitrary formula ψ.

Induction Base. $\mathrm{lcomp}(\psi) = 0$. We only have to consider the case that ψ is of the form p. If $\bot \in \Sigma_{\overrightarrow{n}}$, then, for any formula θ, we have $\Sigma \vdash_{\mathsf{ND}} \theta$ by one application of $(\bot)$. By the construction of D, the domain function for the canonical model, we also have $D(\overrightarrow{n}) = \mathbb{VAR}$ and both sides of the equivalence statement are true for all $p \in \mathbb{VAR}$ as well as $\bot$.
Suppose $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi$, then $\psi \in D(\langle \overrightarrow{n} \rangle)$, so $\overrightarrow{n} \vDash \psi$ holds. For the other direction of the equivalence statement above, we just consult Definition 2.2.9 and immediately recognize that any atomic ψ has to be in $D(\overrightarrow{n})$, whenever $\overrightarrow{n} \vDash \psi$ holds. Since $D(\langle \overrightarrow{n} \rangle)$ is a subset of $\Sigma_{\overrightarrow{n}}$, the atomic formula ψ is an element of the prime theory $\Sigma_{\overrightarrow{n}}$.

Induction Hypothesis. Suppose $m \geq 0$ and the above equivalence statement holds for all ψ with $\mathrm{lcomp}(\psi) \leq m$.

Induction Step. Consider a ψ with $\mathrm{lcomp}(\psi) = m + 1$. We consider the outermost logical connective of ψ and an arbitrary $\overrightarrow{n}$.

ψ is of the form $\psi_1 \wedge \psi_2$. Suppose $\overrightarrow{n} \vDash \psi_1 \wedge \psi_2$ holds. By Definition 2.2.9, this is equivalent to the fact that $\overrightarrow{n} \vDash \psi_1$ holds and $\overrightarrow{n} \vDash \psi_2$ holds. Now we can apply our induction hypothesis and conclude that there is a proof for ψ_1 and there is a proof for ψ_2, both with the hypotheses in $\Sigma_{\overrightarrow{n}}$. So we can easily create a proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1 \wedge \psi_2$.

For the other direction, when we have a proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1 \wedge \psi_2$, we add the two variants of the rule $(\wedge\mathsf{E})$ to create two proofs $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1$

and $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_2$, apply the induction hypothesis to get $\overrightarrow{n} \vDash \psi_1$ and $\overrightarrow{n} \vDash \psi_2$ and then consult Definition 2.2.9 to obtain $\overrightarrow{n} \vDash \psi_1 \wedge \psi_2$.

ψ is of the form $\psi_1 \vee \psi_2$. Suppose $\overrightarrow{n} \vDash \psi_1 \vee \psi_2$ holds. Then, by Definition 2.2.9, we have $\overrightarrow{n} \vDash \psi_1$ or $\overrightarrow{n} \vDash \psi_2$. Now, by our induction hypothesis, we get a proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1$ or a proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_2$ depending on which of the above forcing relations hold. But no matter which proof we obtain, once we add the rule $(\vee\mathsf{I})$ we get a proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1 \vee \psi_2$.

For the other direction, let us consider the proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1 \vee \psi_2$. $\Sigma_{\overrightarrow{n}}$ is a prime theory, so by item (i) of Definition 2.2.21, we have $\psi_1 \vee \psi_2 \in \Sigma_{\overrightarrow{n}}$ and by item (ii) of Definition 2.2.21, we know that $\psi_1 \in \Sigma_{\overrightarrow{n}}$ or $\psi_2 \in \Sigma_{\overrightarrow{n}}$. Thus we can create at least one of the trivial proofs $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1$ or $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_2$. By induction hypothesis we can conclude that $\overrightarrow{n} \vDash \psi_1$ or $\overrightarrow{n} \vDash \psi_2$ holds, thus, by Definition 2.2.9, $\overrightarrow{n} \vDash \psi_1 \vee \psi_2$ holds.

ψ is of the form $\psi_1 \to \psi_2$. Suppose $\overrightarrow{n} \vDash \psi_1 \to \psi_2$ and $\Sigma_{\overrightarrow{n}} \nvdash_{\mathsf{ND}} \psi_1 \to \psi_2$, then clearly $\Sigma_{\overrightarrow{n}}, \psi_1 \nvdash_{\mathsf{ND}} \psi_2$, because otherwise, $(\to\mathsf{I})$ would result in $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1 \to \psi_2$. So $\langle \psi_1, \psi_2 \rangle$ is a pair in the construction of $\mathscr{M}$ (e.g., $\langle \psi_i, \theta_i \rangle$ for some i) and there exists a node $\langle \overrightarrow{n}, i \rangle$, with associated prime theory $\Sigma_{\langle \overrightarrow{n}, i \rangle}$ such that $\Sigma_{\langle \overrightarrow{n}, i \rangle} \supseteq \Sigma_{\overrightarrow{n}} \cup \{\psi_1\}$ and $\Sigma_{\langle \overrightarrow{n}, i \rangle} \nvdash_{\mathsf{ND}} \psi_2$. So, by the induction hypothesis, $\langle \overrightarrow{n}, i \rangle \vDash \psi_1$ holds. If now $\overrightarrow{n} \vDash \psi_1 \to \psi_2$ holds, then $\langle \overrightarrow{n}, i \rangle \vDash \psi_2$ must hold, because $\overrightarrow{n} \leq \langle \overrightarrow{n}, i \rangle$ also holds. This is a contradiction to $\langle \overrightarrow{n}, i \rangle \nvDash \psi_2$.

For the other direction, suppose there is a proof $\Sigma_{\overrightarrow{n}} \vdash_{\mathsf{ND}} \psi_1 \to \psi_2$. It can be transformed to a proof $\Sigma_{\overrightarrow{n}}, \psi_1 \vdash_{\mathsf{ND}} \psi_2$ by assuming ψ_1 and an application of $(\to\mathsf{E})$. Now, for all $\Sigma_{\overrightarrow{m}}$ with $\Sigma_{\overrightarrow{m}} \supseteq \Sigma_{\overrightarrow{n}} \cup \{\psi_1\}$, we have $\Sigma_{\overrightarrow{m}} \vdash_{\mathsf{ND}} \psi_2$. By the induction hypothesis, we get that, for all $\overrightarrow{m}$ where $\overrightarrow{m} \vDash \psi_1$ holds, $\overrightarrow{m} \vDash \psi_2$ also holds. By the construction of $\Sigma_{\overrightarrow{m}}$, we have $\Sigma_{\overrightarrow{m}} \supseteq \Sigma_{\overrightarrow{n}}$ as well as $\overrightarrow{n} \leq \overrightarrow{m}$, and this, by Definition 2.2.9, coincides with the fact that $\overrightarrow{n} \vDash \psi_1 \to \psi_2$.

ψ is of the form $\neg\psi_1$. Once again we just view this as a special case of the implication $\psi = \psi_1 \to \bot$. For the above equivalence statement, it is irrelevant that $\bot$ must not be forced in any world.

This concludes the proof of the lemma. $\square$

In the following lemma, we show that everything that can be proven by Natural Deduction can also be proven by the Heyting calculus. (That is, the Heyting calculus simulates the calculus of Natural Deduction.)

Lemma 2.2.24. *If $\Sigma \vdash_{\mathsf{ND}} \varphi$ holds, then $\Sigma \vdash_{\mathsf{H}} \varphi$ holds.*

Proof. We have to show that every formula φ, which has an **ND** proof where all uncanceled are hypotheses in Σ, also has a Heyting proof $\Sigma \vdash_{\mathsf{H}} \varphi$. This can be shown by induction on the length of the **ND** proof.

Induction Base. $n = 1$. When there is a proof of length 1 in the calculus **ND** for the formula φ, φ has to be in Σ. So the proof $\Sigma \vdash_{\mathsf{H}} \varphi$ is the trivial proof of length 1.

Induction Hypothesis. Suppose $n > 0$. For each proof $\Sigma \vdash_{\mathsf{ND}} \varphi$ of length $\leq n$, there is a corresponding proof $\Sigma \vdash_{\mathsf{H}} \varphi$.

Induction Step. Consider an **ND**-proof of length $n + 1$. We perform a case analysis on the last rule application of the **ND** proof. The proofs of the premises of the last rule application are of length $\leq n$ and thus the induction hypothesis applies to them.

Case ($\wedge$I): By the induction hypothesis, we have two Heyting proofs $\Sigma \vdash_{\mathsf{H}} \varphi$ and $\Sigma \vdash_{\mathsf{H}} \psi$. Using H1 (i.e., take $\varphi \to ((\theta \to \theta) \to \varphi$ with an arbitrary θ) and (MP), we also have Heyting proofs $\Sigma \vdash_{\mathsf{H}} (\theta \to \theta) \to \varphi$ and $\Sigma \vdash_{\mathsf{H}} (\theta \to \theta) \to \psi$. We instantiate H3 to

$$((\theta \to \theta) \to \varphi) \to (((\theta \to \theta) \to \psi) \to ((\theta \to \theta) \to \varphi \wedge \psi))$$

and apply (MP) twice. We obtain $\Sigma \vdash_{\mathsf{H}} (\theta \to \theta) \to \varphi \wedge \psi$. Now we use the proof of Example 2.2.4 to get $\vdash_{\mathsf{H}} \theta \to \theta$ and by (MP), we obtain $\Sigma \vdash_{\mathsf{H}} \varphi \wedge \psi$.

Case ($\wedge$E): For the first variant of ($\wedge$E), the induction hypothesis provides a proof $\Sigma \vdash_{\mathsf{H}} \psi \wedge \theta$, which gives us a proof $\Sigma \vdash_{\mathsf{H}} \psi$ when we add the axiom H4 and apply (MP). The second variant of ($\wedge$E) is treated analogously with H5.

Case ($\to$I): When ($\to$I) is the last applied rule, the **ND** proof up to this rule corresponds to a proof $\Sigma, \varphi \vdash_{\mathsf{ND}} \psi$, so, by the induction hypothesis, we have a Heyting proof $\Sigma, \varphi \vdash_{\mathsf{H}} \psi$. By applying the Deduction Theorem (Theorem 2.2.6), we get a proof $\Sigma \vdash_{\mathsf{H}} \varphi \to \psi$.

Case ($\to$E): If this is the last applied rule, then, by the induction hypothesis, there are two proofs $\Sigma \vdash_{\mathsf{H}} \varphi \to \psi$ and $\Sigma \vdash_{\mathsf{H}} \varphi$, which can be concatenated to a proof of $\Sigma \vdash_{\mathsf{H}} \psi$ with one additional application of (MP).

Case ($\vee$I): This case is similar to ($\wedge$E). For the first variant of ($\vee$I), the induction hypothesis provides a proof $\Sigma \vdash_{\mathsf{H}} \psi$, which gives us a proof

$\Sigma \vdash_{\mathsf{H}} \psi \vee \theta$ when we add the axiom H6 and apply (MP). The second variant of ($\vee$I) is treated analogously with H7.

Case ($\vee$E): By the induction hypothesis, we have three proofs corresponding to the premises of the ($\vee$E)-rule, namely $\Sigma \vdash_{\mathsf{H}} \varphi \vee \psi$, $\Sigma, \varphi \vdash_{\mathsf{H}} \theta$ and $\Sigma, \psi \vdash_{\mathsf{H}} \theta$. To the last two proofs, we apply the Deduction Theorem to get $\Sigma \vdash_{\mathsf{H}} \varphi \to \theta$ and $\Sigma \vdash_{\mathsf{H}} \psi \to \theta$. Then we use these two proofs, together with axiom H8, apply (MP) twice and get $\Sigma \vdash_{\mathsf{H}} \varphi \vee \psi \to \theta$. Using the proof $\Sigma \vdash_{\mathsf{H}} \varphi \vee \psi$, we apply (MP) and get $\Sigma \vdash_{\mathsf{H}} \theta$ as desired.

Case ($\perp$): By the induction hypothesis, we have a Heyting proof of $\perp$ from Σ. Take the formula $\perp \to \varphi$ (established in Example 2.2.5) and use (MP) to derive φ.

Case ($\neg$I) and ($\neg$E): These rules are just specializations of ($\to$I) and ($\to$E). We just have to replace the atomic formula ψ with $\perp$ and can reuse our above arguments.

This concludes the proof of the lemma. □

Theorem 2.2.25 (Completeness of intuitionistic propositional logic)**.**

$$\Sigma \vdash_{\mathsf{H}} \varphi \Rightarrow \Sigma \models \varphi \Rightarrow \Sigma \vdash_{\mathsf{ND}} \varphi \Rightarrow \Sigma \vdash_{\mathsf{H}} \varphi$$

Proof. The first implication $\Sigma \vdash_{\mathsf{H}} \varphi \Rightarrow \Sigma \models \varphi$ is the soundness of Kripke models with respect to Heyting proofs (cf. Lemma 2.2.12).

To show that $\Sigma \models \varphi \Rightarrow \Sigma \vdash_{\mathsf{ND}} \varphi$ holds, we suppose that there is no proof $\Sigma \vdash_{\mathsf{ND}} \varphi$ (i.e., $\Sigma \nvdash_{\mathsf{ND}} \varphi$ holds). Then, by Lemma 2.2.23, we have a model $\mathscr{M}$ which does not force φ at some world a, but forces all formulae of Σ at every world. So $\mathscr{M} \not\models \varphi$ and $\Sigma \not\models \varphi$ hold. Thus, we have $\Sigma \nvdash_{\mathsf{ND}} \varphi \Rightarrow \Sigma \not\models \varphi$ and $\Sigma \models \varphi \Rightarrow \Sigma \vdash_{\mathsf{ND}} \varphi$ holds.

The third implication $\Sigma \vdash_{\mathsf{ND}} \varphi \Rightarrow \Sigma \vdash_{\mathsf{H}} \varphi$ is shown in Lemma 2.2.24. □

Although it was not necessary for proving the completeness of propositional intuitionistic logic and is now a by-product of Theorem 2.2.25, we will give a direct proof that the calculus **ND** simulates **H**.

Lemma 2.2.26. *If $\Sigma \vdash_{\mathsf{H}} \varphi$ holds, then $\Sigma \vdash_{\mathsf{ND}} \varphi$ holds.*

Proof. As a preparatory step, we show that H1, ..., H10 are provable in **ND**.

