DISSERTATION

D-branes and Superstring BRST Cohomology

A Thesis
Presented to the Faculty of Science and Informatics
Vienna University of Technology

Under the Supervision of A. o. Prof. Maximilian Kreuzer
Institute for Theoretical Physics

In Partial Fulfillment
Of the Requirements for the Degree
Doctor of Technical Sciences

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Vienna, May, 2001

Zusammenfassung

Die Suche nach einer vereinheitlichten Theorie aller Naturkräfte ist eines der Hauptziele der modernen Physik. Die Stringtheorie ist zur Zeit der vielversprechendste Kandidat für eine solche Theorie. Sie vereint nicht nur die Gravitation mit den Kräften des Elektromagnetismus und der starken und schwachen Wechselwirkung, sondern sie vereint auch die fundamentalen Bausteine der Natur in ein einziges Objekt, nämlich eine schwingende Saite, den String.

Mitte der neunziger Jahre hat die "zweite String-Revolution" gezeigt, dass sich die fünf bis dahin bekannten und als unterschiedlich betrachteten Stringtheorien als verschiedene Limiten einer einzigen fundamentalen Theorie verstehen lassen. Die verschiedenen Versionen dieser fundamentalen Theorie sind durch ein Netz von Abbildungen, genannt Dualitäten, miteinander verbunden. In diesem Zusammenhang spielen nicht störungstheoretische, ausgedehnte Objekte, sogenannte D-branes, eine herausragende Rolle. Von besonderem Interesse ist die sogenannte Selbst-Dualität der Typ II-B Stringtheorie. Die zugehörige Dualitätstransformation ($SL(2,\mathbb{Z})$) bildet eine Theorie geschlossener (fundamentaler) Strings auf eine Theorie von D(irichlet)-Strings ab.

In dieser Arbeit werden allgemeine Modelle von fundamentalen Strings und Dirichlet-Strings in der NSR Formulierung mit den Methoden der BRST Kohomologie untersucht. Diese Methode hat sich im Zusammenhang mit der Beschreibung von Eichtheorien als äußerst nützlich erwiesen. Sie faßt wesentliche Eigenschaften der Eichsymmetrie in eine einzige nilpotente Antiderivation zusammen. Diese Antiderivation wird BRST Differential genannt. In den Kohomologiegruppen des BRST Differentials sind wichtige physikalische Informationen über einer Eichtheorie enthalten, sowohl die Ebene der klassischen Physik als auch die der Quantenphysik betreffend.

Die Klasse der untersuchten Theorien wird durch ihren Feldinhalt und die auferlegten Eichsymmetrien definiert. Im besonderen wird Invarianz unter lokaler $N = 1$ Supersymmetrie verlangt. Durch die Einfürung einer geeigneten Basis der Felder läßt sich die Berechnung der Kohomologiegruppen auf die Betrachtung superkonformer Tensorfelder und geeignet definierter Geistfelder einschränken. In einem ersten Schritt der Kohomologie wird die allgemeinste Wirkung für diese Klasse von Modellen berechnet. Im Falle der

D-String Modelle wird die bekannte Superstring Wirkung um einen $U(1)$ Anteil erweitert. Die so ergänzte Wirkung läßt sich in einem erweiterten Target-Raum interpretieren.

Die allgemeinste Wirkung bestimmt die Transformationen der für die weitere Analyse wichtigen Antifelder unter der Einwirkung des BRST Differentials. Mit den vollständigen BRST Transformationen der Felder und Antifelder werden die globalen Symmetrien der betrachteten Modelle klassifiziert und an einem vereinfachten Beispiel veranschaulicht. Es zeigt sich, dass im Falle der D-String Modelle die globalen Symmetrien durch jene der fundamentalen Superstring-Modelle nicht ausgeschöpft werden. Es treten nichttriviale Symmetrien der zusätzlichen Target-Raum Dimensionen auf und die Isometrien des "standard Target-Raumes" werden um Dilatationen erweitert.

Im folgenden wird gezeigt, daß alle physikalisch relevanten Kohomologiegruppen der betrachteten supersymmetrischen Modelle zu jenen der rein bosonischen Modelle (Modelle ohne Supersymmetrie) isomorph sind. Dieses Ergebnis ist überraschend, zeigt es doch, daß die lokale $N = 1$ Supersymmetrie keinen Einfluß auf die Kohomologie des BRST Differentials hat und somit auch auf wesentliche physikalische Eigenschaften der betrachteten Modelle nicht einwirkt. Dies steht im Gegensatz zu Theorien mit "mehr" Supersymmetrie. So ist bekannt, dass lokal $N = 2$ supersymmetrische Stringtheorien die Struktur der zugrundeliegenden Raum-Zeit Mannigfaltigkeiten auf sogenannte Kähler-Mannigfaltigkeiten einschränken.

cknowledgements

I am deeply indebted to my advisor Prof. Maximilian Kreuzer for his encouragement to tackle the vast field of string theory and guiding me through its beautiful though sometimes bewildering, but always challenging landscape. Apart from his guidance through physics he also provided the necessary financial support in the form of ÖNB and FWF grants. In this regard I would also like to thank Prof. Manfred Schweda, who assisted with his resources in difficult times.

I am grateful to my collaborator Friedemann Brandt for never abandoning the hope that our project will be finished some times and for teaching me almost everything I know about BRST cohomology.

It is a pleasure to express my gratitude to the members of the institute for providing a stimulating atmosphere especially during coffee breaks. Special thanks go to Herbert Balasin for his help with physical and mathematical problems, but also for sharing many thoughts in political and social matters and of course for his "experimental triple espresso". It is always great fun to enjoy a movie with him or a walking-tour in the Mediterranean rain. Never give up, never surrender!

Thanks go to Martin Ertl, Peter Fischer, Daniel Grumiller and Axel Schwarz, with whom I shared the office, for their humor and their patience. It is always a pleasure to have you around. I also want to thank Erwin Riegler, with whom I spent many hours on algebra and algebraic geometry. Thanks go to Manfred Herbst, with whom I enjoy a fruitful collaboration on noncommutative geometry.

I want to thank my parents and my sister for their constant support during the many years of my studies. Without them these studies would not have been possible.

Finally and most of all, I want to thank Barbara for her love and patience. Her belief in me and her encouragement during difficult hours are the source of my hope and confidence.

This work was supported by the ÖNB under grant number 7731 and by the Austrian Research Fund FWF under grant number P14639-TPH.

Contents

Chapter 1

Introduction

Why Strings?

String theory [1–3] is a promising candidate for a consistent theory of all forces of nature. It combines a number of ideas that have been put forward in search for a unified theory, like compactification of extra dimensions (Kaluza-Klein mechanism), grand unification and supersymmetry. Moreover, string theory necessarily contains a massless spin-2 state, i.e., it contains gravity. All of these features of the theory arise from the simple idea to replace the standard point particle by one dimensional objects, namely strings. This might raise the question, why not two dimensional objects, called membranes, or even higher dimensional objects ("p-branes")? The answer to this question is, as in most cases in string theory, given by mathematical consistency. Only for one-dimensional objects the mathematical structure seems to control the difficulties arising from divergences, both space-time and internal. Nevertheless, the idea of higher dimensional objects reappears in several ways and plays an outstanding role in the description of string theory at strong coupling.

Until the mid nineties the existence of five different consistent string theories puzzled the scientific community and disappointed those, who claimed the absolute uniqueness of string theory. This was related to the limited understanding of string theory in terms of perturbation theory, the interaction of few strings at weak coupling. The increasing insight into the dynamics of strings at strong coupling resolved this unsatisfactory situation in an elegant way. It turned out that the seemingly different consistent string theories at weak coupling are merely different limits in the space of vacua of a single underlying theory, thereby relating different weakly coupled string theories by dualities. By now a web of dualities connects the different string theories and the eleven dimensional "M-theory".

Why D-branes?

An essential ingredient in the understanding of nonperturbative effects in string theory is the appearance of new extended objects, D-branes [5, 6]. These dynamical objects have the simple interpretation as objects on which strings can end. The massless states, which correspond to D-brane modes arising from an open string attached to it, give rise to a vector field living on the world volume of the D-brane and a number of scalars describing the embedding of the brane into space-time. Thus D-branes are closely related to gauge theories and a fruitful interplay between gauge theory, the geometry of D-branes and string theory has been the origin of many insights in recent years. D-branes provide a remarkably simple description of nonperturbative phenomena, since they have the correct properties to fill out duality multiplets and a highlight of "D-brane physics" is the application to the quantum mechanics of black holes.

An especially interesting case of strong-weak duality in string theory is the conjectured self duality of type IIB theory. The dual objects to the fundamental string are conjectured to be D-strings. They have the same massless excitations (recall that a gauge field in two dimensions has no dynamics), but they are different objects. Especially their tensions are different with their quotient given by the string coupling. At weak coupling the fundamental string is much lighter than the D-string, while at strong coupling the situation is reversed. Thus one is naturally led to the conclusion that the theory at weak coupling is the same as at strong coupling, with the rôle of the fundamental string and the D-string reversed. The corresponding duality transformation is conjectured to be the integer subgroup of the $SL(2,\mathbb{R})$ symmetry of the low energy IIB supergravity. It acts on (p,q) strings, i.e, the bound states of p fundamental strings with q D-strings, and is believed to be an exact symmetry of the theory.

It is well known that the tension of a super-p-brane may be generated dynamically as the flux of a world volume p-form gauge field [7, 8]. This suggests to combine the gauge field of the D-string and the tension-generating gauge field into an $SL(2,\mathbb{R})$ doublet [9, 10]. The result is a twelve dimensional theory. The idea to construct manifestly duality invariant actions for strings and branes has been taken up by several authors in a variety of contexts [11–18].

Why BRST cohomology?

Gauge invariance is a basic principle in models of fundamental interactions. The BRST formalism, first established by Becchi, Rouet and Stora [19–21], provides an extremely useful tool for dealing with gauge symmetries. It encodes the gauge symmetry and its properties in a single antiderivative, usually denoted by s, which is strictly nilpotent on all the fields and in its extension to the so-called field-antifield formalism also on the antifields. This antiderivative is called BRST differential. The nilpotency of the BRST differential defines the BRST cohomology in the space of local functions of the fields and antifields, which is the space of all BRST closed functions ω, $s\omega = 0$, modulo BRST exact functions. A function ω is called BRST exact, if it lies in the image of s, i.e., $\omega = s\eta$. Due to the nilpotency of s BRST exact functions are automatically closed.

The cohomology of the BRST differential captures important physical information on the quantum level as well as on the classical. In fact this was realized at first at the quantum level, where it turns out to be a useful tool in the perturbative renormalization of quantum field theories. Quantizing a gauge theory usually starts with fixing a gauge. The BRST symmetry then becomes a substitute for gauge invariance. The applications of BRST methods at the quantum level include the classification of candidate anomalies, the determination of gauge invariant counter terms, and the renormalization of composite, gauge-invariant operators in the context of the operator product expansion.

The relevance of the BRST cohomology at the classical level has been realized more recently. At negative ghost number the BRST cohomology is isomorphic to the "characteristic cohomology". This cohomology generalizes the notion of conserved currents and involves necessarily antifields, since these are the only elements with negative ghost number. Another important application of BRST methods at the classical level is the relation to deformation theory. This is of interest for the construction of consistent interactions and the proof of their uniqueness up to field redefinitions.

For all of the physical questions above, a complete treatment of the problem in the language of the BRST formalism requires the consideration of antifields. In the following fifteen years after the initiation of the investigation of the BRST cohomology with the seminal papers [19–21] many results on the antifield independent cohomology were established. However, the antifield dependent problem remained largely untouched. Originally the antifields were considered as sources for the BRST variations of the fields. This point of view was apt for the purposes of renormalization of gauge theories but obscured their central rôle for cohomological calculations. The novel interpretation of the antifields as being associated to the equations of motion and thereby implementing them into the cohomological problem in an algebraically well defined way opened the road to new progress. This interpretation originates from the Hamiltonian formulation of the BRST symmetry [22–24]. There the antifields are regarded as the momenta conjugate to the ghosts. The implementation of the equations of motion via the so-called Koszul-Tate differential is essential for the generalization of the BRST construction to the case where the gauge algebra closes only on-shell. In its present form the antifield formalism was established in [25–27].

For an introduction to the BRST formalism see the book of Henneaux and Teitelboim [28] and the reviews on the antifield formalism [29] and on the applications of BRST cohomology in the context of gauge theories of Yang-Mills type [30], where also an extensive list of the relevant literature can be found.

Outline of the thesis

Motivated by the considerations discussed above we present in this thesis a BRST cohomological analysis of superstring models in the NSR formulation [31–33] with local (1,1) supersymmetry [34, 35] including an arbitrary number of abelian gauge fields. The class of models under study is quite general since it is characterized only by requirements on the field content and the gauge symmetries. In particular it contains both, models of fundamental superstrings and of their $SL(2,\mathbb{Z})$ dual D-strings, but it is not restricted to them.

As a first step of the cohomological analysis all local world-sheet actions compatible with these requirements are determined. This analysis is accomplished by a cohomological computation in the space of local functions which do not depend on antifields (this is possible because we use a formulation in which the commutator algebra of the gauge transformations closes off-shell). Its result has been reported and discussed already in [36]: when abelian gauge fields are absent, the cohomological analysis reproduces the general superstring action found already in [37]; in presence of abelian gauge fields, it yields locally supersymmetric extensions of the purely bosonic actions derived in [38, 39] and may be interpreted in terms of an enlarged target space with one 'frozen' extra dimension for each gauge field. In particular there are locally supersymmetric actions of the Born-Infeld type among these actions [36].

The second step of the cohomological analysis investigates the local BRST cohomology, denoted by $H(s)$ throughout the thesis, for the models whose world-sheet actions were determined by the first step. The action is needed to fix the BRST transformations of the antifields. Our analysis is general except for a very mild assumption (invertibility) on the "target space metric".

We explicitly compute the cohomology groups with ghost numbers 0 and 1, which contain the information on the rigid symmetries and dynamical conservation laws and discuss the results for a simplified model. In view of a possible interpretation of the actions in terms of a twelve dimensional theory (in the case of two abelian world sheet gauge fields), it is interesting that the symmetries of the super-D-string action are not exhausted by the isometries of the ten-dimensional standard superstring target space. Additional symmetries are possible, acting nontrivially also on the extra dimensions. Interestingly the solutions to the superstring BRST cohomology at ghost numbers 0 and 1 are already characterized by their purely bosonic counterparts. This suggests the conjecture that the cohomology groups of the supersymmetric models are in one to one correspondence with those of the purely bosonic models.[1]

That this is indeed the case, at least for the physically interesting cohomology groups, is the subject of the last part of this thesis. We shall prove that the cohomology groups of $H(s)$ at ghost numbers $g < 4$ are isomorphic to their counterparts in the corresponding bosonic string models[2] [the bosonic model corresponding to a particular superstring model is obtained from the latter simply by setting all fermions to zero in the world-sheet action]. Furthermore, the correspondence is very explicit: the representatives of the s-cohomology of a superstring model are simply extensions of their "bosonic" counterparts, i.e., they contain the representatives of the s-cohomology of the corresponding bosonic string model and complete them to s-cocycles of the superstring model [analogously to the superstring action itself, which contains the bosonic string action and completes it to a locally supersymmetric one].

This result provides a complete characterization of the cohomology groups $H^g(s)$, $g < 4$ because the cohomology $H(s)$ for the bosonic string models has been completely determined in [40] (ordinary bosonic strings) and [39] (bosonic strings with world-sheet gauge fields). In particular, this implies that the nontrivial Noether currents, global symmetries, consistent deformations, background charges and candidate gauge anomalies of an NSR superstring model with (1,1) supersymmetry are in one-to-one to correspondence with those of the bosonic string model. The results for the bosonic models were derived and discussed in detail in [38–42]. We shall not repeat or summarize these results here, but we shall briefly comment on the relevance of our results to the deformation problem at the end of section 4.1.

The result is quite remarkable and surprising since it means that the local (1,1) supersymmetry of the models under study has no effect on the structure of the cohomology at all! We note that our analysis and result applies analogously to models with less supersymmetry, notably heterotic strings with local (1,0) supersymmetry (by switching off one of the supersymmetries). However, we do not expect that it extends to superstrings with two or more local supersymmetries of the same chirality, such as heterotic strings with local (2,0) supersymmetry. These supersymmetries restrict already the world-sheet action to special backgrounds [43–45]. Accordingly, we expect that the local BRST cohomology of such superstring models is "smaller" than the one for corresponding bosonic strings.

The thesis is organized as follows. In section 2 we give a lightning review

[1] Indeed already the different supersymmetric world sheet actions are parametrized by the same "target space functions" as the bosonic actions.

[2] We believe that the isomorphism extends to all higher ghost number sectors as well since most parts of our proof (in fact, everything except for the case-by-case study in appendix A.2) hold for all ghost numbers.

of string theory with special emphasis on the relevance in view of "D-brane" physics. In particular we explain that the existence of D-branes is required by consistency of string theory with T-duality. Furthermore, we summarize some well known results on strings in background fields and finally we discuss the emergence of noncommutative geometry from open strings in background fields. For the case of general backgrounds we give a Kontsevich type product and discuss its properties in the context of conformal invariance. In the following sections we turn to the BRST cohomological analysis of superstrings and D-strings.

In section 3.2 we specify the field content and the gauge symmetries of the models under consideration. The BRST transformations of the fields corresponding to the gauge symmetries are given. In section 3.3 we construct field variables (jet space coordinates) that are well suited for the cohomological analysis. This involves the super-Beltrami parametrization for the gravitational multiplet and a construction of superconformal tensor fields for the matter and gauge multiplets. In section 4 the first part of the cohomological analysis is carried out. We determine the most general action for the field content and gauge transformations introduced before by computing $H^2(s)$ in the space of antifield independent local functions.

In section 5 we introduce the antifields, give their BRST transformations and extend the superconformal tensor calculus by constructing superconformal antifield variables. The explicit analysis of the antifield dependent cohomology at ghost numbers 0 and 1 is carried out in chapter 6. A detailed calculation is given and the results are discussed for a simplified model.

Then we turn to the general proof of the one to one correspondence of the BRST cohomology $H(s)$ to the purely bosonic one at ghost numbers $g < 4$. In section 7.1 we define and analyze an on-shell BRST cohomology $H(\sigma)$; in section 7.2 we show that $H^g(\sigma)$ is isomorphic to $H^g(s)$ and to the cohomology of the corresponding bosonic string model when $g < 4$. Some details of the analysis of sections 7.1 and 7.2 are collected in the appendices A.1 and A.2. The remaining appendices give a short summary of the derivation of the gauge transformations from the supergravity Bianchi identities and contain a collection of the s-transformations of the covariant (= superconformal) field and antifield variables.

Chapter 2

String theory in a nutshell

This chapter is devoted to a lightning review of string theory. Due to its rich structure it is hopeless to cover the subject in a self contained way without restricting to certain subareas of the theory. Thus, we will focus mainly on the basic concepts relevant for the topics discussed in the rest of the thesis. Most of the material presented here can be found in any introductory lectures on string theory and D-branes [1–4].

2.1 Open and closed strings

A bosonic string propagating in a D dimensional flat space-time is described by the embedding functions $X^\mu(\tau, \sigma)$, with $\mu = 0, 1, \ldots, D-1$, of the two dimensional "world-sheet" parameterized by τ and σ into "space-time". In analogy to the point particle case one can write down an action proportional to the area of the world-sheet measured by the induced metric on the world sheet. This action is called the Nambu-Goto action. It has the awkward property of containing derivatives under the square root and is thus not well suited for quantization. There is a fairly easy way to circumvent this problem, by introducing an additional auxiliary metric g_{mn} on the world-sheet, which is the analog to the einbein introduced for the point particle. The resulting world sheet action, which is usually called Polyakov[1] action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-g}\, g^{mn} \partial_m X^\mu \partial_n X^\nu \eta_{\mu\nu}, \tag{2.1.1}$$

where g denotes the determinant of the world-sheet metric. The factor in front of the integral is proportional to the tension of the string written in terms of the Regge[2] slope α', which has the dimension of (space-time) length

[1]The action was in fact found by Brink, Di-Vecchia and Howe and Deser and Zumino. Polyakov pointed out its relevance in the path integral quantization.

[2]String theory was originally proposed to be a theory of strong interactions. Meson resonances obey a linear spin-mass relation, with $\alpha' \sim (1GeV)^{-2}$ being the slope of the

squared. $\eta_{\mu\nu}$ is the space-time metric and g^{mn} is the inverse of the world-sheet metric. This action is classically equivalent to the Nambu-Goto action, i.e., it gives rise to the same equations of motion.

The Polyakov action possesses a large number of symmetries, namely D-dimensional Poincaré invariance

$$\begin{aligned} X'^{\mu} &= \Lambda^{\mu}{}_{\nu} X^{\nu} + A^{\mu} \\ g'_{mn} &= g_{mn}, \end{aligned} \tag{2.1.2}$$

where $\Lambda^{\mu}{}_{\nu}$ is a Lorentz transformation and A^{μ} is a translation, two dimensional diffeomorphism invariance

$$\begin{aligned} \delta X^{\mu} &= \xi^{m} \partial_{m} X^{\mu} \\ \delta g_{mn} &= \xi^{l} \partial_{l} g_{mn} + (\partial_{m} \xi^{l}) g_{ln} + (\partial_{n} \xi^{l}) g_{lm}, \end{aligned} \tag{2.1.3}$$

for two parameters $\xi^{l}(\tau, \sigma)$ and Weyl invariance

$$\begin{aligned} X'^{\mu} &= X^{\mu} \\ g'_{mn} &= e^{2\omega(\tau,\sigma)} g_{mn}. \end{aligned} \tag{2.1.4}$$

Poincaré invariance is a consequence of taking space-time to be flat and is a global symmetry in the world-sheet sense. Equation (2.1.2) states that the embedding functions $X^{\nu}(\tau, \sigma)$ simply transform as a vector, while the world sheet metric is invariant. The invariance under two dimensional world sheet diffeomorphisms and the invariance under local rescalings of the world-sheet metric are nontrivial gauge symmetries. From equations (2.1.3) it follows that $X^{\nu}(\tau, \sigma)$ transforms as a scalar under local reparametrizations of the world-sheet, while the metric transforms of course as a covariant rank two tensor. These symmetries are essential features of the theory and in section 3.2 we will use them, extended by additional gauge symmetries and supplemented with a prescribed field content, to characterize the whole class of models considered in the BRST cohomological problem. Moreover, they are features of the classical theory and give interesting results when one tries to retain them in a quantum theory. We will come back to this point later, when we discuss strings propagating in background fields.

The equations of motion following from the variation of the Polyakov action are

$$T^{mn} = 0 \tag{2.1.5}$$

$$\partial_{m} \left(\sqrt{g} g^{mn} \partial_{n} X^{\mu} \right) = 0, \tag{2.1.6}$$

trajectories. In string theory the parameter is of the order of the natural scale determined by the fundamental constants of gravity and quantum mechanics, i.e., the inverse Planck mass squared M_P^{-2}.

where T^{mn} is the world sheet energy momentum tensor. It is conserved $\nabla_n T^{nm} = 0$ as a consequence of reparametrization invariance and moreover, Weyl invariance requires the energy momentum tensor to be traceless, $T_m{}^m = 0$. The second equation has to be supplemented with appropriate boundary conditions. Taking the world-sheet to be parameterized such that $0 \leq \sigma \leq \pi'$ one has

$$\begin{aligned}
\text{open string :} \qquad & X'^{\mu}(\tau, 0) = X'^{\mu}(\tau, \pi) = 0, \\
\text{closed string :} \qquad & X'^{\mu}(\tau, 0) = X'^{\mu}(\tau, \pi) \\
& X^{\mu}(\tau, 0) = X^{\mu}(\tau, \pi) \\
& g_{mn}(\tau, 0) = g_{mn}(\tau, \pi),
\end{aligned} \tag{2.1.7}$$

where a prime denotes the derivative with respect to σ. Note that we have introduced closed strings by imposing periodicity. The boundary conditions for the open string are the standard Neumann boundary conditions stated more covariantly $n^m \partial_m X^\mu = 0$, where n^m is a vector normal to the boundary. The boundary conditions (2.1.7) are the only ones that are compatible with space-time Poincaré invariance and the equations of motion. If the condition of Poincaré invariance is relaxed there are certain other possibilities, which will become important in the context of D-branes, see section 2.2. Their relevance and consistency was discovered in the context of T-duality.

