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Towards Consistent Non-Commutative Gauge Theories

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Kurzfassung

Betrachtet man die perturbative Entwicklung einer Quantenfeldtheorie in nichtkommutativer Raumzeit, so ist es notwendig, deformierte "Sternprodukte" für die Feldoperatoren einzuführen. Im einfachsten Fall sind das die Groenewold-Moyal-Weyl Sternprodukte (siehe z.B. die Review-Artikel [9, 10]). Dies führt zu neuen Feynman-Regeln für die Vertizes der Wechselwirkung: Sie erhalten zusätzliche Phasen, die vom Deformationsparameter abhängen und zu "planaren" und "nichtplanaren" Feynman-Diagrammen führen. Diese modifizierten Feynman-Regeln, insbesondere die zusätzlichen Phasen in den Vertizes, regularisieren einige à priori UV-divergente (ultraviolett-divergente) Schleifendiagramme nichtkommutativer Quantenfeldtheorien, sodass diese zwar UV-endlich aber auf der anderen Seite IR-divergent (infrarot-divergent) für verschwindende externe Impulse werden. Dieses Phänomen bezeichnet man als UV/IR-Mischungsproblem, denn es stellt ein echtes Hindernis für das Renormierungsprogramm dar.

Die vorliegende Arbeit beschäftigt sich mit möglichen Lösungen dieses Problems bei nichtkommutativen Eichfeldtheorien:

A. A. Slavnov [11, 12] hat eine Erweiterung nichtkommutativer Eichfeldtheorien vorgeschlagen, um derartige IR-Divergenzen in Schleifenkorrekturen zu umgehen. Bei dieser Modifizierung handelt es sich um einen topologischen Term, der in bestimmten Eichungen neue Symmetrien zur Folge hat: In zwei Publikation [4, 6], die das Resultat einer internationalen Kooperation mit François Gieres von der Université Claude Bernard (Lyon 1), Olivier Piguet von der Universidade Federal do Espírito Santo (UFES, Vitória, Brazil), Stefan Hohenegger vom CERN (Theorie-Abteilung, Genf) und natürlich mit meinem Betreuer Manfred Schweda darstellten, haben wir die Existenz einer Vektorsupersymmetrie und einer weiteren vektoriell-bosonischen Symmetrie in einer axialen Eichfixierung diskutiert. Wie allgemein bekannt ist, haben Symmetrien, insbesondere lineare Vektorsupersymmetrien in topologischen Feldtheorien (vom Schwarz-Typ), zur Folge, dass diese am Quantenniveau UV-endlich werden. Daher wurde das Studium dieses "Slavnov-Modells" zum Schwerpunkt dieser Arbeit.

Ein weiterer Ansatz, um das UV/IR-Mischungsproblem zu lösen wurde von Grosse und Wulkenhaar [13, 14] vorgeschlagen: Durch das Hinzufügen eines harmonischen Oszillatorpotentials in der Wirkung einer nichtkommutativen skalaren ϕ^4 Theorie im euklidischen $\mathbb{R}^4$ konnte nicht nur die IR-Endlichkeit des Modells sondern sogar dessen Renormierbarkeit gezeigt werden. Dieser Erfolg war Motivation genug, um eine ähnliche Erweiterung bei

nichtkommutativen $U(1)$ Eichfeldtheorien zu studieren. In einer neuen Kooperation mit Harald Grosse von der Universität Wien und Manfred Schweda entstand die Publikation [7].

Als Abschluss dieser Zusammenfassung sei noch die folgende Liste der im Rahmen dieser Dissertation entstandenen Publikationen, Preprints und Konferenz-Proceedings gegeben: Eine ausführliche Diskussion der in nichtkommutativen Eichfeldtheorien (mit und ohne Slavnov-Term) auftretenden IR-Divergenzen und deren Eichunabhängigkeit findet sich in den Referenzen [1, 2, 3]. Symmetrien und topologische Aspekte wurden in den Publikationen [4, 6] und den Proceedings [5, 8] behandelt. Schließlich entstand aus der Diskussion eines nichtkommutativen Eichfeldmodells mit harmonischem Oszillatorpotential die Publikation [7].

Abstract

The perturbative realization of any quantum field theory on non-commutative space-time is based on the fact that one has to use a deformed "star" product for the field operators, which in the simplest case is the so-called Groenewold-Moyal-Weyl star product (see [9, 10] for a review). This modification leads to new Feynman rules for the interaction vertices, namely additional phases depending on the deformation parameter which lead to "planar" and "non-planar" Feynman diagrams. In using these modified Feynman rules for non-commutative quantum field theories (NCQFTs), some à priori ultraviolet (UV) divergent loop diagrams become UV finite due to the regulating effect of the additional phases in the interaction vertices. But on the other hand, new infrared (IR) divergences appear in these graphs for vanishing external momenta. This is the so-called UV/IR mixing problem, which presents a real obstacle when it comes to renormalization of NCQFTs.

This dissertation is devoted to studying two of the most promising ideas to cure the UV/IR mixing problems in non-commutative gauge field theories:

A. A. Slavnov [11, 12] proposed an extension to non-commutative gauge theories which could render them IR finite. In fact, this extension represents a topological term which, in certain gauges, introduces new symmetries to the model: In two peer-reviewed publications [4, 6], which were the result of an international cooperation with François Gieres of Université Claude Bernard (Lyon 1), Olivier Piguet of Universidade Federal do Espírito Santo (UFES, Vitória, Brazil), Stefan Hohenegger of CERN (Theory Department, Geneva) and, of course, my supervisor Manfred Schweda, we showed the appearance of a vector supersymmetry and an additional bosonic vectorial symmetry when using an axial gauge fixing. It is well known that symmetries, especially linear vector supersymmetries in topological field theories (of the Schwarz type), lead to remarkable ultraviolet finiteness properties at the quantum level. Therefore, the main focus of this doctoral thesis is on a variety of aspects of this "Slavnov model".

A further ansatz to eliminate the UV/IR mixing problem was proposed by Grosse and Wulkenhaar [13, 14]: By adding a harmonic oscillator potential to the action of a non-commutative scalar ϕ^4 theory in Euclidian space, they were able to show not only IR finiteness but even complete renormalizability of the model. Motivated by this success, we considered a similar extension for a non-commutative $U(1)$ gauge field action. A cooperation with Harald Grosse of the University of Vienna and Manfred Schweda led to the peer-reviewed publication [7].

To complete this abstract, I include the following list of publications, preprints and conference proceedings which emerged from this thesis: An analysis of the IR divergences appearing in non-commutative gauge theories (with and without the Slavnov term) and their gauge fixing independence is given in references [1, 2, 3]. Symmetries and topological aspects of the Slavnov term were discussed in the publications [4, 6] and in the conference proceedings [5, 8]. Finally, a non-commutative gauge field model with harmonic oscillator potential was discussed in [7].

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Contents

Chapter 1

Introduction

1.1 Motivation and background

The idea that some "minimal length" of space-time should exist dates back as far as 1946/47 when Snyder formalized the idea of using a non-commutative structure for space-time coordinates in his articles "Quantized Space-Time" [15] and "The Electromagnetic Field in Quantized Space-Time" [16]. His motivation was to "smear out" point-like interactions of particles in order to regularize ultraviolet divergences which are typical for quantum field theories. Further pioneers of non-commutative geometry and non-commutative quantum field theory were Groenewold (1946) [17], Moyal (1949) [18], Madore (1992) [19], Connes (1994) [20] and Filk (1996) [21]. Especially the extensive work of Connes, which even involved a reformulation of the standard model of particle physics [20, 22, 23] as a "spectral model of space-time" based on ideas of non-commutative geometry[1], received much attention. However, it

[1] In Connes' version of the standard model there are indeed fewer free parameters and hence there is a concrete prediction for the mass of the so far undiscovered Higgs particle

was not until Seiberg and Witten [24] discovered that a non-commutative field theory action arises in string theory as an effective action on a D-brane with a strong B-field background, that non-commutative quantum field theory enjoyed greater interest among high-energy physicists. Furthermore, one should also mention the work of Connes, Douglas and Schwarz [25, 26], who studied toroidal compactification of Matrix theory in the framework of non-commutative geometry. Today, several extensive reviews exist on this field, e.g. by Douglas and Nekrasov (2001) [9], Szabo (2001) [10], Zachos (2001) [27], Landi (1997) [28] and many others.

However, despite initial hopes, non-commutativity failed to eliminate UV divergences in quantum field theory: In general, new types of Feynman graphs, so-called *non-planer* graphs, appear in addition to the "old" *planar* graphs. Planar graphs suffer from the same ultraviolet problems as regular quantum field theories. Hence, renormalization is still required to get a finite theory. Additionally, however, one has to deal with new types of divergences in the IR regime. These new singularities appear in the non-planar graphs and are due to additional phase factors[2] in the Feynman integrals. These phases depend on exceptional momenta p_μ and have the effect of UV regularization [29, 30]. However, as these momenta become smaller and finally approach zero, the regularizing effect becomes weaker and eventually fails. Hence, the initial UV divergence reappears, manifesting itself as an IR divergence for $p_\mu \to 0$. This mechanism is commonly referred to as *UV/IR mixing*. Non-locality of the IR divergences presents a major problem when renormalizing a non-commutative field theory and in the past decade the main effort in this area of physics was to find a way to handle the UV/IR mixing problem. In a recent review article [31] the present situation of the renormalization of NCQFT is elucidated very elegantly.

But why put so much effort into a theory that has failed to regularize the UV divergences of point-like interactions, which seemed to be the initial motivation to study non-commutative field theories in the first place? Well, as already mentioned, the discovery of non-commutative field theories within string theory [24] provided a strong motivation. But in my opinion the strongest argument for space-time non-commutativity as a valid description of space-time geometry at small distances is the apparent inconsistency between Einstein's theory of general relativity [32] and the standard model of particle physics (see for example [33] and references therein): While according

of about 170 GeV. It will be very interesting to see if the LHC at CERN will confirm this result when it goes online towards the end of 2008.

[2] The origin of these additional phases will be explained in Section 1.2.

to general relativity, the gravitational "force" is actually a purely classical effect of space-time curvature caused by the presence of matter fields, the same matter fields are described by quantum field theory within the standard model. Therefore, we are dealing with an equation whose left hand side consists of the classical Einstein tensor[3]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

and whose right hand side is given by the energy momentum tensor of quantum fields denoted by $T_{\mu\nu}$. Of course, most problems of gravity deal with large distances where quantum effects are completely negligible and a classical version of the energy momentum tensor is perfectly sufficient. However, at very small distances, e.g. of the order of the Planck length $\lambda_p \simeq 10^{-33}$cm, the classical notion of geometry completely breaks down due to quantum effects. For instance, consider the following gedanken experiment [34, 35, 36]: According to Heisenberg's uncertainty relation, measuring the position of a point particle with high accuracy a will cause an uncertainty in momentum of the order $\frac{1}{a}$ (in natural units $\hbar = c = G = 1$). Therefore, an energy of the order $\frac{1}{a}$ will be concentrated in the localized region, and the associated energy-momentum tensor $T_{\mu\nu}$ will generate a gravitational field which will be determined by solving Einstein's equations for the metric $g_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{1.1}$$

If the uncertainties Δx_μ in the measurement of coordinates become sufficiently small, which is the case near the Planck length

$$\Delta x_\mu \simeq \lambda_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 10^{-33}\text{cm}, \tag{1.2}$$

the gravitational field generated by the measurement will become so strong as to prevent light or other signals from leaving the region in question. In order to avoid black holes from being produced in the course of measurement, one is more than tempted to introduce quantum, or non-commutative, space-time. This implies that one should introduce a non-vanishing commutator for the space-time coordinates themselves, namely

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}. \tag{1.3}$$

[3]The space-time metric is denoted by $g_{\mu\nu}$, and $R_{\mu\nu}$ and R denote the Ricci tensor and the Ricci scalar, respectively.

This means that the initially classical coordinates of space-time are promoted to operators acting on a Hilbert space. A non-vanishing commutator relation, such as (1.3), always implies an uncertainty relation, in this case

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}| \sim (\lambda_p)^2 \,. \tag{1.4}$$

Note that $(\lambda_p)^2$ is just a lower bound motivated by our gedanken experiment above. The actual value for $|\theta^{\mu\nu}|$ might well be much larger and has to be determined by future experiments. As for the explicit form of the matrix $\theta^{\mu\nu}$, we will only consider the simplest case where its entries are constants, since the (more realistic) case of x-dependence becomes far more complicated and needs to be studied once one has (successfully) constructed a field theory with constant θ. At this point one should also mention that attempts have been made to construct a deformed version of Einstein gravity, i.e. see [37, 38, 39, 40] and references therein.

In the following section, however, a particularly popular formulation of a flat non-commutative space-time, namely θ-deformed space-time and the so-called Weyl-Moyal correspondence will be introduced.

1.2 θ-deformed space-time

Following the work of Filk [21], where the (commuting) coordinates of flat Minkowski space $\mathbb{M}^d$ are replaced by Hermitian operators $\hat{x}^\mu$ (with $\mu = 0, 1, \ldots, (d-1)$), we consider a canonical structure defined by the following algebra:

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= \mathrm{i}\theta^{\mu\nu}, \\ [\theta^{\mu\nu}, \hat{x}^\rho] &= 0. \end{aligned} \tag{1.5}$$

In the simplest case the matrix $\theta^{\mu\nu}$ is constant. Furthermore, it is real and antisymmetric. In natural units, where $\hbar = c = 1$, its mass dimension $[\theta] = -2$. At this point one also has to mention that the commutation relations (1.5) between the coordinates explicitly break Lorentz invariance [41, 42, 43]. Other possibilities for θ are e.g. $\theta^{\mu\nu} = C^{\mu\nu}{}_\rho x^\rho$ (Lie algebra) or $\theta^{\mu\nu} = R^{\mu\nu}{}_{\rho\sigma} x^\rho x^\sigma$ (quantum space structure) — see, for example, reference [44] for a detailed discussion. However, for the sake of simplicity we will only consider constant θ throughout this work. We call a space with the commutation relations (1.5) a *non-commutative* space $\mathbb{M}^d_{\mathrm{NC}}$.

In order to construct the perturbative field theory formulation, it is more convenient to use fields $\Phi(x)$ (which are functions of ordinary commuting coordinates) instead of operator valued objects like $\hat{\Phi}(\hat{x})$. To be able to pass to such fields, in respecting the properties (1.5), one must redefine the multiplication law of functional (field) space. One therefore defines the linear map $\hat{f}(\hat{x}) \mapsto S[\hat{f}](x)$, called the "symbol" of the operator $\hat{f}$, and can then represent the original operator multiplication in terms of so-called *star products* of symbols as

$$\hat{f}\hat{g} = S^{-1}\left[S[\hat{f}] \star S\left[\hat{g}\right]\right], \tag{1.6}$$

(see for instance references [9, 44, 45]). In using the Weyl-ordered symbol (which corresponds to the Weyl-ordering prescription of the operators) one arrives at the following definitions (with $S[\hat{f}](x) \to \Phi(x)$):

$$\begin{aligned}
\hat{\Phi}(\hat{x}) &\longleftrightarrow \Phi(x), \\
\hat{\Phi}(\hat{x}) &= \int \frac{d^dk}{(2\pi)^d} e^{\mathrm{i}k\hat{x}} \widetilde{\Phi}(k), \\
\widetilde{\Phi}(k) &= \int d^dx e^{-\mathrm{i}kx} \Phi(x),
\end{aligned} \tag{1.7}$$

where k and x are real variables. For two arbitrary scalar fields $\hat{\Phi}_1$, $\hat{\Phi}_2$ one therefore has[4]

$$\begin{aligned}
\hat{\Phi}_1(\hat{x})\hat{\Phi}_2(\hat{x}) &= \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \widetilde{\Phi}_1(k_1)\widetilde{\Phi}_2(k_2) e^{\mathrm{i}k_1\hat{x}} e^{\mathrm{i}k_2\hat{x}} \\
&= \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \widetilde{\Phi}_1(k_1)\widetilde{\Phi}_2(k_2) e^{\mathrm{i}(k_1+k_2)\hat{x} - \frac{1}{2}[\hat{x}^\mu, \hat{x}^\nu]k_{1,\mu}k_{2,\nu}}.
\end{aligned} \tag{1.8}$$

Hence one has the following *Weyl-Moyal correspondence*[5] [47, 48]:

$$\hat{\Phi}_1(\hat{x})\hat{\Phi}_2(\hat{x}) \longleftrightarrow \Phi_1(x) \star \Phi_2(x), \tag{1.9}$$

where, in using relation (1.5) to replace the commutator in the exponent of (1.8), the Groenewold-Moyal-Weyl star product is given by

$$\Phi_1(x) \star \Phi_2(x) = e^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial^x_\mu \partial^y_\nu} \Phi_1(x)\Phi_2(y)\big|_{x=y}. \tag{1.10}$$

[4] One has to use the Baker-Campbell-Hausdorff formula, as well as relation (1.5).

[5] Weyl's contribution dates back to 1927 when he introduced a correspondence between quantum-mechanical operators and ordinary $\mathbb{C}$-number phase-space functions [46].

This means that we can work on a usual commutative space for which the multiplication operation is modified by the star product (1.10). For the ordinary commuting coordinates this implies[6]

$$
\begin{aligned}
[x^\mu \stackrel{\star}{,} x^\nu] &= \mathrm{i}\theta^{\mu\nu}, \\
[\theta^{\mu\nu} \stackrel{\star}{,} x^\rho] &= 0.
\end{aligned} \tag{1.11}
$$

A natural generalization of (1.10) is given by

$$
\begin{aligned}
\Phi_1(x) \star \Phi_2(x) \star \cdots \star \Phi_m(x) = \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d} \cdots \int \frac{d^dk_m}{(2\pi)^d} e^{\mathrm{i}\sum\limits_{i=1}^{m} k_i^\mu x_\mu} \\
\times \widetilde{\Phi}_1(k_1)\widetilde{\Phi}_2(k_2)\cdots\widetilde{\Phi}_m(k_m) e^{-\frac{\mathrm{i}}{2}\sum\limits_{i<j}^{m} k_i \times k_j},
\end{aligned} \tag{1.12}
$$

where $k \times k'$ is an abbreviation for $k \times k' \equiv k_\mu \theta^{\mu\nu} k'_\nu \equiv k_\mu \tilde{k}'^\mu$. Furthermore, one can easily verify the following properties of the star product:

$$
\int d^dx \, (\Phi_1 \star \Phi_2)(x) = \int d^dx \; \Phi_1(x)\Phi_2(x), \tag{1.13a}
$$

$$
\int d^dx \, (\Phi_1 \star \Phi_2 \star \cdots \star \Phi_m)(x) = \int d^dx \, (\Phi_2 \star \cdots \star \Phi_m \star \Phi_1)(x), \tag{1.13b}
$$

$$
\frac{\delta}{\delta\Phi_1(y)} \int d^dx \, (\Phi_1 \star \Phi_2 \star \cdots \star \Phi_m)(x) = (\Phi_2 \star \cdots \star \Phi_m)(y). \tag{1.13c}
$$

Equations (1.12) and (1.13a) demonstrate that in a θ-deformed quantum field theory the free field part is not modified and therefore the corresponding propagators remain unchanged[7]. Only the interaction terms in the action are equipped with additional phases leading to completely new features in the perturbative realization of the corresponding non-commutative quantum field theories (NCQFTs), i.e. Feynman graphs now split into UV divergent *planar* and UV finite *non-planar* contributions. As already mentioned in Section 1.1, UV finiteness of the non-planar graphs is due to the presence of regularizing phases. The downside, however, is that the non-planar graphs exhibit IR divergences for small external momenta, which is commonly referred to as the *UV/IR mixing problem* of NCQFT.

[6]The Weyl bracket is defined as $[A \stackrel{\star}{,} B] = A \star B - B \star A$.

[7]Actually, there is one exception where this is not true: In Minkowski space when $\theta^{i0} \neq 0$, i.e. time is non-commutative, one needs to heavily modify the Feynman rules in order to restore unitarity of the S-matrix and replace the propagators with so-called contractors. Section 4.1 will deal with this question in more detail.

1.3 Non-commutative space-time and strings

As already anticipated, non-commutative field theory arises in a certain low-energy limit of string theory with D-branes and a strong B-field background. The main ideas that lead to this conclusion, as discovered by Seiberg and Witten [24], are the following:

Consider type II strings in flat space in the presence of a constant Neveu-Schwarz B-field, with Dp-branes and with couplings of gauge fields A_i to the string worldsheet. The matrix B_{ij} should have even rank $r \leq p+1$. The worldsheet action is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} g_{ij} \partial_a x^i \partial^a x^j - \frac{\mathrm{i}}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j - \mathrm{i} \int_{\partial\Sigma} A_i(x) \partial_t x^i, \tag{1.14}$$

where Σ is the string worldsheet with Euclidian signature and ∂_t is a tangential derivative along the worldsheet boundary $\partial\Sigma$.

For slowly varying fields one may write the following *effective* Lagrangian for the gauge fields A_i on the Dp-brane

$$\mathcal{L}_{\mathrm{DBI}} = \frac{1}{g_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det(g + 2\pi\alpha'(B+F))}, \tag{1.15}$$

which is the well-known Dirac-Born-Infeld Lagrangian. In reference [24] it was shown that this action is equivalent to a non-commutative action

$$\mathcal{L}(\hat{F}) = \frac{1}{G_s (2\pi)^p (\alpha')^{\frac{p+1}{2}}} \sqrt{\det G + 2\pi\alpha' \hat{F}}\,, \tag{1.16}$$

where field products are now replaced by Groenewold-Moyal-Weyl star products, and that in the zero slope limit with

$$\begin{aligned} &\alpha' \sim \epsilon^{\frac{1}{2}} \to 0\,, \\ &g^{ij} \sim \epsilon \to 0 \qquad \text{for } i,j = 1,\ldots,r\,, \end{aligned} \tag{1.17}$$

and everything else (including the B-field) held fixed, this action essentially reduces to the action of non-commutative Yang Mills theory

$$\widehat{\mathcal{L}} \simeq \frac{(\alpha')^{\frac{3-p}{2}}}{4(2\pi)^{p-2} G_s} \sqrt{\det G} G^{im} G^{jn} \hat{F}_{ij} \star \hat{F}_{mn}\,. \tag{1.18}$$

The variables G, θ and G_s are related to g, B and g_s in the following way:

$$\begin{aligned} G_s &= g_s \left(\frac{\det G}{\det(g + 2\pi\alpha' B)} \right)^{\frac{1}{2}} , \\ G_{ij} &= -(2\pi\alpha')^2 \left(B g^{-1} B \right)_{ij} , \\ \theta^{ij} &= \left(B^{-1} \right)^{ij} \qquad \text{for } i, j = 1, \dots, r . \end{aligned} \tag{1.19}$$

This means that non-commutative field theories describe an "intermediate regime" which energetically lies between the validity of regular QFT and the regime where string theory could become important. It is therefore likely that non-commutative effects will be discovered before actual stringy effects in future experiments (unless, of course, nature has something completely else in store for us).

1.4 Attempts at eliminating the UV/IR mixing problem

1.4.1 The Seiberg-Witten map

A first idea to eliminate the UV/IR mixing problem was to expand the star-products in the action up to a given order (for simplicity, usually first order) in θ. In doing this one arrives at the so-called *Seiberg-Witten map* [24], which in the simplest case maps a non-commutative $U(1)$ gauge field A_μ to a commuting $U(1)$ Maxwell field a_μ. The key relation here is

$$A(a) + \delta_\alpha A(a) = A(a + \delta_\epsilon a) , \tag{1.20}$$

referred to as *gauge equivalence*[8]. It means that doing a gauge transformation of the non-commutative gauge field A with non-commutative gauge parameter α is equivalent to a gauge transformation of the commuting field a with commuting gauge parameter ϵ. In this framework the deformation parameter $\theta^{\mu\nu}$ plays the role of a constant, unquantized and external field. In this way, a θ-expanded deformed non-commutative Maxwell theory can be obtained [49] where the photon receives a self-interaction via the background field $\theta^{\mu\nu}$.

[8]The existence of such a map, especially the gauge equivalence relation, may be motivated by the fact that the zero slope limit of string theory with D-branes and a B-field, as described in Section 1.3, can lead either to a commutative or a non-commutative effective field theory depending on the regularization scheme used.

At this point one has to stress that gauge field theories formulated via the Seiberg-Witten map are manifestly IR *finite* in the sense of UV/IR mixing. Only the usual UV divergences are present. This fact has been an encouragement for further investigations. Unfortunately, this first impression of optimism was quenched soon after, since such theories are *non-renormalizable* if one also adds fermions to the pure gauge sector as was proven by Wulkenhaar in 2001 [50] (although the pure gauge sector by itself would be renormalizable — cf. ref. [51]). However, the Seiberg-Witten map still has some interesting properties worth discussing: For instance, it was shown in reference [52] that non-commutative field theories formulated in flat space-time appear to be equivalent to ordinary commuting field theories in *curved* space-time. In other words, corrections due to non-commutativity through the Seiberg-Witten map have an effect similar to gravitation in the weak field expansion of the gravitational fields.

1.4.2 Slavnov's trick

In 2003, Slavnov [11, 12] suggested another way of dealing with arising IR singularities in non-commutative gauge theories by adding a further term in the action. This *Slavnov term* has the form

$$\frac{1}{2}\int d^4x\lambda\star\theta^{\mu\nu}F_{\mu\nu}\,, \tag{1.21}$$

where $\theta^{\mu\nu}$ is once again the deformation parameter of non-commutative space-time and λ is a *dynamical* multiplier field[9] leading to a new kind of constraint. This constraint modifies the gauge field propagator $\Delta^A_{\mu\nu}(k)$ in such a way that it becomes transverse with respect to $\tilde{k}^\mu=\theta^{\mu\nu}k_\nu$. This is important, since the vacuum polarization $\Pi^{\mu\nu}$ of (4-dimensional) gauge theories is characterized by the quadratically IR singular structure:

$$\Pi^{\mu\nu}_{\text{IR-div}}(k)\sim\frac{\tilde{k}^\mu\tilde{k}^\nu}{(\tilde{k}^2)^2}, \tag{1.22}$$

where k_μ represents the external momentum. Higher loop insertions of these IR divergent $\Pi^{\mu\nu}_{\text{IR-div}}$ into internal gauge boson loops therefore vanish. Slavnov's idea was motivated by the results of one loop calculations of non-commutative gauge theories previously done by i.e. Hayakawa [53] and others revealing that the leading IR divergent term has the form (1.22).

[9] We will clarify what is meant by "dynamical multiplier field" in a moment.

Gauge independence of this term, or to be more precise, independence of the gauge parameter α of a general covariant gauge fixing, was shortly afterwards shown by Ruiz [54]. In Chapter 2 we will furthermore show that the quadratic IR divergences are, in fact, independent also from an axial gauge fixing and that this gauge independence survives after adding the Slavnov term [1, 2].

Notice that for a general non-commutative $U(N)$ gauge field theory the Slavnov term is, in fact, not needed outside of the $U(1)$ subsector. The reason for this is that the dangerous IR divergences stem solely from the $U(1)$ subsector, as was shown by Armoni and Lopez in reference [55]. To be more precise: Non-planar Feynman diagrams with $U(1)$ boson external lines are infrared singular, whereas non-planar diagrams with only $SU(N)$ boson external lines *do not* exhibit IR poles — at least at one-loop order (see also the discussion in [11]).

However, at this stage one cannot be completely sure that higher loop orders are free of IR divergences due to the well-known phenomenon of *overlapping divergences*. Furthermore, as Chapter 2.2 will reveal, the Slavnov term leads to new Feynman rules involving propagators and vertices of the multiplier field λ (which is why we previously have emphasized that it is a dynamical field). This means one has to deal with additional (and potentially divergent) Feynman graphs. Chapter 3 will deal with these problems and possible solutions involving new symmetries in more detail.

Finally, the new constraint coming from the Slavnov term is not yet fully understood. An analysis using the Hamilton formalism á la Dirac [56] will hopefully resolve the mystery of physically interpreting Slavnov's constraint. A further question concerning this matter is whether this constraint is also fulfilled on the quantum level. A first step in these directions will be presented in Chapter 5.

1.4.3 The Grosse-Wulkenhaar model

At about the same time that Slavnov presented his solution to the UV/IR mixing problem, Grosse and Wulkenhaar [13, 14] came up with a different idea, which proved to be very successfull for non-commutative scalar ϕ^4 theory in Euclidian space: They suggested adding a harmonic oscillator term in the bilinear part of the action

$$S[\phi] = \int d^4x \left[\frac{1}{2} \partial_\mu \phi \star \partial_\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \star (\tilde{x}_\mu \phi) + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right], \tag{1.23}$$

where $\tilde{x}_\mu = (\theta^{-1})_{\mu\nu} x_\nu$ and Ω is a constant parameter. Note that the deformation parameter $\theta_{\mu\nu}$ has full rank in this model.

The four-dimensional model described by (1.23) turns out to be renormalizable to all orders of perturbation theory, as was proven in references [57, 58, 59]. Two further proofs have been worked out in [60, 61]. The renormalization group flow for the coupling constant turned out to be bounded, which was shown by a first order calculation in [57], extended to three loops in [58] and recently to all orders by the Paris group in [59]. This might lead to a constructive procedure for a non-commutative scalar ϕ^4 theory. A recent review on this matter can be found in [31] and references therein.

The propagator of this model is essentially the inverse of the operator $(-\Delta_4 + \Omega^2 \tilde{x}^2 + m^2)$ which is commonly known as the *Mehler kernel* [62]. The crucial feature of the Mehler kernel and also the action under consideration, is its invariance under a *Langmann-Szabo duality* [63]. This means that, apart from a scaling factor, they have the same appearance in position space and momentum space. This is due to the "symmetric" occurrences of partial derivatives ∂_μ and coordinates $\tilde{x}_\mu$ in the action. Although this duality is easily implemented in ϕ^4 theory, finding a way to construct a non-commutative gauge theory making use of this trick is not so straightforward. In Chapter 6 a corresponding model is presented. A further open (and highly non-trivial) question concerns the extension to Minkowski space-time.

Chapter 2

Gauge Independence of IR Singularities

2.1 Non-commutative $U(1)$ Maxwell theory

2.1.1 Preliminaries

It has already been argued in the introduction that non-commutative gauge theories suffer from problematic IR singularities which must be dealt with if one hopes to be able to renormalize such models. In this context it is important to know whether and how these IR singularities depend on the gauge fixing. Several authors (Hayakawa, Ruiz) have discussed this question in connection with a covariant gauge fixing. According to their work [53, 54, 64], the quadratic IR singularities of the vacuum polarization of a non-commutative $U(1)$ gauge theory are independent of the covariant gauge fixing parameter. However, the discussion of other gauge fixings, e.g. the axial gauge, are missing. Therefore, the following section is devoted to verifying the gauge fixing independence of the quadratic IR divergence in non-commutative $U(1)$ Maxwell theory at one-loop level [3] using the following interpolating gauge fixing for the gauge field A_μ:

$$N_\mu A^\mu = 0, \qquad \text{with} \qquad N_\mu = \partial_\mu - \xi \frac{(n\partial)}{n^2} n_\mu, \tag{2.1}$$

which was originally proposed in ref. [65] and used e.g. in [66, 67]. The constant vector n^μ and the real variable ξ are gauge parameters — ξ taking values between $(-\infty, +1)$. This makes it possible to interpolate between a linear class of gauges: the covariant one ($\xi = 0$) and the axial gauge ($\xi \to -\infty$).

