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Convolutions and Multiplier Transformations
of Convex Bodies

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Kurzfassung

Die vorliegende Arbeit befasst sich mit Abbildungen des Raumes der konvexen Körper $\mathcal{K}^n$ in sich, die mit gewissen algebraischen Strukturen auf $\mathcal{K}^n$ verträglich sind. Das Studium von Operatoren, welche mit der wichtigsten Addition auf $\mathcal{K}^n$, der Minkowski Addition, verträglich sind, ist ein natürliches Anliegen. Aus geometrischer Sicht besonders interessant sind Abbildungen, welche mit der Gruppe der Rotationen $SO(n)$ vertauschen. Minkowski Endomorphismen, das sind stetige Selbstabbildungen von $\mathcal{K}^n$, die Minkowski additiv und drehäquivariant sind, wurden von Schneider [43], [44] und Kiderlen [20] systematisch untersucht. Kiderlen gibt in [20] ebenfalls eine vollständige Charakterisierung von Blaschke Endomorphismen (stetige, drehäquivariante und additive Abbildungen bzgl. der Blaschke Addition konvexer Körper) an und zeigt, dass diese als Adjungierte von schwach monotonen Minkowski Endomorphismen aufgefasst werden können.

In dieser Dissertation werden Blaschke Minkowski Homomorphismen, das sind stetige, drehäquivariante und Blaschke Minkowski gemischt additive Abbildungen, untersucht. Eines der Hauptresultate zeigt, dass diese Operatoren eine Darstellung mit Hilfe eines sphärischen Faltungsoperators erlauben, in Analogie zu Minkowski und Blaschke Endomorphismen. Wir geben eine vollständige Charakterisierung gerader Blaschke Minkowski Homomorphismen an und stellen Verbindungen zu der von Schneider und Kiderlen entwickelten Theorie her. Der bekannteste Vertreter dieser Operatoren ist die Abbildung, welche einem konvexen Körper seinen Projektionenkörper zuordnet. Die erzielten Ergebnisse zeigen, dass allgemeine Blaschke Minkowski Homomorphismen in verschiedener Hinsicht ein diesem Prototyp ähnliches Verhalten aufweisen. Diese Resultate sind [48] entnommen.

Motiviert durch wichtige Volumsungleichungen für Projektionenkörper untersuchen wir das Verhalten des Volumens (und allgemeinerer Quermaßintegrale) der Bilder von Blaschke Minkowski Homomorphismen. Wir zeigen, dass diese Operatoren ein dem Volumen analoges Verhalten in Bezug auf Minkowski Linearkombinationen aufweisen und für die wesentlichen Ungleichungen der Brunn Minkowski Theorie analoge Relationen für das Volumen der Bilder von Blaschke Minkowski Homomorphismen gelten. Die erzielten Resultate verallgemeinern Ergebnisse von Lutwak [28], [33] für Projektionenkörper und entstammen [49].

In den letzten Jahren wurde eine zur Brunn Minkowski Theorie duale Theorie für Sternkörper entwickelt. Für Ungleichungen der klassischen Theorie konvexer Körper gelten (oft einfacher zu beweisende) analoge Ungleichungen für Sternkörper. Für viele unserer Ergebnisse können solche dualen Resultate gezeigt werden. Motiviert durch Eigenschaften der wohlbekannten Schnittkörper, die das duale Gegenstück zu Projektionenkörpern darstellen, definieren wir radiale Blaschke Minkowski Homomorphismen. Wir geben eine vollständige Charakterisierung dieser Abbildungen an und zeigen, dass zu den von uns bewiesenen Volumensungleichungen für Blaschke Minkowski Homomorphismen duale Relationen gelten.

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Der Aufbau der vorliegenden Arbeit gestaltet sich wie folgt: Im ersten Kapitel präsentieren wir die Grundlagen zur Faltung sphärischer Funktionen und Maße, sowie das benötigte Material über Kugelfunktionen. Danach geben wir eine kurze Einführung in die Brunn Minkowski Theorie konvexer Körper und in die dazu duale Theorie für Sternkörper.

Im zweiten Kapitel erläutern wir zunächst bekannte Ergebnisse zu Minkowski und Blaschke Endomorphismen und beweisen anschließend den Darstellungssatz für allgemeine Blaschke Minkowski Homomorphismen. Danach zeigen wir, wie sich daraus eine vollständige Charakterisierung aller geraden Blaschke Minkowski Homomorphismen ergibt. Als weitere Anwendung geben wir Charakterisierungen des Projektionenkörpers, sowie des Minkowski und Blaschke Differenzenkörpers an. Schließlich folgern wir, dass die Bilder von Minkowski Linearkombinationen unter Blaschke Minkowski Homomorphismen ein dem Volumen analoges Verhalten aufweisen, speziell erfüllen diese Operatoren eine Steiner Formel. Analoge Resultate für radiale Blaschke Minkowski Homomorphismen von Sternkörpern beweisen wir am Ende dieses Abschnitts.

Im dritten Kapitel wenden wir uns geometrischen Ungleichungen für die Bilder der betrachteten Abbildungen zu. Wir zeigen zunächst ein Resultat für schwach monotone Minkowski Endomorphismen, welches eine Schar von verschärften Ungleichungen zwischen den zwei aufeinanderfolgenden Quermaßintegralen W_{n-1} und W_{n-2} impliziert. Danach beweisen wir zu klassichen Ungleichungen der Brunn Minkowski Theorie analoge Relationen für das Volumen der Bilder von Blaschke Minkowski Homomorphismen und deren Polarkörper. Auch dieses Kapitel beschließen wir mit dualen Resultaten für Abbildungen von Sternkörpern.

Abstract

In this thesis we study mappings of the space of convex bodies $\mathcal{K}^n$ into itself which are compatible with certain natural algebraic structures on $\mathcal{K}^n$. The investigation of mappings, which are compatible with the most important addition on $\mathcal{K}^n$, the Minkowski addition, is a natural concern. Of a particular interest from a geometrical point of view are maps which intertwine the group of rotations $SO(n)$. Minkowski endomorphisms, i.e. continuous, rotation intertwining and Minkowski additive maps of $\mathcal{K}^n$ into itself, have been investigated systematically by Schneider [43], [44] and Kiderlen [20]. Kiderlen also establishes in [20] a complete classification of all Blaschke endomorphisms, i.e. continuous, rotation intertwining and additive maps with respect to Blaschke addition of convex bodies, and shows that these can be interpreted as adjoint maps of weakly monotone Minkowski endomorphisms.

In this work we investigate Blaschke Minkowski homomorphisms, i.e. continuous, rotation intertwining and Blaschke Minkowski mixed additive maps. One of the main results shows that these operators admit a representation via a spherical convolution operator in analogy to Minkowski and Blaschke endomorphisms. Moreover we give a complete classification of all even Blaschke Minkowski homomorphisms and form connections to the theory of endomorphisms developed by Schneider and Kiderlen. The most widely known example of these maps is the projection body operator. The established results show that general Blaschke Minkowski homomorphisms behave in many respects similar to this prototype. These results are taken from [48].

Motivated by important volume inequalities for projection bodies we study the behavior of the volume (and more general quermassintegrals) of the images of Blaschke Minkowski homomorphisms. We show that these operators behave similar to the volume functional with respect to Minkowski linear combinations and that for fundamental inequalities of the Brunn Minkowski theory there are analogous inequalities satisfied by the volume of the images of Blaschke Minkowski homomorphisms. The established theorems generalize results by Lutwak [28], [33] for projection bodies and are taken from [49].

In recent years a theory for star bodies dual to the Brunn Minkowski Theory of convex bodies was developed. For inequalities of the classical theory of convex bodies there are analogous relations (often easier to prove) satisfied by star bodies. For many of our results there are corresponding dual counterparts. Motivated by properties of the well known intersection body operator, the dual to the projection body operator, we define radial Blaschke Minkowski homomorphisms. We give a complete classification of these operators and show that they satisfy volume inequalities analogous to the inequalities we proved for Blaschke Minkowski homomorphisms.

This thesis is organized as follows: In the first chapter we collect the basic material on spherical convolution of functions and measures as well as spherical harmonics. We also give a brief introduction to the Brunn Minkowski Theory of convex bodies and the dual theory of star bodies.

In the second chapter we first explain known results on Minkowski and Blaschke endomorphisms. Then we prove the representation theorem for general Blaschke Minkowski homomorphisms and show how this makes a complete classification of all even Blaschke Minkowski homomorphisms possible. As further applications we obtain characterizations of the projection body operator and the Minkowski and Blaschke difference body operators. Finally we deduce that the image of a Minkowski linear combination of convex bodies under Blaschke Minkowski homomorphisms behaves similar to the volume functional, in particular these operators satisfy a Steiner formula. Analogous results for radial Blaschke Minkowski homomorphisms of star bodies will also be proved at the end of this part.

In the third chapter we turn to geometric inequalities for the images of the mappings under consideration. We first show a result for weakly monotone Minkowski endomorphisms which implies a strengthened version of the classical inequality between the two consecutive quermassintegrals W_{n-1} and W_{n-2}. We will then prove analogs of the classical inequalities from the Brunn Minkowski Theory for the volume of the images of Blaschke Minkowski homomorphisms and their polar bodies. We conclude also this chapter with corresponding results for mappings of star bodies.

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Contents

Chapter 1

Introduction

1.1 Harmonic Analysis on the Unit Sphere

In the following sections we collect the material from harmonic analysis and convex geometry that will be needed later. Let $SO(n)$ denote the group of rotations in n dimensions. We will deal with different kinds of analytical representations of convex bodies by functions and measures on the unit sphere S^{n-1} of n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 3$. As we will often identify S^{n-1} with the homogeneous space $SO(n)/SO(n-1)$, where $SO(n-1)$ denotes the group of rotations leaving the point $\widehat{e}$ (the pole) of S^{n-1} fixed, we will first introduce some basic notions connected to $SO(n)$ and S^{n-1}. Using the identification $S^{n-1} \cong SO(n)/SO(n-1)$, it is possible to introduce a convolution structure on $C(S^{n-1})$, the space of continuous functions with the uniform topology. A special role play convolution operators generated by $SO(n-1)$ invariant (or zonal) functions and measures. In particular these are an important notion in the theory of spherical harmonics, for which we will collect the most important material. As general reference for this section we recommend the article by Grinberg and Zhang [15] and the book by Groemer [16].

1.1.1 Convolution of Spherical Functions and Measures

The identification of S^{n-1} with $SO(n)/SO(n-1)$ is for $u \in S^{n-1}$ given by

$$u = \vartheta\widehat{e} \mapsto \vartheta SO(n-1).$$

The projection from $SO(n)$ onto S^{n-1} is $\vartheta \mapsto \widehat{\vartheta} := \vartheta\widehat{e}$. The unity $e \in SO(n)$ is mapped to the pole of the sphere $\widehat{e} \in S^{n-1}$. $SO(n)$ and S^{n-1} will be equipped with the invariant probability measures denoted by $d\vartheta$ and du.

Let $C(SO(n))$ denote the set of continuous functions on $SO(n)$ with the uniform topology and $\mathcal{M}(SO(n))$ its dual space of signed finite measures on $SO(n)$ with the weak* topology. Let $\mathcal{M}^+(SO(n))$ be the set of nonnegative measures on $SO(n)$. For $\mu \in \mathcal{M}(SO(n))$ and $f \in C(SO(n))$, the canonical pairing is

$$\langle \mu, f \rangle = \langle f, \mu \rangle = \int_{SO(n)} f(\vartheta) d\mu(\vartheta).$$

Sometimes we will identify a continuous function f with the absolute continuous measure with density f and thus view $C(SO(n))$ as a subspace of $\mathcal{M}(SO(n))$. The canonical pairing is then consistent with the usual inner product on $C(SO(n))$.

For $\vartheta \in SO(n)$, the left translation ϑf of $f \in C(SO(n))$ is defined by

$$\vartheta f(\eta) = f(\vartheta^{-1}\eta). \tag{1.1}$$

For $\mu \in \mathcal{M}(SO(n))$, we set

$$\langle \vartheta\mu, f \rangle = \langle \mu, \vartheta^{-1}f \rangle, \tag{1.2}$$

then $\vartheta\mu$ is just the image measure of μ under the rotation ϑ. For $f \in C(SO(n))$ the function $\hat{f} \in C(SO(n))$ is defined by

$$\hat{f}(\vartheta) = f(\vartheta^{-1}). \tag{1.3}$$

For a measure $\mu \in \mathcal{M}(SO(n))$, we set

$$\langle \hat{\mu}, f \rangle = \langle \mu, \hat{f} \rangle. \tag{1.4}$$

As $SO(n)$ is a compact Lie group the space $C(SO(n))$ carries a natural convolution structure. For $f, g \in C(SO(n))$, the convolution $f * g \in C(SO(n))$ is defined by

$$(f * g)(\eta) = \int_{SO(n)} f(\eta\vartheta^{-1})g(\vartheta)d\vartheta = \int_{SO(n)} f(\vartheta)g(\vartheta^{-1}\eta)d\vartheta.$$

For $\mu \in \mathcal{M}(SO(n))$, the convolutions $\mu * f \in C(SO(n))$ and $f * \mu \in C(SO(n))$ with a function $f \in C(SO(n))$ are defined by

$$(f * \mu)(\eta) = \int_{SO(n)} f(\eta\vartheta^{-1})d\mu(\vartheta), \quad (\mu * f)(\eta) = \int_{SO(n)} \vartheta f(\eta)d\mu(\vartheta). \tag{1.5}$$

With these definitions $f * \mu$ and $\mu * f$ are real analytic if f is real analytic.

Using (1.5), one easily checks that for $\sigma \in \mathcal{M}(SO(n))$ and $f, g \in C(SO(n))$

$$\langle g * \sigma, f \rangle = \langle g, f * \hat{\sigma} \rangle. \tag{1.6}$$

This leads to the definition of the convolution of two measures $\mu, \sigma \in \mathcal{M}(SO(n))$

$$\langle \mu * \sigma, f \rangle = \langle \sigma, \hat{\mu} * f \rangle = \langle \mu, f * \hat{\sigma} \rangle. \tag{1.7}$$

The convolution on $\mathcal{M}(SO(n))$ has the usual properties of a convolution structure, it is associative, and if $\|\cdot\|_{TV}$ denotes the total variation norm of measures, then for nonnegative measures μ, σ, we have $\|\mu * \sigma\|_{TV} = \|\mu\|_{TV}\|\sigma\|_{TV}$. Since for $n \geq 3$, the group of rotations is not abelian, the convolution on $\mathcal{M}(SO(n))$ is not commutative. For the following lemma see [15], p.85.

Lemma 1.1 *Let $\mu_m, \mu \in \mathcal{M}(SO(n))$, $m = 1, 2, \ldots$ and let $f \in C(SO(n))$. If $\mu_m \to \mu$ weakly, then $f * \mu_m \to f * \mu$ and $\mu_m * f \to \mu * f$ uniformly.*

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In order to define a convolution structure on $C(S^{n-1})$, we will use the method from Grinberg and Zhang [15] identifying S^{n-1} with $SO(n)/SO(n-1)$. This leads to the identification of $C(S^{n-1})$ with right $SO(n-1)$-invariant functions in $C(SO(n))$ by setting

$$\breve{f}(\vartheta) = f(\vartheta \hat{e}), \qquad f \in C(S^{n-1}). \tag{1.8}$$

Conversely, every $f \in C(SO(n))$ induces a continuous function $\hat{f}$ on S^{n-1}, defined by

$$\hat{f}(\eta) = \int_{SO(n-1)} f(\eta \vartheta) d\vartheta.$$

If $f \in C(SO(n))$ is right $SO(n-1)$ invariant and $g \in C(S^{n-1})$, then $f = \breve{\hat{f}}$ and $g = \hat{\breve{g}}$. Thus $C(S^{n-1})$ is isomorphic to the subspace of right $SO(n-1)$ invariant functions in $C(SO(n))$. For a measure $\mu \in \mathcal{M}(S^{n-1})$ and a function $f \in C(SO(n))$, we set

$$\langle \breve{\mu}, f \rangle = \langle \mu, \hat{f} \rangle.$$

In this way the one-to-one correspondence of functions on S^{n-1} with right $SO(n-1)$ invariant functions on $SO(n)$ carries over to the space $\mathcal{M}(S^{n-1})$ and right $SO(n-1)$ invariant measures in $\mathcal{M}(SO(n))$. The following relation between integrals over $SO(n)$ and spherical integration will be used frequently:

$$\int_{SO(n)} f(\vartheta) d\vartheta = \int_{S^{n-1}} \int_{SO(n-1)} f(\vartheta \eta) d\eta d\hat{\vartheta}. \tag{1.9}$$

Note that definitions (1.1), (1.2) and (1.3), (1.4) become now meaningful for spherical functions and measures. Convolution on $C(S^{n-1})$ can be defined via the identification (1.8). For example the convolution of a function $f \in C(S^{n-1})$ with a measure $\mu \in \mathcal{M}(S^{n-1})$ is given by

$$(f * \mu)(\hat{\eta}) = (\breve{f} * \breve{\mu})(\eta) = \int_{SO(n)} f(\eta \vartheta^{-1} \hat{e}) d\breve{\mu}(\vartheta).$$

In an analogous way, convolutions of functions or measures can be defined. In particular the convolution $\mu * f \in C(S^{n-1})$ of a measure $\mu \in \mathcal{M}(SO(n))$ and a function $f \in C(S^{n-1})$ is defined by

$$(\mu * f)(u) = \int_{SO(n)} f(\vartheta^{-1} u) d\mu(\vartheta). \tag{1.10}$$

Thus, if $\mu \in \mathcal{M}(SO(n))$ is a nonnegative measure, $\mu * f$ can be interpreted as a weighted rotation mean of f.

Note that the Dirac measure $\delta_{\hat{e}}$ is the unique rightneutral element for the convolution on S^{n-1} while the convolution with $\delta_{-\hat{e}}$ represents the reflection in the origin, i.e., for $f \in C(S^{n-1})$

$$(f * \delta_{-\hat{e}})(u) = f(-u). \tag{1.11}$$

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1.1.2 Zonal Functions and Measures

An essential role among spherical functions play $SO(n-1)$ invariant functions. Such a function with the property that $\vartheta f = f$ for every $\vartheta \in SO(n-1)$, is called zonal. A zonal function is by definition constant on every parallel circle

$$S_t^{n-2} := \{u \in S^{n-1} : u = \vartheta v, \vartheta \in SO(n-1)\}, \tag{1.12}$$

where $v \in S^{n-1}$ with $v \cdot \hat{e} = t \in [-1,1]$. Thus these functions depend only on the distance of u to $\hat{e}$, i.e., on the value $u \cdot \hat{e}$. Zonal functions can be identified with $SO(n-1)$-biinvariant functions on $SO(n)$.

Of course the notion of $SO(n-1)$ invariance carries over to measures as well. We call a measure $\mu \in \mathcal{M}(S^{n-1})$ zonal, if $\vartheta\mu = \mu$ for every $\vartheta \in SO(n-1)$. The space of all continuous, zonal functions will be denoted by $\mathcal{C}(S^{n-1}, \hat{e})$, and $\mathcal{M}(S^{n-1}, \hat{e})$ denotes the space of zonal measures on S^{n-1}.

Spherical convolution becomes simpler for zonal measures. Using (1.9) we get for $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$

$$(f * \mu)(\hat{\eta}) = \langle f, \eta\mu \rangle = \int_{S^{n-1}} f(\eta u)d\mu(u). \tag{1.13}$$

For $f \in \mathcal{C}(S^{n-1})$, the rotational symmetrization $\bar{f} \in \mathcal{C}(S^{n-1}, \hat{e})$ is defined by

$$\bar{f} = \delta_{\hat{e}} * f = \int_{SO(n-1)} \vartheta f d\vartheta.$$

Since $\delta_{\hat{e}}$ is the right invariant element for the convolution on S^{n-1}, we get

$$f * g = f * \delta_{\hat{e}} * g = f * \bar{g}. \tag{1.14}$$

Thus, for spherical convolution from the right, it suffices to consider zonal functions and measures. Note that, if $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$, then by (1.13) for every $f \in \mathcal{C}(S^{n-1})$

$$(\vartheta f) * \mu = \vartheta(f * \mu) \tag{1.15}$$

for every $\vartheta \in SO(n)$. Thus the spherical convolution from the right is a rotation intertwining operator on $\mathcal{C}(S^{n-1})$ and $\mathcal{M}(S^{n-1})$.

We note here that several authors, see [1], [20], [41], used spherical convolution in a disguised version as a generalized Radon transform (with Parameter $t \in [-1,1]$), defined in our notation for $f \in \mathcal{C}(S^{n-1})$ by

$$R_t f = f * \mu_{S_t^{n-2}},$$

where $\mu_{S_t^{n-2}}$ is the invariant probability measure concentrated on the parallel circle S_t^{n-2} given by (1.12). For $t = 0$, this becomes the classical spherical Radon transform. The connection between this integral transform and spherical convolution is for $f \in \mathcal{C}(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$ given by, see [20],

$$f * \mu = \int_{S^{n-1}} R_{\hat{e} \cdot v} f d\mu(v).$$

As a zonal function on S^{n-1} depends only on the value of $u \cdot \widehat{e}$, there is a natural isomorphism between functions and measures on $[-1,1]$ and zonal functions and measures on S^{n-1}. Define a map $\Lambda : C(S^{n-1}, \widehat{e}) \to C([-1,1]), f \mapsto \Lambda f$, by

$$\Lambda f(t) = f(t\widehat{e} + \sqrt{1-t^2}v), \qquad v \in \widehat{e}^{\perp} \cap S^{n-1}. \tag{1.16}$$

Then it is easy to see that Λ is an isomorphism with inverse

$$\Lambda^{-1} : C([-1,1]) \to C(S^{n-1}, \widehat{e}), f \mapsto f(\widehat{e} \cdot \,).$$

For a zonal measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and a function $f \in C([-1,1])$ define

$$\langle \Lambda\mu, f \rangle = \langle \mu, \Lambda^{-1}f \rangle.$$

The map $\Lambda : \mathcal{M}(S^{n-1}, \widehat{e}) \to \mathcal{M}([-1,1])$ is the extension of the map defined in (1.16) and it is again an isomorphism between $\mathcal{M}(S^{n-1}, \widehat{e})$ and $\mathcal{M}([-1,1])$ with inverse

$$\langle \Lambda^{-1}\mu, f \rangle = \langle \mu, \Lambda \bar{f} \rangle, \qquad \mu \in \mathcal{M}([-1,1]), f \in C(S^{n-1}).$$

The isomorphism Λ allows us to identify the dual space of $C(S^{n-1}, \widehat{e})$ with the space $\mathcal{M}(S^{n-1}, \widehat{e})$. Using this identification, we obtain for $\mu, \nu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and $f \in C(S^{n-1}, \widehat{e})$,

$$\langle \mu * \nu, f \rangle = \int_{S^{n-1}} \int_{S^{n-1}} \Lambda f(u \cdot v) d\mu(u) d\nu(v) = \langle \nu * \mu, f \rangle. \tag{1.17}$$

Thus, the convolution of zonal functions and measures is abelian and $\mathcal{M}(S^{n-1}, \widehat{e})$, with the convolution structure, becomes an abelian Banach algebra.

