Diplomarbeit

Preserving Non–Null with Suslin$^+$ forcings

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Zusammenfassung

Ich stelle eine vereinfachte Version von Shelah's "preserving a little implies preserving much" vor: Sei I ein Ideal, das von einem Suslin ccc forcing Q erzeugt wird (z.B. das Ideal der Lebesque Nullmengen oder das der mageren Mengen). Ein solches Ideal ist notwendigerweise σ–vollständig und ccc, und $\mathrm{ro}(Q) \cong \mathrm{Borel}/I$.

Wir nennen ein forcing P preserving, wenn P keine positive Borelmenge klein macht, d.h. wenn für jede Borelmenge A in V mit $A \notin I$ gilt: in der P–Erweiterung $V[G_P]$ ist $A^V \notin I$. P ist strongly preserving, wenn P gar keine Menge klein macht, das heißt im Fall Q=random insbesondere auch daß keine nichtmeßbaren Mengen Nullmengen werden.

Im allgemeinen ist preserving stärker als nur die Aussage daß ${}^{\omega}\omega \cap V$ nicht klein wird, und zumindest konsistenterweise gibt es ein proper forcing P das preserving aber nicht strongly preserving ist. Wenn P allerdings Suslin$^+$ ist, gilt: Ist P preserving, dann erhält P generische Elemente über Kandidaten, und ist daher strongly generic.

Die Äquivalenz zur Erhaltung der Generizität ist auch für die Limesschritte von proper Iterationen nützlich: Es ist völlig unklar wie eine Eigenschaft wie preserving erhalten werden sollte, aber Erhaltung von Generizität ist (zumindest in den wichtigsten Fällen, null und mager) iterierbar. Siehe dazu das Kapitel über Erhaltungssätze in [BJ95]. Mehr zu (kurzen) Suslin proper Iterationen steht in [GJ92].

Für mager wurde die Equivalenz von preserving und strongly preserving von Goldstern and Shelah in [She98, Lem 3.11, p.920] gezeigt. Pawlikowski, auf [JS90] aufbauend, bewies in [Paw95] die Äquivalenz für P=Laver und I=Null. Shelah hat die Äquivalenz allgemein im Zusammenhang mit nep forcing in [She04] bewiesen. Die Definition und die grundlegenden Eigenschaften des zu einem forcing Q gehörenden Ideals wurden schon seit langem verwendet, z.B. im Arbeiten von Judah, Bartoszyński und Rosłanowski, die in [BJ95] zitiert sind. Im Zusammenhang damit steht auch [Sik64, §31].

Abstract

We (i.e. I) present a simplified version of Shelah's "preserving a little implies preserving much": If I is the ideal generated by a Suslin ccc forcing (e.g. Lebesque–null or meager), and P is a Suslin$^+$ forcing, and P is I–preserving (i.e. it doesn't make any positive *Borel*–set small), then P preserves generics over candidates and therefore is strongly I–preserving (i.e. doesn't make *any* positive set small).

This is also useful for preservation in limit–steps of iterations $(P_\alpha)_{\alpha<\delta}$: while it is not clear how one could argue directly that P_δ still is weakly I–preserving, the equivalent "preservation of generics" can often be shown to be iterable (see e.g. the chapter on preservation theorems for proper iterations in [BJ95] for the case of I=Lebesque–null or meager). For (short) iterations of Suslin forcings, see [GJ92].

For the ideal of meager sets the equivalence of weak preservation and preservation was done by Goldstern and Shelah [She98, Lem 3.11, p.920]. Pawlikowski [Paw95] showed the equivalence for P=Laver and I=Null, building on [JS90]. Shelah proved the theorem in the context of nep forcing [She04]. The definition and basic properties of the ideal belonging to Q have been used for a long time, e.g. in works of Judah, Bartoszyński and Rosłanowski, cited in [BJ95], also related is [Sik64, §31].

Contents

1 Review of Suslin$^+$ forcing

In this section, we will recall the definition and basic properties of Suslin proper, Suslin ccc and Suslin$^+$ forcing, and introduce an "effective" version on Axiom A. Despite it's name, Suslin$^+$ is a *generalization* of Suslin proper.

1.1 Candidates, Suslin and Suslin$^+$ forcing

We assume that the forcing Q is defined by formulas $\varphi_{\in Q}(x)$ and $\varphi_{\leq}(x,y)$, using a real parameter p_Q. Fixing ZFC*, we call M a "candidate" if it is a countable transitive ZFC* model and $p_Q \in M$.

Assume that $\varphi_{\in Q}$ and $\varphi_{\leq}$ are absolute between candidates and V. Let M be a candidate, and let Q^M, $\leq^M$ be the evaluation of $\varphi_{\in Q}$, $\varphi_{\leq}$ in M (i.e. $Q^M = Q \cap M$ and $\leq^M = \leq \cap M$). We assume that in M, Q^M is a set and $\leq^M$ a (quasi) p.o. on this set. We will also assume that $p \perp q$ is absolute between M and V. Then we call $q \in Q$ M–generic (or: Q–generic over M), if $q \Vdash$ "$G_Q \cap Q^M$ is Q^M–generic over M".

Since in $V[G_Q]$, $G_Q \cap Q^M$ is closed upwards, and $\perp$ is absolute, "$q \in Q$ is M–generic" is equivalent to: For all $D \in M$ s.t. $D \subset Q^M$ dense (or max a.c. etc): $q \Vdash G \cap D \neq \emptyset$.

Definition 1.1. A (definition of a) forcing Q is Suslin (or: strongly Suslin) in the parameter $p_Q \in \mathbb{R}$ w.r.t. ZFC*, if:

1. p_Q codes three $\underset{\sim}{\Sigma}^1_1$ relations, $R^{\in}_Q$, $R^{\leq}_Q$ and $R^{\perp}_Q$.
2. $R^{\leq}_Q$ is a (quasi) p.o. on $Q = \{x \in {}^\omega\omega : R^{\in}_Q(x)\}$, and $p \perp_Q q$ iff $R^{\perp}_Q(p,q)$.

Q is Suslin proper, if in addition

3. for every candidate M, and every $p \in Q^M$, there is a $q \leq p$ M–generic.

Remarks:

- If Q is Suslin, then (2) holds in all candidates as well, since (2) can be written as a conjunction of a $\underset{\sim}{\Sigma}^1_1$ and a $\underset{\sim}{\Pi}^1_1$ formula.

- However, the formula "$(\in_Q, \leq_Q, p_Q, \mathrm{ZFC}^*)$ codes a Suslin proper forcing" is a $\underset{\sim}{\Pi}^1_3$ statement, so in general (3) will not hold any more in candidates, i.e. a Suslin forcing Q that is Suslin proper in V is not necessarily proper in a candidate M.
- If Q is Suslin, then $\perp$ is a Borel relation, and therefore the statement "$\{q_i : i \in \omega\}$ is predense below p" (i.e. $p \Vdash G \cap \{q_i : i \in \omega\} \neq \emptyset$) is $\underset{\sim}{\Pi}^1_1$ (i.e. relatively $\underset{\sim}{\Pi}^1_1$ in the $\underset{\sim}{\Sigma}^1_1$ set ${}^{(\omega+1)}Q$).