In order to quantize non-commutative $U(1)$ Maxwell theory consistently, one has to use the BRST procedure entailing the introduction of the Faddeev-Popov ($\Phi\Pi$) ghost and antighost fields c and $\bar{c}$. Additionally, in order to be more general, we can also introduce a further gauge parameter α leading to the following classical action in 4-dimensional Minkowski space-time $\mathbb{M}^4_{\mathrm{NC}}$:

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{\alpha}{2} B \star B + B \star N^\mu A_\mu - \bar{c} \star N^\mu D_\mu c \right], \tag{2.2}$$

where the non-commutative field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} A_\nu \right], \tag{2.3}$$

and the covariant derivative D_μ is defined as

$$D_\mu \cdot = \partial_\mu \cdot - \mathrm{i}g \left[A_\mu \stackrel{\star}{,} \cdot \right]. \tag{2.4}$$

In order to preserve the unitarity of the S-matrix [29, 68, 69], we assume $\theta^{\mu\nu}$ to be space-like, i.e. $\theta^{0i} = 0$. B is the multiplier field implementing the gauge constraint

$$\frac{\delta S}{\delta B} = \alpha B + N^\mu A_\mu = 0, \tag{2.5}$$

which for $\alpha = 0$ reduces to (2.1). Some choices for the two gauge parameters α and ξ are quite prominent in the literature[1]:

- $\xi = 0$ and α=0, normally called Landau gauge

[1] See for example references [70, 71] for a review on quantum field theories and ref. [72, 73] for a review on gauge field theories in non-covariant gauges.

- $\xi = 0$ and α=1, usually known as Feynman gauge
- $\xi \to -\infty$ (or $N_\mu = n_\mu$) and $\alpha = 0$, leading to the homogeneous axial gauge.

In this section, however, we will use generic gauge parameters in order to investigate the dependence of the highest IR poles on ξ and α. The gauge vector n_μ will be more or less generic as well, but with the restriction $n^2 \neq 0$, i.e. light-like gauges will not be considered.

The action (2.2) is invariant with respect to the BRST symmetry [74]

$$\begin{aligned} &sA_\mu = D_\mu c = \partial_\mu c - \mathrm{i}g\left[A_\mu \stackrel{\star}{,} c\right], && sc = \mathrm{i}gc \star c, \\ &s\bar{c} = B, && sB = 0, \\ &s^2\phi = 0, \qquad \text{for } \phi = \{A^\mu, B, c, \bar{c}\}. && \end{aligned} \tag{2.6}$$

As usual, the transformations (2.6) are nilpotent, non-linear and supersymmetric. For describing the symmetry content encoded by equations (2.6) one has to add a term of the form

$$S_{\text{ext}} = \int d^4x \left[\rho^\mu \star sA_\mu + \sigma \star sc\right] \tag{2.7}$$

to the action (2.2), where ρ^μ and σ are unquantized external BRST invariant sources for the non-linear contributions of the BRST transformations. The symmetry content of

$$S_{\text{tot}} = S + S_{\text{ext}} \tag{2.8}$$

is now described by the non-linear Slavnov-Taylor identity

$$\mathcal{S}\left(S_{\text{tot}}\right) = \int d^4x \left(\frac{\delta S_{\text{tot}}}{\delta \rho^\mu} \star \frac{\delta S_{\text{tot}}}{\delta A_\mu} + \frac{\delta S_{\text{tot}}}{\delta \sigma} \star \frac{\delta S_{\text{tot}}}{\delta c} + B \star \frac{\delta S_{\text{tot}}}{\delta \bar{c}}\right) = 0. \tag{2.9}$$

The use of the star product (1.10) in the bilinear action has no effect. Thus, the free field theory remains unchanged and therefore the propagators of the $U(1)$ Maxwell theory are not touched by non-commutativity.

In momentum representation the gauge field propagator becomes

$$\mathrm{i}\Delta^A_{\mu\nu}(k) = -\frac{\mathrm{i}}{k^2}\left[g_{\mu\nu} - ak_\mu k_\nu + b(n_\mu k_\nu + n_\nu k_\mu)\right], \tag{2.10}$$

with

$$a = \frac{(1-\alpha)k^2 - \zeta^2 n^2 (nk)^2}{\left[k^2 - \zeta(nk)^2\right]^2}, \tag{2.11}$$

and

$$b = \frac{\zeta(nk)}{k^2 - \zeta(nk)^2}, \tag{2.12}$$

where $\zeta = \frac{\xi}{n^2}$ and $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$ is the Lorentz metric. In the limit $\zeta \to 0$ ($\xi \to 0$) one recovers the usual gauge field propagator for a covariant gauge fixing:

$$\mathrm{i}\Delta^{\mathrm{cov}}_{\mu\nu}(k) = -\frac{\mathrm{i}}{k^2}\left[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right]. \tag{2.13}$$

In the limit $\zeta \to -\infty$ ($\xi \to -\infty$) and $n^2 \neq 0$ one has the corresponding gauge field propagator in the axial gauge:

$$\mathrm{i}\Delta^{\mathrm{ax}}_{\mu\nu}(k) = -\frac{\mathrm{i}}{k^2}\left[g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{(nk)} + n^2\frac{k_\mu k_\nu}{(nk)^2}\right]. \tag{2.14}$$

The remaining ghost-antighost propagator is given by

$$\mathrm{i}\Delta^{c\bar{c}}(k) = \frac{\mathrm{i}}{k^2 - \zeta(nk)^2}, \tag{2.15}$$

and the mixed propagator between the gauge field A_μ and the multiplier field B becomes

$$\mathrm{i}\Delta^{B}_{\mu}(k) = -\frac{k_\mu}{k^2 - \zeta(nk)^2}. \tag{2.16}$$

One observes that also the propagators (2.15) and (2.16) depend on the gauge parameter ζ. Additionally, it must be mentioned that the vertex for the interaction of the gauge field and the ghosts is gauge dependent as well:

$$V_{\mu c\bar{c}}(q_1, q_2, k) = 2g\,(q_{2\mu} - \zeta(nq_2)n_\mu)\sin\left(\frac{q_1\tilde{q}_2}{2}\right), \tag{2.17}$$

where $\tilde{q}_2^\mu$ is defined by $\tilde{q}_2^\mu = \theta^{\mu\nu}q_{2\nu}$. k_μ denotes the gauge field momentum and $q_{i\mu}$ ($i = 1,2$) are the momenta of the ghost fields.

The other couplings describing the self-interactions of the bosons (stemming from the invariant part of the action) are gauge independent and are well-known in the literature [53, 54, 55]. The three-photon vertex is given by

$$\begin{aligned} V^{3A}_{\rho\sigma\tau}(k_1, k_2, k_3) = &-2g\Big[(k_3 - k_2)_\rho g_{\sigma\tau} + (k_1 - k_3)_\sigma g_{\rho\tau} + \\ &+ (k_2 - k_1)_\tau g_{\rho\sigma}\Big]\sin\left(\frac{k_1\tilde{k}_2}{2}\right), \end{aligned} \tag{2.18}$$

and the four-boson vertex reads

$$V^{4A}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4) = -4\mathrm{i}g^2 \Bigg[(g_{\rho\tau}g_{\sigma\epsilon} - g_{\rho\epsilon}g_{\sigma\tau}) \sin\left(\frac{k_1\tilde{k}_2}{2}\right) \sin\left(\frac{k_3\tilde{k}_4}{2}\right) + (g_{\rho\sigma}g_{\tau\epsilon} - g_{\rho\epsilon}g_{\sigma\tau}) \sin\left(\frac{k_1\tilde{k}_3}{2}\right) \sin\left(\frac{k_2\tilde{k}_4}{2}\right) + (g_{\rho\sigma}g_{\tau\epsilon} - g_{\rho\tau}g_{\sigma\epsilon}) \sin\left(\frac{k_2\tilde{k}_3}{2}\right) \sin\left(\frac{k_1\tilde{k}_4}{2}\right) \Bigg]. \tag{2.19}$$

One observes that the Feynman rules for the vertices contain phases. Due to this fact, the behaviour for high internal momenta of the corresponding Feynman integrals in momentum representation is modified in a new fashion: For high internal momenta, the phases act as a regularization induced by the oscillating phase factors. This implies that non-planar one-particle irreducible (1PI) graphs, which are à priori UV divergent by naïve power counting, become finite but develop a new singularity for vanishing external momenta. This interplay between the expected UV divergences — which are not present in non-planar Feynman graphs — and the existence of the real IR singularity represents the so-called UV/IR mixing problem [48, 29], which has already been mentioned in the introduction.

2.1.2 IR divergences at one-loop level

The aim of this section is to investigate the gauge independence of IR singularities emerging from the one-loop corrections to the vacuum polarization in the framework of the interpolating gauge mentioned above. For this reason one has to consider the following three amputated one-loop graphs presented in Fig. 2.1.

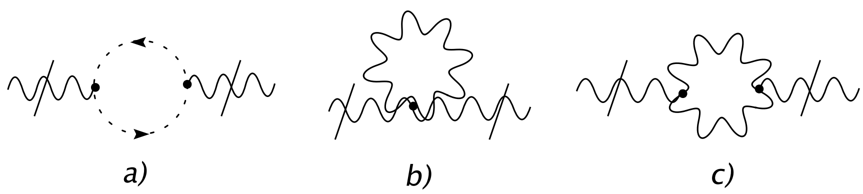


Figure 2.1: Gauge boson self-energy — amputated graphs

Corresponding to the Feynman rules given in Section 2.1.1, the vacuum polarization tensor $\Pi_{\mu\nu}(p)$ in the one-loop approximation is a Feynman integral of the following form:

$$\mathrm{i}\Pi_{\mu\nu}(p) = \int d^4k I_{\mu\nu}(k,p)\sin^2\left(\frac{k\tilde{p}}{2}\right). \tag{2.20}$$

Details are given in Appendix B. Additionally, one also has the transversality condition

$$p^\mu \Pi_{\mu\nu}(p) = 0, \tag{2.21}$$

which follows from the Slavnov-Taylor identity (2.9).

In order to isolate the expected IR singularities of the non-planar sector, one proceeds the same way as in the standard renormalization program for planar graphs (i.e. graphs without phases) in considering the expansion

$$\begin{aligned}\mathrm{i}\Pi_{\mu\nu}(p) = &\int d^4k I_{\mu\nu}(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) + p^\rho \int d^4k \frac{\partial}{\partial p^\rho} I_{\mu\nu}(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) + \\ &+ \frac{1}{2}p^\sigma p^\rho \int d^4k \frac{\partial^2}{\partial p^\sigma \partial p^\rho} I_{\mu\nu}(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) + \ldots\end{aligned} \tag{2.22}$$

Due to the fact that the naïve degree of divergence $D = 2$ ($D = 4 - E = 2$, where E denotes the number of external bosons) for high internal k, one is inclined to believe that the first term of (2.22) is a candidate for a quadratic non-commutative IR singularity. The second may be linearly divergent. However, for dimensional reasons no linear IR divergences occur. The third term in (2.22) may contain logarithmic divergences.

Calculation of the first term of (2.22) leads to

$$\begin{aligned}\int d^4k I^{\mu\nu}(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) = 4g^2 \int \frac{d^4k}{(2\pi)^4}\sin^2\left(\frac{k\tilde{p}}{2}\right)\frac{1}{k^2}\Bigg\{ &-2g^{\mu\nu} - \\ - b\left(n^\mu k^\nu + k^\mu n^\nu\right)&\left[1 + (nk)b - \frac{k^2}{k^2 - \zeta(nk)^2}\right] + \\ + \frac{k^\mu k^\nu}{k^2}&\left[5 + 2(nk)b + (nk)^2 b^2 - \frac{k^4}{\left[k^2 - \zeta(nk)^2\right]^2}\right]\Bigg\},\end{aligned} \tag{2.23}$$

(cf. Appendix B). This expression is obviously independent of a, which was defined in (2.11), and hence independent of the gauge parameter α. With the definition (2.12) one finally obtains

$$\int d^4k I_{\mu\nu}(k,0) \sin^2\left(\frac{k\tilde{p}}{2}\right) = 4g^2 \int \frac{d^4k}{(2\pi)^4}\left[4\frac{k_\mu k_\nu}{k^4} - 2\frac{g_{\mu\nu}}{k^2}\right] \sin^2\left(\frac{k\tilde{p}}{2}\right). \tag{2.24}$$

One observes that the gauge dependent tensor structure based on the existence of the gauge directions n_μ and the dependence on the gauge parameters α and ζ (or ξ with $\zeta = \xi/n^2$) cancel completely. Using the identity

$$\sin^2\left(\frac{k\tilde{p}}{2}\right) = \frac{1}{2}\left(1 - \cos\left(k\tilde{p}\right)\right) , \tag{2.25}$$

the Feynman integral (2.24) splits into a non-planar contribution with phases $\cos(k\tilde{p})$ and a planar part without phases. Performing the integration of the non-planar part we reproduce the known result [53, 54]

$$-\frac{1}{2}\int d^4k I_{\mu\nu}(k,0) \cos\left(k\tilde{p}\right) = \mathrm{i}\frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^4}. \tag{2.26}$$

The final result of the non-planar contributions to the vacuum polarization at the one-loop level is finite but shows the expected quadratic IR singularity for vanishing external momentum. The pole term in equation (2.26) is the manifestation of the UV/IR mixing which is a typical new feature of non-commutative quantum field theories. The origin of these singularities is the UV regime which seems to influence the IR behaviour of the field model. These pole terms create serious problems in the renormalization procedure, since when the non-planar singular IR contributions are inserted into higher loop diagrams they create new divergences. One possible way to bypass these difficulties is to use the Slavnov trick [11, 12], which will be introduced in the next section.

2.2 The Slavnov term

2.2.1 Introducing the extension

As mentioned previously, A. A. Slavnov introduced an additional term [11] in the gauge field action, so as to eliminate the UV/IR mixing problems. Extending the action (2.2) and including this (gauge invariant) "Slavnov term" we now have

$$S = \int d^4x \Bigg(-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{\alpha}{2} B \star B + B \star N^\mu A_\mu - \bar{c} \star N^\mu D_\mu c + \\ + \frac{\lambda}{2} \star \theta^{\mu\nu} F_{\mu\nu} \Bigg)(x), \tag{2.27}$$

where $\theta^{\mu\nu}$ is the parameter describing non-commutativity (1.11) and λ is a new multiplier field imposing the following constraint:

$$\frac{\delta S}{\delta \lambda} = \frac{1}{2} \theta^{\mu\nu} F_{\mu\nu} = 0. \tag{2.28}$$

This relation changes the gauge field propagator in a drastic manner: The propagator becomes transverse with respect to $\tilde{k}^\mu$. In momentum space this means:

$$\mathrm{i}\Delta^A_{\mu\nu}(k) = -\frac{\mathrm{i}}{k^2} \left[g_{\mu\nu} - a k_\mu k_\nu + b \left(n_\mu k_\nu + k_\mu n_\nu \right) - b' \left(\tilde{k}_\mu k_\nu + k_\mu \tilde{k}_\nu \right) - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right] \tag{2.29}$$

with

$$\begin{aligned} a &= \frac{(1-\alpha)k^2 - \zeta^2 (nk)^2 \left[n^2 - \frac{(n\tilde{k})^2}{\tilde{k}^2} \right]}{\left[k^2 - \zeta (nk)^2 \right]^2}, \\ b &= \frac{\zeta (nk)}{k^2 - \zeta (nk)^2}, \qquad b' = \frac{(n\tilde{k})}{\tilde{k}^2} b, \\ \zeta &= \frac{\xi}{n^2}, \qquad n^2 \neq 0\,. \end{aligned} \tag{2.30}$$

In the limit $\zeta \to 0$ ($\xi \to 0$) one recovers the gauge field propagator for a covariant gauge fixing characterized by the gauge parameter α:

$$\mathrm{i}\Delta^{\mathrm{cov}}_{\mu\nu} = -\frac{\mathrm{i}}{k^2} \left[g_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right]. \tag{2.31}$$

In the limit $\zeta \to -\infty$ ($\xi \to -\infty$) and $n^2 \neq 0$ one has the corresponding gauge field propagator in the axial gauge:

$$\mathrm{i}\Delta^{\mathrm{ax}}_{\mu\nu} = -\frac{\mathrm{i}}{k^2} \Bigg[g_{\mu\nu} + \left(n^2 - \frac{(n\tilde{k})^2}{\tilde{k}^2} \right) \frac{k_\mu k_\nu}{(nk)^2} - \frac{n_\mu k_\nu + n_\nu k_\mu}{(nk)} + \\ + \frac{(n\tilde{k})}{\tilde{k}^2} \frac{\left(\tilde{k}_\mu k_\nu + k_\mu \tilde{k}_\nu \right)}{(nk)} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \Bigg]. \tag{2.32}$$

From (2.29) follows

$$\tilde{k}^\mu \Delta^A_{\mu\nu}(k) = -\frac{1}{k^2}\left[\tilde{k}_\nu + b(\tilde{k}n)k_\nu - b'\tilde{k}^2 k_\nu - \tilde{k}_\nu\right] = 0, \tag{2.33}$$

where the definition of b' in (2.30) was used. This new kind of transversality is actually encoded in equation (2.28), if one considers only the bilinear parts of the action responsible for calculating the gauge field propagator, and will be very useful in avoiding the UV/IR mixing at higher loop order: In the previous section we have seen [64, 54, 1] that the troublesome IR singularities of self-energy insertions of the gauge boson are gauge fixing independent and of the form

$$\Pi^{\mathrm{IR}}_{\mu\nu}(k) \propto \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2}. \tag{2.34}$$

If one inserts these structures into higher order diagrams, the loop integrations lead to problems around $k = 0$. With the new transversality property of the propagator this problem is circumvented, as is shown with the help of Figure 2.2. As is clearly seen, the inserted self-energy is multiplied by two in-

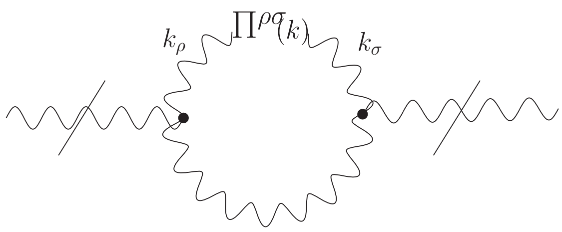


Figure 2.2: A possible example for an insertion for (amputated) higher order graphs

ternal propagators, i.e. $\Delta^A_{\mu\rho}(k)\Pi^{\rho\sigma}(k)\Delta^A_{\sigma\nu}(k)$, before integrating. Especially, for the IR divergent parts we have

$$\Delta^A_{\mu\rho}(k)\frac{\tilde{k}^\rho \tilde{k}^\sigma}{(\tilde{k}^2)^2}\Delta^A_{\sigma\nu}(k) = 0. \tag{2.35}$$

This means the insertions of the troublesome IR divergent parts vanish even without integration. This procedure, of course, requires that no further IR divergences with a different tensor structure appear in the model. There are some convincing arguments in [11] that only terms like (2.34) are obtained, though an explicit one-loop calculation is missing. It is one of the tasks of this section to show the outcome of such calculations.

There is another new feature in connection with the multiplier field λ: It becomes a dynamical field with a non-vanishing propagator and also induces new polynomial interactions with the gauge fields. This might cause further problems which will also be discussed.

2.2.2 Symmetries of the model

In addition to the BRST transformation laws of (2.6) we now have

$$s\lambda = -\mathrm{i}g\,[\lambda \stackrel{\star}{,} c] \;, \quad s^2\lambda = 0\,. \tag{2.36}$$

It is obvious that this transformation law for λ renders the Slavnov term invariant, since the field tensor $F_{\mu\nu}$ transforms covariantly (i.e. $sF_{\mu\nu} = -\mathrm{i}g\,[F_{\mu\nu} \stackrel{\star}{,} c]$), as can easily be verified. Since λ transforms non-linearly under BRST, one must introduce a further external source, which we denote γ. Hence, S_{ext} as defined in (2.7), now becomes

$$S_{\text{ext}} = \int d^4x \left[\rho^\mu \star sA_\mu + \sigma \star sc + \gamma \star s\lambda\right], \tag{2.37}$$

with $s\rho^\mu = s\sigma = s\gamma = 0$. At the classical level, one has the following non-linear identity for the classical vertex functional $S_{\text{tot}} = S + S_{\text{ext}}$

$$\begin{aligned}\mathcal{S}\left(S_{\text{tot}}\right) = \int d^4x \Bigg(&\frac{\delta S_{\text{tot}}}{\delta\rho^\mu} \star \frac{\delta S_{\text{tot}}}{\delta A_\mu} + \frac{\delta S_{\text{tot}}}{\delta\sigma} \star \frac{\delta S_{\text{tot}}}{\delta c} + \frac{\delta S_{\text{tot}}}{\delta\gamma} \star \frac{\delta S_{\text{tot}}}{\delta\lambda} + \\ &+ B \star \frac{\delta S_{\text{tot}}}{\delta\bar{c}}\Bigg) = 0.\end{aligned} \tag{2.38}$$

This equation describes the symmetry content with respect to (2.6) and (2.36). Together with the linearized BRST operator [75, 73]

$$\begin{aligned}\mathcal{S}_S = \int d^4x \Bigg(&\frac{\delta S_{\text{tot}}}{\delta\rho^\mu} \star \frac{\delta}{\delta A_\mu} + \frac{\delta S_{\text{tot}}}{\delta A_\mu} \star \frac{\delta}{\delta\rho^\mu} + \frac{\delta S_{\text{tot}}}{\delta\sigma} \star \frac{\delta}{\delta c} + \frac{\delta S_{\text{tot}}}{\delta c} \star \frac{\delta}{\delta\sigma} + \\ &+ \frac{\delta S_{\text{tot}}}{\delta\gamma} \star \frac{\delta}{\delta\lambda} + \frac{\delta S_{\text{tot}}}{\delta\lambda} \star \frac{\delta}{\delta\gamma} + B \star \frac{\delta}{\delta\bar{c}}\Bigg),\end{aligned} \tag{2.39}$$

one obtains from (2.38)

$$\frac{\delta}{\delta A_\rho(y)}\mathcal{S}(S) = \mathcal{S}_S \frac{\delta S_{\text{tot}}}{\delta A_\rho(y)} = 0. \tag{2.40}$$

When taking the functional derivative of (2.40) with respect to c and then setting all fields to zero, one obtains the transversality condition for the one-particle irreducible (1PI) two-point graph

$$\partial^y_\mu \frac{\delta^2 S_{\text{tot}}}{\delta A_\mu(y)\delta A_\nu(y)} = 0. \tag{2.41}$$

The central task of the perturbative analysis is to study the behaviour of the symmetry content in the presence of radiative corrections. One important question is the validity of (2.41) at the perturbative level.

2.2.3 Gauge boson self-energy at the one-loop level

Due to the presence of the scalar field λ in the action (2.27), additional Feynman rules are introduced: a λ-λ propagator, in momentum space given by

$$\mathrm{i}\Delta^{\lambda\lambda}(k) = \mathrm{i}\frac{k^2}{\tilde{k}^2}, \tag{2.42}$$

a mixed λ-A propagator

$$\mathrm{i}\Delta^{\lambda A}_\mu(k) = \frac{\tilde{k}_\mu}{\tilde{k}^2}, \tag{2.43}$$

and a λ-A-A vertex, given by

$$V^{\lambda AA}_{\mu\nu}(p, k_1, k_2) = 2\mathrm{i}g\theta_{\mu\nu}\sin\left(\frac{k_1\tilde{k}_2}{2}\right). \tag{2.44}$$

Therefore, compared to the non-commutative model without the Slavnov term, we have many additional contributions to the gauge boson self-energy (see Figure 2.3) at one-loop level. All Feynman rules, including the ones introduced in the previous chapter, are collected in Appendix A.

In order to isolate the expected IR singularities of the non-planar sector, we consider the following expansion of the momentum representation of the two point self-energy corrections

$$\begin{aligned}\mathrm{i}\Pi_{\mu\nu}(p) = \int d^4k \sin^2\left(\frac{k\tilde{p}}{2}\right)\Bigg\{ & I_{\mu\nu}(k,0) + p^\rho \frac{\partial}{\partial p^\rho} I_{\mu\nu}(k,p)\Big|_{p=0} + \\ & + \frac{1}{2}p^\sigma p^\rho \frac{\partial^2}{\partial p^\sigma \partial p^\rho} I_{\mu\nu}(k,p)\Big|_{p=0} + \ldots \Bigg\},\end{aligned} \tag{2.45}$$

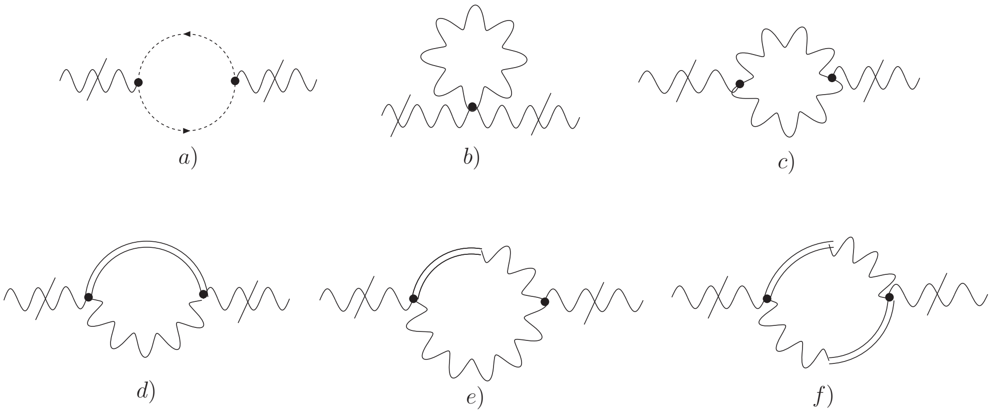


Figure 2.3: Gauge boson self-energy — amputated graphs

where the worst expected IR divergence will appear in the non-planar part of the first term of this expansion. For the first term of equation (2.45) one arrives at the following expression (see Appendix B for details):

$$\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{\mathrm{IR}}(p) &\equiv \int d^4k I^{\mu\nu}(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) = \\
&= 4g^2 \int \frac{d^4k}{(2\pi)^4}\left\{\left(2\frac{k^\mu k^\nu}{k^4} - \frac{g^{\mu\nu}}{k^2}\right) + \theta^{\mu\tau}\left(\frac{g_{\tau\sigma}}{\tilde{k}^2} - 2\frac{\tilde{k}_\tau\tilde{k}_\sigma}{\tilde{k}^4}\right)\theta^{\sigma\nu}\right\}\sin^2\left(\frac{k\tilde{p}}{2}\right).
\end{aligned} \tag{2.46}$$

This result shows some very interesting features: First of all, we notice that all gauge dependent terms have obviously cancelled, leaving (2.46) completely *gauge independent*.

Let us compare (2.46) with the corresponding expression of a model *without* the Slavnov term: In this case one only has three graphs at the one-loop level, namely Figures 2.3a), 2.3b) and 2.3c). The result is (where p describes the external momentum):

$$\mathrm{i}\Pi^{\mu\nu}_{\mathrm{IR,noSl}}(p) = 8g^2 \int \frac{d^4k}{(2\pi)^4}\left[2\frac{k^\mu k^\nu}{k^4} - \frac{g^{\mu\nu}}{k^2}\right]\sin^2\left(\frac{k\tilde{p}}{2}\right). \tag{2.47}$$

(2.47) is, of course, also gauge independent, as has already been derived in the previous section. Comparing (2.46) with (2.47) we see that adding the Slavnov term changes the overall factor of the terms appearing in both models and, coming from the graphs depicted in Figures 2.3d) and 2.3f), additional terms including $\theta_{\mu\nu}$ occur. (All other θ-dependent terms, including those coming from the other graphs, cancel.)

An important question now is whether these additional terms in the integrand also lead to IR divergent terms of the form (2.34), since Slavnov's trick is based on the fact that the gauge propagator is transverse with respect to $\tilde{k}_\mu$ (see equation (2.33)). Actually, one sees immediately that in case $\theta_{\mu\nu}$ does not have full rank, part of the integrand of (2.46) will be independent of certain directions and hence produce additional UV divergences. However, one may still hope that these divergences are proportional to $\tilde{p}_\mu \tilde{p}_\nu$ in which case Slavnov's trick would still work.

2.2.4 Higher loop orders

We have now shown that the IR divergent terms at one-loop order are

- proportional to $\Pi^{\mathrm{IR}}_{\mu\nu} \propto \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}$ and
- that these IR singularities are gauge independent [1, 2, 54].
- Furthermore, previous studies have shown that the structure of the IR divergences is independent of couplings to fermions and scalar fields — except that the overall factor is changed when coupling to the latter [2, 76, 77].

Therefore, the graph depicted in Figure 2.4a) is free of non-integrable IR singularities. However, we have also found out that, in case $\theta_{\mu\nu}$ does not have full rank, new UV divergences in both planar as well as non-planar graphs appear. Those coming from non-planar graphs also have the structure $\tilde{p}_\mu \tilde{p}_\nu$ and hence drop out before integrating out Figure 2.4a). UV singularities coming from the planar graphs may still cause problems.

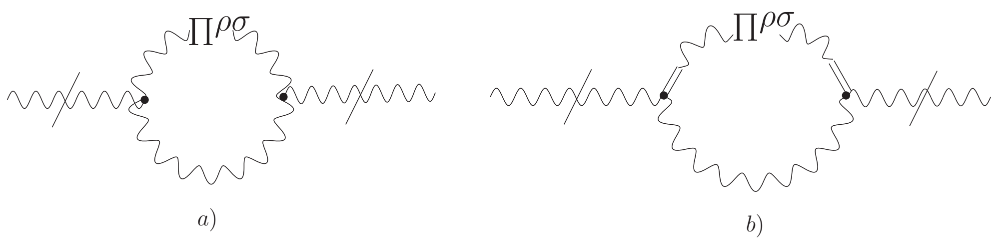


Figure 2.4: Some (amputated) 2-loop graphs

Unfortunately, due to the existence of the λ-vertex, the number of graphs is greatly increased and at 2-loop level one can also construct graphs for which Slavnov's trick does not work. An example is depicted in Figure 2.4b). If one computes this graph in 4-dimensional Euclidian space (for simplicity in Feynman gauge $\alpha = 1$, $\xi = 0$) with the choice

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

one finds

$$\begin{aligned} \Pi^{IR,b}_{\mu\nu,2l}(p) &= \frac{8g^4}{\pi^2} \int \frac{d^4k}{(2\pi)^4} \sin^2\left(\frac{k\tilde{p}}{2}\right) \theta_{\mu\rho} \left(\delta_{\rho\sigma} - \frac{(\tilde{k}-\tilde{p})_\rho (\tilde{k}-\tilde{p})_\sigma}{(\tilde{k}-\tilde{p})^2} \right) \frac{\theta_{\sigma\nu}}{(k-p)^2 \tilde{k}^4} \\ &= \lim_{\epsilon \to 0} \frac{-\ln\epsilon}{16\pi^2\theta^2\tilde{p}^2} \theta_{\rho\mu} \left(\delta_{\mu\nu} - \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} \right) \theta_{\nu\sigma}. \end{aligned} \tag{2.48}$$

This expression seems problematic in two ways: The last parameter integral diverged, producing $\lim\limits_{\epsilon\to 0} \ln\epsilon$. This was expected, since the integral over k-space in (2.48) superficially showed a logarithmic divergence at $k = 0$. Secondly, we observe that $\tilde{p}_\mu \tilde{p}_\nu$ is obviously not the only IR divergent structure in the model: (2.48) is also transversal with respect to p_ρ and shows a quadratic IR divergence as well. Furthermore, letting (2.48) act on the photon propagator in Feynman gauge yields zero, but this need not be true in a more general gauge.

However, there are still many other graphs at 2-loop level which could produce similar results. There are in fact some convincing arguments why all these problematic terms should cancel: Slavnov, for instance, considered a special axial gauge $n_\mu = (0, 1, 0, 0)$ in [12] fixing $A_1 = 0$. In this special gauge the λ-A-A vertex term in the action $\lambda\theta^{\mu\nu} [A_\mu \stackrel{\star}{,} A_\nu]$ vanishes when choosing $\theta_{12} = -\theta_{21} = \theta$ as the only non-vanishing components. One can then easily work out that, assuming asymptotic boundary conditions, $A_2 = 0$ follows from Slavnov's constraint $\widetilde{\partial}^\mu A_\mu = 0$. Hence, none of the graphs including the λ-field exists and none of the problems discussed earlier is present. Chapter 3 will deal with this interesting feature in more detail.