Case H1:

$$\cfrac{\cfrac{\cfrac{\cfrac{[\varphi]_2 \quad [\psi]_1}{\varphi \wedge \psi}\ (\wedge\mathsf{I})}{\varphi}\ (\wedge\mathsf{E})}{\psi \to \varphi}\ (\to\mathsf{I})_1}{\varphi \to (\psi \to \varphi)}\ (\to\mathsf{I})_2$$

Case H2:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\varphi]_1 \quad [\varphi \to \psi]_2}{\psi}\ (\to\mathsf{E}) \quad \cfrac{[\varphi]_1 \quad [\varphi \to (\psi \to \theta)]_3}{\psi \to \theta}\ (\to\mathsf{E})}{\theta}\ (\to\mathsf{E})}{\varphi \to \theta}\ (\to\mathsf{I})_1}{(\varphi \to \psi) \to (\varphi \to \theta)}\ (\to\mathsf{I})_2}{(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))}\ (\to\mathsf{I})_3$$

Case H3:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[\theta]_1 \quad [\theta \to \varphi]_3}{\varphi}\ (\to\mathsf{E}) \quad \cfrac{[\theta]_1 \quad [\theta \to \psi]_2}{\psi}\ (\to\mathsf{E})}{\varphi \wedge \psi}\ (\wedge\mathsf{I})}{\theta \to \varphi \wedge \psi}\ (\to\mathsf{I})_1}{(\theta \to \psi) \to (\theta \to \varphi \wedge \psi)}\ (\to\mathsf{I})_2}{(\theta \to \varphi) \to ((\theta \to \psi) \to (\theta \to \varphi \wedge \psi))}\ (\to\mathsf{I})_3$$

Case H4:

$$\cfrac{\cfrac{[\varphi \wedge \psi]}{\varphi}\ (\wedge\mathsf{E})}{\varphi \wedge \psi \to \varphi}\ (\to\mathsf{I})$$

Case H5: H5 is shown analogously to the case of H4.

Case H6:

$$\cfrac{\cfrac{[\varphi]}{\varphi \vee \psi}\ (\vee\mathsf{I})}{\varphi \to \varphi \vee \psi}\ (\to\mathsf{I})$$

Case H7: H7 is shown analogously to the case of H6.

Case H8:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \vee \psi]_2 \qquad \dfrac{[\varphi]_1 \quad [\varphi \to \theta]_4}{\theta}\,(\to\mathsf{E}) \qquad \dfrac{[\psi]_1 \quad [\psi \to \theta]_3}{\theta}\,(\to\mathsf{E})}{\theta}\,(\vee\mathsf{E})_1}{\varphi \vee \psi \to \theta}\,(\to\mathsf{I})_2}{(\psi \to \theta) \to (\varphi \vee \psi \to \theta)}\,(\to\mathsf{I})_3}{(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))}\,(\to\mathsf{I})_4$$

Case H9:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]_1 \quad [\varphi \to \neg\psi]_3}{\neg\psi}\,(\to\mathsf{E}) \qquad [\psi]_2}{\bot}\,(\neg\mathsf{E})}{\neg\varphi}\,(\neg\mathsf{I})_1}{\psi \to \neg\varphi}\,(\to\mathsf{I})_2}{(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)}\,(\to\mathsf{I})_3$$

Case H10:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]_1 \quad [\neg\varphi]_2}{\bot}\,(\neg\mathsf{E})}{\psi}\,(\bot)}{\varphi \to \psi}\,(\to\mathsf{I})_1}{\neg\varphi \to (\varphi \to \psi)}\,(\to\mathsf{I})_2$$

We will proceed by induction on the length of the Heyting proof $\Sigma \vdash_{\mathsf{H}} \varphi$ to show that, for every Heyting proof of length n, there is a **ND** proof $\Sigma \vdash_{\mathsf{ND}} \varphi$.

Induction Base. $n = 1$. When we have a Heyting proof $\Sigma \vdash_{\mathsf{H}} \varphi$ of length 1, the proven formula φ is either an element of Σ or (the substitution instance of) an axiom.

If $\varphi \in \Sigma$, then the **ND** proof is trivial. If φ is one of the axioms of **H**, there is a **ND** proof for φ, as we have shown above.

Induction Hypothesis. Suppose $n > 0$. For each proof $\Sigma \vdash_{\mathsf{H}} \varphi$ of length $\leq n$, we have a corresponding proof $\Sigma \vdash_{\mathsf{ND}} \varphi$.

Induction Step. Consider a Heyting proof of length $n + 1$. We perform a case analysis according to the type of φ, the last formula of the proof $\Sigma \vdash_{\mathsf{H}} \varphi$.

Case 1: If the last formula is an axiom of the calculus **H**, there is an **ND** proof as we have shown above in the preparatory step.

Case 2: If the last formula is an element of Σ, the corresponding **ND** proof is the trivial proof of length 1.

Case 3: If, on the other hand, φ is the result of an application of (MP), there are two formulae $\psi \to \varphi$ and ψ with Heyting proofs of length $\leq n$, which, by the induction hypothesis, have corresponding **ND** proofs $\Sigma \vdash_{\mathbf{ND}} \psi \to \varphi$ and $\Sigma \vdash_{\mathbf{ND}} \psi$. These two proofs can be concatenated by the rule $(\to\mathsf{E})$ to a proof $\Sigma \vdash_{\mathbf{ND}} \varphi$.

This concludes the proof of the lemma. □

Chapter 3

Modal Logics

3.1 History and Motivation

Modal logic was first devised by C. I. Lewis in his work "A Survey of Symbolic Logic" in 1918. He criticized the paradoxes of (material) implication, such as "ex falso quodlibet" or $\varphi \to \varphi$ and developed a *strict implication* $\varphi \succ \psi$ to avoid (some) paradoxes.

In contrast to intuitionistic logic, which is a subset of classical logic, modal logic is an extension of classical logic. The language of modal logic contains the language of classical logic and the two modalities $\Box$ (sometimes denoted as L) and $\Diamond$ (sometimes denoted as M). In classical modal logics, $\Box$ and $\Diamond$ are inter-definable, i.e., $\Diamond\varphi$ can be defined to be $\neg\Box\neg\varphi$. The formula $\Box\varphi$ is often read as "necessarily φ", whereas $\Diamond\varphi$ is read as "possibly φ". In a philosophical context or in temporal logics (which are special modal logics), the modalities $\Box$ and $\Diamond$ also may denote other dual concepts like "It is obligatory to ..." vs. "It is allowed to ..." or "It will always be that ..." vs. "At some point, it will be the case that ...".

Calculi for modal logic have all the rules and axioms of the base logic (which usually is classical logic) and some more rules and axioms to treat the modal operators.
Depending on which rules and axioms are added to the base logic, we get different modal logics. We discuss *normal modal logics* and the *possible worlds semantic* which has been introduced by Saul Kripke in 1959. Normal modal logics contain all classical tautologies, the modal axiom K and admit the rule of necessitation; the latter two are introduced in Definition 3.2.1.

3.2 Propositional Modal Logic

To obtain propositional modal logic, we need a proof system for classical logic. We will take the system for intuitionistic logic as in Definition 2.2.1, extended by the *double negation elimination* axiom DN $\neg\neg\varphi \to \varphi$, mentioned in Remark 2.2.2. Therefore, we obtain an axiomatization of propositional classical logic.

To spell it out, here are the axioms of the calculus **C**. C1,…,C10 correspond to H1,…,H10.

C1 $\varphi \to (\psi \to \varphi)$

C2 $(\varphi \to (\psi \to \theta)) \to ((\varphi \to \psi) \to (\varphi \to \theta))$

C3 $(\theta \to \varphi) \to ((\theta \to \psi) \to (\theta \to \varphi \wedge \psi))$

C4 $(\varphi \wedge \psi) \to \varphi$

C5 $(\varphi \wedge \psi) \to \psi$

C6 $\varphi \to (\varphi \vee \psi)$

C7 $\psi \to (\varphi \vee \psi)$

C8 $(\varphi \to \theta) \to ((\psi \to \theta) \to (\varphi \vee \psi \to \theta))$

C9 $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$

C10 $\neg\varphi \to (\varphi \to \psi)$

DN $\neg\neg\varphi \to \varphi$

The only rule of **C** is (MP).

In the following, we sometimes use well-known facts of propositional classical logic (like de Morgan's Laws, the truth table semantics, the deduction theorem, etc.) and suppose that the reader is familiar with them.

Definition 3.2.1 (Basic normal modal logic calculus)**.** *The weakest normal modal logic* **K** *is obtained by adding to* **C** *the axiom*

K $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

and the rule of necessitation

$$\frac{\varphi}{\Box\varphi}\ \textsf{(N)}.$$

As the fact that **K** is called the weakest normal modal logic suggests, there will be other systems of normal modal logics under our consideration. These normal modal logics have additional axioms, other than C1–C10, DN and K.

Definition 3.2.2 (Proofs in normal modal logic)**.** *A proof* $\vdash_{\mathbf{L}} \varphi$ *in the calculus of a normal modal logic* **L** *is a sequence* $(\psi_1, \psi_2, \ldots, \psi_n)$ *of formulae, where* $\psi_n = \varphi$ *and, for all* ψ_i*, either:*

- ψ_i *is a substitution instance of the axioms C1,. . . ,C10, DN, K or one of the axioms specific to* **L***,*
- ψ_i *is the conclusion of (MP) whose premises are* ψ_j *and* ψ_k *with* $j, k < i$*, or*
- ψ_i *is the conclusion of (N) whose premise is* ψ_j *with* $j < i$*.*

The length of the proof $\vdash_{\mathbf{L}} \varphi$ *is the length of the sequence* $(\psi_1, \psi_2, \ldots, \psi_n)$*, which is n in this case.*

Example 3.2.3 ($\Box$-distribution over $\wedge$)**.** *A proof of the formula*

$$\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$$

(written as a sequence of formulae) is as follows:

1. *C4:* $(\varphi \wedge \psi) \to \varphi$
2. *C4:* $(\varphi \wedge \psi) \to \psi$
3. *(N) 1:* $\Box((\varphi \wedge \psi) \to \varphi)$
4. *(N) 2:* $\Box((\varphi \wedge \psi) \to \psi)$
5. *K:* $\Box((\varphi \wedge \psi) \to \varphi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi)$
6. *K:* $\Box((\varphi \wedge \psi) \to \psi) \to (\Box(\varphi \wedge \psi) \to \Box\psi)$
7. *(MP) 3, 5:* $\Box(\varphi \wedge \psi) \to \Box\varphi$
8. *(MP) 4, 6:* $\Box(\varphi \wedge \psi) \to \Box\psi$

9. *C3:* $(\Box(\varphi\wedge\psi) \to \Box\varphi) \to ((\Box(\varphi\wedge\psi) \to \Box\psi) \to (\Box(\varphi\wedge\psi) \to \Box\varphi\wedge\Box\psi))$

10. *(MP) 7, 9:* $(\Box(\varphi \wedge \psi) \to \Box\psi) \to (\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi)$

11. *(MP) 8, 10:* $\Box(\varphi \wedge \psi) \to \Box\varphi \wedge \Box\psi$

All formulae are marked with the axiom, which they instantiate, or the rule, of which they are the conclusion of. The last formula is the one which we wanted to prove.

Definition 3.2.4 (Kripke structures, Kripke models for normal modal logics)**.** *A* Kripke structure *(or* Kripke frame*) for modal logic is a pair* (N, R)*, where* N *is a non-empty set of nodes or worlds and* R *is a binary relation on* N*, the* accessibility relation.
A Kripke model $\mathscr{M}$ *is a triple* (N, R, D)*, where* (N, R) *is a Kripke structure and* D *is a function* $D : N \to \mathscr{P}(\mathbb{VAR})$*.*
According to the properties of the accessibility relation R*, the Kripke frames (and thus also the Kripke models) can be divided in classes. For now, we will only consider* $\mathscr{K}$*, the class of all Kripke frames, i.e., the class of Kripke frame, where there is no restriction on* R*.*

When we reconsider the Kripke structures for intuitionistic logic from Chapter 2 (cf. Remark 2.2.8), the most remarkable difference is the accessibility relation. The relation $\leq$ from Definition 2.2.7 is reflexive and transitive, whereas R from the definition above is a general relation with no restrictions at all.

When we look at the Kripke models, we notice another difference in the definition of the domain function D: it is unrestricted for modal Kripke models, but it has to satisfy

$$\forall a, b \in N : a \leq b \Rightarrow D(a) \subseteq D(b)$$

in the intuitionistic case. In other words, D is non-decreasing, when we "follow" the relation towards the successors.

Later in this chapter, we see that stronger normal modal logics (that is logics where we add some further axioms) impose restrictions on the accessibility relations of the models. In particular, the models for the modal logic **S4** have a reflexive and transitive accessibility relation R.

The translation by Kurt Gödel (cf. [Göd33]) form intuitionistic logic to **S4**, which we mentioned in the introduction to Chapter 2, simulates the property of the intuitionistic domain function by adding the modality $\Box$ to certain sub-formulae. The rule (N) takes care of intuitionistic heredity (cf.

Lemma 2.2.11), which is a direct result of the property of the intuitionistic domain function.

Remark 3.2.5 (Abbreviations and inter-definable connectives for normal modal logics)**.** *The following abbreviations will be used:*

- $\Diamond\varphi$ *is defined to be* $\neg\Box\neg\varphi$.
- $\varphi \leftrightarrow \psi$ *is defined to be* $(\varphi \to \psi) \wedge (\psi \to \varphi)$.
- $\top$ *is defined to be* $\neg\bot$.