In [IHJS88] it is proven that if a forcing Q is Suslin and ccc (in short: Suslin ccc), then Q is Suslin proper in a very absolute way:

Lemma 1.2. Assume Q is Suslin ccc. Then

1. Q is Suslin proper: even 1_Q is generic for every candidate.
2. In every candidate, Q is ccc.
3. This still holds in any extension on V.

Actually this requires that ZFC^* contains a certain sentence φ_0, the completeness theorem for Keisler–logic. However, since it is provable in ZFC that φ_0 holds in $H(\chi)$ for large regular χ, this requirement is easily met, see the section on normality on page 5.

Note that this lemma is trivially true for a Q that is definable without parameters (e.g. Cohen, random, amoeba, Hechler), assuming of course for (2) that ZFC^* is strong enough to prove that Q is ccc, and for (3), that $\mathrm{ZFC} \vdash Q$ ccc.

Cohen, random, Hechler and Amoeba forcing are Suslin ccc and Mathias forcing is Suslin proper. Miller or Sacks forcing, however, are not, since incompatibility is not Borel.

This motivated a generalization of Suslin proper, Suslin$^+$ (see [Gol93, p. 357]): here, we do not require $\perp$ to be $\underset{\sim}{\Sigma}^1_1$ any more, so "$\{q_i : i \in \omega\}$ is predense below p" will generally be $\underset{\sim}{\Pi}^1_2$. However, we require that there is a $\underset{\sim}{\Sigma}^1_2$ relation epd ("effectively predense") that holds for "enough" predense sequences:

Definition 1.3. A (definition of a) forcing Q is Suslin$^+$ in the parameter p_Q w.r.t. ZFC^*, if:

1. p_Q codes two $\underset{\sim}{\Sigma}^1_1$ relations, $R^{\in}_Q$, $R^{\leq}_Q$ and an $(\omega+1)$–place $\underset{\sim}{\Sigma}^1_2$ relation epd.
2. In V and every candidate M, $R^{\leq}_Q$ is a (quasi) p.o. on $Q = \{x \in {}^\omega\omega : R^{\in}_Q(x)\}$, and if $\mathrm{epd}(q_i, p)$, the $\{q_i : i \in \omega\}$ is predense below p.
3. for every candidate M, and every $p \in Q^M$, there is an $q \leq p$ s.t. for all $D \in M$, $D \subseteq Q^M$ dense: $\mathrm{epd}(d_i, q)$ holds for some enumeration $\{d_i : i \in \omega\}$ of D.

Clearly, every Suslin proper forcing is Suslin$^+$: epd can just be defined by "$\{q_i : i \in \omega\}$ is predense below p", which is even a conjunction of $\underset{\sim}{\Pi}^1_1$ and $\underset{\sim}{\Sigma}^1_1$, and then 1.3.(3) is just a reformulation of 1.1(3).

1.2 Effective Axiom A

The usual tree–like forcings are Suslin$^+$. Here, we consider the following forcings consisting of trees on ${}^{<\omega}\omega$ ordered by $\subseteq$ (usually, Sacks is defined on ${}^{<\omega}2$, but this is equivalent by a simple density argument):

- Sacks (perfect trees: $\forall s \in T\ \exists t \geq_T s\ \exists^{\geq 2} n :\ t^\frown n \in T$)
- Miller (superperfect trees: every node has either exactly one or infinitely many immeditate successors, and $\forall s \in T\ \exists t \geq_T s\ \exists^\infty n :\ t^\frown n \in T$)
- Laver (let s be the stem of T. Then $\forall t \geq_T s\ \exists^\infty n :\ t^\frown n \in T$)
- Rosłanowski ($\forall s \in T\ (\exists! n \in \omega :\ s^\frown n \in T) \vee (\forall n \in \omega :\ s^\frown n \in T)$ and $\forall s \in T\ \exists t \geq_T s\ \forall n \in \omega :\ t^\frown n \in T$)

Clearly, "$p \in Q$" and "$q \leq p$" are Borel (but $p \perp q$ is not).

(Alternatively, Q could of course be defined as the set of trees just *containing* a corresponding set, then $x \in Q$ is $\utilde{\Sigma}^1_1$, and two compatible elements p, q have a canonical lower bound, $p \cap q$).

In the following, we call Sacks, Miller and Rosłanowski "Miller–like". For Sacks, there is a proof of the Suslin$^+$ property in [Gol93], using games. Here we prove Suslin$^+$ using an effective version of Axiom A:

Baumgartner's Axiom A for a forcing $(Q, \leq)$ (see e.g. [Bau83]) can be formulated as follows: There are relations $\leq_n$ s.t.

1. $\leq_{n+1} \subseteq \leq_n \subseteq \leq$
2. $\forall (a_n) \in {}^\omega Q :\ a_{n+1} \leq_n a_n\ \rightarrow\ \exists a_\omega \forall n\, a_\omega \leq a_n$ (fusion)
3. $\forall p\, \forall n\, \forall D \subseteq Q$ dense $\exists q \leq_n p\, \exists B \subseteq D$ countable, predense $\leq q$

Remarks:

- Actually, this is a weak version of Axiom A, usually even something like $a_\omega \leq_n a_n$ or $a_\omega \leq_{n-1} a_n$ will hold.
- It is easy to see that in (3), instead of dense predense we can use open dense or maximal antichain.

Now for "effective Axiom A" it is required that the $B \subseteq D$ in (3) is *effectively* predense below q, not just predense. Then Suslin$^+$ follows from 1.3(1)&(2) and the effective version of Axiom A. To be more exact:

Lemma 1.4. Q is Suslin$^+$ in the parameter p_Q w.r.t. ZFC*, if

1. p_Q codes $\underset{\sim}{\Sigma}^1_1$ relations, $\in_Q$, $\leq_Q$, and $\underset{\sim}{\Sigma}^1_2$ relations $\leq^n_Q$ ($n \in \omega$) and a $(\omega+1)$–place $\underset{\sim}{\Sigma}^1_2$ relation epd.
2. In V and every candidate M: $R^{\leq}_Q$ is a (quasi) p.o. on $Q = \{x \in {}^\omega\omega : R^{\in}_Q(x)\}$, and $\mathrm{epd}(q_i, p) \;\;\rightarrow\;\; (q_i)$ is predense below p
3. In V, $\forall (a_n) \in {}^\omega Q : a_{n+1} \leq_n a_n \;\rightarrow\; \exists a_\omega \forall n\, a_\omega \leq a_n$ (fusion).
4. In all candidates, $\forall p\, \forall n\, \forall D \subseteq Q$ dense $\exists q \leq_n p\; \exists \{b_i : i \in \omega\} \subseteq D$ s.t. $\mathrm{epd}(b_i, q)$

Proof. First we define $\mathrm{epd}'(p'_i, q')$ to mean $\exists q \geq q'\, \exists \{p_i\} \subseteq \{p'_i\} : \mathrm{epd}(p_i, q)$. Clearly, this is a $\underset{\sim}{\Sigma}^1_2$ relation coded by p_Q satisfying 1.3.(2). Let M be a candidate, let $\{A_i : i \in \omega\}$ list the maximal antichains of Q^M in M, and let $a_0 = p \in Q^M$ arbitrary. We have to find a $q \leq p$ satisfying 1.3.(3) w.r.t. epd$'$. In M, find to each a_n an $a_{n+1} \leq_n a_n$ according to (4), using A_n as A. In V, find $q = a_\omega$ according to (3). Now, for each n, $M \vDash \mathrm{epd}(b_i, a_{n+1})$, so this holds in V, and $q \leq a_{n+1}$, so by the definition of epd$'$, $\mathrm{epd}'(p_i, q)$, where $\{p_i : i \in \omega\} = A_n$. $\square$

The usual proofs that the forcings defined above satisfy axiom A also show that they satisfy the effective version.