Chapter 3

Symmetries in Non-Commutative Gauge Theories

3.1 A vector supersymmetry in Slavnov-modified NCGFT

3.1.1 BF models and the Slavnov term

In this chapter, we present a new approach by identifying the Slavnov term (1.21) with a topological term. As before, in order to preserve the unitarity of the S-matrix [29, 68, 69], we assume $\theta^{\mu\nu}$ to be space-like, i.e. $\theta^{0i} = 0$ in suitable space-time coordinates. Furthermore, we can choose the spatial coordinates in such a way that the only non-vanishing components of the θ-matrix are $\theta^{12} = -\theta^{21} = \theta$. Thus, the components θ^{ij} with $i, j \in \{1, 2\}$ can be written as $\theta^{ij} = \theta\epsilon^{ij}$, where ϵ^{ij} is the two-dimensional Levi-Civita symbol[1]. The Slavnov term (1.21) then reads $\frac{\theta}{2}\int d^4x\,\lambda\star\epsilon^{ij}F_{ij}$ so that it resembles the action for a 2-dimensional BF model with Abelian gauge group [78]

$$S_{\rm BF} = \frac{1}{2}\int d^2x\,\phi\,\epsilon^{ij}F_{ij}\,. \tag{3.1}$$

The latter model represents a topological quantum field theory and it is well known that such theories exhibit remarkable ultraviolet finiteness properties at the quantum level. In particular, the 3-dimensional Chern-Simons theory and the BF models in arbitrary space-time dimension represent fully finite quantum field theories. Their perturbative finiteness relies on the existence of a linear vector supersymmetry (VSUSY for short) which is generated by a set of fermionic charges forming a Lorentz vector [79, 80, 81, 75]. Together with the scalar fermionic charge of the BRST symmetry, they form a superalgebra of the Wess-Zumino type, i.e. a graded algebra which closes on-shell on space-time translations. More precisely, one has the following graded commutation relations between the BRST operator s and the operator δ_μ describing VSUSY:

$$\{s, \delta_\mu\}\,\Phi = \partial_\mu\Phi \;+\; \text{contact terms.} \tag{3.2}$$

Here, Φ collectively denotes the basic fields appearing in the topological model under consideration and *contact terms* are expressions which vanish if the equations of motion are used. In this context, the axial gauge plays a special role, since the topological field theories mentioned above are characterized by the complete absence of radiative corrections at the loop level in this gauge.

[1] We have $\epsilon_{ij}\epsilon^{kl} = \delta_i^k\delta_j^l - \delta_i^l\delta_j^k$.

We note that the non-commutative 2-dimensional BF model is characterized, at least in the Lorentz gauge, by a VSUSY of the same form as in the commutative case [82, 83].

3.1.2 Symmetries of NCGFT with Slavnov term in the axial gauge

We start with the action given in (2.27) in 3+1 dimensional Minkowski spacetime $\mathbb{M}^4_{\text{NC}}$. This time, however, we choose an *axial* gauge fixing by replacing N^μ with the constant axial gauge-fixing vector n^μ. Furthermore, we set $\alpha = 0$ and hence have no $B \star B$-term. With $\theta^{12} = -\theta^{21} = \theta$ as the only non-vanishing components of the θ-matrix, the Slavnov term reduces to

$$\frac{\theta}{2} \int d^4x\, \lambda \star \epsilon^{ij} F_{ij}\,, \tag{3.3}$$

i.e. (3.1) written as an integral over 4-dimensional non-commutative spacetime. The axial gauge-fixing vector n^μ appearing in the action will be chosen to lie in the plane of non-commuting coordinates, i.e. the (x^1, x^2)-plane, hence $n^0 = n^3 = 0$. We will see below that this allows us to find a VSUSY which is analogous to the one characterizing the 2-dimensional non-commutative BF model.

In order to distinguish the x^1, x^2-components from the other ones, we will use the following *notation for the remainder of this chapter:*
Greek indices $\mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}$ correspond to the 4-dimensional spacetime, Latin indices $i, j, k, l \in \{1, 2\}$ label the x^1, x^2-components and capital Latin indices $I, J, K, L \in \{0, 3\}$ denote the x^0, x^3-components.

For the particular choices of the axial gauge-fixing vector n^μ and the deformation matrix that we specified above, the action (2.27) reads

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \frac{\theta}{2} \lambda \star \epsilon^{ij} F_{ij} + B \star n^i A_i - \bar{c} \star n^i D_i c \right) . \tag{3.4}$$

It is worthwhile recalling that the star product is associative and has the properties (1.13) which allow us to perform cyclic permutations under an integral. This property will often be used in the following.

In order to simplify the notation, we will not spell out the star product symbol in the sequel: *all products between fields (or functions of fields) are understood to be star products.* Furthermore, we assume that the algebra of fields is graded by the ghost number. Accordingly, *all commutators are considered to be graded* with respect to this degree, e.g. $\frac{1}{2}[c, c]$ stands for $\frac{1}{2}\{c \overset{\star}{,} c\} = c \star c$ and $[A_\mu, c]$ stands for $[A_\mu \overset{\star}{,} c] = A_\mu \star c - c \star A_\mu$.

The action functional (3.4) is invariant under the *BRST transformations*

$$
\begin{aligned}
sA_\mu &= D_\mu c\,, & s\bar{c} &= B\,, \\
s\lambda &= -\mathrm{i}g\,[\lambda, c]\,, & sB &= 0\,, \\
sc &= \frac{\mathrm{i}g}{2}\,[c, c]\,,
\end{aligned}
\tag{3.5}
$$

which are nilpotent, i.e. $s^2\Phi = 0$ for $\Phi \in \{A_\mu, \lambda, c, \bar{c}, B\}$. The functional (3.4) is also invariant under the following *VSUSY transformations* which are labelled by a vector index $i \in \{1, 2\}$ and which only involve the x^1, x^2-components of the fields:

$$
\begin{aligned}
\delta_i A_J &= 0\,, & \delta_i c &= A_i\,, \\
\delta_i A_j &= 0\,, & \delta_i \bar{c} &= 0\,, \\
\delta_i \lambda &= \frac{\epsilon_{ij}}{\theta} n^j \bar{c}\,, & \delta_i B &= \partial_i \bar{c}\,.
\end{aligned}
\tag{3.6}
$$

The noteworthy feature of these transformations is that they relate the invariant and the gauge-fixing parts of the action (3.4). Since the operator δ_i lowers the ghost number by one unit, it represents an antiderivation (very much like the BRST operator s which raises the ghost number by one unit). Note that it is only the interplay of appropriate choices for $\theta^{\mu\nu}$ and n^μ which leads to the existence of the VSUSY. The crucial point is the choice of the vector n^μ lying in the plane of non-commuting coordinates.

The invariance of the action functional (3.4) under the transformations (3.6) is described by the *Ward identity*

$$
\mathcal{W}_i S \equiv \int d^4x \left(\partial_i \bar{c}\, \frac{\delta S}{\delta B} + A_i\, \frac{\delta S}{\delta c} + \frac{\epsilon_{ij}}{\theta} n^j \bar{c}\, \frac{\delta S}{\delta \lambda} \right) = 0\,. \tag{3.7}
$$

For later reference, we determine the equations of motion associated to the action (3.4). They are given by $\frac{\delta S}{\delta \Phi} = 0$ where Φ denotes a generic field. One finds that

$$\frac{\delta S}{\delta c} = -n^i D_i \bar{c} \,, \qquad \frac{\delta S}{\delta \bar{c}} = -n^i D_i c \,, \tag{3.8a}$$

$$\frac{\delta S}{\delta A_i} = D_\mu F^{\mu i} + \theta \epsilon^{ij} D_j \lambda + n^i B - \mathrm{i} g n^i [\bar{c}, c] \,, \tag{3.8b}$$

$$\frac{\delta S}{\delta A_I} = D_\mu F^{\mu I} \,, \qquad \frac{\delta S}{\delta \lambda} = \frac{\theta}{2} \epsilon^{ij} F_{ij} = \theta F_{12} \,, \tag{3.8c}$$

$$\frac{\delta S}{\delta B} = n^i A_i \,. \tag{3.8d}$$

The equation of motion for λ implements the Slavnov condition $\epsilon^{ij} F_{ij} = 0$, i.e. the vanishing of the third component of the magnetic field: $B_3 = 0$. The equation of motion for the Lagrange multiplier field B implements the axial gauge condition $n^i A_i = 0$.

From equations (3.5) and (3.6), we can deduce the graded commutation relations of the BRST and the VSUSY transformations. In using expressions (3.8), the results can be cast into the following form:

$$[s, s]\,\Phi = [\delta_i, \delta_j]\,\Phi = 0 \qquad \text{for } \Phi \in \{A_\mu, \lambda, c, \bar{c}, B\} \tag{3.9}$$

and

$$[s, \delta_i]\,\Phi = \partial_i \Phi \qquad \text{for } \Phi \in \{c, \bar{c}, B\} \,, \tag{3.10a}$$

$$[s, \delta_i]\,A_J = \partial_i A_J - F_{iJ} \,, \tag{3.10b}$$

$$[s, \delta_i]\,A_j = \partial_i A_j - \frac{\epsilon_{ij}}{\theta} \frac{\delta S}{\delta \lambda} \,, \tag{3.10c}$$

$$[s, \delta_i]\,\lambda = \partial_i \lambda + \frac{\epsilon_{ij}}{\theta} \frac{\delta S}{\delta A_j} - \frac{1}{\theta^2} D_i \frac{\delta S}{\delta \lambda} - \frac{\epsilon_{ij}}{\theta} D_K F^{Kj}. \tag{3.10d}$$

Since contact terms appear in the graded commutators, the algebra can only close on-shell. Note that, apart from the translations, the commutators (3.10b) and (3.10d) involve some extra contributions which are not related to equations of motion. One can readily verify that these terms represent a *new symmetry* of the action (3.4) defined by the following field variations:

$$\hat{\delta}_i A_J = -F_{iJ} \,, \qquad \hat{\delta}_i \lambda = -\frac{\epsilon_{ij}}{\theta} D_K F^{Kj} \,,$$
$$\hat{\delta}_i \Phi = 0 \qquad \text{for all other fields} \,. \tag{3.11}$$

Noting that the transformations (3.11) and the Bianchi identity imply

$$\hat{\delta}_i F_{JK} = -D_i F_{JK} \,, \qquad \hat{\delta}_i F_{jK} = -D_i F_{jK} - D_K F_{ij} \,,$$

the proof of this new symmetry becomes straightforward. Also observe that the operator $\hat{\delta}_i$ does not change the ghost number and we therefore refer to this symmetry as a *bosonic vectorial symmetry*.

Together with the BRST transformations, the VSUSY and the translations in the (x^1, x^2)-plane $\partial_i\Phi$, this new bosonic vectorial symmetry forms an algebra which actually closes on-shell: the translations commute with all transformations and

$$\left.\begin{aligned} [s,s]\,\Phi = [s,\hat{\delta}_j]\,\Phi = 0 \\ [\delta_i,\delta_j]\,\Phi = [\delta_i,\hat{\delta}_j]\,\Phi = 0 \end{aligned}\right\} \quad \text{for all fields } \Phi\,, \tag{3.12}$$

$$[s,\delta_i]\,\Phi = \partial_i\Phi + \hat{\delta}_i\Phi \qquad \text{for } \Phi \in \{A_J, c, \bar{c}, B\}\,, \tag{3.13a}$$

$$[s,\delta_i]\,A_j = \partial_i A_j + \hat{\delta}_i A_j - \frac{\epsilon_{ij}}{\theta}\frac{\delta S}{\delta\lambda}\,, \tag{3.13b}$$

$$[s,\delta_i]\,\lambda = \partial_i\lambda + \hat{\delta}_i\lambda + \frac{\epsilon_{ij}}{\theta}\frac{\delta S}{\delta A_j} - \frac{1}{\theta^2}D_i\frac{\delta S}{\delta\lambda}\,, \tag{3.13c}$$

and

$$\begin{aligned} [\hat{\delta}_i,\hat{\delta}_j]\,A_J &= \frac{\epsilon_{ij}}{\theta}D_J\frac{\delta S}{\delta\lambda}\,, \\ [\hat{\delta}_i,\hat{\delta}_j]\,\lambda &= \frac{\epsilon_{ij}}{\theta}D_J\frac{\delta S}{\delta A_J}\,, \\ [\hat{\delta}_i,\hat{\delta}_j]\,\Phi &= 0 \qquad\qquad \text{for } \Phi \in \{A_i, c, \bar{c}, B\}\,. \end{aligned} \tag{3.14}$$

3.1.3 Generalized BRST operator

We can combine the various symmetry operators defined above into a generalized BRST operator that we denote by $\triangle$:

$$\triangle \equiv s + \xi\cdot\partial + \varepsilon^i\delta_i + \mu^i\hat{\delta}_i \qquad \text{with } \; \xi\cdot\partial \equiv \xi^i\partial_i\,. \tag{3.15}$$

Here, the constant parameters ξ^i and μ^i have ghost number 1 and ε^i has ghost number 2. The induced field variations read

$$\triangle A_i = D_i c + \xi \cdot \partial A_i \,, \tag{3.16a}$$

$$\triangle A_J = D_J c + \xi \cdot \partial A_J + \mu^i F_{Ji} \,, \tag{3.16b}$$

$$\triangle \lambda = -\mathrm{i}g\,[\lambda, c] + \xi \cdot \partial \lambda + \varepsilon^i \frac{\epsilon_{ij}}{\theta} n^j \bar{c} + \mu^i \frac{\epsilon_{ij}}{\theta} D_K F^{jK} \,, \tag{3.16c}$$

$$\triangle c = \frac{\mathrm{i}g}{2}\,[c, c] + \xi \cdot \partial c + \varepsilon^i A_i \,, \tag{3.16d}$$

$$\triangle \bar{c} = B + \xi \cdot \partial \bar{c} \,, \tag{3.16e}$$

$$\triangle B = \xi \cdot \partial B + \varepsilon \cdot \partial \bar{c} \,, \tag{3.16f}$$

and imply

$$\triangle F_{iJ} = -\mathrm{i}g\,[F_{iJ}, c] + \xi \cdot \partial F_{iJ} - \mu^k D_i F_{kJ} \,.$$

Imposing that the parameters ξ^i, ε^i and μ^i transform as

$$\triangle \xi^i = \triangle \mu^i = -\varepsilon^i \,, \qquad \triangle \varepsilon^i = 0 \,, \tag{3.17}$$

we conclude that the operator (3.15) is nilpotent on-shell:

$$\triangle^2 A_i = \varepsilon^j \frac{\epsilon_{ij}}{\theta} \frac{\delta S}{\delta \lambda} \,, \tag{3.18a}$$

$$\triangle^2 A_J = \frac{\mu^i \mu^j}{2} \frac{\epsilon_{ij}}{\theta} D_J \frac{\delta S}{\delta \lambda} \,, \tag{3.18b}$$

$$\triangle^2 \lambda = \frac{\mu^i \mu^j}{2} \frac{\epsilon_{ij}}{\theta} D_J \frac{\delta S}{\delta A_J} + \varepsilon^i \frac{\epsilon_{ij}}{\theta} \frac{\delta S}{\delta A_j} - \varepsilon^i \frac{1}{\theta^2} D_i \frac{\delta S}{\delta \lambda} \,, \tag{3.18c}$$

$$\triangle^2 c = \triangle^2 \bar{c} = \triangle^2 B = 0 \,. \tag{3.18d}$$

3.1.4 Slavnov-Taylor and Ward identities

The Ward identities corresponding to the various symmetries of the action can be gathered into a Slavnov-Taylor (ST) identity expressing the invariance of an appropriate total action S_{tot} under the generalized BRST transformations (3.16), (3.17). In this respect, we introduce an external field Φ^* (i.e. an antifield in the terminology of Batalin and Vilkovisky [84, 85]) for each field $\Phi \in \{A_\mu, \lambda, c\}$ since the latter transform non-linearly under the BRST variations — see e.g. reference [75]. We note that the external sources $A^{*\mu}$ and λ^* have ghost number -1 whereas c^* has ghost number -2. (Appendix C gives a short introduction to the BV-formalism. See also [86, 87, 88] for a recent review.)

ST identity

In view of the transformation laws (3.16) and (3.17), the *ST identity* reads

$$0 = \mathcal{S}(S_{\rm tot}) \equiv \int d^4x \left\{ \sum_{\Phi \in \{A_\mu, \lambda, c\}} \frac{\delta S_{\rm tot}}{\delta \Phi^*} \frac{\delta S_{\rm tot}}{\delta \Phi} + (B + \xi \cdot \partial \bar{c}) \, \frac{\delta S_{\rm tot}}{\delta \bar{c}} \right.$$
$$\left. + (\xi \cdot \partial B + \varepsilon \cdot \partial \bar{c}) \, \frac{\delta S_{\rm tot}}{\delta B} \right\} - \varepsilon^i \left(\frac{\partial S_{\rm tot}}{\partial \xi^i} + \frac{\partial S_{\rm tot}}{\partial \mu^i} \right) \, . \quad (3.19)$$

This functional equation is supplemented with the *gauge-fixing condition*

$$\frac{\delta S_{\rm tot}}{\delta B} = n^i A_i \, . \quad (3.20)$$

By differentiating the ST identity with respect to the field B, one finds

$$0 = \frac{\delta}{\delta B} \mathcal{S}(S_{\rm tot}) = \mathcal{G} S_{\rm tot} - \xi \cdot \partial \, \frac{\delta S_{\rm tot}}{\delta B} \, , \qquad \text{with} \quad \mathcal{G} \equiv \frac{\delta}{\delta \bar{c}} + n^i \frac{\delta}{\delta A^{*i}} \, ,$$

i.e., by virtue of (3.20), the so-called *ghost equation*:

$$\mathcal{G} S_{\rm tot} = \xi \cdot \partial \, (n^i A_i) \, . \quad (3.21)$$

The associated homogeneous equation $\mathcal{G} \bar{S} = 0$ is solved by functionals which we denote $\bar{S}[\hat{A}^{*i}, \dots]$ and which depend on the variables A^{*i} and $\bar{c}$ only through the shifted antifield

$$\hat{A}^{*i} \equiv A^{*i} - n^i \bar{c} \, . \quad (3.22)$$

Thus, the functional $S_{\rm tot}[A, \lambda, c, \bar{c}, B \, ; A^*, \lambda^*, c^*; \xi, \mu, \varepsilon]$ which solves both the ghost equation (3.21) and the gauge-fixing condition (3.20) has the form

$$S_{\rm tot} = \int d^4x \, (B + \xi \cdot \partial \bar{c}) n^i A_i \; + \; \bar{S}[A, \lambda, c \, ; \hat{A}^{*i}, A^{*J}, \lambda^*, c^*; \, \xi, \mu, \varepsilon] \, , \quad (3.23)$$

where the B-dependent term ensures the validity of condition (3.20).

By substituting expression (3.23) into the ST identity (3.19), we conclude that the latter equation is satisfied if $\bar{S}$ solves the *reduced ST identity*

$$0 = \mathcal{B}(\bar{S}) \equiv \sum_{\Phi \in \{A_\mu, \lambda, c\}} \int d^4x \, \frac{\delta \bar{S}}{\delta \hat{\Phi}^*} \frac{\delta \bar{S}}{\delta \Phi} \; - \; \varepsilon^i \, \left(\frac{\partial \bar{S}}{\partial \xi^i} \, + \, \frac{\partial \bar{S}}{\partial \mu^i} \right) \, . \quad (3.24)$$

Here, $\hat{\Phi}^*$ collectively denotes all antifields, but with A^{*i} replaced by the shifted antifield (3.22). Following standard practice [75], we introduce the following notation for the external sources in order to make the formulas clearer:

$$\rho^\mu \equiv A^{*\mu}\,, \quad \gamma \equiv \lambda^*\,, \quad \sigma \equiv c^*\,, \quad \hat{\rho}^i = \hat{A}^{*i}\,.$$

It can be verified along the usual lines (e.g. see [75]) that the *solution of the reduced ST identity* (3.24) is given by[2]

$$\begin{aligned}
\bar{S} = \int d^4x \Bigg\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{2}\lambda\epsilon^{ij}F_{ij} \\
&+ \hat{\rho}^i\left(D_i c + \xi\cdot\partial A_i\right) + \rho^J\left(D_J c + \xi\cdot\partial A_J + \mu^i F_{Ji}\right) \\
&+ \gamma\left(-\mathrm{i}g[\lambda,c] + \xi\cdot\partial\lambda + \mu^i\frac{\epsilon_{ij}}{\theta}D_K F^{jK}\right) + \sigma\left(\frac{\mathrm{i}g}{2}\left[c,c\right] + \xi\cdot\partial c + \varepsilon^i A_i\right) \\
&+ \left(\frac{\mu^i\mu^j}{2}\,\frac{\epsilon_{ij}}{\theta}(D_J\rho^J) + \varepsilon^i\frac{\epsilon_{ij}}{\theta}\,\hat{\rho}^j - \varepsilon^i\frac{1}{2\theta^2}\left(D_i\gamma\right)\right)\gamma\Bigg\}\,. \qquad (3.25)
\end{aligned}$$

Note that

$$\bar{S} = S_{\mathrm{inv}} + S_{\mathrm{antifields}} + S_{\mathrm{quadratic}}\,,$$

where S_{inv} is the gauge invariant part (i.e. the first two terms) of the action (3.4), $S_{\mathrm{antifields}}$ represents the linear coupling of the shifted antifields $\hat{\Phi}^*$ to the generalized BRST transformations (3.16a-d) (the $\bar{c}$-dependent term being omitted) and $S_{\mathrm{quadratic}}$, which is quadratic in the shifted antifields, reflects the contact terms appearing in the closure relations (3.18).

The antighost and ghost equations

Differentiating the total action (3.23)-(3.25) with respect to the ghost field, one obtains

$$\frac{\delta S_{\mathrm{tot}}}{\delta c} = D_i(\rho^i - n^i\bar{c}) + D_J\rho^J - \mathrm{i}g[\lambda,\gamma] + \mathrm{i}g[c,\sigma] + \xi\cdot\partial\sigma\,.$$

By substituting the gauge-fixing condition (3.20) in the $n^i A_i$-dependent term on the right-hand side, we obtain the functional identity

$$\frac{\delta S_{\mathrm{tot}}}{\delta c} + \mathrm{i}g\left[\bar{c}, \frac{\delta S_{\mathrm{tot}}}{\delta B}\right] + n\cdot\partial\bar{c} = D_\mu\rho^\mu - \mathrm{i}g[\lambda,\gamma] + \mathrm{i}g[c,\sigma] + \xi\cdot\partial\sigma\,, \qquad (3.26)$$

[2]Simply insert (3.25) into (3.24) to check that it really solves the ST identity.

which is called the *antighost equation* [75, 89]. This equation makes sense as an identity for the action functional since the right-hand side is linear in the quantum fields. Moreover it is local, i.e. not integrated, in space-time.

Similarly, differentiating the total action with respect to the antighost field, one obtains the *ghost field equation* in functional form:

$$\frac{\delta S_{\text{tot}}}{\delta \bar{c}} + \mathrm{i}g\left[c, \frac{\delta S_{\text{tot}}}{\delta B}\right] + n \cdot \partial c - \xi \cdot \partial \frac{\delta S_{\text{tot}}}{\delta B} = -\varepsilon^i \frac{\epsilon_{ij}}{\theta} n^j \gamma\,. \tag{3.27}$$

The fact that both the ghost and the antighost field equations can be cast as such local functional identities is an expression of the ghost freedom of gauge theories quantized in an axial gauge [90, 91].

Ward identities

The Ward identities describing the (non-)invariance of S_{tot} under the VSUSY-variations δ_i, the vectorial symmetry transformations $\hat{\delta}_i$ and the translations ∂_i can be derived from the ST identity (3.19) by differentiating this identity with respect to the corresponding constant ghosts ε^i, μ^i and ξ^i, respectively.

For instance, by differentiating (3.19) with respect to ξ^i and by taking the gauge-fixing condition (3.20) into account, we obtain the *Ward identity for translation symmetry*:

$$0 = \frac{\partial}{\partial \xi^i} \mathcal{S}(S_{\text{tot}}) = \sum_{\varphi} \int d^4x\; \partial_i \varphi\, \frac{\delta S_{\text{tot}}}{\delta \varphi}\,, \tag{3.28}$$

where $\varphi \in \{A_\mu, \lambda, c, \bar{c}, B; A^*_\mu, \lambda^*, c^*\}$.

By differentiating (3.19) with respect to ε^i, we obtain

$$\begin{aligned} 0 = \frac{\partial}{\partial \varepsilon^i} \mathcal{S}(S_{\text{tot}}) = -\frac{\partial S_{\text{tot}}}{\partial \xi^i} - \frac{\partial S_{\text{tot}}}{\partial \mu^i} + \int d^4x \Bigg\{ \partial_i \bar{c}\, \frac{\delta S_{\text{tot}}}{\delta B} + (B + \xi \cdot \partial \bar{c})\, \frac{\delta}{\delta \bar{c}} \frac{\partial S_{\text{tot}}}{\partial \varepsilon^i} \\ + \sum_{\Phi} \left[\left(\frac{\delta}{\delta \Phi^*} \frac{\partial S_{\text{tot}}}{\partial \varepsilon^i} \right) \frac{\delta S_{\text{tot}}}{\delta \Phi} + \frac{\delta S_{\text{tot}}}{\delta \Phi^*} \left(\frac{\delta}{\delta \Phi} \frac{\partial S_{\text{tot}}}{\partial \varepsilon^i} \right) \right] \Bigg\}. \end{aligned} \tag{3.29}$$

From (3.23) and (3.25), we deduce that

$$\frac{\partial S_{\text{tot}}}{\partial \varepsilon^i} = \int d^4x \left\{ \sigma A_i + \frac{\epsilon_{ij}}{\theta}\hat{\rho}^j\gamma + \frac{1}{2\theta^2}\gamma D_i\gamma \right\} , \tag{3.30a}$$

$$\frac{\partial S_{\text{tot}}}{\partial \xi^i} = \int d^4x \left\{ -\rho^\mu \partial_i A_\mu - \gamma\partial_i\lambda + \sigma\partial_i c \right\} , \tag{3.30b}$$

$$\frac{\partial S_{\text{tot}}}{\partial \mu^i} = \int d^4x \left\{ F_{iJ}\rho^J + \frac{\epsilon_{ij}}{\theta}\left(D_K F^{Kj}\right)\gamma + \frac{\epsilon_{ij}}{\theta}\mu^j\left(D_J\rho^J\right)\gamma \right\} . \tag{3.30c}$$

Notice that the right-hand sides of the first two equations are linear in the quantum fields, which is not the case for the third one. Insertion of these expressions into equation (3.29) yields a *broken Ward identity for the VSUSY:*

$$\mathcal{W}_i S_{\text{tot}} = \Delta_i \,, \tag{3.31}$$

with

$$\mathcal{W}_i S_{\text{tot}} = \int d^4x \left\{ \partial_i\bar{c}\,\frac{\delta S_{\text{tot}}}{\delta B} + A_i\,\frac{\delta S_{\text{tot}}}{\delta c} + \left(\frac{\epsilon_{ij}}{\theta}\left(n^j\bar{c} - \rho^j\right) + \frac{1}{\theta^2}D_i\gamma\right)\frac{\delta S_{\text{tot}}}{\delta\lambda} \right.$$
$$\left. + \gamma\frac{\epsilon_{ij}}{\theta}\,\frac{\delta S_{\text{tot}}}{\delta A_j} + \left(\sigma + \frac{\mathrm{i}g}{\theta^2}\gamma\gamma\right)\frac{\delta S_{\text{tot}}}{\delta\rho^i} \right\} \tag{3.32}$$

and

$$\Delta_i = \frac{\partial S_{\text{tot}}}{\partial \xi^i} + \frac{\partial S_{\text{tot}}}{\partial \mu^i} + \int d^4x\,\frac{\epsilon_{ij}}{\theta}n^j\left(B + \xi\cdot\partial\bar{c}\right)\gamma\,. \tag{3.33}$$

More explicitly, $\Delta_i = \Delta_i\Big|_{\xi=\mu=0} + B_i[\xi,\mu]$ with

$$\Delta_i\Big|_{\xi=\mu=0} = \int d^4x \left\{ \sigma\partial_i c - \rho^\mu\partial_i A_\mu - \gamma\partial_i\lambda - \rho^J F_{Ji} + \gamma\frac{\epsilon_{ij}}{\theta}\left(n^j B - D_K F^{jK}\right) \right\}$$
$$B_i[\xi,\mu] = \int d^4x \left\{ \xi\cdot\partial\bar{c}\,\frac{\epsilon_{ij}}{\theta}n^j\gamma + \frac{\epsilon_{ij}}{\theta}\mu^j\left(D_J\rho^J\right)\gamma \right\} . \tag{3.34}$$

Several remarks concerning the results (3.31)-(3.34) are in order. First, we note that the field variations given by (3.32) extend the VSUSY transformations (3.6) by source dependent terms. It is the presence of the sources which leads to a breaking Δ_i of the VSUSY (cf. the unbroken Ward identity (3.7) for the gauge-fixed action). Secondly, we remark that the breaking of the VSUSY is *non-linear* in the quantum fields: The non-linear contributions are contained in $\Delta_i\Big|_{\xi=\mu=0}$ and given by

$$-\int d^4x \left\{ \rho^J F_{Ji} + \gamma\frac{\epsilon_{ij}}{\theta}D_K F^{jK} \right\} = -\int d^4x \left\{ \rho^J\left(\hat{\delta}_i A_J\right) + \gamma\left(\hat{\delta}_i\lambda\right) \right\} ,$$

where $\hat{\delta}_i$ are the vectorial symmetry transformations (3.11). However, these non-linear breakings (which could jeopardize a non-ambiguous definition of the theory) are contained in the derivative $\partial S_{\rm tot}/\partial\mu^i$ and are therefore functionally well-defined.

Finally, we come to the Ward identity for the bosonic vectorial symmetry $\hat{\delta}_i$. In differentiating the ST identity (3.19) with respect to μ^i and using (3.30c), one finds

$$\begin{aligned}\int d^4x \Bigg\{ & - F_{iJ}\,\frac{\delta S_{\rm tot}}{\delta A_J} - \frac{\epsilon_{ij}}{\theta}\left(D_K F^{Kj} + \mu^j\, D_K\rho^K\right)\frac{\delta S_{\rm tot}}{\delta\lambda} + D_K\rho^K\,\frac{\delta S_{\rm tot}}{\delta\rho^i} \\ & + \frac{\epsilon_{ij}}{\theta} D_K D^K \gamma\,\frac{\delta S_{\rm tot}}{\delta\rho_j} - \left(D_i\rho^I + \frac{\epsilon_{ij}}{\theta} D^j D^I\gamma + \mathrm{i}g\frac{\epsilon_{ij}}{\theta}\left[F^{Ij},\gamma\right]\right)\frac{\delta S_{\rm tot}}{\delta\rho^I} \\ & + \mathrm{i}g\frac{\epsilon_{ij}}{\theta}\mu^j\left[\rho^I,\gamma\right]\frac{\delta S_{\rm tot}}{\delta\rho^I}\Bigg\} = -\int d^4x\,\frac{\epsilon_{ij}}{\theta}\,\varepsilon^j\left(D_K\rho^K\right)\gamma\,. \end{aligned} \tag{3.35}$$

In this case we have a breaking which is linear in the quantum fields and vanishes if we set the external sources to zero.