The Polyakov action (2.1.1) defines a two dimensional field theory on the string world-sheet. It describes D massless scalar fields X^μ coupled to the metric g_{mn}. From the world-sheet point of view Poincaré invariance is an internal symmetry acting on fields at fixed τ and σ. Amplitudes for space-time processes are given in terms of matrix elements of this two dimensional quantum field theory. In section 2.3 we will consider generalizations of the Polyakov action, namely nonlinear sigma models.

2.2 D-branes

D-branes are extended objects defined by the fact that open strings may end on them. The existence of such extended objects in string theory has been uncovered in the context of T-duality [46, 47]. Let us review some of the arguments.

Using two dimensional diffeomorphism invariance and Weyl symmetry, which are three local or gauge symmetries, to fix the three independent degrees of freedom of the world-sheet metric, we can at least locally choose it to be of the form $g_{mn} = \delta_{mn}$. Furthermore, choosing complex coordinates on the world sheet and mapping it to the complex plane one can write the Polyakov action as

$$S = \frac{1}{2\pi\alpha'} \int_\Sigma \mathrm{d}^2 z \, \partial X^\mu \bar{\partial} X_\mu . \tag{2.2.8}$$

The equations of motion then take the form

$$\partial\bar{\partial}X^\mu = 0, \tag{2.2.9}$$

which implies that ∂X^μ is holomorphic and $\bar{\partial}X^\mu$ is antiholomorphic. In terms of the corresponding Laurent expansions the general solution is given by

$$\begin{aligned} X^\mu &= x^\mu - \mathrm{i}\frac{\alpha'}{2}p^\mu \ln|z|^2 + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}(\alpha_n^\mu z^{-n} + \tilde{\alpha}_n^\mu \bar{z}^{-n}) \\ &\equiv X_L^\mu + X_R^\mu \end{aligned} \tag{2.2.10}$$

for closed strings and for open strings

$$X^\mu = x^\mu - \mathrm{i}\frac{\alpha'}{2}p^\mu \ln|z|^2 + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu(z^{-n} + \bar{z}^{-n}). \tag{2.2.11}$$

The overall motion of the string is described by its center of mass position x^μ and its momentum p^μ, which is identified with the zero mode of the Laurent expansion of ∂X^μ and $\bar{\partial}X^\mu$. The mode expansions describe the oscillatory degrees of freedom of the string.

T-duality for closed strings

Now consider closed strings in a target space with one compact dimension, say X^{25}. Let us work out the implications of the periodicity $X^{25} = X^{25} + 2\pi R$ for the solution to the equations of motion. We focus on the zero mode contributions written in terms of the original variables

$$X^\mu = x^\mu + \tilde{x}^\mu - \mathrm{i}\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)\tau + \sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu)\sigma + (\text{oscillators}). \tag{2.2.12}$$

In the case of a non-compact dimension the term proportional to σ has to vanish so that $\alpha_0^\mu = \tilde{\alpha}_0^\mu$. The compact dimension allows an additional solution. Running once around the closed string we get

$$X^\mu(z,\bar{z}) \to X^\mu(z,\bar{z}) + 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0^\mu - \tilde{\alpha}_0^\mu). \tag{2.2.13}$$

But now X^μ need not be single valued under the change $\sigma \to \sigma + 2\pi$. It can change by an integer multiple of $2\pi R$. Furthermore the momentum identified with $p^\mu = \sqrt{\frac{1}{2\alpha'}}(\alpha_0^\mu + \tilde{\alpha}_0^\mu)$ has to be an integer multiple of the inverse radius of the compact dimension to ensure the single valuedness of

$\exp(\mathrm{i}\, p \cdot X)$. Solving the two resulting equations for the compact dimension X^{25} one finds

$$\begin{aligned} \alpha_o^{25} &= \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R}+\frac{wR}{\alpha'}\right) \equiv \sqrt{\frac{\alpha'}{2}}p_L \\ \tilde{\alpha}_o^{25} &= \sqrt{\frac{\alpha'}{2}}\left(\frac{n}{R}-\frac{wR}{\alpha'}\right) \equiv \sqrt{\frac{\alpha'}{2}}p_R, \end{aligned} \tag{2.2.14}$$

where n and w are integers. We conclude that in the case of a compact dimension a whole tower of new states appears corresponding to a closed string wound w times around the compact dimension. For a large radius R of the compact dimension the momentum states are light and the winding states are heavy, i.e., it costs much energy to excite them in the spectrum. In the case of a small radius the situation is reversed. The momentum states are heavy while the winding states are light.

One can push this further and ask what happens in the decompactification limit $R \to \infty$ and in the limit of $R \to 0$. In the decompactification limit the winding modes become infinitely massive and decouple from the spectrum. The momentum states go over to a continuum of states. Indeed this perfectly fits with what one should intuitively expect, namely to recover the uncompactified situation. But what happens in the $R \to 0$ limit? The momentum states become infinitely heavy and decouple from the spectrum. In the case of point particles this would be all we observe. The compactified dimension vanishes and we are left with one dimension less. But closed strings behave quite differently. The winding states now form a continuum and the uncompactified dimension reappears! In fact a theory of closed strings compactified on a circle of radius R is dual to a theory compactified on a radius $1/R$, i.e., the spectrum is invariant under the exchange of $n \leftrightarrow w$ and $R \leftrightarrow \alpha'/R$. The fully interacting theory can be described in terms of the T-dualized coordinate $X'(z,\bar{z}) = X(z) - X(\bar{z})$, which is a parity transformation acting on the right moving part only. It has the same operator products and energy momentum tensor, since the minus sign enters in all these cases in pairs. The dual coordinate accounts only for the change in the sign of the right moving zero mode in the conformal field theory, which changes the spectrum from the theory with radius R to that of the theory with radius $1/R$. The theories are identical, one being written in terms of X and one in terms of X'.

This duality is called T-duality and it is an exact symmetry of perturbative closed string theory. This gives evidence to the idea of a minimal length in string theory, namely the self dual radius $R = \sqrt{\alpha'}$. The same considerations hold for toroidal compactification of several dimensions and even for more general compactifications.

T-duality and open strings

Something different has to happen in the case of open strings. This is clear, since there is no conserved winding number for open strings. So in the $R \to 0$ limit there is no tower of winding states, which effectively generates a dimension. Rather the situation is similar to the field theory case: the states with nonzero momentum become infinitely heavy and decouple from the spectrum and we are left with one dimension less. Now the puzzling point in this story is that a theory of open strings necessarily contains closed strings. After taking the $R \to 0$ the open strings live in one dimension less than the closed strings! The solution to this puzzle is that the endpoints of the open strings are confined to a $D-1$ dimensional hyperplane. Indeed the interior of an open string cannot be distinguished from a closed string and thus should still vibrate in all D dimensions just like a "real" closed string.

Let us work this out in more detail starting from the open string mode expansion $X^\mu(z,\bar{z}) = X^\mu(z) + X^\mu(\bar{z})$

$$\begin{aligned} X^\mu(z) &= \frac{1}{2}x^\mu + \frac{1}{2}x'^\mu - \mathrm{i}\alpha' p^\mu \ln z + \mathrm{i}\sqrt{\frac{a'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu z^{-n} \\ X^\mu(\bar{z}) &= \frac{1}{2}x^\mu - \frac{1}{2}x'^\mu - \mathrm{i}\alpha' p^\mu \ln \bar{z} + \mathrm{i}\sqrt{\frac{a'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \bar{z}^{-n} \end{aligned} \qquad (2.2.15)$$

and consider the coordinate X^{25} compactified on a circle of radius R. The T-dual coordinate is $X'^{25}(z,\bar{z}) = X^{25}(z) - X^{25}(\bar{z})$. Thus we get[3]

$$\begin{aligned} X'^{25}(z,\bar{z}) &= x'^{25} - \mathrm{i}\alpha' p^{25}\ln(\frac{z}{\bar{z}}) + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}(z^{-n} - \bar{z}^{-n}) \\ &= x'^{25} + 2\mathrm{i}\alpha' p^{25}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin n\sigma \\ &= x'^{25} + 2\mathrm{i}\alpha'\frac{n}{R}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-in\tau}\sin n\sigma. \end{aligned} \qquad (2.2.16)$$

The essential point is the absence of a τ dependence in the zero mode sector, i.e. there is no momentum in the X'^{25} direction. The Neumann boundary conditions $\partial_\sigma X = 0$ are replaced by Dirichlet boundary conditions $\partial_t X = 0$! The oscillator terms vanish at the endpoints $\sigma = 0, \pi$ and the ends are confined to

$$X'^{25}(\pi) - X'^{25}(0) = 2\pi\alpha'\frac{n}{R} = 2\pi\alpha' n R'. \qquad (2.2.17)$$

[3]After Wick rotating to Euclidean time $\tau \to -it$ and mapping the cylinder to the complex plane one has $z = \exp(t + i\sigma) = \exp(i\sigma^+)$.

The difference is an integer multiple of the radius of the dual dimension. Thus we conclude that under T-duality the normal and the tangential derivative are exchanged

$$\begin{aligned}\partial_n X^{25}(z,\bar{z}) &= \partial_z X^{25}(z) + \partial_{\bar{z}} X^{25}(\bar{z}) = \partial_t X'^{25}(z,\bar{z})\\ \partial_t X^{25}(z,\bar{z}) &= \partial_z X^{25}(z) - \partial_{\bar{z}} X^{25}(\bar{z}) = \partial_n X'^{25}(z,\bar{z})\end{aligned} \qquad (2.2.18)$$

This gives a consistent picture of what happens in the T-dualized direction. In all other directions the situation is not changed and the string endpoints are still free to move. The 24 dimensional hyperplane to which the string ends are confined are called a Dirichlet 24-brane or D24-brane for short. The same picture goes through for any number of coordinates giving D-branes of higher codimension.

It is natural to expect that these objects are really dynamical objects, because in a theory containing gravity perfectly rigid objects do not exist. Rather one expects the D-branes to fluctuate in shape and position. One can work this out by looking at the massless spectrum of the theory. Massless states arise from non-winding states because the string tension contributes an energy to a stretched string. Sticking to the example of the D-24 brane we find

$$\begin{aligned}&\alpha^{\mu}_{-1}|k>, \qquad V = \partial_t X^{\mu} e^{\mathrm{i}kX}\\ &\alpha^{25}_{-1}|k>, \qquad V = \partial_t X^{25} e^{\mathrm{i}kX} = \partial_n X'^{25} e^{\mathrm{i}kX'}.\end{aligned} \qquad (2.2.19)$$

These are of course the same massless states as those of the original theory but viewed from the dual theory. The first line in (2.2.19) is a gauge field living on the D-brane with 25 components tangent to the brane depending on the world volume coordinates of the brane. The second line, representing the gauge field in the compact direction in the original theory, becomes the position of the brane in the dual picture. From the D-brane world volume point of view it is simply a scalar living there. Again this picture goes through for several T-dualized directions. Now let us consider the meaning of these modes. Let the value of the scalar vary while we move along the brane. This corresponds to an embedding of the brane into the transverse dimensions and thus determines the shape of the brane. The scalar thus plays the same role as the coordinate function X^{μ}, which describes a string. Recall that from the world-sheet point of view the X^{μ}'s are scalar fields!

The values of the gauge field background describe the shape of the D-brane as a (possibly solitonic) background for the gauge degrees of freedom and their quanta describe fluctuations about that background.

World-volume actions for D-branes

We started from open strings compactified on a circle and were naturally led to the existence of extended objects on which open strings are allowed to end. Moreover, due to the presence of gravity we conclude that these are in fact dynamical objects. Thus one might ask how the low energy effective world-volume action looks like. This is easily answered taking into account the discussion of the previous paragraph. The massless fields on the brane world-volume are given by a gauge field A_m and a number of transverse scalars corresponding to the position of the brane. Introducing the corresponding world-volume fields ξ^a one is led by direct analogy to the string case to the following action for a single D-brane

$$S_p = -T_p \int_\Sigma \mathrm{d}\xi^{p+1} e^{-\Phi} \sqrt{\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{2.2.20}$$

where G_{ab} and B_{ab} denote the pullback of the space-time metric and the antisymmetric tensor field of the closed string background to the $(p+1)$ dimensional D-brane world-volume. This is nothing but the analog of the Nambu-Goto string action and is known as the Born-Infeld action for non-linear electrodynamics. The dependency on $B + 2\pi\alpha' F$ [4] can be understood by the fact that in the open string action the B-field and the boundary gauge field A are related by a space-time gauge invariance

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \qquad A_\mu \to A_\mu - \frac{1}{2\pi\alpha'}\Lambda_\mu, \tag{2.2.21}$$

which is preserved by the combination $2\pi\alpha' \mathcal{F}_{\mu\nu} = B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}$. This is invariant under both space-time gauge symmetries, the one mentioned above and the $U(1)$ gauge symmetry of A.

An interesting modification arises when one considers a number of coincident D-branes. Intuitively it is clear that additional massless degree of freedom arise from strings ending on different branes. The $U(1)$ gauge symmetry is enhanced and becomes a non-abelian $U(N)$ gauge symmetry, where N is the number of coincident branes, and the gauge field becomes an $N \times N$ matrix. The same happens to the collective coordinates for the embedding of the D-branes. This is the first appearance of "noncommutative geometry" in terms of matrix coordinates. Again some insight into the form of the low energy effective action can be gained by T-duality starting from the Born-Infeld action.

As a concluding remark to this section we comment on the dilaton factor $e^{-\Phi}$ in the Born-Infeld action and the brane tension T_p. The dilaton dependency can be understood from the the fact that this is an open string tree level effective action computed on the disk. The Dp-brane tension is determined by T-duality (by a recursion relation) up to an overall normalization. The actual value of the D-brane tension can be computed from the exchange of a closed string between two D-branes and is of the order of the inverse string coupling.

[4] For convenience we will set $2\pi\alpha' = 1$ in the following and reintroduce the explicit dependency on α' where it is necessary.

2.3 Strings in background fields

We have written down the Polyakov action (2.1.1) assuming that the strings are propagating in an uncompactified flat target space with a Minkowskian metric $\eta_{\mu\nu}$. A first step towards a generalization is to consider the nonlinear sigma model

$$S = -\frac{1}{4\pi\alpha'} \quad d\tau d\sigma \sqrt{-g}\, g^{mn} \partial_m X^\mu \partial_n X^\nu G_{\mu\nu}(X), \tag{2.3.22}$$

with a nontrivial space-time metric $G_{\mu\nu}(X)$. From the two-dimensional world-sheet point of view this corresponds to a theory of D scalar fields with field dependent couplings. That this is indeed a sensible choice can be seen by considering an expansion around the flat background $G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$, where $h_{\mu\nu}(X)$ is a small deviation from flat space. Inserting this into the (Euclidean) path integral one finds a term

$$\frac{1}{4\pi\alpha'} \quad \mathrm{d}^2 z\, h_{\mu\nu}(X) \partial X^\mu \bar{\partial} X_\nu. \tag{2.3.23}$$

Setting $h_{\mu\nu}(X) \propto \zeta_{\mu\nu} \exp(\mathrm{i}k \cdot X)$ with a symmetric polarization matrix $\zeta_{\mu\nu}$ one is simply inserting a graviton vertex operator into the path integral. The insertion of the full metric $G_{\mu\nu}(X)$ corresponds to a coherent state of gravitons. Generalizing this procedure to include other backgrounds of the massless sting states one obtains for the closed string sector

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'} \quad & d\tau d\sigma \sqrt{-g} \left[g^{mn} \partial_m X^\mu \partial_n X^\nu G_{\mu\nu}(X) \right. \\ & \left. + \varepsilon^{mn} \partial_m X^\mu \partial_n X^\nu B_{\mu\nu}(X) + \alpha' R^{(2)} \Phi(X) \right], \end{aligned} \tag{2.3.24}$$

where $R^{(2)}$ denotes the two-dimensional Ricci scalar associated with the world-sheet metric g_{mn}. We have added terms corresponding to the antisymmetric tensor field $B_{\mu\nu}(X)$ and the dilaton $\Phi(X)$. In the limit of small B and Φ the vertex operators for these backgrounds are $B_{\mu\nu}(X) \propto a_{\mu\nu} \exp(\mathrm{i}k \cdot X)$ and $\Phi(X) \propto \phi \exp(\mathrm{i}k \cdot X)$ with an antisymmetric polarization matrix $a_{\mu\nu}$. Here some remarks are in order concerning the coupling of the dilaton. Firstly one observes that this action is classically invariant under global scale transformations but not under local Weyl transformations. The dilaton term breaks this invariance unless the dilaton is constant. Let us consider a constant dilaton for the moment. Then the first and the third term together look like the action for D massless scalars minimally coupled to gravity in two dimensions. But there is no dynamics associated with the world-sheet Ricci scalar appearing in the dilaton term. This is easily seen from the Einstein equations in two dimensions, because $R_{mn} - \frac{1}{2} g_{mn} R$ vanishes identically. However, the Hilbert action has a topological meaning. In the path integral a term

$$-\frac{\lambda}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-g}\,R^{(2)}, \tag{2.3.25}$$

where λ for now is an arbitrary parameter, will give rise to a factor $\exp(-\lambda\chi)$. χ denotes the Euler number of the string world sheet $\chi = 2-2h-b-c$, where h, b, c are the numbers of handles, boundaries and crosscaps, respectively. For open strings (2.3.25) is in fact modified. One then has to include the extrinsic curvature on the boundary. For instance an open string tree level diagram has the topology of the disk and will thus be weighted with a factor $\exp(-\lambda)$. The emission and reabsorption of an open string will be related to a change in the Euler number of $\delta\chi = -1$. Relative to the tree level open string diagram the amplitude for emitting an open string will be weighted by a factor $\exp(\lambda/2)$, which we thus regard as the open string coupling. In the same way one gets for the amplitude for emitting a closed string a factor $\exp(\lambda)$, which is regarded as the closed string coupling. Hence the coupling constants in string theory are controlled by the Euler term in the action. Now let us return to the situation for the constant dilaton background. From the discussion above one might suspect that the string coupling is a free parameter, but this is not the case. Different values for the string coupling do not correspond to different theories but to different backgrounds in a single theory and the only free parameter in the theory remains the string tension.

Now before we turn to implications resulting from Weyl invariance let us inspect some possible extensions for open string backgrounds. The most general action for open strings coupling to massless background fields[5] is

$$\begin{aligned} S = -\frac{1}{4\pi\alpha'}\int_\Sigma d\tau d\sigma\sqrt{-g}\,\big[&g^{mn}\partial_m X^\mu \partial_n X^\nu G_{\mu\nu}(X) \\ &+\varepsilon^{mn}\partial_m X^\mu \partial_n X^\nu B_{\mu\nu}(X) + \alpha' R^{(2)}\Phi(X)\big] \\ -\frac{1}{2\pi\alpha'}\int_{\partial\Sigma} ds\big[&2\pi\alpha' A_\mu \partial_t X^\mu + \alpha' K^{(2)}\Phi(X)\big] \end{aligned} \tag{2.3.26}$$

We have included the extrinsic curvature of the boundary $K^{(2)}$ and the open string gauge field A_μ with the vertex operator $\int_{\partial\Sigma} ds\zeta_\mu\partial_t X^\mu \exp(ik\cdot X)$, where ∂_t denotes the tangential derivative to the world-sheet boundary $\partial\Sigma$. The Gauss-Bonnet term, which gives the Euler number, is now

$$\frac{1}{4\pi}\int_\Sigma R^{(2)} + \frac{1}{2\pi}\int_{\partial\Sigma} K^{(2)} = \chi, \tag{2.3.27}$$

which explains the necessity to include the boundary curvature because the dilaton determining the coupling constant must multiply the entire Euler density.

[5] It is also possible to include other backgrounds, for instance for the tachyon field or higher order tensor fields corresponding to massive spin > 2 modes. For closed strings the term corresponding to the tachyon is $S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\sqrt{-g}\,T(X)$.

To define a consistent string theory the action (2.3.26) has to be Weyl invariant, both classically and as a quantum theory. This is related to the tracelessness property of the two dimensional energy momentum tensor. For the closed string sector one finds [1–3]

$$T_m^{\ m} = -\frac{1}{2\alpha'}\beta^G_{\mu\nu} g^{mn}\partial_m X^\mu \partial_n X^\nu - \frac{1}{2\alpha'}\beta^B_{\mu\nu}\varepsilon^{mn}\partial_m X^\mu \partial_n X^\nu - \frac{1}{2}\beta^\Phi R^{(2)}, \tag{2.3.28}$$

where the coefficient functions are the renormalization group beta functionals associated with the coupling functions indicated as superscripts.

Scale invariance in a quantum field theory is related to the vanishing of the renormalization group β functions, which arise from ultraviolet divergences in Feynman diagrams. Since Weyl invariance implies scale invariance, which in turn is related to the vanishing of the beta function, the ultraviolet finiteness of the two dimensional quantum field theory and Weyl invariance are intimately related[6]. The breakdown of scale invariance in the quantum theory can be understood by the fact that there is no regularization scheme preserving conformal invariance. The subtraction of contributions of a massive regulator field as in the Pauli-Villars regularization breaks scale invariance whereas dimensional regularization violates scale invariance because the sigma model is only scale invariant in two dimensions.

There have been a large number of contributions to this subject, most of them in the 80's [48–53]. The remarkable result is that the vanishing of the beta functions gives rise to space-time field equations. Explicitly the beta functions for the closed string sector are given by

$$\begin{aligned}
\beta^G_{\mu\nu} &= \alpha' \left(R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - \tfrac{1}{4} H_{\mu\lambda\rho} H_\nu^{\ \lambda\rho} \right) + O(\alpha'^2) \\
\beta^B_{\mu\nu} &= \alpha' \left(-\tfrac{1}{2} \nabla^\lambda H_{\lambda\mu\nu} + \nabla^\lambda \Phi H_{\lambda\mu\nu} \right) + O(\alpha'^2) \\
\beta^\Phi &= \tfrac{D-26}{6} - \alpha' \left(\nabla^2 \Phi - \nabla_\lambda \Phi \nabla^\lambda \Phi + \tfrac{1}{24} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right) + O(\alpha'^2)
\end{aligned} \tag{2.3.29}$$

For the gauge field background the beta function is given by [51, 52]

$$\begin{aligned}
\beta^A_\mu = & \left(\frac{G}{G-(B+F)}\right)^{\lambda\nu} \nabla_\lambda (B+F)_{\nu\mu} \\
& + \frac{1}{2} \left(\frac{(B+F)}{G-(B+F)}\right)^{\lambda\nu} H_{\lambda\nu}^{\ \ \rho} (B+F)_{\rho\mu} + \frac{1}{2}\nabla^\nu \Phi (B+F)_{\mu\nu},
\end{aligned} \tag{2.3.30}$$

which is valid to all orders in α' and to lowest order in derivatives of $B+F$. [7] Recall that only the combination $\mathcal{F}$ which is invariant under both space-time gauge transformations, the $U(1)$ gauge transformation of A and the

[6] Considerations concerning full ultraviolet finiteness also have to take into account wave function renormalization not just the renormalization of the couplings.