To be more explicit: Assume Q is any of the forcings defined above. We define (for p, $q \in Q$, $n \in \omega$):

- $\mathrm{split}(p) = \{s \in p : \exists^{\geq 2}\; s^\frown n \in p\}$
- $\mathrm{split}(p,n) = \{s \in \mathrm{split}(p) : \forall t \subsetneq s : t \notin \mathrm{split}(p,n)\}$
- $q \leq_n p$, if $q \leq p$ and $\mathrm{split}(q,n) = \mathrm{split}(p,n)$
  (so $q \leq_0 p$ if $q \leq p$ and q has the same stem as p).
- for $s \in p$, $p^{[s]} = \{t \in p : t \subseteq s \vee s \subseteq t\}$
- $F \subseteq p$ is a front, if it is an antichain meeting every branch of p.
- $\mathrm{epd}(p_i, q)$ is defined by: There is a front $F \subseteq q$ such that $\forall f \in F\, \exists i \in \omega : p_i = q^{[f]}$.
- For Miller–like forcings, effectively predense could also be define as $\mathrm{epd}'(p_i, q) \;\; :\leftrightarrow$ $\exists n\, \forall s \in \mathrm{split}(q,n)\, \exists i : p_i = q^{[s]}$.

Clearly, $\mathrm{split}(p)$, $\mathrm{split}(p,n)$, $p^{[s]}$ and epd$'$ are Borel, "F is a front" is $\underset{\sim}{\Pi}^1_1$, therefore epd is $\underset{\sim}{\Sigma}^1_2$.

The following facts are easy to check ($p, q \in Q$):

- if $s \in p$, then $p^{[s]} \in Q$
- if $F \subset p$ is a front, and $q \parallel p$, then $\exists s \in F\ q \parallel p^{[s]}$
- split(p,n) is a front in p
- For $(q_n)_{n\in\omega}$ s.t. $q_{n+1} \leq_n q_n$, there is a canonical limit q_ω and $q_\omega \leq_n q_n$.
- If Q is Miller–like, and if $F \subset p$ is a front, and $\forall s \in F, p_s \in Q$, $p_s \subseteq p^{[s]}$, then $\bigcup_{s\in F} p_s \in Q$.
- If Q is Laver, and if $F \subset p$ is a front, and $\forall s \in F, p_s \in Q$ has stem s, then $\bigcup_{s\in F} p_s \in Q$.

Then effective Axiom A for Miller–like forcings is proven as follows: Assume, $D \leq Q$ is dense, $p \in Q$. For all $s \in \text{split}(s,n)$, $p^{[s]} \in Q$, so there is a $p_s \leq p^{[s]}$ s.t. $p_s \in D$. Now let $q = \bigcup_{s\in F} p_s \in Q$. Then $q \leq_n p$, and $\{p_s\} \subseteq D$ are effectively predense below q according to the definition of epd$'$ (or epd).

For Laver, we have to define a rank of nodes: If D is a dense set and p a condition with stem s_0, $s \geq_p s_0$, define $\text{rk}_D(p,s)$ as follows: If there is a $q \leq p$, $q \in D$, q has stem s, then $\text{rk}_D(p,s) = 0$. Otherwise $\text{rk}(p,s)$ is the minimal α s.t. $\exists^\infty t \succ s : \text{rk}_D(t) < \alpha$. rk_D is well–defined for all nodes above the stem of p: Otherwise, the set of nodes not in $\text{dom}(\text{rk}_D)$ form a Laver condition $q \leq p$, then pick $q' \leq q$ s.t. $q' \in D$, let s be the stem of q', then $\text{rk}_D(p,s) = 0$, a contradiction. Now the value of rk_D is strictly decreasing along branches, therefore $F = \{s \in p : \text{rk}_D(p,s) = 0\}$ is a front. So for each $s \in F$ there is a $p_s \leq p$ in D with stem s. So $\{p_s : s \in F\}$ is effectively predense below $\bigcup_{s\in F} p_s$.

1.3 Normality

ZFC* is called normal if for regular χ large enough, $H(\chi) \vDash \text{ZFC}^*$. We will only be interested in Suslin forcings that are defined with respect to a normal ZFC*: If e.g. ZFC* contains $0 = 1$, then every Q that is Suslin is trivially Suslin proper, but this is of course not the right spirit.

ZFC* will definitely be normal if ZFC* follows from e.g. ZFC minus powerset plus $\beth_{630}$ exists. In such a case, ZFC* is called absolutely normal (since it will still be normal in any ZFC–model V'). We can find a fixed absolutely normal ZFC^*_0 that will work for all Suslin ccc forcings (ZFC^*_0 just has to contain the completenes theorem for Keisler–logic, see lemma 1.2). So if Q is Suslin ccc, then it is automatically Suslin proper w.r.t. the fixed normal ZFC^*_0, and this still holds in all forcing extensions of V. (Note however, that Q can not be normal w.r.t. ZFC^*_0 in every ZFC^*_0–candidate, otherwise there would be an infinite descending chain of candidates.) We can also fix a stronger, still absolutely normal ZFC^*_1 s.t. $\text{ZFC}^*_1 \vdash \forall P \Vdash_P \text{ZFC}^*_0$, i.e. for all $\varphi \in \text{ZFC}^*_0$,

$\mathrm{ZFC}_1^* \vdash (\forall P$ forcing: $\Vdash_P \varphi)$. Then any ZFC_1^*–candidate remains a ZFC_0^* candidate after forcing with any set forcing P. We can also assume that ZFC_1^*, ZFC_0^* are definable in (and therefore element of) any ZFC_0^*–candidate. So if M_1 is a candidate, it makes sense to say $M_1 \vDash$ "M_2 is a candidate" (and this clearly implies that M_2 really is a candidate).

In the normal case, a Suslin$^+$ forcing is proper: Assume $N \prec H(\chi)$, $p \in N$. Let M be the transitive collapse of N. Since $Q \subseteq {}^{\omega}\omega$, Q^N isn't changed by the collapse. If $q \leq p$ M–generic, then q is N–generic.

We will frequently and without mentioning use the well known fact that for large (w.r.t. τ), regular χ: $H(\chi) \vDash p \Vdash \varphi(\tau)$ iff $p \Vdash H(\chi)^{V[G]} \vDash \varphi(\tau)$.

As an example how we will use normality, assume that for a name $\underset{\sim}{\eta} \in H(\aleph_1)$ and all candidates M, $M \vDash \Vdash \underset{\sim}{\eta} \notin V$. Then this is true in V as well. Otherwise, if $V \vDash p \Vdash \underset{\sim}{\eta} = r$, then for some regular χ, $H(\chi) \vDash p \Vdash \underset{\sim}{\eta} = r$ (since $\underset{\sim}{\eta}[G] = r$ is absolute between $H(\chi)^{V[G]}$ and $V[G]$). Take $N \prec H(\chi)$ countable, M its transitive collapse. Then M is a candidate, and $M \vDash p \Vdash \underset{\sim}{\eta} = r$, a contradiction. We will usually abbrevate arguments of this kind by just refering to normality.