In classical logic, some connectives are inter-definable. The following statements hold classically—and as the modal calculus K *is an extension of the calculus* C*—also in* K *and all stronger normal modal logics.*

- $\neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
- $\neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$
- $(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi)$

These equivalences can be used as rewrite rules, and so all formulae in classical logic can be rewritten to contain only $\to$ *and* $\bot$ *(or* $\neg$*) as connectives, or alternatively to contain only* $\vee$ *(or* $\wedge$*) and* $\neg$*. This rewriting of formulae might make some proofs easier, in which we proceed by induction on the formula structure, but decreases readability—so we will not take advantage of this possibility.*

Definition 3.2.6 (Forcing relation for modal Kripke models)**.** *The forcing relation* $\vDash$ *on a Kripke model* $\mathscr{M} = (N, \mathrm{R}, D)$ *for normal modal logics is inductively defined for individual worlds* $a \in N$ *by:*

1. $a \vDash p :\Leftrightarrow p \in D(a)$;
2. $a \vDash \neg\varphi :\Leftrightarrow a \nvDash \varphi$;
3. $a \vDash \psi \wedge \varphi :\Leftrightarrow a \vDash \psi$ *and* $a \vDash \varphi$;
4. $a \vDash \psi \vee \varphi :\Leftrightarrow a \vDash \psi$ *or* $a \vDash \varphi$;
5. $a \vDash \psi \to \varphi :\Leftrightarrow$ *If* $a \vDash \psi$*, then also* $a \vDash \varphi$;
6. $a \vDash \Box\varphi :\Leftrightarrow$ *for all* b *with* $a \,\mathrm{R}\, b$*:* $b \vDash \varphi$;
7. $a \vDash \Diamond\varphi :\Leftrightarrow$ *there exists* b *with* $a \,\mathrm{R}\, b$ *and* $b \vDash \varphi$.

We say $\mathscr{M} \vDash \varphi$, if $a \vDash \varphi$ for all $a \in N$, and we say $\mathscr{M} \vDash \Sigma$, if, for all $\sigma \in \Sigma$, $\mathscr{M} \vDash \sigma$. Furthermore, we say that $\mathscr{C} \vDash \varphi$ holds, if, for all models $\mathscr{M}$ of a certain class $\mathscr{C}$ of Kripke frames, $\mathscr{M} \vDash \varphi$ holds.

When we take a closer look at Definition 3.2.6, we recognize that, also on the semantical side, modal logic is only an extension of classical logic. Items 1 to 5 would be defined in exactly the same way for the forcing relation of classical propositional logic. When we erase all axioms and rules which contain modalities, we get a calculus for classical logic. Analogously, when we erase all notions of other worlds in the definition of the Kripke models and the forcing relation, we get models suitable for classical logic.

Lemma 3.2.7 (Soundness for normal modal logics (with respect to modal Kripke models))**.** *If $\vdash_{\mathsf{K}} \varphi$ holds, then $\mathscr{K} \vDash \varphi$ holds.*

Proof. We have to show that, whenever there exists a proof $\vdash_{\mathsf{K}} \varphi$, i.e., a sequence of formulae $(\alpha_1, \alpha_2 \ldots \alpha_n)$ with $\alpha_n = \varphi$ (as in Definition 3.2.2), all models $\mathscr{M}$ force φ or *a fortiori* all α_i.

We will proceed by induction on the length n of the proof $\Sigma \vdash_{\mathsf{K}} \varphi$.

Induction Base. Here we will consider all proofs (α_1) of α_1 with length 1. We distinguish whether φ is an axiom of classical logic or φ is K.

Case 1: α_1 is an instance of a "classical" axiom (i.e., the formula α_1 is an instance of $\mathsf{C1}$-$\mathsf{C10}$ or DN). These axiom schemata contain no modalities. Consider an arbitrary world a to find out whether $a \vDash \alpha_1$. When we have a look at the forcing relation for modal Kripke models, only items 1 to 5 of Definition 3.2.6 are relevant, because of the lack of any $\Box$ and $\Diamond$.[1] Thus, the soundness of this axioms coincides exactly with the soundness of classical propositional logic. This is not surprising, because these axioms are the ones we directly took over from classical logic. So we can make use of the well known fact that classical logic is sound to prove the soundness of these axioms.

Case 2: α_1 is an instance of K. Suppose there is a world a in an arbitrary model $\mathscr{M}$ with $a \nvDash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$. Then $a \vDash \Box(\varphi \to \psi)$ and $a \nvDash \Box\varphi \to \Box\psi$ holds. $a \nvDash \Box\varphi \to \Box\psi$ means that $a \vDash \Box\varphi$ and $a \nvDash \Box\psi$. So there is a world b with $a \mathrm{R} b$ and $b \nvDash \psi$, but, for all c with $a \mathrm{R} c$, $c \vDash \varphi \to \psi$ and $c \vDash \varphi$ hold. By item 5 in Definition 3.2.6,

[1] The fact that there are no modalities in the axioms does not imply that there are no modalities in their substitution instances, but the sub-formulae are not relevant to establish the truth of an axiom.

$c \vDash \psi$ holds. Hence, we have a contradiction as b coincides with some c. Therefore the axiom K is forced in (all worlds a of) all models.

Induction Hypothesis. Suppose $n \geq 0$ and $\mathscr{K} \vDash \varphi$ holds for all formulas φ, which have a proof $\vdash_{\mathbf{K}} \varphi$ of length $\leq n$.

Induction step: When we have a proof $\vdash_{\mathbf{K}} \varphi$ of length $n+1$, the formula α_{n+1} (i.e., the formula φ) can either be an axiom, which essentially corresponds to the base case, or it can be the conclusion of a rule, namely (MP) and (N).

Case (MP): α_{n+1} is the conclusion of an application of (MP). There exist two premise formulae α_i and α_j, with $i, j \leq n$. Without loss of generality we assume that $\alpha_j = \alpha_i \to \alpha_{n+1}$. Then, by the induction hypothesis, $\mathscr{K} \vDash \alpha_i \to \alpha_{n+1}$ and $\mathscr{K} \vDash \alpha_i$ hold. Thus, by Definition 3.2.6, it follows that $\mathscr{K} \vDash \alpha_{n+1}$ holds.

Case (N): α_{n+1} is the conclusion of an application of (N). Then there exists a formula α_i with $i \leq n$, $\alpha_{n+1} = \Box\alpha_i$ and, for any $\mathscr{M}$ in $\mathscr{K}$, all worlds in $\mathscr{M}$ force α_i by the induction hypothesis. So, for any world a, all worlds b with $a \,\mathrm{R}\, b$ force α_i, so $a \vDash \Box\alpha_i$ and $\mathscr{K} \vDash \alpha_{n+1}$ hold.

This concludes the proof of the lemma. □

To establish completeness of normal modal logics (and at first of the system K) with respect to modal Kripke models, we will construct a canonical model for each formula set Σ, which only forces formulae that can be proven by Σ.

To achieve this we introduce the notion of maximal consistent extensions.

We will mainly follow [HC96].

Definition 3.2.8 (Consistency, maximality)**.** *A set Σ of formulae is* $\mathbf{L}$-consistent*, if there is no proof $\vdash_{\mathbf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_n)$ for $\alpha_1, \ldots, \alpha_n \in \Sigma$, where $\vdash_{\mathbf{L}}$ is the proof relation of some normal modal logic $\mathbf{L}$.*
A set Σ of formulae is maximal *if, for every formula φ, either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$.*

Definition 3.2.9 (Maximal $\mathbf{L}$-consistent extensions)**.** *A set of formulae $\Sigma' \supseteq \Sigma$ is a maximal $\mathbf{L}$-consistent extension of Σ, if Σ' is $\mathbf{L}$-consistent and maximal.*

Lemma 3.2.10 (Existence of maximal $\mathbf{L}$-consistent extensions)**.** *For every $\mathbf{L}$-consistent set Σ, there is a maximal $\mathbf{L}$-consistent extension Σ' of Σ.*

Proof. In order to construct such a Σ', we enumerate all formulae of the language of the normal modal logic $\mathbf{L}$ and create a chain of $\mathbf{L}$-consistent formula sets $\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \ldots$ and define $\Sigma' = \bigcup_{i \in \mathbb{N}} \Sigma_i$.
To do this, we enumerate all formulae φ by an index n. We create the Σ_n as follows: We start with $\Sigma_0 = \Sigma$; the other Σ_n are constructed inductively by the following rule:

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if } \Sigma_n \cup \{\varphi_n\} \text{ is } \mathbf{L}\text{-consistent,} \\ \Sigma_n \cup \{\neg\varphi_n\} & \text{otherwise.} \end{cases}$$

We have to show that Σ' is maximal and $\mathbf{L}$-consistent.

Σ' is maximal. For every formula φ, there is an $n \in \mathbb{N}$ such that $\varphi = \varphi_n$. Therefore, $\varphi \in \Sigma_{n+1}$ or $\neg\varphi \in \Sigma_{n+1}$ and thus $\varphi \in \Sigma'$ or $\neg\varphi \in \Sigma'$.

Σ' is $\mathbf{L}$-consistent. Suppose Σ' is $\mathbf{L}$-inconsistent, i.e., there is a proof $\vdash_{\mathbf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_k)$ for $\alpha_1, \ldots, \alpha_k \in \Sigma'$. As there are only finitely many α_i, we can find an n such that $\alpha_1, \ldots, \alpha_k \in \Sigma_n$, i.e., a Σ_n which is $\mathbf{L}$-inconsistent.

When we are able to show that all Σ_n are $\mathbf{L}$-consistent, we have produced a contradiction thus proving the $\mathbf{L}$-consistency of Σ'.
We proceed by induction on n to show that all Σ_n are $\mathbf{L}$-consistent.

Induction Base. $n = 0$. Then $\Sigma_0 = \Sigma$ is $\mathbf{L}$-consistent by the assumptions of this lemma.

Induction Hypothesis. Suppose $n \geq 0$ and assume that, for all $i \leq n$, Σ_i is $\mathbf{L}$-consistent.

Induction Step. By the induction hypothesis, we have that Σ_n is $\mathbf{L}$-consistent. We have to show that Σ_{n+1} is $\mathbf{L}$-consistent. To achieve this, we will use an indirect approach and assume that Σ_{n+1} is $\mathbf{L}$-inconsistent.
Since Σ_{n+1} is considered to be $\mathbf{L}$-inconsistent and Σ_n is $\mathbf{L}$-consistent, Σ_{n+1} must be $\Sigma_n \cup \{\neg\varphi_n\}$ by construction. Moreover, $\Sigma_n \cup \{\varphi_n\}$ is $\mathbf{L}$-inconsistent. So we have a proof $\vdash_{\mathbf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \varphi_n)$ and a proof $\vdash_{\mathbf{L}} \neg(\beta_1 \wedge \ldots \wedge \beta_l \wedge \neg\varphi_n)$. Clearly φ_n (and $\neg\varphi_n$ respectively) must occur in the proof, since otherwise, Σ_n would be $\mathbf{L}$-inconsistent as well.

Since the logical connective $\wedge$ is left-associative, $(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \varphi_n)$ is actually $(\ldots(\alpha_1 \wedge \alpha_2) \wedge \ldots \wedge \alpha_k) \wedge \varphi_n)$. We abbreviate $(\ldots(\alpha_1 \wedge \alpha_2) \wedge \ldots \wedge \alpha_k)$ by $\bar{\alpha}$ in the rest of the proof (analogously, we write $\bar{\beta}$ for the conjunction of all β_j).

$$
\dfrac{
  \dfrac{[\bar\alpha \wedge \bar\beta]_2}{\bar\beta}\,(\wedge\mathsf{E})
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{[\bar\alpha \wedge \bar\beta]_2}{\bar\alpha}\,(\wedge\mathsf{E})
        \quad [\varphi_n]_1
      }{\bar\alpha \wedge \varphi_n}\,(\wedge\mathsf{I})
      \quad \neg(\bar\alpha \wedge \varphi_n)
    }{\bot}\,(\neg\mathsf{E})
  }{\neg\varphi_n}\,(\neg\mathsf{I})_1
}{
  \dfrac{
    \dfrac{\bar\beta \wedge \neg\varphi_n \qquad \neg(\bar\beta \wedge \neg\varphi_n)}{\bot}\,(\neg\mathsf{E})
  }{\neg(\bar\alpha \wedge \bar\beta)}\,(\neg\mathsf{I})_2
}\,(\wedge\mathsf{I})
$$

Figure 3.1: The proof $\neg(\bar\alpha \wedge \varphi_n), \neg(\bar\beta \wedge \neg\varphi_n) \vdash_{\mathsf{ND}} \neg(\bar\alpha \wedge \bar\beta)$.

In Figure 3.1, we show that there is a proof

$$\neg(\bar\alpha \wedge \varphi_n), \neg(\bar\beta \wedge \neg\varphi_n) \vdash_{\mathsf{ND}} \neg(\bar\alpha \wedge \bar\beta)$$

in intuitionistic logic. Therefore $\neg(\bar\alpha\wedge\bar\beta)$ is also provable from $\neg(\bar\alpha\wedge\varphi_n)$ and $\neg(\bar\beta \wedge \neg\varphi_n)$ in classical logic and modal logic. Here we use the system **ND** for convenience, as such proofs are rather lengthy and tedious in a Hilbert style calculus.

By Theorem 2.2.25, when $\neg(\bar\alpha \wedge \varphi_n), \neg(\bar\beta \wedge \neg\varphi_n) \vdash_{\mathsf{ND}} \neg(\bar\alpha \wedge \bar\beta)$ holds, there also is a **H**-proof $\neg(\bar\alpha \wedge \varphi_n), \neg(\bar\beta \wedge \neg\varphi_n) \vdash_{\mathsf{H}} \neg(\bar\alpha \wedge \bar\beta)$. By the deduction theorem for **H** (cf. Theorem 2.2.6), we have $\vdash_{\mathsf{H}} \neg(\bar\alpha\wedge\varphi_n) \rightarrow (\neg(\bar\beta\wedge\neg\varphi_n) \rightarrow \neg(\bar\alpha\wedge\bar\beta))$. Clearly, the above **H**-proof is also a proof in the modal calculus **K** (or any stronger normal modal logic **L**), namely $\vdash_{\mathsf{K}} \neg(\bar\alpha \wedge \varphi_n) \rightarrow (\neg(\bar\beta \wedge \neg\varphi_n) \rightarrow \neg(\bar\alpha \wedge \bar\beta))$. By our assumption that Σ_{n+1} is **L**-inconsistent, we have obtained two more **L**-proofs $\vdash_{\mathsf{L}} \neg(\bar\alpha\wedge\varphi_n)$ and $\vdash_{\mathsf{L}} \neg(\bar\beta\wedge\neg\varphi_n)$. Via two applications of (MP), these three proofs can be combined to a proof $\vdash_{\mathsf{L}} \neg(\bar\alpha \wedge \bar\beta)$ or—to spell it out—a proof $\vdash_{\mathsf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \beta_1 \wedge \ldots \wedge \beta_l)$. But $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l \in \Sigma_n$, so we have obtained **L**-inconsistency of Σ_n by assuming that Σ_{n+1} is **L**-inconsistent. This is a contradiction to the induction hypothesis and concludes the induction proof of the **L**-consistency of Σ'.