3.1.5 Consequences of VSUSY

The generating functional Z^c of the connected Green functions is given by the Legendre transform[3]

$$Z^c[j_{\{A,\lambda,B,c,\bar{c}\}}] = \Gamma[A,\lambda,B,c,\bar{c}] + \int d^4x\,(j_A^\mu A_\mu + j_\lambda\lambda + j_B B + j_c c + j_{\bar{c}}\bar{c})\,. \tag{3.36}$$

Thus, we have the usual relations

$$\begin{aligned} &\frac{\delta Z^c}{\delta j_A^\mu} = A_\mu\,, \quad \frac{\delta Z^c}{\delta j_\lambda} = \lambda\,, \quad \frac{\delta Z^c}{\delta j_B} = B\,, \quad \frac{\delta Z^c}{\delta j_c} = c\,, \quad \frac{\delta Z^c}{\delta j_{\bar{c}}} = \bar{c}\,, \\ &\frac{\delta\Gamma}{\delta A_\mu} = -j_A^\mu\,, \quad \frac{\delta\Gamma}{\delta\lambda} = -j_\lambda\,, \quad \frac{\delta\Gamma}{\delta B} = -j_B\,, \quad \frac{\delta\Gamma}{\delta c} = j_c\,, \quad \frac{\delta\Gamma}{\delta\bar{c}} = j_{\bar{c}}\,, \end{aligned} \tag{3.37}$$

and

$$\frac{\delta Z^c}{\delta\Phi^*} = \frac{\delta\Gamma}{\delta\Phi^*}\,, \qquad \frac{\partial Z^c}{\partial\xi^i} = \frac{\partial\Gamma}{\partial\xi^i}\,, \qquad \frac{\partial Z^c}{\partial\varepsilon^i} = \frac{\partial\Gamma}{\partial\varepsilon^i}\,, \qquad \frac{\partial Z^c}{\partial\mu^i} = \frac{\partial\Gamma}{\partial\mu^i}\,. \tag{3.38}$$

[3] In the "classical approximation", the generating functional Γ of the one-particle-irreducible Green functions is equal to the total classical action $S_{\rm tot}$. Its Legendre transform Z^c generates the connected Green functions in the tree graph approximation.

For vanishing antifields, the *Ward identity* describing the VSUSY (3.31) becomes in terms of Z^c:

$$\mathcal{W}_i Z^c = \int d^4x \left\{ j_B \, \partial_i \frac{\delta Z^c}{\delta j_{\bar{c}}} - j_c \, \frac{\delta Z^c}{\delta j^i_A} + \frac{\epsilon_{ij}}{\theta} n^j j_\lambda \, \frac{\delta Z^c}{\delta j_{\bar{c}}} \right\} = 0 \,. \tag{3.39}$$

By varying (3.39) with respect to the appropriate sources, one obtains the following relations for the two-point functions (i.e. the free propagators):

$$\left.\frac{\delta^2 Z^c}{\delta j^i_A \delta j_\lambda}\right|_{j=0} = -\frac{\epsilon_{ij}}{\theta} n^j \left.\frac{\delta^2 Z^c}{\delta j_{\bar{c}} \delta j_c}\right|_{j=0} , \qquad \left.\frac{\delta^2 Z^c}{\delta j^i_A \delta j^\nu_A}\right|_{j=0} = 0 \,. \tag{3.40}$$

The *gauge-fixing condition* (3.20) is equivalent to

$$-j_B = n^i \frac{\delta Z^c}{\delta j^i_A} \,,$$

from which it follows that

$$n^i \left.\frac{\delta^2 Z^c}{\delta j_B(y) \delta j^i_A(x)}\right|_{j=0} = -\delta^{(4)}(x-y) \,. \tag{3.41}$$

For vanishing antifields, the *antighost equation* (3.26) can be written as

$$-n \cdot \partial \frac{\delta Z^c}{\delta j_{\bar{c}}} - \mathrm{i} g \left[j_B, \frac{\delta Z^c}{\delta j_{\bar{c}}} \right] = j_c \,,$$

and by varying this equation with respect to j_c, one concludes that

$$n \cdot \partial \left.\frac{\delta^2 Z^c}{\delta j_c(x) \delta j_{\bar{c}}(y)}\right|_{j=0} = -\delta^{(4)}(x-y) \,. \tag{3.42}$$

Note that the same result may be obtained from the *ghost equation* (3.27) which, in terms of Z^c (for vanishing antifields and $\xi^i = 0$), reads:

$$-n \cdot \partial \frac{\delta Z^c}{\delta j_c} - \mathrm{i} g \left[j_B, \frac{\delta Z^c}{\delta j_c} \right] = j_{\bar{c}} \,. \tag{3.43}$$

In momentum space, the free propagators of the theory are given by (cf. Appendix A, Table A.1 with $\xi \to -\infty$)

$$\mathrm{i}\Delta^{c\bar{c}}(k) = -\frac{1}{nk} \,, \qquad \mathrm{i}\Delta^{AB}_\mu(k) = \frac{\mathrm{i}k_\mu}{nk} \,, \tag{3.44a}$$

$$\mathrm{i}\Delta^{A\lambda}_\mu(k) = \frac{1}{\tilde{k}^2} \left(\tilde{k}_\mu - k_\mu \frac{n\tilde{k}}{nk} \right) , \tag{3.44b}$$

$$\mathrm{i}\Delta^{A}_{\mu\nu}(k) = \frac{-\mathrm{i}}{k^2} \left[g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{nk} + a \frac{k_\mu k_\nu}{(nk)^2} + b'(k_\mu \tilde{k}_\nu + k_\nu \tilde{k}_\mu) - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right] , \tag{3.44c}$$

with $(g_{\mu\nu}) = \mathrm{diag}\,(1,-1,-1,-1)$ and[4]

$$\tilde{k}_i \equiv \theta\epsilon_{ij}k^j\,, \qquad \tilde{k}_J \equiv 0\,,$$
$$a \equiv n^2 - \frac{(n\tilde{k})^2}{\tilde{k}^2}\,, \qquad b' \equiv \frac{n\tilde{k}}{(nk)\tilde{k}^2}\,. \tag{3.45}$$

One can easily verify that these propagators obey the conditions (3.40), (3.41) and (3.42).

As we are now going to show, the remarkable outcome of the identities (3.40), (3.41) and (3.42) is that they are sufficient for killing all possible IR divergences in the radiative corrections. The second relation in (3.40), which states that the photon propagators $\Delta^A_{i\nu}$ vanish, has an important consequence. Indeed, since the λAA-vertex is proportional to θ^{ij}, all Feynman graphs which include a λAA-vertex contracted with an internal photon line *must cancel* (cf. Figure 3.1). But since it is obviously impossible to construct

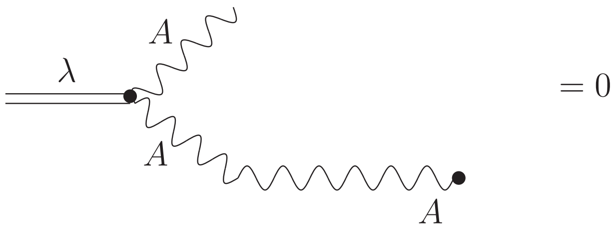


Figure 3.1: The λAA-vertex contracted with a photon propagator vanishes.

a Feynman graph (except for a tree graph) including λAA-vertices which do not couple to internal photon propagators, *all loop corrections involving the λAA-vertex have to vanish*! Note that a mixed photon-λ propagator contracted with a λAA-vertex leads to the necessity of another λAA-vertex, and so in order to build a closed loop, photon propagators are necessary (see Figure 3.2). Hence, the Feynman rules involving the λ-field do not enter the loop corrections of the photon n-point function. In particular, the IR-problematic graph mentioned in Chapter 2.2.4 and depicted in Figure 3.3 is absent for our choice of gauge. Now that we have shown that the λ-field plays no role in the radiative corrections of the gauge field, the absence of IR divergences follows from the line of arguments given in reference [12].

From these considerations, it should also become obvious that all loop corrections to the λ-propagator and the mixed λ-photon propagator vanish,

[4] We have $\tilde{k}^2 = -\theta^2(k_1^2+k_2^2)$, $nk = -(n_1k_1+n_2k_2)$, $n\tilde{k} = \theta(n_1k_2-n_2k_1)$ and $\mathrm{i}\Delta^{AB}_{\mu}(x-y) = -\mathrm{i}\frac{\delta^2 Z^c}{\delta j_B(y)\delta j^\mu_A(x)}\Big|_{j=0}$.

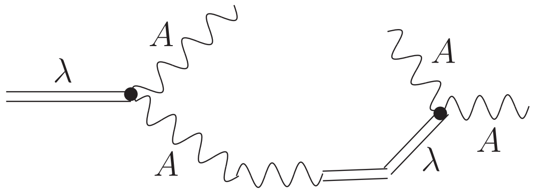


Figure 3.2: Building a Feynman loop graph with a λAA-vertex is impossible without a photon propagator.

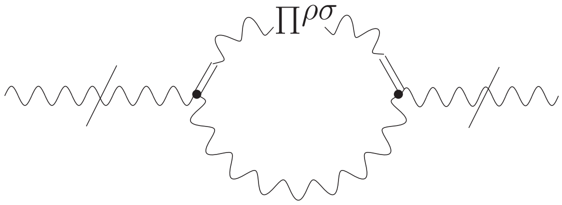


Figure 3.3: The "problematic" 2-loop graph vanishes in this case.

leaving the tree approximation as the exact solution for this sector. Furthermore, equations (3.41) and (3.42) provide exact solutions to the AB propagator and the ghost propagator [78]. Notice also that the first of equations (3.40) is consistent with the considerations above: it gives us the exact solution for the mixed λ-photon propagator once the solution for the ghost propagator is found from (3.42).

3.1.6 Conclusion

As discussed in Section 3.1.2, the $U(1)$-NCGFT with Slavnov term and with an appropriate axial gauge-fixing exhibits a far richer symmetry structure than initially expected. In particular, it admits a linear VSUSY which is similar to the one present in the 2-dimensional BF model, provided one chooses the deformation matrix $\theta^{\mu\nu}$ to be space-like and the axial gauge-fixing vector n^μ to lie in the plane of the non-commuting coordinates. While this VSUSY yields a superalgebra (which includes the BRST operator s and the translation generator in the non-commutative plane), it differs from the one present in the non-commutative 2-dimensional BF model by the fact that it contains an additional non-linear vectorial symmetry (given by the transformation laws (3.11)).

As a consequence of the identities for the free propagators, which follow from the VSUSY, all loop corrections become independent of the λAA-vertex. This is the reason why the theory in our particular space-like axial gauge is finite, as pointed out by Slavnov in reference [12].

Thus, the absence of IR singularities in a NCGFT can be achieved by other means than extending it to a Poincaré supersymmetric theory[5] (as was already emphasized by Slavnov [12]), namely by modifying it physically by adding the Slavnov term (which leads to the presence of VSUSY that is characteristic for a class of gauge-fixings). One may note that a supersymmetry is again responsible for the cancellation of IR singularities. But, in contrast to the Poincaré supersymmetry which is physical, VSUSY is not physical: Its existence follows from the specific choice we have made for the gauge-fixing[6].

3.2 A generalization of Slavnov-extended NCGFT

We recall that the (gauge-)invariant action for a non-commutative $U(1)$ gauge field, enhanced by the extension proposed by Slavnov [11], is given by

$$S_{\text{inv}} = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\lambda}{2} \theta^{\mu\nu} F_{\mu\nu} \right], \tag{3.46}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g \left[A_\mu, A_\nu \right] \tag{3.47}$$

denotes the field strength of the gauge connection and the signature of space-time is once more given by $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Furthermore, we keep the simplified notation of omitting the star product symbols and using graded commutators. In the previous chapter (cf. [4]) the action (3.46) was interpreted as a topological 2-dimensional BF model coupled to Maxwell theory. However, the price which had to be paid for this identification was a restric-

[5]The role of Poincaré supersymmetry for the cancellation of IR singularities has been extensively studied in the literature — see [92] for a review and further references.

[6]It has been shown [47, 93] for non-commutative $\mathbb{R}^3$ that Chern-Simons models without Poincaré supersymmetry may also be free of the IR singularities, depending on the gauge-fixing choice and on the coupling with matter.

tion of the (matrix-valued) deformation parameter to the special form,

$$\theta^{\mu\nu} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.48}$$

which made it possible to write the Slavnov term as $\frac{\lambda}{2}\epsilon^{ab}F_{ab}$ with $a, b \in \{1, 2\}$. In this section, however, we propose a possibility to consider a more general $\theta_{\mu\nu}$ without spoiling the topological nature of the theory. To this end we take $\theta^{\mu\nu}$ to be completely arbitrary, at least in its spatial components[7],

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^{12} & \theta^{13} \\ 0 & -\theta^{12} & 0 & \theta^{23} \\ 0 & -\theta^{13} & -\theta^{23} & 0 \end{pmatrix}, \tag{3.49}$$

and remember that the Slavnov term was originally designed to introduce the following constraint on the field strength:

$$\theta^{12}F_{12} + \theta^{13}F_{13} + \theta^{23}F_{23} = 0. \tag{3.50}$$

We now impose the more restrictive constraint that each of the three terms vanishes by itself and implement this new restriction with the help of three multiplier fields $U_i(x)$ with $i \in \{1, 2, 3\}$ in the following way:

$$\int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + U_3\theta^{12}F_{12} + U_2\theta^{13}F_{13} + U_1\theta^{23}F_{23} \right]. \tag{3.51}$$

Upon introducing the rescaled fields

$$\lambda_1 \equiv \theta^{23}U_1, \qquad \lambda_2 \equiv -\theta^{13}U_2, \qquad \lambda_3 \equiv \theta^{12}U_3, \tag{3.52}$$

the invariant action can be rewritten in the form [6]

$$S_{\text{inv}} = \int d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon^{ijk}F_{ij}\lambda_k \right], \tag{3.53}$$

which is analogous to a 3-dimensional BF model coupled to Maxwell theory. Throughout this chapter we will use the convenient notation where Greek indices μ, ν, ρ, σ once more take the values $0, 1, 2, 3$ while, in contrast to the previous section, Latin indices denote all spatial directions $i, j, k, l \in \{1, 2, 3\}$.

[7] We assume the spatial coordinates commute with time ($\theta^{0\mu} = 0$) in order to avoid problems with unitarity, as already mentioned.

In fact, the action (3.53) is invariant under two gauge symmetries. The first one is given by

$$\begin{aligned}\delta_{g1}A_\mu &= D_\mu\Lambda,\\ \delta_{g1}\lambda_k &= -\mathrm{i}g[\lambda_k,\Lambda],\end{aligned}\tag{3.54}$$

and the second gauge symmetry reads

$$\begin{aligned}\delta_{g2}A_\mu &= 0,\\ \delta_{g2}\lambda_k &= D_k\Lambda',\end{aligned}\tag{3.55}$$

where Λ, Λ' are arbitrary gauge parameters. The covariant derivative D_μ has already been defined in (2.4).

Observe that Λ' is a scalar and hence this model does not contain any so-called *zero modes*, which are typical for $n \geq 4$-dimensional BF models (where Λ' would be a $(n-3)$-form, cf. [94, 75]). For the gauge fixing procedure we assume once more that the algebra of fields is graded by the ghost number and, accordingly, *all commutators are considered to be graded*. At this point we would also like to draw attention to the fact that the deformation parameter $\theta^{\mu\nu}$ does not appear explicitly in the Slavnov term of the action (3.53) (apart from its appearance in the star products, of course). Therefore, it will make no difference which explicit form is chosen for $\theta^{\mu\nu}$ in the upcoming considerations (i.e. we are free to choose any value for the entries θ_{12}, θ_{13} and θ_{23} in (3.49)). The only restriction we need to take into account is that $\theta^{0\mu}=0$ for the reasons already mentioned.

We now continue by adding gauge fixing terms to our model in a BRST invariant way. To this end we fix both gauge symmetries using axial gauges following [95]:

$$\begin{aligned}S = \int d^4x \Big\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon^{ijk}F_{ij}\lambda_k + Bn^iA_i + dm^i\lambda_i - \bar{c}n^iD_ic \\ &- \bar{\phi}m^i\big(D_i\phi - \mathrm{i}g\,[\lambda_i, c]\,\big)\Big\}.\end{aligned}\tag{3.56}$$

The multiplier fields B and d implement axial gauge fixings for the gauge symmetries (3.54) and (3.55), respectively. Both gauge fixings are chosen to be space-like ($n^0 = m^0 = 0$), which we will find necessary in order to make the action invariant under a vector supersymmetry in the 3-dimensional subspace, as we will show in the next subsection. The remaining terms in (3.56) denote the ghost part of the action introducing the ghosts/antighosts $c, \bar{c}, \phi, \bar{\phi}$. The canonical dimensions and ghost numbers for the various fields are summarized in Table 3.1.

	A_μ	λ_k	B	d	c	$\bar{c}$	ϕ	$\bar{\phi}$
dimension	1	2	3	2	0	3	1	2
$\phi\pi$-charge	0	0	0	0	1	-1	1	-1

Table 3.1: Canonical dimensions and ghost numbers of fields

Before we discuss the symmetries of the action (3.56), let us consider the following: It is well-known in the literature (see e.g. [73] for a review) that axial gauge fixings render gauge theories "ghost-free", i.e. appropriate redefinitions of the multiplier fields implementing the gauge fixing lead to a decoupling of the ghost fields from the gauge fields. However, for us it will turn out to be more convenient to merely decouple the ghosts from each other and choose $n^k = m^k$, as this will render the action invariant with respect to a *linear* vector supersymmetry. The necessary field redefinition is

$$d \quad \to \quad d' = d - \mathrm{i}g\left[\bar{\phi}, c\right] . \tag{3.57}$$

Hence, the action we will continue to work with is given by

$$S = \int d^4x \Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\epsilon^{ijk}F_{ij}\lambda_k + Bn^iA_i + d'm^i\lambda_i - \bar{c}n^iD_ic - \bar{\phi}m^iD_i\phi \Big\}, \tag{3.58}$$

with $n^k = m^k$.

3.2.1 BRST & VSUSY

The action (3.58) is invariant under the following BRST transformations, as can be easily verified:

$$\begin{aligned}
&sA_\mu = D_\mu c, && s\lambda_i = D_i\phi - \mathrm{i}g\left[\lambda_i, c\right], \\
&sc = \frac{\mathrm{i}g}{2}\left[c, c\right], && s\phi = \mathrm{i}g\left[\phi, c\right], \\
&s\bar{c} = B, && s\bar{\phi} = d' + \mathrm{i}g\left[\bar{\phi}, c\right], \\
&sB = 0, && sd' = -\mathrm{i}g\left[d', c\right], \\
&s^2\varphi = 0, \qquad \text{for} && \varphi \in \{A_\mu, \lambda, B, d', c, \bar{c}, \phi, \bar{\phi}\}.
\end{aligned} \tag{3.59}$$

The reason why $\bar{\phi}$ and d' do not form a BRST doublet similar to $\bar{c}$ and B lies in the field redefinition $d' = d - \mathrm{i}g\left[\bar{\phi}, c\right]$. However, this will not disturb us.

Furthermore, as already alluded to, the action is also invariant under the following fermionic symmetry

$$\begin{aligned}
&\delta_i A_\mu = 0, && && \delta_i \lambda_j = -\epsilon_{ijk} n^k \bar{c}, \\
&\delta_i c = A_i, && && \delta_i \phi = 0, \\
&\delta_i \bar{c} = 0, && && \delta_i \bar{\phi} = 0, \\
&\delta_i B = \partial_i \bar{c}, && && \delta_i d' = 0, \\
&\delta_i \delta_j \varphi = \delta_0 \varphi = 0, && \text{for} && \varphi \in \{A_\mu, \lambda, B, d', c, \bar{c}, \phi, \bar{\phi}\},
\end{aligned} \tag{3.60}$$

provided $n^k = m^k$. Besides the fact that the operator for this symmetry carries a space-time index, it is crucial to notice that it is a *linear* symmetry. In order to provide a connection with Chapter 3.1 (and reference [4]) as well as with the (non-commutative) 3-dimensional BF model, we will hence refer to (3.60) as *vector supersymmetry*, or VSUSY for short. The reason for the fact that the symmetry has a different form from the familiar one of BF models is obviously the presence of the $F_{\mu\nu}F^{\mu\nu}$-term in the action (3.58) — and of course the fact that we are dealing with $3+1$ dimensional space-time. As already anticipated, linearity of this symmetry was achieved through the field redefinition $d' = d - \mathrm{i}g\left[\bar{\phi}, c\right]$, while the initial multiplier field d would have transformed non-linearly under VSUSY. Yet, linearity of the VSUSY will turn out to be crucial for our considerations.

The invariance of the action functional (3.58) under the VSUSY transformations (3.60) is described by the Ward identity

$$\mathcal{W}_i S \equiv \int d^4x \left(\partial_i \bar{c} \frac{\delta S}{\delta B} + A_i \frac{\delta S}{\delta c} + \epsilon_{ijk} n^j \bar{c} \frac{\delta S}{\delta \lambda_k} \right) = 0, \tag{3.61}$$

which will play an important role when considering loop corrections (cf. Section 3.2.3).

As we have seen, the VSUSY depends crucially on our choice of gauge. Moreover, the interplay of the form of $\theta^{\mu\nu}$ (as given by equation (3.49)) together with the space-like nature of the chosen gauge vector gives rise to even more symmetries, as we are about to show right now. Let us take a look at the algebra satisfied by the BRST symmetry and the VSUSY: From relations (3.59) and (3.60) it follows that[8]

$$[s, s]\,\varphi = [\delta_i, \delta_j]\,\varphi = 0, \qquad \text{for } \varphi = \{A_\mu, \lambda_j, B, d', c, \bar{c}, \phi, \bar{\phi}\}, \tag{3.62a}$$

$$[s, \delta_i]\, A_j = \partial_i A_j - \epsilon_{ijk} \frac{\delta S}{\delta \lambda_k} + \hat{\delta}_i A_j, \tag{3.62b}$$

[8]The corresponding equations of motion are displayed in Appendix D.1.

$$[s,\delta_i]\,A_0 = \partial_i A_0 + \hat{\delta}_i A_0, \tag{3.62c}$$
$$[s,\delta_i]\,c = \partial_i c, \tag{3.62d}$$
$$[s,\delta_i]\,\bar{c} = \partial_i \bar{c}, \tag{3.62e}$$
$$[s,\delta_i]\,B = \partial_i B, \tag{3.62f}$$
$$[s,\delta_i]\,\lambda_j = \partial_i \lambda_j - \epsilon_{ijk}\frac{\delta S}{\delta A_k} - D_i\frac{\delta S}{\delta \lambda^j} + \hat{\delta}_i \lambda_j, \tag{3.62g}$$
$$[s,\delta_i]\,d' = \partial_i d' + \hat{\delta}_i d', \tag{3.62h}$$
$$[s,\delta_i]\,\phi = \partial_i \phi + \hat{\delta}_i \phi, \tag{3.62i}$$
$$[s,\delta_i]\,\bar{\phi} = \partial_i \bar{\phi} + \hat{\delta}_i \bar{\phi}, \tag{3.62j}$$

implying a new bosonic vectorial symmetry of the action (3.58) whose action on the fields is given by the transformation laws

$$\begin{aligned}
\hat{\delta}_i A_j &= \epsilon_{ijk} n^k d'\,, \\
\hat{\delta}_i A_0 &= -F_{i0}\,, \\
\hat{\delta}_i \lambda_j &= \epsilon_{ijk} D_0 F^{0k} - D_j \lambda_i + \frac{1}{2}\epsilon_{lmi} D_j F^{lm} + n_j D_i d' - \mathrm{i}g\epsilon_{ijk} n^k \left[\bar{\phi}, \phi\right]\,, \\
\hat{\delta}_i d' &= -D_i d'\,, \\
\hat{\delta}_i \phi &= -D_i \phi\,, \\
\hat{\delta}_i \bar{\phi} &= -D_i \bar{\phi}\,, \\
\hat{\delta}_0 \varphi &= 0, \qquad \text{for all fields } \varphi.
\end{aligned} \tag{3.63}$$

From the right hand side of (3.62) we already see that the algebra of symmetries can only close *on-shell*. Apart from the new symmetry (3.63) we also notice that the space translations ∂_i appear.

3.2.2 Differences compared to the 2-dimensional BF-type Slavnov term

In Chapter 3.1 (cf. ref. [4]) it was shown that the algebra of BRST, VSUSY, the vectorial bosonic symmetry and the translation symmetry closes on-shell for non-commutative Maxwell theory with a Slavnov term resembling the 2-dimensional BF model. Here, however, things are slightly more complicated:

Computing further commutators, we readily find that

$$\left[s, \hat{\delta}_i\right] \varphi = 0 \tag{3.64}$$

for all fields. However, in trying to work out the complete symmetry algebra, one encounters even more symmetries, e.g.

$$
\begin{aligned}
\left[\delta_i, \hat{\delta}_j\right] \lambda_k &= \epsilon_{ijl} n^l D_k \bar{c}\,, \\
\left[\delta_i, \hat{\delta}_j\right] c &= \hat{\delta}_i A_j\,, \\
\left[\delta_i, \hat{\delta}_j\right] \varphi &= 0, \qquad \text{for all other fields } \varphi\,.
\end{aligned}
\tag{3.65}
$$

The right hand sides of these expressions represent new symmetry transformations of the action (3.58), as can be easily checked. Similarly one obtains

$$
\left[\hat{\delta}_i, \hat{\delta}_j\right] \varphi = \text{further symmetry transf. of } \varphi\,. \tag{3.66}
$$

In fact, computation of even more commutators between the new symmetries reveals numerous further ones which, however, will not be discussed here. We are primarily interested in the linear vector supersymmetry denoted by δ_i and will discuss its consequences in the next section.

But first we would like to draw attention to an interesting feature of the new bosonic vectorial symmetry (3.63): Inspired by its counterpart in Chapter 3.1 (cf. reference [4]), which was a symmetry of the gauge invariant action, we easily find the corresponding symmetry for the gauge invariant action (3.53) in our present model:

$$
\begin{aligned}
\hat{\delta}_i^{(1a)} A_j &= 0, \\
\hat{\delta}_i^{(1a)} A_0 &= -F_{i0}, \\
\hat{\delta}_i^{(1a)} \lambda_j &= \epsilon_{ijk} D_0 F^{0k}.
\end{aligned}
\tag{3.67}
$$

In contrast to the situation in Chapter 3.1, the gauge fixing of (3.58) *breaks* this symmetry. Instead (due to our space-like axial gauge fixing) it is replaced by[9]

$$
\begin{aligned}
\hat{\delta}_i^{(1)} A_j &= \epsilon_{ijk} n^k d', \\
\hat{\delta}_i^{(1)} A_0 &= -F_{i0}, \\
\hat{\delta}_i^{(1)} \lambda_j &= \epsilon_{ijk} D_0 F^{0k} - D_j \lambda_i + \frac{1}{2}\epsilon_{lmi} D_j F^{lm} + n_j D_i d', \\
\hat{\delta}_i^{(1)} d' &= -D_i d', \\
\hat{\delta}_i^{(1)} \varphi &= 0, \qquad \text{for} \qquad \varphi \in \{B, c, \bar{c}, \phi, \bar{\phi}\}.
\end{aligned}
\tag{3.68}
$$

[9]Notice that the replacement (3.68) is not unique: The gauge fixed action (3.58) is also invariant under $\hat{\delta}'_i \lambda_j = \delta_{ij} n^k D_k d' + \frac{1}{2}\epsilon_{lmi} D_j F^{lm}$ (where $\hat{\delta}'_i \varphi = 0$ for all other fields φ) and hence (3.68) might as well be replaced by an arbitrary linear combination of both, e.g. $\hat{\delta}_i^{(1)} \to \hat{\delta}_i^{(1)} - \hat{\delta}'_i$.

A further nice observation is that the transformations $\hat{\delta}_i^{(1b)}\lambda_j = -D_j\lambda_i$ and $\hat{\delta}_i^{(1c)}\lambda_j = \frac{1}{2}\epsilon_{lmi}D_jF^{lm}$ leave the gauge invariant action (3.53) invariant as well. In fact, looking at $\hat{\delta}_i^{(1b)}\lambda_j$ one is strongly reminded of the second gauge symmetry (3.55). The remaining field transformations of (3.63) form yet another symmetry[10] of the gauge fixed action (3.58) which does not involve the gauge field A_μ:

$$\begin{aligned}
&\hat{\delta}_i^{(2)}\lambda_j = -\mathrm{i}g\epsilon_{ijk}n^k\left[\bar{\phi},\phi\right],\\
&\hat{\delta}_i^{(2)}\phi = -D_i\phi,\\
&\hat{\delta}_i^{(2)}\bar{\phi} = -D_i\bar{\phi},\\
&\hat{\delta}_i^{(2)}\varphi = 0, \qquad \text{for} \qquad \varphi \in \{A_\mu, c, \bar{c}, B, d'\}\,.
\end{aligned} \tag{3.69}$$

Thus, in contrast to the simpler model of non-commutative Maxwell theory with a Slavnov term resembling the 2-dimensional BF model, the right hand sides of the commutators $[s, \delta_i]$ reveal a linear combination of *two* symmetries ($\hat{\delta}_i = \hat{\delta}_i^{(1)} + \hat{\delta}_i^{(2)}$), one of which is a modified version of (3.67) due to gauge fixing, namely (3.68). Furthermore, the algebra does not close immediately, but instead, many additional symmetries appear.

In conclusion of this subsection, it should be noted that the appearance of an additional bosonic vectorial symmetry of the gauge invariant action seems to be typical for Yang Mills theories with a BF-type Slavnov term. However, its survival after gauge fixing is in general not compatible with the existence of a linear VSUSY.