For every countable transitive model, $M \vDash$ "$q \Vdash \varphi(\tau)$" iff for all M–generic G, $M[G] \vDash$ "$\varphi(\tau[G])$". If Q is Suslin$^+$ and M a candidate, then $M \vDash$ "$p \Vdash \varphi(\tau)$" iff for all M- *and* V–generic G, $M[G] \vDash$ "$\varphi(\tau[G])$". (One direction is clear. For the other, assume $M \vDash$ "$p' \leq p$, $p' \Vdash \neg\varphi(\tau)$". Let $q \leq p'$ be M generic. Then for any V–generic G containing q, G is M–generic as well and $M[G] \vDash$ "$\neg\varphi(\tau[G])$".)

Lemma 1.5. Let $V_1 \subseteq V_2$ be two transitive models of ZFC, $\omega_1 \subset V_1$, $V_1 \vDash x \in H(\aleph_1)$. Then "there is a candidate M containing x s.t. $M \vDash \varphi(x)$" is absolute between V_1 and V_2.

This is shown exactly as Σ_1 (Shoenfield–Levy) absoluteness.

Lemma 1.6. If Q is Suslin ccc, M_1 a candidate, $M_2 \supset M_1$ is either V or another candidate, and G is Q–generic over M_2, then G is Q–generic over M_1.

Proof. If $A \in M_1$, $M_1 \vDash$ "A max a.c.", then $M_1 \vDash$ "A countable" because of lemma 1.2, so "A is maximal" is a $\underset{\sim}{\Pi}_1^1$ statement, therefore absolute, so $G \cap A \neq 0$. $\square$

2 The Ideals

In this section we will introduce the class of ideals to which the main theorem will apply.

2.1 The Forcing Q

If Q is ccc, then a name τ for an element of ${}^\omega\omega$ can clearly be transformed into an equivalent countable name $\underset{\sim}{\eta}$: for every n, pick a maximal antichain A_n deciding $\tau(n)$, then $\underset{\sim}{\eta}$:=$\{(p,(n,m)) : p \in A_n, p \Vdash \tau(n) = m\}$.

From now on, we will assume the following (and M will always denote a candidate):

Assumption 2.1. Q is a Suslin ccc forcing, $\underset{\sim}{\eta}$ is a countable name coded by p_Q, $\Vdash_Q \underset{\sim}{\eta} \in {}^\omega\omega \setminus V$, and in all candidates: $\{[\![\underset{\sim}{\eta}(n) = m]\!], n, m \in \omega\}$ generates $\mathrm{ro}(Q)$.

"X generates $\mathrm{ro}(Q)$" means that there is no proper sub–Boolean–algebra $Y \supseteq X$ of $\mathrm{ro}(Q)$ s.t. for all $A \subseteq Y$, $\sup_{\mathrm{ro}(Q)}(A) \in Y$.

Lemma 2.2. This assumption is absolute between $V \subseteq V'$ transitive models of ZFC s.t. ${\omega_1}^{V'} \subseteq V$. Also, the assumption is downwards absolute between V and candidates M.

Proof.

- "Q is a Suslin ccc" is absolute anyway (see 1.2).
- $\Vdash_Q \underset{\sim}{\eta} \in {}^\omega\omega \setminus V$ is true in V iff it holds in all candidates:
If $M \vDash p \Vdash \underset{\sim}{\eta} = r$, take $q \leq p$ M–generic, then in V, $q \Vdash \underset{\sim}{\eta} = r$. The other direction follows from normality.
- A statement of the form "every candidate thinks $\varphi(x)$" for an $x \in H(\aleph_1)^V$ is absolute between V and V' by 1.5, and downwards absolute between V and candidates, since $M_1 \vDash$ "M_2 candidate" $\rightarrow$ $V \vDash$ "M_2 candidate.

$\square$

Lemma 2.3. For A Borel, "$q \Vdash \underset{\sim}{\eta} \in A$" is $\underset{\sim}{\Delta}^1_2$, absolute between V, V' as in 2.2, and absolute between candidates and V.

Remark: [BB97, 2.7] gives a general result for Q ccc and $\underset{\sim}{\Sigma}^1_n$.

Proof. We assume A is built up along a wellfounded tree T_A from basic clopen sets of the form $\{x : x(n) = m\}$ using countable unions and intersections (but no complements). If $A = \bigcup\{A_i : i \in \omega\}$, then wlog we can assume that every member of the sequence $(A_i)_{i\in\omega}$ occurs infinitely often in this sequence.

So there is a tree T_A together with a mapping a that assigns a basic clopen set to each leaf, and "$\bigcup$" or "$\bigcap$" to all other nodes. This determines a canonical assignment from the nodes $s \in T_A$ to Borel sets A_s s.t. $A_{\langle\rangle} = A$. (And if $a(s) = \bigcup$, $s^\frown n \in T_A$, then there are infinitely many m s.t. $A_{s^\frown n} = A_{s^\frown m}$.) Then $q \Vdash \underset{\sim}{\eta} \in A$ iff

$(*)$ $\exists b : T_A \to Q \cup \{\mathbb{F}\}$ s.t.

- $b(\langle\rangle) = q$,
- $\forall s \in T_A :\ a(s) = \text{“}\bigcup\text{”} \quad \to \quad \{b(t) \neq \mathbb{F} : t \succ (s)\}$ is predense $\leq b(s)$,
- $\forall s \in T_A :\ a(s) = \text{“}\bigcap\text{”} \quad \to \quad \forall t \succ s : b(s) = b(t)$
- if s is a leaf, and $a(s) = A_s = \{x : x(n) = m\}$, then $\{p : (p,(n,m)) \in \underset{\sim}{\eta}\}$ is predense$\leq b(s)$