[7] Displaying the α' dependency of the beta functions in the case of a pure gauge field background gives $\beta^A_\mu = 2\pi\alpha' \left(1 - (2\pi\alpha' F)^2\right)^{-1\,\lambda\nu} \nabla_\lambda F_{\nu\mu}$.

combined transformation (2.2.21) of B and A, enters in these expressions. $H_{\lambda\mu\nu}$ is the corresponding field strength and is given by

$$H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}. \tag{2.3.31}$$

Indeed it is possible to derive these space-time equations of motion from a space-time action. For the closed string background this is

$$\begin{aligned} S_{eff}^{closed} = \frac{1}{2\kappa^2} \quad d^D X \sqrt{-G} e^{-2\Phi} \Big[& R + 4\nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \\ & - \frac{2(D-26)}{3\alpha'} + O(\alpha') \Big]. \end{aligned} \tag{2.3.32}$$

By a field redefinition one can remove the dilaton factor in front of the space-time Ricci scalar and thus obtain the standard Einstein-Hilbert action. This is usually referred to as going to the "Einstein frame". In this terminology the action (2.3.32) is written in the "string frame". To lowest order in α' the effective action corresponding to the open string sector is given by the Yang-Mills action

$$S_{eff}^{YM} = -\frac{C}{4} \quad d^D X e^{-\Phi} F_{\mu\nu} F^{\mu\nu} + O(\alpha'). \tag{2.3.33}$$

Note that the appearance of the factor $\exp \Phi$ in the actions above is consistent with the factors one would expect for the appearance of the string coupling.

In (2.3.30) we gave the space-time equation of motion for the gauge field A to all orders in α' but the Yang-Mills action comprises only leading order terms in α'. One might ask if one can do better and indeed the space-time effective action including all orders in α' and lowest order in derivatives of $B + F$ is given by the Born-Infeld action

$$S_{eff}^{open} = \quad d^D X e^{-\Phi} \quad \overline{\det(G + B + F)}. \tag{2.3.34}$$

One might propose that the proper way to describe interacting open and closed strings is to simply add the space-time effective actions for the closed and open string sector. The equations of motion arising from the combined action reproduce correctly the beta functions for the gauge field but the closed string beta functions are extended by additional terms corresponding to gauge field source terms. This is quite reasonable since the gauge fields should act as a source for gravity. But the presence of a boundary does not change the beta functions of the closed string massless fields. Nevertheless one can argue that the corresponding equations of motion are interpretable as string loop corrected beta functions [52].

We will not be concerned with higher order loop corrections to the beta functions, but the Born-Infeld action will once again show up in the context of noncommutative geometry arising from D-branes in nontrivial background fields.

2.4 Noncommutative Geometry

This section is devoted to an old idea in a new guise and also contains some comments on recent work done in collaboration with Manfred Herbst [54]. The idea[8] that the structure of space-time changes at short distances and thereby provides an effective ultraviolet cut-off, which regularizes the notorious infinities present in quantum field theory, was already proposed by Heisenberg in the 1930's. He suggested a lattice structure, which of course breaks Lorentz invariance. Nevertheless, for practical and numerical reasons this lattice version of space-time is quite satisfactory, when random lattices are used. In this Lattice approximation Lorentz symmetry is a classical symmetry and is broken at the microscopic level. Some time later Snyder [57] proposed the idea to use a noncommutative structure at small length scales. It was von Neumann who introduced the term "noncommutative geometry" for a general geometry in which the algebra of functions is replaced by a noncommutative algebra.

The argument that a noncommutative structure provides an effective cut-off can be seen from analogy with the quantization of the classical phase-space, where coordinates are replaced by generators of the algebra. Since these do not commute they cannot be diagonalized simultaneously and thus it is no longer justified to speak of the phase-space in terms of points. Rather the points of phase-space have to be replaced with Bohr cells. In the same way one replaces the points of space-time with Planck cells with the dimension of the Planck area.[9] In a coherent description this "pointlessness" eliminates the ultraviolet divergences of quantum field theory by coarse-graining space-time just like an ultraviolet cut-off Λ prevents a theory to probe length scales smaller the Λ^{-1}. The question is how does this coherent description of space-time look like? The simplest but by far not the only possibility is to introduce noncommuting space-time coordinates, i.e. to replace the coordinates by generators satisfying commutation relations

$$[q^\mu, q^\nu] = \mathrm{i}k\theta^{\mu\nu}. \tag{2.4.35}$$

The parameter k is a fundamental area scale.

A simple and probably the most prominent example of a noncommutative "space" covariant under the action of a continuous symmetry group is provided by the "fuzzy sphere" [56]. Let us review how noncommutative geometry makes its appearance in string theory.

[8] For a historical review of noncommutative geometry see for instance [55]. The standard reference for a rigorous mathematical presentation of noncommutative geometry is the book of A. Connes [58], but see also for instance the introductory lectures of Giovanni Landi [59].

[9] This assumption is in fact not mandatory, since experimental bounds would allow much larger scales.

Open strings in a constant B-field

The most prominent example for the appearance of noncommutative geometry in string theory, which is also most extensively covered in the literature, arises from open strings in the background of a constant antisymmetric tensor field $B_{\mu\nu}$ [60–63]. The simplest case is to consider bosonic open strings moving in a flat Euclidean background

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} [g_{\mu\nu}\partial_m X^\mu \partial^m X^\nu + \mathrm{i}\varepsilon^{mn} B_{\mu\nu}\partial_m X^\mu \partial_n X^\nu]. \tag{2.4.36}$$

The term involving the B-field background can be rewritten as a boundary term

$$S_B = \frac{\mathrm{i}}{4\pi\alpha'} \int_{\partial\Sigma} B_{\mu\nu} X^\mu \partial_t X^\nu, \tag{2.4.37}$$

where ∂_t denotes the derivative tangential to the world-sheet boundary. The only effect of this boundary action is that it modifies the boundary conditions to

$$G_{\mu\nu}\partial_n X^\nu + \mathrm{i}B_{\mu\nu}\partial_t X^\nu \Big|_{\partial\Sigma} = 0, \tag{2.4.38}$$

with ∂_n denoting the normal derivative. For $B = 0$ these are simply Neumann boundary conditions, whereas for large B (or $g \to 0$) the boundary conditions become Dirichlet. Thus (2.4.38) interpolates between these two cases. By conformally mapping the string world sheet to the upper half plane (we will only be concerned with the tree level approximation) and choosing complex coordinates the propagator consistent with the boundary conditions (2.4.38) is [51, 52]

$$\begin{aligned} < X^\mu(z,\bar z) X^\nu(w,\bar w) > = -\alpha' \Big[& g^{\mu\nu} \ln|z-w| - g^{\mu\nu} \ln|z-\bar w| \\ & + G^{\mu\nu} \ln|z-\bar w|^2 + \Theta^{\mu\nu} \ln\left(\frac{\bar w - z}{\bar z - w}\right) \Big], \end{aligned} \tag{2.4.39}$$

where the following quantities are introduced

$$\begin{aligned} G^{\mu\nu} &= \Big[\frac{1}{g - B^2}\Big]^{\mu\nu} \\ \Theta^{\mu\nu} &= \Big[\frac{B}{g - B^2}\Big]^{\mu\nu}. \end{aligned} \tag{2.4.40}$$

These quantities are to be understood as series in g and B. Note that $G^{\mu\nu}$ is symmetric and $\Theta^{\mu\nu}$ is antisymmetric. In fact these quantities have already appeared in a different context. The beta functions for the gauge field background (2.3.30) contain exactly these quantities generalized to a nonconstant B field and a possibly curved metric. Moreover the effect of the gauge field is taken into account by replacing the B field by the gauge invariant quantity $B+F$. We will keep this in mind, when we try to generalize the setting used in (2.4.36) to arbitrary backgrounds.

Restricting the propagator (2.4.39) to boundary values of z and w, i.e., $z = \bar{z} = \tau$ and $w = \bar{w} = \tau'$, one gets the propagator relevant for open string vertex operators

$$< X^\mu(\tau) X^\nu(\tau') >= -\alpha' G^{\mu\nu} \ln(\tau - \tau')^2 + \mathrm{i}\pi\alpha' \Theta^{\mu\nu} \epsilon(\tau - \tau'), \qquad (2.4.41)$$

with $\epsilon(\tau)$ denoting the sign function being 1 or -1 for positive or negative τ. This suggests a simple intuitive interpretation of the objects defined in (2.4.40), namely that $G_{\mu\nu}$ is the metric effectively seen by the open strings. This is justified by the way $G^{\mu\nu}$ appears in the boundary propagator.

The interpretation of $\Theta^{\mu\nu}$ becomes clear, when one computes the commutator interpreting τ as time

$$\begin{aligned} [X^\mu(\tau), X^\nu(\tau)] &= T(X^\mu(\tau), X^\nu(\tau^-) - X^\mu(\tau), X^\nu(\tau^+)) \\ &= \mathrm{i}\Theta^{\mu\nu}, \end{aligned} \qquad (2.4.42)$$

which is exactly the relation (2.4.35) for noncommutative coordinates! This suggests that we should be able to describe the theory in terms of a noncommutative deformed product defined on functions. Indeed this can be accomplished by taking the zero slope limit $\alpha' \to 0$ to decouple the string behavior, while keeping the open string parameters G and Θ fixed. In this limit one is left with a topological action for the bulk and the boundary degrees of freedom are governed by the boundary action (2.4.37). The product of functions is identified as the Moyal-Weyl product

$$f(x) * g(x) = e^{\frac{\mathrm{i}}{2}\Theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\mu}} f(y) g(z)\big|_{y=z=x}. \qquad (2.4.43)$$

An interesting thing happens when an abelian gauge field is added by coupling it to the boundary in the usual way. Due to the presence of divergences in the quantum field theory, the theory has to be regularized. Choosing a point splitting regularization on finds that the usual gauge transformation has to be modified to the gauge invariance of noncommutative Yang-Mills theory. On the other hand, if one would have chosen a Pauli-Villars regularization the ordinary gauge transformation would have been preserved. But ambiguities arising from different choices of regularization schemes should be related to field redefinitions in the effective action. This has led Seiberg and Witten [63] to propose a map from "ordinary" gauge theory to noncommutative gauge theory, which by now is well-known as the Seiberg-Witten map. The natural question arises if the effects of a more general (nonconstant) B-field can still be described in the elegant way of replacing ordinary products by a star product.

Open stings in general backgrounds

Physically the situation described in the previous subsection corresponds to the embedding of a flat brane into flat background. The first step towards a generalization of this situation is to allow for a varying B-field and field strength F of the boundary gauge field, but to demand that the field strength $H = dB$ should vanish. The physical picture in this situation is the embedding of a curved brane into a flat background. This situation is closely related to the problem of deformation quantization of Poisson manifolds. A typical example of a Poisson manifold is provided by a symplectic manifold, i.e., a differentiable manifold endowed with a nondegenerate closed two form. This two form is provided by the B field due to the vanishing of the field strength. It was shown by Kontsevich [64] that every finite dimensional Poisson manifold can be quantized in the sense of deformation quantization. Stated without mathematical rigour this means that there exists an isomorphism from equivalence classes of associative algebras (we think of them as the algebras of functions) to the equivalence classes of Poisson manifolds. This boils down to the problem of identifying an appropriate star product on the space of formal power series in a deformation parameter, suggestively denoted by $\hbar$, with coefficients in the space of smooth functions $C^\infty(M)$ on a differentiable manifold M. We denote this algebra with $\mathcal{A}[[\hbar]]$. Appropriate means in this context that the star product is associative and reduces to

$$f * g = fg + \frac{\mathrm{i}\hbar}{2}\{f, g\} + O(\hbar^2), \tag{2.4.44}$$

where $\{f, g\}$ denotes the Poisson bracket on the manifold M. More generally a star product is defined in terms of bidifferential operators B_i, where the subscript i indicates the order in the deformation parameter $\hbar$. There is a natural gauge group acting on star products, which consists of automorphisms of the algebra $\mathcal{A}[[\hbar]]$ of the form $f \to f + \sum_{n>0} \hbar^n D_n(f)$, where the D_n are differential operators. It is natural to consider star products up to this gauge equivalence. Kontsevich showed that every Poisson bracket comes from a canonically defined star product modulo equivalence. In doing so he took advantage of ideas from string theory. This was clarified by a work of Cattaneo and Felder [65], who showed that the formula given by Kontsevich can be interpreted in terms of the perturbative expansion of the path integral of a topological model of open strings.

From the sigma model point of view the symplectic case is similar to the constant case in the sense that locally one can choose Darboux coordinates. The algebra of functions on the D-brane world-volume is deformed to a noncommutative (but still associative) algebra in terms of the Kontsevich star product. In the field theory limit $\alpha' \to 0$ correlators can still be computed using the star product. So the structure obtained for the constant case persists for the more general symplectic case. But is this true for the more venturous case of a B field with nonvanishing field strength? This clearly corresponds to the embedding of a curved brane into a curved space-time. The open string sigma model with general background fields defines a highly nonlinear field theory. Thus one can hardly expect to get exact results. One can think of two conceptually rather different approaches to this problem. One is to look for certain controllable settings, for instance strings on group manifolds [66–69], most prominently on the group manifold of $SU(2)$ [70]. In this setting there exists an exact conformal field theory description for certain maximally symmetric branes on S^3, namely those wrapped on conjugacy classes of $SU(2)$, which are generically 2-spheres. The algebra of functions on the brane correspond to the well known "fuzzy-spheres". The exact form of the algebra depends on the size of the 3-spheres, i.e. the level of the corresponding WZW model, in which they are embedded. D-branes on the group manifold of $SU(2)$ have been studied intensively. In [71] and subsequent work [72, 73] it has been argued that the spherical branes are stabilized due to the interplay between the nontrivial B field and the quantized $U(1)$ world volume flux. An interesting feature present in the $SU(2)$ WZW model are the nonassociative deformations of the algebras of functions on the worldvolume at finite level k. In the limit where the level k is sent to infinity, i.e., when the background becomes flat (remember the level k is associated to the radius of the S^3), these algebras become associative. We will also find nonassociative algebras by taking a different route.

A rather different approach, though conceptually more straightforward, is to generalize the methods used in [60] to the situation of curved backgrounds by using a perturbative expansion [54, 78]. The starting point for these calculations is the open string sigma model with generic background for the space-time metric $g_{\mu\nu}(X)$ and the gauge potentials $B_{\mu\nu}(X)$ and $A_\mu(X)$. Then one employs the standard background field method [48, 74–77] to expand around the zero modes $X^\mu = x^\mu + \zeta^\mu$. This allows to expand the action into a free part and additional interaction terms. The propagator for the free field theory is then given by (2.4.39), which in turn can be used to perturbatively calculate correlation functions of the interacting theory. Carrying out these calculations one can read off a noncommutative and even nonassociative product from the correlators [54]

$$\begin{aligned} f(x) \circ g(x) &= f\, g - \frac{i}{2}\Theta^{\mu\nu}\, D_\mu f\, D_\nu g - \frac{1}{8}\Theta^{\mu\nu}\Theta^{\rho\sigma}\, D_\mu D_\rho f\, D_\nu D_\sigma g \\ &- \frac{1}{12}\Theta^{\mu\rho} D_\rho \Theta^{\nu\sigma}\left(D_\mu D_\nu f\, D_\sigma g + D_\sigma f\, D_\mu D_\nu g\right) \\ &+ O(\Theta^3), \end{aligned} \tag{2.4.45}$$

where Θ is essentially of the same form as in (2.4.40) with B replaced by the fully gauge invariant combination $\mathcal{F} = \mathcal{B} + \in\pi\alpha'\mathcal{F}$. The important difference is, however, that in this case Θ is *not* constant but depends on the zero modes x^μ. The product defined in (2.4.45) has the same structure as the formula given by Kontsevich, but the partial derivatives are replaced by covariant derivatives compatible with the metric $g_{\mu\nu}$ and most notably Θ does in general not define a Poisson structure.

The key properties of this product are that it is noncommutative and nonassociative, but inserted into an integral it becomes associative and enjoys a cyclic symmetry

$$\mathrm{d}^D x \quad \overline{g - \mathcal{F}} f_1 \circ \ldots \circ f_{n-1} \circ f_n \approx \ \mathrm{d}^D x \quad \overline{g - \mathcal{F}} f_n \circ f_1 \circ \ldots \circ f_{n-1}, \tag{2.4.46}$$

which is usually referred to as trace property. Here some discussion is in order. Both properties, associativity as well as the trace property (2.4.46) hold only in a certain sense, namely if the space-time background fields fulfill their equations of motion. In the approximation above (i.e., to second order in Θ) this means that we have to use the beta function for the background gauge field (2.3.30). By virtue of this equation and due to the contribution of the measure, the additional terms give a total divergence and thus the claimed properties indeed hold. At first sight this may seem to be an ad hoc assumption, but let us give some arguments that this is indeed a sensible result.

Both properties are closely related to conformal invariance. This is easily explained for the trace property. Take the world-sheet of the open string to be the disk. Open string vertex operators are inserted on the boundary and thus correlation functions have to be invariant under cyclic permutations of the operator insertions. In fact the correlators have to be invariant under the conformal Killing group of the disk, which is $SL(2,\mathbb{R})$. Nevertheless we cannot expect conformal invariance to hold, if we do not impose the restrictions on the space-time background fields arising from the beta functions. There are, however, some subtleties to be taken care of. First of all, if one insists to describe correlation functions in terms of the generalized star product (2.4.45) one has either to deal with the logarithmic divergences, which come from the $G^{\mu\nu}$ term of the boundary propagator (2.4.41), by an appropriate renormalization procedure or one has to consider a certain decoupling limit, similar to that of Seiberg and Witten. The second solution is definitely the less involved way, but it is not quite clear in which sense the beta functions should be interpreted in a field theory limit of $\alpha' \to 0$. On the other hand, studying the problem of renormalization in this context is an interesting question by itself. Thus we plan to investigate this topic in future work.

a ter 3

r riz i ls

3.1 The cohomological problem

After exploring the playground provided by string theory we turn to the hard facts of the BRST cohomological analysis of superstring models. This analysis will be carried out in the framework of the NSR formulation [31–33] with local (1,1) supersymmetry [34, 35] including an arbitrary number of abelian gauge fields. The class of models under consideration is quite general since it is characterized only by requirements on the field content and the gauge symmetries. The field content is given by the component fields of three types of supersymmetry multiplets: the 2d supergravity multiplet, 'matter multiplets' containing the 'target space coordinates', and abelian gauge field multiplets. The number of matter multiplets and gauge field multiplets is not fixed. Thus our results apply to any target space dimension (1,2, ...) and an arbitrary number (0,1, ...) of abelian world-sheet gauge fields. The supersymmetry transformations are obtained from an analysis of the Bianchi identities of 2d supergravity in presence of abelian gauge fields.

Before starting with the technical part let us summarize some basic facts about the BRST cohomology we are going to analyze. Here and throughout this thesis $H(s)$ denotes the cohomology of the BRST differential in the space of local functions, which neither depend explicitly on the world-sheet coordinates nor on the world-sheet differentials, but only on the fields, antifields and their derivatives. This cohomology is the most important one for the models under study because the other local BRST cohomology groups can be easily derived from it. This is due to the invariance of the models under world-sheet diffeomorphisms, owing to a general property of diffeomorphism invariant theories discussed in detail in sections 5 and 6 of [79] (see also [80–82]).

In particular, $H(s)$ yields directly the cohomology in form-degree 2 of s

modulo the "world-sheet exterior derivative" d. [1] This cohomology is the most relevant one for physical applications and denoted by $H^{g,2}(s|d)$, where g specifies the ghost number sector. Cocycles of $H^{g,2}(s|d)$ are denoted by $\omega^{g,2}$ and the cocycle condition is

$$s\omega^{g,2} + d\omega^{g+1,1} = 0, \tag{3.1.1}$$

where $\omega^{g+1,1}$ is some local 1-form with ghost number $g+1$. $\omega^{g,2}$ is a coboundary in $H^{g,2}(s|d)$ if $\omega^{g,2} = s\omega^{g-1,2} + d\omega^{g,1}$ for some local forms $\omega^{g-1,2}$ and $\omega^{g,1}$. $H^{g,2}(s|d)$ is related to $H(s)$ through the descent equations as explained in [79–82]. The physically interesting cohomology groups $H^{g,2}(s|d)$ are those with ghost numbers $g < 2$: $H^{-1,2}(s|d)$ yields the nontrivial Noether currents and global symmetries [83], $H^{0,2}(s|d)$ and $H^{1,2}(s|d)$ determine the consistent deformations [85], background charges [41] and candidate gauge anomalies (see, e.g., [86]). The corresponding cohomology groups of s are $H^g(s)$ with $g < 4$. These will be the objects of interest in the remainder of this thesis.

3.2 Field content and gauge symmetries

The field content of the models we are going to study is given by the supergravity multiplet consisting of the vielbein $e_m{}^a$, the gravitino χ_m^α and an auxiliary scalar field S.[2] Furthermore we consider a set of scalar multipets $\{X^M, \psi_\alpha^M, F^M\}$ corresponding to the string "target space coordinates" and their superpartners and a set of abelian gauge multiplets $\{A_m^i, \lambda_\alpha^i, \phi^i\}$. On Minkowskian world-sheets all fields are real and the fermions are Majorana-Weyl spinors. The number of scalar multiplets and gauge multiplets is not specified, i.e. our approach covers any number of such fields. As gauge symmetries we impose world-sheet diffeomorphisms, local $2d$ Lorentz transformations, Weyl and super-Weyl transformations and of course local (1,1) world-sheet supersymmetry. Furthermore we require invariance under abelian gauge transformations of the A_m^i and under arbitrary local shifts of the auxiliary field S. The gauge symmetries entail the corresponding ghost fields, which fixes the field content to

$$\Phi^A = \{e_m{}^a, \chi_m{}^\alpha, S, X^M, \psi_\alpha^M, F^M, A_m^i, \lambda_\alpha^i, \phi^i, \xi^m, \xi^\alpha, C^{ab}, C^W, \eta^\alpha, W, c^i\},$$

where ξ^m denote the world sheet diffeomorphism ghosts, ξ^α are the supersymmetry ghosts and C^{ab} is the Lorentz ghost. C^W and η^α are the Weyl and super-Weyl ghosts, respectively. c^i are the ghosts associated with the $U(1)$ transformations of the gauge fields and W denotes the ghost corresponding to the local shifts of the auxiliary field S. The gauge transformation of the

[1] Actually d is defined on the jet space of the fields and antifields [30].