3.2.3 Consequences of the vector supersymmetry

The generating functional Z^c of the connected Green functions is again given by the Legendre transform of the generating functional Γ of the one-particle-irreducible Green functions:

$$Z^c = \Gamma + \int d^4x \left(j_A^\mu A_\mu + j_B B + j_\lambda^i \lambda_i + j_{d'}d' + j_c c + j_{\bar{c}}\bar{c} + j_\phi \phi + j_{\bar{\phi}}\bar{\phi}\right), \tag{3.70}$$

[10] Remember that the BRST transformations were already made up of two separate symmetries, namely those corresponding to the two gauge symmetries (3.54) and (3.55).

where in the classical approximation Γ essentially equals the total classical action S_{tot}. This leads to the usual relations

$$
\begin{aligned}
&\frac{\delta Z^c}{\delta j_A^\mu} = A_\mu, && \frac{\delta Z^c}{\delta j_B} = B, && \frac{\delta Z^c}{\delta j_\lambda^j} = \lambda_j, && \frac{\delta Z^c}{\delta j_{d'}} = d',\\
&\frac{\delta Z^c}{\delta j_c} = c, && \frac{\delta Z^c}{\delta j_{\bar c}} = \bar c, && \frac{\delta Z^c}{\delta j_\phi} = \phi, && \frac{\delta Z^c}{\delta j_{\bar\phi}} = \bar\phi,\\
&\frac{\delta \Gamma}{\delta A_\mu} = -j_A^\mu, && \frac{\delta \Gamma}{\delta B} = -j_B, && \frac{\delta \Gamma}{\delta \lambda_j} = -j_\lambda^j, && \frac{\delta \Gamma}{\delta d'} = -j_{d'},\\
&\frac{\delta \Gamma}{\delta c} = j_c, && \frac{\delta \Gamma}{\delta \bar c} = j_{\bar c}, && \frac{\delta \Gamma}{\delta \phi} = j_\phi, && \frac{\delta \Gamma}{\delta \bar\phi} = j_{\bar\phi}.
\end{aligned}
\tag{3.71}
$$

In the tree graph approximation, the Ward identity (3.61) describing the linear vector supersymmetry in terms of Z^c is given by

$$
\mathcal{W}_i Z^c = \int d^4x \Big[j_B \partial_i \frac{\delta Z^c}{\delta j_{\bar c}} - j_c \frac{\delta Z^c}{\delta j_A^i} + \epsilon_{ijk} n^j j_\lambda^k \frac{\delta Z^c}{\delta j_{\bar c}} \Big] = 0. \tag{3.72}
$$

Varying (3.72) with respect to the appropriate sources one obtains the following relations:

$$
\left.\frac{\delta^2 Z^c}{\delta j_A^i \delta j_\lambda^j}\right|_{j=0} = \epsilon_{ijk} n^k \left.\frac{\delta^2 Z^c}{\delta j_{\bar c} \delta j_c}\right|_{j=0}, \tag{3.73a}
$$

$$
\left.\frac{\delta^2 Z^c}{\delta j_A^i \delta j_A^\nu}\right|_{j=0} = 0\,. \tag{3.73b}
$$

Furthermore, one has the gauge fixing conditions (cf. (3.58), $m^i = n^i$)

$$
-j_B = n^i \frac{\delta Z^c}{\delta j_A^i}, \tag{3.74a}
$$

$$
-j_{d'} = n^i \frac{\delta Z^c}{\delta j_\lambda^i}, \tag{3.74b}
$$

and the (anti)ghost equations

$$
-n^i \partial_i \frac{\delta Z^c}{\delta j_{\bar c}} - \mathrm{i}g \left[j_B, \frac{\delta Z^c}{\delta j_{\bar c}} \right] = j_c, \qquad -n^i \partial_i \frac{\delta Z^c}{\delta j_c} - \mathrm{i}g \left[j_B, \frac{\delta Z^c}{\delta j_c} \right] = j_{\bar c}\,, \tag{3.74c}
$$

$$
-n^i \partial_i \frac{\delta Z^c}{\delta j_{\bar\phi}} - \mathrm{i}g \left[j_B, \frac{\delta Z^c}{\delta j_{\bar\phi}} \right] = j_\phi, \qquad -n^i \partial_i \frac{\delta Z^c}{\delta j_\phi} - \mathrm{i}g \left[j_B, \frac{\delta Z^c}{\delta j_\phi} \right] = j_{\bar\phi}\,, \tag{3.74d}
$$

from which follow

$$n^i \frac{\delta^2 Z^c}{\delta j_B(y)\delta j^i_A(x)}\bigg|_{j=0} = -\delta^4(x-y), \tag{3.75a}$$

$$n^i \frac{\delta^2 Z^c}{\delta j_{d'}(y)\delta j^i_\lambda(x)}\bigg|_{j=0} = -\delta^4(x-y), \tag{3.75b}$$

$$n^i \partial_i \frac{\delta^2 Z^c}{\delta j_c(y)\delta j_{\bar{c}}(x)}\bigg|_{j=0} = -\delta^4(x-y), \tag{3.75c}$$

$$n^i \partial_i \frac{\delta^2 Z^c}{\delta j_\phi(y)\delta j_{\bar{\phi}}(x)}\bigg|_{j=0} = -\delta^4(x-y). \tag{3.75d}$$

In momentum space, the free propagators of the theory with $m^i = n^i$ are given by (see Appendix D.2)

$$\mathrm{i}\Delta^{c\bar{c}}(k) = -\frac{1}{(nk)}, \qquad \mathrm{i}\Delta^{\phi\bar{\phi}}(k) = -\frac{1}{(nk)}, \tag{3.76a}$$

$$\mathrm{i}\Delta^{AB}_i(k) = \frac{\mathrm{i}k_i}{(nk)}, \qquad \mathrm{i}\Delta^{d'\lambda}_i(k) = \frac{\mathrm{i}k_i}{(nk)}, \tag{3.76b}$$

$$\mathrm{i}\Delta^{\lambda\lambda}_{ij}(k) = \frac{-k^2}{\vec{k}^2}\left(g_{ij} - \frac{k_i n_j + n_i k_j}{(nk)} + n^2 \frac{k_i k_j}{(nk)^2}\right), \tag{3.76c}$$

$$\mathrm{i}\Delta^{A\lambda}_{ij}(k) = \frac{-\mathrm{i}}{\vec{k}^2}\left(\epsilon_{ilj}k^l - \epsilon_{ilr}\frac{k^l n^r k_j}{(nk)} + \epsilon_{jlr}\frac{k^l n^r k_i}{(nk)}\right), \tag{3.76d}$$

$$\mathrm{i}\Delta^{A\lambda}_{i0}(k) = \frac{-\mathrm{i}}{\vec{k}^2}\left(-\epsilon_{ilr}\frac{k^l n^r k_0}{(nk)}\right), \tag{3.76e}$$

$$\mathrm{i}\Delta^{AA}_{00}(k) = -\frac{1}{k^2}\left(g_{00} - \frac{k_0^2}{\vec{k}^2}\right) = \frac{1}{\vec{k}^2}, \qquad \mathrm{i}\Delta^{AA}_{ij}(k) = \mathrm{i}\Delta^{AA}_{i0}(k) = 0, \tag{3.76f}$$

and one easily sees that the relations (3.73) and (3.75) hold[11]. Furthermore, by virtue of equation (3.76f) and $\theta^{\mu 0} = 0$, the gauge field propagator is still transverse with respect to $\tilde{k}_\mu \equiv \theta_{\mu\nu}k^\nu$ despite the modification of the Slavnov term (cf. (1.21) and (3.58)).

Finally, the vector supersymmetry leads to the following nice features for loop calculations: Obviously, the combination of the λA-vertex $V^{\lambda A}_{ijk} \propto \epsilon_{ijk}$ with a gauge field propagator $\Delta^{AA}_{\mu\nu}$ is always zero (see eq. (3.73b)). But

[11] $\tilde{k}^2 = -(k_1^2 + k_2^2)$, $(nk) = -(n_1 k_1 + n_2 k_2)$, $(n\tilde{k}) = (n_1 k_2 - n_2 k_1)$, and similarly for $n^i \leftrightarrow m^i$. Furthermore, $\mathrm{i}\Delta^{\varphi_1\varphi_2}(x-y) = -\mathrm{i}\frac{\delta^2 Z^c}{\delta j_{\varphi_1}(x)\delta j_{\varphi_2}(y)}\Big|_{j=0}$ for all fields φ.

since it is impossible to have λA-vertices in arbitrary loop graphs (except for tree graphs) unless some of them couple to gauge field propagators [4], such graphs will not contribute to any quantum corrections. *Hence, neither the λ-vertex nor the $\lambda/\lambda A$-propagators contribute to the gauge field self-energy corrections at any loop order!* Together with the transversality condition of the gauge field propagator, it therefore follows that no IR divergences from one-loop graph insertions are passed on to higher loop orders.

Note that we have only discussed the IR behaviour of our model, and the UV sector, especially the planar graphs, remain to be thoroughly analyzed. Due to the VSUSY, the λ-field does not play a role in the UV sector either, and therefore we do not expect any major problems. Still one needs to take care when computing the Feynman graphs due to the axial gauge fixing, i.e. an appropriate prescription for the $(nk)^{-1}$ poles is needed (see for example [72] and references therein).

3.2.4 Generalization to arbitrary dimensions

Re-interpretation of the action

At the beginning of Chapter 3.2, we modified the original Slavnov term proposed in [11, 12] by changing the scalar field λ into a set of fields λ_i, labelled by an index corresponding to the non-commutative subsector of space-time. In order to show that the Slavnov trick works, we have taken a rather pragmatic point of view and have not inquired further about the true nature of λ_i. In fact, an intriguing observation can be made when returning to the action (3.58) and explicitly writing out the field strength $F_{\mu\nu}$ in the Slavnov term:

$$\begin{aligned} S = \int d^4x \Big\{ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \epsilon^{ijk}\lambda_i\partial_j A_k - \mathrm{i}g\epsilon^{ijk}\lambda_i A_j A_k + Bn^i A_i + d'm^i\lambda_i - \\ &- \bar{c}n^i D_i c - \bar{\phi}m^i D_i\phi \Big\}. \end{aligned} \tag{3.77}$$

Written in this way, the generalized Slavnov term has certain similarities with a Chern-Simons type term if λ_i is interpreted as a second gauge field. In order to make this observation even more striking, we rescale the fields according to

$$\lambda_i \equiv \mu\lambda'_i, \qquad\qquad d' \equiv \frac{d''}{\mu}, \tag{3.78}$$

where μ is a constant with mass dimension 1. For the action, we then find

$$S = \int d^4x \Big\{ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \mu\epsilon^{ijk}\lambda'_i\partial_j A_k - \mathrm{i}g\mu\epsilon^{ijk}\lambda'_i A_j A_k + Bn^i A_i + d''m^i\lambda'_i - \\ - \bar{c}n^i D_i c - \bar{\phi}'m^i D_i\phi' \Big\}. \quad (3.79)$$

Note that ϕ' and $\bar{\phi}'$ differ from ϕ and $\bar{\phi}$ by their canonical dimension, which can be seen from Table 3.2.

	A_μ	λ'_k	B	d''	c	$\bar{c}$	ϕ'	$\bar{\phi}'$
dimension	1	1	3	3	0	3	0	3
$\phi\pi$-charge	0	0	0	0	1	-1	1	-1

Table 3.2: Canonical dimensions and ghost numbers of redefined fields

Thus, the two sets of fields $(A_\mu, B, c, \bar{c})$ and $(\lambda'_i, d'', \phi', \bar{\phi}')$ not only appear in a rather similar way in the action, but also their canonical dimensions match precisely. This provides some evidence that $(\lambda'_i, d'', \phi', \bar{\phi}')$ should really be interpreted as another gauge field together with a second ghost system. In addition to the classical dynamics, a striking difference is the absence of a λ'_0 component. Indeed, λ'_i has only components corresponding to potentially non-commutative directions. As we will see, this is a general feature when considering similar examples in a different number of dimensions, as we will do in the next section.

Topological terms in higher dimensions

In Chapter 3.1 (cf. reference [4]) it was shown that the interpretation of the Slavnov-term as a topological-type term (resembling a 2-dimensional BF model) is fruitful in studying the fate of the IR divergences in more detail. In Section 3.2.3, we also discovered that modifying the Slavnov term to resemble a 3-dimensional BF model teaches us interesting lessons in this respect. In doing so, however, we had to add an index to the λ field, which (as we have just seen) allows for its interpretation in terms of a gauge field. It is expected that increasing the dimension of the non-commutative subspace (which necessarily also involves increasing the dimension of space-time) will lead to objects with yet more indices whose interpretations remain to be seen. Therefore, besides being interesting in its own right, we might learn a good deal about λ (whatever its "form degree" might be), by introducing

Slavnov terms in higher dimensions, which can again be interpreted as being topological in the same sense as before.

To this end, consider a $D > 2$ dimensional space-time $\mathcal{M}$, which we write as the product of a $(D-n)$ dimensional (commutative) Minkowski space-time and a n-dimensional non-commutative Euclidean space

$$\mathcal{M} = \mathbb{M}^{D-n} \times \mathbb{R}^n_{\mathrm{NC}}. \tag{3.80}$$

We restrict n to be $2 \leq n < D$, since we want to have at least two non-commutative dimensions and we furthermore want to interpret one dimension as time. Space-time indices of the whole $\mathcal{M}$ are denoted by Greek letters, $\mu, \nu \in \{0, 1, \ldots, D-1\}$, while the non-commutative directions are labelled by Latin indices $i, j \in \{D-n, \ldots, D-1\}$. In this setup, the analog to the constraint (3.50) is a sum of $\frac{n(n-1)}{2}$ terms and in the following we will impose the stronger demand that each of them vanishes separately. Let us consider this in somewhat more detail:

<u>**D = 3**</u> :
In this simplest case, the only possibility is to choose $n = 2$, which renders $\theta^{\mu\nu}$ of the form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix}, \qquad \text{with } \theta \neq 0. \tag{3.81}$$

The Slavnov constraint (3.50) consists of a single term

$$\theta F_{12} = 0, \tag{3.82}$$

which is implemented in the action by a scalar field λ:

$$\int d^3x \lambda \theta^{\mu\nu} F_{\mu\nu} = \int d^3x \lambda \theta \epsilon^{ij} F_{ij}. \tag{3.83}$$

<u>**D = 4**</u> :
Here there are two possibilities for n, namely 2 and 3, as can be seen from the following table:

n	$\theta^{\mu\nu}$	constraints	λ-field	action term
2	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}$	$F_{23} = 0$	λ	$\int d^4x \lambda \theta^{\mu\nu} F_{\mu\nu}$ $\propto \int d^4x \lambda \epsilon^{ij} F_{ij}$
3	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^{12} & \theta^{13} \\ 0 & -\theta^{12} & 0 & \theta^{23} \\ 0 & -\theta^{13} & -\theta^{23} & 0 \end{pmatrix}$	$\theta^{12}F_{12} = 0$ $\theta^{13}F_{13} = 0$ $\theta^{23}F_{23} = 0$	λ_i	$\int d^4x \epsilon^{ijk} F_{ij} \lambda_k$

where $\theta^{ij} \neq 0$ for all $i \neq j$. In the case $n = 2$ (which is essentially the one studied in [4]), λ is obviously a scalar once more, while for $n = 3$, λ_i may be interpreted as a vector field with components only in the $\mathbb{R}^3_{\text{NC}}$, as we have already pointed out[12].

Generic D:

From the two previous examples, we can easily generalize the case of generic D and n: Let us again start out with the most generic $\theta^{\mu\nu}$

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \\ & \theta^{ij} \end{pmatrix}, \tag{3.84}$$

with $\theta^{ij} \neq 0$ for all $i \neq j$. The Slavnov constraint one has to impose on this model reads

$$\theta^{ij} F_{ij} = 0, \qquad \text{with } D - n \leq i < j \leq D - 1, \tag{3.85}$$

and the stronger constraints, where each term in the sum is zero, are implemented with the help of $\frac{n(n-1)}{2}$ Lagrange multipliers, which can be arranged into a field $\lambda_{i_1 \ldots i_{n-2}}$ which is totally antisymmetric in all its indices. The corresponding action term is of the form

$$\int d^D x \epsilon^{ijk_1 \ldots k_{n-2}} F_{ij} \lambda_{k_1 \ldots k_{n-2}}, \tag{3.86}$$

resembling a n-dimensional BF model (see e.g. [75, 94, 96]). Note that the field $\lambda_{i_1 \ldots i_{n-2}}$ may be interpreted as a $(n-2)$ form which only has components in $\mathbb{R}^n_{\text{NC}}$.

[12] In general, the number of Lagrange multipliers λ_i might just as well be greater than the number of non-vanishing θ^{ij}. However, in this section we are primarily interested in the case where they are equal.

We would, however, like to stress the following points:

- although we started with a deformation parameter matrix of non-commutativity (3.84) where $\theta^{ij} \neq 0$ for all $i \neq j$ to give the Slavnov constraint an implicative form, the action term (3.86) is in principle valid for any choice of the θ_{ij} ;
- we chose the maximum number of constraints compatible with the Slavnov trick.

Before closing this subsection, let us comment on a special case where we set some of the $\theta_{ij} = 0$ in (3.84) in a special way and see if we find alternatives to the constraints (3.86) resembling topological terms. We hence consider the matrix $\theta^{\mu\nu}$ having the block-diagonal structure

$$\theta^{\mu\nu} = \begin{pmatrix} 0_{D-n} & & & \\ & \theta_{n_1} & & \\ & & \ddots & \\ & & & \theta_{n_p} \end{pmatrix}, \qquad \text{with } \sum_{a=1}^{p} n_a = n, \tag{3.87}$$

where 0_{D-n} stands for a $(D-n) \times (D-n)$ square matrix with 0 entries everywhere, and θ_{n_a} are antisymmetric $n_a \times n_a$ matrices (with $2 \leq n_a \leq n$) with non-zero off-diagonal entries. In other words, we consider a space with p non-commutative subspaces which, however, commute among each other.

If we now label the indices of the a-th non-commutative block[13] by $i^{(a)}$, we can impose the following set of (alternative) constraints

$$\begin{aligned}
&\theta^{i_1^{(1)} i_2^{(1)}} F_{i_1^{(1)} i_2^{(1)}} = 0, \quad \text{with } D-n-1 < i_1^{(1)} < i_2^{(1)} < D-n+n_1\,, \\
&\quad\vdots \\
&\theta^{i_1^{(a)} i_2^{(a)}} F_{i_1^{(a)} i_2^{(a)}} = 0, \quad \text{with } D-n-1+\sum_{b=1}^{a-1} n_b < i_1^{(a)} < i_2^{(a)} < D-n+\sum_{b=1}^{a} n_b, \\
&\quad\vdots \\
&\theta^{i_1^{(p)} i_2^{(p)}} F_{i_1^{(p)} i_2^{(p)}} = 0, \quad \text{with } D-n-1+\sum_{b=1}^{p-1} n_b < i_1^{(p)} < i_2^{(p)} < D-n+\sum_{b=1}^{p} n_b,
\end{aligned} \tag{3.88}$$

where we suspended summation over repeated indices. These constraints

[13] They take values $D-n-1+\sum_{b=1}^{a-1} n_b < i^{(a)} < D-n+\sum_{b=1}^{a} n_b$.

suggest that we consider the term

$$\sum_{a=1}^{p} \int d^D x \epsilon^{i_1^{(a)} \ldots i_{n_a}^{(a)}} \lambda^{(a)}_{i_1^{(a)} \ldots i_{n_a-2}^{(a)}} F_{i_{n_a-1}^{(a)}, i_{n_a}^{(a)}} \tag{3.89}$$

in the action, which can be interpreted as a sum of n_a-dimensional BF terms, and the $\lambda^{(a)}_{i_1^{(a)} \ldots i_{n_a-2}^{(a)}}$ can be identified as $(n_a - 2)$ forms with components in the a-th non-commutative subspace. The symbol $\epsilon^{i_1^{(a)} \ldots i_{n_a}^{(a)}}$ is defined similarly to the Levi-Civita symbol, i.e. it is $+1$ (-1) for even (odd) permutations of its indices. The only difference here is that the range of the indices $i_l^{(a)}$ is given by (3.88) rather than being $1, \ldots, n_a$.

It is also important to stress that the superscript "(a)" of the $\lambda^{(a)}_{i_1^{(a)} \ldots i_{n_a-2}^{(a)}}$ is not an index but only a label for the various multiplier fields.

Generalized Slavnov terms and VSUSY

After having gained some intuitive understanding about the nature of the λ field and having generalized the actions considered throughout this Chapter 3, we might now ask which further notions we are able to generalize to higher dimensions. One interesting point is what happens to the VSUSY in higher dimensions.

We have seen that the action (3.58) is invariant under the vector supersymmetry described by (3.61). On the other hand, if one replaces the gauge invariant part of (3.58) with (3.46) and (3.49), hence implementing the weaker Slavnov constraint (3.50), one cannot find VSUSY. In a first step we therefore try to clarify whether we can find a gauge fixing, so that an action including Slavnov terms of the form (3.89) becomes invariant under a vector supersymmetry. From all we know so far, such a gauge fixing has to be of an axial type. Let us consider a simple example, namely $(D = 5, n = 4)$ with a deformation parameter of the form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 & 0 \\ 0 & -\theta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta_2 \\ 0 & 0 & 0 & -\theta_2 & 0 \end{pmatrix}, \tag{3.90}$$

and the following (gauge fixed) action

$$S = \int d^5x\bigg(- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^{(1)}}{2}\epsilon^{i^{(1)}j^{(1)}}F_{i^{(1)}j^{(1)}} + \frac{\lambda^{(2)}}{2}\epsilon^{i^{(2)}j^{(2)}}F_{i^{(2)}j^{(2)}} + Bn^\mu A_\mu - \\ - \bar{c}n^\mu D_\mu c\bigg). \tag{3.91}$$

We choose n^μ to only have spatial components (i.e. $n^0 = 0$). The action is BRST invariant, with

$$s\lambda^{(1,2)} = -\mathrm{i}g\left[\lambda^{(1,2)}, c\right]$$

and the transformations of the other fields given by (3.59). Since the two Slavnov terms represent 2-dimensional BF terms in the (x^1, x^2)-plane and the (x^3, x^4)-plane respectively, one could naïvely assume invariance of the action under the VSUSY transformations

$$\begin{aligned}
&\delta_\mu A_\nu = \delta_\mu \bar{c} = 0, && \delta_{i^{(1)}}\lambda^{(1)} = \epsilon_{i^{(1)}j^{(1)}}n^{j^{(1)}}\bar{c},\\
&\delta_{i^{(1)}}c = A_{i^{(1)}}, && \delta_{i^{(2)}}\lambda^{(1)} = 0,\\
&\delta_{i^{(2)}}c = A_{i^{(2)}}, && \delta_{i^{(1)}}\lambda^{(2)} = 0,\\
&\delta_{i^{(1)}}B = \partial_{i^{(1)}}\bar{c}, && \delta_{i^{(2)}}\lambda^{(2)} = \epsilon_{i^{(2)}j^{(2)}}n^{j^{(2)}}\bar{c},\\
&\delta_{i^{(2)}}B = \partial_{i^{(2)}}\bar{c}, && \delta_0\varphi = 0 \quad \text{for all fields.}
\end{aligned} \tag{3.92}$$

However, direct calculations show that

$$\begin{aligned}
\delta_{i^{(1)}}S &= \int d^5x\left(\bar{c}n^{j^{(2)}}F_{j^{(2)}i^{(1)}}\right) \neq 0,\\
\delta_{i^{(2)}}S &= \int d^5x\left(\bar{c}n^{j^{(1)}}F_{j^{(1)}i^{(2)}}\right) \neq 0.
\end{aligned} \tag{3.93}$$

So obviously, we have invariance under $\delta_{i^{(1)}}$ if we choose $n^{j^{(2)}} = 0$, or invariance under $\delta_{i^{(2)}}$ if we choose $n^{j^{(1)}} = 0$, but never under both. For higher dimensional models with arbitrary Slavnov terms of the type (3.89) it therefore makes sense to assume that, depending on the choice of the axial gauge fixing vector n^μ, one can at most have invariance under a vector supersymmetry whose operator acts non-trivially *in only one* of the n_a-dimensional subspaces corresponding to the a-th BF term.

In fact, the transformations for VSUSY in the a-th non-commutative subspace (i.e. the a-th summand in equation (3.89)) of an arbitrary dimensional BF-Slavnov model are always the same, namely the only non-trivial transformations are[14]

$$
\begin{aligned}
&\delta_{i^{(a)}} c = A_{i^{(a)}}, &&\delta_{i^{(a)}} \lambda^{(a)}_{j_1^{(a)}\dots j_{n_a-2}^{(a)}} = \epsilon_{i^{(a)}k^{(a)}j_1^{(a)}\dots j_{n_a-2}^{(a)}} n^{k^{(a)}} \bar{c}, \\
&\delta_{i^{(a)}} B = \partial_{i^{(a)}} \bar{c}, && \qquad (3.94)
\end{aligned}
$$

with the range of indices given in (3.88). For the sake of clarity we will drop the superscripts "(a)" in the following and keep in mind that we are referring to the a-th BF term. The linear VSUSY (3.94) exists only after appropriate redefinition of the multiplier fields fixing the gauge symmetries: Let the collection of $2(n_a-2)$ fields

$$
\begin{aligned}
&\left\{\phi, \phi_{j_1}, \phi_{j_1 j_2}, \dots, \phi_{j_1\dots j_{n_a-3}}\right\}, \\
&\left\{\bar\phi, \bar\phi_{j_1}, \bar\phi_{j_1 j_2}, \dots, \bar\phi_{j_1\dots j_{n_a-3}}\right\}, \qquad (3.95)
\end{aligned}
$$

be the tower of ghosts[15] we need to introduce. For $n_a = 2$ no ghosts are needed since λ is a scalar in that case. Furthermore, let the $n_a - 2$ objects

$$
\left\{d, d_{j_1}, \dots, d_{j_1\dots j_{n_a-3}}\right\} \qquad (3.96)
$$

be Lagrange multipliers fixing the gauge freedom of[16]

$$
\left\{\lambda_{j_1\dots j_{n_a-2}}, \phi_{j_1}, \dots, \phi_{j_1\dots j_{n_a-3}}\right\}. \qquad (3.97)
$$

In order to have a linear VSUSY we must redefine the multipliers d according to

$$
\begin{aligned}
&d' = d - \mathrm{i}g\left[\bar\phi, c\right], \\
&d'_{j_1\dots j_{m_a}} = d_{j_1\dots j_{m_a}} - \mathrm{i}g\left[\bar\phi_{j_1\dots j_{m_a}}, c\right], \qquad \forall\, 1 \le m_a \le n_a - 3. \qquad (3.98)
\end{aligned}
$$

leading to the BRST transformations[17]

$$
\begin{aligned}
&s\bar\phi = d' + \mathrm{i}g\left[\bar\phi, c\right], \\
&s\bar\phi_{j_1\dots j_{m_a}} = d'_{j_1\dots j_{m_a}} + \mathrm{i}g\left[\bar\phi_{j_1\dots j_{m_a}}, c\right], \qquad \forall\, 1 \le m_a \le n_a - 3, \\
&sd' = -\mathrm{i}g\left[d', c\right], \\
&sd'_{j_1\dots j_{m_a}} = -\mathrm{i}g\left[d'_{j_1\dots j_{m_a}}, c\right], \qquad \forall\, 1 \le m_a \le n_a - 3. \qquad (3.99)
\end{aligned}
$$

[14] This, of course, includes the case $p = 1$ in (3.89).
[15] See, for example, [75, 94, 96] and references therein.
[16] There is no gauge freedom for the scalar ϕ.
[17] Concerning the BRST transformations for the other fields we refer once again to the literature [75, 94, 96].

We should also stress that the vector supersymmetry operator (3.94) acts non-trivially only on the a-th Slavnov term and the gauge fixing part for the gauge field A_i of the action, provided, of course, its axial gauge fixing vector is chosen to be non-zero only in the n_a-dimensional subspace where it is identical to the axial gauge fixing vector for $\lambda^{(a)}_{j_1 \cdots j_{n_a-2}}$.

Obviously, we would not completely lose VSUSY if we wrote the gauge fixing part of the action in terms of d rather than d', but the VSUSY would become *non-linear*, e.g. the following non-linear VSUSY transformations would have to be added to (3.94):

$$\begin{aligned} \delta_i d &= \mathrm{i}g \left[A_i, \bar{\phi}\right], \\ \delta_i d_{j_1 \dots j_{m_a}} &= \mathrm{i}g \left[A_i, \bar{\phi}_{j_1 \dots j_{m_a}}\right], \qquad \forall\ 1 \leq m_a \leq n_a - 3, \end{aligned} \tag{3.100}$$

for all Lagrange multipliers.

An important point to mention, however, is that the presence of a linear vector supersymmetry alone is not sufficient to guarantee the complete absence of all IR divergences in the loop calculations. In fact, since we have found the VSUSY to act non-trivially only in a certain subspace of the non-commutative space, the argument at the end of Section 3.2.3 cannot be applied here, which means we are not able to prove IR finiteness of the model in this way.

3.2.5 Conclusions

Inspired by the results of Chapter 3.1 concerning Slavnov-extended gauge theories (cf. [4]), we discussed a step-by-step generalization of the Slavnov term. In Sections 3.1.2 and 3.2.3 we considered the more restrictive version (3.53) of the Slavnov term resembling a 3-dimensional BF model. We found numerous new symmetries of the gauge fixed action, one of which is (3.60), a *linear vector supersymmetry* (VSUSY) which (although it is gauge dependent and hence non-physical) allowed us to show that the model is free of quadratic IR divergences.

Section 3.2.4 was then dedicated to possible generalizations to higher dimensional space-times of the form (3.80). We were able to show that in a specific setup the λ field in higher dimensions can be interpreted as a $(n-2)$ form with only components in the n-dimensional non-commutative subspace of space-time. We furthermore discussed various other possibilities of implementing the Slavnov constraint(s) and also gave one version which (upon choosing an appropriate gauge fixing) features the existence of a vector supersymmetry. However, in the general D-dimensional case this is not sufficient to show IR finiteness of the model.

Chapter 4

Further considerations

4.1 Making time non-commutative

Throughout the previous chapters we have kept $\theta^{0\mu} = 0$ in order to preserve unitarity of the S-matrix. The difficulty with handling $\theta^{0\mu} \neq 0$ lies in the fact that, due to the star products, the interaction part of the Lagrangian depends on infinitely many time derivatives acting on the fields. A workaround has been proposed by Doplicher et. al. [35] and further developed for non-commutative scalar ϕ^4 theory by several authors [97, 98, 99, 100]. It is termed "interaction point time ordered perturbation theory" (IPTOPT) and is based on the following idea: Consider the Gell-Mann-Low formula applied to the field operators ϕ of a scalar ϕ^4 theory

$$\langle 0|T\{\phi_H(x_1)\dots\phi_H(x_n)\}|0\rangle = \sum_{m=0}^{\infty}\frac{(-\mathrm{i})^m}{m!}\int\limits_{-\infty}^{\infty}dt_1\int\limits_{-\infty}^{\infty}dt_2\dots\int\limits_{-\infty}^{\infty}dt_m\times \\ \times\langle 0|T\{\phi_I(x_1)\dots\phi_I(x_n)V(t_1)\dots V(t_m)\}|0\rangle\,. \tag{4.1}$$

The subscripts H and I denote the Heisenberg picture and the interaction picture, respectively. V is the interaction part of the Hamiltonian

$$V(z^0) = \int d^3z \frac{\kappa}{4!} \phi(z) \star \phi(z) \star \phi(z) \star \phi(z). \tag{4.2}$$

The idea is that the time-ordering operator T acts on the time components of the x_i and on the so-called *time stamps* $t_1, \ldots, t_m$. For example, considering the interaction (4.2) with an alternative representation for the star products

$$\begin{aligned} V(z^0) = \frac{\kappa}{4!} \prod_{i=1}^{3} \int \frac{d^4 s_i d^4 l_i}{(2\pi)^4} e^{\mathrm{i} s_i l_i} \times \\ \times \phi(z - \frac{1}{2}\tilde{l}_1)\phi(z + s_1 - \frac{1}{2}\tilde{l}_2)\phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3)\phi(z + s_1 + s_2 + s_3), \end{aligned}$$

the time ordering only affects z^0 and no other time components (like e.g. l_i^0 etc.). This leads to modified Feynman rules. For example, the propagator of ϕ^4 theory

$$\Delta(x, x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{\mathrm{i}k(x-x')}}{k^2 + m^2 - \mathrm{i}\epsilon}, \tag{4.3}$$

is generalized to the so-called *contractor*

$$\begin{aligned} \Delta_C(x, t; x', t') = \int \frac{d^4k}{(2\pi)^4} \frac{\exp\left[\mathrm{i}k(x - x') + \mathrm{i}k^0(x^0 - t - (x'^0 - t'))\right]}{k^2 + m^2 - \mathrm{i}\epsilon} \times \\ \times \left[\cos\left(\omega_k(x^0 - t - (x'^0 - t'))\right) - \frac{\mathrm{i}k^0}{\omega_k} \sin\left(\omega_k(x^0 - t - (x'^0 - t'))\right)\right], \end{aligned} \tag{4.4}$$

which for $x^0 = t$ and $x'^0 = t'$ (being the case when $\theta^{0\mu} = 0$) reduces to (4.3). On the one hand, this approach seems promising in some respects, meaning that one may extend the formalism to non-commutative gauge fields, eventually even including the Slavnov term. On the other hand, however, there remains the possibility that due to the modification of the corresponding propagator, i.e. the contractor, the Slavnov trick might no longer work because the contractor could fail to be transversal with respect to $\tilde{k}^\mu$. These open questions will be tackled in a work in progress [101].

Finally, one should also remark that similar work, i.e. considerations concerning proper time ordering when dealing with non-commutative time, has been done by Bahns et al. [102, 103]. There even have been claims that in Minkowski space-time with proper time ordering, no inconsistencies related to UV/IR mixing are present [104]. If this is true, the Slavnov term would of course be obsolete in that framework. However, the question would remain as to how the special case of commuting time ($\theta^{0\mu} = 0$) would fit in, since the time ordering would reduce to the usual commutative version and we already know from the previous chapters that UV/IR mixing does in fact appear there.