Q is Suslin proper, therefore "$\{a_i : i \in \omega\}$ is predense below p" is (relatively) $\underset{\sim}{\Pi}^1_1$, so the statement is $\underset{\sim}{\Sigma}^1_2$. Also, it is equivalent to $q \Vdash \underset{\sim}{\eta} \in A$: If there is such an assignment b, then for all nodes s, $b(s) \Vdash \underset{\sim}{\eta} \in A_s$ (by induction starting at the nodes). For the other direction we construct the assignement b starting at the root $b(\langle\rangle) = q$. If $b(s) = p$, then by induction $p \Vdash \underset{\sim}{\eta} \in A_s$. If $a(s) = \bigcap$, then $A_s = \bigcap_{t\succ s} A_t$, i.e. $p \Vdash \underset{\sim}{\eta} \in \bigcap_{t\succ s} A_t$, then clearly for all $t \succ s$, $p \Vdash \underset{\sim}{\eta} \in A_t$, so $b(t) = p$ works for all successors t of s. If $a(s) = \bigcup$, then $A_s = \bigcup_{t\succ s} A_t$, i.e. $p \Vdash \underset{\sim}{\eta} \in \bigcup_{t\succ s} A_t$. Now consider $X = \bigcup_{t\succ s} X_t$, $X_t = \{p' \in Q : p' \Vdash \underset{\sim}{\eta} \in A_t\}$. Then X is dense below p. Let X' be a maximal antichain in X. Then X' is a countable predense set below p. Now distribute all the $p' \in X'$ to the according $t \succ s$. To be more exact, let $X'_t = X_t \cap X'$. We did assume that for each $t \succ s$, the set $\mathcal{A}_t = \{t' \succ s : A_t = A_{t'}\}$ is infinite. If $t' \in \mathcal{A}_t$, then $X'_t = X'_{t'}$. If $X'_t \neq \emptyset$, then there is a surjective mapping c from X'_t to $\mathcal{A}_t$, since X'_t is at most countable and $\mathcal{A}_t$ infinite. So for $t' \in \mathcal{A}_t$, define $b(t')$ to be $c(t')$. If $X'_t = \emptyset$, define $b(t') = \mathbb{F}$ for $t' \in \mathcal{A}_t$. Now clearly $b(t) \Vdash \underset{\sim}{\eta} \in A_t$, and $\{b(t) \neq \mathbb{F} : t \succ s\}$ is predense below $b(s)$.

Since $q \Vdash \underset{\sim}{\eta} \in A$ is $\underset{\sim}{\Sigma}^1_2$, it is absolute between V, V' and upwards absolute between candidates and V. To see that it is downwards absolute as well, assume that "$q \Vdash \underset{\sim}{\eta} \in A$" holds in V but not in M. Then in M, there is a $q' \leq q$ s.t. $M \vDash$ "$q' \Vdash \underset{\sim}{\eta} \notin A$". If $q'' \leq q'$ M–generic, and G V–generic s.t. $q'' \in G$, then $M[G] \vDash \underset{\sim}{\eta}[G] \notin A$, so $V[G] \vDash \underset{\sim}{\eta}[H] \notin A$ by absoluteness. On the other hand, $q'' \leq q$, a contradiction to $q \Vdash \underset{\sim}{\eta} \in A$.

So $q \Vdash \underset{\sim}{\eta} \in A$ iff for all candidates M s.t. $A, q \in M$, $M \vDash q \Vdash \underset{\sim}{\eta} \in A$, which is $\underset{\sim}{\Pi}^1_2$. So we get $\underset{\sim}{\Delta}^1_2$. $\square$

Lemma 2.4. The statement "$\{[\![\underset{\sim}{\eta}(n) = m]\!], n, m \in \omega\}$ generates $\mathrm{ro}(Q)$" in M is equivalent to: for all $G_1 \neq G_2$ Q–generic over M, $\underset{\sim}{\eta}[G_1] \neq \underset{\sim}{\eta}[G_2]$.

Proof. If $\{[\![\underset{\sim}{\eta}(n) = m]\!], n, m \in \omega\}$ generates $\mathrm{ro}(Q)$, then $G \cap Q^M$ can be calculated (in $M[G]$) from $\underset{\sim}{\eta}[G]$. On the other hand, let (in M) $B = \mathrm{ro}(Q)$, C the proper complete subalgebra generated by $[\![\underset{\sim}{\eta}(n) = m]\!]$. Take $b_0 \in B$ s.t. no $b' \leq b_0$ is in C, and let $c = \inf\{c' \in C : c' \geq b_0\}$, $b_1 = c \setminus b_0$. So for all $c' \in C$, $c' \parallel b_0$ iff $c' \parallel b_1$. Let G_0 be B–generic over M s.t. b_0 in G, then $H = G_0 \cap C$ is C–generic. In $M[H]$, $b_1 \in B/H$. So there is a $G_1 \supset H$ containing b_1. □

Definition 2.5. For $q \in Q^M$, η^* is called $(Q, \underset{\sim}{\eta})$–generic over M containing q ($\eta^* \in \mathrm{Gen}(M, q)$), if there is a $G \in V$ M–generic s.t. $q \in G$ and $\underset{\sim}{\eta}[G] = \eta^*$.

$\mathrm{Gen}(M, 1_Q)$ will be denoted by $\mathrm{Gen}(M)$.

Lemma 2.6. $\mathrm{Gen}(M, q)$ is (uniformly) Borel.

Uniformly means that $x \in \mathrm{Gen}(M, q)$ is absolute between candidates M' and V s.t. $M' \models M$ is a candidate, and between V and V' as in 2.2.

Proof. Let $X = Q^M$ (countable, with discrete topology), $A = \{G \subset X : G\ M\text{–generic (containing } q)\}$. Then A is a $\underset{\sim}{\Pi}^0_2$ subset of ${}^X 2$, i.e. a Borel set. $f : A \to {}^\omega\omega$ defined by $G \mapsto \underset{\sim}{\eta}[G]$ is continuous (since $\underset{\sim}{\eta}[G](n) = m$ iff there is a $p \in G$ s.t. $M \models$ "$p \Vdash \underset{\sim}{\eta}(n) = m$"). f is injective, therefore $\mathrm{Gen}(M, q)$ is Borel. □

2.2 The Q–Ideal

Definition 2.7.
1. $I = \{X \subseteq {}^\omega\omega : \exists A \supseteq X \text{ Borel s.t. } \Vdash_Q \underset{\sim}{\eta} \notin A\}$ (where A is interpreted as a Borel–name evaluated in $V[G]$, not as a set of V).
2. $X \in I^+$ means $X \notin I$, and X is co–I means ${}^\omega\omega \setminus X \in I$.

For example, if $\mathbb{B}$ is the random algebra and $\mathbb{C}$ Cohen forcing, then $I_{\mathbb{B}}$ are the null- and $I_{\mathbb{C}}$ the meager sets.

An immediate consequence of 2.3 is

Corollary 2.8. For A Borel, $A \in I$ is absolute.

Lemma 2.9. I is a σ–complete ccc ideal containing singletons, and there is a surjective σ–Boolean–algebra homomorphism ϕ : Borel $\to \mathrm{ro}(Q)$ with kernel I, i.e. $\mathrm{ro}(Q)$ is isomorphic to Borel$/I$ as a complete Boolean algebra.

ccc means: there is no uncountable family $\{A_i\}$ s.t. $A_i \in I^+$ and $i \neq j \to A_i \cap A_j \in I$ (or equivalently: $A_i \cap A_j = \emptyset$).