Now we have shown that Σ' is maximal and **L**-consistent. □

For the next lemma, we need to know if $\vdash_{\mathsf{K}} \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ holds. The other "direction" of the implication has been shown in Example 3.2.3. This proof in Example 3.2.3 was rather lengthy in the Hilbert style calculus

H. Since the proof for this formula would be even longer, we make use of Theorem 2.2.25 and use the much nicer calculus **ND** for some parts of the next example.

Example 3.2.11 ($\vdash_{\mathbf{K}} \Box\varphi \land \Box\psi \to \Box(\varphi \land \psi)$)**.** *In order to show that* $\vdash_{\mathbf{K}} \Box\varphi \land \Box\psi \to \Box(\varphi \land \psi)$ *holds, we need some preliminary results. First we show that* $\vdash_{\mathbf{K}} \varphi \to (\psi \to (\varphi \land \psi))$ *holds. We present an* **ND***-proof for the formula, which has an analogous* **H***-proof by Theorem 2.2.25, which in turn is a* **K***-proof as* **K** *contains all rules and axioms of* **H***. So here is the proof* $\vdash_{\mathbf{ND}} \varphi \to (\psi \to (\varphi \land \psi))$*:*

$$\dfrac{\dfrac{\dfrac{[\varphi]_2 \quad [\psi]_1}{\varphi \land \psi}\,(\land I)}{\psi \to (\varphi \land \psi)}\,(\to I)_1}{\varphi \to (\psi \to (\varphi \land \psi))}\,(\to I)_2$$

Furthermore, we have to make sure that $\vdash_{\mathbf{K}} (\varphi \to \psi) \to (\psi \to (\theta \to \kappa)) \to ((\varphi \land \theta) \to \kappa)$ *holds. As this formula is intuitionistically valid as well, we again present an* **ND***-proof:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \land \theta]_1}{\theta}\,(\land E) \qquad \dfrac{\dfrac{\dfrac{[\varphi \land \theta]_1}{\varphi}\,(\land E) \quad [\varphi \to \psi]_3}{\psi}\,(\to E) \qquad [\psi \to (\theta \to \kappa)]_2}{\theta \to \kappa}\,(\to E)}{\kappa}\,(\to E)}{(\varphi \land \theta) \to \kappa}\,(\to I)_1}{(\psi \to (\theta \to \kappa)) \to ((\varphi \land \theta) \to \kappa)}\,(\to I)_2}{(\varphi \to \psi) \to ((\psi \to (\theta \to \kappa)) \to ((\varphi \land \theta) \to \kappa))}\,(\to I)_3$$

We will not use this formula directly, but the following substitution instance, where we substitute φ *by* $\Box\varphi$*,* ψ *by* $\Box(\psi \to \varphi \land \psi)$*,* θ *by* $\Box\psi$ *and* κ *by* $\Box(\varphi \land \psi)$*. We obtain:*

$$(\Box\varphi \to \Box(\psi \to \varphi \land \psi)) \to ((\Box(\psi \to \varphi \land \psi) \to (\Box\psi \to \Box(\varphi \land \psi))) \to (\Box\varphi \land \Box\psi \to \Box(\varphi \land \psi)))$$

Here we have removed some parentheses, which we do not need because of the precedence rules for logical connectives, which we introduced in Section 1.1.2.

In the following, we use abbreviations for some formulae:

- *The above formula will be abbreviated by* α*.*

$$
\dfrac{\mathsf{K}_2 \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\text{by } \vdash_{\mathbf{H}} \varphi \to (\psi \to (\varphi \wedge \psi)) \\ \triangledown \\ \varphi \to (\psi \to (\varphi \wedge \psi))\end{array}}{\Box(\varphi \to (\psi \to (\varphi \wedge \psi)))}\,(\mathsf{N}) \qquad \mathsf{K}_1}{\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))}\,(\mathsf{MP}) \qquad \begin{array}{c}\text{by } \vdash_{\mathbf{H}} \alpha \\ \triangledown \\ \alpha\end{array}}{\beta}\,(\mathsf{MP})}{\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)}\,(\mathsf{MP})
$$

Figure 3.2: The proof $\vdash_{\mathbf{K}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$.

- *The above formula, with the principal implication and the formula* $(\Box\varphi \to \Box(\psi \to \varphi \wedge \psi))$ *removed, is abbreviated as* β *(so* $\alpha = (\Box\varphi \to \Box(\psi \to \varphi \wedge \psi)) \to \beta$*).*
- K_1 *is the following substitution instance of* K*:*

$$\Box(\varphi \to (\psi \to (\varphi \wedge \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \wedge \psi))).$$

- K_2 *is the following substitution instance of* K*:*

$$\Box(\psi \to (\varphi \wedge \psi)) \to (\Box\psi \to \Box(\varphi \wedge \psi)).$$

In Figure 3.2 a proof

$$\vdash_{\mathbf{K}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi))$$

is shown, which uses the two proofs $\vdash_{\mathbf{K}} \alpha$ *and* $\vdash_{\mathbf{K}} \varphi \to (\psi \to (\varphi \wedge \psi))$*, which clearly exist, as the* **ND***-proofs have corresponding* **H***-proofs, which are* **K***-proofs.*

Corollary 3.2.12 (Corollary to Example 3.2.11)**.** *The following holds for all* $n > 0$*:*

$$\vdash_{\mathbf{K}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n).$$

Proof. Since we have established that $\vdash_{\mathbf{K}} \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ holds, it is easy to show that the same is true for the conjunction of n formulae, i.e., that $\vdash_{\mathbf{K}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n)$ holds. We proceed by induction on n, the number of conjuncts, to show that the corollary is true.

Induction Base. Suppose $n = 1$. Then the formula which we want to prove is of the form $\Box\varphi_1 \to \Box\varphi_1$. This is a substitution instance of the formula $\varphi \to \varphi$. In Example 2.2.4, we have presented a proof $\vdash_{\mathsf{H}} \varphi \to \varphi$. This proof is also a K proof.

Induction Hypothesis. Suppose $n \geq 1$ and assume that, for all $i \leq n$, we have that $\vdash_{\mathsf{K}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n)$ holds.

Induction Step. Consider $\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_{n+1} \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_{n+1})$ By the induction hypothesis $\vdash_{\mathsf{K}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n)$ holds. From this we obtain $\vdash_{\mathsf{K}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \wedge \Box\varphi_{n+1} \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \wedge \Box\varphi_{n+1}$ with this substitution instance

$$(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n)) \to \\ (((\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n) \wedge \Box\varphi_{n+1}) \to (\Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \wedge \Box\varphi_{n+1}))$$

of the formula we have proven in Example 2.2.18 and one application of (MP). When we take the substitution instance $\Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \wedge \Box\varphi_{n+1} \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi_{n+1})$ of Example 3.2.11, we can use "transitivity" of $\to$ (which was shown in Example 2.2.17) and obtain

$$\vdash_{\mathsf{K}} \Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n \wedge \Box\varphi_{n+1} \to \Box(\varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi_{n+1}).$$

This concludes the proof of the corollary. $\Box$

Lemma 3.2.13. *If an L-consistent set Σ contains $\neg\Box\varphi$, then $\Box^-(\Sigma) \cup \{\neg\varphi\}$ is L-consistent, where*

$$\Box^-(\Sigma) = \{\varphi \mid \Box\varphi \in \Sigma\}.$$

Proof. In the following, we need a proof of $\neg(\psi \wedge \neg\theta) \to (\psi \to \theta)$ and a proof of $(\psi \to \theta) \to \neg(\psi \wedge \neg\theta)$. The first formula is not valid in intuitionistic logic, but in classical logic. As the Hilbert style calculus of K is rather cumbersome, we will present a proof in the calculus ND extended by the rule (RAA) (cf. Remark 2.2.14), which suffices to show that this formula is provable in classical logic. By the completeness of classical logic with respect to the natural deduction calculus extended by the rule (RAA) and the calculus C, this is also provable in the classical fragment of K (and thus any normal modal logic L).
The second formula is intuitionistically valid, so we only present an ND-proof, which has a corresponding H-proof by Theorem 2.2.25. This H-proof is also a K-proof. The two proofs are presented in Figure 3.3.

Suppose $\Box^-(\Sigma) \cup \{\neg\varphi\}$ is L-inconsistent, then there is a proof

$$\vdash_{\mathsf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\varphi).$$

We can safely assume that $\neg\varphi$ is in the proof for the **L**-inconsistency of $\Box^{-}(\Sigma) \cup \{\neg\varphi\}$. When we only have a proof

$$\vdash_{\mathsf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_n)$$

by transitivity of $\rightarrow$ (cf. Example 2.2.17) and with this substitution instance of C4:

$$(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\varphi) \rightarrow (\alpha_1 \wedge \ldots \wedge \alpha_n)$$

we obtain the the desired proof $\vdash_{\mathsf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\varphi)$.
On the other hand, if there are no α_i in the proof for the **L**-inconsistency of $\Box^{-}(\Sigma) \cup \{\neg\varphi\}$, we can add some in the same fashion as we added $\neg\varphi$ above, if $\Box^{-}(\Sigma)$ is not empty. If $\Box^{-}(\varphi)$ is empty and $\vdash_{\mathsf{L}} \neg\neg\varphi$ holds, we also have a proof $\vdash_{\mathsf{L}} \varphi$ and, by necessitation, $\vdash_{\mathsf{L}} \Box\varphi$ and $\vdash_{\mathsf{L}} \neg\neg\Box\varphi$. But then Σ is **L**-inconsistent as $\neg\Box\varphi \in \Sigma$. This is a contradiction to the assumption of the lemma. Now we have established that there has to be a proof of the form $\vdash_{\mathsf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\varphi)$, if we suppose that $\Box^{-}(\Sigma) \cup \{\neg\varphi\}$ is **L**-inconsistent.

This proof $\vdash_{\mathsf{L}} \neg(\alpha_1 \wedge \ldots \wedge \alpha_n \wedge \neg\varphi)$ can be viewed as $\vdash_{\mathsf{L}} \neg(\bar{\alpha} \wedge \neg\varphi)$, where $\bar{\alpha} = \alpha_1 \wedge \ldots \wedge \alpha_n$. Using the formula $\neg(\psi \wedge \neg\theta) \rightarrow (\psi \rightarrow \theta)$, we obtain $\vdash_{\mathsf{L}} (\bar{\alpha} \rightarrow \varphi)$ by an application of (MP). We continue to derive $\Box\bar{\alpha} \rightarrow \Box\varphi$ as follows:

$$\dfrac{\dfrac{\dfrac{\overset{\text{By assumption}}{\overset{\triangledown}{\neg(\bar{\alpha} \wedge \neg\varphi)}} \qquad \overset{\text{By classical logic}}{\overset{\triangledown}{\neg(\bar{\alpha} \wedge \neg\varphi) \rightarrow (\bar{\alpha} \rightarrow \varphi)}}}{\bar{\alpha} \rightarrow \varphi}\,(\mathsf{MP})}{\Box(\bar{\alpha} \rightarrow \varphi)}\,(\mathsf{N}) \qquad \overset{\mathsf{K}}{\Box(\bar{\alpha} \rightarrow \varphi) \rightarrow (\Box\bar{\alpha} \rightarrow \Box\varphi)}}{\Box\bar{\alpha} \rightarrow \Box\varphi}\,(\mathsf{MP})$$

This is a proof of $\vdash_{\mathsf{K}} \Box\bar{\alpha} \rightarrow \Box\varphi$ using the two proofs $\vdash_{\mathsf{H}} \neg(\bar{\alpha} \wedge \neg\varphi)$ and $\vdash_{\mathsf{H}} \neg(\bar{\alpha} \wedge \neg\varphi) \rightarrow (\bar{\alpha} \rightarrow \theta)$. Clearly, these **H**-proofs exist by the completeness of intuitionistic logic and are **L**-proofs, for any normal modal logic **L**.

By the following substitution instance of the formula proven in Corollary 3.2.12

$$\vdash_{\mathsf{K}} \Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n \rightarrow \Box(\alpha_1 \wedge \ldots \wedge \alpha_n)$$

and transitivity of $\rightarrow$, we can transform the proof $\vdash_{\mathsf{K}} \Box\bar{\alpha} \rightarrow \Box\varphi$ to

$$\vdash_{\mathsf{L}} (\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n) \rightarrow \Box\varphi.$$

Using the intuitionistically provable formula $(\psi \rightarrow \theta) \rightarrow \neg(\psi \wedge \neg\theta)$ from Figure 3.3 and (MP), we obtain

$$\vdash_{\mathsf{L}} \neg(\Box\alpha_1 \wedge \ldots \wedge \Box\alpha_n \wedge \neg\Box\varphi).$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\psi]_2 \quad [\neg\theta]_1}{\psi \wedge \neg\theta}\,(\wedge\mathsf{I}) \qquad [\neg(\psi\wedge\neg\theta)]_3}{\bot}\,(\neg\mathsf{E})}{\theta}\,(\mathsf{RAA})_1}{(\psi\to\theta)}\,(\to\mathsf{I})_2}{\neg(\psi\wedge\neg\theta)\to(\psi\to\theta)}\,(\to\mathsf{I})_3$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\psi\wedge\neg\theta]_1}{\psi}\,(\wedge\mathsf{E}) \qquad [\psi\to\theta]_2}{\theta}\,(\to\mathsf{E}) \qquad \dfrac{[\psi\wedge\neg\theta]_1}{\neg\theta}\,(\wedge\mathsf{E})}{\bot}\,(\neg\mathsf{E})}{\neg(\psi\wedge\neg\theta)}\,(\neg\mathsf{I})_1}{(\psi\to\theta)\to\neg(\psi\wedge\neg\theta)}\,(\to\mathsf{I})_2$$

Figure 3.3: The proof $\vdash_{\mathbf{ND}\cup\{(\mathsf{RAA})\}} \neg(\psi \wedge \neg\theta) \to (\psi \to \theta)$ and the proof $\vdash_{\mathbf{ND}} (\psi \to \theta) \to \neg(\psi \wedge \neg\theta)$.

Since $\Box\alpha_1, \ldots, \Box\alpha_n, \neg\Box\varphi \in \Sigma$, Σ is $\mathbf{L}$-inconsistent. This is a contradiction to the prerequisite of the lemma, namely that Σ is $\mathbf{L}$-consistent.