Another important property of zonal measures $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ is

$$\hat{\mu} = \mu. \tag{1.18}$$

As a consequence of (1.6) and (1.18) we obtain the following very useful lemma.

Lemma 1.2 *Let* $\mu, \nu \in \mathcal{M}(S^{n-1})$ *and* $f \in C(S^{n-1})$, *then*

$$\langle \mu * \nu, f \rangle = \langle \mu, f * \nu \rangle.$$

Using Lemma 1.2 and (1.17) we get for $\mu \in \mathcal{M}(S^{n-1})$ and $f \in C(S^{n-1}, \widehat{e})$

$$(\mu * f)(u) = \int_{S^{n-1}} \Lambda f(u \cdot v) d\mu(v). \tag{1.19}$$

We will frequently use zonal approximate identities $(\varphi_k)_{k \in \mathbb{N}}$. These are non-negative functions in $C^{\infty}(S^{n-1})$. They have already been considered by Berg [1] and we just briefly recall their construction.

Let $\|\cdot\|$ denote the standard Euclidean norm in $\mathbb{R}^n$ and let $(f_k)_{k\in\mathbb{N}}$ be nonnegative functions in $C^\infty(\mathbb{R}^n)$ such that for each $k \in \mathbb{N}$

(a) $f_k(x) = 0$ if $\|x\| \geq \frac{1}{k}$,

(b) $f_k(x) = f_k(y)$ if $\|x\| = \|y\|$,

(c) $\int_{\mathbb{R}^n} f_k(x)dx = 1$.

Then the sequence of functions $(\varphi_k)_{k\in\mathbb{N}}$ from $C^\infty(S^{n-1}, \widehat{e})$ defined by

$$\varphi_k(u) = \int\limits_0^\infty f_k(u - r\widehat{e})r^{n-1}dr$$

is called zonal approximate identity. We summarize their most important properties in the following lemma, see [1]:

Lemma 1.3 *Let $(\varphi_k)_{k\in\mathbb{N}}$ be a zonal approximate identity. Then*

(a) $f * \varphi_k \in C^\infty(S^{n-1})$ and $\lim_{k\to\infty} f * \varphi_k = f$ uniformly for every $f \in C(S^{n-1})$.

(b) $\mu * \varphi_k \in C^\infty(S^{n-1})$ and $\lim_{k\to\infty} \mu * \varphi_k = \mu$ weakly for every $\mu \in \mathcal{M}(S^{n-1})$.

1.1.3 Spherical Harmonics

We now collect some facts from the theory of spherical harmonics. A spherical harmonic of dimension n and order k is the restriction to S^{n-1} of a harmonic polynomial of order k in n variables. Let $\mathcal{H}_k^n$ denote the space of spherical harmonics of dimension n and order k. $\mathcal{H}^n$ will denote the space of all finite sums of spherical harmonics of dimension n. $\mathcal{H}_k^n$ is a finite dimensional vector space of dimension

$$N(n, k) = \frac{n + 2k - 2}{n + k - 2}\binom{n + k - 2}{k}.$$

The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to the usual inner product on $C(S^{n-1})$. By definition, $\mathcal{H}_k^n$ is invariant with respect to rotations. Moreover, $\mathcal{H}_k^n$ is irreducible, i.e. $\{0\}$ and $\mathcal{H}_k^n$ are the only subspaces invariant under $SO(n)$. As a consequence we have the following version of Schur's Lemma for spherical harmonics.

Lemma 1.4 *Let $\Phi : \mathcal{H}_k^n \to \mathcal{M}(S^{n-1})$ be a linear map that intertwines rotations. Then Φ is either injective or the zero map.*

If $H_1, \ldots, H_{N(n,k)}$ is an orthonormal basis of $\mathcal{H}_k^n$, then there is a unique polynomial $P_k^n \in C([-1, 1])$ of degree k such that

$$\sum_{i=1}^{N(n,k)} H_i(u)H_i(v) = N(n, k)P_k^n(u \cdot v). \tag{1.20}$$

The polynomial P_k^n is called the Legendre polynomial of dimension n and order k. The zonal function $u \mapsto P_k^n(\widehat{e} \cdot u)$ is up to a multiplicative constant the unique zonal spherical harmonic in $\mathcal{H}_k^n$. Moreover for any given n and k there are $N(n,k)$ points $v_i \in S^{n-1}$ such that

$$\text{span}\{P_k^n(v_1 \cdot .), \ldots, P_k^n(v_{N(n,k)} \cdot .)\} = \mathcal{H}_k^n. \qquad (1.21)$$

The collection $\{H_1, \ldots, H_{N(n,k)} : k \in \mathbb{N}\}$ forms a complete orthogonal system in $\mathcal{L}^2(S^{n-1})$, i.e. for every square integrable function f the series

$$f \sim \sum_{k=0}^{\infty} \pi_k f$$

converges in quadratic mean to f, where $\pi_k f \in \mathcal{H}_k^n$ is the orthogonal projection of f on the space $\mathcal{H}_k^n$. Using (1.20) and (1.13), we obtain

$$\pi_k f = \sum_{i=1}^{N(n,k)} \langle f, H_i \rangle H_i = N(n,k)(f * P_k^n(\widehat{e} \cdot .)).$$

This leads to the definition of the spherical expansion of a measure $\mu \in \mathcal{M}(S^{n-1})$

$$\mu \sim \sum_{k=0}^{\infty} \pi_k \mu, \qquad (1.22)$$

where $\pi_k \mu \in \mathcal{H}_k^n$ is defined by

$$\pi_k \mu = N(n,k)(\mu * P_k^n(\widehat{e} \cdot .)). \qquad (1.23)$$

We note here two special cases of (1.23)

$$\pi_0 \mu = \mu * 1 \qquad \text{and} \qquad \pi_1 \mu = n\mu * (\widehat{e} \cdot .). \qquad (1.24)$$

By Lemma 1.2, we have for every $f \in \mathcal{C}(S^{n-1})$

$$\langle \pi_k \mu, f \rangle = N(n,k)\langle \mu * P_k^n(\widehat{e} \cdot .), f \rangle = N(n,k)\langle \mu, f * P_k^n(\widehat{e} \cdot .) \rangle = \langle \mu, \pi_k f \rangle,$$

which, by the completeness of the system of spherical harmonics, immediately gives:

Lemma 1.5 Let $\mu \in \mathcal{M}(S^{n-1})$. If $\mu * P_k^n(\widehat{e} \cdot .) = 0$ for every $k \in \mathbb{N}$ then $\mu = 0$.

By Lemma 1.5, $\mu \in \mathcal{M}(S^{n-1})$ is uniquely determined by its series expansion (1.22). Zonal functions and measures are even determined by a sequence of real numbers. To see this, note that

$$\delta_{\widehat{e}} * P_k^n(u \cdot .) = P_k^n(\widehat{e} \cdot u) P_k^n(\widehat{e} \cdot .)$$

and thus by (1.19) and (1.7)

$$(\mu * P_k^n(\widehat{e} \cdot .))(u) = \langle \mu, P_k^n(u \cdot .) \rangle = \langle \mu, \delta_{\widehat{e}} * P_k^n(u \cdot .) \rangle = \langle \mu, P_k^n(\widehat{e} \cdot .) \rangle P_k^n(\widehat{e} \cdot u).$$

Hence the series expansion of a zonal measure μ becomes

$$\mu \sim \sum_{k=0}^{\infty} N(n,k) \langle \mu, P_k^n(\widehat{e} \cdot \,) \rangle P_k^n(\widehat{e} \cdot \,).$$

The numbers $\mu_k := \langle \mu, P_k^n(\widehat{e} \cdot \,) \rangle$ are called Legendre coefficients of $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$. Using $\pi_k H = H$ for every $H \in \mathcal{H}_k^n$ and the fact that spherical convolution of zonal measures is commutative, we get a version of the Funk-Hecke Theorem.

Corollary 1.6 *If $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and $H \in \mathcal{H}_k^n$ then $H * \mu = \mu_k H$.*

We are now ready to give the definition of multiplier operators.

Definition 1.7 *We call a map $\Phi : \mathcal{Q} \subseteq \mathcal{M}(S^{n-1}) \to \mathcal{M}(S^{n-1})$ a multiplier transformation if there is a sequence of real numbers c_k such that, for every $k \in \mathbb{N}$,*

$$\pi_k \Phi \mu = c_k \pi_k \mu, \qquad \forall \mu \in \mathcal{Q}. \tag{1.25}$$

The numbers $c_0, c_1, c_2, \ldots$ are called the multipliers of Φ.

Using again the fact that spherical convolution of zonal measures is commutative, we see that for $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ the map $\Phi_\mu : \mathcal{M}(S^{n-1}) \to \mathcal{M}(S^{n-1})$

$$\nu \mapsto \nu * \mu$$

is a multiplier transformation. The sequence of multipliers of these convolution operators is just the sequence of Legendre coefficients of the measure μ.

By definition (1.23) of the orthogonal projection π_k and (1.15), it is easy to see that multiplier transformations intertwine rotations and that, by definition (1.25), they are linear on the space $\mathcal{H}^n$. The following corollary to Schur's Lemma establishes the converse statement, see [43], p.67.

Theorem 1.8 *If $\Phi : \mathcal{H}^n \to \mathcal{M}(S^{n-1})$ is an intertwining linear map, then Φ is a multiplier transformation.*

Proof: Let Φ_m be the restriction of Φ to $\mathcal{H}_m^n$. The map $H \mapsto \pi_k \Phi_m H$ from $\mathcal{H}_m^n$ to $\mathcal{H}_k^n$ is intertwining and linear. By Lemma 1.4, $\pi_k \Phi_m$ is either injective or the zero map. Since $\pi_k \Phi_m \mathcal{H}_m^n$ is invariant under rotations and $\mathcal{H}_k^n$ is irreducible it follows from $N(n,k) \neq N(n,m)$ that $\pi_k \Phi_m = 0$ or $k = m$ and $\pi_k \Phi_m$ is an isomorphism.

Using the fact that $P_k^n(u \cdot \,)$ is up to a multiplicative constant the unique function in $\mathcal{H}_k^n$ invariant under rotations leaving the point $u \in S^{n-1}$ fixed, it is easy to see that there is a constant $c_k(u)$ such that

$$\pi_k \Phi_k P_k^n(u \cdot \,) = c_k(u) P_k^n(u \cdot \,).$$

By replacing u with ϑu for $\vartheta \in SO(n)$, it follows that c_k is independent of $u \in S^{n-1}$. Thus (1.21) implies

$$\pi_k \Phi_k H = c_k H$$

for every $H \in \mathcal{H}_k^n$. The linearity of Φ and π_k finally gives the desired result. ∎

Note that in Theorem 1.8 we did not impose any continuity assumptions on Φ.

1.2 Convex Bodies and Star Bodies

This section collects material on convex bodies and star bodies, see the books by Schneider [46] and Gardner [10] and the article [27] by Lutwak. We present the notion of support functions and area measures and introduce the Steiner point map and projection bodies. We discuss mixed volumes and state for quick reference the most important relations among them. Then we turn to star bodies and introduce radial functions, the intersection body operator and dual mixed volumes, for which we state the inequalities from the dual Brunn Minkowski Theory.

1.2.1 Support Functions and Surface Area Measures

For $n \geq 3$ let $\mathcal{K}^n$ be the space of convex bodies in $\mathbb{R}^n$, i.e., nonempty, compact, convex sets, equipped with the Hausdorff topology. Let $\mathcal{K}_i^n$ be the subset of $\mathcal{K}^n$ consisting of convex bodies whose dimension is at least $n - i$. Then $\mathcal{K}_0^n$ are the convex bodies with interior points. If $K \in \mathcal{K}_0^n$ contains the origin in its interior, the convex body

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in K\}$$

is called the polar body of K. A convex body $K \in \mathcal{K}^n$ is uniquely determined by the values of its support function $h(K, \cdot)$, defined on $\mathbb{R}^n$ by

$$h(K, x) = \max\{x \cdot y : y \in K\}.$$

Support functions are positively homogeneous of degree one and sublinear. Conversely, every function with these properties is the support function of a convex body. We consider mostly their restrictions to S^{n-1} which are elements of $C(S^{n-1})$. The uniform topology on $\mathcal{K}^n$ induced by identifying a convex body with its support function on the sphere coincides with the Hausdorff topology. By (1.1), we have $\vartheta h(K, \cdot) = h(\vartheta K, \cdot)$ for $\vartheta \in SO(n)$. Thus, the support function of a convex body K is zonal if and only if K is invariant under rotations of $SO(n-1)$. We will then call K a body of revolution.

The most important algebraic structure on the set of convex bodies is Minkowski or vector addition. For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$, the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is

$$\lambda_1 K_1 + \lambda_2 K_2 = \{\lambda_1 x + \lambda_2 y : x \in K_1, y \in K_2\}.$$

Using support functions, the Minkowski linear combination can be defined by

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot).$$

By Minkowski's existence theorem, a convex body $K \in \mathcal{K}_0^n$ is also uniquely determined up to translation by its surface area measure $S_{n-1}(K, \cdot)$. The measure of a Borel set $\omega \subseteq S^{n-1}$ is the $n - 1$ dimensional Hausdorff measure of the set of

all boundary points of K at which there exists a normal vector of K belonging to ω. $S_{n-1}(K, \cdot)$ is an element of $\mathcal{M}_o^+(S^{n-1})$, the space of nonnegative measures on S^{n-1} having their center of mass in the origin, equipped with the weak* topology. The topology on the set of translation classes of convex bodies with nonempty interior $[\mathcal{K}_0^n]$ induced by identifying a convex body with its surface area measure again coincides with the Hausdorff topology. Every element of $\mathcal{M}_o^+(S^{n-1})$ that is not concentrated on any great sphere is the surface area measure of a convex body with interior points. For $\vartheta \in SO(n)$, we have $\vartheta S_{n-1}(K, \cdot) = S_{n-1}(\vartheta K, \cdot)$ and the surface area measure of a convex body K is zonal if and only if K is a body of revolution.

For $K_1, K_2 \in \mathcal{K}_0^n$ and $\lambda_1, \lambda_2 \geq 0$ (not both 0), the Blaschke linear combination $\lambda_1 \cdot K_1 \# \lambda_2 \cdot K_2$ is defined (up to translation) by

$$S_{n-1}(\lambda_1 \cdot K_1 \# \lambda_2 \cdot K_2, \cdot) = \lambda_1 S_{n-1}(K_1, \cdot) + \lambda_2 S_{n-1}(K_2, \cdot).$$

With Minkowski and Blaschke addition, $\mathcal{K}^n$ and $[\mathcal{K}_0^n]$ are abelian semi-groups.

The surface area measure of a Minkowski linear combination of convex bodies $K_1, \ldots, K_m$ can be expressed as a polynomial homogeneous of degree $n - 1$:

$$S_{n-1}(\lambda_1 K_1 + \ldots + \lambda_m K_m, \cdot) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} S(K_{i_1}, \ldots, K_{i_{n-1}}, \cdot). \tag{1.26}$$

The coefficients $S(K_{i_1}, \ldots, K_{i_{n-1}}, \cdot) \in \mathcal{M}_o^+(S^{n-1})$ are called the mixed area measures of $K_{i_1}, \ldots, K_{i_n}$. They are symmetric in their arguments and multilinear with respect to Minkowski addition. The measures $S_j(K, \cdot) := S(K, \ldots, K, B, \ldots, B, \cdot)$, where K appears j times and the Euclidean unit ball B appears $n - 1 - j$ times, are called the area measures of order j of K. The area measure of order one $S_1(K, \cdot)$ is related to the support function $h(K, \cdot)$ by the linear second order differential operator

$$\Delta_1 = \Delta_0 + (n - 1),$$

where Δ_0 denotes the Laplace Beltrami operator on S^{n-1}, see [15], p.87. We have

$$\Delta_1 h(K, \cdot) = S_1(K, \cdot), \tag{1.27}$$

where this equality is understood in the sense of distributions. From (1.27), it follows that

$$S_1(K_1 + K_2, \cdot) = S_1(K_1, \cdot) + S_1(K_2, \cdot). \tag{1.28}$$

The Steiner point map $s : \mathcal{K}^n \to \mathbb{R}^n$ is defined by

$$s(K) = n \int_{S^{n-1}} h(K, u) u \, du.$$

Using Lemma 1.8, it is not difficult to show that s is up to a multiplicative constant the unique vector valued continuous, rotation intertwining and Minkowski additive map, see [43]. Since vector addition in $\mathbb{R}^n$ coincides with Minkowski addition of singletons, it is possible to give an equivalent definition of the Steiner point map

$$h(\{s(K)\}, \cdot) = nh(K, \cdot) * (\widehat{e} \cdot \,. \,) = \pi_1 h(K, \cdot). \tag{1.29}$$

There is no nonzero vector valued map from the set of translation classes of convex bodies $[\mathcal{K}_0^n] = \mathcal{K}_0^n / \mathbb{R}^n$ that is continuous, rotation intertwining and additive with respect to Blaschke addition. This fact is reflected by the relation

$$\pi_1 S_{n-1}(K, \cdot) = n S_{n-1}(K, \cdot) * (\widehat{e} \cdot \,. \,) = 0. \tag{1.30}$$

1.2.2 Mixed Volumes and Projection Bodies

The volume of a Minkowski linear combination $\lambda_1 K_1 + \ldots + \lambda_m K_m$ of convex bodies $K_1, \ldots, K_m$ is a homogeneous polynomial of degree n in the λ_i

$$V(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n} V(K_{i_1}, \ldots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are called mixed volumes of $K_{i_1}, \ldots, K_{i_n}$. These functionals are symmetric in their arguments, nonnegative, translation invariant and monotone (with respect to set inclusion). Moreover, they have the following properties:

(i) They are multilinear with respect to Minkowski linear combinations.

(ii) Their diagonal form reduces to ordinary volume:

$$V(K, \ldots, K) = V(K).$$

(iii) They are invariant under simultaneous volume preserving linear transformations, i.e. if $A \in SL(n)$, then

$$V(AK_1, \ldots, AK_n) = V(K_1, \ldots, K_n).$$

Denote by $V_i(K, L)$ the mixed volume $V(K, \ldots, K, L, \ldots, L)$, where K appears $n - i$ times and L appears i times. For $0 \leq i \leq n - 1$, we write $W_i(K, L)$ for the mixed volume $V(K, \ldots, K, B, \ldots, B, L)$, where K appears $n - i - 1$ times and B appears i times. The mixed volume $W_i(K, K)$ will be written as $W_i(K)$ and is called the ith quermassintegral of K. If $\mathbf{C} = (K_1, \ldots, K_i)$, then $V_i(K, \mathbf{C})$ denotes the mixed volume $V(K, \ldots, K, K_1, \ldots, K_i)$ with $n - i$ copies of K.

For any convex body K, we have the following integral representation:

$$V(K, K_1, \ldots, K_{n-1}) = \frac{1}{n} \langle h(K, \cdot), S(K_1, \ldots, K_{n-1}, \cdot) \rangle. \tag{1.31}$$

In particular, for the functional $V_1(K, L)$, we have

$$V_1(K, L) = \frac{1}{n} \langle h(L, \cdot), S_{n-1}(K, \cdot) \rangle. \tag{1.32}$$

Hence $V_1 : [\mathcal{K}_0^n] \times \mathcal{K}^n \to \mathbb{R}$ is bilinear with respect to Blaschke and Minkowski addition. By (1.24) and (1.31), we get for $K \in \mathcal{K}^n$

$$W_{n-1}(K) = \kappa_n \pi_0 h(K, \cdot) \quad \text{and} \quad W_1(K) = \frac{1}{n} \pi_0 S_{n-1}(K, \cdot). \tag{1.33}$$

We will now present the fundamental inequalities of the Brunn Minkowski Theory. In order to simplify the equality cases, we will state most of them only for convex bodies with interior points. One of the most general and fundamental inequalities

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for mixed volumes is the Aleksandrov Fenchel inequality: If $K_1, \ldots, K_n \in \mathcal{K}_0^n$ and $1 \leq m \leq n$, then

$$V(K_1, \ldots, K_n)^m \geq \prod_{j=1}^{m} V(K_j, \ldots, K_j, K_{m+1}, \ldots, K_n). \qquad (1.34)$$

Unfortunately, the equality conditions of this inequality are, in general, unknown.

An important special case of inequality (1.34), where the equality conditions are known, is the Minkowski inequality: If $K, L \in \mathcal{K}_0^n$, then

$$V_1(K, L)^n \geq V(K)^{n-1} V(L), \qquad (1.35)$$

with equality if and only if K and L are homothetic. In fact, a more general version of Minkowski's inequality holds: If $0 \leq i \leq n - 2$ and $K, L \in \mathcal{K}_i^n$, then

$$W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1} W_i(L), \qquad (1.36)$$

with equality if and only if K and L are homothetic.

The classical inequality between two consecutive quermassintegrals states that for $K \in \mathcal{K}^n$ and $0 \leq i \leq n - 2$,

$$W_{i+1}(K)^{n-i} \geq \kappa_n W_i(K)^{n-i-1}, \qquad (1.37)$$

where κ_n is the volume of the Euclidean unit ball B. If $K \in \mathcal{K}_{i+1}^n$, there is equality in (1.37) if and only if K is a ball. By repeated application of (1.37) one obtains: If $K \in \mathcal{K}_0^n$ and $0 \leq i < j \leq n - 1$, then

$$W_j(K)^{n-i} \geq \kappa_n^{j-i} W_i(K)^{n-j}, \qquad (1.38)$$

with equality if and only if K is a ball.

A consequence of the Minkowski inequality is the Brunn Minkowski inequality: If $K, L \in \mathcal{K}_0^n$, then

$$V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \qquad (1.39)$$

with equality if and only if K and L are homothetic. This is a special case of the more general inequality: If $0 \leq i \leq n - 2$, then

$$W_i(K + L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \qquad (1.40)$$

with equality if and only if K and L are homothetic.

A further generalization of inequality (1.39) is also known (but without equality conditions): If $0 \leq i \leq n - 2$, $K, L, K_1, \ldots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1, \ldots, K_i)$, then

$$V_i(K + L, \mathbf{C})^{1/(n-i)} \geq V_i(K, \mathbf{C})^{1/(n-i)} + V_i(L, \mathbf{C})^{1/(n-i)}. \qquad (1.41)$$

The projection body ΠK of $K \in \mathcal{K}^n$ is the convex body whose support function is given for $u \in S^{n-1}$ by

$$h(\Pi K, u) = \mathrm{vol}_{n-1}(K | u^\perp), \qquad (1.42)$$

where vol_{n-1} denotes $(n-1)$-dimensional volume and $K | u^\perp$ is the image of the orthogonal projection of K onto the subspace orthogonal to u.