Proof. σ–complete is clear: If $X_i \subseteq A_i \in I$, and $\forall i : \Vdash \underset{\sim}{\eta} \notin A_i$, then $\Vdash \underset{\sim}{\eta} \notin \bigcup A_i \supseteq \bigcup X_i$. For each A Borel, define $\phi(A) = [\![\underset{\sim}{\eta} \in A]\!] \in \mathrm{ro}(Q)$. Then $\phi({}^\omega\omega \setminus A) = \neg\phi(A)$, $\phi(\bigcup A_i) = \sup\{\phi(A_i)\}$, and if $A \subseteq B$, $\phi(A) \leq \phi(B)$. If $\phi(A) \leq \phi(B)$, then $\Vdash \underset{\sim}{\eta} \notin (A \setminus B)$, so $A \setminus B \in I$. Since $\underset{\sim}{\eta}$ generates $\mathrm{ro}(Q)$ (in all candidates, and therefore in V as well by normality) and since Q is ccc, $\mathrm{ro}(Q) = \phi''\mathrm{Borel}$. So $\phi : \mathrm{Borel} \to \mathrm{ro}(Q)$ is a surjective σ–Boolean–algebra homomorphism. The kernel is the $\sigma - closed$ Ideal I, so Borel/I is isomorphic to $\mathrm{ro}(Q)$ as a σ–Boolean–algebra, and since $\mathrm{ro}(Q)$ is ccc, even as complete Boolean algebra. ☐

If $q \in Q$, and B_q Borel s.t. $\phi(B_q) = q$, then for all A Borel, $q \Vdash \underset{\sim}{\eta} \in A$ iff $\Vdash (\underset{\sim}{\eta} \in B \rightarrow \underset{\sim}{\eta} \in A)$ iff $\Vdash \underset{\sim}{\eta} \notin B \setminus A$. Therefore

Corollary 2.10. If $\phi(B_q) = q$, then $q \Vdash \underset{\sim}{\eta} \notin A$ iff $A \cap B_q \in I$.

Lemma 2.11. "$\phi(B_q) = q$" is absolute between V, V' and candidates

Proof. $\phi(B_q) = q$ is equivalent to $(q \Vdash \underset{\sim}{\eta} \in B_q)$ & $(p \perp q \rightarrow p \Vdash \underset{\sim}{\eta} \notin B_q)$. Because of lemma 2.3 and since $p \perp q$ is Borel, this is a $\underset{\sim}{\Sigma}^1_2$ statement, therefore absolute between V and V' and upwards absolute between M and V.

If $V \vDash \phi(B_q) = q$, then $M \vDash q \Vdash \underset{\sim}{\eta} \in B_q$ (2.3 again). Assume, in M there is a $p \perp q$ s.t. $p \nVdash \underset{\sim}{\eta} \notin B_q$. So in V, $p \perp q$, and wlog $p \Vdash \underset{\sim}{\eta} \in B_q$ in M and therefore in V, a contradiction. ☐

Lemma 2.12.
1. $\mathrm{Gen}(M) = {}^\omega\omega \setminus \bigcup\{A^V : A \in \mathrm{Borel} \cap I \cap M\}$
2. If $B_q \in M$, $q \in Q^M$, $\phi(B_q) = q$, then $\mathrm{Gen}(M, q) = {}^\omega\omega \setminus \bigcup\{A^V : q \Vdash \underset{\sim}{\eta} \in A\} = \mathrm{Gen}(M) \cap B_q$.

I.e. η^* is generic over M iff for all A Borel s.t. $M \vDash$ "$A \in I$": $\eta^* \notin A^V$.

Proof. Assume $\eta^* \in \mathrm{Gen}(M, q)$. Let G be M–generic s.t. $q \in G$ and $\underset{\sim}{\eta}[G] = \eta^*$. If $M \vDash q \Vdash \underset{\sim}{\eta} \notin A$, then $M[G] \vDash \eta^* \notin A^{M[G]}$, i.e. $V \vDash \eta^* \notin A^V$.

For the other direction, we define in M $\phi : \mathrm{Borel} \to \mathrm{ro}(Q)$ as in the proof of 2.9. If $\phi(A) \leq \phi(B)$, then $\Vdash \underset{\sim}{\eta} \notin (A \setminus B)$, so by our assumption, $\eta^* \notin (A \setminus B)$. Given η^*, define G by $\phi(A) \in G$ iff $\eta^* \in A$. G is a well defined: If $\eta^* \in A \setminus B$, then $\phi(A) \neq \phi(B)$. We have to show that G is a generic filter over M: If $\phi(A_1), \phi(A_2) \in G$, then $\eta^* \in A_1 \cap A_2$, so $\phi(A_1) \wedge \phi(A_2) \in G$. If $\phi(A) \leq \phi(B)$, then $\eta^* \notin (A \setminus B)$, so $\phi(A) \in G \rightarrow \phi(B) \in G$. Since $\phi(\emptyset) = 0$, and $\eta^* \notin \emptyset$, $0 \notin G$. If $\sup(\phi(A_i)) \in G$, $(A_i) \in M$, then $\eta^* \in \bigcup A_i$, i.e. for some i, $\phi(A_i) \in G$.

If $\phi(B) = q$, then $q \Vdash \underset{\sim}{\eta} \in B$ by definition of ϕ, i.e. $q \Vdash \underset{\sim}{\eta} \notin {}^\omega\omega \setminus B$, so so $\eta^* \notin {}^\omega\omega \setminus B$ by the assumption, so $q \in G$.

It remains to be shown that $\bigcup\{A^V : A \in M, q \Vdash \underset{\sim}{\eta} \notin A\} = {}^\omega\omega \setminus (B_q \cap \mathrm{Gen}(M))$. By 2.10, $q \Vdash \underset{\sim}{\eta} \notin A$ iff $A \cap B_q \in I$. So if $q \Vdash \underset{\sim}{\eta} \notin A$ and $\eta^* \in A \cap B_q$, then $\eta^* \notin \mathrm{Gen}(M)$, i.e. if $\eta^* \in A$ then $\eta^* \notin B_q$ or $\eta^* \notin \mathrm{Gen}(M)$. On the other hand, if $\eta^* \notin B_q$, then for $A = {}^\omega\omega \setminus B_q$, $q \Vdash \underset{\sim}{\eta} \notin A$ and $\eta^* \in A$. And $\mathrm{Gen}(M, q) \subseteq \mathrm{Gen}(M)$ is clear. $\square$

Lemma 2.13. $\mathrm{Gen}(M) \in$ co–I, and $\mathrm{Gen}(M, q)$ is relatively co–I in B_q.

Proof. $\mathrm{Gen}(M)$ is Borel. Let $A = {}^\omega\omega \setminus \mathrm{Gen}(M)$. Assume $A \in I^+$, i.e. $p \Vdash \underset{\sim}{\eta} \in A$. So if $p \in G$ V–generic, then $V[G] \models$ "$\underset{\sim}{\eta}[G]$ not $(Q, \underset{\sim}{\eta})$–generic/$M$". But if G is V–generic, then it is M–generic (see 1.6), a contradiction. $\square$

Note that if Q is not ccc, then our definition of I does not lead to anything useful. For example, if Q is Sacks forcing, then I_Q is the ideal of countable sets, and clearly lemma 2.12 does not hold any more. There seem to be a few possible definitions for a similar I generated by a non–ccc Q, see e.g. [She04].

3 Preservation

Definition 3.1. 1. P is I–preserving, if for all $A \in I^+$ Borel, $\Vdash_P A^V \in I^+$.

2. P is strongly I–preserving, if for all $X \in I^+$, $\Vdash_P \widehat{X} \in I^+$.

For example, $\mathbb{B}$ is strongly $I_\mathbb{B}$–preserving, but not $I_\mathbb{C}$–preserving. $\mathbb{C}$ is strongly $I_\mathbb{C}$–preserving, but not $I_\mathbb{B}$–preserving.