This concludes the proof of the lemma. $\square$

Lemma 3.2.14 (Properties of $\mathbf{L}$-consistent sets)**.** *Let Σ be a maximal $\mathbf{L}$-consistent set. Then the following properties hold:*

1. *For all formulae φ, either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$.*
2. *$\varphi \vee \psi \in \Sigma$, if and only if $\varphi \in \Sigma$ or $\psi \in \Sigma$.*
3. *$\varphi \wedge \psi \in \Sigma$, if and only if $\varphi \in \Sigma$ and $\psi \in \Sigma$.*
4. *If $\varphi \to \psi \in \Sigma$, then, if $\varphi \in \Sigma$, also $\psi \in \Sigma$.*
5. *If $\vdash_{\mathbf{L}} \varphi$ holds, then $\varphi \in \Sigma$.*

Proof. We will prove the above properties one by one.

1. By the maximality of Σ, we have $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$. If both φ and $\neg\varphi$ were elements of Σ, then Σ is $\mathbf{L}$-inconsistent by the following substitution instance of the formula proven in Example 2.2.16:

$$\varphi \wedge (\varphi \to \bot) \to \bot$$

Since this formula is provable in intuitionistic logic, it is also provable in **K**.

2. If $\varphi \vee \psi \in \Sigma$ and neither $\varphi \in \Sigma$ nor $\psi \in \Sigma$, then, by the maximality of Σ, we have $\neg\varphi \in \Sigma$ and $\neg\psi \in \Sigma$. But then, by Example 2.2.19, the formula
$$\neg((\varphi \vee \psi) \wedge \neg\varphi \wedge \neg\psi)$$
is intuitionistically valid. This means that it is also provable in **K** or any stronger normal modal logic **L**. Thus, Σ is **L**-inconsistent.

   For the other direction, without loss of generality, suppose that $\varphi \in \Sigma$ and $\varphi \vee \psi \notin \Sigma$. By the maximality of Σ, $\neg(\varphi \vee \psi)$ must be in Σ. With axiom C6: $(\varphi \to (\varphi \vee \psi))$ and this substitution instance
$$(\varphi \to (\varphi \vee \psi)) \to \neg(\varphi \wedge \neg(\varphi \vee \psi))$$
of the second formula proven in Figure 3.3, we obtain a proof $\vdash_{\mathbf{K}} \neg(\varphi \wedge \neg(\varphi \vee \psi))$. But then, Σ is **L**-inconsistent.

3. Suppose $\varphi \wedge \psi \in \Sigma$ and $\varphi \notin \Sigma$. Then, by the maximality of Σ, $\neg\varphi$ must be in Σ. With axiom C4: $((\varphi \wedge \psi) \to \varphi)$ and the substitution instance
$$((\varphi \wedge \psi) \to \varphi) \to \neg((\varphi \wedge \psi) \wedge \neg\varphi)$$
of the second formula proven in Figure 3.3, we obtain a proof $\vdash_{\mathbf{K}} \neg((\varphi \wedge \psi) \wedge \neg\varphi)$. But then, Σ is **L**-inconsistent.

   For the other direction, suppose $\varphi \in \Sigma$ and $\psi \in \Sigma$, but $\varphi \wedge \psi \notin \Sigma$. Then $\neg(\varphi \wedge \psi) \in \Sigma$. By Example 2.2.16, we have the following proof
$$\vdash_{\mathbf{L}} ((\varphi \wedge \psi) \wedge ((\varphi \wedge \psi) \to \bot) \to \bot.$$
Since the proven formula can be read as $\neg(\varphi \wedge \psi \wedge \neg(\varphi \wedge \psi))$, Σ is **L**-inconsistent.

4. Suppose $\varphi \to \psi \in \Sigma$ and $\varphi \in \Sigma$, but $\psi \notin \Sigma$. Then $\neg\psi \in \Sigma$. By Example 2.2.20, we have the following proof:
$$\vdash_{\mathbf{L}} \neg(\varphi \wedge (\varphi \to \psi) \wedge \neg\psi).$$
This is a contradiction to the **L**-consistency of Σ.

5. Suppose $\vdash_{\mathbf{L}} \varphi$ holds, but $\varphi \notin \Sigma$. Then $\neg\varphi \in \Sigma$. In the following proof tree, we see that any proof $\vdash_{\mathbf{L}} \varphi$ can easily be transformed to a proof $\vdash_{\mathbf{L}} \neg\neg\varphi$, thus Σ is **L**-inconsistent.

$$\cfrac{\begin{array}{c}\vdash_{\mathsf{L}}\varphi\\ \triangledown\\ \varphi\end{array} \qquad \cfrac{\begin{array}{c}\text{By Example 2.2.4}\\ \triangledown\\ \neg\varphi\to\neg\varphi\end{array} \qquad \begin{array}{c}\mathsf{C9}\\ (\neg\varphi\to\neg\varphi)\to(\varphi\to\neg\neg\varphi)\end{array}}{\varphi\to\neg\neg\varphi}\,(\mathsf{MP})}{\neg\neg\varphi}\,(\mathsf{MP})$$

This concludes the proof of the lemma. □

Definition 3.2.15 (Canonical model for L)**.** *The* canonical model $\mathscr{C}$ *for a normal modal logic* L *is a triple* (N, R, D)*, where* N *is the set of all maximal* L*-consistent sets* Σ*,* R *is defined by*

$$\Sigma\,\mathrm{R}\,\Sigma' \Leftrightarrow: \Box^{-}(\Sigma) \subseteq \Sigma'$$

and $D(\Sigma) = \Sigma \cap \mathbb{VAR}$*.*

Lemma 3.2.16 (Canonical model for K)**.** *There is a canonical models* $\mathscr{C} = (N, \mathrm{R}, D)$ *for the normal modal logic* K*.*

Proof. By the existence of maximal L-consistent extensions (Lemma 3.2.10), the set N is not empty, but it contains all maximal L-consistent extensions of $\emptyset$. As there are no restrictions on the properties of R and D, other than the range of the domain function being the set of variables, we have model satisfying the conditions of Definition 3.2.4. □

Lemma 3.2.17. *Let* $\mathscr{C} = (N, \mathrm{R}, D)$ *be the canonical model for a normal modal logic* L*, then, for any* $\Sigma \in N$*,* $\Sigma \vDash \varphi$ *holds, if and only if* $\varphi \in \Sigma$*.*

Proof. We proceed by induction on the logical complexity, $\mathrm{lcomp}(\varphi)$, of φ to show that $\Sigma \vDash \varphi$ holds, if and only if $\varphi \in \Sigma$.

Induction Base. $\mathrm{lcomp}(\varphi) = 0$. As $\bot$ is never forced in any world, and any set containing $\bot$ is immediately L-inconsistent (by $\vdash_{\mathsf{L}} \bot \to \bot$ for any normal modal logic L), we only have to consider here the case where $\varphi = p$. For propositional variables, the statement holds, because $D(\Sigma) = \Sigma \cap \mathbb{VAR}$ in the construction of the canonical model (cf. Definition 3.2.15).

Induction Hypothesis. Suppose $n \geq 0$ and for all φ with $\mathrm{lcomp}(\varphi) \leq n$ and all $\Sigma \in N$, the following equivalence holds:

$$\Sigma \vDash \varphi \Leftrightarrow \varphi \in \Sigma$$

Induction Step. Let us consider φ with $\mathrm{lcomp}(\varphi) = n + 1$. We perform a case distinction with respect to the top-level connective.

φ is of the form $\neg\varphi_1$: By Definition 3.2.6, $\Sigma \vDash \neg\varphi_1$ holds, if and only if $\Sigma \nvDash \varphi_1$. By the induction hypothesis, this is the case, if and only if $\varphi_1 \notin \Sigma$. By Lemma 3.2.14 (1), $\varphi_1 \notin \Sigma$ if and only if $\neg\varphi_1 \in \Sigma$.

φ is of the form $\varphi_1 \wedge \varphi_2$: Suppose $\Sigma \vDash \varphi$ holds. Then, by Definition 3.2.6, $\Sigma \vDash \varphi_1$ and $\Sigma \vDash \varphi_2$ holds. By the induction hypothesis, we have $\varphi_1, \varphi_2 \in \Sigma$. By Lemma 3.2.14 (3), this is the case if and only if $\varphi_1 \wedge \varphi_2 \in \Sigma$.

φ is of the form $\varphi_1 \vee \varphi_2$: If $\Sigma \vDash \varphi$ holds, then, by Definition 3.2.6, $\Sigma \vDash \varphi_1$ holds or $\Sigma \vDash \varphi_2$ holds. Without loss of generality, we assume that $\Sigma \vDash \varphi_1$ holds. By the induction hypothesis, we obtain $\varphi_1 \in \Sigma$. By Lemma 3.2.14 (2), then also $\varphi_1 \vee \varphi_2 \in \Sigma$.

For the other direction, if $\varphi_1 \vee \varphi_2 \in \Sigma$, then $\varphi_1 \in \Sigma$ or $\varphi_2 \in \Sigma$, by Lemma 3.2.14 (2). Without loss of generality, assume $\varphi_1 \in \Sigma$. Then, by the induction hypothesis, $\Sigma \vDash \varphi_1$ holds. By Definition 3.2.6, $\Sigma \vDash \varphi_1 \vee \varphi_2$ also holds.

φ is of the form $\varphi_1 \to \varphi_2$: Suppose $\Sigma \vDash \varphi$ holds. There are two cases depending on whether φ_1 is forced in Σ.
Sub-case 1: If $\Sigma \nvDash \varphi_1$ holds, then, by the induction hypothesis and the maximality of Σ, $\varphi_1 \notin \Sigma$ and $\neg\varphi_1 \in \Sigma$. If $\varphi_1 \to \varphi_2 \notin \Sigma$, then $\neg(\varphi_1 \to \varphi_2) \in \Sigma$. But then Σ is **L**-inconsistent, as can be seen in the following proof tree:

$$\dfrac{\overset{\textsf{C9}}{\neg\varphi_1 \to (\varphi_1 \to \varphi_2)} \quad \overset{\text{By Figure 3.3}}{(\neg\varphi_1 \to (\varphi_1 \to \varphi_2)) \to \neg(\neg\varphi_1 \wedge \neg(\varphi_1 \to \varphi))}}{\neg(\neg\varphi_1 \wedge \neg(\varphi_1 \to \varphi))}\ (\textsf{MP})$$

Therefore, $\varphi_1 \to \varphi_2 \in \Sigma$ holds.
Sub-case 2: If $\Sigma \vDash \varphi_1$ holds, by Definition 3.2.6, $\Sigma \vDash \varphi_2$ also holds, as we suppose $\Sigma \vDash \varphi_1 \to \varphi_2$ holds. By the induction hypothesis, $\varphi_1, \varphi_2 \in \Sigma$. If $\varphi \notin \Sigma$ and, thus, $\neg\varphi \in \Sigma$, Σ is **L**-inconsistent, as can be seen in the following **ND**-proof tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg(\varphi_1 \to \varphi_2) \wedge \varphi_2]_1}{\varphi_2}\ (\wedge\mathsf{E})}{\varphi_1 \to \varphi_2}\ (\to\mathsf{I}) \qquad \dfrac{[\neg(\varphi_1 \to \varphi_2) \wedge \varphi_2]_1}{\neg(\varphi_1 \to \varphi_2)}\ (\wedge\mathsf{E})}{\bot}\ (\neg\mathsf{E})}{\neg(\neg(\varphi_1 \to \varphi_2) \wedge \varphi_2)}\ (\to\mathsf{I})_1$$

For the other direction, suppose $\varphi \in \Sigma$. If $\varphi_1 \in \Sigma$, then, by Lemma 3.2.14 (4), $\varphi_2 \in \Sigma$. So by the induction hypothesis, $\Sigma \vDash \varphi_1$ and $\Sigma \vDash \varphi_2$

hold. Then, by Definition 3.2.6, $\Sigma \vDash \varphi_1 \to \varphi_2$ holds.
If $\varphi_1 \notin \Sigma$, then by the induction hypothesis $\Sigma \nvDash \varphi_1$ holds, and so, $\Sigma \vDash \varphi_1 \to \varphi_2$ holds trivially.

φ is of the form $\Box\varphi_1$: Suppose $\varphi \notin \Sigma$. Then, by the maximality of Σ, the formula $\neg\Box\varphi_1 \in \Sigma$. By Lemma 3.2.13 and Lemma 3.2.10, there is a maximal L-consistent set Σ' with $\Box^-(\Sigma) \cup \{\neg\varphi_1\} \subseteq \Sigma'$. By Definition 3.2.15, $\Sigma \,\mathrm{R}\, \Sigma'$ holds, but, by Lemma 3.2.14 (1), $\varphi_1 \notin \Sigma'$ and by the induction hypothesis Σ' does not force φ_1. By the definition of the forcing relation, it follows that $\Sigma \nvDash \varphi$.

Suppose $\varphi \in \Sigma$. Then, for all Σ' with $\Sigma \,\mathrm{R}\, \Sigma'$, we have $\varphi_1 \in \Sigma'$, by the construction of the canonical model (cf. Definition 3.2.15). So by the induction hypothesis, for all Σ' with $\Sigma \,\mathrm{R}\, \Sigma'$, $\Sigma' \vDash \varphi_1$ holds. By Definition 3.2.6, $\Sigma \vDash \Box\varphi_1$ holds.

This concludes the proof of the lemma. □

Lemma 3.2.18 (Completeness of canonical models)**.** *Let $\mathscr{C} = (N, \mathrm{R}, D)$ be the canonical model for a normal modal logic L. If $\mathscr{C} \vDash \varphi$ holds, then $\vdash_{\mathsf{L}} \varphi$ holds.*

Proof. We use an indirect approach to prove this lemma and show that $\nvdash_{\mathsf{L}} \varphi$ implies $\mathscr{C} \nvDash \varphi$.

Suppose $\nvdash_{\mathsf{L}} \varphi$ holds, then the set $\{\neg\varphi\}$ is L-consistent, as any proof of its inconsistency, that is $\vdash_{\mathsf{L}} \neg\neg\varphi$, can be transformed to a proof $\vdash_{\mathsf{L}} \varphi$ via the axiom DN. Such a proof must not exist by the assumption that $\nvdash_{\mathsf{L}} \varphi$ holds.

In the canonical model $\mathscr{C}$, there is a maximal L-consistent set Σ by Lemma 3.2.10, which is an extension of $\{\neg\varphi\}$, i.e., $\{\neg\varphi\} \subseteq \Sigma$. As $\neg\varphi \in \Sigma$, by the properties of maximal L-consistent sets—as established in item 1 of Lemma 3.2.14—we have $\varphi \notin \Sigma$.