Projection bodies and their polars have received considerable attention over the last decades due to their connection to different areas such as geometric tomography, stereology, combinatorics, computational and stochastic geometry, see [2], [3], [10], [14], [15], [25], [26], [28], [33], [47], [54].

Definition (1.42) can be rewritten using mixed volumes and spherical convolution:

$$h(\Pi K, u) = V_1(K, [-u, u]) = \frac{1}{2}(S_{n-1}(K, \cdot) * h([-\widehat{e}, \widehat{e}], \cdot))(u), \qquad (1.43)$$

where $[-u, u]$ denotes the segment with endpoints $-u$ and u. Note that by (1.10), this can be interpreted as a weighted (Minkowski) rotation mean of the segment $[-\widehat{e}, \widehat{e}]$. In fact, Π maps polytopes to finite Minkowski linear combinations of rotated and dilated copies of $[-\widehat{e}, \widehat{e}]$, and general convex bodies to zonoids, i.e., limits of Minkowski sums of line segments.

From formula (1.43), Lemma 1.1 and (1.15), we obtain the following wellknown properties of the projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$:

(a) Π is continuous.

(b) Π is Blaschke Minkowski additive, i.e. $\Pi(K \# L) = \Pi K + \Pi L$ for all $K, L \in \mathcal{K}_0^n$.

(c) Π intertwines rotations, i.e. $\Pi(\vartheta K) = \vartheta \Pi K$ for all $K \in \mathcal{K}^n$ and all $\vartheta \in SO(n)$.

An important volume inequality for the polars of projection bodies is the Petty projection inequality [39]: If $K \in \mathcal{K}_0^n$, then

$$V(\Pi^* K) \leq \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n V(K)^{1-n}, \qquad (1.44)$$

with equality if and only if K is an ellipsoid. To prove the corresponding result for the volume of the projection body itself is a major open problem in convex geometry, see [34]. Petty conjectured that

$$\frac{\kappa_n^{n-1}}{\kappa_{n-1}^n} V(\Pi K) \geq \kappa_n V(K)^{n-1}, \qquad (1.45)$$

with equality if and only if K is an ellipsoid.

The polarization of volume under Minkowski linear combinations and (1.42) imply an analogous behavior of the projection body operator

$$\Pi(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Pi(K_{i_1}, \ldots, K_{i_{n-1}}),$$

where the sum is with respect to Minkowski addition. The bodies $\Pi(K_{i_1}, \ldots, K_{i_{n-1}})$ are called mixed projection bodies and were introduced already in the classic volume of Bonnesen-Fenchel [4]. Mixed projection bodies are symmetric in their arguments and are multilinear with respect to Minkowski addition.

In [28] and [33], Lutwak considered the volume of mixed projection bodies and their polars and established analogs of the classical mixed volume inequalities. We will show in Chapter 3 that properties (a), (b) and (c) of the projection body operator are responsible not only for its behavior under Minkowski linear combinations, but also for most of the inequalities established in [28] and [33].

1.2.3 Radial Functions, Dual Mixed Volumes and Intersection Bodies

A compact set L in $\mathbb{R}^n$ which is starshaped with respect to the origin o is uniquely determined by the values of its radial function $\rho(L, \cdot)$, defined on $\mathbb{R}^n \backslash \{o\}$ by

$$\rho(L, x) = \max\{\lambda \geq 0 : \lambda x \in L\}.$$

Radial functions are positively homogeneous of degree -1. Thus we can identify them with their restriction on S^{n-1}. If $\rho(L, \cdot)$ is continuous on S^{n-1}, we call L a star body. With this definition the set of radial functions of star bodies coincides with the nonnegative continuous functions on S^{n-1}. Every $K \in \mathcal{K}_0^n$ containing the origin in its interior is a star body. We have the relation

$$h(K, \cdot) = \rho^{-1}(K^*, \cdot). \tag{1.46}$$

Let $\mathcal{S}^n$ denote the space of star bodies endowed with the uniform topology induced by identifying a star body with its radial function on the sphere. For $L_1, L_2 \in \mathcal{S}^n$ and $\lambda_1, \lambda_2 \geq 0$, the radial Minkowski linear combination $\lambda_1 L_1 \tilde{+} \lambda_2 L_2$ is the star body defined by

$$\rho(\lambda_1 L_1 \tilde{+} \lambda_2 L_2, \cdot) = \lambda_1 \rho(L_1, \cdot) + \lambda_2 \rho(L_2, \cdot). \tag{1.47}$$

If $\lambda_1, \lambda_2 \geq 0$, then the radial Blaschke linear combination $\lambda_1 \cdot L_1 \tilde{\#} \lambda_2 \cdot L_2$ of the star bodies L_1 and L_2 is the star body whose radial function satisfies

$$\rho^{n-1}(\lambda_1 \cdot L_1 \tilde{\#} \lambda_2 \cdot L_2, \cdot) = \lambda_1 \rho^{n-1}(L_1, \cdot) + \lambda_2 \rho^{n-1}(L_2, \cdot). \tag{1.48}$$

For star bodies $L_1, \ldots, L_m$ and $\lambda_1, \ldots, \lambda_m \geq 0$, it is obvious from (1.47) that $\rho^{n-1}(\lambda_1 \cdot L_1 \tilde{+} \ldots \tilde{+} \lambda_m \cdot L_m, \cdot)$ can be expressed as a polynomial homogeneous of degree $n - 1$

$$\rho^{n-1}(\lambda_1 L_1 \tilde{+} \ldots \tilde{+} \lambda_m L_m, \cdot) = \sum_{i_1, \ldots, i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \rho(L_{i_1}, \cdot) \cdots \rho(L_{i_{n-1}}, \cdot). \tag{1.49}$$

The volume of a radial Minkowski linear combination $\lambda_1 L_1 \tilde{+} \ldots \tilde{+} \lambda_m L_m$ of star bodies $L_1, \ldots, L_m$ admits a polarization formula of the form

$$V(\lambda_1 L_1 \tilde{+} \ldots \tilde{+} \lambda_m L_m) = \sum_{i_1, \ldots, i_n} \tilde{V}(L_{i_1}, \ldots, L_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $\tilde{V}(L_{i_1}, \ldots, L_{i_n})$ are called dual mixed volumes of $L_{i_1}, \ldots, L_{i_n}$. They are nonnegative, symmetric and monotone (with respect to set inclusion). They are also multilinear with respect to radial Minkowski addition, $\tilde{V}(L, \ldots, L) = V(L)$, and they are invariant under simultaneous volume preserving transformations of their arguments. The following integral representation of dual mixed volumes holds:

$$\tilde{V}(L_1, \ldots, L_n) = \frac{1}{n} \int_{S^{n-1}} \rho(L_1, u) \cdots \rho(L_n, u) du,$$

where du is the spherical Lebesgue measure of S^{n-1}. The definitions of $\tilde{V}_i(K,L)$, $\tilde{W}_i(K,L)$, etc. are analogous to the ones for mixed volumes. A slight extension of the notation $\tilde{V}_i(K,L)$ is for $r \in \mathbb{R}$

$$\tilde{V}_r(K,L) = \frac{1}{n} \int_{S^{n-1}} \rho^{n-r}(K,u)\rho^r(L,u)du. \tag{1.50}$$

Obviously, we have $\tilde{V}_r(L,L) = V(L)$ for every $r \in \mathbb{R}$ and every $L \in \mathcal{S}^n$.

The most general inequality for dual mixed volumes is the dual Aleksandrov Fenchel inequality: If $L_1, \ldots, L_n \in \mathcal{S}^n$ and $1 \le m \le n$, then

$$\tilde{V}(L_1, \ldots, L_n)^m \le \prod_{j=1}^{m} \tilde{V}(L_j, \ldots, L_j, L_{m+1}, \ldots, L_n) \tag{1.51}$$

with equality if and only if $L_1, \ldots, L_m$ are dilates. A special case of inequality (1.51) is the dual Minkowski inequality: If $K, L \in \mathcal{S}^n$, then

$$\tilde{V}_1(K,L)^n \le V(K)^{n-1}V(L),$$

with equality if and only if K and L are dilates. A more general version of the dual Minkowski inequality is: If $0 \le i \le n-2$, then

$$\tilde{W}_i(K,L)^{n-i} \le \tilde{W}_i(K)^{n-i-1}\tilde{W}_i(L), \tag{1.52}$$

with equality if and only if K and L are dilates.

We will also need the following Minkowski type inequality: If $K, L \in \mathcal{S}^n$, then

$$\tilde{V}_{-1}(K,L)^n \ge V(K)^{n+1}V(L)^{-1}, \tag{1.53}$$

with equality if and only if K and L are dilates.

The inequality between two consecutive dual quermassintegrals states that, for $L \in \mathcal{S}^n$ and $0 \le i \le n-2$,

$$\tilde{W}_{i+1}(L)^{n-i} \le \kappa_n \tilde{W}_i(L)^{n-i-1},$$

with equality if and only if L is a centered ball.

A consequence of the dual Minkowski inequality is the dual Brunn Minkowski inequality: If $K, L \in \mathcal{S}^n$ then

$$V(K \tilde{+} L)^{1/n} \le V(K)^{1/n} + V(L)^{1/n},$$

with equality if and only if K and L are dilates. Using Minkowski's integral inequality, this can be further generalized: If $0 \le i \le n-2$, then

$$\tilde{W}_i(K \tilde{+} L)^{1/(n-i)} \le \tilde{W}_i(K)^{1/(n-i)} + \tilde{W}_i(L)^{1/(n-i)},$$

with equality if and only if K and L are dilates. If $0 \le i \le n-2$, $K, L, L_1, \ldots, L_i \in \mathcal{S}^n$ and $\mathbf{C} = (L_1, \ldots, L_i)$, then

$$\tilde{V}_i(K+L, \mathbf{C})^{1/(n-i)} \le \tilde{V}_i(K, \mathbf{C})^{1/(n-i)} + \tilde{V}_i(L, \mathbf{C})^{1/(n-i)},$$

with equality if and only if K and L are dilates.

The intersection body IL of $L \in \mathcal{S}^n$ is the star body whose radial function is given for $u \in S^{n-1}$ by

$$\rho(IL,u) = \mathrm{vol}_{n-1}(L \cap u^{\perp}). \tag{1.54}$$

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Using spherical convolution, we can rewrite definition (1.54) as

$$\rho(IL, u) = \kappa_{n-1} R_0 \rho(L, \cdot)^{n-1} = \rho(L, \cdot)^{n-1} * \sigma_{S_0^{n-2}}, \qquad (1.55)$$

where $\sigma_{S_0^{n-2}}$ is the invariant measure concentrated on the parallel circle S_0^{n-2} with total mass κ_{n-1}. Intersection bodies appear already in a paper by Busemann [6], but were first explicitly defined and named by Lutwak [30]. Intersection bodies turned out to be critical for the solution of the Busemann-Petty problem, see [8], [9], [11], [19], [21], [22], [56]. The fundamental volume inequality for intersection bodies is the Busemann intersection inequality [6]: Among bodies of given volume the intersection bodies have maximal volume precisely for ellipsoids centered in the origin. To prove a corresponding result for the minimal volume of intersection bodies of a given volume is another major open problem in convex geometry.

From (1.55), we can deduce the following properties of the operator $I : \mathcal{S}^n \to \mathcal{S}^n$:

(a)$_\mathbf{d}$ I is continuous.

(b)$_\mathbf{d}$ $I(K \,\tilde{\#}\, L) = IK \,\tilde{+}\, IL$ for all $K, L \in \mathcal{S}^n$.

(c)$_\mathbf{d}$ I intertwines rotations.

Moreover (1.55) implies the following behavior of the intersection body with respect to radial Minkowski addition

$$I(\lambda_1 L_1 \,\tilde{+}\, \ldots \,\tilde{+}\, \lambda_m L_m) = \widetilde{\sum} \lambda_{i_1} \cdots \lambda_{i_{n-1}} I(L_{i_1}, \ldots, L_{i_{n-1}}),$$

where the sum is with respect to radial Minkowski addition. The star bodies $I(L_{i_1}, \ldots, L_{i_{n-1}})$ are called mixed intersection bodies. They were introduced by Zhang [55]. Mixed intersection bodies are symmetric in their arguments and are multilinear with respect to radial Minkowski addition.

In [23] and [24], it was shown that for the fundamental inequalities of the dual Brunn Minkowski Theory, the volume of these mixed intersection bodies satisfies analogous inequalities. In Chapter 3 we will generalize these results to operators satisfying properties (a)$_\mathbf{d}$, (b)$_\mathbf{d}$ and (c)$_\mathbf{d}$ of the intersection body operator.

Chapter 2

Intertwining Additive Maps

2.1 Endomorphisms of $\mathcal{K}^n$

In this section, we give an overview of the known results on Minkowski and Blaschke endomorphisms of convex bodies by Schneider [43], [44] and Kiderlen [20]. We first collect a few lemmas on intertwining maps of continuous functions and measures and prove a strengthened version of a very useful lemma by Schneider and Kiderlen. This will allow us to give short proofs of Kiderlen's characterization theorems of weakly monotone Minkowski and general Blaschke endomorphisms. Finally, we will explain Kiderlen's notion of adjointness of Minkowski and Blaschke endomorphisms and make a few remarks on the open problem concerning the classification of general Minkowski endomorphisms.

2.1.1 Some Useful Lemmas

In [7], Dunkl considered continuous, rotation intertwining and linear maps of $\mathcal{C}(S^{n-1})$ into itself. We call a map $\Phi : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ with these properties an endomorphism of $\mathcal{C}(S^{n-1})$ and state the following result due to Dunkl [7] as a lemma:

Lemma 2.1 *A map Φ is an endomorphism of $\mathcal{C}(S^{n-1})$ if and only if there is a unique measure $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$ such that*

$$\Phi f = f * \mu, \qquad f \in \mathcal{C}(S^{n-1}). \tag{2.1}$$

Proof: From Lemma 1.1 and (1.15) it follows that mappings of the form (2.1) are endomorphisms. The uniqueness of the measure μ follows from the multiplier property of zonal convolution and the completeness of the system of spherical harmonics.

For an endomorphism Φ of $\mathcal{C}(S^{n-1})$ consider the map

$$\varphi : \mathcal{C}(S^{n-1}) \to \mathbb{R}, \ f \mapsto \Phi f(\hat{e}).$$

By the properties of Φ, the functional φ is continuous and linear on $C(S^{n-1})$. Thus, by the Riesz representation theorem, there is a measure $\mu \in \mathcal{M}(S^{n-1})$ such that

$$\varphi(f) = \langle f, \mu \rangle.$$

Since φ is $SO(n-1)$ invariant, the measure μ is zonal. Thus, we have for $\widehat{\eta} \in S^{n-1}$

$$\Phi f(\widehat{\eta}) = \Phi(\eta^{-1}f)(\widehat{e}) = \varphi(\eta^{-1}f) = \langle \eta^{-1}f, \mu \rangle = \int_{S^{n-1}} f(\eta u)d\mu(u).$$

The theorem follows now from (1.13). ∎

Note that, in the proof of Lemma 2.1, we only need the continuity of the map $f \mapsto \Phi f(\widehat{e})$. Lemma 2.1 illustrates the use of classifications of linear functionals to obtain results on intertwining linear maps. We will draw on this idea in a more geometric context later on. The following corollary shows that also endomorphisms of $\mathcal{M}(S^{n-1})$, i.e., continuous, rotation intertwining and linear maps of $\mathcal{M}(S^{n-1})$ into itself, are generated by zonal measures, see [20].

Corollary 2.2 *A map Ψ is an endomorphism of $\mathcal{M}(S^{n-1})$ if and only if there is a unique measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that*

$$\Psi\nu = \nu * \mu, \qquad \nu \in \mathcal{M}(S^{n-1}). \tag{2.2}$$

Proof: By Lemma 1.1 and (1.15), mappings of the form of (2.2) are endomorphisms. The uniqueness of the measure μ follows again from the multiplier property of zonal convolution and from the completeness of the system of spherical harmonics.

Conversely, let Ψ be an endomorphism of $\mathcal{M}(S^{n-1})$. It is easy to see that also the adjoint Ψ^* of Ψ is rotation intertwining. Moreover, we have

$$|\Psi^* f(u)| = |\langle \Psi^* f, \delta_u \rangle| = |\langle f, \Psi^* \delta_u \rangle| \leq \|f\|_\infty \|\Psi \delta_u\| = \|f\|_\infty \|\Psi \delta_{\widehat{e}} \cdot\|$$

Thus, Ψ^* is continuous and hence an endomorphism of $C(S^{n-1})$. By Lemma 2.1, there is a unique measure μ such that $\Psi^* f = f * \mu$. Thus, by Lemma 1.2, we have

$$\langle \Psi\nu, f \rangle = \langle \nu, \Psi^* f \rangle = \langle \nu, f * \mu \rangle = \langle \nu * \mu, f \rangle. \qquad ∎$$

We call a map $\Phi : C(S^{n-1}) \to C(S^{n-1})$ monotone if it maps nonnegative functions to nonnegative ones. The following lemma is a slight variation of Lemma 2.1. It can be proved, using the Riesz representation theorem for positive linear forms:

Lemma 2.3 *A map $\Phi : C(S^{n-1}) \to C(S^{n-1})$ is a monotone, linear map that intertwines rotations if and only if there is a unique measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ such that*

$$\Phi f = f * \mu, \qquad f \in C(S^{n-1}).$$

We now turn to mappings of convex bodies. By (1.29), the Steiner point map can be interpreted as a convolution operator on the cone of support functions. In the following we will consider more general transformations of convex bodies induced by convolution operators.

By (1.5), the convolution from the left with measures $\mu \in \mathcal{M}^+(SO(n))$ can be interpreted as (weighted) rotation means. In particular we have for every convex body $K \in \mathcal{K}^n$

$$\mu * h(K, \cdot) = \int_{SO(n)} h(\vartheta K, \cdot) d\mu(\vartheta), \qquad \mu * S_{n-1}(K, \cdot) = \int_{SO(n)} S_{n-1}(\vartheta K, \cdot) d\mu(\vartheta).$$

The following consequences of this interpretation appear in [15].

Lemma 2.4 *Let* $\mu \in \mathcal{M}^+(SO(n))$.

(a) *For* $K \in \mathcal{K}^n$, *the function* $\mu * h(K, \cdot)$ *is the support function of a convex body.*

(b) *For* $L \in \mathcal{K}_0^n$ *and* $\mu \neq 0$, *the measure* $\mu * S_{n-1}(L, \cdot)$ *is the surface area measure of a convex body with interior points.*

Let $K \in \mathcal{K}_0^n$. Then, by (1.30) we have $\pi_1 S_{n-1}(K, \cdot) = 0$, i.e., the center of mass of surface area measures is the origin. By (1.14) and the remarks after Definition 1.7, spherical convolution operators from the right are multiplier transformations. Thus, we have for a nonnegative measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$

$$\pi_1(S_{n-1}(K, \cdot) * \mu) = 0.$$

Hence the convolution of surface area measures with nonnegative zonal measures from the right again give nonnegative measures with center of mass in the origin. It is also not hard to see that $S_{n-1}(K, \cdot) * \mu$ is not concentrated on any great sphere. Thus, the measure $S_{n-1}(K, \cdot) * \mu$ is again a surface area measure of a convex body. Noting (1.30), we see that in fact it is sufficient that the measure μ is positive up to addition of a measure with density $c(\widehat{e} \cdot \,)$. We capture this property of a measure in the following definition:

Definition 2.5 *A measure* $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ *is called a linear measure if* μ *has a density of the form* $c(\widehat{e} \cdot \,)$, $c \in \mathbb{R}$.
The measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ *is called weakly positive if it is nonnegative up to addition of a linear measure.*

It was shown in [20] that also the cone of support functions is invariant under convolution of zonal weakly positive measures. We summarize these results in

Lemma 2.6 *Let* $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ *be weakly positive.*

(a) *For* $K \in \mathcal{K}^n$, *the function* $h(K, \cdot) * \mu$ *is the support function of a convex body.*

(b) *For* $L \in \mathcal{K}_0^n$ *and* μ *not linear, the measure* $S_{n-1}(L, \cdot) * \mu$ *is the surface area measure of a convex body with interior points.*

We will need a criterion to determine if a measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ is weakly positive. Let $\mathcal{L} = \{h(K, \cdot) - h(L, \cdot) : K, L \in \mathcal{K}^n\}$ denote the vector space of differences of support functions. The following lemma is in a slightly weaker form due to Schneider [44] for $n = 2$ and Kiderlen [20] for $n \geq 3$.

Lemma 2.7 *Let $g \in \mathcal{L}$ and let $\mathcal{N}$ be a dense subset of $\mathcal{M}_o^+(S^{n-1})$. Then*

$$\langle g, \mu \rangle \geq 0 \qquad \forall \mu \in \mathcal{N} \tag{2.3}$$

if and only if there is an $x \in \mathbb{R}^n$ such that

$$g(u) + x \cdot u \geq 0 \qquad \forall u \in S^{n-1}. \tag{2.4}$$

Proof: Obviously, (2.4) for some $x \in \mathbb{R}^n$ implies (2.3). Conversely, assume that (2.3) holds. Since $\mathcal{N}$ is dense in $\mathcal{M}_o^+(S^{n-1})$, (2.3) holds for every measure in $\mathcal{M}_o^+(S^{n-1})$. Let

$$g = h(L, \cdot) - h(M, \cdot)$$

with convex bodies $L, M \in \mathcal{K}_0^n$. Define the inradius of L relative to M by

$$r(L, M) = \max\{\lambda \geq 0 : \lambda M \subseteq L + x \text{ for some } x \in \mathbb{R}^n\}.$$

Choose $x \in \mathbb{R}^n$, with $r(L, M)M \subseteq L + x$. By the definition of $r(L, M)$, the contact points of $r(L, M)M$ and $L + x$ are distributed on their respective boundaries such that

$$o \in \text{conv}\{N(L, y) \cap S^{n-1} : y \in r(L, M)M \cap L + x\},$$

where $N(L, y)$ is the normal cone of L in y. Otherwise we could move the body $r(L, M)M$ inside $L + x$ away from the contact points and blow it up, in contradiction to the definition of $r(L, M)$. Let $\mu \in \mathcal{M}_o^+(S^{n-1})$ be concentrated in the set $\{N(L, y) \cap S^{n-1} : y \in r(L, M)M \cap L + x\}$. By (2.3),

$$r(L, M)\langle h(M, \cdot), \mu \rangle = \langle h(L, \cdot), \mu \rangle = \langle g + h(M, \cdot), \mu \rangle \geq \langle h(M, \cdot), \mu \rangle.$$

Thus $r(L, M) \geq 1$, and hence we have for every $u \in S^{n-1}$

$$g(u) + h(M, u) + x \cdot u = h(L + x, u) \geq r(M, L)h(M, u) \geq h(M, u). \qquad \blacksquare$$

Using (1.32), and noting that the set of surface area measures of convex bodies is a dense subset of $\mathcal{M}_o^+(S^{n-1})$, we obtain the following geometric consequence of Lemma 2.7 which was proved differently by Weil in [50].