Note that being preserving is stronger than just "$\Vdash_P V \cap {}^\omega\omega \notin I$". For example, let $X = \{x \in {}^\omega\omega : x(0) = 0\}$, $Y = {}^\omega\omega \setminus X$. Let Q be the forcing that adds a real $\underset{\sim}{\eta}$ s.t. $\underset{\sim}{\eta}$ is random if $\underset{\sim}{\eta} \in X$, and $\underset{\sim}{\eta}$ is Cohen otherwise. Clearly, Q is Suslin ccc. $A \in I$ iff ($A \cap X$ null and $A \cap Y$ meager). So if P is random forcing, then $\Vdash_P ({}^\omega\omega^V \notin I \;\&\; Y^V \in I)$. Note that in this case, for any candidate M, a Q–generic real η^* over M will still be generic after forcing with P if $\eta^* \in X$, but not if $\eta^* \in Y$.

However, if P is homogeneous in a certain way, then weakly preserving and preserving are equivalent (see [She04] for a sufficient condition).

Also, preserving and strongly preserving are generally not equivalent, not even for P ccc. The standard example is the following: Let Q be $\mathbb{C}$ (Cohen), i.e. I is the ideal of meager sets. We will construct a forcing extension V' of V and a forcing $P \in V'$ s.t. P is preserving but not strongly preserving (in V'):

Let $\mathbb{C}_{\omega_1}$ be the forcing adding $\aleph_1$ many Cohen reals $(c_i)_{i\in\omega_1}$, i.e. $\mathbb{C}_{\omega_1} = \{f : \omega \times \omega_1 \to 2 \text{ partial, finite}\}$. Then for any $\mathbb{C}_{\omega_1}$–extension $V[c_i]$, $\{c_i : i \in \omega_1\}$ is a Luzin set (i.e. for all X meager, $X \cap \{c_i : i \in \omega_1\}$ is countable), and for all A Borel non–meager, $A \cap \{c_i : i \in \omega_1\}$ is uncountable. If r is random over V, and $(c_i)_{i\in\omega_1}$ is $\mathbb{C}_{\omega_1}$–generic over $V[r]$, then (c_i) is $\mathbb{C}_{\omega_1}$–generic over V as well. So the ccc forcing $\mathbb{B} * \mathbb{C}_{\omega_1}^{V[G_C]}$ can be factored as $\mathbb{C}_{\omega_1}^V * \underset{\sim}{P}$, where $\underset{\sim}{P}$ is a name for a ccc forcing. Let $V' = V[(c_i)]$, $V'' = V'[G_P] = V[r][(c_i)]$. Then in V', $P = \underset{\sim}{P}[(c_i)]$ is ccc and preserving, ${}^\omega\omega \cap V \notin I$, but $P \Vdash {}^\omega\omega \cap V \in I$.

Definition 3.2. 1. For $q \in Q^M$, $p \in P^M$, η^* is called absolutely $(Q, \underset{\sim}{\eta})$–generic over M containing q w.r.t. p ($\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M, q, p)$), if there is a $p' \leq p$ P–generic over M s.t. (in V), $p' \Vdash_P \eta^* \in \mathrm{Gen}(M[G], q)$.

2. P preserves generics for M, if $\mathrm{Gen}(M) = \mathrm{Gen}^{\mathrm{abs}}(M, 1_Q, p)$, i.e. every M–generic real could still be $M[G]$–generic in an extension.

$\mathrm{Gen}^{\mathrm{abs}}(M, q, p) \subseteq \mathrm{Gen}(M, q)$ by 1.6.

Lemma 3.3. If P preserves generics for (the transitive collapse of) cofinally many countable $N \prec H(\chi)$, then P is strongly preserving.

Here, cofinally many means that for all $X \subset {}^\omega\omega$ there is a $N \prec H(\chi)$ countable containing X and P with the required property.

Remark: The lemma still holds if Q is any ccc forcing (then N is not collapsed but used directly as in usual proper forcing).

Proof. Assume, $p \Vdash_P X \subseteq \underset{\sim}{A}[G_P] \in I$, i.e. $p \Vdash_P \Vdash_Q \underset{\sim}{\eta} \notin \underset{\sim}{A}[G_P]^{V[G_P][G_Q]}$. Let $N \prec H(\chi)$ containing $P, X, \underset{\sim}{A}, Q, p$, let M be thecollapse of N and $\eta^* \in \mathrm{Gen}(M)$, $p' \leq p$ M–generic s.t. $p' \Vdash \eta^* \in \mathrm{Gen}(M[G_P])$. Let G be V–generic, $p' \in G$.

Then $V[G] \vDash M[G_P][G_Q] \vDash \eta^* \notin A \supseteq X$, so $V \vDash \eta^* \notin X$. Therefore $\mathrm{Gen}(M) \cap X = \emptyset$. $\mathrm{Gen}(M)$ is co–I, therefore $V \vDash X \in I$. □

Theorem 3.4. Assume that P is Q–preserving in V, and that this property is preserved by set forcing. Then P is strongly preserving.

Remarks: Actually, it is not necessary that "P is Q–preserving" is preserved by any set forcing, it is just required that "P is Q–preserving" holds after collapsing some regular cardinal. However, since collapses seems to be nearly as complicated as any forcing can get, this condition doesn't seem to be much weaker.

We will show that for many $N \prec H(\chi)$, $q \in Q^N$, $p \in P^N$, M the transitive collapse: $\mathrm{Gen}^{\mathrm{abs}}(M, q, p)$ is equal to $\mathrm{Gen}(M, q)$. We will start with showing that it is not empty:

Lemma 3.5. If P is preserving, then for all candidates M, $q \in Q^M$, $p \in P^M$, $\mathrm{Gen}^{\mathrm{abs}}(M, q, p) \neq \emptyset$.

Proof. In V, $B_q \in I^+$ (since $q \Vdash \underset{\sim}{\eta} \in B_q$). Assume that G is M– and V–generic. Then in $V[G]$, $\mathrm{Gen}(M[G], q)$ is relatively co–I in B_q, and $B_q^V \in I^+$, so $Y = \mathrm{Gen}(M[G], q) \cap V \in I^+$.

If $V \vDash \mathrm{Gen}^{\mathrm{abs}}(M, q, p) = \emptyset$, then for all $\eta^* \in V$, $p' \leq p$ M–generic: $p' \Vdash \eta^* \notin Y$, i.e. $p' \Vdash Y = \emptyset$, a contradiction. □

Now choose χ_1, χ_2 regular s.t. $2^{\aleph_0} < \chi_1$, $2^{\chi_1} < \chi_2$, and $H(\chi_i) \vDash \mathrm{ZFC}^*_1, {\mathrm{ZFC}^*}_P$.

Let $N \prec H(\chi)$ be a (P– and Q–) candidate, χ_1, χ_2 in N. Choose any $p_0 \in P^N$, $q_0 \in Q^N$, and let $\phi : N \to M$ be the transitive collapse of N, $\phi(H(\chi_1)){=}{:}H_1$. Let R_i^- be the collapse of $H(\chi_i)$ to a countable ordinal, and $R_i{:=}\phi(R_i^-)$ ($i \in \{1, 2\}$). So M is a candidate, $M \vDash H_1 = H(\phi(\chi_1))$, and $M \vDash R_1 \Vdash 2^{\aleph_0} \leq \phi(\chi_2)$.