Applying Lemma 3.2.17, we obtain that, for all Σ, $\Sigma \nvDash \varphi$ holds, if $\varphi \notin \Sigma$ holds. Thus, since $\Sigma \in N$, the canonical model $\mathscr{C}$ for the logic L does not force φ. □

Theorem 3.2.19 (Completeness of K)**.** K *is sound and complete with respect to $\mathscr{K}$, the class of all Kripke frames. In symbols:*

$$\mathscr{K} \vDash \varphi \Leftrightarrow \vdash_{\mathsf{K}} \varphi$$

Proof. In Lemma 3.2.7, we have shown that $\mathscr{K} \vDash \varphi$ holds, if there is a proof $\vdash_{\mathsf{K}} \varphi$. To show that there is a proof $\vdash_{\mathsf{K}} \varphi$, if $\mathscr{K} \vDash \varphi$ holds, we assume $\nvdash_{\mathsf{K}} \varphi$ holds. Since $\nvdash_{\mathsf{K}} \varphi$ holds, the canonical model $\mathscr{C}$ for K does not force φ. Clearly, the class $\mathscr{K}$ (cf. Definition 3.2.4) contains the Kripke model $\mathscr{C}$. So $\mathscr{K} \nvDash \varphi$ holds. $\square$

3.2.1 Other Modal Logics

We have introduced the weakest normal modal logic K, but there are many other normal modal logics, which have some extra axioms and impose constraints on the modal accessibility relation R. Consequently, their models are in a smaller class of Kripke frames, as opposed to the class $\mathscr{K}$ of all Kripke frames.
These logics are stronger than K in the sense that the set of valid formulae is a super-set of the valid formulae of K.

Definition 3.2.20 (Axioms for stronger normal modal logics)**.** *The following axioms can be added to C1-C10, DN and K (i.e., the axioms of the calculus* K*) to obtain modal logics stronger than* K*:*

T $\Box\varphi \to \varphi$

D $\Box\varphi \to \Diamond\varphi$

4 $\Box\varphi \to \Box\Box\varphi$

B $\varphi \to \Box\Diamond\varphi$

Definition 3.2.21 (Stronger normal modal logics)**.**
The system T *has the rules and axioms of* K *and the axiom T.*
The system D *has the rules and axioms of* K *and the axiom D.*
The system $\mathsf{S4}$ *has the rules and axioms of* K *and the axioms T and 4.*
The system $\mathsf{S5}$ *has the rules and axioms of* K *and the axioms T, 4 and B.*
For proofs in $\mathsf{K}, \mathsf{T}, \mathsf{D}, \mathsf{S4}$ *or* $\mathsf{S5}$*, we use the respective proof relations* $\vdash_{\mathsf{K}}, \vdash_{\mathsf{T}}, \vdash_{\mathsf{D}}, \vdash_{\mathsf{S4}}$ *or* $\vdash_{\mathsf{S5}}$*.*

The proof relation $\vdash_{\mathsf{L}}$ for $\mathsf{L} \in \{\mathsf{T}, \mathsf{D}, \mathsf{S4}, \mathsf{S5}\}$ is an extension of the relation $\vdash_{\mathsf{K}}$ as defined in Definition 3.2.2.

As we have already mentioned, the stronger logics are larger in the sense that more formulae are derivable, but, on the semantic side, the additional axioms impose constraints on the accessibility relation between the worlds of the Kripke models, thus allowing only models of a certain class of Kripke frames.

Lemma 3.2.22 (Soundness for stronger normal modal logics)**.**

1. *If* T *is one of the axioms of a normal modal logic* $\mathbf{L}$ *and* $\vdash_{\mathbf{L}} \varphi$ *holds, then* $\mathscr{M} \vDash \varphi$ *holds for all models* $\mathscr{M}$*, where the accessibility relation* R *is reflexive (i.e., for every world* x*,* $x \,\mathrm{R}\, x$ *holds).*

2. *If* D *is one of the axioms of a normal modal logic* $\mathbf{L}$ *and* $\vdash_{\mathbf{L}} \varphi$ *holds, then* $\mathscr{M} \vDash \varphi$ *holds for all models* $\mathscr{M}$*, where the accessibility relation* R *is serial (i.e., for every world* x*, there is a world* y*, such that* $x \,\mathrm{R}\, y$ *holds).*

3. *If* $\mathsf{4}$ *is one of the axioms of a normal modal logic* $\mathbf{L}$ *and* $\vdash_{\mathbf{L}} \varphi$ *holds, then* $\mathscr{M} \vDash \varphi$ *holds for all models* $\mathscr{M}$*, where the accessibility relation* R *is transitive (i.e., for three worlds* x, y, z*, if* $x \,\mathrm{R}\, y$ *and* $y \,\mathrm{R}\, z$ *hold, then* $x \,\mathrm{R}\, z$ *also holds).*

4. *If* B *is one of the axioms of a normal modal logic* $\mathbf{L}$ *and* $\vdash_{\mathbf{L}} \varphi$ *holds, then* $\mathscr{M} \vDash \varphi$ *holds for all models* $\mathscr{M}$*, where the accessibility relation* R *is symmetric (i.e., for two worlds* x, y*, if* $x \,\mathrm{R}\, y$ *holds, then* $y \,\mathrm{R}\, x$ *also holds).*

Proof. This lemma is an extension of Lemma 3.2.7, which states that, if $\vdash_{\mathbf{K}} \varphi$ holds, then $\mathscr{K} \vDash \varphi$. In Lemma 3.2.7, we have shown that the existence of a proof $\vdash_{\mathbf{K}} \varphi$ implies that all models force φ.

When we take into account that the lemma is no longer applicable to all Kripke model, but only to those, whose accessibility relation satisfy some restrictions, it suffices to extend the induction base and the induction step with the cases that, either α_1 or—for the induction step—α_{n+1}, is one of the four axioms treated in this lemma.

In the induction base, we treated the case that α_1 is an axiom of $\mathbf{C}$ (Case 1) and the case that α_1 is the axiom K. We will extend these cases by a new case for each $\alpha_1 \in \{\mathsf{T}, \mathsf{D}, \mathsf{4}, \mathsf{B}\}$.

In those four new induction base cases, we have to make use of the respective restrictions to the accessibility relation R.

Case 3: α_1 is an instance of T. Suppose there is a model $\mathscr{M}$ with a reflexive accessibility relation R. Let us consider an arbitrary world a with $a \vDash \Box\varphi$. Then, for all worlds b with $a \,\mathrm{R}\, b$, we have $b \vDash \varphi$. As R is reflexive, we get $a \vDash \varphi$. So, for any world a, $a \vDash \Box\varphi \rightarrow \varphi$ holds and thus all substitution instances of T are forced in models with a reflexive relation R.

Case 4: α_1 is an instance of D. Consider a model $\mathscr{M}$, where the accessibility relation R is serial. When $a \vDash \Box\varphi$ holds, then $b \vDash \varphi$ holds for all b with $a\,\mathrm{R}\,b$. By the seriality of R, there is a successor b. So, by Definition 3.2.6, a forces $\Diamond\varphi$, whenever $a \vDash \Box\varphi$ holds or—in symbols—$a \vDash \Box\varphi \to \Diamond\varphi$ for any world a, whenever R is serial.

Case 5: α_1 is an instance of $\mathsf{4}$. Consider an arbitrary model $\mathscr{M}$, where the accessibility relation R is transitive. Suppose there are worlds a, b with $a\,\mathrm{R}\,b$ and $a \vDash \Box\varphi$. So b forces φ. All successors of b also force φ, as they are also successors of a, because of the transitivity of R. As for all worlds c with $a\,\mathrm{R}\,b$ and $b\,\mathrm{R}\,c$, where the relation $c \vDash \varphi$ holds, $a \vDash \Box\Box\varphi$ also holds and so, by Definition 3.2.6, we get $a \vDash \Box\varphi \to \Box\Box\varphi$.

Case 6: α_1 is an instance of B. Suppose there is a model $\mathscr{M}$, where R is symmetric. If an arbitrary world a forces φ, all successors b of a force $\Diamond\varphi$, since $b\,\mathrm{R}\,a$ and $a \vDash \varphi$ hold. As $b \vDash \Diamond\varphi$ holds for all successors b of a, a forces $\Box\Diamond\varphi$. Therefore, for all a, $a \vDash \varphi \to \Box\Diamond\varphi$ holds.

Once again we will omit the induction step for the respective axioms, as they are analogous to the induction base.

As all the other cases of the induction base and the induction step can be directly taken from Lemma 3.2.7, this concludes the proof of the extension of Lemma 3.2.7. □

Definition 3.2.23 (Classes of Kripke frames). *We denote by $\mathscr{T}$ the class of all Kripke frames, which have a reflexive accessibility relation.*
We denote by $\mathscr{D}$ the class of all Kripke frames, which have a serial accessibility relation.
We denote by $\mathscr{S}4$ the class of all Kripke frames, which have a reflexive and transitive accessibility relation.
We denote by $\mathscr{S}5$ the class of all Kripke frames, which have a reflexive, symmetric and transitive accessibility relation, i.e., the class of Kripke frames whose accessibility relation is an equivalence relation.

We write $\mathscr{C} \vDash \varphi$, if φ is forced in all models of a class $\mathscr{C}$ of Kripke frames.

In the proofs of the following lemmata, Σ, Σ' and Σ'' will denote maximal $\mathbf{L}$-consistent sets.

Lemma 3.2.24. *If T is in the set of axioms of a normal modal logic $\mathbf{L}$, then, in the canonical model $\mathscr{C}$ for $\mathbf{L}$, the relation* R *is reflexive.*

Proof. Recall that, in a canonical model, $\Sigma \, \mathrm{R} \, \Sigma'$ is defined to hold, if and only if $\Box^-(\Sigma) \subseteq \Sigma'$. We have to show that $\Box^-(\Sigma) \subseteq \Sigma$ holds for any maximal $\mathbf{L}$-consistent set Σ. We use an indirect approach. Suppose there is a world Σ in the canonical model $\mathscr{C}$ and a formula φ with $\Box\varphi \in \Sigma$ and $\varphi \notin \Sigma$, i.e. $\Box^-(\Sigma) \nsubseteq \Sigma$. Then, by the maximality of Σ, $\neg\varphi \in \Sigma$. But $\Box\varphi \to \varphi$ is in the set of axioms, we have $\vdash_{\mathbf{L}} \Box\varphi \to \varphi$ and by maximality of Σ also $\Box\varphi \to \varphi \in \Sigma$. By item 4 of Lemma 3.2.14, $\varphi \in \Sigma$. Therefore, $\Box^-(\Sigma) \subseteq \Sigma$ holds. $\Box$

Lemma 3.2.25. *If* D *is in the set of axioms of a normal modal logic* $\mathbf{L}$*, then, in the canonical model* $\mathscr{C}$ *for* $\mathbf{L}$*, the relation* R *is serial.*

Proof. If D is in the set of axioms of some calculus $\mathbf{L}$, then $\vdash_{\mathbf{L}} \Box\neg\bot \to \neg\Box\neg\neg\bot$—or shorter $\vdash_{\mathbf{L}} \Box\top \to \Diamond\top$—holds. Since $\vdash_{\mathbf{L}} \top$ can easily be proven in intuitionistic (as well as classical, or modal) logic (cf. Example 2.2.4), by one application of the rule (N), we can prove $\vdash_{\mathbf{L}} \Box\top$. Now, for all maximal $\mathbf{L}$-consistent sets Σ, we have $\Box\top \in \Sigma$ and $\Box\top \to \Diamond\top \in \Sigma$, thus also $\Diamond\top \in \Sigma$. By Lemma 3.2.13, $\Box^-(\Sigma) \cup \{\neg\neg\top\}$ is $\mathbf{L}$-consistent. Now, since $\Box^-(\Sigma)$ is $\mathbf{L}$-consistent (as a subset of a $\mathbf{L}$-consistent set), there is a maximal $\mathbf{L}$-consistent extension $\Sigma' \supseteq \Box^-(\Sigma)$. So there is a Σ' in the canonical model $\mathscr{C}$ with $\Sigma \, \mathrm{R} \, \Sigma'$, if D is in the set of axioms of the calculus $\mathbf{L}$. $\Box$

Lemma 3.2.26. *If* $\mathsf{4}$ *is in the set of axioms of a normal modal logic* $\mathbf{L}$*, then, in the canonical model* $\mathscr{C}$ *for* $\mathbf{L}$*, the relation* R *is transitive.*

Proof. We have to show that, whenever we have $\Sigma \, \mathrm{R} \, \Sigma'$ and $\Sigma' \, \mathrm{R} \, \Sigma''$ in a canonical model, constructed as in Definition 3.2.15 for some logic $\mathbf{L}$ which includes axiom $\mathsf{4}$, $\Sigma \, \mathrm{R} \, \Sigma''$ holds.
By Definition 3.2.15, we have $\Box^-(\Sigma) \subseteq \Sigma'$ and thus $\Box^-(\Box^-(\Sigma)) \subseteq \Box^-(\Sigma') \subseteq \Sigma''$. Now suppose that $\Sigma \, \mathrm{R} \, \Sigma''$ does not hold. Then there is a φ, such that $\varphi \in \Box^-(\Sigma)$ and $\varphi \notin \Sigma''$. But since $\Box\varphi \in \Sigma$, then, by using the axiom $\Box\varphi \to \Box\Box\varphi$, we can show that also $\Box\Box\varphi \in \Sigma$. So it must also hold that $\varphi \in \Box^-(\Box^-(\Sigma)) \subseteq \Sigma''$, and thus R is transitive. $\Box$

Lemma 3.2.27. *If* B *is in the set of axioms of a normal modal logic* $\mathbf{L}$*, then, in the canonical model* $\mathscr{C}$ *for* $\mathbf{L}$*, the relation* R *is symmetric.*

Proof. We have to show that, if the relation $\Box^-(\Sigma) \subseteq \Sigma'$ holds, then $\Box^-(\Sigma') \subseteq \Sigma$ also holds. Now suppose there is a formula $\varphi \in \Box^-(\Sigma')$, such that $\varphi \notin \Sigma$. By the maximality of Σ, we have that $\neg\varphi \in \Sigma$. With the instantiation $\Sigma \vdash_{\mathbf{L}} \neg\varphi \to \Box\neg\Box\neg\neg\varphi$ of the axiom B, we obtain $\neg\Box\neg\neg\varphi \in \Box^-(\Sigma) \subseteq \Sigma'$. Now we have $\neg\Box\neg\neg\varphi \in \Sigma'$ and $\Box\neg\varphi \in \Sigma'$. By the (even intuitionistically) valid formula $\varphi \to \neg\neg\varphi$, the rule (N) and the axiom K, we get $\vdash_{\mathbf{L}} \Box\varphi \to \Box\neg\neg\varphi$ (The proof is shown in in Figure 3.4).