Corollary 2.8 *Let $K, L \in \mathcal{K}^n$. If $V_1(M, K) \leq V_1(M, L)$ for every $M \in \mathcal{K}_0^n$, then there is a vector $x \in \mathbb{R}^n$ such that $K + x \subseteq L$.*

Note that, if the function g in Lemma 2.7 is zonal, then the vector x in (2.4) can be chosen as a multiple of $\hat{e}$. The following consequence of Lemma 2.7, which we will use frequently, is due to Kiderlen [20].

Corollary 2.9 *Let $\mu \in \mathcal{M}(S^{n-1}, \hat{e})$, and let $\mathcal{N}$ be a dense subset of $\mathcal{M}_o^+(S^{n-1})$. Then*

$$\nu * \mu \in \mathcal{M}_o^+(S^{n-1}) \qquad \forall \nu \in \mathcal{N} \tag{2.5}$$

if and only if μ is weakly positive.

Proof: It is clear that (2.5) holds if μ is weakly positive. Conversely, assume that (2.5) holds. Let $(\varphi_k)_{k\in\mathbb{N}}$ be a zonal approximate identity. Then $\nu * \mu * \varphi_k \geq 0$, and by Lemma 1.3, we have $\mu * \varphi_k \in C^\infty(S^{n-1})$. Using (1.13), we see that $(\nu * \mu * \varphi_k)(\widehat{e}) = \langle \mu * \varphi_k, \nu \rangle \geq 0$ for every $\nu \in \mathcal{N}$. As $C^\infty(S^{n-1}) \subseteq \mathcal{L}$, see [46] p.27, by Lemma 2.7 and the remark after Corollary 2.8, there are $c_k \in \mathbb{R}$ such that

$$(\mu * \varphi_k)(u) + c_k(\widehat{e} \cdot u) \geq 0.$$

Thus, for nonnegative $f \in C(S^{n-1})$, we have by (1.24)

$$f * \mu * \varphi_k \geq -c_k f * (\widehat{e} \cdot .) = -\frac{c_k}{n} \pi_1 f.$$

By Lemma 1.3, we have $\mu * \varphi_k \to \mu$ weakly, and thus $f * \mu * \varphi_k \to f * \mu$ uniformly by Lemma 1.1. Hence there exists $b \in \mathbb{R}$ such that $b \geq -c_k \pi_1 f$. Since $\pi_1 f$ is a linear functional, the sequence c_k is bounded. Therefore we can assume that $c_k \to c$. ∎

2.1.2 Kiderlen's Characterization Theorems

We now use the tools that were prepared in the last subsection to give short proofs of the classification results on rotation intertwining and additive maps from [20].

Definition 2.10 *We call a map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ that is continuous, rotation intertwining and Minkowski additive a Minkowski endomorphism. A Blaschke endomorphism is a map $\Psi : [\mathcal{K}_0^n] \to [\mathcal{K}_0^n]$ that is continuous, rotation intertwining and additive with respect to Blaschke addition.*

Let $K, L \in \mathcal{K}^n$. Then $K \subseteq L$ if and only if $h(K, \cdot) \leq h(L, \cdot)$. Thus by (1.29) a map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ defined by

$$h(\Phi K, \cdot) = h(K, \cdot) * \mu,$$

with a weakly positive measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ is monotone (with respect to set inclusion) on the set of convex bodies having their Steiner point in the origin. We call a Minkowski endomorphism with this property *weakly monotone*.

A classification of weakly monotone Minkowski and general Blaschke endomorphisms was established by Kiderlen in [20]. We summarize his results in

Theorem 2.11 *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a weakly monotone Minkowski endomorphism if and only if there is a unique weakly positive measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that*

$$h(\Phi K, \cdot) = h(K, \cdot) * \mu, \qquad K \in \mathcal{K}^n. \tag{2.6}$$

A map $\Psi : [\mathcal{K}_0^n] \to [\mathcal{K}_0^n]$ is a Blaschke endomorphism if and only if there is a weakly positive measure $\nu \in \mathcal{M}(S^{n-1}, \widehat{e})$, unique up to addition of a linear measure, such that

$$S_{n-1}(\Psi K, \cdot) = S_{n-1}(K, \cdot) * \nu, \qquad K \in \mathcal{K}_0^n. \tag{2.7}$$

Proof: By Lemma 2.6, mappings of the form of (2.6) and (2.7) are intertwining endomorphisms of the respective additive structures. Thus, we only need to show that there are weakly positive measures $\mu, \nu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that (2.6) and (2.7) hold for every weakly monotone Minkowski and general Blaschke endomorphism.

Let Φ be a weakly monotone Minkowski endomorphism. Since the Steiner point map is uniformly continuous, it is easy to show that the monotonicity property of Φ implies uniform continuity of Φ. Thus the induced map on the vector space of differences of support functions $\tilde{\Phi} : \mathcal{L} \to C(S^{n-1})$ defined by

$$\tilde{\Phi}(h(K_1, \cdot) - h(K_2, \cdot)) = h(\Phi K_1, \cdot) - h(\Phi K_2, \cdot)$$

is continuous, additive and intertwining. Since $\mathcal{L}$ is dense in $C(S^{n-1})$, the map $\tilde{\Phi}$ can be extended uniquely to an endomorphism of $C(S^{n-1})$. By Theorem 2.1, there is a unique zonal measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that

$$\tilde{\Phi}f = f * \mu, \qquad f \in C(S^{n-1}).$$

Since Φ is weakly monotone, for every nonnegative $f \in C(S^{n-1})$ with $\pi_1 f = 0$ we have

$$(f * \mu)(\widehat{e}) = \langle f, \mu \rangle \geq 0.$$

These functions form a dense subset in the weak* topology of $\mathcal{M}_o^+(S^{n-1})$. Hence, by Lemma 2.7, the measure μ is weakly positive.

Let Ψ be a Blaschke endomorphism. Then Ψ induces a linear map $\tilde{\Psi}$ on the vector space $\mathcal{M}_o(S^{n-1})$ of differences of surface area measures of convex bodies by

$$\tilde{\Psi}(S_{n-1}(K_1, \cdot) - S_{n-1}(K_2, \cdot)) = S_{n-1}(\Psi K_1, \cdot) - S_{n-1}(\Psi K_2, \cdot).$$

$\tilde{\Psi}$ can be further extended to the space $\mathcal{M}(S^{n-1})$ by

$$\tilde{\Psi}\sigma = \tilde{\Psi}(\sigma - \pi_1 \sigma), \qquad \sigma \in \mathcal{M}(S^{n-1}).$$

Define a map $\Gamma : C(S^{n-1}) \to C(S^{n-1})$

$$(\Gamma f)(u) := \langle f, \tilde{\Psi}\delta_u \rangle.$$

By the properties of $\tilde{\Psi}$ and Theorem 2.1, the map Γ is an endomorphism of $C(S^{n-1})$, hence there is a measure $\nu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that $\Gamma f = f * \nu$ for $f \in C(S^{n-1})$.

We thus have for every $f \in C(S^{n-1})$ and every $u \in S^{n-1}$

$$\langle \Gamma f, \delta_u \rangle = \langle f * \nu, \delta_u \rangle = \langle f, \delta_u * \nu \rangle = \langle f, \tilde{\Psi}\delta_u \rangle.$$

By the linearity of $\tilde{\Psi}$ and of the convolution, it follows that $\tilde{\Psi}\sigma = \sigma * \nu$ for every $\sigma \in \mathcal{M}_o(S^{n-1})$ with finite support. Hence, for every polytope $P \in \mathcal{K}_0^n$,

$$\tilde{\Psi}(S_{n-1}(P, \cdot)) = S_{n-1}(\Psi P, \cdot) = S_{n-1}(P, \cdot) * \nu.$$

By the continuity of Ψ, of the convolution and by the weak continuity of surface area measures, this implies (2.7) for every $K \in \mathcal{K}_0^n$. By Corollary 2.9 and (1.30), the measure ν is weakly positive and unique up to addition of a linear measure. ∎

Kiderlen noted in [20] the following nice application of Theorem 2.11:

Corollary 2.12 *Let* $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ *be a weakly monotone Minkowski endomorphism, which is not a combination of the identity and the reflection in the origin. If $K \in \mathcal{K}^n$ is homothetic to ΦK, then K is a ball.*

The major open problem concerning Minkowski endomorphisms is a classification without the extra assumption of weak monotonicity. For $n = 2$, Schneider obtained in [44] such a result by showing that every Minkowski endomorphism is weakly monotone. The following conjecture appears implicitly in [44] and [20].

Conjecture 2.13 *Every Minkowski endomorphism is weakly monotone.*

In the course of this chapter we will give several reformulations of this conjecture.

In [20], a natural notion of adjointness between Minkowski and Blaschke endomorphisms was introduced.

Definition 2.14 *A Minkowski endomorphism Φ and a Blaschke endomorphism Ψ are called adjoint if for every $K \in \mathcal{K}_0^n$ and every $L \in \mathcal{K}^n$*

$$V_1(\Psi K, L) = V_1(K, \Phi L).$$

Using (1.32), Lemma 1.2 and Theorem 2.11, we see that a Blaschke and a Minkowski endomorphism are adjoint if and only if they have the same generating measure up to addition of a linear measure. By Theorem 2.11, every Blaschke endomorphism has an adjoint weakly monotone Minkowski endomorphism. The converse statement is equivalent to Conjecture 2.13:

Conjecture 2.13$_1$ *Every Minkowski endomorphism has an adjoint Blaschke endomorphism.*

2.2 Blaschke Minkowski Homomorphisms

As a continuation of the work by Schneider and Kiderlen, we investigate Blaschke Minkowski homomorphisms, i.e., continuous mappings from the space $\mathcal{K}^n$ into itself, that are rotation intertwining and Blaschke Minkowski mixed additive. The main results in this section are a representation theorem for general Blaschke Minkowski homomorphisms and a complete classification of all even Blaschke Minkowski homomorphisms. Moreover, we give characterizations of the projection body operator and the Minkowski and Blaschke difference body operators. In the last part of this section, we investigate the behavior of Blaschke Minkowski homomorphisms under Minkowski linear combinations and show that they admit a polarization formula analogous to that of the ordinary volume functional. The results of this section are mainly taken from [48].

2.2.1 Representation and Characterization Theorems

Since endomorphisms of $\mathcal{K}^n$ are more or less well understood, the question arises how homomorphisms between the two natural additive structures, Minkowski and Blaschke addition, look like. This motivates the following definition:

Definition 2.15 *A map* $\Phi : [\mathcal{K}_0^n] \to \mathcal{K}^n$ *is called a Blaschke Minkowski homomorphism if it satisfies the following conditions:*

(i) Φ *is continuous.*

(ii) For all $K, L \in [\mathcal{K}_0^n]$

$$\Phi(K \# L) = \Phi K + \Phi L. \tag{2.8}$$

(iii) Φ *is rotation intertwining, i.e. for all* $K \in [\mathcal{K}_0^n]$ *and every* $\vartheta \in SO(n)$

$$\Phi(\vartheta K) = \vartheta \Phi K.$$

By the properties (a), (b) and (c) of the projection body operator, see Section 1.2.2, Π is a first example of a Blaschke Minkowski homomorphism. Later on we will see that there are many more examples, see also [13], [18] and [48]. We call the operator that maps every convex body to the origin *trivial*.

The representation formulas for Minkowski and Blaschke endomorphisms obtained in Theorem 2.11 show that the respective endomorphisms induce multiplier transformations on the cone of support functions and surface area measures, respectively. This fact has been deduced before for Minkowski endomorphisms by Schneider in [43] using a different method. In the following we will adapt the technique by Schneider to show that also Blaschke Minkowski homomorphisms induce multiplier transformations.

Every Blaschke Minkowski homomorphism $\Phi : [\mathcal{K}_0^n] \to \mathcal{K}^n$ induces a map on the set of surface area measures by

$$\Phi S_{n-1}(K, \cdot) = h(\Phi K, \cdot), \qquad K \in \mathcal{K}_0^n. \tag{2.9}$$

Using Theorem 1.8, we obtain:

Theorem 2.16 *Let* $\Phi : [\mathcal{K}_0^n] \to \mathcal{K}^n$ *be a Blaschke Minkowski homomorphism. Then the induced map on the set of surface area measures is a multiplier transformation, i.e., there is a sequence* $c_k \in \mathbb{R}$ *such that, for every* $K \in \mathcal{K}_0^n$,

$$\pi_k h(\Phi K, \cdot) = \pi_k \Phi S_{n-1}(K, \cdot) = c_k \pi_k S_{n-1}(K, \cdot).$$

For the proof of Theorem 2.16, we need some well known facts on the vector space of differences of surface area measures, see [53] and [16], p.70.

Lemma 2.17 *Let* $\mathcal{Q} \subseteq \mathcal{M}_o^+(S^{n-1})$ *denote the set of surface area measures of convex bodies with interior points. Then*

(a) $\mathcal{Q}$ *is dense in* $\mathcal{M}_o^+(S^{n-1})$ *and* $\mathcal{M}_o(S^{n-1}) = \mathcal{Q} - \mathcal{Q}$.

(b) $\mathcal{Q} \cap \mathcal{H}^n$ *is dense in* $\mathcal{Q}$.

PROOF OF THEOREM 2.16: By the additivity property of Blaschke Minkowski homomorphisms, the induced map (2.9) on the cone $\mathcal{Q}$ of surface area measures of convex bodies is linear, and hence by Lemma 2.17 (a), there is a unique linear extension $\tilde{\Phi}$ to the vector space $\mathcal{M}(S^{n-1})$ given by

$$\tilde{\Phi}(\mu) = \Phi S_{n-1}(K_+, \cdot) - \Phi S_{n-1}(K_-, \cdot),$$

where $\mu - \pi_1\mu = S_{n-1}(K_+, \cdot) - S_{n-1}(K_-, \cdot) \in \mathcal{M}_0(S^{n-1})$ for some $K_+, K_- \in \mathcal{K}_0^n$.

The restriction of $\tilde{\Phi}$ to $\mathcal{H}^n$ is by definition linear and intertwines rotations. Thus, by Theorem 1.8, it is a multiplier transformation. The result follows since $\tilde{\Phi}$ and Φ coincide on the set $\mathcal{Q} \cap \mathcal{H}^n$ which is dense in $\mathcal{Q}$ by Lemma 2.17 (b). ∎

By Cauchy's surface area formula, the mean width of the projection body of a convex body $K \in \mathcal{K}_0^n$ is a constant multiple of the surface area of K. The following corollary to Theorem 2.16 is a generalization of this fact.

Corollary 2.18 *Let* $\Phi : [\mathcal{K}_0^n] \to \mathcal{K}^n$ *be a Blaschke Minkowski homomorphism. Then*

$$W_{n-1}(\Phi K) = r_\Phi W_1(K),$$

where $r_\Phi \in \mathbb{R}^+$ *is the radius of the ball* ΦB.

Proof: We will first show that ΦB is a ball. To see this note that $\pi_k S_{n-1}(B, \cdot) = 0$ for $k \geq 1$. Thus by Theorem 2.16, we have $\pi_k h(\Phi B, \cdot) = 0$ for $k \geq 1$, hence ΦB is a ball. By Theorem 2.16, the radius r_Φ of ΦB is given by

$$r_\Phi = \pi_0 h(\Phi B, \cdot) = \pi_0 \Phi S_{n-1}(B, \cdot) = c_0 \pi_0 S_{n-1}(B, \cdot) = c_0 \omega_n,$$

where c_0 denotes the first multiplier of Φ and ω_n is the surface area of B. By (1.33), we have $W_{n-1}(\Phi K) = \kappa_n \pi_0 h(\Phi K)$ and thus, again by Theorem 2.16 and (1.33),

$$W_{n-1}(\Phi K) = \kappa_n \pi_0 \Phi S_{n-1}(K, \cdot) = \frac{r_\Phi}{n} \pi_0 S_{n-1}(K, \cdot) = r_\Phi W_1(K). \quad ∎$$

From now on we will view a map $\Phi : [\mathcal{K}_0^n] \to \mathcal{K}^n$ via the obvious identification as a translation invariant map on $\mathcal{K}_0^n$. The next lemma shows that every Blaschke Minkowski homomorphism has a continuous extension to $\mathcal{K}^n$.

Lemma 2.19 *Let* $\Phi : \mathcal{K}_0^n \to \mathcal{K}^n$ *be a Blaschke Minkowski homomorphism. Then there is a unique continuous extension of* Φ *to* $\mathcal{K}^n$.

Proof: Let $K_m \in \mathcal{K}_0^n$ be a sequence converging to $K \in \mathcal{K}^n$. Then we define

$$\Phi K = \lim_{m \to \infty} \Phi K_m.$$

To see that this limit exists, note that, by Corollary 2.18, $W_{n-1}(\Phi K_m) = r_\Phi W_1(K_m)$. Thus, $W_{n-1}(\Phi K_m) \to r_\Phi W_1(K)$ as $m \to \infty$. Hence the sequence ΦK_m is bounded.

Let ΦK_{m_j} be a convergent subsequence of ΦK_m with limit $L \in \mathcal{K}^n$. By Theorem 2.16 and (1.23),

$$\pi_k h(\Phi K_{m_j}, \cdot) = c_k \pi_k S_{n-1}(K_{m_j}, \cdot) = c_k N(n, k) S_{n-1}(K_{m_j}, \cdot) * P_k^n(\widehat{e} \cdot \ ).$$

By Lemma 1.1, this converges uniformly to $c_k \pi_k S_{n-1}(K, \cdot)$. On the other hand, $\pi_k h(\Phi K_{m_j}, \cdot) \to \pi_k h(L, \cdot)$ as $j \to \infty$. By the completeness of spherical harmonics, the limits of every convergent subsequence ΦK_{m_j} of ΦK_m coincide and thus ΦK_m itself is convergent. ∎

In the following, we will not distinguish between a Blaschke Minkowski homomorphism $\Phi : \mathcal{K}_0^n \to \mathcal{K}^n$ and its continuous extension to $\mathcal{K}^n$.

In order to establish a representation theorem for Blaschke Minkowski homomorphisms, we will draw on the idea by Dunkl [7] and use a classification of real valued functionals to obtain a representation for rotation intertwining linear maps, see also [44]. A map Φ defined on $\mathcal{K}^n$ and taking values in an abelian semigroup is called a valuation if for all $K, L \in \mathcal{K}^n$ such that also $K \cup L \in \mathcal{K}^n$,

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L.$$

Since the map $K \mapsto S_{n-1}(K, \cdot)$ is a translation invariant valuation, see [46], p.201, we obtain from the definition of Blaschke addition that for all $K, L \in \mathcal{K}_0^n$ such that $K \cup L \in \mathcal{K}_0^n$ and $K \cap L \in \mathcal{K}_0^n$,

$$(K \cup L) \# (K \cap L) = K \# L.$$

Thus, if Φ is a Blaschke Minkowski homomorphism, we have by Lemma 2.19 for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi K + \Phi L. \tag{2.10}$$

Hence, Φ is a valuation with respect to Minkowski addition. For further information on valuations of this type, see [25] and [26].

The following characterization is due to Hadwiger [17] and McMullen [35]:

Theorem 2.20 *A map $\varphi : \mathcal{K}^n \to \mathbb{R}$ is a continuous translation invariant valuation homogeneous of degree $n - 1$ if and only if there is a function $g \in C(S^{n-1})$, unique up to addition of a linear function, such that*

$$\varphi(K) = \langle g, S_{n-1}(K, \cdot) \rangle.$$

Using Theorem 2.20 and (2.10), we can derive a representation theorem for Blaschke Minkowski homomorphisms.

Theorem 2.21 *If $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is a Blaschke Minkowski homomorphism, then there is a weakly positive $g \in C(S^{n-1}, \widehat{e})$, unique up to addition of a linear function, such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * g. \tag{2.11}$$

Proof: Define a functional $\varphi : \mathcal{K}^n \to \mathbb{R}$ by

$$\varphi(K) = h(\Phi K, \widehat{e}).$$

Since $S_{n-1}(\lambda K, \cdot) = \lambda^{n-1} S_{n-1}(K, \cdot)$ for $\lambda \geq 0$ and $K \in \mathcal{K}^n$, we have by (2.8)

$$\Phi \lambda K = \lambda^{n-1} \Phi K. \tag{2.12}$$

Using (2.12) and (2.10), we see that the map φ is a continuous valuation on $\mathcal{K}^n$ homogeneous of degree $n-1$. By Theorem 2.20, there is a function $g \in \mathcal{C}(S^{n-1})$, unique up to addition of a linear function, such that

$$\varphi(K) = \langle g, S_{n-1}(K, \cdot) \rangle.$$

Since φ is invariant under rotations leaving $\widehat{e}$ fixed, the function g is zonal, and thus, by (1.1) and (1.2),

$$h(\Phi K, \widehat{\eta}) = h(\Phi K, \eta \widehat{e}) = \langle g, S_{n-1}(\eta^{-1} K, \cdot) \rangle = \langle \eta g, S_{n-1}(K, \cdot) \rangle. \tag{2.13}$$

(2.11) follows now from (1.13) and (1.19). To see that g is weakly positive, note that by (1.29), (1.30) and the commutativity of the convolution of zonal functions,

$$h(\{s(\Phi K)\}, \cdot) = n h(\Phi K, \cdot) * (\widehat{e} \cdot .) = n S_{n-1}(K, \cdot) * (\widehat{e} \cdot .) * g = 0.$$

Since $s(\Phi K) \in \operatorname{relint} \Phi K$, see [46], p.43, we have $h(\Phi K, \cdot) \geq 0$. Thus, noting that the set of surface area measures is a dense subset of $\mathcal{M}_o^+(S^{n-1})$, it follows from Corollary 2.9 that g is weakly positive. ∎

For later applications, we state further properties of the generating functions of Blaschke Minkowski homomorphisms in the following lemma.

Lemma 2.22 *Let $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ be the generating function of a Blaschke Minkowski homomorphism.*

(a) g is a difference of support functions, i.e. $g \in \mathcal{L}$.

(b) There is a symmetric body of revolution $L \in \mathcal{K}^n$ such that, for every $u \in S^{n-1}$,

$$g(u) + g(-u) = h(L, u).$$

Proof: By Lemma 2.17 (a), there are convex bodies $K_+, K_- \in \mathcal{K}_0^n$ such that

$$\delta_{\widehat{e}} - \pi_1 \delta_{\widehat{e}} = S_{n-1}(K_+, \cdot) - S_{n-1}(K_-, \cdot).$$

Since the Dirac measure $\delta_{\widehat{e}}$ is the neutral element for zonal convolution, and as $(\pi_1 \delta_{\widehat{e}})(u) = n \widehat{e} \cdot u$ by (1.24) we obtain

$$(\delta_{\widehat{e}} - \pi_1 \delta_{\widehat{e}}) * g = g - \pi_1 g = h(\Phi K_+, \cdot) - h(\Phi K_-, \cdot).$$

Since $\pi_1 g$ is a linear functional on $\mathbb{R}^n$, there is a vector $x \in \mathbb{R}^n$ such that

$$(\pi_1 g)(u) = x \cdot u = h(\{x\}, u).$$

Hence $g = h(\Phi K_+ + x, \cdot) - h(\Phi K_-, \cdot)$, which proves (a).