[2] m, a, α denote $2d$ world-sheet, Lorentz and spinor indices, respectively.

supergravity multiplet written as BRST transformations are

$$
\begin{aligned}
se_m^{\ a} &= \xi^n\partial_n e_m^{\ a} + (\partial_m\xi^n)e_n^{\ a} - 2\mathrm{i}\xi^\alpha\chi_m^{\ \beta}(\gamma^a C)_{\alpha\beta} + C_b{}^a e_m^{\ b} + C^W e_m^{\ a} \\
s\chi_m^{\ \alpha} &= \xi^n\partial_n\chi_m^{\ \alpha} + (\partial_m\xi^n)\chi_n^{\ \alpha} + \nabla_m\xi^\alpha - \tfrac{1}{4}\,\xi^\beta e_m^{\ a} S(\gamma_a)_\beta{}^\alpha + \tfrac{1}{2}\,C^W\chi_m^{\ \alpha} \\
&\quad + \mathrm{i}\eta^\beta(\gamma_m)_\beta^{\ \alpha} - \tfrac{1}{4}\,C^{ab}\chi_m^{\ \beta}\varepsilon_{ab}(\gamma_*)_\beta^{\ \alpha} \\
sS &= \xi^n\partial_n S - C^W S + W \\
&\quad -4\xi^\gamma(\gamma_* C)_{\gamma\alpha}\varepsilon^{nm}\nabla_n\chi_m^{\ \alpha} + \mathrm{i}\,\xi^\gamma(\gamma^m C)_{\gamma\alpha}\chi_m^{\ \alpha} S,
\end{aligned}
\tag{3.2.2}
$$

where $C_{\alpha\beta}$ is the charge conjugation matrix satisfying $-(\gamma^a)^T = C^{-1}(\gamma^a)C$. γ_* is defined through $\gamma^a\gamma^b = \eta^{ab}\mathbb{1} + \varepsilon^{ab}\gamma_*$ and $\varepsilon^{01} = \varepsilon_{10} = 1$. ∇_m denotes the Lorentz covariant derivative

$$
\nabla_m = \partial_m - \tfrac{1}{2}\,\omega_m^{\ ab} l_{ab}
$$

in terms of the Lorentz generator l_{ab} and the spin connection

$$
\begin{aligned}
\omega_m^{\ ab} &= E^{an}E^{bk}(\omega_{[mn]k} - \omega_{[nk]m} + \omega_{[km]n}) \\
\omega_{[mn]k} &= e_{kd}\partial_{[n}e_{m]}^{\ d} - \mathrm{i}\chi_n\gamma_k\chi_m, \qquad E_a^{\ m}e_m^{\ b} = \delta_a^{\ b}.
\end{aligned}
\tag{3.2.3}
$$

The BRST transformations of the scalar multiplets read

$$
\begin{aligned}
sX^M &= \xi^m\partial_m X^M + \xi^\alpha\psi_\alpha^M \\
s\psi_\alpha^M &= \xi^m\partial_m\psi_\alpha^M - \mathrm{i}\xi^\beta(\gamma^m C)_{\beta\alpha}(\partial_m X^M - \chi_m^{\ \gamma}\psi_\gamma^M) + \xi^\beta C_{\beta\alpha}F^M \\
&\quad + \tfrac{1}{4}\,C^{ab}\varepsilon_{ab}(\gamma_*)_\alpha^{\ \beta}\psi_\beta^M - \tfrac{1}{2}\,C^W\psi_\alpha^M \\
sF^M &= \xi^m\partial_m F^M + \xi^\alpha(\gamma^m)_\alpha^{\ \beta}\{\nabla_m\psi_\beta^M + \mathrm{i}\chi_m^{\ \gamma}(\gamma^n C)_{\gamma\beta}(\partial_n X^M - \chi_n^{\ \delta}\psi_\delta^M) \\
&\quad -\chi_m^{\ \gamma}C_{\gamma\beta}F^M\} - C^W F^M.
\end{aligned}
\tag{3.2.4}
$$

The BRST transformations of the $U(1)$ multiplets are

$$
\begin{aligned}
s\phi^i &= \xi^n\partial_n\phi^i + \xi^\alpha(\gamma_*)_\alpha^{\ \beta}\lambda_\beta^i - C^W\phi^i \\
s\lambda_\beta^i &= \xi^n\partial_n\lambda_\beta^i + \xi^\alpha\big(\mathrm{i}(\gamma_* C)_{\alpha\beta}\varepsilon^{mn}(\partial_m A_n^i + \chi_m\gamma_n\lambda^i - \mathrm{i}\chi_n\gamma_* C\chi_m\phi^i) \\
&\quad -\mathrm{i}(\gamma_*\gamma^m C)_{\alpha\beta}(\partial_m\phi^i - \chi_m\gamma_*\lambda^i) + \mathrm{i}(\gamma_* C)_{\alpha\beta}S\phi^i\big) \\
&\quad + \tfrac{1}{4}\,C^{ab}\varepsilon_{ab}(\gamma_*)_\beta^{\ \gamma}\lambda_\gamma^i + 2\eta^\alpha(\gamma_* C)_{\alpha\beta}\phi^i - \tfrac{3}{2}\,C^W\lambda_\beta^i \\
sA_m^i &= \xi^n\partial_n A_m^i + (\partial_m\xi^n)A_n^i + \partial_m c^i \\
&\quad -2\mathrm{i}\,\xi^\alpha\chi_m^{\ \beta}(\gamma_* C)_{\beta\alpha}\phi^i - \xi^\alpha(\gamma_m)_\alpha^{\ \beta}\lambda_\beta^i.
\end{aligned}
\tag{3.2.5}
$$

These transformations were obtained by analyzing the $2d$ supergravity algebra in presence of the scalar matter and gauge multiplets [91] analogously to the superspace analysis of [92]. A short summary of the analysis is given in appendix B. In the supergravity sector we used the constraints

$$
T_{\alpha\beta}{}^a = 2\mathrm{i}(\gamma^a C)_{\alpha\beta}, \quad T_{ab}{}^c = T_{\alpha\beta}{}^\gamma = 0 \tag{3.2.6}
$$

and in the $U(1)$ sector

$$F^i_{\alpha\beta} = 2\mathrm{i}(\gamma_* C)_{\alpha\beta}\phi^i. \tag{3.2.7}$$

All constraints are conventional, i.e., can be achieved by redefinitions of the connections. The transformations of the ghosts are such that the BRST differential s squares to zero,

$$\begin{aligned}
s\xi^n &= \xi^m\partial_m\xi^n + \mathrm{i}\xi^\alpha\xi^\beta(\gamma^n C)_{\alpha\beta} \\
s\xi^\alpha &= \xi^n\partial_n\xi^\alpha - \mathrm{i}\xi^\gamma\xi^\beta(\gamma^m C)_{\beta\gamma}\chi_m{}^\alpha - \tfrac{1}{4}\,C^{ab}\xi^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha + \tfrac{1}{2}\,C^W\xi^\alpha \\
sC^{ab} &= \xi^m\partial_m C^{ab} - \tfrac{\mathrm{i}}{4}\,\xi^\alpha\xi^\beta S(\gamma_* C)_{\alpha\beta}\varepsilon^{ab} - \mathrm{i}\xi^\alpha\xi^\beta(\gamma^m C)_{\beta\alpha}\omega_m{}^{ab} \\
&\quad -2\eta^\beta\xi^\alpha(\gamma_* C)_{\alpha\beta}\varepsilon^{ab} \\
sC^W &= \xi^n\partial_n C^W + 2\eta^\beta\xi_\beta \\
s\eta^\alpha &= \xi^n\partial_n\eta^\alpha - \tfrac{1}{4}\,C^{ab}\eta^\beta\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha + \mathrm{i}\xi^\beta(\gamma^n)_\beta{}^\alpha\left(\tfrac{1}{2}\,\partial_n C^W - \eta^\gamma(\chi_n C)_\gamma\right) \\
&\quad -\tfrac{1}{2}\,C^W\eta^\alpha + \xi^\alpha W \\
sW &= \xi^n\partial_n W - 4\mathrm{i}\xi^\beta(\gamma^m C)_{\beta\alpha}\left(\nabla_m\eta^\alpha - \tfrac{1}{4}\,\chi_m{}^\alpha W - \tfrac{i}{2}\,\chi_m{}^\gamma(\gamma^n)_\gamma{}^\alpha(\partial_n C^W)\right) \\
&\quad -4\xi^\beta\chi_m{}^\alpha(\gamma^m\gamma^n C)_{\alpha\beta}\eta^\gamma(\chi_n C)_\gamma - C^W W \\
sc^i &= \xi^m\partial_m c^i + \mathrm{i}\xi^\alpha\xi^\beta(\gamma_* C)_{\alpha\beta}\phi^i - \mathrm{i}\xi^\alpha\xi^\beta(\gamma^m C)_{\alpha\beta}A^i_m.
\end{aligned} \tag{3.2.8}$$

We remark that the use of Weyl, super-Weyl and Lorentz transformations, as well as the shift symmetry associated with the auxiliary field S are artefacts of the formulation and disappear in an equivalent formulation based on a Beltrami parametrization of the world-sheet zweibein (see sections 3.3 and 4). Of course we could have used the Beltrami approach from the very beginning, but we decided to start from the more familiar formulation presented above.

3.3 Superconformal tensor calculus

The first part of our cohomological analysis consists in the construction of a suitable "basis" for the fields and their derivatives (more precisely: suitable coordinates of the jet space associated with the fields). The goal is to find a basis $\{u^\ell, v^\ell, w^I\}$ with as many s-doublets (u^ℓ, v^ℓ) as possible and complementary (local) variables w^I such that sw^I can be expressed solely in terms of the w's, i.e.,

$$su^\ell = v^\ell, \quad sw^I = r^I(w). \tag{3.3.9}$$

On general grounds, such a basis is related to a tensor calculus [82, 93, 94]. In the present case the tensor calculus is a superconformal one, generalizing the conformal tensor calculus in bosonic string models found in [40] (see also [39]). The w's with ghost number 1 are specific ghost variables corresponding to the superconformal algebra, the w's with ghost number 0 are "superconformal tensor fields" on which this algebra is represented.

3.3.1 Super-Beltrami parametrization

The superconformal structure of the models under consideration is related to the supersymmetric generalization of the so-called Beltrami parametrization [95, 96]. Beltrami differentials parametrize conformal classes of $2d$ metrics, and this makes them natural quantities to be used as basic variables in the present context. Since Beltrami differentials change only under worldsheet reparametrizations but not under Weyl or Lorentz transformations, their use leads to a simpler formulation of the models under study (cf. remarks at the end of section 3.2, and in section 4). In the following we choose a Euclidean notation and parametrize the worldsheet with independent variables z and $\bar{z}$ rather than with light cone coordinates, because this simplifies the notation and avoids some factors of i.[3]

As it is not hard to guess the supersymmetric generalization of the Beltrami parametrization involves in addition to the bosonic Beltrami differential μ a fermionic partner α, the Beltramino. The starting point is the parametrization of the vielbein

$$\begin{aligned} e^z &= (dz + d\bar{z}\mu_{\bar{z}}{}^z)e_z{}^z \\ e^{\bar{z}} &= (d\bar{z} + dz\mu_z{}^{\bar{z}})e_{\bar{z}}{}^{\bar{z}}. \end{aligned} \tag{3.3.10}$$

The coefficients $\mu_{\bar{z}}{}^z$ and $\mu_z{}^{\bar{z}}$ are the Beltrami differentials

$$\begin{aligned} \mu &:= \mu_{\bar{z}}{}^z = \frac{e_{\bar{z}}{}^z}{e_z{}^z}, \\ \bar{\mu} &:= \mu_z{}^{\bar{z}} = \frac{e_z{}^{\bar{z}}}{e_{\bar{z}}{}^{\bar{z}}}, \end{aligned} \tag{3.3.11}$$

whereas the factors $e_z{}^z$ and $e_{\bar{z}}{}^{\bar{z}}$ are referred to as conformal factors. One should note that the Beltrami differentials transform under diffeomorphisms but do not change under Weyl or Lorentz transformations. The latter "structure group transformations" are carried solely by the conformal factors which form s-doublets (u^ℓ, v^ℓ) with ghost variables substituting (in the new basis) for the Lorentz ghost and the Weyl ghost.

The fermionic superpartners of the Beltrami differentials are suitable combinations of the gravitino fields

$$\begin{aligned} \alpha &:= \sqrt{\frac{8}{e_z{}^z}}\left(\chi_{\bar{z}}{}^2 - \mu\chi_z{}^2\right) \\ \bar{\alpha} &:= \sqrt{\frac{8}{e_{\bar{z}}{}^{\bar{z}}}}\left(\chi_z{}^1 - \bar{\mu}\chi_{\bar{z}}{}^1\right). \end{aligned} \tag{3.3.12}$$

The Beltraminos are also invariant under structure group transformations. Especially they do not change under super-Weyl transformations. Again one

[3] Note that reality conditions of spinors are subtle after Wick rotation to Euclidean space: In our left-right symmetric case of (1,1) supersymmetry we could define $(\psi)^* = \bar{\psi}$ and work with manifestly real actions, but obviously this would not be possible for heterotic theories. This is, however, irrelevant in our algebraic context.

can find complementary combinations of the gravitinos forming s-doublets with ghost variables that substitute for the super-Weyl ghosts. The fact that Weyl, Lorentz and super-Weyl ghosts (and not just their derivatives) occur in s-doublets as we just described reflects that Weyl, Lorentz and super-Weyl invariance are artefacts of the formulation.

The Beltrami parametrization involves also a redefinition of the diffeomorphism ghosts, sometimes called the Beltrami ghost fields. This again has to be supplemented with a redefinition of the supersymmetry ghosts. The new ghost variables, which replace the diffeomorphism ghosts ξ^z and $\xi^{\bar z}$ and the supersymmetry ghosts ξ^1 and ξ^2 are

$$\begin{aligned} \eta &:= (\xi^z + \mu\xi^{\bar z}) \\ \bar\eta &:= (\xi^{\bar z} + \bar\mu\xi^z) \\ \varepsilon &:= \tfrac{1}{2}\,(\hat\xi^2 + \xi^{\bar z}\alpha), \qquad \hat\xi^2 := \overline{\tfrac{8}{e_z{}^z}}\,\xi^2 \\ \bar\varepsilon &:= \tfrac{1}{2}\,(\hat\xi^1 + \xi^{z}\bar\alpha), \qquad \hat\xi^1 := \overline{\tfrac{8}{e_{\bar z}{}^{\bar z}}}\,\xi^1 \end{aligned} \tag{3.3.13}$$

In terms of the new ghost variables the BRST transformations of "right-moving" and "left-moving" quantities decouple from each other [95],

$$\begin{aligned} s\mu &= \big(\bar\partial - \mu\partial + (\partial\mu)\big)\,\eta + \alpha\varepsilon \\ s\alpha &= \big(2\bar\partial - 2\mu\partial + (\partial\mu)\big)\,\varepsilon + \eta\partial\alpha + \tfrac{1}{2}\,\alpha\partial\eta \\ s\eta &= \eta\partial\eta - \varepsilon\varepsilon \\ s\varepsilon &= \eta\partial\varepsilon - \tfrac{1}{2}\,\varepsilon\partial\eta, \end{aligned} \tag{3.3.14}$$

with analogous transformations for the right movers.

3.3.2 Superconformal ghost variables and algebra

We have now paved the road for the construction of field variables $\{u^\ell, v^\ell, w^I\}$ fulfilling (3.3.9). In fact we have already identified some s-doublets (u^ℓ, v^ℓ), namely the u's given by the conformal factors and their fermionic counterparts and the corresponding v's given by ghost fields substituting in the new basis for the Weyl, Lorentz and super-Weyl ghosts. Furthermore, the field S obviously forms an s-doublet with a ghost field substituting for W. The derivatives of these u's and v's form s-doublets as well. The Beltrami differentials $\mu, \bar\mu$ and their derivatives are u's too. From (3.3.14) one observes that $s\mu$ and $s\bar\mu$ contain derivatives $\bar\partial\eta$ and $\partial\bar\eta$ and of the reparametrization ghosts, respectively. Taking derivatives of these transformations, one sees that the m-th derivatives of the Beltrami differentials pair off with ghost variables that substitute in the new basis for all $(m+1)$-th derivatives of the reparametrization ghosts except for $\partial^{m+1}\eta$ and $\bar\partial^{m+1}\bar\eta$. Analogously, the s-transformations of the Beltraminos contain derivatives $\bar\partial\varepsilon$ and $\partial\bar\varepsilon$ of the supersymmetry ghosts. Thus the m-th derivatives of α and $\bar\alpha$ pair off with ghost variables substituting for all $(m+1)$-th derivatives of ϵ and $\bar{\epsilon}$ except for $\partial^{m+1}\varepsilon$ and $\bar{\partial}^{m+1}\bar{\varepsilon}$. We introduce the following notation for those ghost variables which do not sit in s-doublets:

$$\{C^N\} = \{\eta^p, \bar{\eta}^p, \varepsilon^{p+\frac{1}{2}}, \bar{\varepsilon}^{p+\frac{1}{2}} : p = -1, 0, 1, \dots\}, \tag{3.3.15}$$

with

$$\begin{aligned}
\eta^p &= \frac{1}{(p+1)!}\,\partial^{p+1}\eta \\
\bar{\eta}^p &= \frac{1}{(p+1)!}\,\bar{\partial}^{p+1}\bar{\eta} \\
\varepsilon^{p+\frac{1}{2}} &= \frac{1}{(p+1)!}\,\partial^{p+1}\varepsilon \\
\bar{\varepsilon}^{p+\frac{1}{2}} &= \frac{1}{(p+1)!}\,\bar{\partial}^{p+1}\bar{\varepsilon}.
\end{aligned} \tag{3.3.16}$$

These ghost variables fulfill the requirement imposed in (3.3.9) on w's. Indeed, using (3.3.14), one easily computes their s-transformations:

$$\begin{aligned}
s\eta^p &= -\frac{1}{2}\eta^q\eta^r f_{rq}{}^p + \frac{1}{2}\varepsilon^a\varepsilon^b f_{ab}{}^p \\
&= \frac{1}{2}\eta^q\eta^r(r-q)\delta^p_{r+q} - \frac{1}{2}\varepsilon^a\varepsilon^b 2\delta^p_{a+b}
\end{aligned} \tag{3.3.17}$$

$$\begin{aligned}
s\varepsilon^a &= -\frac{1}{2}\eta^p\varepsilon^c f_{cp}{}^a + \frac{1}{2}\varepsilon^c\eta^p f_{pc}{}^a \\
&= -\varepsilon^c\eta^p\left(\frac{p}{2} - c\right)\delta^a_{p+c}.
\end{aligned} \tag{3.3.18}$$

The f's which occur in these transformations are the structure constants of a graded commutator algebra of operators Δ_N to be represented on tensor fields constructed of the component fields of the matter and $U(1)$ multiplets,

$$\{\Delta_N\} = \{L_p, \bar{L}_p, G_{p+\frac{1}{2}}, \bar{G}_{p+\frac{1}{2}} : p = -1, 0, 1, \dots\}. \tag{3.3.19}$$

This graded commutator algebra is nothing but the NS superconformal algebra

$$[L_p, L_q] = (p-q)L_{p+q}, \qquad \{G_a, G_b\} = 2L_{a+b}, \qquad [L_p, G_a] = \left(\frac{p}{2} - a\right)G_{p+a}, \tag{3.3.20}$$

with the analogous formulas for the $\bar{L}$'s and $\bar{G}$'s and the usual property that the holomorphic and antiholomorphic generators (anti-)commute,

$$\begin{aligned}
[L_p, \bar{L}_q] &= 0, \qquad & \{G_a, \bar{G}_b\} &= 0, \\
[L_p, \bar{G}_a] &= 0, \qquad & [\bar{L}_p, G_a] &= 0.
\end{aligned}$$

The representation of this algebra on superconformal tensor fields, and the explicit construction of these tensor fields, will be given in the following subsection.

3.3.3 Superconformal tensor fields

We shall now summarize the representation of the algebra (3.3.20) on superconformal tensor fields constructed of the fields and their derivatives (the representation on antifields is discussed in section 5) such that the BRST transformation of these tensor fields reads[4]

$$s\mathcal{T} = \sum_{p\geq -1} \left(\eta^p L_p + \bar{\eta}^p \bar{L}_p + \varepsilon^{p+\frac{1}{2}}\, G_{p+\frac{1}{2}} + \bar{\varepsilon}^{p+\frac{1}{2}}\, \bar{G}_{p+\frac{1}{2}} \right) \mathcal{T}. \tag{3.3.21}$$

The superconformal tensor fields corresponding to the fields X^M, ψ^M_α, F^M and their derivatives are denoted by $X^M_{m,n}$, $\psi^M_{m,n}$, $\bar{\psi}^M_{m,n}$, $F^M_{m,n}$, where the subscripts take the values $(m,n \in \{0,1,2,\dots\})$ and denote the number of operations L_{-1} and $\bar{L}_{-1}$ acting on $X^M_{0,0}$, $\psi^M_{0,0}$, $\bar{\psi}^M_{0,0}$, $F^M_{0,0}$, respectively. L_{-1} and $\bar{L}_{-1}$ will be identified with covariant derivatives (see below),

$$\begin{gathered} X^M_{0,0} \equiv X^M,\ \psi^M_{0,0} \equiv (e_z{}^z/2)^{\frac{1}{2}}\, \psi^M_2,\ \bar{\psi}^M_{0,0} \equiv (e_{\bar{z}}{}^{\bar{z}}/2)^{\frac{1}{2}}\, \psi^M_1, \\ F^M_{0,0} \equiv \tfrac{1}{2}\, (e_z{}^z)^{\frac{1}{2}}\, (e_{\bar{z}}{}^{\bar{z}})^{\frac{1}{2}}\, F^M, \\ X^M_{m,n} = (L_{-1})^m (\bar{L}_{-1})^n X^M_{0,0} \quad (m,n \in \{0,1,2,\dots\}) \quad \text{etc.} \end{gathered}$$

The representation on these tensor fields can be inductively deduced from the algebra (3.3.20) using that all operations L_m, $\bar{L}_m$, G_a, $\bar{G}_a$ vanish on $X^M_{0,0}$ except for L_{-1}, $\bar{L}_{-1}$, $G_{-1/2}$ and $\bar{G}_{-1/2}$, with $G_{-1/2} X^M_{0,0} = \psi^M_{0,0}$ and $\bar{G}_{-1/2} X^M_{0,0} = \bar{\psi}^M_{0,0}$ (as can be read off from sX^M). This gives on $X^M_{m,n}$:

$$\begin{aligned} L_p X^M_{m,n} &= \begin{cases} \frac{m!}{(m-p-1)!} X^M_{m-p,n} & \text{for } p < m \\ 0 & \text{for } p \geq m \end{cases} \\ \bar{L}_q X^M_{m,n} &= \begin{cases} \frac{n!}{(n-q-1)!} X^M_{m,n-q} & \text{for } q < n \\ 0 & \text{for } q \geq n \end{cases} \\ G_{p+\frac{1}{2}}\, X^M_{m,n} &= \begin{cases} \frac{m!}{(m-p-1)!} \psi^M_{m-p-1,n} & \text{for } p < m \\ 0 & \text{for } p \geq m \end{cases} \\ \bar{G}_{q+\frac{1}{2}}\, X^M_{m,n} &= \begin{cases} \frac{n!}{(n-q-1)!} \bar{\psi}^M_{m,n-q-1} & \text{for } q < n \\ 0 & \text{for } q \geq n \end{cases} \end{aligned}$$

The action on the other fields is then easily obtained using

$$[L_p, G_{-\frac{1}{2}}] = \tfrac{1}{2}\, (p+1)\, G_{p-\frac{1}{2}}\,, \qquad \{G_{p+\frac{1}{2}}, G_{-\frac{1}{2}}\} = 2L_p$$

[4] $\mathcal{T}$ stands for any of these superconformal tensor fields; η's and ε's are the ghost variables (3.3.16).

and the analogous formulas for $\bar{L}$ and $\bar{G}$ in (3.3.20). One obtains

$$
\begin{aligned}
L_p\psi^M_{m,n} &= \begin{cases} \frac{m!}{(m-p)!}\left(m-p+\frac{1}{2}(p+1)\right)\psi^M_{m-p,n} & \text{for } p\leq m \\ 0 & \text{for } p>m \end{cases} \\
G_{p+\frac{1}{2}}\,\psi^M_{m,n} &= \begin{cases} \frac{m!}{(m-p-1)!}X^M_{m-p,n} & \text{for } p<m \\ 0 & \text{for } p\geq m \end{cases} \\
\bar{G}_{q+\frac{1}{2}}\,\psi^M_{m,n} &= \begin{cases} -\frac{n!}{(n-q-1)!}F^M_{m,n-q-1} & \text{for } q<n \\ 0 & \text{for } q\geq n \end{cases} \\
\bar{L}_q\psi^M_{m,n} &= \begin{cases} \frac{n!}{(n-q-1)!}\psi^M_{m,n-q} & \text{for } q<n \\ 0 & \text{for } q\geq n \end{cases} \\
L_pF^M_{m,n} &= \begin{cases} \frac{m!}{(m-p)!}\left(m-p+\frac{1}{2}(p+1)\right)F^M_{m-p,n} & \text{for } p\leq m \\ 0 & \text{for } p>m \end{cases} \\
G_{p+\frac{1}{2}}\,F^M_{m,n} &= \begin{cases} \frac{m!}{(m-p-1)!}\,\bar{\psi}^M_{m-p,n} & \text{for } p<m \\ 0 & \text{for } p\geq m \end{cases}
\end{aligned}
$$

and analogous formulas for L's, G's, $\bar{L}$'s and $\bar{G}$'s acting on $\bar{\psi}^M_{m,n}$, and $\bar{L}$'s and $\bar{G}$'s acting on $F^M_{m,n}$.