So $\neg\Box\neg\neg\varphi \in \Sigma'$ and $\Box\neg\neg\varphi \in \Sigma'$. This is a contradiction to the consistency of Σ'. Therefore, R is symmetric. $\Box$

Theorem 3.2.28 (Soundness and completeness of stronger normal modal logics)**.**

$$\begin{aligned} \mathscr{T} \models \varphi &\quad\Leftrightarrow\quad \vdash_{\mathbf{T}} \varphi \\ \mathscr{D} \models \varphi &\quad\Leftrightarrow\quad \vdash_{\mathbf{D}} \varphi \\ \mathscr{S}4 \models \varphi &\quad\Leftrightarrow\quad \vdash_{\mathbf{S4}} \varphi \\ \mathscr{S}5 \models \varphi &\quad\Leftrightarrow\quad \vdash_{\mathbf{S5}} \varphi \end{aligned}$$

Proof. The soundness part, i.e., everything that can be proven holds in the corresponding models, in symbols

$$\vdash_{\mathbf{L}} \varphi \Rightarrow \mathscr{L} \models \varphi,$$

where $\mathscr{L}$ is the class of Kripke frames corresponding to $\mathbf{L}$, was proven in Lemma 3.2.22.

For the other direction, we use an indirect approach and show that $\mathscr{L} \not\models \varphi$ holds, when there is no proof $\vdash_{\mathbf{L}} \varphi$. In particular the canonical model $\mathscr{C}$ for $\mathbf{L}$ does not force φ. In Lemmata 3.2.24–3.2.27 we have shown that the canonical model of a logic $\mathbf{L}$ is an element of the corresponding class $\mathscr{L}$ of Kripke frames. By the completeness of the canonical model (cf. Lemma 3.2.18), $\mathscr{C} \models \varphi$ holds, if and only if a proof $\vdash_{\mathbf{L}} \varphi$ exists. Since there is no such proof, the canonical model does not force φ and so the respective class of Kripke frames does not force φ either. So when $\mathscr{L} \models \varphi$ holds, there has to be a proof $\vdash_{\mathbf{L}} \varphi$. This concludes the completeness proof for stronger normal modal logics.

Therefore, the normal modal logics $\mathbf{T}$, $\mathbf{D}$, $\mathbf{S4}$ and $\mathbf{S5}$ are sound and complete with respect to the corresponding class of Kripke frames. $\Box$

$$
\dfrac{\mathsf{K}\!:\ \Box(\varphi \to \neg\neg\varphi) \to (\Box\varphi \to \Box\neg\neg\varphi) \qquad \dfrac{\dfrac{\overset{\text{by Example 2.2.4}}{\overset{\bigtriangledown}{\neg\varphi \to \neg\varphi}} \qquad \mathsf{C9}\!:\ (\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\neg\varphi)}{\varphi \to \neg\neg\varphi}\ (\mathsf{MP})}{\Box(\varphi \to \neg\neg\varphi)}\ (\mathsf{N})}{\Box\varphi \to \Box\neg\neg\varphi}\ (\mathsf{MP})
$$

Figure 3.4: The proof of the formula $\Box\varphi \to \Box\neg\neg\varphi$

Chapter 4

Intuitionistic Modal Logic

Intuitionistic modal logic usually is a system with an intuitionistic base logic and one or two modal operators. R. A. Bull and G. Fischer Servi were among the first to come up with an intuitionistic modal logic in 1966 and 1977 respectively (see [Bul66] and [FS77] for details).

Since there are Kripke models for intuitionistic logic as well as for modal logic, it is reasonable to assume that we can use Kripke models for intuitionistic modal logics as well. Of course, these models must have at least two accessibility relations, one for the intuitionistic part, and one for the modal part. In contrast to modal logics with a classical base, the two modalities $\Box$ and $\Diamond$ are not inter-definable in general and there might be even more accessibility relations, one for each modality and one for the "intuitionism". The interplay of these two relations has to be carefully defined in order to create feasible models.

Although intuitionistic modal logics have been devised more than 40 years ago, there are not many applications, most notably [DP96], which use intuitionistic modal logic to model staged computation in the λ-calculus. Another example is lax logic, which is an intuitionistic logic with one modality. Lax logic is used in hardware verification (see [Men93] for details). It has been shown in [Egl02] that lax logic can be faithfully embedded into intuitionistic logic. Hence, from a provability viewpoint, lax logic is "intuitionistic logic with added syntactic sugar".

In the following, we will introduce another intuitionistic modal logic with only one modality.

4.1 Modal Logic with an Intuitionistic Base Defined with the Modality $\Box$

In the following section, we will introduce a modal logic with intuitionistic base defined with the single modal operator $\Box$. This particular logic was first defined in [BD84] and we mostly follow this paper.

The logic $\mathbf{HK}\Box$ is a natural extension of $\mathbf{H}$: natural in the sense that removing all occurrences of $\Box$ from a valid formula derived by $\mathbf{HK}\Box$ gives us a valid formula, which can be proven in the calculus $\mathbf{H}$.
Furthermore $\mathbf{HK}\Box$ is a sub-logic of $\mathbf{K}$, because its modal rule can be viewed as a derived rule in $\mathbf{K}$ and its modal axioms can be proven in $\mathbf{K}$.

4.1.1 Calculi for Intuitionistic Modal Logics Defined with $\Box$

Definition 4.1.1 ($\mathbf{HK}\Box$)**.** *When we add the following axioms and the rule (R$\Box$) to Heyting's propositional calculus $\mathbf{H}$ (from Definition 2.2.1), we obtain the system $\mathbf{HK}\Box$.*

$\Box$*1* $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$

$\Box$*2* $\Box(\varphi \to \varphi)$

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}\ (R\Box)$$

We call (R$\Box$) the rule of regularity*.*

Definition 4.1.2 (Proofs in $\mathbf{HK}\Box$)**.** *A proof* $\Sigma \vdash_{\mathbf{HK}\Box} \varphi$ *is a sequence* $(\psi_1, \psi_2, \dots, \psi_n)$ *of formulae, where* $\psi_n = \varphi$ *and, for all* ψ_i*, either:*

- ψ_i *is a substitution instance of one of the axioms of* $\mathbf{HK}\Box$*,*
- $\psi_i \in \Sigma$*,*
- ψ_i *is the conclusion of (MP), whose premises are* ψ_j *and* ψ_k *with* $j, k < i$*, or*
- ψ_i *is the conclusion of (R$\Box$) whose premise is* ψ_j *with* $j < i$*.*

When Σ *is the empty set, we sometimes write* $\vdash_{\mathbf{HK}\Box} \varphi$ *or* $\mathbf{HK}\Box\vdash \varphi$ *instead of* $\emptyset \vdash_{\mathbf{HK}\Box} \varphi$ *As usual, the length of a* $\mathbf{HK}\Box$*-proof is defined as the number of formulae occurring in that sequence.*

Remark 4.1.3 ($\mathsf{HK}\Box$-proofs in [BD84]). *In [BD84], $\mathsf{HK}\Box$-proofs are defined such that only (MP) is a valid inference rule and (R$\Box$) is not allowed to be used in a proof. The motivation for this is the fact that the Deduction Theorem (Theorem 2.2.6) is of the same form and has the same proof as it has for intuitionistic logic. But this decreases the expressiveness of $\mathsf{HK}\Box$, and $\mathsf{HK}\Box$ is no longer an intuitionistic modal logic in the sense of [WZ99][1].*

In their completeness proof for $\mathsf{HK}\Box$ with respect to H-models (cf. Definition 4.1.5), the authors of [BD84] use the Deduction Theorem and also the fact that $\mathsf{HK}\Box$ produces the same set of formulae as the calculus H (Definition 2.2.1) extended by the axiom K and the rule (N) (as in Definition 3.2.1). Unfortunately, we have not been able to prove the pairwise simulation of these two modal intuitionistic calculi—especially when we do not permit (R$\Box$) in $\mathsf{HK}\Box$-proofs.

Relation to H and K We expect from an intuitionistic analogue of the the weakest normal modal logic K that, when we remove all occurrences of $\Box$, we just get intuitionistic propositional logic. By removing all $\Box$, the modal axioms of $\mathsf{HK}\Box$ are just instances of the intuitionistically provable formula $\varphi \to \varphi$ (cf. Example 2.2.4). So all modal axioms become non-modal formulae which are provable in H and the rule (R$\Box$) disappears.

Another expectation we have for the logic $\mathsf{HK}\Box$ is that once we add the axiom of double negation[2], we get the system K. Since there is no proof $\vdash_{\mathsf{HK}\Box} \mathsf{K}$ in [BD84] this remains an open question. We believe that

$$\vdash_{\mathsf{HK}\Box} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

is not provable. But $\mathsf{HK}\Box$ extended by the axiom DN is a sub-system of K.

4.1.2 Semantics for $\mathsf{HK}\Box$

Definition 4.1.4 (Composite relation). *Let R_1 and R_2 be arbitrary binary relations. We define the composite relation $\mathrm{R}_1 \circ \mathrm{R}_2$ as follows:*

$$\mathrm{R}_1 \circ \mathrm{R}_2 = \{(x, z) \mid \exists y : (x, y) \in \mathrm{R}_1 \ \ \textit{and} \ \ (y, z) \in \mathrm{R}_2\}$$

Definition 4.1.5 ($\mathsf{H}\Box$ frames and models). *A triple $\mathscr{F} = (N, \leq, \mathrm{R})$ is a* $\mathsf{H}\Box$ frame, *if and only if*

1. *N is a non-empty set of nodes or worlds,*

[1]By [WZ99] an intuitionistic modal logic must contain intuitionistic logic, has to be closed under modus ponens, substitution and the rule of regularity for all modalities.

[2]DN as in the system C of classical logic, see Section 3.2

2. $\leq$ *is a partial ordering over* N*, i.e., a reflexive and transitive relation,*

3. R *is an arbitrary relation over* N *and*

4. *the following restriction holds:* $(\leq \circ \mathrm{R}) \subseteq (\mathrm{R} \circ \leq)$.

Furthermore $\mathscr{M} = (N, \leq, \mathrm{R}, D)$ *is a* $\mathbf{H}\Box$ model *if and only if* $(N, \leq, \mathrm{R})$ *is a* $\mathbf{H}\Box$ *frame and* D *is the domain function* $D : N \to \mathscr{P}(\mathbb{VAR})$*, which satisfies*

$$\forall a, b \in N : \; a \leq b \Rightarrow D(a) \subseteq D(b).$$

Definition 4.1.6 (Forcing relation for $\mathbf{H}\Box$ models)**.** *The forcing relation* $\vDash$ *establishes truth in a model* $\mathscr{M} = (N, \leq, \mathrm{R}, D)$ *and is defined inductively as follows:*

1. $a \vDash p \Leftrightarrow: p \in D(a)$*;*

2. $a \vDash \psi \wedge \varphi \Leftrightarrow: a \vDash \psi$ *and* $a \vDash \varphi$*;*

3. $a \vDash \psi \vee \varphi \Leftrightarrow: a \vDash \psi$ *or* $a \vDash \varphi$*;*

4. $a \vDash \psi \to \varphi \Leftrightarrow:$ *for all* b *with* $a \leq b : b \vDash \psi \Rightarrow b \vDash \varphi$*;*

5. $a \vDash \neg\varphi \Leftrightarrow:$ *for all* b *with* $a \leq b : a \nvDash \varphi$*;*

6. $a \vDash \Box\varphi \Leftrightarrow:$ *for all* b *with* $a \, \mathrm{R} \, b \; : \; b \vDash \varphi$.

We say φ *holds in a model* $\mathscr{M}$, $\mathscr{M}$ *forces* φ *or* $\mathscr{M} \vDash \varphi$*, if, for all worlds* a *of* $\mathscr{M}$*, the relation* $a \vDash \varphi$ *holds. We say* $\mathscr{M} \vDash \Sigma$ *holds, if* $\mathscr{M} \vDash \sigma$ *holds for all* $\sigma \in \Sigma$*. Furthermore we say* $\Sigma \vDash \varphi$*, if for all models with* $\mathscr{M} \vDash \Sigma$*, also* $\mathscr{M} \vDash \varphi$ *holds.*

As mentioned in Remark 2.2.10 for pure intuitionistic logic, $\bot$ *is never forced in any world, so* $\neg\varphi$ *can be conveniently viewed as* $\varphi \to \bot$*.*

Remark 4.1.7. *We notice that items 1 to 5 of Definition 4.1.6 coincide exactly with items 1 to 5 of Definition 2.2.9, while item 6 coincides with item 6 of Definition 3.2.6.*

The following lemma points out another analogy to intuitionistic logic (cf. Lemma 2.2.11).

Lemma 4.1.8 (Intuitionistic Heredity)**.**

$$a \leq b \Rightarrow (a \vDash \varphi \Rightarrow b \vDash \varphi)$$

Proof. We will prove the lemma by induction on $\mathrm{lcomp}(\varphi)$, the logical complexity of φ. Assume that $a \leq b$ holds.

Induction Base. $\mathrm{lcomp}(\varphi) = 0$, i.e., φ is of the form p, where p is a variable. We need not consider the case that $\varphi = \bot$, because $\bot$ is not forced in any world. Now suppose $a \vDash p$ holds. By Definition 4.1.6, this is equivalent to $p \in D(a)$. By assumption, we have $a \leq b$, which implies $D(a) \subseteq D(b)$ by Definition 4.1.5. Thus, we know that $p \in D(b)$ and obtain $b \vDash \varphi$.

Induction Hypothesis. Suppose $n \geq 0$ and assume that, for all formulas φ with $\mathrm{lcomp}(\varphi) \leq n$ and all worlds a, b with $a \leq b$, the relation $b \vDash \varphi$ holds, if $a \vDash \varphi$ holds.