To see (b), let $\{b_1, \ldots, b_n\}$ be an orthonormal basis in $\mathbb{R}^n$ such that $\widehat{e} = b_n$. For a vector $x \in \mathbb{R}^n$ let $x_1, \ldots, x_n$, denote its coordinates with respect to $b_1, \ldots, b_n$. Choose $\beta \in \mathbb{R}^+$ such that the ellipsoid E_α defined by

$$\frac{x_1^2 + \ldots + x_{n-1}^2}{\alpha^2} + \frac{x_n^2}{\beta^2} \leq 1$$

has surface area $S(E_\alpha) = 1$. It was shown in [15], p.103, that as $\alpha \to \infty$, we have $\beta \to 0$ and

$$S_{n-1}(E_\alpha, \cdot) \to \frac{1}{2}(\delta_{\widehat{e}} + \delta_{-\widehat{e}})$$

weakly. By Lemma 1.1,

$$h(\Phi E_\alpha, u) = (S_{n-1}(E_\alpha, \cdot) * g)(u) \to \frac{1}{2}(g(u) + g(-u))$$

uniformly in $u \in S^{n-1}$. Since $h(\Phi E_\alpha, \cdot)$ converges uniformly, it converges to a support function of a convex body, which proves (b). ∎

We call a map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ even if $\Phi K = \Phi(-K)$ for all $K \in \mathcal{K}^n$. An immediate consequence of Lemma 2.22 is the complete classification of all even Blaschke Minkowski homomorphisms.

Theorem 2.23 *A map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ is an even Blaschke Minkowski homomorphism if and only if there is a centrally symmetric body of revolution $L \in \mathcal{K}^n$, unique up to translation, such that*

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot).$$

Proof: A Blaschke Minkowski homomorphism is even if and only if its generating function is even. Thus the result follows from Lemma 2.22 (b). ∎

The projection body operator $\Pi : \mathcal{K}^n \to \mathcal{K}^n$ is an even Blaschke Minkowski homomorphism. By (1.43) its generating body of revolution is a dilate of the segment $[-\widehat{e}, \widehat{e}]$. The operator Π maps polytopes to Minkowski sums of rotated and dilated copies of the line segment $[-\widehat{e}, \widehat{e}]$. By Theorem 2.23, a general even Blaschke Minkowski homomorphism maps polytopes to finite Minkowski linear combinations of rotated and dilated copies of a symmetric body of revolution L. General convex bodies are mapped to limits of these finite Minkowski linear combinations.

Another wellknown example of an even Blaschke Minkowski homomorphism is provided by the sine transform of the surface area measure of a convex body K, see [18], [42]: Define an operator $\Theta : \mathcal{K}^n \to \mathcal{K}^n$ by

$$h(\Theta K, \cdot) = S_{n-1}(K, \cdot) * h(B \cap \widehat{e}^\perp, \cdot).$$

Then Θ is an even Blaschke Minkowski homomorphism whose images are (limits of) Minkowski sums of rotated and dilated copies of the disc $B \cap \widehat{e}^\perp$. The value $h(\Theta K, u)$ is up to a factor the integrated surface area of parallel hyperplane sections of K in the direction u.

If $g = h(L, \cdot)$ for some body of revolution $L \in \mathcal{K}^n$ is the generating function of a Blaschke Minkowski homomorphism Φ, then by (2.13) and (1.32),

$$h(\Phi K, \widehat{\eta}) = n V_1(K, \eta L). \tag{2.14}$$

Since $K_1 \subseteq K_2$ if and only if $h(K_1, \cdot) \leq h(K_2, \cdot)$, the monotonicity of mixed volumes together with (2.14) implies

Corollary 2.24 *A Blaschke Minkowski homomorphism whose generating function is given by $h(L, \cdot)$ for some $L \in \mathcal{K}^n$, is monotone with respect to set inclusion.*

Note that by Theorem 2.23 and Corollary 2.24, every even Blaschke Minkowski homomorphism is monotone.

By Lemma 1.10, every map $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ of the form

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot)$$

for some $L \in \mathcal{K}^n$ is a Blaschke Minkowski homomorphism, but in general there are generating functions g of Blaschke Minkowski homomorphisms that are not support functions. An example of such a map is the (normalized) second mean section operator M_2 introduced in [13] and further investigated in [18]: Let $\mathcal{E}_2^n$ be the affine Grassmanian of two-dimensional planes in $\mathbb{R}^n$ and μ_2 its motion invariant measure, normalized such that $\mu_2(\{E \in \mathcal{E}_2^n : E \cap B \neq \varnothing\}) = \kappa_{n-2}$. Then

$$h(M_2 K, \cdot) = (n-1) \int_{\mathcal{E}_2^n} h(K \cap E, \cdot) d\mu_2(E) - h(\{z_{n-1}(K)\}, \cdot) = S_{n-1}(K, \cdot) * g_2, \tag{2.15}$$

where $z_{n-1}(K)$ is the $(n-1)$st intrinsic moment vector of K, see [46], p.304, and where Λg_2 is given by

$$\Lambda g_2(t) = \arccos(-t)\sqrt{1 - t^2}.$$

The function g_2 is not a support function. Note that the operator M_2 is not monotone but that it has the following weak monotonicity property: M_2 is monotone on those convex bodies which have their $(n-1)$st intrinsic moment vector in the origin. This is similar to the monotonicity property of weakly monotone Minkowski endomorphisms.

We note here that the function g_2 has appeared in convexity before, but in a different context. In [1], Berg showed that for every $n \geq 2$ there are functions g_n such that, for every $K \in \mathcal{K}^n$ with $s(K) = o$,

$$h(K, \cdot) = S_1(K, \cdot) * g_n. \tag{2.16}$$

The generating function of the operator M_2 is up to a factor precisely the function appearing in (2.16) for dimension 2.

We will give now a complete characterization of generating functions of Blaschke Minkowski homomorphisms in the spirit of a classification result of Weil [52] of generating measures of generalized zonoids. For this, we need the extension of area measures of convex bodies to the space $\mathcal{L}$ of differences of support functions.

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Definition 2.25 *Let $g_i \in \mathcal{L}$, $i = 1, \ldots, n-1$, with $g_i = h(K_i^0, \cdot) - h(K_i^1, \cdot)$. Then the mixed surface area measure of $g_1, \ldots, g_{n-1}$ is defined by*

$$S(g_1, \ldots, g_{n-1}, \cdot) = \sum_{\alpha_1, \ldots, \alpha_{n-1} \in \{0,1\}} (-1)^{\alpha_1 + \cdots + \alpha_{n-1}} S(K_1^{\alpha_1}, \ldots, K_{n-1}^{\alpha_{n-1}}, \cdot) \in \mathcal{M}_o(S^{n-1}).$$

For a function $f \in C(S^{n-1})$, define

$$V(f, g_1, \ldots, g_{n-1}) = \langle f, S(g_1, \ldots, g_{n-1}, \cdot) \rangle.$$

For $g \in \mathcal{L}$ and $j = 1, \ldots, n-1$, the measure $S_j(g, \cdot) = S(g, \ldots, g, 1, \ldots, 1, \cdot)$ where g appears j times and 1 appears $n - j - 1$ times, is called the surface area measure of order j of g.

If Φ is a Blaschke Minkowski homomorphism, then by Lemma 2.22 (a),

$$h(\Phi K, \cdot) = S_{n-1}(K, \cdot) * g = S_{n-1}(K, \cdot) * h(L_+, \cdot) - S_{n-1}(K, \cdot) * h(L_-, \cdot),$$

where $g = h(L_+, \cdot) - h(L_-, \cdot)$. Thus defining Blaschke Minkowski homomorphisms Φ_+ and Φ_- with generating functions $h(L_+, \cdot)$ and $h(L_-, \cdot)$, we get

$$h(\Phi K, \cdot) = h(\Phi_+ K, \cdot) - h(\Phi_- K, \cdot). \tag{2.17}$$

In the light of (2.17), we need a criterion to determine whether a difference of support functions is in fact a support function. This was established by Weil in [51].

Theorem 2.26 *A function $g \in \mathcal{L}$ is the support function of a convex body K if and only if, for all $j \in \{1, \ldots, n-1\}$,*

$$S_j(g, \cdot) \in \mathcal{M}_o^+(S^{n-1}).$$

In order to use Theorem 2.26, we need to determine the area measures $S_j(\Phi K, \cdot)$. In [15], p.105, the area measures of the convex body with support function $\mu * h(K, \cdot)$, $\mu \in \mathcal{M}^+(SO(n))$ were calculated. The result established there extends easily to differences of support functions. Identifying spherical measures with right $SO(n-1)$ invariant measures on $SO(n)$, we get the following lemma.

Lemma 2.27 *Let Φ be a Blaschke Minkowski homomorphism with generating function $g \in \mathcal{L}$. Then $\langle f, S_j(\Phi K, \cdot) \rangle$ is given by*

$$\int_{(S^{n-1})^j} V(f, \Lambda g(u_1 \cdot .), \ldots, \Lambda g(u_j \cdot .), 1, \ldots, 1) dS_{n-1}(K, u_1) \ldots dS_{n-1}(K, u_j).$$

Using Lemma 2.27, Theorem 2.26 and the fact that the set of surface area measures of convex bodies forms a dense subset of $\mathcal{M}_o^+(S^{n-1})$, we obtain the following characterization of generating functions of Blaschke Minkowski homomorphisms.

Theorem 2.28 *A function $g \in \mathcal{L}$ is the generating function of a Blaschke Minkowski homomorphism if and only if, for every $j = 1, \ldots, n-1$,*

$$\int_{(S^{n-1})^j} V(f, \Lambda g(u_1 \cdot .), \ldots, \Lambda g(u_j \cdot .), 1, \ldots, 1) d\mu(u_1) \ldots d\mu(u_j) \geq 0$$

for every nonnegative $f \in C(S^{n-1})$ and every $\mu \in \mathcal{M}_o^+(S^{n-1})$.

2.2.2 Endomorphisms and Homomorphisms

We now establish a connection between adjoint Minkowski and Blaschke endomorphisms and Blaschke Minkowski homomorphisms.

Theorem 2.29 *Let Ψ be a Minkowski and Ψ^* a Blaschke endomorphism. Then the following statements are equivalent:*

(1) Ψ and Ψ^* are adjoint endomorphisms.

(2) For every Blaschke Minkowski homomorphism Φ

$$\Phi \circ \Psi^* = \Psi \circ \Phi. \tag{2.18}$$

(3) (2.18) holds for some injective Blaschke Minkowski homomorphism Φ.

Proof: If Ψ and Ψ^* are adjoint, then Ψ is weakly monotone and they have the same generating measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$. Let Φ be a Blaschke Minkowski homomorphism with generating function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$. From the commutativity of zonal convolution, it follows that

$$
\begin{aligned}
h(\Phi\Psi^* K, \cdot) &= S_{n-1}(\Psi^* K, \cdot) * g = S_{n-1}(K, \cdot) * \mu * g \\
&= S_{n-1}(K, \cdot) * g * \mu = h(\Phi K, \cdot) * \mu = h(\Psi\Phi K, \cdot).
\end{aligned}
$$

Thus (1) implies (2) and obviously (2) implies (3).

By the multiplier property, a Blaschke Minkowski homomorphism Φ is injective if and only if all the multipliers of g are nonzero. Thus, the multipliers of Ψ^* and Ψ can be determined from $\Phi \circ \Psi^*$ and $\Psi \circ \Phi$ and are equal if (2.18) holds. By the completeness of the system of spherical harmonics it follows that (3) implies (1). ∎

Theorem 2.29 shows that the following conjecture is equivalent to Conjecture 2.13:

Conjecture 2.13₂ *There exists an injective Blaschke Minkowski homomorphism whose range is invariant under every Minkowski endomorphism.*

In [13], Goodey and Weil showed that the second mean section operator M_2 is injective. Thus, another formulation of Conjecture 2.13 is:

Conjecture 2.13₃ *For every Minkowski endomorphism Φ there exists a Blaschke endomorphism Ψ such that*

$$M_2 \circ \Psi = \Phi \circ M_2.$$

Motivated by Conjecture 2.13₂, we further investigate the range of Blaschke Minkowski homomorphisms.

Theorem 2.30 *The range of every Blaschke Minkowski homomorphism is nowhere dense in $\mathcal{K}^n$.*

Proof: We call $K \in \mathcal{K}_0^n$ Blaschke decomposable if there exist two bodies $K_1, K_2 \in \mathcal{K}_0^n$ not homothetic to K such that $K = K_1 \# K_2$. By a result of Bronshtein [5], the only Blaschke indecomposable bodies in $\mathcal{K}_0^n$ are the simplices. Thus, every body in the range of a Blaschke Minkowski homomorphism with the only possible exception of the image of simplices is decomposable with respect to Minkowski addition.

Since the image of simplices is nowhere dense in $\mathcal{K}^n$ and since, on the other hand, the indecomposable bodies with respect to Minkowski addition form a dense subset of $\mathcal{K}^n$, the desired result follows. ∎

In the next part of this section we will see that most of the geometric convolution operators we encountered so far do not attain values in the set of polytopes.

Theorem 2.31 *Let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ be a Blaschke Minkowski homomorphism whose generating function is the support function $h(L, \cdot)$ of a body of revolution $L \in \mathcal{K}^n$. If there is a convex body $K \in \mathcal{K}_0^n$ such that ΦK is a polytope, then there is a constant $c \in \mathbb{R}^+$ such that*

$$\Phi = c\Pi.$$

Proof: Let $P = \Phi K = \mathrm{conv}\{x_1, \ldots, x_k\}$ be a polytope with vertices $x_1, \ldots, x_k$. Then

$$h(P, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot).$$

Since the body $L \in \mathcal{K}^n$ is unique up to translation, we can assume that $h(L, \cdot) \geq 0$. Let $\mu = \tilde{S}_{n-1}(K, \cdot) \in \mathcal{M}^+(SO(n))$, then by (2.11)

$$h(P, \cdot) = \int_{SO(n)} h(\vartheta L, \cdot) d\mu(\vartheta). \tag{2.19}$$

From now on, we consider support functions as positive homogeneous functions on $\mathbb{R}^n$. Let $C_1, \ldots, C_k$ denote the normal cones of the vertices of P. Then the support function $h(P, \cdot)$ is linear in every C_i, $i = 1, \ldots, k$. Thus, by (2.19), we have

$$\int_{SO(n)} h(\vartheta L, v_1) + h(\vartheta L, v_2) - h(\vartheta L, v_1 + v_2) d\mu(\vartheta) = 0 \tag{2.20}$$

for all $v_1, v_2 \in C_i$. Since support functions are sublinear, the integrand in (2.20) is nonnegative. Thus, as μ is nonnegative, $h(\vartheta L, v_1) + h(\vartheta L, v_2) = h(\vartheta L, v_1 + v_2)$ for all ϑ in the support of μ. For each such ϑ, we thus have

$$h(L, v_1) + h(L, v_2) = h(L, v_1 + v_2)$$

for all $v_1, v_2 \in \vartheta C_i$. Hence L is a polytope itself. But since L is a body of revolution and the only polytopes that are bodies of revolution are the multiples of the segment $[-\widehat{e}, \widehat{e}]$, the desired result follows from (1.43). ∎

Note that Theorem 2.23 and Theorem 2.31 imply the following characterization of the projection body operator. For a corresponding result in dimension two see [44].

Corollary 2.32 *Let* $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ *be an even Blaschke Minkowski homomorphism. If there exists a convex body* $K \in \mathcal{K}_0^n$ *such that* ΦK *is a polytope, then there is a constant* $c \in \mathbb{R}^+$ *such that*

$$\Phi = c\Pi.$$

The Difference body operator D is the Minkowski endomorphism defined by

$$DK = K + (-K).$$

The Blaschke body operator ∇ is the Blaschke endomorphism defined by

$$\nabla K = K \# (-K).$$

Our characterization of the projection body operator from Corollary 2.32 implies the following characterizations of the Difference and Blaschke body operators:

Corollary 2.33 *The only even Blaschke endomorphisms taking values in the set of polytopes are constant multiples of* ∇.

If an even Minkowski endomorphism maps a zonoid onto a polytope, then it is a constant multiple of D.

Proof: Let Ψ be an even Blaschke endomorphism and let $\Psi K = P$ be a polytope for some $K \in \mathcal{K}_0^n$. By (2.8), the map $\Pi \circ \Psi$ is an even Blaschke Minkowski homomorphism such that $\Pi \Psi K$ is a polytope. By Corollary 2.32 and Theorem 2.23, there is a constant $c \in \mathbb{R}^+$ such that

$$\Pi \circ \Psi = c\Pi. \tag{2.21}$$

Since Π is injective, all the even multipliers of Π are nonzero. Thus, by (2.21), all even multipliers of Ψ are equal to c. Noting that the odd multipliers of even multiplier operators are zero the result follows.

An analogous argument leads to the second statement. ■

2.2.3 Polarization Formulas and Induced Operators

At the end of Section 1.2.2, we noted that the projection body operator admits a polarization formula under Minkowski linear combinations. In this way, it induces mixed projection operators. For the remainder of this section let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ always denote a Blaschke Minkowski homomorphism. The following theorem generalizes the notion of mixed projection bodies:

Theorem 2.34 *There is a continuous operator*

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments such that, for $K_1, \ldots, K_m \in \mathcal{K}^n$ *and* $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\Phi(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi(K_{i_1}, \ldots, K_{i_{n-1}}), \tag{2.22}$$

where the sum is with respect to Minkowski addition.

Proof. Let $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ be the generating function of the Blaschke Minkowski homomorphism Φ. If we define an operator

$$\Phi : \underbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}_{n-1} \to \mathcal{K}^n,$$

by

$$h(\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}, \cdot) * g, \qquad (2.23)$$

then (1.26) and the linearity of convolution together imply (2.22). The mixed operator Φ is well defined, as by Minkowski's existence theorem, the mixed area measure $S(K_1, \ldots, K_{n-1}, \cdot)$ is the surface area measure of a convex body $[K_1, \ldots, K_{n-1}]$, called the mixed body of $K_1, \ldots, K_{n-1}$, see [29], and thus

$$\Phi(K_1, \ldots, K_{n-1}) = \Phi[K_1, \ldots, K_{n-1}]. \qquad (2.24)$$

By Lemma 1.1 and by the weak continuity of mixed area measures, see [46], p.276, the mixed operators defined by (2.23) are continuous and symmetric. ∎

Further properties of mixed Blaschke Minkowski homomorphisms, which are immediate consequences of the corresponding properties of mixed area measures and the convolution representation (2.23), are:

(i) They are multilinear with respect to Minkowski linear combinations.

(ii) Their diagonal form reduces to the Blaschke Minkowski homomorphism:

$$\Phi(K, \ldots, K) = \Phi K.$$

(iii) They intertwine simultaneous rotations, i.e. if $\vartheta \in SO(n)$, then

$$\Phi(\vartheta K_1, \ldots, \vartheta K_{n-1}) = \vartheta \Phi(K_1, \ldots, K_{n-1}).$$

From the fact that $S(K_1, \ldots, K_{n-1}, \cdot) * (\widehat{e} \cdot .) = 0$ for $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, see [46], p.281, we get by (1.29), (2.23) and by the commutativity of zonal convolution

$$h(\{s(\Phi(K_1, \ldots, K_{n-1}))\}, \cdot) = nS(K_1, \ldots, K_{n-1}, \cdot) * (\widehat{e} \cdot .) * g = 0.$$

Hence,

$$s(\Phi(K_1, \ldots, K_{n-1})) = o. \qquad (2.25)$$

Since $s(\Phi(K_1, \ldots, K_{n-1})) \in \operatorname{rel int} \Phi(K_1, \ldots, K_{n-1})$, see [46], p.43, we see that $\Phi(K_1, \ldots, K_{n-1})$ contains the origin.

Lemma 2.35 *If Φ is nontrivial and if $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ have nonempty interior, then $\Phi(K_1, \ldots, K_{n-1}) \in \mathcal{K}_0^n$.*

Proof: Assume that $\Phi(K_1, \ldots, K_{n-1}) \subseteq H$ for some hyperplane H through the origin. Then also $-\Phi(K_1, \ldots, K_{n-1}) \subseteq H$, and thus

$$\Phi(K_1, \ldots, K_{n-1}) + (-\Phi(K_1, \ldots, K_{n-1})) \subseteq H. \tag{2.26}$$

Denote by $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ the generating function of Φ. Then, by (1.11) and (2.23),

$$h(\Phi(K_1, \ldots, K_{n-1}), \cdot) + h(-\Phi(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}, \cdot) * g * (\delta_{\widehat{e}} + \delta_{-\widehat{e}}).$$

By Lemma 2.22, there is a convex body of revolution $L \in \mathcal{K}^n$ such that

$$g * (\delta_{\widehat{e}} + \delta_{-\widehat{e}}) = h(L, \cdot).$$

Since Φ is nontrivial, L is not a singleton, and we have

$$h(\Phi(K_1, \ldots, K_{n-1}), \widehat{\eta}) + h(-\Phi(K_1, \ldots, K_{n-1}), \widehat{\eta}) = V(K_1, \ldots, K_{n-1}, \eta L).$$

Since $K_1, \ldots, K_{n-1} \in \mathcal{K}_0^n$ and L is not a singleton, we have $V(K_1, \ldots, K_{n-1}, \eta L) > 0$ for every $\eta \in SO(n)$, see [46], p.277, which is a contradiction to (2.26). ∎

Let $K_1, \ldots, K_{n-1} \in \mathcal{K}_0^n$, then $\Phi(K_1, \ldots, K_{n-1}) \in \mathcal{K}_0^n$. Thus, $s(\Phi(K_1, \ldots, K_{n-1})) \in$ int $\Phi(K_1, \ldots, K_{n-1})$ and $\Phi(K_1, \ldots, K_{n-1})$ contains the origin in its interior by (2.25). Hence the polar body $\Phi^*(K_1, \ldots, K_{n-1})$, in particular for $K \in \mathcal{K}_0^n$ the body $\Phi^* K$, is well defined.

By (1.46) and (2.23), we get for the polar of a mixed Blaschke Minkowski homomorphism Φ with generating function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ the representation:

$$\rho^{-1}(\Phi^*(K_1, \ldots, K_{n-1}), \cdot) = S(K_1, \ldots, K_{n-1}, \cdot) * g. \tag{2.27}$$

Of particular interest for us is the following special case of Theorem 2.34:

Corollary 2.36 *The map Φ satisfies the Steiner type formula*

$$\Phi(K + \varepsilon B) = \sum_{i=0}^{n-1} \varepsilon^i \binom{n-1}{i} \Phi_i K.$$

The operators $\Phi_i : \mathcal{K}^n \to \mathcal{K}^n$, $i = 0, \ldots, n-1$, are continuous, rotation intertwining with $\Phi_0 = \Phi$.