The relation to the fields and their derivatives is established by identifying the operations L_{-1} and $\bar{L}_{-1}$ with covariant derivatives $\mathcal{D}$ and $\bar{\mathcal{D}}$ along the lines of [82],

$$
\begin{aligned}
L_{-1}\equiv\mathcal{D} &= \frac{1}{1-\mu\bar{\mu}}\Big[\partial-\bar{\mu}\bar{\partial}-\sum_{p\geq 0}(\bar{M}^p\bar{L}_p-\bar{\mu}M^pL_p) \\
&\qquad -\sum_a(\bar{A}^a\bar{G}_a-\bar{\mu}A^aG_a)\Big] \\
\bar{L}_{-1}\equiv\bar{\mathcal{D}} &= \frac{1}{1-\mu\bar{\mu}}\Big[\bar{\partial}-\mu\partial-\sum_{p\geq 0}(M^pL_p-\mu\bar{M}^p\bar{L}_p) \\
&\qquad -\sum_a(A^aG_a-\mu\bar{A}^a\bar{G}_a)\Big] \qquad (3.3.22)
\end{aligned}
$$

where

$$
\begin{aligned}
M^p &= \tfrac{1}{(p+1)!}\,\partial^{p+1}\mu, \quad \bar{M}^p = \tfrac{1}{(p+1)!}\,\bar{\partial}^{p+1}\bar{\mu}, \\
A^{p+\frac{1}{2}} &= \tfrac{1}{(p+1)!2}\,\partial^{p+1}\alpha, \quad \bar{A}^{p+\frac{1}{2}} = \tfrac{1}{(p+1)!2}\,\bar{\partial}^{p+1}\bar{\alpha}.
\end{aligned}
$$

One readily checks that these formulas result in local expressions for the superconformal tensor fields and their s-transformations. Introducing the following notation for the lowest weight superconformal matter fields

$$X^M\equiv X^M_{0,0}\,,\quad \psi^M\equiv\psi^M_{0,0}\,,\quad \bar{\psi}^M\equiv\bar{\psi}^M_{0,0}\,,\quad \hat{F}^M\equiv F^M_{0,0}\,, \qquad (3.3.23)$$

one gets in particular the following supercovariant derivatives

$$\begin{aligned}
\mathcal{D}X^M &= \frac{1}{1-\mu\bar{\mu}}\Big[(\partial-\bar{\mu}\bar{\partial})X^M - \tfrac{1}{2}\,\bar{\alpha}\bar{\psi}^M + \tfrac{1}{2}\,\bar{\mu}\alpha\psi^M\Big] \\
\mathcal{D}\psi^M &= \frac{1}{1-\mu\bar{\mu}}\Big[(\partial-\bar{\mu}\bar{\partial})\psi^M + \tfrac{1}{2}\,\bar{\mu}(\partial\mu)\psi^M + \tfrac{1}{2}\,\bar{\alpha}\hat{F}^M + \tfrac{1}{2}\,\bar{\mu}\alpha\mathcal{D}X^M\Big] \\
\bar{\mathcal{D}}\psi^M &= \frac{1}{1-\mu\bar{\mu}}\Big[(\bar{\partial}-\mu\partial)\psi^M - \tfrac{1}{2}\,(\partial\mu)\psi^M - \tfrac{1}{2}\,\alpha\mathcal{D}X^M - \tfrac{1}{2}\,\mu\bar{\alpha}\hat{F}^M\Big]
\end{aligned} \tag{3.3.24}$$

and analogous expressions for $\bar{\mathcal{D}}X^M$, $\bar{\mathcal{D}}\bar{\psi}^M$ and $\mathcal{D}\bar{\psi}^M$. We do not spell out higher order covariant derivatives explicitly because it turns out that they do not contribute nontrivially to the cohomology. The BRST transformations of the superconformal tensor fields are summarized in appendix C.1.

The construction of the superconformal tensor fields arising from the gauge multiplets is similar, once one has identified the suitable ghost variables and the lowest order tensor fields. The gauge fields A_m^i and their symmetrized derivatives $\partial_{(m_1}\dots\partial_{m_k}A^i_{m_{k+1})}$ $(k=1,2,\dots)$ form s-doublets with ghost variables that substitute for all the derivatives of the ghosts c^i. Therefore one expects that only the undifferentiated ghosts c^i give rise to w-variables. Promising candidates for these w-variables are ghost variables C^i of the same form as in the purely bosonic case [39],

$$C^i = c^i + \xi^m A^i_m\ . \tag{3.3.25}$$

The s-transformations of the gauge fields, written in terms of C^i, and of the C^i themselves read

$$\begin{aligned}
sA^i_m &= \xi^n(\partial_n A^i_m - \partial_m A^i_n) + \partial_m C^i - \xi^\alpha \chi_m{}^\beta F^i_{\alpha\beta} - \xi^\alpha e_m{}^a F^i_{a\alpha} \\
sC^i &= \xi^m\xi^n(\partial_m A^i_n - \partial_n A^i_m) + \tfrac{1}{2}\,\xi^\alpha\xi^\beta F^i_{\alpha\beta} \\
&\quad + \xi^m\xi^\alpha\chi_m{}^\beta F^i_{\alpha\beta} + \xi^m\xi^\alpha F^i_{m\alpha}
\end{aligned} \tag{3.3.26}$$

where we used notation of appendix B. Since we expect C^i to count among the w's, its s-transformation should involve only w's again, see (3.3.9). This suggests a strategy to determine the superconformal tensor fields corresponding to the undifferentiated fields ϕ^i, λ^i_α and to the field strengths of A^i_m: one tries to rewrite sC^i in (3.3.26) in terms of the ghost variables (3.3.16) and to read off from the result the sought superconformal tensor fields. This strategy turns out to be successful; one obtains

$$sC^i = \eta\bar{\eta}F^i_{0,0} + \eta\bar{\varepsilon}\lambda^i_{0,0} + \bar{\eta}\varepsilon\bar{\lambda}^i_{0,0} + \varepsilon\bar{\varepsilon}\phi^i_{0,0}$$

where

$$\begin{aligned}
\phi^i_{0,0} &= \overline{e_z{}^z e_{\bar z}{}^{\bar z}}\,\phi^i \\
\lambda^i_{0,0} &= \overline{\tfrac{e_{\bar z}{}^{\bar z}}{2}}\,\left(-e_z{}^z \lambda^i_2 + \chi^2_z \phi^i\right) \\
\bar\lambda^i_{0,0} &= \overline{\tfrac{e_z{}^z}{2}}\,\left(e_{\bar z}{}^{\bar z} \lambda^i_1 + \chi^1_{\bar z} \phi^i\right) \\
F^i_{0,0} &= \frac{1}{1-\mu\bar\mu}\big(\tfrac{1}{2}\,\varepsilon^{mn}(\partial_m A^i_n - \partial_n A^i_m) \\
&\qquad + \tfrac{1}{2}\,\mu\bar\alpha\lambda^i - \tfrac{1}{2}\,\bar\mu\alpha\bar\lambda^i - \tfrac{1}{4}\,\alpha\bar\alpha\phi^i\big).
\end{aligned} \tag{3.3.27}$$

An explicit computation shows that the s-transformations of these quantities are indeed of the desired form (3.3.21), with

$$\lambda^i_{0,0} = G_{-\frac{1}{2}}\,\phi^i_{0,0}, \qquad \bar\lambda^i = \bar G_{-\frac{1}{2}}\,\phi^i_{0,0}, \qquad F^i_{0,0} = \bar G_{-\frac{1}{2}}\,G_{-\frac{1}{2}}\,\phi^i_{0,0}. \tag{3.3.28}$$

It is now straightforward to construct, along the previous lines, variables $\phi^i_{m,n}$, $\lambda^i_{m,n}$, $\bar\lambda^i_{m,n}$, $F^i_{m,n}$ on which the algebra (3.3.20) is represented and (3.3.21) and (3.3.22) hold. We do not spell out these tensor fields (with m or n different from 0) explicitly because it turns out that they do not contribute nontrivially to the cohomology. The resulting BRST transformations are summarized in appendix C.1 too.

We introduce the following notation for the lowest order (i.e. lowest weight, see below) superconformal tensor fields arising from the gauge multiplet:

$$\hat\phi^i \equiv \phi^i_{0,0}\,, \quad \lambda^i \equiv \lambda^i_{0,0}\,, \quad \bar\lambda^i \equiv \bar\lambda^i_{0,0}\,, \quad \mathcal{F}^i \equiv F^i_{0,0}\,. \tag{3.3.29}$$

Again tensor fields of higher order will be denoted by $\mathcal{D}\hat\phi^i$, $\bar{\mathcal{D}}\hat\phi^i$, $\mathcal{D}\bar{\mathcal{D}}\hat\phi^i$ etc. but as already stated above their explicit form will not be needed.

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We shall now determine the most general action for the field content and gauge transformations specified in section 3.2. The action has vanishing ghost number and is independent of antifields. Furthermore the requirement that the action be gauge invariant translates into BRST invariance up to surface terms. The integrands of the world-sheet actions we are looking for are thus the antifield independent solutions $\omega^{0,2}$ of equation (3.1.1). They are related through the descent equations to the solutions of

$$\begin{gathered} s\omega = 0, \quad \omega \neq s\hat{\omega}, \\ \mathrm{gh}\,(\omega) = 2, \quad \mathrm{agh}\,(\omega) = \mathrm{agh}\,(\hat{\omega}) = 0 \end{gathered} \tag{4.0.1}$$

where gh is the ghost number and agh is the antifield number (="antighost number", see section 5 for the definition). In the previous section we have constructed a basis for the fields and their derivatives satisfying the requirements of (3.3.9). By standard arguments this implies that ω and $\hat{\omega}$ can be assumed to depend only on the w^I, i.e., on superconformal tensor and ghost fields introduced in section 3.3.[1] Furthermore we can restrict the investigation to functions ω and $\hat{\omega}$ with vanishing "conformal weights" by an argument used already in [39, 40]: we extend the definition of L_0 and $\bar{L}_0$ to all w's (including the ghost variables) by

$$\Big\{s\,,\,\frac{\partial}{\partial(\partial\eta)}\Big\}w^I = L_0\,w^I\;,\;\Big\{s\,,\,\frac{\partial}{\partial(\bar{\partial}\bar{\eta})}\Big\}w^I = \bar{L}_0\,w^I\;. \tag{4.0.2}$$

Hence, in the space of local functions of the w's the derivatives with respect to $\partial\eta$ and $\bar{\partial}\bar{\eta}$ are contracting homotopies for L_0 and $\bar{L}_0$, respectively, and the cohomology can be nontrivial only in the intersection of the kernels of L_0 and $\bar{L}_0$.

[1] The u's and v's contribute only "topologically" via the de Rham cohomology of the zweibein manifold to the s-cohomology, cf. theorem 5.1 of [79]. In particular they do not contribute nontrivially to the solutions of (4.0.1).

All w's are eigenfunctions of L_0 and $\bar{L}_0$ with the eigenvalues being their "conformal weights". The only w^I with negative conformal weights are the undifferentiated diffeomorphism ghosts $\eta, \bar{\eta}$ and the undifferentiated supersymmetry ghosts $\varepsilon, \bar{\varepsilon}$; their conformal weights are $(-1,0)$, $(0,-1)$, $(-1/2,0)$ and $(0,-1/2)$, respectively [here (a,b) are the eigenvalues of $(L_0, \bar{L}_0)$]. The only superconformal tensor fields with vanishing conformal weights are the undifferentiated X^M. These properties simplify the analysis enormously.

Our strategy for finding the solutions to (4.0.1) will be based on an expansion in supersymmetry ghosts

$$\begin{aligned} \omega &= \sum_{k=0}^{\bar{k}} \omega_k \ , \quad (N_\varepsilon + N_{\bar{\varepsilon}})\omega_k = k\omega_k \\ s &= s_2 + s_1 + s_0 \ , \quad [N_\varepsilon + N_{\bar{\varepsilon}}, s_k] = k s_k, \end{aligned} \tag{4.0.3}$$

where we have introduced the counting operator N_ε for the susy ghost ε and all its derivatives

$$N_\varepsilon = \sum_{n\geq 0} (\partial^n \varepsilon) \frac{\partial}{\partial(\partial^n \varepsilon)} \tag{4.0.4}$$

and analogously $N_{\bar{\varepsilon}}$ counts $\bar{\varepsilon}$ and derivatives thereof.[2] One observes that s_2 is the simplest piece in the above decomposition of s. It acts nontrivially only on the reparametrization ghosts η, $\bar{\eta}$, derivatives thereof and on C^i,

$$s_2\eta = -\varepsilon\varepsilon \ , \quad s_2\bar{\eta} = -\bar{\varepsilon}\bar{\varepsilon} \ , \quad s_2 C^i = \varepsilon\bar{\varepsilon}\hat{\phi}^i \ .$$

We shall base the investigation on the cohomology of s_2. The cocycle condition $s\omega = 0$ decomposes into

$$s_2\omega_{\bar{k}} = 0, \quad s_1\omega_{\bar{k}} + s_2\omega_{\bar{k}-1} = 0, \quad \ldots \tag{4.0.5}$$

Due to the requirement of ghost number 2 and antifield number 0 in (4.0.1), one is left with $0 \leq \bar{k} \leq 2$. The three possible values for $\bar{k}$ are now analysed case by case.

$\underline{\bar{k}=0}$: The general form of $\omega_{\bar{0}}$ according to the condition of vanishing conformal weight is

$$\begin{aligned} \omega_{\bar{0}} \ = \ & \eta\bar{\eta}A_{(1,1)} + \eta\partial\eta A_{(1,0)} + \bar{\eta}\bar{\partial}\bar{\eta}A_{(0,1)} + \eta\bar{\partial}\bar{\eta}B_{(1,0)} + \bar{\eta}\partial\eta B_{(0,1)} \\ & + \eta\partial^2\eta A_{(0,0)} + \bar{\eta}\bar{\partial}^2\bar{\eta}\bar{A}_{(0,0)} + \partial\eta\bar{\partial}\bar{\eta}B_{(0,0)} + C^iC^jD_{ij(0,0)} \\ & + \eta C^i D_{i(1,0)} + \bar{\eta}C^i D_{i(0,1)} + \partial\eta C^i D_{i(0,0)} + \bar{\partial}\bar{\eta}C^i\bar{D}_{i(0,0)}, \end{aligned}$$

[2] We note that the expansion (4.0.3) holds because we are studying the antifield independent cohomology here. The analogous expansion in presence of antifields is more involved; in fact, it can even involve infinitely many terms. Therefore the strategy applied here to determine the action is not practicable in the same way for analysing the full (antifield dependent) cohomology later.

where the A's, B's and D's do not depend on the ghosts and the subscripts (m,n) indicate their conformal weights. It is easy to verify explicitly that

$$s_2\omega_{\bar{0}} = 0 \;\Leftrightarrow\; \omega_{\bar{0}} = 0. \tag{4.0.6}$$

$\underline{\bar{k}=1.}$ The general form of $\omega_{\bar{1}}$ is

$$\begin{aligned}\omega_{\bar{1}} \;&=\; \eta\varepsilon A_{(3/2,0)} + \bar{\eta}\bar{\varepsilon}A_{(0,3/2)} + \eta\bar{\varepsilon}A_{(1,1/2)} + \bar{\eta}\varepsilon A_{(1/2,1)} \\ &\quad +\eta\partial\varepsilon A_{(1/2,0)} + \bar{\eta}\bar{\partial}\bar{\varepsilon}A_{(0,1/2)} + \varepsilon\partial\eta B_{(1/2,0)} + \bar{\varepsilon}\bar{\partial}\bar{\eta}B_{(0,1/2)} \\ &\quad +\varepsilon\bar{\partial}\bar{\eta}C_{(1/2,0)} + \bar{\varepsilon}\partial\eta C_{(0,1/2)} + \varepsilon C^i D_{i(1/2,0)} + \bar{\varepsilon}C^i D_{i(0,1/2)},\end{aligned}$$

where again the A's, B's and D's do not depend on the ghosts and their conformal weights are indicated in brackets. A straightforward computation shows that $s_2\omega_{\bar{1}} = 0$ imposes

$$\begin{aligned}A_{(3/2,0)} = A_{(0,3/2)} = C_{(1/2,0)} = C_{(0,1/2)} = 0 \\ A_{(1,1/2)} = \hat{\phi}^i D_{i(1/2,0)}\;, \quad A_{(1/2,1)} = \hat{\phi}^i D_{i(0,1/2)} \\ A_{(1/2,0)} = -2B_{(1/2,0)}\;, \quad A_{(0,1/2)} = -2B_{(0,1/2)}\end{aligned}$$

The conformal weights $(1/2,0)$ and $(0,1/2)$ imply

$$\begin{aligned}D_{i(1/2,0)} = \psi^M D_{Mi}(X),\; D_{i(0,1/2)} = \bar{\psi}^M \bar{D}_{Mi}(X) \\ B_{(1/2,0)} = \psi^M B_M(X),\; B_{(0,1/2)} = \bar{\psi}^M \bar{B}_M(X)\end{aligned}$$

where we indicated that the remaining B's and D's are arbitrary functions of the X's. Hence, we get

$$\begin{aligned}\omega_{\bar{1}} \;&=\; (\eta\bar{\varepsilon}\hat{\phi}^i + \varepsilon C^i)\psi^M D_{Mi}(X) + (\bar{\eta}\varepsilon\hat{\phi}^i + \bar{\varepsilon}C^i)\bar{\psi}^M \bar{D}_{Mi}(X) \\ &\quad +(\varepsilon\partial\eta - 2\eta\partial\varepsilon)\psi^M B_M(X) + (\bar{\varepsilon}\bar{\partial}\bar{\eta} - 2\bar{\eta}\bar{\partial}\bar{\varepsilon})\bar{\psi}^M \bar{B}_M(X).\end{aligned}$$

The second equation (4.0.5) requires that $s_1\omega_{\bar{1}}$ be s_2-exact. This imposes

$$\begin{aligned}&B_M = \bar{B}_M = 0, \quad D_{Mi} = \bar{D}_{Mi}\;, \quad \partial_N \bar{D}_{iM} = \partial_M D_{iN} \\ \Leftrightarrow\; &B_M = \bar{B}_M = 0, \quad D_{Mi} = \bar{D}_{Mi} = \partial_M D_i(X)\end{aligned}$$

where we have introduced the notation

$$\partial_M = \frac{\partial}{\partial X^M}\;.$$

Furthermore, the second equation (4.0.5) uniquely determines the function ω_0, which corresponds to $\omega_{\bar{1}}$ [the uniqueness follows from (4.0.6)]. It turns out that the other equations (4.0.5) do not impose further conditions in this case, but are automatically fulfilled. Altogether we find

$$\omega_{\bar{1}} \;=\; [(\eta\bar{\varepsilon}\hat{\phi}^i + \varepsilon C^i)\psi^M + (\bar{\eta}\varepsilon\hat{\phi}^i + \bar{\varepsilon}C^i)\bar{\psi}^M]\partial_M D_i(X) \tag{4.0.7}$$

$$\begin{aligned}\omega_0 \;&=\; -\eta\bar{\eta}[\psi^M\bar{\lambda}^i - \bar{\psi}^M\lambda^i + \hat{F}^M\hat{\phi}^i + \psi^M\bar{\psi}^N\hat{\phi}^i\partial_N]\partial_M D_i(X) \\ &\quad +C^i(\eta\mathcal{D}X^M + \bar{\eta}\bar{\mathcal{D}}X^M)\partial_M D_i(X)\end{aligned} \tag{4.0.8}$$

Using the freedom to add a coboundary we obtain by adding $s[C^i D_i(X)]$ to $\omega_{\bar{1}} + \omega_0$ the equivalent solution

$$\begin{aligned}\eta\bar{\eta}\mathcal{F}^i D_i(X) - \eta\bar{\eta}(\psi^M \bar{\lambda}^i - \bar{\psi}^M \lambda^i + \hat{F}^M \hat{\phi}^i + \psi^M \bar{\psi}^N \hat{\phi}^i \partial_N)\partial_M D_i(X) \\ +\eta\bar{\varepsilon}(\lambda^i + \hat{\phi}^i \psi^M \partial_M) D_i(X) + \bar{\eta}\varepsilon(\bar{\lambda}^i + \hat{\phi}^i \bar{\psi}^M \partial_M) D_i(X) \\ +\varepsilon\bar{\varepsilon}\hat{\phi}^i D_i(X). \qquad (4.0.9)\end{aligned}$$

<u>$\bar{k}$=2.</u> The general form of $\omega_{\bar{2}}$ is given by

$$\omega_{\bar{2}} = \varepsilon\varepsilon A_{(1,0)} + \bar{\varepsilon}\bar{\varepsilon} A_{(0,1)} + \varepsilon\bar{\varepsilon} A_{(1/2,1/2)} + \varepsilon\partial\varepsilon B(X) + \bar{\varepsilon}\bar{\partial}\bar{\varepsilon}\bar{B}(X),$$

where due to the indicated conformal weights one has

$$\begin{aligned} A_{(1,0)} &= \mathcal{D}X^M A_M(X) + \psi^M \psi^N A_{MN}(X) \\ A_{(0,1)} &= \bar{\mathcal{D}}X^M \bar{A}_M(X) + \bar{\psi}^M \bar{\psi}^N \bar{A}_{MN}(X) \\ A_{(1/2,1/2)} &= \hat{F}^M H_M(X) + \hat{\phi}^i H_i(X) + \psi^M \bar{\psi}^N H_{MN}(X)\end{aligned}$$