Induction Step. $\mathrm{lcomp}(\varphi) = n + 1$. Since $n \geq 0$, the formula φ contains at least one connective. We perform a case distinction according to the outermost logical connective, since, for the immediate sub-formulae of the formula under consideration, the desired property holds by the induction hypothesis.

φ is of the form $\Box\psi$. In this case, we suppose a forces $\Box\psi$ and $a \leq b$ holds, so, for all worlds c, the implication $a \mathrm{R} c \Rightarrow c \vDash \psi$ holds by Definition 4.1.6, item 6. By the induction hypothesis, we have $d \vDash \psi$ for all d with $c \mathrm{R} d$. Now let's consider the e with $b \mathrm{R} e$:

$$\{e | b \mathrm{R} e\} = \{e | a \leq b \mathrm{R} e\} \subseteq \{e | a \ (\leq \circ \mathrm{R})\ e\} \overset{\text{by Definition 4.1.5 item 4}}{\subseteq} \{d | a \ (\mathrm{R} \circ \leq)\ d\} = \{d | \exists c : \ a \mathrm{R} c \leq d\}.$$

So when d forces ψ, it also holds that $e \vDash \psi$ for all e with $b \mathrm{R} e$, as the set of those e is a subset of all d with $\{d | a \mathrm{R} \circ \leq d\}$. Thus $b \vDash \Box\psi$ holds for all b with $a \leq b$.

φ is of the form $\neg\psi$ or φ is of the form $\psi \circ \sigma$ with $\circ \in \{\wedge, \vee, \rightarrow\}$. Since the relevant items of Definition 4.1.6 correspond directly to Definition 2.2.9, and since $\leq$ behaves the same way in intuitionistic Kripke models as in **H**$\Box$ models, we can directly use the arguments from Lemma 2.2.11 for the remaining cases of the induction step.

This concludes the induction proof of the lemma. $\Box$

In the next lemma, we show that the condition $(\leq \circ \mathrm{R}) \subseteq (\mathrm{R} \circ \leq)$ is necessary for the usefulness of our models.

Lemma 4.1.9. *Let* $\bar{\mathscr{M}} =< N, \leq, \mathrm{R} >$ *be a "quasi"* **H**$\Box$ *frame, which does not satisfy condition 4. of Definition 4.1.5. Then there is a formula* φ *and* *two worlds $a, b \in N$ such that $a \leq b$, $a \vDash \varphi$ and $b \nvDash \varphi$ for some domain-function D.*

Proof. When $(\leq \circ \mathrm{R}) \nsubseteq (\mathrm{R} \circ \leq)$ holds, there have to be two worlds a, c, such that $a \leq \circ \mathrm{R}\, c$ holds, but $a \,\mathrm{R} \circ \leq c$ does not hold. So there is a world $b \in N$ with $a \leq b \,\mathrm{R}\, c$. Now let us create a domain function as follows:

$$D(e) = \begin{cases} \mathbb{VAR} & \forall e : \, a \;\mathrm{R} \circ \leq e, \\ \mathbb{VAR} \setminus \{p\} & \text{otherwise}, \end{cases}$$

where p is an arbitrary propositional variable.

This is a valid domain function, as it obviously satisfies the condition $a' \leq b' \Rightarrow D(a') \subseteq D(b')$ for all $a' \notin \{e | a \;\mathrm{R} \circ \leq e\}$. For all $a' \in \{e | a \;\mathrm{R} \circ \leq e\}$, we have $b' \in \{e | a \;\mathrm{R} \circ \leq e\}$ by the transitivity of $\leq$.

For all d with $a \,\mathrm{R}\, d$, we have—by reflexivity of $\leq$—$a \,\mathrm{R}\, d \leq d$ and thus $d \vDash p$ and $a \vDash \Box p$ hold. Since $a \,\mathrm{R} \circ \leq c$ does not hold because of the definition of the domain function D, c does not force p. We have $b \,\mathrm{R}\, c$ and so $b \vDash \Box p$ cannot hold.

Since $a \vDash \Box p$, $a \leq b$ and $b \nvDash \Box p$ holds, there is no intuitionistic heredity in this particular model. So item 4 of Definition 4.1.5 is a necessary condition for our $\mathbf{H}\Box$ models to satisfy intuitionistic heredity. $\Box$

Lemma 4.1.10 (Soundness of $\mathbf{HK}\Box$ with respect to $\mathbf{H}\Box$ models))**.**

$$\Sigma \vdash_{\mathbf{HK}\Box} \varphi \Rightarrow \Sigma \vDash \varphi$$

Proof. As a preparatory step, we show that the axioms $\Box 1$ and $\Box 2$ hold in any world a of any Kripke model.

$\Box 1$: $a \vDash \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$
If $a \vDash \Box\varphi \wedge \Box\psi$ holds, then, for all worlds b with $a \,\mathrm{R}\, b$, the relations $b \vDash \varphi$ and $b \vDash \psi$ hold by Definition 4.1.6 (6). Thus, all b force $\varphi \wedge \psi$ and so $a \vDash \Box(\varphi \wedge \psi)$ holds. By Definition 4.1.6, $a \vDash \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$ holds.

$\Box 2$: $a \vDash \Box(\varphi \to \varphi)$
Trivially, in any world a, $\varphi \to \varphi$ is forced. So it is also forced in all R-successors of a. Since we have $b \vDash \varphi \to \varphi$ for all worlds b with $a \,\mathrm{R}\, b$ by Definition 4.1.6 (6), the relation $a \vDash \Box(\varphi \to \varphi)$ holds.

Now we proceed by induction on n, the length of the proof $\Sigma \vdash_{\mathbf{HK}\Box}$, to prove the above implication.

Induction Base. $n = 1$. $\Sigma \vdash_{\mathbf{HK}\Box}$ is of the form (φ). If φ is in Σ, $\Sigma \vDash \varphi$ holds trivially. If φ is one of the non-modal axioms (i.e., H1–H10), φ is forced in all **HK**□-models by the the same arguments, we already used in Lemma 2.2.12. The only difference is that we have to refer to Definition 4.1.6 instead of Definition 2.2.9.
If φ is one of the modal axioms (i.e., □1 or □2), any world a forces φ, as we have shown in the preparatory step.

Induction Hypothesis. Suppose $n > 0$. If there is a proof $\Sigma \vdash_{\mathbf{HK}\Box} \varphi$ of length $\leq n$, $\Sigma \vDash \varphi$ holds.

Induction Step. Suppose there is a proof $\Sigma \vdash_{\mathbf{HK}\Box} \varphi$ of length $n + 1$. The proof has the form $(\alpha_1, \ldots, \alpha_{n+1})$ with $\alpha_{n+1} = \varphi$. The formula α_{n+1} can either be an axiom, an element of Σ or the result of an application of the rules (MP) or (R□). Unless α_{n+1} is the result of a rule application, this essentially corresponds to the base case. We perform a case distinction according to the rule which has α_{n+1} as its conclusion.

Case 1: Suppose α_{n+1} is the result of an application of (MP):

$$\frac{\alpha_i \quad \alpha_i \to \alpha_{n+1}}{\alpha_{n+1}} \; (\mathsf{MP})$$

with $\alpha_j = \alpha_i \to \alpha_{n+1}$ and $i, j \leq n$. By the induction hypothesis, $\Sigma \vDash \alpha_i \to \alpha_{n+1}$ holds and $\Sigma \vDash \alpha_i$ holds. But then—by Definition 4.1.6—$\Sigma \vDash \alpha_{n+1}$ holds.

Case 2: Suppose α_{n+1} is the result of an application of (R□):

$$\frac{\sigma \to \psi}{\Box\sigma \to \Box\psi} \; (\mathsf{R}\Box)$$

with some $\alpha_i = \sigma \to \psi$ and $i \leq n$. By the induction hypothesis, $a \vDash \sigma \to \psi$ holds for all $a \in \mathscr{M}$. Then, for all b with $a \leq b$, if $b \vDash \sigma$ holds, then $b \vDash \psi$ holds.
Now, for all $c \in \mathscr{M}$ with $c \,\mathrm{R}\, a$, if $c \vDash \Box\sigma$ holds, we have $a \vDash \sigma$ by Definition 4.1.6, item 6. Then, by Lemma 4.1.8, $b \vDash \sigma$ holds, and thus, as all worlds force $\sigma \to \psi$, by the induction hypothesis $b \vDash \psi$ holds.
We just have to show that for all $d \in \mathscr{M}$ with $c \leq d$: $d \vDash \Box\psi$ holds, if $c \vDash \Box\sigma$ holds. All e with $d \,\mathrm{R}\, e$ coincide with the $\leq$-successors b of the world a by the restriction imposed on the two relations $\leq$ and R by item 4 of Definition 4.1.5 (i.e., $(\leq \circ \mathrm{R}) \subseteq (\mathrm{R} \circ \leq)$). Since $b \vDash \psi$ holds, $e \vDash \psi$ also holds for all e with $d \,\mathrm{R}\, e$. Thus $d \vDash \Box\psi$ holds. We now have $d \vDash \Box\psi$, when $c \vDash \Box\sigma$. So, by item 4 of Definition 4.1.6, $c \vDash \Box\sigma \to \Box\psi$ holds for all $c \in \mathscr{M}$.

Figure 4.1: $\sigma \to \psi$ in $\mathsf{H}\Box$ models

$$\begin{tikzcd}
a \vDash \sigma \arrow[r, "\leq"] & b \vDash \psi \\
c \vDash \Box\sigma \arrow[u, "R"] \arrow[r, "\leq"] & d \vDash \Box\psi \arrow[u, dotted, "R"']
\end{tikzcd}$$

This concludes the proof of the lemma. $\Box$

Completeness

Unfortunately, we have not been able to retrace the steps taken in [BD84] to prove completeness of the calculus $\mathsf{HK}\Box$ with respect to $\mathsf{H}\Box$-models. The main problem is the fact that the authors of [BD84] switch between two notions of $\mathsf{HK}\Box$-proofs. When it suits their argumentative needs, they allow applications of the rule ($\mathsf{R}\Box$). But when they need the deduction theorem, which does not allow for other rules than (MP), they use it as well. It seems prudent to permit the applications of ($\mathsf{R}\Box$) in proofs, because otherwise $\mathsf{HK}\Box$ is probably not more expressive than intuitionistic logic.

The main problem is a preliminary lemma of the completeness proof (Lemma 8) on page 226 in [BD84], the authors have a proof of the form

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{HK}\Box} \psi,$$

from which they obtain

$$\vdash_{\mathsf{HK}\Box} \varphi_1 \wedge \ldots \wedge \varphi_n \to \psi$$

"by the Deduction Theorem and theorems of $\mathsf{HK}\Box$". But the Deduction Theorem in this form is only permissible, when we do not allow applications of ($\mathsf{R}\Box$). In the next step, they obtain

$$\vdash_{\mathsf{HK}\Box} \Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \Box\psi$$

without any further explanation (other than using the Deduction Theorem and theorems of $\mathsf{HK}\Box$). This looks like an application of ($\mathsf{R}\Box$), although it admittedly could have also been produced if—for example—a formula of the form $(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ was a theorem of $\mathsf{HK}\Box$ (in the notion of the calculus which does not allow for applications of the rule $(\mathsf{R}\Box)$). We have not been able to disprove that one can obtain a proof

$$\vdash_{\mathbf{HK}\Box} \Box(\varphi_1 \wedge \ldots \wedge \varphi_n) \to \Box\psi$$

from a proof

$$\varphi_1, \ldots, \varphi_n \vdash_{\mathbf{HK}\Box} \psi$$

using either notion of $\mathbf{HK}\Box$-proofs.

Furthermore, we have not been able to show that the calculus $\mathbf{HK}\Box$ and the calculus $\mathbf{H}$ extended by the axiom K and the rule (N) produce the same set of formulae, as we have not been able to come up with an $\mathbf{HK}\Box$-proof for the axiom K (cf. Remark 4.1.3). This might have facilitated another approach to proving completeness.

It is likely that there is a completeness proof of $\mathbf{HK}\Box$, but for now it still remains an open question to produce it.

Chapter 5

Conclusion, Outlook and Future Work

The initial intention of this thesis was to investigate modal logics with an intuitionistic base. There are already several papers on this topic, but I decided to start my investigation of this subject with the rather old paper [BD84], which, in contrast to the even earlier first publications on this topic, seemed already quite evolved and "modern".

To put my own investigation on intuitionistic modal logics on a firm base, it seemed necessary to reproduce one of the well-known completeness proofs for intuitionistic logic as well as modal logic. As the authors of [BD84] used a Hilbert style calculus, it seemed the natural choice for the other non-classical logics as well.

As—even for intuitionistic logic—proving completeness with a Hilbert style calculus was rather tedious, at least at the level of detail I envisioned for this thesis, I helped myself by introducing a calculus of natural deduction (as can be found in [vD04]). With this calculus proving completeness was less complicated and it could be used in the following chapters as well. So instead of showing soundness and completeness of the calculus **H** (cf. Definition 2.2.1), I took an unusual approach by showing soundness for **H** and completeness for **ND**. To complete the soundness and completeness proof for both calculi in a circular fashion, I then showed that **H** simulates **ND**.

In Chapter 3, when I examined modal logic, I used the same Hilbert style calculus for the classical base logic (of course extended by an axiom which made the elimination of double negation possible). But whenever the Hilbert style calculus proved to cumbersome and I only needed results which could be proven in classical logic, I conveniently used the natural deduction calculus for intuitionistic logic (again extended by an extra rule to expand the calculus' expressiveness to classical logic).

I used the technique from [HC96] to prove soundness and completeness as well for some stronger normal modal logics without much additional effort.

When I started to investigate intuitionistic modal logic in Chapter 4, the original intention was to improve the rudimentary proof sketches of the soundness and completeness of $\mathbf{HK}\Box$ in [BD84]. But there were severe problems in the argumentation of the completeness proof, which I pointed out in the section *Completeness* on page 63. So there still remain a lot of open questions for future work on $\mathbf{HK}\Box$:

- Is it possible to prove completeness for the calculus $\mathbf{HK}\Box$ with respect to $\mathbf{H}$-models?

- Is the calculus $\mathbf{HK}\Box$ without the rule $(\mathsf{R}\Box)$ essentially intuitionistic logic with some syntactic sugar?

- Do the calculi $\mathbf{HK}\Box$ and $\mathbf{H}$ extended by the axiom K and the rule (N) simulate each other?

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