4.2 A Slavnov term in string theory?

In Chapter 1.3 we outlined how non-commutative field theories arise as effective field theories in string theory [24]. The question is whether a (generalized) Slavnov term can be generated via a similar mechanism from a different term of the effective D-brane action [105]. The first observation in this respect is due to the results of Chapter 3.2.4 (cf. ref. [6]), where we were able to show that under certain circumstances for a $D > 2$ dimensional space-time manifold of the form

$$\mathcal{M} = \mathbb{M}^{D-n} \times \mathbb{R}^n_{\mathrm{NC}}\,, \tag{4.5}$$

where $\mathbb{M}^{D-n}$ denotes a $(D-n)$ dimensional Minkowski space and $\mathbb{R}^n_{\mathrm{NC}}$ is a n-dimensional non-commutative Euclidean space, a generalized Slavnov term of the form

$$\int d^D x \epsilon^{ijk_1\ldots k_{n-2}} F_{ij} \lambda_{k_1,\ldots,k_{n-2}}\,, \tag{4.6}$$

can be written. In this respect, $\lambda_{i_1,\ldots,i_{n-2}}$ can be interpreted as a $(n-2)$ form with components only in the $\mathbb{R}^n_{\mathrm{NC}}$ subspace. Recalling the bosonic spectrum of type II string theory displayed in Table 4.1 (cf. any textbook on string theory, e.g. [106, 107]), the idea is that (depending on the space-time dimen-

sector	**field**	**type IIA**	**type IIB**
NS-NS	dilaton	Φ	Φ
	metric	$G_{\mu\nu}$	$G_{\mu\nu}$
	anti-sym. B-field	$B_{\mu\nu}$	$B_{\mu\nu}$
R-R	n-form potentials	$C_{(1)}, C_{(3)}, C_{(5)}, C_{(7)}$	$C_{(0)}, C_{(2)}, C_{(4)}, C_{(6)}, C_{(8)}$

Table 4.1: The bosonic spectrum of type II string theory

sion), the λ-field which implements the Slavnov constraint(s) corresponds to one of the Ramond-Ramond potentials. This conjecture gets some support from the term of the effective action, with which these fields couple on the string theory side. The simplest term one can write down in the presence of a Dp-brane is the following Wess-Zumino-like term for a coupling to the $(p+1)$-form $C_{(p+1)}$:

$$S_{WZ} = \mu_p \int_{\mathcal{M}_{p+1}} C_{(p+1)} \,, \tag{4.7}$$

where μ_p is the corresponding RR charge and $\mathcal{M}_{p+1}$ is the world-volume of the brane.

In fact, a number of different terms can be added and the most general one is the so-called *Chern-Simons* action[1]

$$S_{CS} = \frac{\mu_p}{g_s} \int_{\mathcal{M}_{p+1}} \left[\sum_p C_{(p+1)} \right] \wedge \mathrm{Tr} e^{2\pi\alpha'(F+B)}, \tag{4.8}$$

where the following is meant: Since the expansion of the exponential involves differential forms of various rank, the integral picks out the summand of the RR forms which is necessary for arriving at the total form degree $(p+1)$.

The link to non-commutative theory is now provided in the following way: If we replace the B-field dependence with a θ-dependence (it should, similarly to the method used in [24], turn all products into star products), then the remaining term is just the Chern character $e^{\frac{\mathrm{i}F}{2\pi}}$ normalized in a somewhat different way. The Chern character generates, in a sense, the Chern classes via the series expansion (see for example [110])

$$\det \left(\mathbb{1} + t \frac{\mathrm{i}F}{2\pi} \right) = \sum_{j=0}^{n} c_j(F) t^j \,, \tag{4.9}$$

where, explicitly, the first Chern classes read

$$c_0(F) = 1 \,, \tag{4.10}$$

$$c_1(F) = \frac{\mathrm{i}}{2\pi} \mathrm{Tr} F \,, \tag{4.11}$$

$$c_2(F) = \frac{1}{2} \left(\frac{\mathrm{i}}{2\pi} \right)^2 \left[\mathrm{Tr} F \wedge \mathrm{Tr} F - \mathrm{Tr}(F \wedge F) \right] \,. \tag{4.12}$$

Note that these quantities depend crucially on the gauge group of F. For example, choosing an Abelian $U(1)$, then $c_2(F) = 0$, while $c_1(F)$ is non-trivial. For a $SU(N)$ group, however, c_1 vanishes due to the relation of the

[1]See in this context, for example, [108, 109] and references therein.

generators T^a,

$$\mathrm{Tr} T^a = 0\,, \tag{4.13}$$

while

$$c_2(F) = -\frac{1}{2}\left(\frac{\mathrm{i}}{2\pi}\right)^2 \mathrm{Tr}(F \wedge F)\,. \tag{4.14}$$

Furthermore, it is worth mentioning that the integrals over these classes give topological invariants (the integer topological class or the instanton number, for example). Since we have seen that the Slavnov term in non-commutative field theories (under certain circumstances) can be written as a topological term, this provides an interesting link to our non-commutative model.

To sum up: *The idea is to identify the (generalized) Slavnov (multiplier) field λ with a Ramond-Ramond potential in type II string theory and interpret the Slavnov term as being proportional to the first Chern class in an expansion of the effective action on a Dp-brane.*

The proposed identification in its current state is, however, hardly more than "pattern matching". Open questions are:

- What other arguments can be found for identifying the Slavnov term with the first relevant term in the expansion of the non-commutative Chern-Simons action?
- Is there a deeper meaning in the possible connection between the λ-field of the Slavnov term and the Ramond-Ramond potentials in type II string theory?
- A further point is that in order to get a Slavnov-extended gauge theory in D space-time dimensions with only $n < D$ non-commutative directions (cf. (4.5)), one would have to consider, on the string theory side, a Dirac-Born-Infeld action for a Dp-brane with $p = D - 1$. But the corresponding Chern-Simons action would have to be considered for a $p' = n - 1$ dimensional brane which would have to be a spatial submanifold of the initial Dp-brane, i.e. the integrand would have to consist of n-forms rather than D-forms, and one would have to integrate over the remaining $(D - n)$ directions (including time). How can we explain/legitimate this procedure from a string theoretic point of view? On the field theory side this action is clear: We don't want to have space-time non-commutativity in order to avoid difficulties with causality and unitarity. Of course one could always use the IPTOPT-formalism (cf. Chapter 4.1) to include space-time non-commutativity, in which case the difficulties on the string theory side could be avoided.

4.3 Non-Commutativity as a gravitational effect

In this section we continue the discussion previously carried out by Rivelles [52] of Seiberg-Witten expanded non-commutative gauge field theories viewed as a coupling to a field dependent gravitational background. We show that such an identification, however, cannot be found for fermionic matter.

Summary of known results

The Seiberg-Witten map has already been introduced in Chapter 1.4.1. As was shown by Rivelles in [52] one can interpret a θ-expanded non-commutative action of scalar fields coupled to gauge fields in flat space-time as a commutative action coupled to a (gauge field dependent) gravitational background in the weak field approximation. In this section we first summarize some results of that paper, especially focusing on the gauge field sector, and then we consider couplings to fermions.

The action for a non-commutative $U(1)$ gauge field is once more given by

$$S_A = -\frac{1}{4}\int d^4x F_{\mu\nu} \star F^{\mu\nu}, \tag{4.15}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} A_\nu]\,.$$

It is invariant under the gauge transformation

$$\delta A_\mu = D_\mu \Lambda = \partial_\mu \Lambda - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} \Lambda]\,.$$

The Seiberg-Witten map is derived by assuming the existence of an Abelian gauge field a_μ with the usual Abelian gauge transformation $\delta a_\mu = \partial_\mu \lambda$ such that

$$A_\mu(a) + \delta_\Lambda A_\mu(a) = A_\mu(a + \delta_\lambda a).$$

Up to first order in θ one finds

$$\begin{aligned} A_\mu &= a_\mu - \frac{1}{2}\theta^{\rho\sigma} a_\rho \left(\partial_\sigma a_\mu + f_{\sigma\mu}\right), \\ \Lambda &= \lambda - \frac{1}{2}\theta^{\rho\sigma} a_\rho \left(\partial_\sigma \lambda\right), \end{aligned} \tag{4.16}$$

where $f_{\sigma\mu} = \partial_\sigma a_\mu - \partial_\mu a_\sigma$. Hence, the first order θ-expanded action is

$$S_A = -\frac{1}{4}\int d^4x \left[f_{\mu\nu}f^{\mu\nu} + 2\theta^{\mu\alpha} f_\alpha^{\ \nu} \left(f_\mu^{\ \beta} f_{\beta\nu} + \frac{1}{4}\eta_{\mu\nu} f^{\alpha\beta} f_{\alpha\beta} \right) \right], \tag{4.17}$$

where $\eta_{\mu\nu} = \text{diag}(+,-,-,-)$ is the Lorentz metric. Note that the tensor inside the parenthesis is traceless.

We compare this action to its commutative counterpart coupled to a gravitational background[2]

$$S_{g,a} = -\frac{1}{4}\int d^4x \sqrt{g} g^{\mu\rho} g^{\nu\sigma} f_{\mu\nu} f_{\rho\sigma}, \tag{4.18}$$

and expand the metric $g_{\mu\nu}$ around the flat metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \hat{h}_{\mu\nu}. \tag{4.19}$$

It will be convenient to separate the (small) deviation $\hat{h}_{\mu\nu}$ from the metric according to

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} + \frac{1}{4}\eta_{\mu\nu} h, \tag{4.20}$$

where h is the trace of $\hat{h}_{\mu\nu}$ and $h_{\mu\nu}$ is traceless. Expanding the action (4.18) up to first order in $\hat{h}$, we find that the trace-part drops out leading to

$$S_{g,a} = -\frac{1}{4}\int d^4x \left(f_{\mu\nu} f^{\mu\nu} + 2h^{\mu\nu} f_\mu^{\ \alpha} f_{\alpha\nu} \right). \tag{4.21}$$

Comparing this result with (4.17) we can deduce that non-commutativity has a similar effect as a deviation from the flat metric with

$$h_{\mu\nu} = \frac{1}{2}\left(\theta_{\mu\alpha} f^\alpha_{\ \nu} + \theta_{\nu\alpha} f^\alpha_{\ \mu} \right) + \frac{1}{4}\eta_{\mu\nu}\theta^{\alpha\beta} f_{\alpha\beta}. \tag{4.22}$$

Notice that so far the trace h is still arbitrary, as it does not appear in the action.

Adding fermions [113, 114]:

The non-commutative fermion action in the massless case is given by

$$S_{\text{ferm}} = \int d^4x \left[\bar{\Psi} \star \mathrm{i}\gamma^\mu D_\mu \Psi \right], \tag{4.23}$$

[2]See [52] and standard textbooks on general relativity, such as [111, 112].

with

$$D_\mu \Psi = \partial_\mu \Psi - \mathrm{i} g A_\mu \star \Psi.$$

The Seiberg-Witten map for Ψ in its simplest form [50] is

$$\Psi = \psi - \frac{1}{2}\theta^{\rho\sigma} A_\rho \partial_\sigma \psi, \tag{4.24}$$

and hence with (4.16) the θ expanded action up to first order becomes

$$S_{\text{ferm}} = \int d^4x \left[\bar{\psi}\mathrm{i}\gamma^\mu D^c_\mu \psi - \frac{1}{2}\left(\theta^{\mu\alpha} f_\alpha{}^\nu + \frac{1}{2}\eta^{\mu\nu}\theta^{\alpha\beta} f_{\alpha\beta}\right)\mathrm{i}\gamma_\nu D^c_\mu \psi \right], \tag{4.25}$$

with[3]

$$D^c_\mu \psi = \partial\psi - \mathrm{i} g a_\mu \psi.$$

The Gamma matrices γ^μ form the usual Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Introducing the "vielbeins" $e^\mu_a(x)$, which form a basis for the tangent space at point x, one can write the metric g_{ab} of curved space-time as

$$g_{ab}(x) = e^\mu_a(x) e^\nu_b(x) \eta_{\mu\nu}\,. \tag{4.26}$$

In order to write down a fermion action in curved space-time, one needs to introduce generalized Gamma matrices as $\gamma^a \equiv \gamma^\mu E^a_\mu$, where $E^a_\mu(x)$ denotes the inverse of the vielbein. The action for fermions in curved space-time coupled to the gauge field is given by

$$S = \int d^4x \sqrt{-g}\bar{\psi}[\mathrm{i}E^a_\mu \gamma^\mu(\partial_a - \mathrm{i}gA_a + \frac{1}{2}\omega_{a\rho\sigma}\Sigma^{\rho\sigma})]\psi, \tag{4.27}$$

where the last term describes the spin connection. $\Sigma^{\rho\sigma} = \frac{\mathrm{i}}{4}[\gamma^\rho, \gamma^\sigma]$ are the generators of the Lorentz group. Using the identities

$$\begin{aligned}
\gamma^\mu[\gamma^\rho, \gamma^\sigma] &= 2(\eta^{\mu\rho}\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho + \mathrm{i}\epsilon^{\mu\rho\sigma\nu}\gamma^5\gamma_\nu), \\
E^a_\mu \omega_{a\rho\sigma} &= -\Omega_{\rho\mu\sigma} + \Omega_{\mu\sigma\rho} - \Omega_{\sigma\rho\mu}, \\
\Omega_{\mu\rho\sigma} \equiv \eta_{\mu\nu}\partial_{[a} e^\nu_{b]} E^a_\rho E^b_\sigma &= \frac{1}{2}\eta_{\mu\nu}\left(\partial_a e^\nu_b - \partial_b e^\nu_a\right) E^a_\rho E^b_\sigma,
\end{aligned} \tag{4.28}$$

we can rewrite the action (4.27) as

$$\begin{aligned}
S = \int d^4x \sqrt{-g}\bar{\psi}\Big[&\mathrm{i}E^a_\mu \gamma^\mu(\partial_a - \mathrm{i}gA_a) + \\
&+ \frac{1}{4}\left(2\eta^{\mu\rho}\gamma^\sigma + \mathrm{i}\epsilon^{\mu\rho\sigma\nu}\gamma^5\gamma_\nu\right)\partial_{[a} e^\alpha_{b]}\left(\eta_{\rho\alpha}E^a_\mu E^b_\sigma - \eta_{\mu\alpha}E^a_\sigma E^b_\rho + \eta_{\sigma\alpha}E^a_\rho E^b_\mu\right)\Big]\psi.
\end{aligned} \tag{4.29}$$

[3]Here D^c_μ denotes the covariant derivative in the commutative world.

We now make a weak field expansion of the metric by using the following ansatz for the vielbein:

$$e_a^\mu \equiv \eta_a^\mu + \tilde{e}_a^\mu \,,$$

where the object η_a^μ looks like the Lorentz metric and $\tilde{e}_a^\mu$ is a small deviation thereof. For the inverse vielbein one hence finds up to first order in $\tilde{e}$:

$$E_\mu^a \approx \eta_\mu^a - \tilde{e}_\mu^a,$$

where the indices of η and $\tilde{e}$ were pulled up/down with $\eta_{\mu\nu}$ and η^{ab}, respectively. From $g_{ab} = \eta_{ab} + \hat{h}_{ab}$ one furthermore finds the relation

$$\hat{h}_{ab} = \tilde{e}_a^\mu \eta_{\mu b} + \tilde{e}_b^\mu \eta_{\mu a}.$$

Expanding the action (4.27) up to first order in $\tilde{e}$ (or $\hat{h}$) one gets

$$\begin{aligned} S = \int d^4x (1+\tilde{e})\bar{\psi}\Big[&\mathrm{i}(\eta_\mu^a - \tilde{e}_\mu^a)\gamma^\mu(\partial_a - \mathrm{i}gA_a) + \frac{1}{2}\gamma^\mu \left(\eta_\mu^a \partial_\nu \tilde{e}_a^\nu - \partial_\mu \tilde{e}\right) \\ &+ \frac{1}{4}\mathrm{i}\epsilon^{\mu\rho\sigma\nu}\gamma^5\gamma_\nu \partial_{[a}\tilde{e}_{b]}^\alpha \left(\eta_{\rho\alpha}\eta_\mu^a\eta_\sigma^b - \eta_{\mu\alpha}\eta_\sigma^a\eta_\rho^b + \eta_{\sigma\alpha}\eta_\rho^a\eta_\mu^b\right)\Big]\psi. \end{aligned} \tag{4.30}$$

with $\tilde{e} \equiv \tilde{e}_a^\mu \eta_\mu^a$. Obviously, the second line vanishes for symmetric vielbeins (i.e. $\eta_{\rho\alpha}\tilde{e}_b^\alpha \eta_\sigma^b$ symmetric in ρ and σ). Additionally assuming tracelessness $h = 0$, which follows from the harmonic gauge condition $\partial^b h_{ba} = 0$, this expression reduces to

$$S = \int d^4x \bar{\psi}\mathrm{i}(\eta_\mu^a - \tilde{e}_\mu^a)\gamma^\mu(\partial_a - \mathrm{i}gA_a)\psi, \tag{4.31}$$

which may be compared to the SW-expanded action (4.25). Unfortunately, the expression we would like to identify with $h_{\mu\nu}$, namely

$$\left(\theta^{\mu\alpha} f_\alpha{}^\nu + \frac{1}{2}\eta^{\mu\nu}\theta^{\alpha\beta} f_{\alpha\beta}\right),$$

is neither symmetric nor traceless, nor does it fulfill the harmonic gauge condition, and hence this identification cannot be made. One could try to make use of some of the freedom in the Seiberg-Witten map (cf. [50]) in combination with a more general $\hat{h}_{ab}$, i.e. one with non-vanishing trace, but unfortunately the result stays the same: the desired identification cannot be made.

However, Steinacker recently proposed a similar ansatz in reference [115] where the $U(1)$ sector of a general non-commutative $U(N)$ gauge theory in the matrix model formulation describes gravity, hence the name "emergent gravity". One may therefore expect to successfully include fermions in that more general framework[4].

[4] In fact, the idea of "induced gravity" emerging from a quantum field theory goes back to Sakharov [116, 117]. Many ideas in that direction have since been explored (cf. e.g. references in [115]).

Chapter 5

Quantization of the Slavnov Model

5.1 Dirac formalism

Before we try to quantize non-commutative $U(1)$ Maxwell theory with the Slavnov term, we will give a short introduction to the quantization-formalism introduced by Dirac [56]. To this end we will mainly follow reference [118]:

5.1.1 Hamilton systems with constraints

We start with an action for classical mechanics:

$$S = \int L(q_n, \dot{q}_n) dt \,, \tag{5.1}$$

where L is the Lagrangian and the $\dot{q}_n$ denote the time-derivatives of the generalized coordinates q_n. Variation of this action leads to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n}\,, \tag{5.2}$$

and using the chain rule for the left hand side of this equation leads to

$$\ddot{q}_{n'}\left(\frac{\partial^2 L}{\partial \dot{q}_{n'}\partial \dot{q}_n}\right) = \frac{\partial L}{\partial q_n} - \dot{q}_{n'}\frac{\partial^2 L}{\partial q_{n'}\partial \dot{q}_n}\,. \tag{5.3}$$

$\ddot{q}_{n'}$ can only be determined from this equation if the determinant

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_{n'}\partial \dot{q}_n}\right) \neq 0\,, \tag{5.4}$$

is unequal zero. In this case, a Legendre transformation leads to the so-called *Hamiltonian*

$$H(q_n, p_n) \equiv p_n\dot{q}_n(q_n, p_n) - L(q_n, \dot{q}_n(q_n, p_n)), \tag{5.5}$$

where the conjugate momenta p_n are defined as

$$p_n \equiv \frac{\partial L}{\partial \dot{q}_n}\,. \tag{5.6}$$

If the p_n do *not* depend on the $\dot{q}_n$, which is exactly the case when the determinant (5.4) vanishes[1], one gets certain relations

$$\phi_m(q, p) = 0, \tag{5.7}$$

out of (5.6), so-called *primary constraints*. But even if the transformation is singular, one can easily show that H depends only on q_n and p_n: Variation of the right hand side of (5.5) yields

$$\delta H = p_n\delta\dot{q}_n + \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n - \frac{\partial L}{\partial \dot{q}_n}\delta\dot{q}_n = \dot{q}_n\delta p_n - \frac{\partial L}{\partial q_n}\delta q_n, \tag{5.8}$$

where the definition (5.6) was used. Hence, from (5.5) follows

$$\left(\dot{q}_n - \frac{\partial H}{\partial p_n}\right)\delta p_n - \left(\frac{\partial L}{\partial q_n} + \frac{\partial H}{\partial q_n}\right)\delta q_n = 0. \tag{5.9}$$

Note that the variations δp_n and δq_n are *not independent* from each other because of the constraints. Since every function G on the phase space, which

[1]With (5.6) one has $\frac{\partial^2 L}{\partial \dot{q}_{n'}\partial \dot{q}_n} = \frac{\partial p_n}{\partial \dot{q}_{n'}}$.

vanishes on the subspace $\phi_m = 0$, can be written as a linear combination of the constraints ($G = g_m \phi_m$), one concludes that (5.9) must have the form

$$u_m \frac{\partial \phi_m}{\partial p_{m'}} \delta p_{m'} + u_m \frac{\partial \phi_m}{\partial q_{m'}} \delta q_{m'} = 0. \tag{5.10}$$

(A proof can be found in e.g. reference [118].) Comparing coefficients finally leads to the generalized Hamiltonian equations of motion

$$\dot{q}_n = \frac{\partial H}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n}, \tag{5.11a}$$

$$\dot{p}_n = -\frac{\partial H}{\partial q_n} - u_m \frac{\partial \phi_m}{\partial q_n}, \tag{5.11b}$$

where the equations (5.2) and (5.6) were used for the left hand side of (5.11b).

We now define the *Poisson bracket* as

$$\{f, g\}_{PB} = \frac{\partial f}{\partial q_n} \frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n} \frac{\partial g}{\partial q_n}. \tag{5.12}$$

It has the following properties:

antisymmetry: $\{f, g\}_{PB} = -\{g, f\}_{PB}$, (5.13a)

linearity: $\{c_1 f_1 + c_2 f_2, g\}_{PB} = c_1 \{f_1, g\}_{PB} + c_2 \{f_2, g\}_{PB}$, (5.13b)

product rule: $\{f_1 f_2, g\}_{PB} = f_1 \{f_2, g\}_{PB} + \{f_1, g\}_{PB} f_2$, (5.13c)

Jacobi identity: $\{f, \{g, h\}_{PB}\}_{PB} + \{g, \{h, f\}_{PB}\}_{PB} + \{h, \{f, g\}_{PB}\}_{PB} = 0.$ (5.13d)

Let g be an arbitrary function of q_n and p_n. Its time derivative is then given by

$$\dot{g} = \frac{\partial g}{\partial q_n} \dot{q}_n + \frac{\partial g}{\partial p_n} \dot{p}_n. \tag{5.14}$$

Using the Hamiltonian equations of motion (5.11) and the definition of the Poisson bracket (5.12) one obtains

$$\dot{g} = \{g, H\}_{PB} + u_m \{g, \phi_m\}_{PB}. \tag{5.15}$$

Following the notation of Dirac [56] we write this expression as a "weak" relation:

$$\dot{g} \approx \{g, H_T\}_{PB}, \tag{5.16}$$

where H_T denotes the "total" Hamiltonian, $H_T \equiv H + u_m\phi_m$. The "$\approx$" means that one has to evaluate all Poisson brackets before setting the constraints to zero ($\phi_m \approx 0$).

Since the constraints ϕ_m are functions of q_n and p_n as well, they must of course fulfill the same equation of motion (5.15) as the functions g. Hence, consistency demands:

$$0 \approx \dot{\phi}_m \approx \{\phi_m, H\}_{PB} + u_{m'}\{\phi_m, \phi_{m'}\}_{PB} \,. \tag{5.17}$$

This equation can be used to determine the u_m unless the second Poisson bracket vanishes ($\{\phi_m, \phi_{m'}\}_{PB}$=0). In that case one gets further constraints, so-called *secondary* constraints:

$$\chi(q,p) \approx 0\,. \tag{5.18}$$

Of course the secondary constraints have to fulfill the equation of motion (5.15), too. This may lead to further secondary constraints and so on.

Once all secondary constraints have been found, one can start to classify them. According to Dirac [56], there are two "classes" of constraints: *first class* and *second class* (not to be confused with *primary* and *secondary*). A phase space function (or constraint) is called *first class* when its Poisson brackets with all other constraints is (weakly) zero. If at least one of these Poisson brackets is unequal zero, one speaks of a *second class* function/constraint.

Let us go back to equation (5.17): As long as the second Poisson bracket does *not* vanish, one gets solutions for u_m:

$$u_m = U_m(p,q) + v_a V_{am}, \tag{5.19}$$

where U_m are special solutions of the inhomogeneous equation and V_{am} are solutions of the homogeneous equation

$$V_{am}\{\phi_j, \phi_m\}_{PB} = 0\,. \tag{5.20}$$

The v_a are arbitrary parameters, which means that some kind of freedom is contained in the theory. Consider a dynamic variable $g(t)$ with an initial value $g(0) \equiv g_0$: After the infinitesimal time interval δt one has

$$g(\delta t) = g_0 + \dot{g}\delta t = g_0 + \{g, H_T\}_{PB}\,\delta t\,, \tag{5.21}$$

where (5.16) was used. Defining

$$H' \equiv H + U_m\phi_m \quad \text{and} \quad \phi_a \equiv V_{am}\phi_m\,, \tag{5.22}$$

(see (5.19)), one may write

$$g(\delta t) = g_0 + \delta t \left(\{g, H'\}_{PB} + v_a \{g, \phi_a\}_{PB}\right) . \tag{5.23}$$

Due to (5.20) and the product rule for the Poisson bracket (5.13c) it is obvious that ϕ_a is a first-class constraint[2]. Furthermore, H' is also a first class function by construction (cf. (5.17)). As noticed earlier, v_a are arbitrary parameters, which means that $g(\delta t)$ is ambiguous as well: Replacing v_a with some v'_a in (5.23) leads to a different value for $g(\delta t)$, the deviation being

$$\Delta g(\delta t) = \delta t(v_a - v'_a) \{g, \phi_a\}_{PB} \equiv \epsilon_a \{g, \phi_a\}_{PB} . \tag{5.24}$$

If one interprets (5.24) as a *gauge transformation*, then the first-class constraints ϕ_a are obviously its generators. In doing two successive gauge transformations of g, one can easily show that the Poisson bracket $\{\phi_a, \phi_{a'}\}_{PB}$ generates a gauge transformation as well. Applying the product rule (5.13c) one can furthermore show that the Poisson bracket of two first class constraints is a first-class constraint itself. Hence, $\{\phi_a, \phi_{a'}\}_{PB}$ must be a linear combination of the first-class constraints in the model under consideration. Therefore, we deduce that all primary and secondary first-class constraints generate gauge transformations[3]. This fact should also be taken into account in the equations of motion. We therefore define the *extended Hamiltonian*

$$H_E \equiv H_T + v'_{a'}\phi_{a'} . \tag{5.25}$$

The generators $\phi_{a'}$ are all those which are not already contained in H_T and are therefore first-class *secondary* constraints. The corresponding equations of motion are now given by

$$\dot{g} \approx \{g, H_E\}_{PB} . \tag{5.26}$$

What about the second class constraints? In order to treat those we first consider the matrix $C_{AB} = \{\phi_A, \phi_B\}_{PB}$ where the ϕ_A now denote *all* constraints, and for simplicity we assume the *irreducible* case, i.e. that all $\phi_A \approx 0$ are independent from each other. Obviously, $\det C_{AB} \approx 0$ if there is at least one first class constraint among the ϕ_A. Redefining the constraints as $\phi_A \to a_A^{\ B}\phi_B$ with an appropriate invertible matrix $a_A^{\ B}$ one can always find an equivalent description of the constraint surface in terms of constraints $\gamma_a \approx 0$, $\chi_\alpha \approx 0$, whose Poisson bracket matrix reads *weakly*

$$\begin{array}{cc} & \begin{array}{cc} \gamma_a & \chi_\alpha \end{array} \\ \begin{array}{c} \gamma_b \\ \chi_\beta \end{array} & \begin{pmatrix} 0 & 0 \\ 0 & C_{\beta\alpha} \end{pmatrix} \end{array} , \tag{5.27}$$

[2] $\{\phi_j, \phi_a\}_{PB} = V_{am}\{\phi_j, \phi_m\}_{PB} + \{\phi_j, V_{am}\}_{PB}\,\phi_m \approx 0$.
[3] Initially, $\phi_a = V_{am}\phi_m$ consisted only of *primary* constraints.

where $C_{\beta\alpha}$ is an antisymmetric matrix that is everywhere invertible on the constraint surface [118]. In this representation, the constraints are completely split into first and second classes, and the number of second class constraints is obviously even.

A possible way of treating the second class constraints was invented by Dirac in introducing the so-called *Dirac bracket*

$$\{f, g\}_D \equiv \{f, g\}_{PB} - \{f, \chi_\alpha\}_{PB} C^{\alpha\beta} \{\chi_\beta, g\}_{PB} \,, \tag{5.28}$$

where $C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$. Since the extended Hamiltonian is first class, one can easily verify that H_E still generates the correct equations of motion in terms of the Dirac bracket:

$$\dot{g} \approx \{g, H_E\}_D \,. \tag{5.29}$$

The original Poisson bracket can be discarded after having served its purpose of distinguishing between first-class and second-class constraints and all the equations of the theory can now be formulated in terms of the Dirac bracket (see ref. [118] for detailed proof).

5.1.2 Field theoretic extension

We are now interested in the field theoretic extension of the formalism developed above and illustrate this with an example: free Maxwell theory (cf. ref. [118]). The action is given by

$$S = -\frac{1}{4} \int dt \int d^3x F_{\mu\nu} F^{\mu\nu} \,, \tag{5.30}$$

with the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{5.31}$$

As usual, Greek indices run from 0 to 3 while Latin indices run from 1 to 3. Furthermore, $x_0 \equiv t$. The fields $A_\mu(t, \vec{x})$ correspond to the $q_n(t)$ in the previous section. The variable $\vec{x}$ can be interpreted as a "continuous" index. According to (5.6) with $\dot{A}_\mu \equiv \partial_0 A_\mu = \frac{\partial A_\mu}{\partial t}$, the conjugate momenta are given by

$$\pi^\mu(\vec{x}) = \frac{\delta}{\delta \dot{A}_\mu(\vec{x})} \left(-\frac{1}{4} \int d^3x' F_{\rho\sigma}(\vec{x}') F^{\rho\sigma}(\vec{x}') \right) = F^{\mu 0}(\vec{x}) \,. \tag{5.32}$$

In analogy to $\{q_n, p_{n'}\}_{PB} = \delta_{nn'}$ we now have

$$\{A_\mu(\vec{x}), \pi^\nu(\vec{x}')\}_{PB} = \delta_\mu^\nu \delta^3 (\vec{x} - \vec{x}') \ . \tag{5.33}$$

Due to antisymmetry of the field tensor, equation (5.32) yields the primary constraint

$$\pi^0(\vec{x}) \approx 0 \, . \tag{5.34}$$

A Legendre transformation, as defined in (5.5), gives us the Hamiltonian of Maxwell theory:

$$H = \int d^3x \left(\pi^\mu \dot{A}_\mu + \frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} F^{r0} F_{r0} \right) \, . \tag{5.35}$$

The constraint (5.34), partial integration and the fact that $F_{r0} = -\pi^r$ yields

$$\begin{aligned} H &= \int d^3x \left(\frac{1}{4} F^{rs} F_{rs} - \frac{1}{2} F^{r0} F_{r0} + F^{r0} \partial_r A_0 \right) = \\ &= \int d^3x \left(\frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} \pi^r \pi^r - A_0 \partial_r \pi^r \right) \, . \end{aligned} \tag{5.36}$$

All time derivatives have now been replaced by conjugate momenta enabling us to use the consistency condition (5.17) to get

$$0 \approx \dot{\pi}^0 \approx \left\{ \pi^0, H \right\}_{PB} = \partial_r \pi^r ,$$

which yields the secondary constraint

$$\partial_r \pi^r \approx 0. \tag{5.37}$$

A further consistency check shows that (5.34) and (5.37) are the only constraints, since $\{\partial_r \pi^r, H\}_{PB} = 0$. Furthermore, they are *first-class* because of

$$\begin{aligned} \left\{ \pi^0(\vec{x}), \pi^0(\vec{x}') \right\}_{PB} &= 0, \\ \left\{ \pi^0(\vec{x}), \partial_r \pi^r(\vec{x}') \right\}_{PB} &= 0, \\ \left\{ \partial_r \pi^r(\vec{x}), \partial_r \pi^r(\vec{x}') \right\}_{PB} &= 0. \end{aligned}$$

Obviously, the Hamiltonian H is first-class as well and therefore can be used for H' from (5.22). The total Hamiltonian H_T hence becomes

$$H_T = \int \left(\frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} \pi^r \pi^r \right) d^3x - \int A_0 \partial_r \pi^r d^3x + \int v(\vec{x}) \pi^0 d^3x \, , \tag{5.38}$$

where $v(\vec{x})$ is arbitrary. Inserting A_0 into the equation of motion (5.16), we see that $v(\vec{x}) = \dot{A}_0(\vec{x})$. This means that the time derivative of A_0 is *ambiguous* and that A_0 as well as its conjugate momentum $\pi^0 = 0$ are *unphysical*. Using the "extended" Hamiltonian H_E one may eliminate these unphysical quantities:

$$H_E = H_T + \int u(x)\partial_r \pi^r d^3x \,. \tag{5.39}$$

Choosing $v(x) = 0$ and $u'(x) = u(x) - A_0$ one arrives at the new (simplified) Hamiltonian (cf. (5.38))

$$H = \int \left(\frac{1}{4} F^{rs} F_{rs} + \frac{1}{2} \pi^r \pi^r \right) d^3x + \int u'(x) \partial_r \pi^r d^3x \,, \tag{5.40}$$

which still produces the correct equations of motion for all physically relevant variables.