We can simplify $\omega_{\bar{2}}$ using the freedom to subtract s-exact pieces from an s-cocycle. In particular, we can therefore neglect pieces in $\omega_{\bar{2}}$ which are of the form $s_1\hat{\omega}_1 + s_2\hat{\omega}_0$ (i.e. we consider $\omega' = \omega - s(\hat{\omega}_1 + \hat{\omega}_0)$ where ω is an s-cocycle arising from $\omega_{\bar{2}}$). Choosing

$$\hat{\omega}_1 = \tfrac{1}{2}\,(\bar{\varepsilon}\bar{\psi}^M - \varepsilon\psi^M) H_M(X)$$

we get

$$\begin{aligned} s_1\hat{\omega}_1 = \varepsilon\bar{\varepsilon}\hat{F}^M H_M(X) + \tfrac{1}{2}\,(\bar{\varepsilon}\bar{\varepsilon}\bar{\mathcal{D}}X^M - \varepsilon\varepsilon\mathcal{D}X^M) H_M(X) \\ -\tfrac{1}{2}\,(\bar{\varepsilon}\bar{\psi}^M - \varepsilon\psi^M)(\bar{\varepsilon}\bar{\psi}^N + \varepsilon\psi^N)\partial_N H_M(X).\end{aligned}$$

This shows that by subtracting $s_1\hat{\omega}_1$ from $\omega_{\bar{2}}$, we can remove the piece $\hat{F}^M H_M(X)$ from $A_{(1/2,1/2)}$, thereby redefining $A_{(1,0)}$, $A_{(0,1)}$ and $H_{MN}(X)$. Furthermore, we have

$$\begin{aligned} \varepsilon\varepsilon A_{(1,0)} + \bar{\varepsilon}\bar{\varepsilon} A_{(0,1)} + \varepsilon\bar{\varepsilon}\hat{\phi}^i H_i(X) + \varepsilon\partial\varepsilon B(X) + \bar{\varepsilon}\bar{\partial}\bar{\varepsilon}\bar{B}(X) = s_2\hat{\omega}_0, \\ \hat{\omega}_0 = -\eta A_{(1,0)} - \bar{\eta} A_{(0,1)} + C^i H_i(X) - \tfrac{1}{2}\,\partial\eta B(X) - \tfrac{1}{2}\,\bar{\partial}\bar{\eta}\bar{B}(X).\end{aligned}$$

Hence, we can also remove the pieces containing $A_{(1,0)}$, $A_{(0,1)}$, $H_i(X)$, $B(X)$ and $\bar{B}(X)$ from $\omega_{\bar{2}}$. Without loss of generality, we can thus restrict the investigation of the case $\bar{k} = 2$ to

$$\omega_{\bar{2}} = \varepsilon\bar{\varepsilon}\psi^M \bar{\psi}^N H_{MN}(X). \qquad (4.0.10)$$

Obviously $\omega_{\bar{2}}$ satisfies the first eqation (4.0.5), since it does not involve η, $\bar{\eta}$ or C^i. One now has to analyze the remaining equations (4.0.5). It is straightforward to compute $s_1\omega_{\bar{2}}$ and to verify that the second equation (4.0.5) is solved by

$$\begin{aligned}\omega_1 &= \eta\bar{\varepsilon}[\mathcal{D}X^M\bar{\psi}^N - \psi^M\hat{F}^N + \psi^M\bar{\psi}^N\psi^K\partial_K]H_{MN}(X)\\ &\quad -\bar{\eta}\varepsilon[\psi^M\bar{\mathcal{D}}X^N + \hat{F}^M\bar{\psi}^N - \psi^M\bar{\psi}^N\bar{\psi}^K\partial_K]H_{MN}(X).\end{aligned} \quad (4.0.11)$$

The third eq. (4.0.5) requires that $s_0\omega_{\bar{2}} + s_1\omega_1$ be s_2-exact. This turns out to be the case (for arbitrary H_{MN}) and determines ω_0. One finds

$$\begin{aligned}\omega_0 &= \eta\bar{\eta}\Omega,\\ \Omega &= (\mathcal{D}X^M\bar{\mathcal{D}}X^N + \hat{F}^M\hat{F}^N + \mathcal{D}\bar{\psi}^M\bar{\psi}^N - \psi^M\bar{\mathcal{D}}\psi^N)H_{MN}(X)\\ &\quad -(\mathcal{D}X^M\bar{\psi}^N\bar{\psi}^K + \bar{\mathcal{D}}X^N\psi^M\psi^K)\partial_K H_{MN}(X)\\ &\quad +(\hat{F}^M\psi^K\bar{\psi}^N - \hat{F}^K\psi^M\bar{\psi}^N + \hat{F}^N\psi^M\bar{\psi}^K)\partial_K H_{MN}(X)\\ &\quad +\psi^M\psi^K\bar{\psi}^N\bar{\psi}^L\partial_K\partial_L H_{MN}(X).\end{aligned} \quad (4.0.12)$$

The remaining two equations (4.0.5) are also satisfied and the functions $H_{MN}(X)$ are completely arbitrary. The symmetrized part $H_{(MN)}(X)$ and the antisymmetrized part $H_{[MN]}(X)$ give rise to the "target space metric" G_{MN} and the "Kalb-Ramond field" B_{MN}, respectively. Despite of our string inspired terminology we stress that there are no conditions imposed on G_{MN} and B_{MN} apart from their symmetry properties. In particular the "metric" G_{MN} need not be invertible (in section 7.1 we shall impose that a submatrix of G_{MN} be invertible). B_{MN} is determined only up to

$$H_{[MN]}(X) \to H_{[MN]}(X) + \partial_{[M}B_{N]}(X)$$

where $B_M(X)$ are arbitrary functions. This originates from the fact that the s-cocycle $\omega = \omega_{\bar{2}} + \omega_1 + \omega_0$ remains form invariant under

$$\omega \to \omega + s[(\varepsilon\psi^M + \bar{\varepsilon}\bar{\psi}^M + \eta\mathcal{D}X^M + \bar{\eta}\bar{\mathcal{D}}X^M)B_M(X) + \ldots]$$

where the dots stand for terms at least bilinear in the fermions. Changing ω by such s-exact pieces results in the above change of $H_{[MN]}(X)$ and modifies the Lagrangian by a total derivative.

4.1 Result

We conclude that up to redefinitions by coboundary terms, the general solution of (4.0.1) is given by the sum of the functions (4.0.9)–(4.0.12). The solution involves arbitrary functions $D_i(X)$ and $H_{MN}(X)$, which parametrize the various possible actions. The antisymmetric part of $H_{MN}(X)$ is determined only up to redefinitions of the form $H_{MN}(X) \to H_{MN}(X) + \partial_{[M}B_{N]}(X)$, which modify the Lagrangian only by total derivatives. The functions $D_i(X)$ are determined up to arbitrary constants, since only derivatives thereof enter in the equivalent solution (4.0.7) and (4.0.8).[3] Owing to general properties of descent equations in diffeomorphism invariant theories [79–82], the integrand of the action is obtained from the solution of (4.0.1) simply by substituting world-sheet differentials for diffeomorphism ghosts ξ^m. The resulting Lagrangian, written in terms of the Beltrami fields, is a generalized version of the one found in [95]:

$$
\begin{aligned}
L &= L_{Matter} + L_{U1} \\
L_{Matter} &= \tfrac{1}{1-\mu\bar{\mu}}\big[(\partial-\bar{\mu}\bar{\partial})X^M(\bar{\partial}-\mu\partial)X^N(G_{MN}+B_{MN}) \\
&\quad -\left((\partial-\bar{\mu}\bar{\partial})X^M\alpha\psi^N+(\bar{\partial}-\mu\partial)X^M\bar{\alpha}\bar{\psi}^N\right)G_{MN} \\
&\quad -\tfrac{1}{2}\,\alpha\bar{\alpha}\psi^M\bar{\psi}^N G_{MN}\big]-(1-\mu\bar{\mu})\hat{F}^M\hat{F}^N G_{MN} \\
&\quad -\left(\bar{\psi}^N(\partial-\bar{\mu}\bar{\partial})\bar{\psi}^M+\psi^N(\bar{\partial}-\mu\partial)\psi^M\right)G_{MN} \\
&\quad -\bar{\psi}^M\bar{\psi}^N(\partial-\bar{\mu}\bar{\partial})X^K(\Gamma_{KNM}-\tfrac{1}{2}\,H_{KNM}) \\
&\quad -\psi^M\psi^N(\bar{\partial}-\mu\partial)X^K(\Gamma_{KNM}+\tfrac{1}{2}\,H_{KNM}) \\
&\quad +\tfrac{1}{6}\,(\bar{\alpha}\bar{\psi}^M\bar{\psi}^N\bar{\psi}^K-\alpha\psi^M\psi^N\psi^K)H_{KMN} \\
&\quad +(1-\mu\bar{\mu})\hat{F}^M\psi^K\bar{\psi}^N(2\Gamma_{KNM}-H_{KNM}) \\
&\quad +\tfrac{1}{2}\,(1-\mu\bar{\mu})\psi^M\psi^K\bar{\psi}^N\bar{\psi}^L R_{KMLN} \\
L_{U1} &= F^i D_i-(1-\mu\bar{\mu})[\psi^M(\bar{\lambda}^i-\tfrac{1}{2}\,\tfrac{1}{1-\mu\bar{\mu}}\,\mu\bar{\alpha}\hat{\phi}^i) \\
&\quad -\bar{\psi}^M(\lambda^i-\tfrac{1}{2}\,\tfrac{1}{1-\mu\bar{\mu}}\,\bar{\mu}\alpha\hat{\phi}^i) \\
&\quad +\hat{F}^M\hat{\phi}^i+\psi^M\bar{\psi}^N\partial_N]\partial_M D_i
\end{aligned}
\tag{4.1.13}
$$

where we have introduced the following notations

$$
\begin{aligned}
G_{MN} &:= H_{(MN)}(X) \qquad B_{MN} := H_{[MN]}(X) \\
D_i &:= D_i(X) \qquad F^i := \varepsilon^{mn}(\partial_m A_n^i - \partial_n A_m^i) \\
\Omega_{KNM} &:= \partial_K H_{MN}(X) - \partial_M H_{KN}(X) + \partial_N H_{KM}(X) \\
&= 2\Gamma_{KNM} - H_{KNM} \\
R_{KLMN} &:= \partial_M \partial_{[K} H_{L]N}(X) - \partial_N \partial_{[K} H_{L]M}(X)
\end{aligned}
$$

The "target space curvature" R_{KLMN} we have introduced is of course not the Riemannian one. The Riemannian curvature appears after eliminating the auxiliary fields from the action.

Of course, the action can be also written in terms of the original fields introduced in section 3.2. One obtains from the matter part the well known

[3] A constant in D_i yields a topological term in the action proportional to the Chern class of the gauge bundle.

superstring action including the B-field background [37]

$$\begin{aligned}
L/e \;=\;& \tfrac{1}{2}\partial_m X^M \partial_n X^N(-h^{mn}G_{MN}+\varepsilon^{mn}B_{MN}) + \tfrac{\mathrm{i}}{2}\overline{\psi}^M\gamma^m\partial_m\psi^N G_{MN}\\
&+\tfrac{1}{2}F^M F^N G_{MN} + \chi_k\gamma^n\gamma^k(\psi^N\partial_n X^M - \tfrac{1}{4}C\chi_n\overline{\psi}^M\psi^N)G_{MN}\\
&+(\tfrac{1}{2}F^M\overline{\psi}^K\psi^N - \mathrm{i}\,\overline{\psi}^N\gamma^m\psi^M\partial_m X^K)\Gamma_{NKM}\\
&+\tfrac{1}{4}(F^M\overline{\psi}^K\gamma_*\psi^N - \mathrm{i}\overline{\psi}^N\gamma^m\gamma_*\psi^M\partial_m X^K)H_{NKM}\\
&-\tfrac{\mathrm{i}}{12}\chi_m\gamma^n\gamma^m\psi^M\overline{\psi}^N\gamma_n\gamma_*\psi^K H_{MNK}\\
&+\tfrac{1}{16}\overline{\psi}^M(\mathbb{1}+\gamma_*)\psi^N\overline{\psi}^K(\mathbb{1}+\gamma_*)\psi^L R_{KMLN}\\
&+\varepsilon^{mn}D_i\partial_m A_n^i + \tfrac{\mathrm{i}}{4}\overline{\psi}^M\psi^N\phi^i\partial_N\partial_M D_i\\
&+\tfrac{1}{2}(\mathrm{i}\overline{\psi}^N\gamma_*\lambda^i - \mathrm{i}F^N\phi^i + \chi_m\gamma^m\psi^N\phi^i)\partial_N D_i.
\end{aligned} \tag{4.1.14}$$

Thus the cohomological analysis shows that in the absence of gauge multiplets the Lagrangian derived in [37] is in fact unique up to total derivatives and choices of the background fields. It should be kept in mind, however, that this uniqueness is tied to the gauge transformations specified in section 3.2. It gets lost when one allows that the gauge transformations get deformed. This deformation problem can be analysed by BRST cohomological means too, but then the relevant cohomological problem includes the antifields [85]. The results which we shall derive in the second part of this work imply that the nontrivial deformations correspond one-to-one to the deformations of the bosonic string models. All deformations of bosonic string models without world-sheet gauge fields were derived in [41]. We can thus conclude that the nontrivial deformations of the standard superstring world-sheet action [37] and its gauge transformations are supersymmetric generalizations of the actions and gauge transformations given in [41]. A full analysis (to all orders in the deformation parameters) of the deformation problem for bosonic models with world-sheet gauge fields is missing so far, but a complete classification of the first order deformations was given in [39]. The latter results extend thus to the superstring models too.

a ter 5

ifi l s

To proceed with our analysis we have to bring the antifields into the game. According to the principles of the field-antifield formalism [25, 26, 28, 29] to each field a corresponding antifield Φ^*_A is introduced with ghost number and statistics

$$\mathrm{gh}\,(\Phi^*_A) = -\mathrm{gh}\,(\Phi^A) - 1, \quad \epsilon(\Phi^*_A) = \epsilon(\Phi^A) + 1 \ (\mathrm{mod}\ 2),$$

such that the statistics of the antifields is opposite to that of the corresponding fields. It is useful to introduce still another grading into the algebra of fields and antifields, namely the already mentioned antifield (or antighost) number. On all the fields (including the ghosts) the antifield number is defined to be zero, i.e., $\mathrm{agh}\,(\Phi^A) = 0$. On the antifields the antifield number equals minus the ghost number, $\mathrm{agh}\,(\Phi^*_A) = -\mathrm{gh}\,(\Phi^*_A)$.

The antibracket for two arbitrary functions of the fields Φ^A and antifields Φ^*_A is defined as

$$(F, G) = \quad \frac{\delta_R F}{\delta \Phi^A}\frac{\delta_L G}{\delta \Phi^*_A} - \frac{\delta_R F}{\delta \Phi^*_A}\frac{\delta_L G}{\delta \Phi^A} \quad .$$

Thus the antibracket has odd statistics and carries ghost number one. The BRST transformations of the antifields are generated via the antibracket by the proper solution $\mathcal{S}$ to the classical master equation $(\mathcal{S}, \mathcal{S}) = 0$ according to

$$s\Phi^*_A = (\mathcal{S}, \Phi^*_A) = \frac{\delta_R \mathcal{S}}{\delta \Phi^A}.$$

Owing to the off-shell closure of the gauge algebra $\mathcal{S}$ simply reads

$$\mathcal{S} = S_0 - \quad (s\Phi^A)\Phi^*_A,$$

where S_0 is the classical action and $s\Phi^A$ are the BRST transformations given in section 3.2. It is useful to decompose the BRST differential according to the grading with respect to the antifield number $s = \quad {}_{k\geq -1}\, s_k$ with agh $(s_k) = k$ (this decomposition should not be confused with the one in (4.0.3) even though we use the same notation). The decomposition starts with the field theoretical Koszul-Tate differential $\delta \equiv s_{-1}$ and the differential $\gamma \equiv s_0$. Contrary to the bosonic case the decomposition does not terminate at this level. An additional part s_1 raising the antifield number by one unit shows up reflecting field dependent gauge transformations in the commutator of supersymmetry transformations. The Koszul-Tate differential acts nontrivially only on the antifields and implements the equations of motion. Hence, the knowlegde of the classical action is necessary to determine the δ-transformations of the antifields. However, the action of the part of the BRST differential leaving the antifield number unchanged is determined solely by the imposed gauge transformations. The γ-transformations of the antifields corresponding to the matter fields and the $U(1)$ multiplet read

$$\begin{aligned}
\gamma X^*_M &= \partial_m(\xi^m X^*_M) - \mathrm{i}\partial_m(\xi^\beta(\gamma^m C)_{\beta\alpha}\psi^{*\alpha}_M) \\
&\quad -\tfrac{1}{2}\,\partial_m(\xi^\alpha(\gamma^n\gamma^m C)_{\alpha\beta}\chi_n^\beta F^*_M) \\
\gamma\psi^{*\alpha}_M &= \partial_m(\xi^m\psi^{*\alpha}_M) + \xi^\alpha X^*_M - \mathrm{i}\xi^\gamma(\gamma^m C)_{\gamma\beta}\chi_m^\alpha\psi^{*\beta}_M - \tfrac{\mathrm{i}}{2}\,\partial_m(\xi^\beta(\gamma^m)_\beta{}^\alpha F^*_M) \\
&\quad -\tfrac{\mathrm{i}}{8}\,\xi^\beta(\gamma^m\gamma_*)_\beta{}^\alpha\omega_m{}^{ab}\varepsilon_{ab}F^*_M - \tfrac{1}{2}\,\xi^\beta\chi_m{}^\delta(\gamma^m\gamma^n C)_{\beta\delta}\chi_n{}^\alpha F^*_M \\
&\quad -\tfrac{1}{4}\,C^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha\psi^{*\beta}_M + \tfrac{1}{2}\,C^W\psi^{*\alpha}_M \\
\gamma F^*_M &= \partial_m(\xi^m F^*_M) - \xi^\beta C_{\beta\alpha}\psi^{*\alpha}_M - \tfrac{\mathrm{i}}{2}\,\xi^\beta(\gamma^m C)_{\beta\alpha}\chi_m^\alpha F^*_M + C^W F^*_M \\
\gamma A^{*m}_i &= \partial_n(\xi^n A^{*m}_i) - (\partial_n\xi^m)A^{*n}_i \\
&\quad +\mathrm{i}\partial_n(\xi^\alpha(\gamma_* C)_{\alpha\beta}\varepsilon^{nm}\lambda^{*\beta}_i) \\
\gamma\phi^*_i &= \partial_m(\xi^m\phi^*_i) - \xi^\alpha(\gamma_* C)_{\alpha\beta}\varepsilon^{mn}\chi_n(\gamma_* C)\chi_m\lambda^{*\beta}_i \\
&\quad -\mathrm{i}\partial_m(\xi^\alpha(\gamma_*\gamma^m C)_{\alpha\beta}\lambda^{*\beta}_i - \tfrac{\mathrm{i}}{2}\,\xi^\alpha(\gamma_* C)_{\alpha\beta}S\lambda^{*\beta}_i \\
&\quad +2\mathrm{i}\xi^\alpha\chi_m^\beta(\gamma_* C)_{\beta\alpha}A^{*m}_i - 2\eta^\gamma(\gamma_* C)_{\gamma\beta}\lambda^{*\beta}_i + C^W\phi^*_i \\
\gamma\lambda^{*\alpha}_i &= \partial_m(\xi^m\lambda^{*\alpha}_i) - \xi^\beta(\gamma_m)_\beta{}^\alpha A^{*m}_i + \xi^\beta(\gamma_*)_\beta{}^\alpha\phi^*_i \\
&\quad -\mathrm{i}\xi^\beta(\gamma_*\gamma^m C)_{\beta\gamma}(\chi\gamma_*)^\alpha\lambda^{*\gamma}_i - \mathrm{i}\xi^\delta(\gamma_* C)_{\delta\beta}\varepsilon^{kl}\left(\chi_k^\gamma(\gamma_l)_\gamma{}^\alpha\right)\lambda^{*\beta}_i \\
&\quad -\tfrac{1}{4}\,C^{ab}\varepsilon_{ab}(\gamma_*)_\beta{}^\alpha\lambda^{*\beta}_i + \tfrac{3}{2}\,C^W\lambda^{*\alpha}_i.
\end{aligned} \tag{5.0.1}$$

s_1 acts nontrivially on A^{*m}_i, ϕ^*_i and on the antifields for the gravitational multiplet χ^{*m}_α, e^{*m}_a and S^*. In particular one finds

$$s_1 A^{*m}_i = \mathrm{i}\xi^\alpha\xi^\beta(\gamma^m C)_{\beta\alpha}c^*_i, \qquad s_1\phi^*_i = -\mathrm{i}\xi^\alpha\xi^\beta(\gamma_* C)_{\beta\alpha}c^*_i,$$

where c^*_i denote the antifields corresponding to $U(1)$ ghosts.

The explicit form of the BRST transformations of the antifields for the gravitational multiplet and the ghosts will not be needed in the following. In section 7.2 it is shown that they do not contribute nontrivially to the cohomology, at least at ghost number $g < 4$.