If $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ again denotes the generating function of Φ then, by (2.23),

$$h(\Phi_i K, \cdot) = S_{n-1-i}(K, \cdot) * g. \tag{2.28}$$

Since the mappings $K \mapsto S_i(K, \cdot)$ are valuations, the Φ_i are valuations with respect to Minkowski addition. We will consider only the operators Φ_i, $i = 0, \ldots, n-2$, since Φ_{n-1} maps every body K to ΦB because $S_0(K, \cdot) = S_{n-1}(B, \cdot)$ is independent of K. Using the argument from the proof of Lemma 2.35, we obtain

Lemma 2.37 *If Φ is nontrivial, then for $K \in \mathcal{K}_i^n$ we have $\Phi_i K \in \mathcal{K}_0^n$ and $\Phi_i L = o$ if $L \in \mathcal{K}_{i+2}^n \backslash \mathcal{K}_{i+1}^n$.*

By (2.28), the Φ_i are multiplier operators, but apart from $\Phi_0 = \Phi$ and Φ_{n-2} they can not be interpreted as additive transformations of convex bodies, since the set of area measures $S_j(K, \cdot)$ of order j does not form a cone in $\mathcal{M}_o^+(S^{n-1})$ for $j = 2, \ldots, n-2$, see [12]. The operator Φ_{n-2} is by (1.28) a Minkowski endomorphism.

As the Laplace Beltrami operator Δ_0 is an intertwining operator, so is the operator Δ_1 appearing in (1.27). Thus by Lemma 1.8, Δ_1 is a multiplier operator. For the following lemma, see [15], p.86, and note that multiplier transformations are obviously commutative.

Lemma 2.38 *Let $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ and $\nu \in \mathcal{M}(S^{n-1})$. Then*

$$\Delta_1(\nu * \mu) = \nu * (\Delta_1 \mu) = (\Delta_1 \nu) * \mu$$

in the sense of distributions.

Using Lemma 2.38, we get the following result.

Theorem 2.39 *The operator Φ_{n-2} is a weakly monotone Minkowski endomorphism.*

Proof. By Theorem 2.11, we have to show that there is a weakly positive measure $\mu \in \mathcal{M}(S^{n-1}, \widehat{e})$ such that $h(\Phi_{n-2}K, \cdot) = h(K, \cdot) * \mu$. If $g \in \mathcal{C}(S^{n-1}, \widehat{e})$ is the generating function of Φ, then by Lemma 2.38 and (1.27),

$$h(\Phi_{n-2}K, \cdot) = S_1(K, \cdot) * g = h(K, \cdot) * \Delta_1 g,$$

thus we need to show that $\Delta_1 g$ is a weakly positive measure. Using Lemma 2.22 (a), we have $g = h(L_1, \cdot) - h(L_2, \cdot)$ for two convex bodies $L_1, L_2 \in \mathcal{K}^n$. Hence by (1.27),

$$\Delta_1 g = S_1(L_1, \cdot) - S_1(L_2, \cdot).$$

Using again Lemma 2.38 and (1.27), we obtain

$$S_1(\Phi K, \cdot) = S_{n-1}(K, \cdot) * \Delta_1 g \in \mathcal{M}_o^+(S^{n-1}).$$

Thus, the desired result follows from Lemma 2.9 and from the fact that the set of surface area measures is a dense subset of $\mathcal{M}_o^+(S^{n-1})$. ∎

2.3 Endomorphisms and Homomorphisms of $\mathcal{S}^n$

In this last section of this chapter, we will discuss rotation intertwining additive maps of star bodies. We give complete classifications of endomorphisms and homomorphisms of star bodies, now with respect to radial Minkowski and radial Blaschke addition. Since the cone of radial functions coincides with the nonnegative continuous functions the proofs are much simpler than the corresponding results for convex bodies. Finally we will see that radial Blaschke Minkowski homomorphisms satisfy a polarization formula analogous to the one for Blaschke Minkowski homomorphisms of convex bodies.

2.3.1 Dual Classifications and Consequences

We call a map $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ that is rotation intertwining and radial Minkowski additive a radial Minkowski endomorphism. The definition of radial Blaschke endomorphisms is analogous. Note that, in the definitions of radial Minkowski and radial Blaschke endomorphisms, we do not assume continuity. The following consequence of Theorem 2.3 is a dual version of Theorem 2.11:

Theorem 2.40 *A map* $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ *is a radial Minkowski endomorphism if and only if there is a unique nonnegative measure* $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ *such that*

$$\rho(\Psi L, \cdot) = \rho(L, \cdot) * \mu. \tag{2.29}$$

A map $\Upsilon : \mathcal{S}^n \to \mathcal{S}^n$ *is a radial Blaschke endomorphism if and only if there is a unique nonnegative measure* $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ *such that*

$$\rho^{n-1}(\Upsilon L, \cdot) = \rho^{n-1}(L, \cdot) * \mu. \tag{2.30}$$

Proof: From Lemma 1.1, (1.15) and the properties of spherical convolution, it is clear that mappings of the form of (2.29) and (2.30) are radial Minkowski respectively radial Blaschke endomorphisms. Thus, it suffices to show that for every such operator, there is a measure $\mu \in \mathcal{M}^+(S^{n-1}, \widehat{e})$ such that (2.29) and (2.30) holds.

Consider first a radial Minkowski endomorphism Ψ. Since the cone of radial functions of star bodies coincides with the set of nonnegative continuous functions on S^{n-1}, the vector space $\{\rho(K, \cdot) - \rho(L, \cdot) : K, L \in \mathcal{S}^n\}$ coincides with $\mathcal{C}(S^{n-1})$. The operator $\bar{\Psi} : \mathcal{C}(S^{n-1}) \to \mathcal{C}(S^{n-1})$ defined by

$$\bar{\Psi} f = \rho(\Psi L_1, \cdot) - \rho(\Psi L_2, \cdot),$$

where $f = \rho(L_1, \cdot) - \rho(L_2, \cdot)$, is a linear extension of Ψ to $\mathcal{C}(S^{n-1})$ that intertwines rotations. Since the cone of radial functions is invariant under $\bar{\Psi}$, it is also monotone. Hence by Theorem 2.3, there is a nonnegative measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ such that $\bar{\Psi} f = f * \mu$. The statement now follows from $\bar{\Psi}\rho(L, \cdot) = \rho(\Psi L, \cdot)$.

If Υ is a radial Blaschke endomorphism, formula (2.30) follows in the same way since the vector space $\{\rho^{n-1}(K, \cdot) - \rho^{n-1}(L, \cdot) : K, L \in \mathcal{S}^n\}$ coincides with $\mathcal{C}(S^{n-1})$ as well, which was the critical point in the argument above. ∎

By Theorem 2.40, radial Minkowski and radial Blaschke endomorphisms are continuous. This is a consequence of the fact that these operators are monotone. We can also introduce a natural notion of adjointness between them:

Definition 2.41 *A radial Minkowski endomorphism* Ψ *and a Blaschke endomorphism* Υ *are called adjoint if, for all* $K, L \in \mathcal{S}^n$,

$$\tilde{V}_1(\Upsilon K, L) = \tilde{V}_1(K, \Psi L).$$

Again it is easy to see that a radial Minkowski and a radial Blaschke endomorphism are adjoint if and only if they have the same generating measure. This time, every radial Minkowski endomorphism has an adjoint radial Blaschke endomorphism and vice versa.

Motivated by the properties $(a)_d, (b)_d, (c)_d$ of the intersection body operator I, we also define radial Blaschke Minkowski homomorphisms:

Definition 2.42 *A map* $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ *is called radial Blaschke Minkowski homomorphism if it satisfies the following conditions:*

$(2)_d$ $\Psi(K \,\widetilde{\#}\, L) = \Psi K \,\widetilde{+}\, \Psi L$ *for all* $K, L \in \mathcal{S}^n$.

$(3)_d$ Ψ *intertwines rotations.*

The same arguments as in the proof of Theorem 2.40 yield:

Theorem 2.43 *A map* $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ *is a radial Blaschke Minkowski homomorphism if and only if there is a nonnegative measure* $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ *such that*

$$\rho(\Psi L, \cdot) = \rho(L, \cdot)^{n-1} * \mu.$$

As a consequence of Theorem 2.43, we see that every radial Blaschke Minkowski homomorphism Ψ also satisfies:

$(1)_d$ Ψ is continuous.

The notion of mixed intersection bodies is generalized by:

Theorem 2.44 *Let* $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ *be a radial Blaschke Minkowski homomorphism, then there is a continuous operator*

$$\Psi : \underbrace{\mathcal{S}^n \times \cdots \times \mathcal{S}^n}_{n-1} \to \mathcal{S}^n,$$

symmetric in its arguments such that, for $L_1, \ldots, L_m \in \mathcal{S}^n$ *and* $\lambda_1, \ldots, \lambda_m \geq 0$,

$$\Psi(\lambda_1 L_1 \widetilde{+} \ldots \widetilde{+} \lambda_m L_m) = \overset{\sim}{\sum} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Psi(L_{i_1}, \ldots, L_{i_{n-1}}),$$

where the sum is with respect to radial Minkowski addition.

Proof. Let $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ be the generating measure of Ψ and define a mixed operator $\Psi : \mathcal{S}^n \times \cdots \times \mathcal{S}^n \to \mathcal{S}^n$ by

$$\rho(\Psi(L_1, \ldots, L_{n-1}), \cdot) = \rho(L_1, \cdot) \cdots \rho(L_{n-1}, \cdot) * \mu.$$

The map defined in this way is symmetric and by Lemma 1.1 continuous. The polarization formula is a direct consequence of Theorem 2.43 and (1.49). ∎

The properties (ii) and (iii) of mixed Blaschke Minkowski homomorphisms also hold for mixed radial Blaschke Minkowski homomorphisms but property (i) has to be replaced by:

$(i)_d$ They are multilinear with respect to radial Minkowski linear combinations.

Chapter 3

Volume Inequalities and Intertwining Additive Maps

3.1 Variants of a Conjecture by Petty

An important open problem in the field of affine isoperimetric inequalities is Petty's conjectured projection inequality (1.45). In this section we investigate analogous problems for general Blaschke Minkowski homomorphisms and their induced operators. We will first collect several useful identities for mixed volumes of mixed Blaschke Minkowski homomorphisms which will be used throughout the chapter. We will then prove an inequality for weakly monotone Minkowski endomorphisms, providing a strengthened version of the classical inequality between the two consecutive quermassintegrals W_{n-1} and W_{n-2}. The proofs are based on techniques developed by Lutwak in [32] and are taken from [48].

3.1.1 Some Useful Identities

Throughout this chapter let $\Phi : \mathcal{K}^n \to \mathcal{K}^n$ denote a *nontrivial* Blaschke Minkowski homomorphism with generating function $g \in \mathcal{C}(S^{n-1}, \widehat{e})$. For $K, L \in \mathcal{K}^n$, denote by $\Phi_i(K, L)$ the mixed operator $\Phi(K, \ldots, K, L, \ldots, L)$ with i copies of L and $n - i - 1$ copies of K. The body $\Phi_i(K, B)$ then simply becomes $\Phi_i K$.

For our further investigations we state the following geometric consequences of Lemma 1.2.

Lemma 3.1 *If $K_1, \ldots, K_{n-1}, L_1, \ldots, L_{n-1} \in \mathcal{K}^n$, then*

$$V(K_1, \ldots, K_{n-1}, \Phi(L_1, \ldots, L_{n-1})) = V(L_1, \ldots, L_{n-1}, \Phi(K_1, \ldots, K_{n-1})). \quad (3.1)$$

In particular, for $K, L \in \mathcal{K}^n$ and $0 \le i, j \le n - 2$,

$$W_i(K, \Phi(L_1, \ldots, L_{n-1})) = V(L_1, \ldots, L_{n-1}, \Phi_i K) \quad (3.2)$$

and

$$W_i(K, \Phi_j L) = W_j(L, \Phi_i K). \quad (3.3)$$

Proof: By (1.31), we have

$$V(K_1, \ldots, K_{n-1}, \Phi(L_1, \ldots, L_{n-1})) = \langle h(\Phi(L_1, \ldots, L_{n-1}), \cdot), S(K_1, \ldots, K_{n-1}) \rangle.$$

Hence, identity (3.1) follows from (2.23) and Lemma 1.2.

For $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = B$, identity (3.1) reduces to (3.2). Finally put $L_1 = \ldots = L_{n-j-1} = L$ and $L_{n-j} = \ldots = L_{n-1} = B$ in (3.2), to obtain identity (3.3). ∎

In the next lemma we summarize further special cases of identity (3.1). These make use of the fact that the image of a ball under a Blaschke Minkowski homomorphism is again a ball; this was shown in the proof of Lemma 2.18. Let r_Φ denote the radius of the ball ΦB.

Lemma 3.2 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then*

$$W_{n-1}(\Phi(K_1, \ldots, K_{n-1})) = r_\Phi V(K_1, \ldots, K_{n-1}, B). \tag{3.4}$$

In particular, for $K, L \in \mathcal{K}^n$,

$$W_{n-1}(\Phi_1(K, L)) = r_\Phi W_1(K, L), \tag{3.5}$$

and, for $0 \leq i \leq n - 2$,

$$W_{n-1}(\Phi_i K) = r_\Phi W_{i+1}(K). \tag{3.6}$$

The Shephard problem asks whether for $K, L \in \mathcal{K}_0^n$,

$$\mathrm{vol}_{n-1}(K|u^\perp) = h(\Pi K, u) \leq h(\Pi L, u) = \mathrm{vol}_{n-1}(L|u^\perp) \tag{3.7}$$

for every $u \in S^{n-1}$, implies

$$V(K) \leq V(L).$$

Obviously (3.7) is equivalent to $\Pi K \subseteq \Pi L$. As was shown independently by Petty [38] and Schneider [40], the answer to Shephard's problem is no in general, but if the body L is a projection body, the answer is yes. The crucial tool in the proof of the latter statement is a special case of identity (3.3). In fact, an analogous result can be shown for general Blaschke Minkowski homomorphisms.

Corollary 3.3 *Let $K \in \mathcal{K}_i^n$ and $L \in \Phi_i \mathcal{K}_i^n$. Then, for $i = 0, \ldots, n - 2$,*

$$\Phi_i K \subseteq \Phi_i L \quad \Rightarrow \quad W_i(K) \leq W_i(L)$$

and $W_i(K) = W_i(L)$ if and only if K and L are translates.

Proof: From the monotonicity of mixed volumes, (3.3) and the fact that $L = \Phi_i L_0$ for some convex body $L_0 \in \mathcal{K}_i^n$, it follows that

$$W_i(K, \Phi_i L_0) = W_i(L_0, \Phi_i K) \leq W_i(L_0, \Phi_i L) = W_i(L, \Phi_i L_0) = W_i(L).$$

Using the generalized Minkowski inequality (1.36), we thus get

$$W_i(K) \leq W_i(L),$$

with equality only if K and L are homothetic. But homothetic bodies of equal ith quermassintegral must be translates of each other. ∎

The special case $i = 0, \Phi = \Pi$ of Corollary 3.3 is the result of Schneider and Petty.

3.1.2 An Inequality for Minkowski Endomorphisms

The following theorem provides an upper bound for the ith quermassintegral of $\Phi_i K$.

Theorem 3.4 *For* $i = 0, \ldots, n-2$ *and* $K \in \mathcal{K}^n$,

$$W_{i+1}(K)^{n-i} \geq \frac{\kappa_n^{n-1-i}}{r_\Phi^{n-i}} W_i(\Phi_i K),$$

with equality if and only if $\Phi_i K$ *is a ball.*

Proof: Let $K \in \mathcal{K}^n$ and $0 \leq i \leq n-2$. From inequality (1.37), we get by repeated application, the inequality

$$W_{n-1}(K)^{n-i} \geq \kappa_n^{n-1-i} W_i(K),$$

where, for $K \in \mathcal{K}_{n-1}^n$, equality holds if and only if K is a ball. By setting $K = \Phi_i K$, (3.6) gives the desired result. ∎

We now turn to a generalized version of Petty's projection problem:

Definition 3.5 *Define* $\psi_i : \mathcal{K}_i^n \to \mathbb{R}$ *by*

$$\psi_i(K) = \frac{W_i(\Phi_i K)}{W_i(K)^{n-1-i}}.$$

Note that $W_i(K) > 0$ if and only if $K \in \mathcal{K}_i^n$. Thus ψ_i is well defined. Moreover, by Lemma 2.37, we have $\psi_i(K) > 0$ for every $K \in \mathcal{K}_i^n$. From the properties of Φ_i, it follows that ψ_i is similarity invariant. Petty's conjectured projection inequality is that for $\Phi = \Pi$ the functional ψ_0 attains a minimum precisely for ellipsoids. The following theorem generalizes results for the projection body operator by Schneider [45] and Lutwak [32]

Theorem 3.6 *If* $K \in \mathcal{K}_i^n$ *and* $0 \leq i \leq n-2$, *then*

$$\psi_i(K) \geq \psi_i(\Phi_i K),$$

with equality if and only if K *and* $\Phi_i^2 K$ *are homothetic.*

Proof: Let $K, L \in \mathcal{K}_i^n$. From the generalized Minkowski inequality (1.36) together with (3.3), we get

$$W_i(L, \Phi_i K)^{n-i} = W_i(K, \Phi_i L)^{n-i} \geq W_i(K)^{n-1-i} W_i(\Phi_i L),$$

with equality if and only if K and $\Phi_i L$ are homothetic. Setting $L = \Phi_i K$ gives

$$W_i(\Phi_i K)^{n-i} \geq W_i(K)^{n-1-i} W_i(\Phi_i^2 K),$$

with equality if and only if K and $\Phi_i^2 K$ are homothetic. ∎

In the case $i = n-2$, Corollary 2.12 and Lemma 2.39 can be used to deduce the following generalization of an inequality for Π_{n-2} by Lutwak [32]:

Theorem 3.7 *If $K \in \mathcal{K}^n$, then*

$$W_{n-1}(K)^2 \geq \frac{\kappa_n}{r_\Phi^2} W_{n-2}(\Phi_{n-2}K) \geq \kappa_n W_{n-2}(K). \tag{3.8}$$

If K is not a singleton, there is equality on the left hand side only if $\Phi_{n-2}K$ is a ball and equality on the right hand side only if K is ball.

Proof: The left hand side of inequality (3.8) is a consequence of Theorem 3.4. From Lemma 2.39 and Corollary 2.12, it follows that K and $\Phi_{n-2}K$ are homothetic if and only if K is a ball. This fact combined with Theorem 3.6 implies that for the functional $\psi_{n-2} : \mathcal{K}^n_{n-2} \to \mathbb{R}$,

$$\psi_{n-2}(K) \geq \psi_{n-2}(\Phi_{n-2}K), \tag{3.9}$$

with equality if and only if K is a ball. Hence, only for balls can ψ_{n-2} attain a minimum on $\mathcal{K}^n_{n-2}$. To complete the proof of the right hand side of (3.8), it suffices to show that ψ_{n-2} actually attains a minimum on $\mathcal{K}^n_{n-2}$. To do this, let

$$c_0 := \inf\{\psi_{n-2}(K) : K \in \mathcal{K}^n_{n-2}\}.$$

From (3.9) and from the fact that $\Phi_{n-2} : \mathcal{K}^n_{n-2} \to \mathcal{K}^n$ by Lemma 2.37, it follows that

$$c_0 = \inf\{\psi_{n-2}(K) : K \in \mathcal{K}^n_0\}.$$

From the translation invariance of ψ_{n-2}, it follows that only bodies with Steiner point in the origin need to be considered, i.e.,

$$c_0 = \inf\{\psi_{n-2}(K) : K \in \mathcal{K}^n_0, s(K) = o\}.$$

For $j > 0$, let

$$C_j := \{K \in \mathcal{K}^n_0 : s(K) = o, j^{-1}B \subseteq K \subseteq jB\},$$

and let

$$c_j := \inf\{\psi_{n-2}(K) : K \in C_j\}. \tag{3.10}$$

Clearly, $c_0 = \lim_{j\to\infty} c_j$. Fix j and choose a sequence $K_m \in C_j$ with $\psi_{n-2}(K_m) \to c_j$. By the Blaschke selection theorem, it may be assumed that K_m converges to some $K_j \in C_j$. Since ψ_{n-2} is continuous on $\mathcal{K}^n_{n-2}$, we have $c_j = \psi_{n-2}(K_j)$. As $j^{-1}B \subseteq K_j \subseteq jB$ and $s(K) = o$, it follows from Lemma 2.39 that $j^{-1}B \subseteq r_\Phi^{-1}\Phi_{n-2}K_j \subseteq jB$. Hence,

$$r_\Phi^{-1}\Phi_{n-2}K \in C_j.$$

Thus, by the dilatation invariance of ψ_{n-2} and by the definition of c_j, we have

$$\psi_{n-2}(\Phi_{n-2}K_j) = \psi_{n-2}(r_\Phi^{-1}\Phi_{n-2}K_j) \geq c_j = \psi_{n-2}(K_j).$$

From (3.9) thus follows that K_j is a ball, and hence, $c_j = r_\Phi^2$. Since each $c_j = r_\Phi^2$, we conclude that $c_0 = r_\Phi^2 = \psi_{n-2}(B)$. Hence, ψ_{n-2} attains a minimum on $\mathcal{K}^n_{n-2}$. ∎

In the proof of Theorem 3.7, we have used only that Φ_{n-2} is a weakly monotone Minkowski endomorphism which is not a combination of the identity and the reflection in the origin. Thus, inequality (3.8) with equality cases is valid for every such operator, compare also [43], p.70, for a related result:

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Corollary 3.8 *Let* $\Psi : \mathcal{K}^n \to \mathcal{K}^n$ *be a weakly monotone Minkowski endomorphism. If* $K \in \mathcal{K}^n$, *then*

$$W_{n-1}(K)^2 \geq \frac{\kappa_n}{r_\Psi^2} W_{n-2}(\Psi K) \geq \kappa_n W_{n-2}(K).$$

If K *is not a singleton and* Ψ *not a combination of the identity and the reflection in the origin, equality holds on the left hand side only if* ΨK *is a ball and on the right hand side only if* K *is ball.*

Note that by (1.37) inequality (3.8) provides a strengthened version of the classical inequality between the two consecutive quermassintegrals W_{n-1} and W_{n-2}. It is the author's belief that Petty's conjectured projection inequality which would provide a strengthened version of the classical isoperimetric inequality, holds in a more general form for every Blaschke Minkowski homomorphism and its induced operators:

Conjecture 3.9 *If* $K \in \mathcal{K}_i^n$, *then*

$$\kappa_n^{n-2-i} W_i(\Phi_i K) \geq r_\Phi^{n-i} W_i(K)^{n-1-i}. \tag{3.11}$$

Inequality (3.11) would yield a family of strengthened versions of the classical inequalities (1.37) between all consecutive quermassintegrals.