Let $\eta^* \in \mathrm{Gen}(M, q_0)$. We have to show that $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(M, q_0, p_0)$. Let $G_Q \in V$ be a M–generic filter containing q_0 s.t. $\underset{\sim}{\eta}[G_Q] = \eta^*$, and $G_R \in V$ R_2–generic over $M[G_Q]$, $M' = M[G_Q][G_R]$.

Lemma 3.6. $M' \vDash$ "H_1 is a candidate, $\eta^* \in \mathrm{Gen}^{\mathrm{abs}}(H_1, q_0, p_0)$"

If this is correct, then theorem 3.4 follows: Assume, $M' \vDash$ "$p' \leq p_0$ H_1–generic, $p' \Vdash \eta^* \in \mathrm{Gen}(H_1[G_P], q_0)$". Let $p'' \leq p$ be M'–generic. Then p'' is H_1 generic and therefore M generic as well (since $\mathfrak{P}(P) \cap M = \mathfrak{P}(P) \cap H_1$), and $p'' \Vdash \eta^* \in \mathrm{Gen}(M[G_P], q_0)$.

Proof of lemma 3.6. It is clear that H_1 is a candidate in M'. Assume towards a contradiction, that $M' \vDash$ "$\eta^* \notin \mathrm{Gen}^{\mathrm{abs}}(H_1, q, p)$", Then this is forced by some $q_1 \in G_Q$ and $r \in R_2$, but since R_2 is homogeneous, we can omit r, and since $q_0 \in G_Q$, wlog $q_1 \leq q_0$, so
$(*)$ $M \vDash$ "$q_1 \Vdash_Q \Vdash_{R_2} \eta^* \notin \mathrm{Gen}^{\mathrm{abs}}(H_1, q_0, p_0)$".
Now we can construct the following diagram:

$$\begin{tikzcd}
M \arrow[rr, "R_1"] \arrow[dr, "Q"'] & & M[G_{R_1}]{:=}M_1 \arrow[dr, "R'", "\tilde{G}_2"'] & \\
& M[\eta^{\otimes}] \arrow[ur, "R_1/Q", "\tilde{G}_1"'] \arrow[rr, dashed, "R_2", "\tilde{G}_1 * \tilde{G}_2"'] & & M[\eta^{\otimes}][G_{R_2}] = M_2
\end{tikzcd}$$

First, choose $G_{R_1} \in V$ R_1–generic over M, and let $M_1 = M[G_{R_1}]$. In M_1, pick $\eta^{\otimes} \in \mathrm{Gen}^{\mathrm{abs}}(H_1, q_1, p_0)$ (using lemma 3.5), so since $\mathrm{Gen}^{\mathrm{abs}} \subseteq \mathrm{Gen}$, $M_1 \vDash$ "$\exists\, G^{\otimes}_Q$ Q–gen$/H_1$ s.t. $q_1 \in G^{\otimes}_Q, \underset{\sim}{\eta}[G^{\otimes}_Q] = \eta^{\otimes}$". This $G^{\otimes}_Q$ clearly is M–generic as well (since $M \cap \mathfrak{P}(Q) = H_1 \cap \mathfrak{P}(Q)$), so we can factorize R_1 as $R_1 = Q * R_1/Q$ s.t. $G_{R_1} = G^{\otimes}_Q * \tilde{G}_1$.

Now we look at the forcing $R_2 = R_2^M$ in $M[G^{\otimes}_Q]$. R_2 forces that R_1 is countable and therefore equivalent to Cohen forcing, R_1/Q is a subforcing of R_1. Also, R_2 adds a Cohen real R_2 adds a Cohen real. So R_2 can be factorized as $R_2 = (R_1/Q) * R'$, where $R' = (R_2/(R_1/Q))$. We already have $\tilde{G}_1$ (R_1/Q)–generic over $M[G^{\otimes}_Q]$, now choose $\tilde{G}_2 \in V$ R'–generic over M_1, and let $G_{R_2} = \tilde{G}_1 * \tilde{G}_2$ So $G_{R_2} \in V$ is R_2–generic over $M[G^{\otimes}_Q]$, $M_2{:=}M[G^{\otimes}_Q][G_{R_2}]$.

Let H_2 be $H(\phi(\chi_2))^{M_1}$. Then $H_2 \vDash$ "$p_1 \leq p_0 H_1$–generic$, p_1 \Vdash \eta^{\otimes} \in \mathrm{Gen}(H_1[G_P])$", and in M_2, H_2 is a candidate. Let in M_2, $p_2 \leq p_1$ be H_2–generic, and in V $p_3 \leq p_2$ M_2–generic. Let G_P be P–generic over V containing p_3. Then G is M_2– and H_2–generic, and since $\mathfrak{P}(P) \cap H_2 = \mathfrak{P}(P) \cap M_1$, G_P is M_1–generic as well. So $M_1[G_P] \vDash$ "$\exists G^{\otimes}_2$ Q–generic$/H_1[G_P]$ s.t. $\underset{\sim}{\eta}[G^{\otimes}_2]$"$] = \eta^{\otimes}$. But this $G^{\otimes}_2$ exists in $M_2[G_P] \supseteq M_1[G_P]$, and being a generic filter for a candidate and the evaluation of names in candidates is absolute. So in M_2, $\eta^{\otimes}$ could be generic for $H_1[G_P]$ after forcing with P, a contradiction to $(*)$. ∎

Bibliography

[Bau83] James E. Baumgartner, *Iterated forcing*, Surveys in set theory, London Math. Soc. Lecture Note Ser, vol. 87, Cambridge Univ. Press, Cambridge-New York, 1983, Proceedings of Symp. in Set Theory, Cambridge, August 1978; ed. Mathias, A.R.D., pp. 1–59.

[BB97] Joan Bagaria and Roger Bosch, *Projective forcing*, Annals of Pure and Applied Logic (1997), no. 86, 237–266.

[BJ95] Tomek Bartoszynski and Haim Judah, *Set theory: On the structure of the real line*, A K Peters, Wellesley, MA, 1995.

[GJ92] Martin Goldstern and Haim Judah, *Iteration of Souslin Forcing, Projective Measurability and the Borel Conjecture*, Israel Journal of Mathematics **78** (1992), 335–362.

[Gol93] Martin Goldstern, *Tools for Your Forcing Construction*, Set Theory of The Reals (Haim Judah, ed.), Israel Mathematical Conference Proceedings, vol. 6, American Mathematical Society, 1993, pp. 305–360.

[IHJS88] Jaime Ihoda (Haim Judah) and Saharon Shelah, *Souslin forcing*, The Journal of Symbolic Logic **53** (1988), 1188–1207.

[JS90] Haim Judah and Saharon Shelah, *The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing)*, The Journal of Symbolic Logic **55** (1990), 909–927.

[Paw95] Janusz Pawlikowski, *Laver's forcing and outer measure*, Proceedings of BEST Conferences 1991–1994 (Tomek Bartoszyński and Marion Scheepers, eds.), American Mathematical Society, Providence, 1995.

[She98] Saharon Shelah, *Proper and improper forcing*, Perspectives in Mathematical Logic, Springer, 1998.

[She04] ———, *Properness Without Elementaricity*, Journal of Applied Analysis **10** (2004), math.LO/9712283.

[Sik64] Roman Sikorski, *Boolean Algebras*, Springer Verlag, 1964.