5.2 Quantization of the free Slavnov model

We consider the action (2.27) introduced in Chapter 2.2.1, this time with a covariant gauge fixing, i.e. $\xi = 0$ (cf. references [119, 120]). The bilinear parts are

$$S_{\text{bi}} = \int d^4x \left(\frac{1}{2} A^\nu \partial^\mu \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) + B \partial_\mu A^\mu + \frac{\alpha}{2} B^2 - \lambda \widetilde{\partial}_\mu A^\mu \right) . \tag{5.41}$$

We find the following equations of motion for the free fields:

$$\partial^\mu \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) - \partial_\nu B + \widetilde{\partial}_\nu \lambda = 0, \tag{5.42a}$$

$$\widetilde{\partial}^\mu A_\mu = 0, \tag{5.42b}$$

$$\partial^\mu A_\mu + \alpha B = 0. \tag{5.42c}$$

In order to find the constraints of the action (5.41) according to the formalism developed in the previous section, we need to do a Legendre transformation which involves 3-dimensional integrals. However, if we choose $\theta^{0i} = 0$, which we must in order to preserve causality, it is easy to see that the same properties of the star product (cyclic permutation, etc.) hold under the 3-dimensional integral as under the 4-dimensional integral. Hence, we find the three *primary constraints*

$$\pi^0 - B = 0, \tag{5.43a}$$

$$\pi_B = 0, \tag{5.43b}$$

$$\pi_\lambda = 0, \tag{5.43c}$$

and the Hamiltonian

$$H_0 = \int d^3x \bigg(\frac{1}{2}\partial_i A_j \left(\partial^i A^j - \partial^j A^i\right) + \frac{1}{2}\pi^i\pi^i + \pi^i\partial^i A^0 + \\ + B\partial^i A^i - \frac{\alpha}{2}B^2 + \lambda\widetilde{\partial}^i A^i \bigg), \tag{5.44}$$

where $\pi^i = (\partial^i A^j - \partial^j A^i)$ are the conjugate momenta of A_i. The consistency conditions (5.17) lead to the two *secondary constraints*

$$(\widetilde{\partial}A) = 0, \tag{5.45a}$$

$$(\widetilde{\partial}\pi) = 0. \tag{5.45b}$$

Furthermore, one has the following Poisson brackets between the constraints:

$$\{\pi_\lambda, (\text{all constraints})\}_{PB} = 0, \tag{5.46a}$$

$$\left\{(\pi^0 - B)(\vec{x}), \pi_B(\vec{x}')\right\}_{PB} = -\delta^3(\vec{x} - \vec{x}'), \tag{5.46b}$$

$$\left\{(\widetilde{\partial}A)(\vec{x}), (\widetilde{\partial}\pi)(\vec{x}')\right\}_{PB} = \widetilde{\Box}_x \delta^3(\vec{x} - \vec{x}'). \tag{5.46c}$$

All other brackets vanish. We therefore have one first-class constraint (π_λ) and four second-class constraints leading to the following Dirac brackets between the fields and their conjugate momenta:

$$\{\lambda, \pi_\lambda\}_D = \{\lambda, \pi_\lambda\}_{PB}\,, \tag{5.47a}$$

$$\{B, \pi_B\}_D = 0, \tag{5.47b}$$

$$\left\{A_0, \pi^0\right\}_D = \left\{A_0, \pi^0\right\}_{PB}\,, \tag{5.47c}$$

$$\left\{A_i(\vec{x}), \pi^j(\vec{x}')\right\}_D = \left(\delta_i^j - \frac{\widetilde{\partial}_i\widetilde{\partial}^j}{\widetilde{\Box}}\right)_x \delta^3(\vec{x} - \vec{x}'). \tag{5.47d}$$

Note that the Dirac brackets (5.47) are independent of α. For simplicity, we now continue our discussion in Feynman gauge ($\alpha = 1$) and make the following ansatz for the gauge field:

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\rho \epsilon_\mu(\rho) \left(a_\rho^+(\vec{k})e^{\mathrm{i}kx} + a_\rho^-(\vec{k})e^{-\mathrm{i}kx}\right)\Big|_{k_0=\omega_k} \\ \equiv A_\mu^+(x) + A_\mu^-(x), \tag{5.48}$$

where the polarization vectors $\epsilon_\mu(\rho)$ must be transversal with respect to $\tilde{k}^\mu$. If we assume that the Dirac brackets for a_ρ^+ and a_ρ^- (which eventually will become commutators when quantizing) are given by

$$\left\{a_\rho^-(\vec{k}), a_\sigma^+(\vec{k}')\right\}_D = 2\omega_k(2\pi)^3 g_{\rho\sigma}\delta^3(\vec{k} - \vec{k}'), \tag{5.49}$$

and consider

$$\pi^k = \partial^k A^0 - \partial^0 A^k,$$

we find that in order to be consistent with the Dirac brackets (5.47c) and (5.47d) derived earlier, the following relation must hold for the sum over the polarization tensors:

$$\sum_\rho \epsilon_\mu(\rho)\epsilon_\nu(\rho) = \left(g_{\mu\nu} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right). \tag{5.50}$$

Let us check this for the simple case where $\theta_{12} = \theta$ is the only non-vanishing component of the deformation matrix and $k^\mu = (k_0, 0, 0, k_3)$: Obviously, $\tilde{k}_\mu$ vanishes identically, leaving us with the solution of ordinary QED in Feynman gauge, since the two secondary constraints (5.45) also vanish identically and no longer pose extra conditions on the gauge field. The far more interesting case is where the gauge field propagates in the (x^1, x^2)-plane, e.g. $k^\mu = (k_0, 0, k_2, 0)$: Now we have $\tilde{k}_\mu = (0, \theta k_2, 0, 0)$ and

$$\left(g_{\mu\nu} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \sum_{\rho=1}^{3} \epsilon_\mu(\rho)\epsilon_\nu(\rho), \tag{5.51}$$

with

$$\begin{aligned} \epsilon_\mu(1) &= (1, 0, 0, 0), \\ \epsilon_\mu(2) &= (0, 0, 1, 0), \\ \epsilon_\mu(3) &= (0, 0, 0, 1). \end{aligned} \tag{5.52}$$

Relation (5.51) reflects the fact that the Slavnov constraint (5.42b) (cf. also (5.45)) eliminates one degree of freedom of the gauge field A_μ whenever it propagates in the (x^1, x^2)-plane, i.e. the plane of non-commuting coordinates.

In the quantized model, the Dirac brackets get replaced by commutators and instead of (5.49) one has

$$\left[\hat{a}^-_\rho(\vec{k}), \hat{a}^+_\sigma(\vec{k}') \right] = 2\mathrm{i}\omega_k (2\pi)^3 g_{\rho\sigma} \delta^3(\vec{k} - \vec{k}'). \tag{5.53}$$

Now we can derive the gauge field two point function from (5.48) using (5.50) and (5.53):

$$
\begin{aligned}
\langle 0|TA_\mu(x)A_\nu(y)|0\rangle &= \big\langle 0\big| \left(\left[A_\mu^-(x), A_\nu^+(y)\right]\Theta(x^0-y^0) + x \leftrightarrow y\right)|0\big\rangle \\
&= -\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(g_{\mu\nu} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right)\Big(\Theta(x^0-y^0)e^{\mathrm{i}k(x-y)} + \\
&\qquad + \Theta(y^0-x^0)e^{-\mathrm{i}k(x-y)}\Big)\Big|_{k_0=\omega_k}, \quad (5.54)
\end{aligned}
$$

where T denotes the time-ordering operator and $\Theta(x^0-y^0)$ is the Heaviside step function. Furthermore, using an integral representation of the step function such as the well-known formula[4]

$$
\Theta(\pm(x^0-y^0)) = \frac{\pm 1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} d\tau \frac{e^{-\mathrm{i}\tau(x^0-y^0)}}{\tau \pm \mathrm{i}\epsilon}, \tag{5.55}
$$

and the substitution

$$
\omega_k - \tau \equiv k^0,
$$

one arrives at (remember $\tilde{k}_0 = 0$)

$$
\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-\mathrm{i}k(x-y)} \frac{-\mathrm{i}}{k^2+\mathrm{i}\epsilon}\left(g_{\mu\nu} - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2}\right), \tag{5.56}
$$

which is consistent with the propagator in Table A.1 of Appendix A with $\xi = 0$ and $\alpha = 1$.

[4]The validity of (5.55) may be easily verified by using the residue theorem of complex analysis.

Chapter 6

NCGFT on $\mathbb{R}^4_\Theta$ with an Oscillator Term

In this chapter we would like to discuss a different ansatz to handle the UV/IR mixing problem. As noted in the introduction, Grosse and Wulkenhaar were able to solve the UV/IR mixing problem [13, 14] by adding an oscillator-like term in the action of the Euclidean scalar ϕ^4 model on deformed $\mathbb{R}^4_\Theta$ space. Inspired by the renormalizability of non-commutative ϕ^4 theory in $\mathbb{R}^4_\Theta$ with an oscillator term, we will now try to construct a renormalizable non-commutative $U(1)$ gauge theory in a similar way [7], for simplicity also in Euclidean space.

6.1 Constructing the action

In ϕ^4 theory, the propagator was modified by the oscillator term in such a way that it essentially became the Mehler kernel, which in momentum space reads

$$\begin{aligned}
K_M(p,q) &= \frac{\omega^3}{2\pi^2}\int\limits_0^\infty d\alpha \frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}\exp\left(-\frac{\frac{\omega}{2}(p^2+q^2)(1+e^{-2\alpha})-2\omega e^{-\alpha}pq}{(1-e^{-2\alpha})}\right) \\
&= \frac{\omega^3}{8\pi^2}\int\limits_0^\infty \frac{d\alpha}{\sinh^2(\alpha)}\exp\left(-\frac{\omega}{4}\coth\left(\frac{\alpha}{2}\right)u^2-\frac{\omega}{4}\tanh\left(\frac{\alpha}{2}\right)v^2\right) ,
\end{aligned} \tag{6.1}$$

where ω is some parameter fixed by the action and where the "short" variable $u=(p-q)$ and the "long" variable $v=(p+q)$ have been introduced. Here, we will try to do the same thing: Since an oscillator term $\int d^4x\Omega^2\tilde{x}^2A_\mu A_\mu$ is not gauge invariant, there are basically two possible ways to construct the model: either one adds further terms in order to make the action gauge invariant (cf. [121, 122]) or one views the oscillator term as part of the gauge fixing part of the action. Here, we will take the latter approach and note that the oscillator term has the form of a mass term with non-constant "mass" $m^2=\Omega^2\tilde{x}^2$. However, we will add a further term to the gauge fixing action in order to simplify the gauge field propagator. Our starting point is hence the following action:

$$S=\int d^4x\left[\frac{1}{4}F_{\mu\nu}F_{\mu\nu}-\frac{1}{2}A_\mu\left(\partial_\mu\partial_\nu-\Omega^2\tilde{x}^2\delta_{\mu\nu}\right)A_\nu\right], \tag{6.2}$$

with

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu-\partial_\nu A_\mu-\mathrm{i}g\left[A_\mu \stackrel{\star}{,} A_\nu\right], \\
\tilde{x}_\mu &= \left(\theta^{-1}\right)_{\mu\nu}x_\nu, \\
\mathrm{i}\theta_{\mu\nu} &= \left[x_\mu \stackrel{\star}{,} x_\nu\right].
\end{aligned} \tag{6.3}$$

$\theta_{\mu\nu}$ is assumed to be a constant skew-symmetric matrix of full rank and Ω is a constant parameter. Some remarks concerning this action are in order. Recall that the Groenewold-Moyal-Weyl star product has the properties (1.13) as well as:

$$\left\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu(x)\right\}=2\tilde{x}_\mu A_\nu(x), \tag{6.4}$$

for the star-anticommutator between $\tilde{x}_\mu$ and an arbitrary field (in this case the gauge field A_ν). Due to this property, one may write for the oscillator term

$$\frac{1}{4}\left\{\tilde{x}_\nu \stackrel{\star}{,} A_\mu\right\}\star\left\{\tilde{x}_\nu \stackrel{\star}{,} A_\mu\right\}=\left(\tilde{x}_\nu A_\mu\right)\star\left(\tilde{x}_\nu A_\mu\right), \tag{6.5}$$

and the remaining star is removed by the integral over space according to (1.13a). Hence, there are only ordinary products left in the oscillator term and A and $\tilde{x}$ may be rearranged to the form written in (6.2). In order to avoid confusion, we will not use the simplified notation of Chapter 3, but explicitly spell out all necessary star product symbols (except for those cases where the star product may be omitted due to relation (1.13a)). Furthermore, we will be accurate about distinguishing between star-commutators $[A_\mu \stackrel{\star}{,} A_\nu]$ and star-anticommutators $\{A_\mu \stackrel{\star}{,} A_\nu\}$.

From the bilinear part of the action (6.2) we easily arrive at the equations of motion for the free fields:

$$\frac{\delta S_{\text{bi}}}{\delta A_\mu} = \left(-\Delta_4 + \Omega^2 \tilde{x}^2\right) A_\mu \,. \tag{6.6}$$

Notice that the terms $\partial_\mu \partial_\nu A_\nu$ have cancelled due to gauge fixing. The inverse of the operator $(\Delta_4 - \Omega^2 \tilde{x}^2)$ gives the Mehler kernel (6.1), which will become the propagator of the gauge field. In case one chooses the block-diagonal form

$$\theta_{\mu\nu} = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{6.7}$$

for the deformation parameter, i.e. the simplest case, one has $\omega = \frac{\theta}{\Omega}$ for the parameter in the Mehler kernel (6.1). Also notice that it has the property

$$\int d^4 q K_M(p,q) = \frac{1}{p^2}\left(1 - e^{-\frac{\omega}{2}p^2}\right) , \tag{6.8}$$

which in the limit $\omega \to \infty$ reduces to $\frac{1}{p^2}$. This means that for $\Omega = 0$ the usual propagator in Feynman gauge is recovered (as should be the case).

Now we need to find the ghost sector for the action. In order to do this, we need to rewrite the gauge fixing in terms of some multiplier field and add ghosts. Since our "mass" $\Omega^2\tilde{x}^2$ is x-dependent, we cannot simply employ the gauge fixing and ghost sector of Curci and Ferrari [123] (see also [124]).

Instead, we suggest the following gauge fixed action in the classical limit:

$$\begin{aligned}
S &= S_{\mathrm{inv}} + S_{\mathrm{m}} + S_{\mathrm{gf}}\,, \\
S_{\mathrm{inv}} &= \frac{1}{4}\int d^4x\, F_{\mu\nu} \star F_{\mu\nu}\,, \\
S_{\mathrm{m}} &= \frac{\Omega^2}{4}\int d^4x \left(\frac{1}{2}\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \star \{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} + \{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \star \{\tilde{x}_\mu \stackrel{\star}{,} c\}\right) = \\
&= \frac{\Omega^2}{8}\int d^4x\, (\tilde{x}_\mu \star \mathcal{C}_\mu)\,, \\
S_{\mathrm{gf}} &= \int d^4x \left[B \star \partial_\mu A_\mu - \frac{1}{2} B \star B - \bar{c} \star \partial_\mu s A_\mu - \frac{\Omega^2}{8}\widetilde{c}_\mu \star s\,\mathcal{C}_\mu\right]
\end{aligned} \tag{6.9}$$

with

$$\mathcal{C}_\mu = \Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}]\Big)\,. \tag{6.10}$$

This action is invariant under the BRST transformations given by

$$\begin{aligned}
&sA_\mu = D_\mu c = \partial_\mu c - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} c]\,, && s\bar{c} = B, \\
&sc = \mathrm{i}g c \star c, && sB = 0, \\
&s\widetilde{c}_\mu = \tilde{x}_\mu, && s^2\varphi = 0\ \forall\ \varphi \in \{A_\mu, B, c, \bar{c}, \widetilde{c}_\mu\}\,.
\end{aligned} \tag{6.11}$$

B is the multiplier field implementing the gauge fixing, which for $\widetilde{c}_\mu \to 0$ reduces to the usual covariant Feynman gauge $\partial_\mu A_\mu - B = 0$. Ω is a constant parameter and c, $\bar{c}$ are the ghost/antighost, respectively. The "mass" term for the ghosts (cf. second term in S_{m}) has been introduced in order to have a Mehler kernel also for the ghost propagator. The field $\widetilde{c}_\mu$ is a multiplier field with mass dimension 1 and ghost number -1, which imposes a constraint, namely on-shell BRST invariance of $\mathcal{C}_\mu$. In fact, because of $s\tilde{x}_\mu = 0$, this constraint also implies on-shell BRST invariance of the mass terms S_{m}. Furthermore, $s^2\,\mathcal{C}_\mu = 0$ vanishes identically, i.e. off-shell. Using the properties of the star product (1.13), (6.4) and (6.5), one may rewrite S_{m} also in the form

$$S_{\mathrm{m}} = \int d^4x\, \Omega^2 \tilde{x}^2 \left(\frac{1}{2}A^2 + \bar{c}c\right)\,, \tag{6.12}$$

which is the most convenient one for determining the propagators.

A further comment we would like to make is that the classical action may be reexpressed by the formula

$$S = \int d^4x \left[\frac{1}{4}F_{\mu\nu} \star F_{\mu\nu} + s\left(\bar{c} \star \partial_\mu A_\mu - \frac{1}{2}\bar{c} \star B\right) + \frac{\Omega^2}{8}\, s\,(\widetilde{c}_\mu \star \mathcal{C}_\mu)\right], \tag{6.13}$$

showing the unphysical character of the s-variations.

The equations of motion at the classical level read:

$$\frac{\delta\Gamma}{\delta B} = \partial_\mu A_\mu - B + \frac{\Omega^2}{8}\Big([\{\tilde{x}_\mu \stackrel{\star}{,} c\} \stackrel{\star}{,} \widetilde{c}_\mu] - \{\tilde{x}_\mu \stackrel{\star}{,} [\widetilde{c}_\mu \stackrel{\star}{,} c]\} \Big) = 0, \tag{6.14a}$$

$$\begin{aligned}\frac{\delta\Gamma}{\delta A_\nu} &= \left(-\Delta_4 + \Omega^2\tilde{x}^2\right) A_\nu + \mathrm{i}g\,[A_\mu \stackrel{\star}{,} F_{\mu\nu}] + \mathrm{i}g\partial_\mu\,[A_\mu \stackrel{\star}{,} A_\nu] + \mathrm{i}g\,\{\partial_\nu \bar{c} \stackrel{\star}{,} c\} + \\ &\quad + \partial_\nu(\partial A) - \partial_\nu B + \frac{\Omega^2}{8}\Big(\{[D_\nu c \stackrel{\star}{,} \widetilde{c}_\mu] \stackrel{\star}{,} \tilde{x}_\mu\} + [\{D_\nu c \stackrel{\star}{,} \tilde{x}_\mu\} \stackrel{\star}{,} \widetilde{c}_\mu] \Big) - \\ &\quad - \mathrm{i}g\frac{\Omega^2}{8}\Big(\{c \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} \{A_\nu \stackrel{\star}{,} \widetilde{c}_\mu\}\}\} + \{c \stackrel{\star}{,} \{\widetilde{c}_\mu \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\}\}\} \Big) = 0,\end{aligned} \tag{6.14b}$$

$$\begin{aligned}\frac{\delta\Gamma}{\delta \bar{c}} &= \left(-\Delta_4 + \Omega^2\tilde{x}^2\right) c - \mathrm{i}g\frac{\Omega^2}{8}\Big(\{\{\tilde{x}_\mu \stackrel{\star}{,} c \star c\} \stackrel{\star}{,} \widetilde{c}_\mu\} + \{\tilde{x}_\mu \stackrel{\star}{,} \{\widetilde{c}_\mu \stackrel{\star}{,} c \star c\}\} \Big) \\ &\quad + \mathrm{i}g\partial_\mu\,[A_\mu \stackrel{\star}{,} c] = 0,\end{aligned} \tag{6.14c}$$

$$\begin{aligned}\frac{\delta\Gamma}{\delta c} &= \left(\Delta_4 - \Omega^2\tilde{x}^2\right) \bar{c} + \frac{\Omega^2}{8}\Big(\{\widetilde{c}_\mu \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} B\}\} + \{\tilde{x}_\mu \stackrel{\star}{,} \{\widetilde{c}_\mu \stackrel{\star}{,} B\}\} \Big) - \\ &\quad - \mathrm{i}g\,[A_\mu \stackrel{\star}{,} \partial_\mu \bar{c}] - \frac{\Omega^2}{8} D_\nu \Big(\{\tilde{x}_\mu \stackrel{\star}{,} \{A_\nu \stackrel{\star}{,} \widetilde{c}_\mu\}\} + \{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} \widetilde{c}_\mu\} \Big) + \\ &\quad + \mathrm{i}g\frac{\Omega^2}{8}\Big([c \stackrel{\star}{,} [\widetilde{c}_\mu \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\}]] - [c \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} [\bar{c} \stackrel{\star}{,} \widetilde{c}_\mu]\}] \Big) = 0,\end{aligned} \tag{6.14d}$$

$$\frac{\delta\Gamma}{\delta \widetilde{c}_\mu} = -\frac{\Omega^2}{8} s\,\mathcal{C}_\mu = 0. \tag{6.14e}$$

Finding solutions to these equations and discussing the non-standard gauge fixing (6.14a) are the tasks of a work in progress [125].

The bilinear parts of the action, however, lead to the following improved propagators:

$$\begin{aligned}G^A_{\mu\nu}(x-y) &= \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1} \delta_{\mu\nu}\delta^4(x-y), \\ G^{\bar{c}c}(x-y) &= \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1} \delta^4(x-y),\end{aligned} \tag{6.15a}$$

$$\begin{aligned}G^{BA}_\mu(x-y) &= \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1} \partial_\mu\delta^4(x-y), \\ G^B(x-y) &= \Big[\partial_\mu \left(-\Delta_4 + \Omega^2\tilde{x}^2\right)^{-1} \partial_\mu - 1\Big]\, \delta^4(x-y).\end{aligned} \tag{6.15b}$$

Both the gauge field and the ghost propagators are essentially the Mehler kernel (6.1), so we may expect improved IR behaviour of the Feynman graphs. Since there are no vertices involving the B field and since the additional multiplier $\widetilde{c}_\mu$ has no propagator, neither field will play a role in loop corrections.

When adding external sources for the non-linear BRST transformations sA_μ and sc, such as

$$S_{\text{ext}} = \int d^4x \left[\rho_\mu \star sA_\mu + \sigma \star sc\right], \tag{6.16}$$

we arrive at the Slavnov-Taylor identity

$$\begin{aligned}\mathcal{S}\left(S_{\text{tot}}\right) = \int d^4x \Bigg(&\frac{\delta S_{\text{tot}}}{\delta\rho_\mu} \star \frac{\delta S_{\text{tot}}}{\delta A_\mu} + \frac{\delta S_{\text{tot}}}{\delta\sigma} \star \frac{\delta S_{\text{tot}}}{\delta c} + B \star \frac{\delta S_{\text{tot}}}{\delta\bar{c}} + \\ &+ \tilde{x}_\mu \star \frac{\delta S_{\text{tot}}}{\delta\widetilde{c}_\mu}\Bigg) = 0,\end{aligned} \tag{6.17}$$

with

$$S_{\text{tot}} = S + S_{\text{ext}}.$$

Finally, we also note the following: Using the equation of motion (6.14a) one may eliminate the B-field[1] from the action (6.9). In that form, it becomes obvious that the bilinear parts of the gauge fixed action are invariant under a Langmann-Szabo duality [63]. As usual, without the B-field the BRST transformation of $\bar{c}$ is nilpotent only on-shell:

$$s^2\bar{c} = \frac{\delta S}{\delta\bar{c}}\,. \tag{6.18}$$

A further comment we should make concerns unitarity of the model: It is known that the S-matrix of gauge theories with non-zero mass terms à la Curci and Ferrari is not unitary (see [126, 127, 128]). The reason for this is that the BRST transformations involving the B field fail to be nilpotent. In our present model, on the other hand, we have a different situation: the BRST transformations (6.11) are indeed nilpotent and our "mass" is $\tilde{x}^2$. Therefore, one may hope for unitarity of this model, which of course remains to be verified.

6.2 Outlook

In a first step (work in progress [125]) one should analyze the one-loop calculation of the vacuum polarization for the $U(1)$-photon with the presented

[1]This is equivalent to integrating out the B-field in the path integral.

concepts, in order to demonstrate that one is able to cure the UV/IR mixing problem. With the improved Mehler propagators it is expected that the troublesome UV/IR contributions

$$\Pi^{\mathrm{IR}}_{\mu\nu}(p) \propto \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2}\,, \qquad \tilde{p}_\mu = \theta_{\mu\nu}\tilde{p}_\nu\,, \tag{6.19}$$

which are gauge fixing independent, will cancel. In a further step more general considerations such as renormalization to all orders, RG-flow, etc. are required.

Conclusion

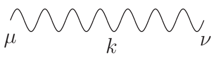
In trying to find an IR finite and consistent non-commutative version of $U(1)$ gauge theory, two very promising candidates were discussed. Both models rely on adding some sort of "improvement term" to the action: The main focus was on the addition of the Slavnov term [11, 12] which was introduced in Chapter 2.2 and removes problematic IR divergences through a new transversality condition. Especially the model's rich symmetry structure (cf. Chapter 3) makes it not only highly interesting, but these symmetries, in particular the topological vector supersymmetry, seem to be the reason for its improved IR behaviour. Further implications of the Slavnov constraint were finally discussed in Chapter 5 using the Hamilton formalism á la Dirac [56].

The second model under consideration was based on the inclusion of a harmonic oscillator potential in the action of non-commutative gauge theory (cf. Chapter 6). This extension was motivated by the Grosse-Wulkenhaar model [13, 14] of non-commutative ϕ^4 theory and improved the IR behaviour by essentially replacing the usual propagators with Mehler kernels and hence breaking translation invariance.

Further ideas, which were outlined in Chapter 4, include the extension to non-commutative Minkowski time, a possible connection between the Slavnov model and string theory and, finally, gravitational effects of non-commutativity.

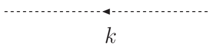
Appendix A

Feynman Rules

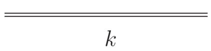
photon propagator 

$$\mathrm{i}\Delta^{A}_{\mu\nu}(k) = -\frac{\mathrm{i}}{k^2}\left[g_{\mu\nu} - a k_\mu k_\nu - \frac{\tilde{k}_\mu \tilde{k}_\nu}{\tilde{k}^2} + b\,(n_\mu k_\nu + k_\mu n_\nu) - b'\left(\tilde{k}_\mu k_\nu + k_\mu \tilde{k}_\nu\right)\right]$$

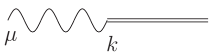
$$\text{with}\quad a = \frac{(1-\alpha)k^2 - \zeta^2 (nk)^2 \left[n^2 - \frac{(n\tilde{k})^2}{\tilde{k}^2}\right]}{[k^2 - \zeta(nk)^2]^2}\,,\quad b = \frac{\zeta(nk)}{k^2-\zeta(nk)^2}\,,\quad b' = \frac{(n\tilde{k})}{\tilde{k}^2} b,\quad \zeta = \frac{\xi}{n^2}$$

ghost propagator 

$\mathrm{i}\Delta^{c\bar{c}}(k) = \frac{\mathrm{i}}{k^2-\zeta(nk)^2}$

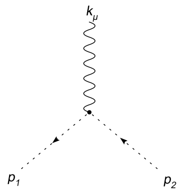
λ propagator 

$\mathrm{i}\Delta^{\lambda\lambda}(k) = \mathrm{i}\frac{k^2}{\tilde{k}^2}$

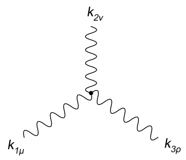
mixed propagator 

$\mathrm{i}\Delta^{\lambda A}_{\mu}(k) = \frac{\tilde{k}_\mu}{\tilde{k}^2}$

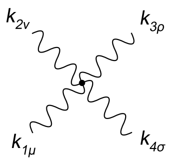
Table A.1: The propagators of the Slavnov model in $\mathbb{M}^d_{\mathrm{NC}}$ with interpolating gauge fixing



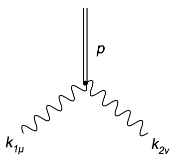
$$V_{\mu c\bar{c}}(p_1,p_2,k) = 2g\left(p_{2\mu} - \zeta(np_2)n_\mu\right)\sin\left(\frac{p_1\tilde{p}_2}{2}\right)$$



$$\widetilde{V}^{3A}_{\mu\nu\rho} = 2g\sin\left(\frac{k_1\tilde{k}_2}{2}\right)\Big(g_{\nu\rho}(k_2-k_3)_\mu + \\ + g_{\mu\rho}(k_3-k_1)_\nu + g_{\mu\nu}(k_1-k_2)_\rho\Big)$$



$$\widetilde{V}^{4A}_{\mu\nu\rho\sigma} = -4\mathrm{i}g^2\Bigg[(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\sin\left(\frac{k_1\tilde{k}_2}{2}\right)\sin\left(\frac{k_3\tilde{k}_4}{2}\right) + \\ +(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\sigma}g_{\nu\rho})\sin\left(\frac{k_1\tilde{k}_3}{2}\right)\sin\left(\frac{k_2\tilde{k}_4}{2}\right) + \\ +(g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma})\sin\left(\frac{k_1\tilde{k}_4}{2}\right)\sin\left(\frac{k_2\tilde{k}_3}{2}\right)\Bigg]$$



$$\widetilde{V}^{\lambda AA}_{\mu\nu} = 2\mathrm{i}g\theta_{\mu\nu}\sin\left(\frac{k_1\tilde{k}_2}{2}\right)$$

Table A.2: The vertices of the Slavnov model in $\mathbb{M}^d_{\mathrm{NC}}$ with interpolating gauge fixing

Appendix B

One-loop Graphs of the Gauge Boson Self-Energy with Slavnov Term

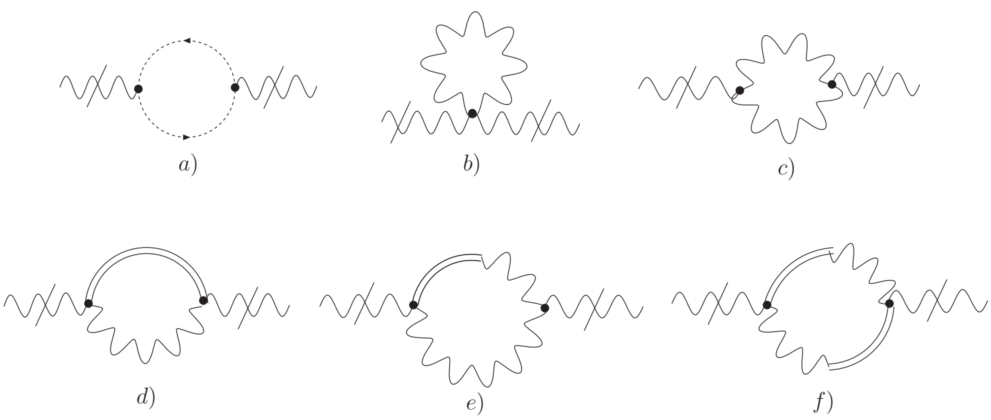


Figure B.1: Gauge boson self-energy — amputated graphs

The gauge boson self-energy at one-loop level consists of six graphs as depicted in Figure B.1: the ghost loop $\Pi^{\mu\nu}_a(p)$ (Fig. B.1a), the tadpole graph $\Pi^{\mu\nu}_b(p)$ (Fig. B.1b), the boson loop $\Pi^{\mu\nu}_c(p)$ (Fig. B.1c), the graph with one inner λ-propagator $\Pi^{\mu\nu}_d(p)$ (Fig. B.1d), the graph with one inner λA-propagator $\Pi^{\mu\nu}_e(p)$ (Fig. B.1e) and the graph with two inner λA-propagators $\Pi^{\mu\nu}_f(p)$ (Fig. B.1f). In order to be more general, all calculations in this appendix will be done in d-dimensional space-time rather than 4-dimensional. This way we will also see how some numerical factors depend on the space-time dimension through the trace of the metric.