If inequality (3.11) is correct in the case $i = 0$, then the conjectured inequality (3.11) holds for all other values of i. To see this, recall that by (2.24) we have for $0 \leq i \leq n - 2$

$$\Phi_i K = \Phi[K]_i, \tag{3.12}$$

where $[K]_i \in \mathcal{K}_0^n$ is the mixed body, see [29], defined by $S_{n-1}([K]_i, \cdot) = S_{n-1-i}(K, \cdot)$. By repeated application of (1.37), we obtain $W_i(K)^n \geq \kappa_n^i V(K)^{n-i}$ and in particular,

$$W_i(\Phi_i K)^n \geq \kappa_n^i V(\Phi_i K)^{n-i}. \tag{3.13}$$

In [29], it was shown that

$$V([K]_i)^{(n-i)(n-1)} \geq \kappa_n^i W_i(K)^{n(n-1-i)}. \tag{3.14}$$

Suppose inequality (3.11) holds for $i = 0$. Using (3.12), we obtain

$$\kappa_n^{n-2} V(\Phi_i K) = \kappa_n^{n-2} V(\Phi[K]_i) \geq r_\Phi^n V([K]_i)^{n-i}. \tag{3.15}$$

Now combine (3.13), (3.14) and (3.15) to get inequality (3.11) for all values of i.

3.2 The Volume of Mixed Blaschke Minkowski Homomorphisms

In this section, analogs of the classical inequalities from the Brunn Minkowski Theory for the volume of mixed Blaschke Minkowski homomorphisms and of their polars are developed. As a corollary, we obtain a new Brunn Minkowski inequality for the volume of polar projection bodies. In order to simplify the equality conditions, we will state all our results only for convex bodies with interior points. In this case, equality holds in our inequalities if and only if equality holds in the classical theorems. The results generalize results of Lutwak [33] and are taken from [49].

3.2.1   A Minkowski Type Inequality

We establish the following Minkowski type inequality: For $K, L \in \mathcal{K}_0^n$,

$$V(\Phi_1(K, L))^{n-1} \geq V(\Phi K)^{n-2} V(\Phi L), \tag{3.16}$$

with equality if and only if K and L are homothetic. In fact, more general inequalities are shown:

Theorem 3.10 *If $K, L \in \mathcal{K}_0^n$ and $0 \leq i \leq n - 1$, then*

$$W_i(\Phi_1(K, L))^{n-1} \geq W_i(\Phi K)^{n-2} W_i(\Phi L), \tag{3.17}$$

with equality if and only if K and L are homothetic.

Proof: By (3.5) and (3.6), the case $i = n - 1$ follows from inequality (1.36). Let therefore $0 \leq i \leq n - 2$ and $Q \in \mathcal{K}_0^n$. By (3.2) and (1.34),

$$\begin{aligned} W_i(Q, \Phi_1(K, L))^{n-1} &= V(K, \ldots, K, L, \Phi_i Q)^{n-1} \geq V_1(K, \Phi_i Q)^{n-2} V_1(L, \Phi_i Q) \\ &= W_i(Q, \Phi K)^{n-2} W_i(Q, \Phi L). \end{aligned}$$

Inequality (1.36) implies

$$W_i(Q, \Phi K)^{(n-2)(n-i)} W_i(Q, \Phi L)^{n-i} \geq W_i(Q)^{(n-1)(n-i-1)} W_i(\Phi K)^{n-2} W_i(\Phi L)$$

and thus,

$$W_i(Q, \Phi_1(K, L))^{(n-1)(n-i)} \geq W_i(Q)^{(n-1)(n-i-1)} W_i(\Phi K)^{n-2} W_i(\Phi L), \tag{3.18}$$

with equality if and only if $Q, \Phi K$ and ΦL are homothetic. Setting $Q = \Phi_1(K, L)$, we obtain the desired inequality. If there is equality in (3.17), we have equality in (3.18). From the fact that the Steiner point of mixed Blaschke Minkowski homomorphisms is the origin, compare (2.25), it follows that there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_1(K, L) = \lambda_1 \Phi K = \lambda_2 \Phi L. \tag{3.19}$$

From the equality in (3.17), it follows that

$$\lambda_1^{n-2} \lambda_2 = 1.$$

Moreover, (3.5), (3.6) and (3.19) give

$$W_1(K, L) = \lambda_1 W_1(K) = \lambda_2 W_1(L).$$

Hence, we have

$$W_1(K, L)^{n-1} = W_1(K)^{n-2} W_1(L),$$

which implies by (1.36) that K and L are homothetic. ∎

Of course, inequality (3.16) is the special case $i = 0$ of Theorem 3.10.

3.2.2 An Aleksandrov Fenchel Type Inequality

Much more general than the Minkowski inequality is the Aleksandrov Fenchel type inequality for the volume of mixed operators: If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, then

$$V(\Phi(K_1, \ldots, K_{n-1}))^m \geq \prod_{j=1}^{m} V(\Phi(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_{n-1})). \qquad (3.20)$$

This is the special case $i = 0$ of

Theorem 3.11 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and $1 \leq m \leq n - 1$, then*

$$W_i(\Phi(K_1, \ldots, K_{n-1}))^m \geq \prod_{j=1}^{m} W_i(\Phi(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})).$$

Proof: The case $i = n - 1$ reduces by (3.4) to inequality (1.34). Hence, we can assume $i \leq n - 2$. From (3.2) and (1.34), it follows that for $Q \in \mathcal{K}^n$,

$$
\begin{aligned}
W_i(Q, \Phi(K_1, \ldots, K_{n-1}))^m &= V(K_1, \ldots, K_{n-1}, \Phi_i Q)^m \\
&\geq \prod_{j=1}^{m} V(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1}, \Phi_i Q) \\
&= \prod_{j=1}^{m} W_i(Q, \Phi(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})).
\end{aligned}
$$

Write $\Phi_{m'}(K_j, \mathbf{C})$ for the mixed operator $\Phi(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})$. Then, by inequality (1.36), we have

$$W_i(Q, \Phi_{m'}(K_j, \mathbf{C}))^{n-i} \geq W_i(Q)^{n-i-1} W_i(\Phi_{m'}(K_j, \mathbf{C})).$$

Hence, we obtain

$$W_i(Q, \Phi(K_1, \ldots, K_{n-1}))^{m(n-i)} \geq W_i(Q)^{m(n-i-1)} \prod_{j=1}^{m} W_i(\Phi_{m'}(K_j, \mathbf{C})).$$

By setting $Q = \Phi(K_1, \ldots, K_{n-1})$, this becomes the desired inequality. ∎

From the case $m = n - 2$ of Theorem 3.11, it follows that

$$W_i(\Phi(K_1, \ldots, K_{n-1}))^{n-2} \geq W_i(\Phi_1(K_1, K_{n-1})) \cdots W_i(\Phi_1(K_{n-2}, K_{n-1})).$$

By combining this inequality and Theorem 3.10, we obtain

Corollary 3.12 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}_0^n$ and $0 \leq i \leq n - 1$, then*

$$W_i(\Phi(K_1, \ldots, K_{n-1}))^{n-1} \geq W_i(\Phi K_1) \cdots W_i(\Phi K_{n-1}),$$

with equality if and only if the K_j are homothetic.

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The special case $K_1 = \ldots = K_{n-1-j} = K$ and $K_{n-j} = \ldots = K_{n-1} = L$ of Corollary 3.12 leads to a generalization of Theorem 3.10:

Corollary 3.13 *If $K, L \in \mathcal{K}_0^n$ and $0 \le i \le n - 1$, $1 \le j \le n - 2$, then*

$$W_i(\Phi_j(K, L))^{n-1} \ge W_i(\Phi K)^{n-j-1} W_i(\Phi L)^j,$$

with equality if and only if K and L are homothetic.

An immediate consequence of Corollary 3.13 is: If $K \in \mathcal{K}_0^n$, $0 \le i \le n - 1$ and $1 \le j \le n - 2$, then

$$W_i(\Phi_j K)^{n-1} \ge r_\Phi^{(n-i)j} \kappa_n^j W_i(\Phi K)^{n-j-1}, \tag{3.21}$$

with equality if and only if K is a ball. However, the equality conditions for a more general inequality can be obtained.

Theorem 3.14 *If $K \in \mathcal{K}_0^n$ and $0 \le i < j \le n - 2$, while $0 \le m \le n - 1$, then*

$$W_m(\Phi_j K)^{n-i-1} \ge r_\Phi^{(n-m)(j-i)} \kappa_n^{j-i} W_m(\Phi_i K)^{n-j-1}, \tag{3.22}$$

with equality if and only if K is a ball.

Proof: From (3.6), it follows that the case $m = n - 1$ of inequality (3.22) reduces to (1.38). Hence, we may assume that $m \le n - 2$. Suppose $Q \in \mathcal{K}_0^n$. From (3.3) and inequality (1.34), it follows that

$$\begin{aligned}
W_m(Q, \Phi_j K)^{n-i-1} &= W_j(K, \Phi_m Q)^{n-i-1} \ge W_{n-1}(\Phi_m Q)^{j-i} W_i(K, \Phi_m Q)^{n-j-1} \\
&= W_{n-1}(\Phi_m Q)^{j-i} W_m(Q, \Phi_i K)^{n-j-1}.
\end{aligned}$$

By (3.6) and inequality (1.37), we have

$$W_{n-1}(\Phi_m Q)^{n-m} = r_\Phi^{n-m} W_{m+1}(Q)^{n-m} \ge r_\Phi^{n-m} \kappa_n W_m(Q)^{n-m-1},$$

with equality if and only if Q is a ball. On the other hand, by inequality (1.36),

$$W_m(Q, \Phi_i K)^{n-m} \ge W_m(Q)^{n-m-1} W_m(\Phi_i K),$$

with equality if and only if Q and $\Phi_i K$ are homothetic. Thus, we obtain

$$W_m(Q, \Phi_j K)^{(n-i-1)(n-m)} \ge r_\Phi^{(n-m)(j-i)} \kappa_n^{j-i} W_m(Q)^{(n-i-1)(n-m-1)} W_m(\Phi_i K)^{n-j-1},$$

with equality if and only if Q and $\Phi_i K$ are balls. Now set $Q = \Phi_j K$, and the result is the promised inequality of the theorem. Suppose there is equality in inequality (3.22). Then $\Phi_i K$ and $\Phi_j K$ must be centered balls. Thus, there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_i K = \lambda_1 B \qquad \text{and} \qquad \Phi_j K = \lambda_2 B. \tag{3.23}$$

From equality in (3.22), it follows that

$$\lambda_2^{n-i-1} = r_\Phi^{j-i} \lambda_1^{n-j-1}.$$

Moreover, (3.6) and (3.23) imply

$$r_\Phi W_{i+1}(K) = \lambda_1 \kappa_n \qquad \text{and} \qquad r_\Phi W_{j+1}(K) = \lambda_2 \kappa_n.$$

Hence, we have

$$W_{j+1}(K)^{n-i-1} = \kappa_n^{j-i} W_{i+1}(K)^{n-j-1},$$

which implies by (1.38) that K and L are homothetic. ∎

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3.2.3 A Brunn Minkowski Type Inequality

The Brunn Minkowski inequality for the volume of Blaschke Minkowski homomorphisms which will be established, is: If $K, L \in \mathcal{K}_0^n$, then

$$V(\Phi(K + L))^{1/n(n-1)} \geq V(\Phi K)^{1/n(n-1)} + V(\Phi L)^{1/n(n-1)}, \qquad (3.24)$$

with equality if and only if K and L are homothetic. In fact, a considerably more general inequality will be proven:

Theorem 3.15 *If $K, L \in \mathcal{K}_0^n$ and $0 \leq i \leq n - 1$, $0 \leq j \leq n - 3$, then*

$$W_i(\Phi_j(K+L))^{1/(n-i)(n-j-1)} \geq W_i(\Phi_j K)^{1/(n-i)(n-j-1)} + W_i(\Phi_j L)^{1/(n-i)(n-j-1)}, \qquad (3.25)$$

with equality if and only if K and L are homothetic.

Proof: By (3.3) and (1.41), we have for $Q \in \mathcal{K}_0^n$,

$$
\begin{aligned}
W_i(Q, \Phi_j(K + L))^{1/(n-j-1)} &= W_j(K + L, \Phi_i Q)^{1/(n-j-1)} \\
&\geq W_j(K, \Phi_i Q)^{1/(n-j-1)} + W_j(L, \Phi_i Q)^{1/(n-j-1)} \\
&= W_i(Q, \Phi_j K)^{1/(n-j-1)} + W_i(Q, \Phi_j L)^{1/(n-j-1)}.
\end{aligned}
$$

By inequality (1.36),

$$W_i(Q, \Phi_j K)^{n-i} \geq W_i(Q)^{n-i-1} W_i(\Phi_j K),$$

with equality if and only if Q and $\Phi_j K$ are homothetic and

$$W_i(Q, \Phi_j L)^{n-i} \geq W_i(Q)^{n-i-1} W_i(\Phi_j L),$$

with equality if and only if Q and $\Phi_j L$ are homothetic. Thus, we obtain

$$
\begin{aligned}
W_i(Q, \Phi_j(K + L))^{1/(n-j-1)} & W_i(Q)^{-(n-i-1)/(n-i)(n-j-1)} \\
&\geq W_i(\Phi_j K)^{1/(n-i)(n-j-1)} + W_i(\Phi_j L)^{1/(n-i)(n-j-1)},
\end{aligned}
$$

with equality if and only if Q, $\Phi_j K$ and $\Phi_j L$ are homothetic. If we set $Q = \Phi_j(K+L)$, we obtain (3.25). If there is equality in (3.25), then, by (2.25), there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_j K = \lambda_1 \Phi_j(K + L) \qquad \text{and} \qquad \Phi_j L = \lambda_2 \Phi_j(K + L). \qquad (3.26)$$

From equality in (3.25), it follows that

$$\lambda_1^{1/(n-j-1)} + \lambda_2^{1/(n-j-1)} = 1.$$

Moreover, (3.6) and (3.26) imply

$$W_{j+1}(K) = \lambda_1 W_{j+1}(K + L) \qquad \text{and} \qquad W_{j+1}(L) = \lambda_2 W_{j+1}(K + L).$$

Hence, we have

$$W_{j+1}(K + L)^{1/(n-j-1)} = W_{j+1}(K)^{1/(n-j-1)} + W_{j+1}(L)^{1/(n-j-1)},$$

which implies by (1.40) that K and L are homothetic. ∎

The most interesting cases of Theorem 3.15 are the cases where $i = 0$ or $j = 0$.

Corollary 3.16 *If* $K, L \in \mathcal{K}_0^n$ *and* $0 \le j \le n - 3$, *then*

$$V(\Phi_j(K + L))^{1/n(n-j-1)} \ge V(\Phi_j K)^{1/n(n-j-1)} + V(\Phi_j L)^{1/n(n-j-1)},$$

with equality if and only if K *and* L *are homothetic.*

Corollary 3.17 *If* $K, L \in \mathcal{K}_0^n$ *and* $0 \le i \le n - 1$, *then*

$$W_i(\Phi(K + L))^{1/(n-i)(n-1)} \ge W_i(\Phi K)^{1/(n-i)(n-1)} + W_i(\Phi L)^{1/(n-i)(n-1)},$$

with equality if and only if K *and* L *are homothetic.*

Of course inequality (3.24) is the special case $i = 0$ and $j = 0$ of Theorem 3.15.

3.2.4 Polar Bodies of Blaschke Minkowski Homomorphisms

In the following we will prove analogs of the inequalities of Sections 3.2.1, 3.2.2 and 3.2.3 for polars of mixed Blaschke Minkowski homomorphisms. To this end, we *restrict* ourselves to Blaschke Minkowski homomorphisms Φ with a generating function of the form $g = h(F, \cdot)$, where $F \in \mathcal{K}^n$ is a body of revolution which is not a singleton. Note that by Lemma 2.4, every support function is generating function of a Blaschke Minkowski homomorphism. In particular, by Theorem 2.23, every even Blaschke Minkowski homomorphism has a generating function of that type.

We now associate with each such Blaschke Minkowski homomorphism Φ two new operators:

Definition 3.18 *Define* $M_\Phi : \mathcal{S}^n \to \mathcal{K}^n$ *by*

$$h(M_\Phi L, \cdot) = \rho^{n+1}(L, \cdot) * h(F, \cdot), \tag{3.27}$$

and let $\Gamma_\Phi : \mathcal{S}^n \to \mathcal{K}^n$ *be defined by*

$$\Gamma_\Phi L = \frac{2}{(n + 1)V(L)} M_\Phi L. \tag{3.28}$$

By Lemma 2.4, the operator M_Φ, and hence also Γ_Φ, is well defined. Note that M_Φ depends, in contrast to Φ, on the position of F but that by Theorem 2.21, we may assume that $s(F) = o$. In this way we associate to each Blaschke Minkowski homomorphism a unique operator M_Φ.

If Φ is the projection body operator Π, the map M_Φ becomes a multiple of the moment body operator. The normalization in (3.28) is chosen in such a way that the body Γ_Π becomes the well known centroid body operator $\Gamma : \mathcal{S}^n \to \mathcal{K}^n$. Centroid bodies were defined and investigated by Petty [37]. They have proven to be an important tool in establishing fundamental affine isoperimetric inequalities, see [10], [31], [36], [39]. The Busemann-Petty centroid inequality, for example, states that

$$V(\Gamma L) \ge \left(\frac{2\kappa_{n-1}}{(n + 1)\kappa_n} \right)^n V(K). \tag{3.29}$$

It is critical for the proof of Petty's projection inequality (1.44). At the end of this section, we will investigate similar problems for Blaschke Minkowski homomorphisms which are generated by support functions. Before that we will prove analogs of the inequalities (3.16), (3.20) and (3.24) for the polars of these operators. Recall that the crucial tool in the proofs of inequalities (3.16), (3.20) and (3.24) is Lemma 3.1. For the proofs in this section, the critical tool will be the following lemma:

Lemma 3.19 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and $L \in \mathcal{S}^n$, then*

$$\tilde{V}_{-1}(L, \Phi^*(K_1, \ldots, K_{n-1})) = V(K_1, \ldots, K_{n-1}, M_\Phi L). \qquad (3.30)$$

In particular, for $K \in \mathcal{K}^n$,

$$\tilde{V}_{-1}(L, \Phi_i^* K) = W_i(K, M_\Phi L). \qquad (3.31)$$

Proof: By (1.50), we have

$$\tilde{V}_{-1}(K, \Phi^*(K_1, \ldots, K_{n-1})) = \langle \rho^{n+1}(K, \cdot), \rho^{-1}(\Phi^*(K_1, \ldots, K_{n-1}), \cdot) \rangle.$$

Hence, identity (3.30) follows from (2.27) and Lemma 1.2. For $K_1 = \ldots = K_{n-i-1} = K$ and $K_{n-i} = \ldots = K_{n-1} = B$, identity (3.30) reduces to (3.31). $\blacksquare$

By definition (3.28) of the operator Γ_Φ, Lemma 3.19 implies:

Lemma 3.20 *If $K \in \mathcal{K}^n$ and $L \in \mathcal{S}^n$, then*

$$V_1(K, \Gamma_\Phi L) = \frac{2}{(n+1)V(L)} \tilde{V}_{-1}(L, \Phi^* K). \qquad (3.32)$$

In particular,

$$V_1(K, \Gamma_\Phi \Phi^* K) = \frac{2}{n+1}. \qquad (3.33)$$

From Lemma 3.19, we immediately get the following Minkowski type inequality for the volume of polar Blaschke Minkowski homomorphisms Φ with a generating function of the form $g = h(F, \cdot)$.

Theorem 3.21 *If $K, L \in \mathcal{K}_0^n$, then*

$$V(\Phi_1^*(K, L))^{n-1} \leq V(\Phi^* K)^{n-2} V(\Phi^* L), \qquad (3.34)$$

with equality if and only if K and L are homothetic.

Proof: Let $Q \in \mathcal{S}^n$. Then, by (3.30) and (1.34),

$$\begin{aligned}
\tilde{V}_{-1}(Q, \Phi_1^*(K, L))^{n-1} &= V(K, \ldots, K, L, M_\Phi Q)^{n-1} \geq V_1(K, M_\Phi Q)^{n-2} V_1(L, M_\Phi Q) \\
&= \tilde{V}_{-1}(Q, \Phi^* K)^{n-2} \tilde{V}_{-1}(Q, \Phi^* L).
\end{aligned}$$

By inequality (1.53), we have

$$\tilde{V}_{-1}(Q, \Phi^* K)^{(n-2)n} \tilde{V}_{-1}(Q, \Phi^* L)^n \geq V(Q)^{(n+1)(n-1)} V(\Phi^* K)^{-(n-2)} V(\Phi^* L)^{-1}.$$

and thus

$$\tilde{V}_{-1}(Q, \Phi_1^*(K,L))^{(n-1)n} \geq V(Q)^{(n+1)(n-1)} V(\Phi^*K)^{-(n-2)} V(\Phi^*L)^{-1},$$

with equality if and only if Q, Φ^*K and Φ^*L are dilates. Setting $Q = \Phi_1^*(K,L)$, we obtain the desired inequality. If there is equality in (3.34), then there exist $\lambda_1, \lambda_2 > 0$ such that

$$\Phi_1^*(K,L) = \lambda_1 \Phi^* K = \lambda_2 \Phi^* L. \tag{3.35}$$

For every convex body $K \in \mathcal{K}^n$ containing the origin and for every $\lambda > 0$, we have $(\lambda K)^* = \lambda^{-1} K^*$, and thus

$$\Phi_1(K,L) = \lambda_1^{-1} \Phi K = \lambda_2^{-1} \Phi L.$$

From the equality in (3.34), it follows that

$$\lambda_1^{-(n-2)} \lambda_2^{-1} = 1.$$

By (3.5), (3.6) and (3.35), we obtain

$$W_1(K,L) = \lambda_1^{-1} W_1(K) = \lambda_2^{-1} W_1(L).$$

Hence, we have

$$W_1(K,L)^{n-1} = W_1(K)^{n-2} W_1(L),$$

which implies, by (1.36), that K and L are homothetic. ∎

Theorem 3.21 is the polar version of inequality (3.16). The next result provides a polar version of the Aleksandrov Fenchel type inequality (3.20).