5.1 Superconformal antifields

We shall now identify "superconformal antifields" whose γ-transformations take the same form as the s-transformations of superconformal tensor fields in (3.3.21). The identification of superconformal antifields is somewhat more involved than the procedure for the fields. From experience with the bosonic case one expects reasonable candidates to arise from redefinitions of the form $\Phi^*_A \to \frac{1}{1-\mu\bar{\mu}}\,\Phi^*_A$, accounting for the fact that antifields transform under diffeomorphisms as tensor densities rather than tensors. In addition we have to take care of their "structure group transformations", i.e., of their conformal weights, their Lorentz transformations and super-Weyl transformations[1]. Yet this does not suffice to obtain γ-transformations of the desired form. It turns out that the antifields have to be mixed among themselves. These considerations lead us to the following definitions of the lowest order matter antifields

$$
\begin{aligned}
\hat{F}^*_M &\equiv F^*_{M(0,0)} = \frac{1}{1-\mu\bar{\mu}}(e_z{}^z e_{\bar{z}}{}^{\bar{z}})^{-\frac{1}{2}}\,F^*_M \\
\hat{\psi}^*_M &\equiv \psi^*_{M(0,0)} = \frac{\mathrm{i}}{\sqrt{2}}\frac{1}{1-\mu\bar{\mu}}(e_z{}^z)^{-\frac{1}{2}}\,\psi^*_M{}^2 + \frac{\mu\bar{\alpha}}{1-\mu\bar{\mu}}\hat{F}^*_M \\
\hat{\bar{\psi}}^*_M &\equiv \bar{\psi}^*_{M(0,0)} = \frac{\mathrm{i}}{\sqrt{2}}\frac{1}{1-\mu\bar{\mu}}(e_{\bar{z}}{}^{\bar{z}})^{-\frac{1}{2}}\,\psi^*_M{}^1 - \frac{\bar{\mu}\alpha}{1-\mu\bar{\mu}}\hat{F}^*_M \\
\hat{X}^*_M &\equiv X^*_{M(0,0)} = \frac{1}{1-\mu\bar{\mu}}X^*_M + \frac{\bar{\mu}\alpha}{1-\mu\bar{\mu}}\hat{\psi}^*_M + \frac{\mu\bar{\alpha}}{1-\mu\bar{\mu}}\hat{\bar{\psi}}^*_M + \frac{\alpha\bar{\alpha}}{1-\mu\bar{\mu}}\hat{F}^*_M.
\end{aligned}
$$

Their γ-transformations are indeed of the desired form (3.3.21) and read explicitly

$$
\begin{aligned}
\gamma\hat{F}^*_M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\hat{F}^*_M - \varepsilon\hat{\bar{\psi}}^*_M + \bar{\varepsilon}\hat{\psi}^*_M + \tfrac{1}{2}\left((\partial\eta) + (\bar{\partial}\bar{\eta})\right)\hat{F}^*_M \\
\gamma\hat{\psi}^*_M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\hat{\psi}^*_M + \varepsilon\hat{X}^*_M + \bar{\varepsilon}\bar{\mathcal{D}}\hat{F}^*_M + (\tfrac{1}{2}(\partial\eta) + (\bar{\partial}\bar{\eta}))\hat{\psi}^*_M + (\bar{\partial}\bar{\varepsilon})\hat{F}^*_M \\
\gamma\hat{\bar{\psi}}^*_M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\hat{\bar{\psi}}^*_M + \bar{\varepsilon}\hat{X}^*_M - \varepsilon\mathcal{D}\hat{F}^*_M + ((\partial\eta) + \tfrac{1}{2}(\bar{\partial}\bar{\eta}))\hat{\bar{\psi}}^*_M - (\partial\varepsilon)\hat{F}^*_M \\
\gamma\hat{X}^*_M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\hat{X}^*_M + \varepsilon\mathcal{D}\hat{\psi}^*_M + \bar{\varepsilon}\bar{\mathcal{D}}\hat{\bar{\psi}}^*_M + ((\partial\eta) + (\bar{\partial}\bar{\eta}))\hat{X}^*_M \\
&\quad + (\partial\varepsilon)\hat{\psi}^*_M + (\bar{\partial}\bar{\varepsilon})\hat{\bar{\psi}}^*_M.
\end{aligned}
\tag{5.1.2}
$$

The expressions above are in fact already complete, since s_1 does not act nontrivially on the matter antifields. Analogously to the situation of the superconformal tensor fields the algebra (3.3.20) is represented on these fields and their derivatives, which we denote by

$$
F^*_{M(m,n)} = (L_{-1})^m(\bar{L}_{-1})^n\hat{F}^*_M \equiv (\mathcal{D})^m(\bar{\mathcal{D}})^n\hat{F}^*_M,
$$

[1] Antifields transform "contragradiently" under structure group transformations as compared to the corresponding fields.

etc, where the operators L_{-1} and $\bar{L}_{-1}$ are identified with supercovariant derivatives as in (3.3.22). In particular one finds on the antifields with lowest conformal weights the following expressions

$$\begin{aligned}
\mathcal{D}\hat{F}^*_M &= \frac{1}{1-\mu\bar{\mu}}\big((\partial-\bar{\mu}\bar{\partial}-\frac{1}{2}(\bar{\partial}\bar{\mu}) \\
&\qquad +\frac{1}{2}\bar{\mu}(\partial\mu))\hat{F}^*_M-\frac{1}{2}\bar{\mu}\alpha\hat{\bar{\psi}}^*_M-\frac{1}{2}\bar{\alpha}\hat{\psi}^*_M\big) \\
\bar{\mathcal{D}}\hat{F}^*_M &= \frac{1}{1-\mu\bar{\mu}}\big((\bar{\partial}-\mu\partial-\frac{1}{2}(\partial\mu) \\
&\qquad +\frac{1}{2}\mu(\bar{\partial}\bar{\mu}))\hat{F}^*_M+\frac{1}{2}\alpha\hat{\bar{\psi}}^*_M+\frac{1}{2}\mu\bar{\alpha}\hat{\psi}^*_M\big) \\
\mathcal{D}\hat{\psi}^*_M &= \frac{1}{1-\mu\bar{\mu}}\big((\partial-\bar{\mu}\bar{\partial}-(\bar{\partial}\bar{\mu})+\frac{1}{2}\bar{\mu}(\partial\mu))\hat{\psi}^*_M \\
&\qquad -\frac{1}{2}\bar{\mu}\alpha\hat{X}^*_M-\frac{1}{2}\bar{\alpha}\bar{\mathcal{D}}\hat{F}^*_M-\frac{1}{2}(\bar{\partial}\bar{\alpha})\hat{F}^*_M\big) \\
\bar{\mathcal{D}}\hat{\psi}^*_M &= \frac{1}{1-\mu\bar{\mu}}\big((\bar{\partial}-\mu\partial-\frac{1}{2}(\partial\mu)+\mu(\bar{\partial}\bar{\mu}))\hat{\psi}^*_M \\
&\qquad +\frac{1}{2}\alpha\hat{X}^*_M+\frac{1}{2}\bar{\mu}\alpha\mathcal{D}\hat{F}^*_M+\frac{1}{2}\bar{\mu}(\partial\alpha)\hat{F}^*_M\big)
\end{aligned}$$

and analogous formulas for $\mathcal{D}\hat{\bar{\psi}}^*_M$ and $\bar{\mathcal{D}}\hat{\bar{\psi}}^*_M$. Again higher order antifields will not be needed.

The construction of the covariant antifields for the gauge multiplet follows the arguments given above, with the additional task to get rid of the super-Weyl transformations. We introduce the redefinitions

$$\begin{aligned}
\hat{\lambda}^*_i &\equiv \lambda^*_{i(0,0)} = -\frac{1}{1-\mu\bar{\mu}}(e_{\bar{z}}{}^{\bar{z}})^{-\frac{1}{2}}(e_z{}^z)^{-1}\lambda^{*2} \\
\hat{\bar{\lambda}}^*_i &\equiv \bar{\lambda}^*_{i(0,0)} = \frac{1}{1-\mu\bar{\mu}}(e_z{}^z)^{-\frac{1}{2}}(e_{\bar{z}}{}^{\bar{z}})^{-1}\lambda^{*1} \\
\hat{\phi}^*_i &\equiv \phi^*_{i(0,0)} = \frac{1}{\sqrt{2}}\frac{1}{1-\mu\bar{\mu}}(e_z{}^z)^{-\frac{1}{2}}(e_{\bar{z}}{}^{\bar{z}})^{-\frac{1}{2}}\phi^*_i \\
&\qquad -\frac{1}{2}\frac{1}{1-\mu\bar{\mu}}\left(\hat{\chi}_z{}^2-\bar{\mu}\hat{\chi}_{\bar{z}}{}^2\right)\hat{\lambda}^*_i-\frac{1}{2}\frac{1}{1-\mu\bar{\mu}}\left(\hat{\chi}_{\bar{z}}{}^1-\mu\hat{\chi}_z{}^1\right)\hat{\bar{\lambda}}^*_i \\
\hat{A}^*_i &\equiv A^*_{i(0,0)} = \frac{1}{\sqrt{2}}\frac{1}{1-\mu\bar{\mu}}\left(\bar{A}^*_i+\bar{\mu}A^*_i\right)-\frac{1}{1-\mu\bar{\mu}}\left(\bar{\alpha}\hat{\lambda}^*_i+\bar{\mu}\alpha\hat{\bar{\lambda}}^*_i\right) \\
\hat{\bar{A}}^*_i &\equiv \bar{A}^*_{i(0,0)} = \frac{1}{\sqrt{2}}\frac{1}{1-\mu\bar{\mu}}\left(A^*_i+\mu\bar{A}^*_i\right)-\frac{1}{1-\mu\bar{\mu}}\left(\alpha\hat{\bar{\lambda}}^*_i+\mu\bar{\alpha}\hat{\lambda}^*_i\right),
\end{aligned}$$

where we have used the shorthand notation for the corrections involving gravitions $\hat{\chi}_z{}^1 = \overline{\frac{8}{e_{\bar{z}}{}^{z}}}\,\chi_z{}^1$ and $\hat{\chi}_z{}^2 = \overline{\frac{8}{e_z{}^{z}}}\,\chi_z{}^2$ with obvious expressions for the $\bar{z}$ components. The γ-transformations then read

$$\begin{aligned}
\gamma\hat{\lambda}_i^* &= (\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\hat{\lambda}_i^* + \tfrac{1}{2}\,\bar{\partial}\bar{\eta}\lambda_i^* + \varepsilon\hat{\phi}_i^* - \bar{\varepsilon}\hat{\bar{A}}_i^* \\
\gamma\hat{\bar{\lambda}}_i^* &= (\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\hat{\bar{\lambda}}_i^* + \tfrac{1}{2}\,\partial\eta\hat{\bar{\lambda}}_i^* + \bar{\varepsilon}\hat{\phi}_i^* - \varepsilon\hat{A}_i^* \\
\gamma\hat{\phi}_i^* &= (\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\hat{\phi}_i^* + \tfrac{1}{2}\,(\partial\eta+\bar{\partial}\bar{\eta})\hat{\phi}_i^* + \varepsilon\mathcal{D}\hat{\lambda}_i^* + \bar{\varepsilon}\bar{\mathcal{D}}\hat{\bar{\lambda}}_i^* \\
\gamma\hat{A}_i^* &= (\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\hat{A}_i^* + \partial\eta\hat{A}_i^* + \bar{\varepsilon}\mathcal{D}\hat{\lambda}_i^* - \varepsilon\mathcal{D}\hat{\bar{\lambda}}_i^* - \partial\varepsilon\hat{\bar{\lambda}}_i^* \\
\gamma\hat{\bar{A}}_i^* &= (\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\hat{\bar{A}}_i^* + \bar{\partial}\bar{\eta}\hat{\bar{A}}_i^* + \varepsilon\bar{\mathcal{D}}\hat{\bar{\lambda}}_i^* - \bar{\varepsilon}\bar{\mathcal{D}}\hat{\lambda}_i^* - \bar{\partial}\bar{\varepsilon}\hat{\lambda}_i^*,
\end{aligned} \tag{5.1.3}$$

and are indeed of the desired form respecting the requirement (3.3.9). Note that the combination of the gravitinos used in the redefinition of $\hat{\phi}_i^*$ transforms into the super-Weyl ghost thereby removing the unwanted transformation properties under the super-Weyl symmetry. Again higher order antifields will not be needed.

The explicit form of the superconformal antifields given above has already been used to derive the results for the rigid symmetries presented in [36]. A complete list of the BRST transformations (including the Koszul-Tate part and the s_1-transformations) of the antifields needed for the cohomological analysis is given in appendix C.2. In the following sections (and also in the appendices) we have dropped the hats on the superconformal antifields, but it is clear from the context which set of variables is meant.

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i i ri s i l s rv i l s

We now turn to the computation of the antifield dependent local BRST cohomology modulo the world-sheet exterior derivative d at negative ghost number $H^{g,2}(s|d)$, $g<0$. As already explained in the introduction the corresponding local BRST cohomology groups $H^g(s)$ are those with $g<2$. They will give us the dynamical conservation laws, rigid symmetries and nontrivial Noether currents of the models under consideration. This is a standard result of local BRST cohomology in the antifield formalism [83] (for a recent review see [30]). It is not surprising that the local BRST cohomology encodes also the constants of motion, since the Koszul-Tate differential implements the equations of motion explicitly.

6.1 The cohomological analysis for $g<2$

The strategy to find solutions to $s\omega=0$ is to expand the local functions with ghost number g into parts with definite antifield number

$$\omega_g=\omega_g^0+\omega_g^1+\dots.$$

Every such decomposition necessarily starts with an antifield independent part, since there are no antifields with negative or vanishing antifield number. Using the decomposition of the BRST differential with respect to the antifield number introduced in chapter 5

$$s=\delta+\gamma+\Sigma_{k>0}s_k$$

starting with the Koszul-Tate differential δ, $\mathrm{agh}\,(\delta)=-1$ the cocycle condition $s\omega_g$ decomposes into

$$\delta s_g^0=0,\quad \gamma\omega_0^0+\delta\omega_0^1=0,\dots \tag{6.1.1}$$

This decomposition is useful, since every nontrivial solution of $s\omega = 0$ is uniquely (up to s-exact terms) characterized by its antifield independent part ω_g^0. This is a standard statement of homological perturbation theory [28] but is intimately tied to the acyclicity of the Koszul-Tate differential, $H_k(\delta) = 0$ for $k > 0$. This again is usually a consequence of certain regularity conditions of the equations of motion. One might wonder, if these standard regularity conditions are fulfilled in the present case and indeed they are not. But fortunately the antifields which do not fulfill the regularity conditions do not contribute at the ghost numbers relevant for the computations in this section. Thus the decomposition still makes sense in our context. This will be discussed in more detail when the isomorphism between the cohomology groups of the bosonic models and their supersymmetric counterparts is established.

In this section we determine the solutions up to antifield number 1 by considering the condition

$$\gamma\omega_0^0 + \delta\omega_0^1 = 0. \tag{6.1.2}$$

This will already give us the nontrivial solutions to the Noether currents and the rigid symmetries. We will explicitly calculate the corresponding cohomology groups $H^g(s)$, $g < 2$, for a simplified model, namely under the assumption that the functions D_i coincide with a subset of the coordinate fields $X^M = \{X^\mu, D_i\} = \{X^\mu, y^i\}$. In fact, this is a rather mild assumption, since it can be achieved by a target space coordinate transformation. We will make this more explicit in section 6.2.

6.1.1 Solution at $g = 0$

The solutions of the BRST cohomology $H^{g,2}(s|d)$ at negative ghost numbers correspond one-to-one to dynamical local conservation laws [83]. At ghost number -2 these are the dynamical conservation laws of second order represented by on-shell closed $(n-k)$-forms (n denotes the dimension of the manifold), which are not weakly locally exact.[1] The corresponding local BRST cohomology group is $H^0(s)$.

As in the computation of the action the starting point will be the most general function with ghost number 0 at most linear in the antifields. Taking into account that the conformal weight has to be zero this reads

$$\begin{aligned}
\omega_0 &= \omega_0^0 + \omega_0^1 \\
\omega_0^0 &= f(X^M) \\
\omega_0^1 &= \eta(A_i^* f^i + \bar{\lambda}_i^* \psi^M f_M^i) + \bar{\eta}(\bar{A}_i^* \bar{f}^i + \lambda_i^* \bar{\psi}^M \bar{f}_M^i) + \varepsilon\bar{\lambda}_i^* g^i + \bar{\varepsilon}\lambda_i^* \bar{g}^i.
\end{aligned}$$

[1] Topological conservation laws are locally but not globally d-exact.

Using the freedom of subtracting trivial parts from ω_0 arising from $\delta\omega^2_{-1}$ and restricting to the case described above, i.e.,

$$\partial_\mu D_i = 0, \qquad \partial^j D_i = \delta^j_i$$

one finds the most general solution to (6.1.2) as

$$\hat{\omega}_0 = f(y_i) - \left(\eta A^*_i - \bar{\eta}\bar{A}^*_i - \varepsilon\bar{\lambda}^*_i + \bar{\varepsilon}\lambda^*_i\right)\partial^i f. \tag{6.1.3}$$

In fact to extract the integrand of the solution it is necessary to complete it to a full solution of the cocycle condition. This can be done easily by observing that the form of the solution suggests a dependence on the special combination of fields and antifields

$$\hat{y}^i = y^i - \eta A^*_i + \bar{\eta}\bar{A}^*_i + \varepsilon\bar{\lambda}^*_i - \bar{\varepsilon}\lambda^*_i + \eta\bar{\eta}C^*,$$

which is a BRST singlet, $s\hat{y}^i = 0$. Thus any function of $\hat{y}^i$ is a solution to the cohomology at ghost number zero and we conclude that there exist infinitely many second order conservation laws $f(\hat{y}^i)$. Here one remark is in order. Expanding the function $f(\hat{y}^i)$ in antifield number one gets up to antifield number one (6.1.3). At antifield number 2 one gets a term linear in the antifield for the $U(1)$ ghost $\partial_i f(y^i)C^*$ and a term quadratic in the antifields for the gauge fields $\partial_i\partial_j f(y^i)A^*_j\bar{A}^*_i$. In the bosonic case this is already the whole story (see section 7 in [39]) and gives the desired integrand[2], but in the supersymmetric case the combinations $\varepsilon\bar{\lambda}^*_i$ and $\bar{\varepsilon}\lambda^*_i$ have vanishing ghost number and conformal weight. Thus they can contribute even nonpolynomially, which is indeed the case for arbitrary functions of $\hat{y}^i$.

6.1.2 Solution at $g = 1$

At ghost number 1 the cohomology group $H^1(s)$ yields nontrivial Noether currents and global ("rigid") symmetries. In particular, we will obtain the target space isometries of the models under consideration.

The most general local function with ghost number 1 independent of antifields and with vanishing conformal weight is

$$\begin{aligned}\omega^0_1 \;&=\; \eta A^0_{(1,0)} + \bar{\eta}A^0_{(0,1)} + \partial\eta A^0_{(0,0)} + \bar{\partial}\bar{\eta}\bar{A}^0_{(0,0)} \\ &\quad +\varepsilon A^0_{(1/2,0)} + \bar{\varepsilon}A^0_{(0,1/2)} + C^i B^0_{i(0,0)}\end{aligned}$$

where the A's and B's do not depend on ghosts and antifields and the subscripts (m,n) indicate their conformal weights. The part of ω_1 with $\mathrm{agh}\,(\omega_1) = 1$ can be expanded into powers of supersymmetry ghosts

$$\omega^1_1 = {}^0\omega^1_1 + {}^1\omega^1_1 + {}^2\omega^1_1$$

[2]The integrand has also a physical interpretation. It generates rigid symmetries of of the proper solution to the master equation via the antibracket [84].

with

$$\begin{aligned}
{}^0\omega_1^1 &= \eta\bar\eta A^1_{(1,1)} + \eta\partial\eta A^1_{(1,0)} + \eta\bar\partial\bar\eta \bar A^1_{(1,0)} + \bar\eta\partial\eta A^1_{(0,1)} + \bar\eta\bar\partial\bar\eta\bar A^1_{(0,1)} \\
&\quad +\eta C^i B^1_{i(1,0)} + \bar\eta C^i B^1_{i(0,1)} \\
{}^1\omega_1^1 &= \eta\varepsilon A^1_{(3/2,0)} + \bar\eta\bar\varepsilon A^1_{(0,3/2)} + \eta\bar\varepsilon A^1_{(1,1/2)} + \bar\eta\varepsilon A^1_{(1/2,1)} \\
&\quad +\eta\partial\varepsilon A^1_{(1/2,0)} + \bar\eta\bar\partial\bar\varepsilon A^1_{(0,1/2)} + \varepsilon\partial\eta C^1_{(1/2,0)} + \varepsilon\bar\partial\bar\eta\bar C^1_{(1/2,0)} \\
&\quad +\bar\varepsilon\partial\eta C^1_{(0,1/2)} + \bar\varepsilon\bar\partial\bar\eta\bar C^1_{(0,1/2)} + \varepsilon C^i B^1_{i(1/2,0)} + \bar\varepsilon C^i \bar B^1_{i(1/2,0)} \\
{}^2\omega_1^1 &= \varepsilon\varepsilon C^1_{(1,0)} + \bar\varepsilon\bar\varepsilon C^1_{(0,1)} + \varepsilon\bar\varepsilon C^1_{(1/2,1/2)}
\end{aligned}$$

where the A's, B's and C's have antifield number 1 as is indicated by the superscripts. These are all possible contributions, since there are no antifields with vanishing conformal weight. Using the freedom to subtract s-exact pieces

$$s(\eta\tilde A^1_{(1,0)} + \bar\eta\tilde A^1_{(0,1)} + \varepsilon\tilde A^1_{(1/2,0)} + \bar\varepsilon\tilde A^1_{(0,1/2)})$$

we remove the terms $\varepsilon\varepsilon C^1{}_{(1,0)}$, $\bar\varepsilon\bar\varepsilon C^1{}_{(0,1)}$ from ${}^2\omega_1^1$ and the terms $\eta\partial\varepsilon A^1{}_{(1/2,0)}$, $\bar\eta\bar\partial\bar\varepsilon A^1{}_{(0,1/2)}$ from ${}^1\omega_1^1$.

As in the computation of the action the analysis will be based on the decomposition of the BRST differential into definite degree with respect to the supersymmetry ghosts. We expand the part of s with antifield number 0 in supersymmetry ghosts, i.e.

$$\gamma = \gamma_0 + \gamma_1 + \gamma_2.$$

The simplest piece in this decomposition γ_2 acts nontrivially only on η, $\bar\eta$ and C^i. In the equations above we have used that $\gamma_2\,\omega_1^0 + \delta\;{}^2\omega_1^1 = 0$ immediately implies

$$A^0_{(0,0)} = \bar A^0_{(0,0)} = 0,$$

since there are no antifield dependent terms containing $\varepsilon\partial\varepsilon$ and $\bar\varepsilon\bar\partial\bar\varepsilon$ that can compensate their contributions. Furthermore, we can immediately conclude that there are no mixed terms $\varepsilon\bar\partial\bar\eta$ and $\bar\varepsilon\partial\eta$ as well as $\eta\bar\partial\bar\eta$ and $\bar\eta\partial\eta$ in $\gamma\omega_1^0$. Thus we are left with the following antifield dependent terms

$$\begin{aligned}
{}^0\omega_1^1 &= \eta\bar\eta A^1_{(1,1)} + \eta\partial\eta A^1_{(1,0)} + \bar\eta\bar\partial\bar\eta\bar A^1_{(0,1)} \\
&\quad +\eta C^i B^1_{i(1,0)} + \bar\eta C^i B^1_{i(0,1)} \\
{}^1\omega_1^1 &= \eta\varepsilon A^1_{(3/2,0)} + \bar\eta\bar\varepsilon A^1_{(0,3/2)} + \eta\bar\varepsilon A^1_{(1,1/2)} + \bar\eta\varepsilon A^1_{(1/2,1)} \\
&\quad +\varepsilon\partial\eta C^1_{(1/2,0)} + \bar\varepsilon\bar\partial\bar\eta\bar C^1_{(0,1/2)} + \varepsilon C^i B^1_{i(1/2,0)} + \bar\varepsilon C^i\bar B^1_{i(1/2,0)} \\
{}^2\omega_1^1 &= \varepsilon\bar\varepsilon C^1_{(1/2,1/2)}
\end{aligned}$$

Next we consider the equation

$$\gamma_2({}^0\omega_1^0) + \gamma_1({}^1\omega_1^0) + \delta\,({}^2\omega_1^1) = 0$$

Using for $A^0_{(1/2,0)}$ and $A^0_{(0,1/2)}$ the expressions

$$A^0_{(1/2,0)} = \psi^M f_M(X) \qquad A^0_{(0,1/2)} = \bar{\psi}^M \bar{f}_M(X),$$

we find

$$\begin{aligned} A^0_{(1,0)} &= \mathcal{D}X^M f_M(X) - \psi^M \psi^N \partial_N f_M(X) \\ A^0_{(0,1)} &= \bar{\mathcal{D}}X^M \bar{f}_M(X) - \bar{\psi}^M \bar{\psi}^N \partial_N \bar{f}_M(X) \end{aligned}$$

Furthermore, using

$$C^1_{(1/2,1/2)} = (F^*_M \mathcal{K}^M + \phi^*_i \mathcal{K}^i + \lambda^*_i \psi^M \mathcal{K}^i_M + \bar{\lambda}^*_i \bar{\psi}^m \bar{\mathcal{K}}^i_m)$$

where we subtracted the trivial part $\delta_{KT}(\bar{\lambda}^*_i \lambda^*_j \hat{\mathcal{K}}^{ij})$ thereby redefining $\mathcal{K}^i_M$ we find that the following equations have to be fulfilled

$$\begin{aligned} B^0_i - \delta_{i,M}\mathcal{K}^M &= 0 \\ \bar{f}_M - f_M + 2G_{MN}\mathcal{K}^N - \delta_{M,i}\mathcal{K}^i &= 0 \\ \partial_M \bar{f}_N - \partial_N f_M + \Omega_{MNK}\mathcal{K}^K - \delta_{N,i}\mathcal{K}^i_M - \delta_{i,M}\delta_{N,n}\bar{\mathcal{K}}^i_n &= 0. \end{aligned} \quad (6.1.4)$$