The first term in the expansion (2.22) of Chapter 2.2.3, this time in d dimensions, is then given by

$$\int d^dk I^{\mu\nu}(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) = \int d^dk \sum_{i=a-f} I^{\mu\nu}_i(k,0)\sin^2\left(\frac{k\tilde{p}}{2}\right) \equiv$$
$$\equiv \mathrm{i}\Pi^{\mu\nu}_{a,\mathrm{IR}}(p) + \mathrm{i}\Pi^{\mu\nu}_{b,\mathrm{IR}}(p) + \mathrm{i}\Pi^{\mu\nu}_{c,\mathrm{IR}}(p) + \mathrm{i}\Pi^{\mu\nu}_{d,\mathrm{IR}}(p) + \mathrm{i}\Pi^{\mu\nu}_{e,\mathrm{IR}}(p) + \mathrm{i}\Pi^{\mu\nu}_{f,\mathrm{IR}}(p). \tag{B.1}$$

Finally, in order to be able to track terms that appear only due to the Slavnov term, we introduce a further parameter ς, which is 1 in the case where the Slavnov term is present and 0 in the case where we consider the model *without* the Slavnov term.

B.1 Ghost loop:

Considering the Feynman rules given in Tables A.1 and A.2 of Appendix A, one obtains for the ghost loop graph depicted in Figure B.1a)

$$\mathrm{i}\Pi^{\mu\nu}_{a,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d}\sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{-\left[k^\mu - \zeta(nk)n^\mu\right]\left[k^\nu - \zeta(nk)n^\nu\right]}{\left[k^2 - \zeta(nk)^2\right]^2} =$$
$$= 4g^2 \int \frac{d^dk}{(2\pi)^d}\sin^2\left(\frac{k\tilde{p}}{2}\right)\left\{\frac{-k^\mu k^\nu}{\left[k^2-\zeta(nk)^2\right]^2} + b\frac{(n^\mu k^\nu + k^\mu n^\nu)}{k^2-\zeta(nk)^2} - b^2 n^\mu n^\nu\right\}. \tag{B.2}$$

B.2 Tadpole:

With the Feynman rules given in Tables A.1 and A.2 one obtains for the graph depicted in Figure B.1b)

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{b,\mathrm{IR}}(p) &= 2g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{k^2} \left(g^{\mu\tau}g^{\sigma\nu} + g^{\mu\sigma}g^{\tau\nu} - 2g^{\mu\nu}g^{\sigma\tau}\right) \times \\
&\quad \times \left[g_{\tau\sigma} - ak_\tau k_\sigma + b(n_\tau k_\sigma + k_\tau n_\sigma) - b'\left(\tilde{k}_\tau k_\sigma + k_\tau \tilde{k}_\sigma\right) - \varsigma \frac{\tilde{k}_\tau \tilde{k}_\sigma}{\tilde{k}^2} \right] \\
&= 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{k^2} \Bigg\{ g^{\mu\nu}\left[k^2 a - d + 1 + \varsigma - 2(nk)b\right] - \\
&\quad - ak^\mu k^\nu + b\left(n^\mu k^\nu + k^\mu n^\nu\right) - b'\left(\tilde{k}^\mu k^\nu + k^\mu \tilde{k}^\nu\right) - \varsigma \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^2} \Bigg\},
\end{aligned}
\tag{B.3}
$$

where $d = g^\mu{}_\mu$ denotes the dimension of space-time and a, b, and b' were defined in (2.30) of Chapter 2.2.1. Note that in terms which are proportional to b', one may drop ς since b' is zero anyway when no Slavnov term is present.

B.3 Photon loop:

Consulting the Feynman rules given in Tables A.1 and A.2 one has for the graph depicted in Figure B.1c)

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{c,\mathrm{IR}}(p) &= 2g^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^4} \left[k^\epsilon g^{\mu\sigma} - 2k^\mu g^{\epsilon\sigma} + k^\sigma g^{\epsilon\mu}\right] \left[k^\rho g^{\nu\tau} - 2k^\nu g^{\rho\tau} + k^\tau g^{\rho\nu}\right] \\
&\quad \times \sin^2\left(\frac{k\tilde{p}}{2}\right) \left[g_{\tau\epsilon} - ak_\tau k_\epsilon + b(n_\tau k_\epsilon + k_\tau n_\epsilon) - b'\left(\tilde{k}_\tau k_\epsilon + k_\tau \tilde{k}_\epsilon\right) - \varsigma \frac{\tilde{k}_\tau \tilde{k}_\epsilon}{\tilde{k}^2} \right] \\
&\quad \times \left[g_{\sigma\rho} - ak_\sigma k_\rho + b(n_\sigma k_\rho + k_\sigma n_\rho) - b'\left(\tilde{k}_\sigma k_\rho + k_\sigma \tilde{k}_\rho\right) - \varsigma \frac{\tilde{k}_\sigma \tilde{k}_\rho}{\tilde{k}^2} \right].
\end{aligned}
\tag{B.4}
$$

Noticing that

$$
\begin{aligned}
&[2k^\mu g^{\epsilon\sigma} - k^\epsilon g^{\mu\sigma} - k^\sigma g^{\epsilon\mu}]\left[g_\tau{}^\epsilon - ak_\tau k_\epsilon + b(n_\tau k_\epsilon + k_\tau n_\epsilon) - b'\left(\tilde{k}_\tau k_\epsilon + k_\tau \tilde{k}_\epsilon\right) - \varsigma\frac{\tilde{k}_\tau \tilde{k}_\epsilon}{\tilde{k}^2}\right] = \\
&= \Big[- k_\tau g^{\mu\sigma} + 2k^\mu g_\tau{}^\sigma - k^\sigma g_\tau{}^\mu + ak_\tau(k^2 g^{\mu\sigma} - k^\mu k^\sigma) + bn_\tau(k^\mu k^\sigma - k^2 g^{\mu\sigma}) + \\
&\quad + bk_\tau(-nkg^{\mu\sigma} + 2k^\mu n^\sigma - n^\mu k^\sigma) + b'k^2\tilde{k}_\tau g^{\mu\sigma} - b'\tilde{k}_\tau k^\mu k^\sigma - 2b'\tilde{k}^\sigma k_\tau k^\mu + b'\tilde{k}^\mu k_\tau k^\sigma - \\
&\quad - 2\varsigma k^\mu \frac{\tilde{k}_\tau \tilde{k}^\sigma}{\tilde{k}^2} + \varsigma k^\sigma \frac{\tilde{k}_\tau \tilde{k}^\mu}{\tilde{k}^2}\Big] = \\
&= \Big[f k_\tau g^{\mu\sigma} - k_\tau k^\sigma(ak^\mu + bn^\mu - b'\tilde{k}^\mu) + 2k^\mu g_\tau{}^\sigma - k^\sigma g_\tau{}^\mu + (k^\mu k^\sigma - k^2 g^{\mu\sigma})(bn_\tau - b'\tilde{k}_\tau) + \\
&\quad + 2bk^\mu k_\tau n^\sigma - 2b'\tilde{k}^\sigma k_\tau k^\mu - 2\varsigma k^\mu \frac{\tilde{k}_\tau \tilde{k}^\sigma}{\tilde{k}^2} + \varsigma k^\sigma \frac{\tilde{k}_\tau \tilde{k}^\mu}{\tilde{k}^2}\Big],
\end{aligned}
\tag{B.5}
$$

with the abbreviation

$$
f = k^2 a - 1 - (nk)b, \tag{B.6}
$$

we get

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{c,\mathrm{IR}}(p) = 2g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) &\Bigg[f k_\tau g^{\mu\sigma} - k_\tau k^\sigma(ak^\mu + bn^\mu - b'\tilde{k}^\mu) + \\
&+ 2k^\mu g_\tau{}^\sigma - k^\sigma g_\tau{}^\mu + 2bk^\mu k_\tau n^\sigma + (k^\mu k^\sigma - k^2 g^{\mu\sigma})(bn_\tau - b'\tilde{k}_\tau) - \\
&- 2b'\tilde{k}^\sigma k_\tau k^\mu - 2\varsigma k^\mu \frac{\tilde{k}_\tau \tilde{k}^\sigma}{\tilde{k}^2} + \varsigma k^\sigma \frac{\tilde{k}_\tau \tilde{k}^\mu}{\tilde{k}^2}\Bigg] \times \Bigg[f k_\sigma g^{\nu\tau} + 2k^\nu g_\sigma{}^\tau - \\
&- k_\sigma k^\tau(ak^\nu + bn^\nu - b'\tilde{k}^\nu) - k^\tau g_\sigma{}^\nu + 2bk^\nu k_\sigma n^\tau - 2b'\tilde{k}^\tau k_\sigma k^\nu + \\
&+ (k^\nu k^\tau - k^2 g^{\nu\tau})(bn_\sigma - b'\tilde{k}_\sigma) - 2\varsigma k^\nu \frac{\tilde{k}_\sigma \tilde{k}^\tau}{\tilde{k}^2} + \varsigma k^\tau \frac{\tilde{k}_\sigma \tilde{k}^\nu}{\tilde{k}^2}\Bigg] \frac{1}{k^4},
\end{aligned}
\tag{B.7}
$$

leading to

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{c,\mathrm{IR}}(p) = 2g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{k^2} &\Bigg\{ 2k^2 b^2 n^\mu n^\nu + 2g^{\mu\nu}\left[(nk)b - f\right] + \\
&+ \frac{k^\mu k^\nu}{k^2}\big[f^2 - 2k^2 af + 4f + 4(nk)bf + k^4 a^2 - 2k^2 a - 4k^2(nk)ab - 3 + \\
&+ 4(d-\varsigma) + 10(nk)b + 5(nk)^2 b^2\big] + 2\varsigma \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^2}\left[f - (nk)b + k^2\tilde{k}^2 b'^2\right] + \\
&+ b\left(n^\mu k^\nu + k^\mu n^\nu\right)\left[k^2 a - f - 5 - 3(nk)b\right] - 2k^2 b' b\left(\tilde{k}^\mu n^\nu + n^\mu \tilde{k}^\nu\right) + \\
&+ b'\left(\tilde{k}^\mu k^\nu + k^\mu \tilde{k}^\nu\right)\left[f - k^2 a + 5 + 3(nk)b\right] \Bigg\},
\end{aligned}
\tag{B.8}
$$

where $d = g^{\mu}{}_{\mu}$ once more. Using (2.30) and (B.6) this expression becomes

$$\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{c,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{k^2} \Bigg\{ & k^2 b^2 n^\mu n^\nu - g^{\mu\nu}\left[k^2 a - 1 - 2(nk)b\right] - \\
& + \frac{k^\mu k^\nu}{k^2}\left[k^2 a + 2(d-\varsigma) - 3 + 2(nk)b + (nk)^2 b^2\right] - k^2 b' b(\tilde{k}^\mu n^\nu + n^\mu \tilde{k}^\nu) + \\
& + \varsigma \frac{\tilde{k}^\mu \tilde{k}^\nu}{\tilde{k}^2}\left(k^2 a - 1 - 2(nk)b + k^2\tilde{k}^2 b'^2\right) - b\left(n^\mu k^\nu + k^\mu n^\nu\right)\left[2 + (nk)b\right] + \\
& + b'(\tilde{k}^\mu k^\nu + k^\mu \tilde{k}^\nu)\left[2 + (nk)b\right] \Bigg\}. \qquad \text{(B.9)}
\end{aligned}$$

B.4 Graph with inner λ-propagator:

With the Feynman rules given in Tables A.1 and A.2 one obtains for the graph depicted in Figure B.1d)

$$\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{d,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{\tilde{k}^2} \theta^{\mu\tau} \Bigg[& g_{\tau\sigma} - a k_\tau k_\sigma + b(n_\tau k_\sigma + k_\tau n_\sigma) - \\
& - b'\left(\tilde{k}_\tau k_\sigma + k_\tau \tilde{k}_\sigma\right) - \frac{\tilde{k}_\tau \tilde{k}_\sigma}{\tilde{k}^2} \Bigg] \theta^{\sigma\nu}. \qquad \text{(B.10)}
\end{aligned}$$

B.5 Graph with one inner λ-A propagator:

The Feynman rules needed for the graph depicted in Figure B.1e) are once more given in Tables A.1 and A.2. Additionally there is also a graph with an inner A-λ propagator instead of a λ-A propagator, but this graph only corresponds to exchanging the external indices. In the following, these additional terms will be abbreviated with "$+\mu \leftrightarrow \nu$". One has

$$\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{e,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) & \left(\tilde{k}_\rho + \tilde{k}^2 b' k_\rho\right)\left[k^\rho g^{\nu\sigma} - 2k^\nu g^{\rho\sigma} + k^\sigma g^{\rho\nu}\right] \times \\
& \times \frac{\theta^{\mu\tau}}{k^2\tilde{k}^2}\left[g_{\tau\sigma} - a k_\tau k_\sigma + b(n_\tau k_\sigma + k_\tau n_\sigma) - b'\left(\tilde{k}_\tau k_\sigma + k_\tau \tilde{k}_\sigma\right) - \frac{\tilde{k}_\tau \tilde{k}_\sigma}{\tilde{k}^2}\right] + \\
& + \mu \leftrightarrow \nu\,. \qquad \text{(B.11)}
\end{aligned}$$

Using (B.5) this expression becomes

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{e,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{\theta^{\mu\tau}}{k^2\tilde{k}^2} \Bigg\{ & - \left[f + k^2\tilde{k}^2 b'^2\right] k_\tau \tilde{k}^\nu + \\
& + k^2 \left(b\tilde{k}^\nu n_\tau - 2b'\tilde{k}^\nu \tilde{k}_\tau + \tilde{k}^2 bb' k_\tau n^\nu + \tilde{k}^2 b' g^\nu{}_\tau\right) - \\
& - \left[2(n\tilde{k})b + \tilde{k}^2 b' \left(f - k^2 a + 2(nk)b\right)\right] k^\nu k_\tau \Bigg\} + \mu \leftrightarrow \nu =
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{e,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \frac{-1}{k^2\tilde{k}^2} \Bigg\{ & 2\left[f + k^2\tilde{k}^2 b'^2\right] \tilde{k}^\mu \tilde{k}^\nu - k^2\tilde{k}^2 bb'(\tilde{k}^\mu n^\nu + n^\mu \tilde{k}^\nu) + \\
& + \left[2(n\tilde{k})b + \tilde{k}^2 b' \left(f - k^2 a + 2(nk)b\right)\right] (\tilde{k}^\mu k^\nu + k^\mu \tilde{k}^\nu) - \\
& - k^2 b(\tilde{k}^\mu \tilde{n}^\nu + \tilde{n}^\mu \tilde{k}^\nu) + 2k^2 b'(\theta^{\mu\tau}\tilde{k}_\tau \tilde{k}^\nu + \tilde{k}^\mu \theta^{\nu\tau}\tilde{k}_\tau) \Bigg\} \sin^2\left(\frac{k\tilde{p}}{2}\right),
\end{aligned}
\tag{B.12}
$$

and inserting (2.30) and (B.6) finally leads to

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{e,\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \Bigg\{ & 2\left[1 + (nk)b - k^2 a - k^2\tilde{k}^2 b'^2\right] \tilde{k}^\mu \tilde{k}^\nu + \\
& + k^2\tilde{k}^2 bb'(\tilde{k}^\mu n^\nu + n^\mu \tilde{k}^\nu) - \tilde{k}^2 b' \left[1 + (nk)b\right] (\tilde{k}^\mu k^\nu + k^\mu \tilde{k}^\nu) + \\
& + k^2 b(\tilde{k}^\mu \tilde{n}^\nu + \tilde{n}^\mu \tilde{k}^\nu) - 2k^2 b'(\theta^{\mu\tau}\tilde{k}_\tau \tilde{k}^\nu + \tilde{k}^\mu \theta^{\nu\tau}\tilde{k}_\tau) \Bigg\} \frac{1}{k^2\tilde{k}^2}.
\end{aligned}
\tag{B.13}
$$

B.6 Graph with two inner λ-A propagators:

For the graph depicted in Figure B.1f) we get

$$
\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{f,\mathrm{IR}}(p) &= 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{1}{\tilde{k}^4} \theta^{\mu\tau}\theta^{\nu\sigma} \left(\tilde{k}_\tau + \tilde{k}^2 b' k_\tau\right) \left(\tilde{k}_\sigma + \tilde{k}^2 b' k_\sigma\right) = \\
&= 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \frac{-1}{\tilde{k}^4} \theta^{\mu\tau} \left(\tilde{k}_\tau \tilde{k}_\sigma + \tilde{k}^2 b'(\tilde{k}_\tau k_\sigma + k_\tau \tilde{k}_\sigma + \tilde{k}^2 b' k_\tau k_\sigma)\right) \theta^{\sigma\nu}.
\end{aligned}
\tag{B.14}
$$

B.7 The sum of all graphs

With $\varsigma = 1$ (and considering $k_\sigma \theta^{\sigma\nu} = -\tilde{k}^\nu$) the sum of all six graphs is given by:

$$\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \Bigg\{ & \theta^{\mu\tau}\left(\frac{g_{\tau\sigma}}{\tilde{k}^2} - 2\frac{\tilde{k}_\tau \tilde{k}_\sigma}{\tilde{k}^4}\right)\theta^{\sigma\nu} - \frac{(d-3)}{k^2} g^{\mu\nu} + \\
& + \frac{k^\mu k^\nu}{k^4}\left[2d - 5 + 2(nk)b + (nk)^2 b^2 - \frac{k^4}{[k^2 - \zeta(nk)^2]^2}\right] - \\
& - \frac{b}{k^2}\left(n^\mu k^\nu + k^\mu n^\nu\right)\left[1 + (nk)b - \frac{k^2}{k^2 - \zeta(nk)^2}\right]\Bigg\}, \qquad \text{(B.15)}
\end{aligned}$$

and finally

$$\begin{aligned}
\mathrm{i}\Pi^{\mu\nu}_{\mathrm{IR}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \Bigg\{ & (d-3)\left(2\frac{k^\mu k^\nu}{k^4} - \frac{g^{\mu\nu}}{k^2}\right) + \\
& + \theta^{\mu\tau}\left(\frac{g_{\tau\sigma}}{\tilde{k}^2} - 2\frac{\tilde{k}_\tau \tilde{k}_\sigma}{\tilde{k}^4}\right)\theta^{\sigma\nu}\Bigg\}. \qquad \text{(B.16)}
\end{aligned}$$

If, on the other hand, one does the calculation *without* the Slavnov term, only the first three graphs appear (Appendices B.1, B.2 and B.3). Furthermore, considering $b' = \varsigma = 0$ in this case[1], the sum of these three graphs is given by

$$\mathrm{i}\Pi^{\mu\nu}_{\mathrm{IR,noSl}}(p) = 4g^2 \int \frac{d^dk}{(2\pi)^d} \sin^2\left(\frac{k\tilde{p}}{2}\right) \left\{(d-2)\left(2\frac{k^\mu k^\nu}{k^4} - \frac{g^{\mu\nu}}{k^2}\right)\right\}. \qquad \text{(B.17)}$$

Two comments are in order:

1. Obviously, the first term in (B.16) vanishes in three-dimensional space-time and
2. (B.17) vanishes completely in 2-dimensional space-time, hence leaving the model *free of IR divergences*.

Now, if a $U(1)$ gauge theory in 2 space-time dimensions is IR finite, is this still the case if one adds the (now unnecessary) Slavnov term? (B.16) does not vanish unless one considers a Euclidian space and a full-rank θ-matrix in the simplest block-diagonal form. For all other choices (B.16) suggests a logarithmic IR divergence.

[1] Note that the parameter b is defined the same way in both cases (cf. equations (2.12) and (2.30)).

Appendix C

BV-Formalism

Let Φ^A be a set of bosonic and fermionic fields which contains the fields Φ^i occurring in the "classical" action S_0 under study and the ghost fields c^α corresponding to the non-trivial gauge symmetries of this action. To each Φ^A one introduces an "antifield" Φ^*_A having opposite statistics[1]:

$$\epsilon(\Phi^A) \equiv \epsilon_A \ , \quad \epsilon(\Phi^*_A) = \epsilon_A + 1. \tag{C.1}$$

The antifields Φ^*_A have ghost numbers related to those of the fields Φ^A:

$$\mathrm{gh}\,(\Phi^*_A) = -\mathrm{gh}\left(\Phi^A\right) - 1\,.$$

For any two functions on the phase space of Φ, Φ^* one defines an operation called "antibracket":

$$\{X, Y\}_{\mathrm{ab}} \equiv \frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A}, \tag{C.2}$$

where the subscripts r, l denote right/left derivation, respectively. Note that the antibracket increases the ghost number by 1 and therefore changes parity:

$$\epsilon\left(\{X, Y\}_{\mathrm{ab}}\right) = \epsilon(X) + \epsilon(Y) + 1\,.$$

Obviously one has

$$\left\{\Phi^A, \Phi^*_B\right\}_{\mathrm{ab}} = \delta^A_B\,, \qquad \left\{\Phi^A, \Phi^B\right\}_{\mathrm{ab}} = 0\,, \qquad \{\Phi^*_A, \Phi^*_B\}_{\mathrm{ab}} = 0\,,$$

[1] ϵ_A denotes the Grassmann parity.

i.e. the fields and antifields are canonically conjugate. The gauge symmetry of the action is fixed by restriction to the following class of surfaces in the phase space:

$$\Sigma: \ \Phi^*_A = \frac{\partial\psi(\Phi)}{\partial\Phi^A}, \tag{C.3}$$

where $\psi(\Phi)$ is some fermionic function, which for obvious reasons is called *gauge fermion*.

Let the bosonic function $\Gamma(\Phi, \Phi^*)$ satisfy the equation[2]

$$\frac{1}{2}\{\Gamma, \Gamma\}_{\mathrm{ab}} = \mathrm{i}\hbar\triangle\Gamma \ , \quad \triangle \equiv \frac{\partial_r}{\partial\Phi^A}\frac{\partial_l}{\partial\Phi^*_A}, \tag{C.4}$$

and $\Gamma_\Sigma(\Phi)$ be the restriction of $\Gamma(\Phi, \Phi^*)$ to the surface (C.3). The solution of (C.4) can be expanded in powers of $\hbar$:

$$\Gamma = S + \sum_{p=1}^{\infty} \hbar^p \Gamma_p, \tag{C.5}$$

the classical part, the so-called *nonminimal action* S, satisfying the *master equation*

$$\{S, S\}_{\mathrm{ab}} = 0. \tag{C.6}$$

The master equation (C.6) tells us that S is invariant with respect to the BRST transformations given by

$$\begin{aligned} s\Phi^A &= \left\{S, \Phi^A\right\}_{\mathrm{ab}}, \\ s\Phi^*_A &= \{S, \Phi^*_A\}_{\mathrm{ab}}, \\ sS &= \{S, S\}_{\mathrm{ab}} = 0, \\ s^2\Phi^A &= s^2\Phi^*_A = 0. \end{aligned} \tag{C.7}$$

The nilpotency of these transformations follows from the master equation (C.6) and the (graded) Jacobi identity for the antibracket. Notice that S can be given by an expansion in powers of antifields. In Chapter 3.1 we used an extended version of this formalism, taking into account the extended BRST operator defined in equation (3.15), the nonminimal action S being denoted as S_{tot} and the master equation being given by the Slavnov-Taylor identity (3.19).

For further details we refer to the extensive literature [84]-[87] and [129]-[135].

[2] also referred to as *quantum master equation*

Appendix D

Supplementary Calculations to the Extended Slavnov Model

D.1 Equations of motion

The equations of motion associated with the action (3.58) are given by:

$$\frac{\delta S}{\delta c} = -n^i D_i \bar{c}\,, \qquad \frac{\delta S}{\delta \bar{c}} = -n^i D_i c\,, \tag{D.1a}$$

$$\frac{\delta S}{\delta \phi} = -m^i D_i \bar{\phi}\,, \qquad \frac{\delta S}{\delta \bar{\phi}} = -m^i D_i \phi\,, \tag{D.1b}$$

$$\frac{\delta S}{\delta B} = n^i A_i\,, \qquad \frac{\delta S}{\delta d'} = m^i \lambda_i\,, \tag{D.1c}$$

$$\frac{\delta S}{\delta A_i} = D_\mu F^{\mu i} + \epsilon^{ijk} D_j \lambda_k + n^i \left(B - \mathrm{i}g\left[\bar{c}, c\right]\right) - \mathrm{i}g m^i \left[\bar{\phi}, \phi\right]\,, \tag{D.1d}$$

$$\frac{\delta S}{\delta A_0} = D_k F^{k0}\,, \qquad \frac{\delta S}{\delta \lambda_i} = \frac{1}{2}\epsilon^{ijk} F_{jk} + m^i d'\,. \tag{D.1e}$$

Note that the symmetries discussed in Section 3.2.1 only exist if $m^i = n^i$.

D.2 Propagators

The equations of motion associated with the bilinear part of the action (3.58) including sources (and, for now, neglecting the ghosts) read:

$$\frac{\delta S_{\mathrm{bi}}}{\delta A^\mu} = \Box A_\mu - \partial_\mu(\partial A) + \delta^i_\mu \epsilon_{ijk}\partial^j\lambda^k + n_\mu B = -j^A_\mu, \tag{D.2a}$$

$$\frac{\delta S_{\mathrm{bi}}}{\delta \lambda^i} = \epsilon_{ijk}\partial^j A^k + m_i d' = -j^\lambda_i, \tag{D.2b}$$

$$\frac{\delta S_{\mathrm{bi}}}{\delta B} = (nA) = -j_B, \tag{D.2c}$$

$$\frac{\delta S_{\mathrm{bi}}}{\delta d'} = (m\lambda) = -j_{d'}. \tag{D.2d}$$

By letting ∂^μ (and ∂^i) act on relations (D.2a) and (D.2b), respectively, one obtains

$$B = -\frac{(\partial j_A)}{(n\partial)}, \tag{D.3}$$

$$d' = -\frac{(\partial j_\lambda)}{(m\partial)}. \tag{D.4}$$

Application of $\epsilon_{ilm}\partial^m$ to (D.2a) then yields[1]

$$-\Box j^\lambda_l + \Box\frac{(\partial j_\lambda)}{(m\partial)}m_l + \partial_l(\partial\lambda) - \Delta\lambda_l - \epsilon_{lmi}\partial^m n^i \frac{(\partial j_A)}{(n\partial)} = -\epsilon_{lmi}\partial^m j^i_A, \tag{D.5}$$

where equations (D.2b), (D.3) and (D.4) were inserted. Multiplying this expression with m^l and using (D.2d) provides an expression for $(\partial\lambda)$ and after reinserting the latter into (D.5) one finds

$$\begin{aligned}\lambda_l = \frac{\Box}{\Delta}\left(-j^\lambda_l + \frac{(\partial j_\lambda)}{(m\partial)}m_l\right) + \frac{1}{\Delta}\epsilon_{lki}\partial^k\left(j^i_A - n^i\frac{(\partial j_A)}{(n\partial)}\right) + \\ + \frac{\partial_l}{(m\partial)}\left[\frac{\Box}{\Delta}\left((mj_\lambda) - m^2\frac{(\partial j_\lambda)}{(m\partial)}\right) - j_{d'} + \frac{1}{\Delta}\epsilon_{ijk}m^i\partial^j\left(\frac{(\partial j_A)}{(n\partial)}n^k - j^k_A\right)\right].\end{aligned} \tag{D.6}$$

Finally, multiplication of (D.2a) with n^i and the use of equations (D.2c), (D.3) and (D.6) provides an expression for (∂A), and after reinserting the

[1] In this context $\Delta \equiv \partial^i\partial_i = \Box - \partial^0\partial_0$.

latter into (D.2a) one arrives at

$$A_i = \frac{1}{\Box}\Bigg\{ - j_i^A + \frac{\partial_i}{(n\partial)}\left(\epsilon_{jkl} n^j \partial^k \lambda^l - \Box j_B - n^2 \frac{(\partial j_A)}{(n\partial)} + (n j_A)\right) + \frac{(\partial j_A)}{(n\partial)} n_i - \\ - \epsilon_{ijl} \partial^j \lambda^l \Bigg\}, \tag{D.7}$$

where λ^l is given by (D.6). The expression for A_0 is similar to (D.7), except for the fact that the last two terms are missing.

By varying equations (D.3), (D.4), (D.6) and (D.7) with respect to the sources, passing over to momentum space and considering the case $m^k = n^k$ one obtains the propagators given in equations (3.44).

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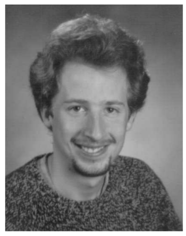
Curriculum Vitae



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Nationality	Austria
Date of birth	21 June 1980
Birthplace	Vienna

Education / Scholarships

March 2007 – March 2008	DOC fellowship for final year of doctoral studies, awarded by the Austrian Academy of Sciences
2 Nov. – 13 Dec. 2006	Six-week sojourn at the Theory Division of CERN (Switzerland) on scholarship from the Vienna Univ. of Technology
March 2005 – March 2008	Doctoral studies in Theoretical Physics at the Vienna University of Technology
May 2005	Bachelor's degree in Astronomy with distinction
November 2004	Dipl.-Ing. in Physics (= master's degree) with distinction
	Diploma thesis: *Non-Commutative Two Plus One Dimensional Quantum Electrodynamics with Chern Simons Term*
22 – 26 September 2003	Triangle Graduate School 2003 in Particle Physics, held in Raach (Austria)
2001 – 2005	Studies in Astronomy at the University of Vienna
1999 – 2000	Audio engineering course at SAE Technology College, Vienna
1998 – 2004	Studies in Physics at the Vienna University of Technology
June 1998	Graduation (*Matura*) with distinction from the secondary school Realgymnasium Polgarstr. 24, 1220 Vienna
1994 – 1997	French courses at Berlitz International, Vienna

Additional Skills

Languages	German (native), English (fluent), French
Computer & Programming	Windows, Linux; C/C++, Mathematica, Fortran; LaTeX

Conferences

30 Nov. – 2 Dec. 2007	4th Vienna Central European Seminar on Particle Physics and QFT: University of Vienna (Austria), contrib.: poster
11 – 14 May 2007	Bayrischzell Workshop 2007 on Non-Commutativity and Physics: Bayrischzell (Germany)
23 – 27 April 2007	International Conference on Non-Commutative Geometry and Physics: Laboratoire de physique théorique d'Orsay (France), contributions: poster and talk
17 – 19 Sept. 2006 and 24 – 26 Sept. 2007	"FAKT" conference of the ÖPG (Austrian Physical Society): Maria Lankowitz & Krems (Austria), contrib.: talk (2006)