Theorem 3.22 *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$ and $1 \leq m \leq n-1$, then*

$$V(\Phi^*(K_1, \ldots, K_{n-1}))^m \leq \prod_{j=1}^m V(\Phi^*(\underbrace{K_j, \ldots, K_j}_{m}, K_{m+1}, \ldots, K_{n-1})).$$

Proof: From (3.30), it follows that for $Q \in \mathcal{S}^n$,

$$\begin{aligned}
\tilde{V}_{-1}(Q, \Phi^*(K_1, \ldots, K_{n-1}))^m &= V(K_1, \ldots, K_{n-1}, M_\Phi Q)^m \\
&\geq \prod_{j=1}^m V(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1}, M_\Phi Q) \\
&= \prod_{j=1}^m \tilde{V}_{-1}(Q, \Phi^*(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})).
\end{aligned}$$

Write $\Phi_{m'}^*(K_j, \mathbf{C})$ for the mixed operator $\Phi^*(K_j, \ldots, K_j, K_{m+1}, \ldots, K_{n-1})$. Then, by inequality (1.53), we have

$$\tilde{V}_{-1}(Q, \Phi_{m'}^*(K_j, \mathbf{C}))^n \geq V(Q)^{n+1} V(\Phi_{m'}^*(K_j, \mathbf{C}))^{-1}.$$

Hence, we obtain

$$V(Q, \Phi^*(K_1, \ldots, K_{n-1}))^{mn} \geq V(Q)^{m(n+1)} \prod_{j=1}^{m} V(\Phi_{m'}^*(K_j, C))^{-1}.$$

Setting $Q = \Phi^*(K_1, \ldots, K_{n-1})$, this becomes the desired inequality. ∎

Combine the special case $m = n - 2$ of Theorem 3.22 and Theorem 3.21, to obtain:

Corollary 3.23 *If* $K_1, \ldots, K_{n-1} \in \mathcal{K}_0^n$, *then*

$$V(\Phi^*(K_1, \ldots, K_{n-1}))^{n-1} \leq V(\Phi^*K_1) \cdots V(\Phi^*K_{n-1}),$$

with equality if and only if the K_j *are homothetic.*

The special case, $K_1 = \ldots = K_{n-1-j} = K$ and $K_{n-j} = \ldots = K_{n-1} = L$, of Corollary 3.23 leads to a generalization of Theorem 3.21:

Corollary 3.24 *If* $K, L \in \mathcal{K}_0^n$ *and* $1 \leq j \leq n - 2$, *then*

$$V(\Phi_j^*(K, L))^{n-1} \leq V(\Phi^*K)^{n-j-1} V(\Phi^*L)^j,$$

with equality if and only if K *and* L *are homothetic.*

A polar version of Theorem 3.14 in the case $m = 0$ is provided by:

Theorem 3.25 *If* $K \in \mathcal{K}_0^n$ *and* $0 \leq i < j \leq n - 2$, *then*

$$V(\Phi_j^*K)^{n-i-1} \leq r_\Phi^{-n(j-i)} \kappa_n^{j-i} V(\Phi_i^*K)^{n-j-1}, \tag{3.36}$$

with equality if and only if K *is a ball.*

Proof: Suppose $Q \in \mathcal{S}^n$. From (3.31) and inequality (1.34), it follows that

$$\begin{aligned}
\tilde{V}_{-1}(Q, \Phi_j^*K)^{n-i-1} &= W_j(K, M_\Phi Q)^{n-i-1} \geq W_{n-1}(M_\Phi Q)^{j-i} W_i(K, M_\Phi Q)^{n-j-1} \\
&= W_{n-1}(M_\Phi Q)^{j-i} \tilde{V}_{-1}(Q, \Phi_i^*K)^{n-j-1}.
\end{aligned}$$

From the definition of W_{n-1} and Lemma 1.2, it follows that

$$W_{n-1}(M_\Phi Q) = \frac{1}{n} \langle \rho^{n+1}(Q, \cdot) * h(F, \cdot), 1 \rangle = r_\Phi \tilde{V}_{-1}(Q, B).$$

Thus, we obtain, by (1.53),

$$W_{n-1}(M_\Phi Q)^n = r_\Phi^n \tilde{V}_{-1}(Q, B)^n \geq r_\Phi^n \kappa_n^{-1} V(Q)^{n+1},$$

with equality if and only if Q is a centered ball. Also, by inequality (1.53),

$$\tilde{V}_{-1}(Q, \Phi_i^*K)^n \geq V(Q)^{n+1} V(\Phi_i^*K)^{-1},$$

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with equality if and only if Q and $\Phi_i^* K$ are dilates. Thus, we obtain

$$\tilde{V}_{-1}(Q, \Phi_j^* K)^{(n-i-1)n} \geq r_\Phi^{n(j-i)} \kappa_n^{i-j} V(Q)^{(n-i-1)(n+1)} V(\Phi_i^* K)^{-(n-j-1)},$$

with equality if and only if Q and $\Phi_i^* K$ are centered balls. Set $Q = \Phi_j^* K$, to obtain the desired inequality. Suppose that equality holds in (3.36). Then, $\Phi_i^* K$ and $\Phi_j^* K$ are centered balls. Thus, there exist $\lambda_1, \lambda_2 > 0$ such that $\Phi_i^* K = \lambda_1 B$ and $\Phi_j^* K = \lambda_2 B$, and hence,

$$\Phi_i K = \lambda_1^{-1} B \qquad \text{and} \qquad \Phi_j K = \lambda_2^{-1} B. \tag{3.37}$$

From the equality in (3.36), it follows that

$$\lambda_2^{n-i-1} = r_\Phi^{i-j} \lambda_1^{n-j-1}.$$

Moreover, (3.6) and (3.37) imply

$$r_\Phi W_{i+1}(K) = \lambda_1^{-1} \kappa_n \qquad \text{and} \qquad r_\Phi W_{j+1}(K) = \lambda_2^{-1} \kappa_n.$$

Hence, we have

$$W_{j+1}(K)^{n-i-1} = \kappa_n^{j-i} W_{i+1}(K)^{n-j-1},$$

which implies, by (1.38), that K and L are homothetic. ∎

The next theorem provides a Brunn Minkowski inequality for the volume of the polar Blaschke Minkowski homomorphisms under consideration:

Theorem 3.26 *If $K, L \in \mathcal{K}_0^n$ and $0 \leq j \leq n - 3$, then*

$$V(\Phi_j^*(K + L))^{-1/n(n-j-1)} \geq V(\Phi_j^* K)^{-1/n(n-j-1)} + V(\Phi_j^* L)^{-1/n(n-j-1)}, \tag{3.38}$$

with equality if and only if K and L are homothetic.

Proof: By (3.31) and (1.41), we have for $Q \in \mathcal{S}^n$,

$$\begin{aligned}
\tilde{V}_{-1}(Q, \Phi_j^*(K + L))^{1/(n-j-1)} &= W_j(K + L, M_\Phi Q)^{1/(n-j-1)} \\
&\geq W_j(K, M_\Phi Q)^{1/(n-j-1)} + W_j(L, M_\Phi Q)^{1/(n-j-1)} \\
&= \tilde{V}_{-1}(Q, \Phi_j^* K)^{1/(n-j-1)} + \tilde{V}_{-1}(Q, \Phi_j^* L)^{1/(n-j-1)}.
\end{aligned}$$

By inequality (1.53),

$$\tilde{V}_{-1}(Q, \Phi_j^* K)^n \geq V(Q)^{n+1} V(\Phi_j^* K)^{-1},$$

with equality if and only if Q and $\Phi_j^* K$ are dilates and

$$\tilde{V}_{-1}(Q, \Phi_j^* L)^n \geq V(Q)^{n+1} V(\Phi_j^* L)^{-1},$$

with equality if and only if Q and $\Phi_j^* L$ are dilates. Thus, we obtain

$$\tilde{V}_{-1}(Q, \Phi_j^*(K + L))^{1/(n-j-1)} V(Q)^{-(n+1)/n(n-j-1)}$$
$$\geq V(\Phi_j^* K)^{-1/n(n-j-1)} + V(\Phi_j^* L)^{-1/n(n-j-1)},$$

with equality if and only if $Q, \Phi_j K$ and $\Phi_j L$ are dilates. If we set $Q = \Phi_j^*(K + L)$, we obtain (3.38). Suppose equality holds in (3.38), then there exist $\lambda_1, \lambda_2 > 0$ such that $\Phi_j^* K = \lambda_1 \Phi_j^*(K + L)$ and $\Phi_j^* L = \lambda_2 \Phi_j^*(K + L)$, and thus,

$$\Phi_j K = \lambda_1^{-1} \Phi_j (K + L) \qquad \text{and} \qquad \Phi_j L = \lambda_2^{-1} \Phi_j (K + L). \tag{3.39}$$

From the equality in (3.38), it follows that

$$\lambda_1^{-1/(n-j-1)} + \lambda_2^{-1/(n-j-1)} = 1,$$

and (3.6) and (3.39) imply

$$W_{j+1}(K) = \lambda_1^{-1} W_{j+1}(K + L) \qquad \text{and} \qquad W_{j+1}(L) = \lambda_2^{-1} W_{j+1}(K + L).$$

Hence, we have

$$W_{j+1}(K + L)^{1/(n-j-1)} = W_{j+1}(K)^{1/(n-j-1)} + W_{j+1}(L)^{1/(n-j-1)}$$

which implies, by (1.40), that K and L are homothetic. ∎

Theorem 3.26 is the polar version of inequality Corollary 3.16. Note that the special case $\Phi = \Pi$ of Theorem 3.21 provides a new Brunn Minkowski inequality for the volume of polar projection bodies:

Corollary 3.27 *If $K, L \in \mathcal{K}_0^n$, then*

$$V(\Pi^*(K + L))^{-1/n(n-1)} \geq V(\Pi^* K)^{-1/n(n-1)} + V(\Pi^* L)^{-1/n(n-1)},$$

with equality if and only if K and L are homothetic.

In Theorems 3.21 to 3.26, we restrict ourselves to Blaschke Minkowski homomorphisms Φ with a generating function g that is a support function. We do this to ensure that star bodies are mapped to convex bodies by the operators M_Φ. An example of a Blaschke Minkowski homomorphism whose generating function is not a support function is provided by the second mean section operator (2.15). The question arises if Theorems 3.21 to 3.26 hold for general Blaschke Minkowski homomorphisms. In view of Lemma 2.22, inequalities for the extended mixed volumes from Definition 2.25 might be used to obtain such results. Unfortunately, very little is known in this direction.

We now turn to the investigation of the volume of the images under the operators Γ_Φ. To this end, we introduce the following notation: Let $F \in \mathcal{K}^n$ again be a body of revolution with $s(F) = o$ whose support function generates the Blaschke Minkowski homomorphism Φ. For $x \in \mathbb{R}^n \backslash \{o\}$, we denote by F_x the body $\|x\| \vartheta F$, where $\vartheta \hat{e} = \frac{x}{\|x\|}$. By using spherical polar coordinates it is easy to see that

$$h(\Gamma_\Phi L, \cdot) = \frac{2}{V(L)} \int_L h(F_x, \cdot) dx \tag{3.40}$$

is an equivalent definition of the operator Γ_Φ. The next lemma establishes a formula for the volume of the images under Γ_Φ.

Lemma 3.28 *If $L \in \mathcal{S}^n$, then*

$$V(\Gamma_\Phi L) = \frac{2^n}{V(L)^n} \int_L \cdots \int_L V(F_{x_1}, \ldots, F_{x_n}) dx_1 \ldots dx_n.$$

Proof: Let $\varepsilon > 0$ and let $\{E_1, \ldots, E_m\}$ be a partition of L into nonempty disjoint Borel sets of diameters less than ε. Since $SO(n)$ and L are compact, the integral (3.40) can be approximated uniformly by a Riemann sum, i.e., there are $x_j \in E_j, 1 \leq j \leq m$ such that

$$\left| h(\Gamma_\Phi L, u) - \frac{2}{V(L)} \sum_{j=1}^m V(E_j) h(F_{x_j}, u) \right| < \varepsilon,$$

for all $u \in S^{n-1}$. The convex body K_ε with

$$h(K_\varepsilon, \cdot) = \frac{2}{V(L)} \sum_{j=1}^m V(E_j) h(F_{x_j}, \cdot),$$

is up to the normalization factor the Minkowski sum of the bodies $V(E_j) F_{x_j}, 1 \leq j \leq m$. As $\varepsilon \to 0$, the body K_ε converges to $\Gamma_\Phi L$. Using the polarization formula for volume and taking the limit $\varepsilon \to 0$, we obtain the desired expression. ∎

If Φ is the projection body operator, then the body F is a segment, and the mixed volume $V(F_{x_1}, \ldots, F_{x_n})$ becomes, up to a factor, the absolute value of the determinant of the x_i. Using Steiner symmetrization and the multilinearity of determinants, the Busemann Petty centroid inequality can be derived from Lemma 3.28. It is the author's belief that a result corresponding to (3.29) holds for the operators Γ_Φ:

Conjecture 3.29 *If $K \in \mathcal{K}_0^n$, then*

$$V(\Gamma_\Phi K) \geq \left(\frac{2r_\Phi}{(n+1)\kappa_n} \right)^n V(K).$$

A positive answer to Conjecture 3.29 would immediately provide a generalization of Petty's projection inequality (1.44) to Blaschke Minkowski homomorphisms generated by support functions:

$$V(\Phi^* K) \leq \left(\frac{\kappa_n}{r_\Phi} \right)^n V(K)^{1-n}.$$

To see this, use identity (3.33) and Minkowski's inequality (1.36) to obtain

$$\begin{aligned}
V(K)^{n-1} V(\Phi^* K) &\leq \left(\frac{(n+1)\kappa_n}{2r_\Phi} \right)^n V(K)^{n-1} V(\Gamma_\Phi \Phi^* K) \\
&\leq \left(\frac{(n+1)\kappa_n}{2r_\Phi} \right)^n V_1(K, \Gamma_\Phi \Phi^* K)^n \\
&= \left(\frac{(n+1)\kappa_n}{2r_\Phi} \right)^n \left(\frac{2}{n+1} \right)^n = \left(\frac{\kappa_n}{r_\Phi} \right)^n.
\end{aligned}$$

3.3 The Dual Inequalities

In analogy to the inequalities of Sections 3.2.1, 3.2.2 and 3.2.3, we will establish dual inequalities for radial Blaschke Minkowski homomorphisms in this last part. As Lutwak shows in [30], see also [10], there is a duality between projection and intersection bodies which, at present, appears to be not well understood. Our results illustrate that there is a similar duality for general Blaschke Minkowski and radial Blaschke Minkowski homomorphisms. The inequalities of this section generalize results of Leng and Zhao [23], [24] for the intersection body operator.

3.3.1 Dual Identities and Consequences

In the following let $\Psi : \mathcal{S}^n \to \mathcal{S}^n$ denote a radial Blaschke Minkowski homomorphism. For $K, L \in \mathcal{S}^n$, the definitions of $\Psi_i(K, L)$ and $\Psi_i K$ are analogous to the ones for mixed Blaschke Minkowski homomorphisms. The main tools in the proofs of Sections 3.2.1, 3.2.2 and 3.2.3 are Lemmas 3.1 and 3.2. These were immediate consequences of the convolution representation of Blaschke Minkowski homomorphisms provided by Theorem 2.21. In Section 2.3.1, we have shown that there is a corresponding representation for radial Blaschke Minkowski homomorphisms, which will now lead to dual versions of Lemmas 3.1 and 3.2. In the same way as Lemmas 3.1 and 3.2 were consequences of Theorem 2.21 and Lemma 1.2, we get from Theorem 2.43:

Lemma 3.30 *If $K_1, \dots, K_{n-1}, L_1, \dots, L_{n-1} \in \mathcal{S}^n$, then*

$$\tilde{V}(K_1, \dots, K_{n-1}, \Psi(L_1, \dots, L_{n-1})) = \tilde{V}(L_1, \dots, L_{n-1}, \Psi(K_1, \dots, K_{n-1})). \quad (3.41)$$

In particular, for $K, L \in \mathcal{S}^n$ and $0 \le i, j \le n - 2$,

$$\tilde{W}_i(K, \Psi(L_1, \dots, L_{n-1})) = \tilde{V}(L_1, \dots, L_{n-1}, \Psi_i K)$$

and

$$\tilde{W}_i(K, \Psi_j L) = \tilde{W}_j(L, \Psi_i K). \quad (3.42)$$

It follows from Theorem 2.43 that the image of the Euclidean unit ball under a radial Blaschke Minkowski homomorphism Ψ is again a ball. Let r_Ψ denote the radius of this ball. Then the dual version of Lemma 3.2 is:

Lemma 3.31 *If $L_1, \dots, L_{n-1} \in \mathcal{S}^n$, then*

$$\tilde{W}_{n-1}(\Psi(L_1, \dots, L_{n-1})) = r_\Psi \tilde{V}(L_1, \dots, L_{n-1}, B).$$

In particular, for $K, L \in \mathcal{S}^n$,

$$\tilde{W}_{n-1}(\Psi_1(K, L)) = r_\Psi \tilde{W}_1(K, L)$$

and, for $0 \le i \le n - 2$,

$$\tilde{W}_{n-1}(\Psi_i L) = r_\Psi \tilde{W}_{i+1}(L). \quad (3.43)$$

The Busemann-Petty problem asks whether for centered convex bodies $K, L \in \mathcal{K}_0^n$,

$$\text{vol}_{n-1}(K \cap u^\perp) = \rho(IK, u) \leq \rho(IL, u) = \text{vol}_{n-1}(L \cap u^\perp) \qquad (3.44)$$

for every $u \in S^{n-1}$, implies

$$V(K) \leq V(L).$$

Obviously, (3.44) is equivalent to $IK \subseteq IL$. As was shown by Lutwak [30], the answer to the Busemann-Petty problem is yes, if the body K is an intersection body. This is the special case $i = 0, \Psi = I$ of

Corollary 3.32 *Let $K \in \Psi_i \mathcal{S}^n$ and $L \in \mathcal{S}^n$. Then, for $i = 0, \ldots, n - 2$,*

$$\Psi_i K \subseteq \Psi_i L \quad \Rightarrow \quad \tilde{W}_i(K) \leq \tilde{W}_i(L)$$

and $\tilde{W}_i(K) = \tilde{W}_i(L)$ if and only if $K = L$.

Proof: From the monotonicity of dual mixed volumes, (3.42) and $K = \Psi_i K_0$ for some star body $K_0 \in \mathcal{S}^n$, it follows that

$$\tilde{W}_i(K) = \tilde{W}_i(K, \Psi_i K_0) = \tilde{W}_i(K_0, \Psi_i K) \leq W_i(K_0, \Psi_i L) = W_i(L, \Psi_i K_0).$$

Using the generalized dual Minkowski inequality (1.52), we thus get

$$\tilde{W}_i(K) \leq \tilde{W}_i(L),$$

with equality only if K and L are dilates, but dilated bodies of equal ith dual quermassintegrals must be equal. ∎

The proofs of the following theorems are analogous to the proofs of the results from Sections 3.2.1, 3.2.2 and 3.2.3. We just have to replace Lemmas 3.1 and 3.2 by Lemmas 3.30 and 3.31, and to use the inequalities for dual mixed volumes from Section 1.2.3 instead of the inequalities for mixed volumes from Section 1.2.2. For this reason we will omit all proofs except one in this section:

Theorem 3.33 *If $L_1, \ldots, L_{n-1} \in \mathcal{S}^n$ and $2 \leq m \leq n - 1$, then*

$$\tilde{W}_i(\Psi(L_1, \ldots, L_{n-1}))^m \leq \prod_{j=1}^{m} \tilde{W}_i(\Psi(\underbrace{L_j, \ldots, L_j}_{m}, L_{m+1}, \ldots, L_{n-1})),$$

with equality if and only if $L_1, \ldots, L_m$ are dilates.

Proof: The case $i = n - 1$ reduces by (3.30) to inequality (1.51). Hence, assume $i \leq n - 2$. From (3.43), it follows that for $Q \in \mathcal{S}^n$,

$$\tilde{W}_i(Q, \Psi(L_1, \ldots, L_{n-1}))^m = \tilde{V}(L_1, \ldots, L_{n-1}, \Psi_i Q)^m$$

$$\leq \prod_{j=1}^{m} \tilde{V}(L_j, \ldots, L_j, L_{m+1}, \ldots, L_{n-1}, \Psi_i Q)$$

$$= \prod_{j=1}^{m} \tilde{W}_i(Q, \Psi(L_j, \ldots, L_j, L_{m+1}, \ldots, L_{n-1})),$$

with equality if and only if $L_1, \ldots, L_m$ are dilates. Let $\Psi_{m'}(L_j, \mathbf{C})$ denote the body $\Psi(L_j, \ldots, L_j, L_{m+1}, \ldots, L_{n-1})$. Then, by inequality (1.52), we have

$$\tilde{W}_i(Q, \Psi_{m'}(L_j, \mathbf{C}))^{n-i} \leq \tilde{W}_i(Q)^{n-i-1} \tilde{W}_i(\Psi_{m'}(L_j, \mathbf{C})),$$

with equality if and only if Q and $\Psi_{m'}(L_j, \mathbf{C})$ are dilates. Hence,

$$\tilde{W}_i(Q, \Psi(L_1, \ldots, L_{n-1}))^{m(n-i)} \leq \tilde{W}_i(Q)^{m(n-i-1)} \prod_{j=1}^{m} \tilde{W}_i(\Psi_{m'}(L_j, \mathbf{C})).$$

By setting $Q = \Psi(L_1, \ldots, L_{n-1})$, the statement follows. ∎

Special cases of Theorem 3.33 are summarized in the following two corollaries which are dual counterparts of Theorems 3.13 and 3.14.

Corollary 3.34 *If $K, L \in \mathcal{S}^n$ and $0 \leq i \leq n - 1$, $1 \leq j \leq n - 2$, then*

$$\tilde{W}_i(\Psi_j(K, L))^{n-1} \leq \tilde{W}_i(\Psi K)^{n-j-1} \tilde{W}_i(\Psi L)^j,$$

with equality if and only if K and L are dilates.

Corollary 3.35 *If $K \in \mathcal{S}^n$ and $0 \leq i < j \leq n - 2$, while $0 \leq m \leq n - 1$, then*

$$\tilde{W}_m(\Psi_j K)^{n-i-1} \leq r_{\Psi}^{(n-m)(j-i)} \kappa_n^{j-i} \tilde{W}_m(\Psi_i K)^{n-j-1},$$

with equality if and only if K is a centered ball.

A further consequence of Theorem 3.33 is the dual version of Corollary 3.12:

Corollary 3.36 *If $L_1, \ldots, L_{n-1} \in \mathcal{S}^n$ and $0 \leq i \leq n - 1$, then*

$$\tilde{W}_i(\Psi(L_1, \ldots, L_{n-1}))^{n-1} \leq \tilde{W}_i(\Psi L_1) \cdots \tilde{W}_i(\Psi L_{n-1}),$$

with equality if and only if the L_j are dilates.

The dual counterpart of Theorem 3.15 is:

Theorem 3.37 *If $K, L \in \mathcal{S}^n$ and $0 \leq i \leq n - 1$, $0 \leq j \leq n - 3$, then*

$$\tilde{W}_i(\Psi_j(K + L))^{1/(n-i)(n-j-1)} \leq \tilde{W}_i(\Psi_j K)^{1/(n-i)(n-j-1)} + \tilde{W}_i(\Phi_j L)^{1/(n-i)(n-j-1)},$$

with equality if and only if K and L are dilates.

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