In order to save some writing we have introduced the Kronecker symbol $\delta_{M,i}$, which should not be confused with the Koszul-Tate differential. It is useful to introduce the following combinations of the coefficient functions f and $\bar{f}$

$$f^+_M = f_M + \bar{f}_M \qquad f^-_M = f_M - \bar{f}_M.$$

Then one obtains by symmetrization and antisymmetrization of the last equation in (6.1.4) the following conditions

$$\begin{aligned} \mathcal{L}_\mathcal{K} G_{MN} &= -\delta_{i,(N}\mathcal{K}^i_{M)} - \delta_{i,(M}\delta_{N),n}\bar{\mathcal{K}}^i_n + \delta_{i,(N}\partial_{M)}\mathcal{K}^i \\ \mathcal{L}_\mathcal{K} B_{MN} &= \partial_{[M} f'^+_{N]} + \partial_{[M}\delta_{N],i}\mathcal{K}^i - \delta_{i,[N}\mathcal{K}^i_{M]} - \delta_{i,[M}\delta_{N],n}\bar{\mathcal{K}}^i_n \\ B^0_i &= \delta_{i,M}\mathcal{K}^M \end{aligned}$$

with $\mathcal{L}_\mathcal{K}$ denoting the usual Lie-derivative along $\mathcal{K}$

$$\mathcal{L}_\mathcal{K} G_{MN} = \mathcal{K}^K \partial_K G_{MN} + (\partial_M \mathcal{K}^K) G_{KN} + (\partial_N \mathcal{K}^K) G_{MK}$$

and where f'_N is given by

$$f'^+_N = f_N - 2B_{NK}\mathcal{K}^K.$$

From these results it follows that

$$A^1_{(1,0)} = 0 \qquad \bar{A}^1_{(0,1)} = 0$$
$$C^1_{(1/2,0)} = 0 \qquad \bar{C}^1_{(0,1/2)} = 0.$$

Next we turn to contributions containing $U(1)$ ghosts. Using the following expressions for the coefficient functions dictated by the conformal weight condition

$$B^1_{i(1/2,0)} = \bar{\lambda}^*_j B^j_i \qquad \bar{B}^1_{i(0,1/2)} = \lambda^*_j \bar{B}^j_i$$
$$B^1_{i(1,0)} = A^*_j b^j_i + \bar{\lambda}^*_j \psi^M b^j_{iM} \qquad \bar{B}^1_{i(0,1)} = \bar{A}^*_j \bar{b}^j_i + \lambda^*_j \bar{\psi}^M \bar{b}^j_{iM}$$

one finds

$$\partial_M B^0_i = (\delta_{M,j}) B^j_i \qquad \partial_M B^0_i = -(\delta_{M,j}) \bar{B}^j_i$$
$$\partial_M B^0_i = (\delta_{M,b}) b^j_i \qquad \partial_M B^0_i = -(\delta_{M,b}) \bar{b}^j_i.$$

This implies that $B^0_i(X)$ is a function of the y^i's only, $B^0_i(X) = B^0_i(y)$. The other terms give only trivial contributions. Thus one ends up with the following terms in the antifield dependent part of the solution

$$\begin{aligned}
{}^0\omega^1_1 &= \eta\bar{\eta} A^1_{(1,1)} + \eta C^i B^1_{i(1,0)} + \bar{\eta} C^i B^1_{i(0,1)} \\
{}^1\omega^1_1 &= \eta\bar{\varepsilon} A^1_{(1,1/2)} + \bar{\eta}\varepsilon A^1_{(1/2,1)} + \varepsilon C^i B^1_{i(1/2,0)} + \bar{\varepsilon} C^i \bar{B}^1_{i(0,1/2,)} \\
{}^2\omega^1_1 &= \varepsilon\bar{\varepsilon} C^1_{(1/2,1/2)}
\end{aligned}$$

where $B^1_{a(1,0)}, B^1_{a(0,1)}, B^1_{a(1/2,0)}, B^1_{a(0,1/2,)}, C^1_{(1/2,1/2)}$ are given in the equations above. To determine the complete solution we make the general ansatz for the antifield dependent part $A^1_{(1,1)}$

$$\begin{aligned}
A^1_{(1,1)} = {} & X^*_M \mathcal{H}^M_{(0,0)} + \psi^*_M \mathcal{H}^M_{(1/2,0)} + \bar{\psi}^*_M \mathcal{H}^M_{(0,1/2)} + F^*_M \mathcal{H}^M_{(1/2,1/2)} + \\
& \phi^*_i \mathcal{G}^i_{(1/2,1/2)} + \lambda^*_i \mathcal{G}^i_{(1,1/2)} + \bar{\lambda}^*_i \mathcal{G}^i_{(1/2,1)} + A^*_i \mathcal{G}^i_{(0,1)} + \bar{A}^*_i \mathcal{G}^i_{(1,0)} + \\
& \mathcal{D}\lambda^*_i \mathcal{G}^i_{(0,1/2)} + \bar{\mathcal{D}}\bar{\lambda}^*_i \mathcal{G}^i_{(1/2,0)} + \bar{\mathcal{D}} A^*_i \mathcal{G}^i_{(0,0)} + \mathcal{D}\bar{A}^*_i \bar{\mathcal{G}}^i_{(0,0)} \qquad (6.1.5)
\end{aligned}$$

where the coefficient functions contained in (6.1.5) are constrained by their conformal weights to be of the form

$$\begin{aligned}
\mathcal{H}^M_{(0,0)} &= \mathcal{H}^M \\
\mathcal{H}^M_{(1/2,0)} &= \psi^N \mathcal{H}_N{}^M \\
\mathcal{H}^M_{(0,1/2)} &= \bar{\psi}^N \bar{\mathcal{H}}_N{}^M \\
\mathcal{H}^M_{(1/2,1/2)} &= F^N h_N{}^N + \psi^N \bar{\psi}^K h_{NK}{}^M + \phi^i h_i{}^M \\
\mathcal{G}^i_{(1/2,1/2)} &= F^N \mathcal{G}_N{}^i + \psi^N \bar{\psi}^K \mathcal{G}_{NK}{}^i + \phi^j \mathcal{G}_j{}^i \\
\mathcal{G}^i_{(1,1/2)} &= \mathcal{D}\bar{\psi}^M \mathcal{P}_M{}^i + \mathcal{D}X^M \bar{\psi}^N \mathcal{P}_{MN}{}^i + \psi^M \psi^N \bar{\psi}^K \mathcal{P}_{MNK}{}^i \\
&\quad + F^M \psi^N \mathcal{Q}_{MN}{}^i + \phi^j \psi^M \mathcal{Q}_{Mj}{}^i + \lambda^j \mathcal{Q}_j{}^i \\
\mathcal{G}^i_{(1/2,1)} &= \bar{\mathcal{D}}\psi^M \bar{\mathcal{P}}_M{}^i + \bar{\mathcal{D}}X^M \psi^N \bar{\mathcal{P}}_{MN}{}^i + \bar{\psi}^M \bar{\psi}^N \psi^K \bar{\mathcal{P}}_{MNK}{}^i \\
&\quad + F^M \bar{\psi}^N \bar{\mathcal{Q}}_{MN}{}^i + \phi^j \bar{\psi}^M \bar{\mathcal{Q}}_{Mj}{}^i + \bar{\lambda}^j \bar{\mathcal{Q}}_j{}^i \\
\mathcal{G}^i_{(0,1)} &= \bar{\mathcal{D}}X^M \bar{\mathcal{R}}_M{}^i + \bar{\psi}^M \bar{\psi}^N \bar{\mathcal{R}}_{MN}{}^i \\
\mathcal{G}^i_{(1,0)} &= \mathcal{D}X^M \mathcal{R}_M{}^i + \psi^M \psi^N \mathcal{R}_{MN}{}^i \\
\mathcal{G}^i_{(1/2,0)} &= \psi^M g_M{}^i \\
\mathcal{G}^i_{(0,1/2)} &= \bar{\psi}^M \bar{g}_M{}^i \\
\mathcal{G}^i_{(0,0)} &= \mathcal{G}^i \\
\bar{\mathcal{G}}^i_{(0,0)} &= \bar{\mathcal{G}}^i
\end{aligned}$$

We still have the freedom to remove trivial parts by using the nilpotency of the Koszul-Tate differential. To this end we examine how the coefficient functions are redefined under $\omega \to \omega - \delta_{KT}\hat{\omega}$, where

$$\begin{aligned}
\hat{\omega} &= \psi^*_M \bar{\lambda}^*_i \hat{\mathcal{H}}^{Mi} + \bar{\psi}^*_M \lambda^*_i \hat{\bar{\mathcal{H}}}^{Mi} + F^*_M \bar{\lambda}^*_i \bar{\psi}^N \hat{\mathcal{H}}_N{}^{Mi} + F^*_M \lambda^*_i \psi^N \hat{\bar{\mathcal{H}}}_N{}^{Mi} \\
&\quad + F^*_M F^*_N \hat{\mathcal{H}}^{[MN]} + F^*_M \phi^*_i \hat{\tilde{\mathcal{H}}}^{Mi} + \phi^*_i \lambda^*_j \psi^N \hat{\mathcal{G}}_N{}^{ij} + \phi^*_i \bar{\lambda}^*_j \bar{\psi}^N \hat{\bar{\mathcal{G}}}_N{}^{ij} \\
&\quad + \phi^*_i \phi^*_j \hat{\mathcal{G}}^{[ij]} + \lambda^*_i \lambda^*_j \hat{\mathcal{G}}^{(ij)}_{(1,0)} + \bar{\lambda}^*_i \bar{\lambda}^*_j \hat{\mathcal{G}}^{(ij)}_{(0,1)} + \lambda^*_i \bar{\lambda}^*_j \hat{\mathcal{G}}^{ij}_{(1/2,1/2)} \\
&\quad + A^*_i \lambda^*_j \hat{\mathcal{G}}^{ij}_{(0,1/2)} + \bar{A}^*_i \bar{\lambda}^*_j \hat{\mathcal{G}}^{ij}_{(1/2,0)} + A^*_i \bar{A}^*_j \hat{\tilde{\mathcal{G}}}^{ij}_{(0,0)} \\
&\quad + \mathcal{D}\lambda^*_i \lambda^*_j \hat{\mathcal{G}}^{ij}_{(0,0)} + \bar{\mathcal{D}}\bar{\lambda}^*_i \bar{\lambda}^*_j \hat{\bar{\mathcal{G}}}^{ij}_{(0,0)} + C^*_i \mathcal{F}^i_{(0,0)}.
\end{aligned}$$

Th coefficient functions contained in this expression are constrained to be of the form

$$\begin{aligned}
\hat{\mathcal{G}}^{(ij)}_{(1,0)} &= \mathcal{D}X^M \hat{g}_M{}^{(ij)} + \psi^K \psi^L \hat{g}_{[KL]}{}^{(ij)} \\
\hat{\mathcal{G}}^{(ij)}_{(0,1)} &= \bar{\mathcal{D}}X^M \hat{\bar{g}}_M{}^{(ij)} + \bar{\psi}^K \bar{\psi}^L \hat{\bar{g}}_{[KL]}{}^{(ij)} \\
\hat{\mathcal{G}}^{ij}_{(1/2,1/2)} &= F^N \hat{\tilde{g}}_N{}^{ij} + \phi^k \hat{\tilde{g}}_k{}^{ij} + \psi^K \bar{\psi}^L \hat{\tilde{g}}_{KL}{}^{ij} \\
\hat{\mathcal{G}}^{ij}_{(0,1/2)} &= \bar{\psi}^K \hat{\bar{\mathcal{F}}}_K{}^{ij} \\
\hat{\mathcal{G}}^{ij}_{(1/2,0)} &= \psi^K \hat{\mathcal{F}}_K{}^{ij}
\end{aligned}$$

$$
\begin{aligned}
\hat{\tilde{\mathcal{G}}}^{ij}_{(0,0)} &= \hat{\tilde{\mathcal{G}}}^{ij} \\
\hat{\mathcal{G}}^{ij}_{(0,0)} &= \hat{g}^{ij} \\
\hat{\bar{\mathcal{G}}}^{ij}_{(0,0)} &= \hat{\bar{g}}^{ij} \\
\mathcal{F}^{i}_{(0,0)} &= \mathcal{F}^{i}.
\end{aligned}
$$

This induces the following redefinitions of the coefficient functions in (6.1.6)

$$
\begin{aligned}
\mathcal{H}^M &\rightarrow \mathcal{H}^M \\
\mathcal{H}_N{}^M &\rightarrow \mathcal{H}_N{}^M + \delta_{N,i}\hat{\mathcal{H}}^{Mi} \\
\bar{\mathcal{H}}_N{}^M &\rightarrow \bar{\mathcal{H}}_N{}^M - \delta_{N,i}\hat{\bar{\mathcal{H}}}^{Mi} \\
h_N{}^M &\rightarrow h_N{}^M + 4G_{NK}\hat{\mathcal{H}}^{[MK]} - \delta_{N,i}\hat{\tilde{\mathcal{H}}}^{Mi} \\
h_i{}^M &\rightarrow h_i{}^M - 2\delta_{N,i}\hat{\mathcal{H}}^{[MN]} \\
h_{NK}{}^M &\rightarrow h_{NK}{}^M - \delta_{N,i}\hat{\mathcal{H}}_K{}^{Mi} - \delta_{K,i}\hat{\bar{\mathcal{H}}}_N{}^{Mi} + 2\Omega_{NKL}\hat{\mathcal{H}}^{[ML]} \\
\mathcal{G}_N{}^i &\rightarrow \mathcal{G}_N{}^i - 2G_{NM}\hat{\tilde{\mathcal{H}}}^{Mi} - 2\delta_{N,j}\hat{\mathcal{G}}^{[ij]} \\
\mathcal{G}_j{}^i &\rightarrow \mathcal{G}_j{}^i + \delta_{M,j}\hat{\tilde{\mathcal{H}}}^{Mi} \\
\mathcal{G}_{NK}{}^i &\rightarrow \mathcal{G}_{NK}{}^i - \Omega_{NKM}\hat{\tilde{\mathcal{H}}}^{Mi} - \delta_{K,j}\,\hat{\mathcal{G}}_N{}^{ij} - \delta_{N,j}\,\hat{\bar{\mathcal{G}}}_K{}^{ij} \\
\mathcal{P}_M{}^i &\rightarrow \mathcal{P}_M{}^i - 2G_{MN}\hat{\bar{\mathcal{H}}}^{Ni} - \delta_{M,j}\,\hat{g}^{ji} \\
\mathcal{P}_{MN}{}^i &\rightarrow \mathcal{P}_{MN}{}^i - \Omega_{MNK}\hat{\bar{\mathcal{H}}}^{Ki} - 2\delta_{N,j}\,\hat{g}_M{}^{(ij)} + \delta_{Mj}\hat{\bar{\mathcal{F}}}_N{}^{ji} \\
\mathcal{P}_{MNK}{}^i &\rightarrow \mathcal{P}_{MNK}{}^i - R_{NMLK}\hat{\bar{\mathcal{H}}}^{Li} - \Omega_{NKL}\hat{\bar{\mathcal{H}}}_M{}^{Li} \\
&\quad -2\delta_{K,j}\,\hat{g}_{[MN]}{}^{(ij)} + \delta_{M,j}\,\hat{\tilde{g}}_{NK}{}^{ij} \\
\mathcal{Q}_{MN}{}^i &\rightarrow \mathcal{Q}_{MN}{}^i - \Omega_{NKM}\hat{\bar{\mathcal{H}}}^{Ki} - 2G_{MK}\hat{\bar{\mathcal{H}}}_N{}^{Ki} \\
&\quad +\delta_{M,j}\,\hat{\mathcal{G}}_N{}^{ji} + \delta_{N,j}\,\hat{\tilde{g}}_M{}^{ij} \\
\mathcal{Q}_{Mj}{}^i &\rightarrow \mathcal{Q}_{Mj}{}^i + \delta_{K,j}\hat{\bar{\mathcal{H}}}_M{}^{Ki} + \delta_{M,k}\,\hat{\tilde{g}}_j{}^{ki} \\
\mathcal{Q}_j{}^i &\rightarrow \mathcal{Q}_j{}^i + \delta_{M,j}\hat{\bar{\mathcal{H}}}^{Mi} \\
\bar{\mathcal{P}}_M{}^i &\rightarrow \bar{\mathcal{P}}_M{}^i - 2G_{MN}\hat{\mathcal{H}}^{Ni} + \delta_{M,j}\,\hat{\bar{g}}^{ji} \\
\bar{\mathcal{P}}_{MN}{}^i &\rightarrow \bar{\mathcal{P}}_{MN}{}^i - \Omega_{NMK}\hat{\mathcal{H}}^{Ki} + 2\delta_{N,j}\,\hat{\bar{g}}_M{}^{(ij)} - \delta_{Mj}\hat{\mathcal{F}}_N{}^{ji} \\
\bar{\mathcal{P}}_{MNK}{}^i &\rightarrow \bar{\mathcal{P}}_{MNK}{}^i + R_{KLNM}\hat{\mathcal{H}}^{Li} - \Omega_{KML}\hat{\mathcal{H}}_N{}^{Li} \\
&\quad +2\delta_{K,j}\,\hat{\bar{g}}_{[MN]}{}^{(ij)} + \delta_{M,j}\,\hat{\tilde{g}}_{KN}{}^{ji} \\
\bar{\mathcal{Q}}_{MN}{}^i &\rightarrow \bar{\mathcal{Q}}_{MN}{}^i + \Omega_{KNM}\hat{\mathcal{H}}^{Ki} - 2G_{MK}\hat{\mathcal{H}}_N{}^{Ki} \\
&\quad +\delta_{M,j}\,\hat{\bar{\mathcal{G}}}_N{}^{ji} - \delta_{N,j}\,\hat{\tilde{g}}_M{}^{ji} \\
\bar{\mathcal{Q}}_{Mj}{}^i &\rightarrow \bar{\mathcal{Q}}_{Mj}{}^i + \delta_{K,j}\hat{\mathcal{H}}_M{}^{Ki} - \delta_{M,k}\,\hat{\tilde{g}}_j{}^{ki} \\
\bar{\mathcal{Q}}_j{}^i &\rightarrow \bar{\mathcal{Q}}_j{}^i - \delta_{M,j}\hat{\mathcal{H}}^{Mi}
\end{aligned}
$$

$$
\begin{array}{rcl}
\mathcal{R}_M{}^i & \rightarrow & \mathcal{R}_M{}^i + \delta_{M,j}\,\hat{\bar{\mathcal{G}}}^{ji} \\
\mathcal{R}_{MN}{}^i & \rightarrow & \mathcal{R}_{MN}{}^i - \delta_{M,j}\hat{\mathcal{F}}_N{}^{ij} \\
\bar{\mathcal{R}}_M{}^i & \rightarrow & \bar{\mathcal{R}}_M{}^i + \delta_{M,j}\,\hat{\bar{\mathcal{G}}}^{ij} \\
\bar{\mathcal{R}}_{MN}{}^i & \rightarrow & \bar{\mathcal{R}}_{MN}{}^i + \delta_{M,j}\hat{\bar{\mathcal{F}}}_N{}^{ij} \\
g_M{}^i & \rightarrow & g_M{}^i + \delta_{M,j}\,\hat{\bar{g}}^{ij} \\
\bar{g}_M{}^i & \rightarrow & \bar{g}_M{}^i - \delta_{M,j}\,\hat{g}^{ij} \\
\mathcal{G}^i & \rightarrow & \mathcal{G}^i + \mathcal{F}^i \\
\bar{\mathcal{G}}^i & \rightarrow & \bar{\mathcal{G}}^i + \mathcal{F}^i
\end{array}
$$

which allows us to remove certain parts of the functions occurring in $A^1_{(1,1)}$. Choosing, according to the decomposition $\mathcal{H}_N{}^M = \delta_{N,\nu}\mathcal{H}_\nu{}^M + \delta_{N,i}\mathcal{H}^{iM}$,

$$\hat{\mathcal{H}}^{Mi} = -\mathcal{H}^{iM}$$

we see that we can remove $\mathcal{H}^{iM}$ from the cohomology. In the same way we remove without loss of generality $\bar{\mathcal{H}}^{iM}$ by the appropriate choice of $\hat{\bar{\mathcal{H}}}^{Mi}$. Furthermore we remove h^{iM}, $h_i{}^\mu$, $h^i{}_k{}^M$, $h_\nu{}^{iM}$, h^{ijM}, $\mathcal{G}^j{}_k{}^i$, $\mathcal{G}_n{}^{ji}$, $\mathcal{G}^{jki}$, $\mathcal{P}^{ji}$, $\bar{\mathcal{P}}^{ji}$, $\mathcal{Q}^k{}_j{}^i$, $\bar{\mathcal{Q}}^k{}_j{}^i$, $\bar{\mathcal{R}}^{ji}$, $\mathcal{R}_\mu{}^{ji}$, $\mathcal{R}^j{}_\nu{}^i$, $\mathcal{R}^{[jk]i}$, $\bar{\mathcal{R}}_\mu{}^{ji}$, $\bar{\mathcal{R}}^j{}_\nu{}^i$, $\bar{\mathcal{R}}^{[jk]i}$ and $\bar{\mathcal{G}}^i$. Thus the coefficient functions take the following form

$$
\begin{array}{rcl}
\mathcal{H}_N{}^M & \rightarrow & \delta_{N,\nu}\,\mathcal{H}_\nu{}^M \\
\bar{\mathcal{H}}_N{}^M & \rightarrow & \delta_{N,\nu}\,\bar{\mathcal{H}}_\nu{}^M \\
h_N{}^M & \rightarrow & \delta_{N,\nu}\,h_\nu{}^M \\
h_i{}^M & \rightarrow & \delta^{M,j}\,h_{ij} \\
h_{NK}{}^M & \rightarrow & \delta_{N,\nu}\delta_{K,\kappa}\,h_{\nu\lambda}{}^M \\
\mathcal{G}_{NK}{}^i & \rightarrow & \delta_{N,\nu}\delta_{K,\kappa}\,\mathcal{G}_{\nu\kappa}{}^i \\
\mathcal{P}_M{}^i & \rightarrow & \delta_{M,\mu}\,\mathcal{P}_\mu{}^i \\
\bar{\mathcal{P}}_M{}^i & \rightarrow & \delta_{M,\mu}\,\bar{\mathcal{P}}_\mu{}^i \\
\mathcal{Q}_{Mj}{}^i & \rightarrow & \delta_{M,\mu}\,\mathcal{Q}_{\mu j}{}^i \\
\bar{\mathcal{Q}}_{Mj}{}^i & \rightarrow & \delta_{M,\mu}\,\bar{\mathcal{Q}}_{\mu j}{}^i \\
\bar{\mathcal{R}}_M{}^i & \rightarrow & \delta_{M,\mu}\,\bar{\mathcal{R}}_\mu{}^i \\
\mathcal{R}_{MN}{}^i & \rightarrow & \delta_{M,\mu}\delta_{N,\nu}\,\mathcal{R}_{[\mu\nu]}{}^i \\
\bar{\mathcal{R}}_{MN}{}^i & \rightarrow & \delta_{M,\mu}\delta_{N,\nu}\,\bar{\mathcal{R}}_{[\mu\nu]}{}^i,
\end{array}
$$

where for simplicity we keep the old symbols for the new functions. This imposes the following conditions on the functions

$$
\begin{aligned}
0 &= B_i^0 + \delta_{M,i}\,\mathcal{H}^M \\
0 &= (f_M - \bar{f}_M) + 2G_{MN}\,\mathcal{H}^N + \delta_{M,i}\,\mathcal{G}^i \\
0 &= (\partial_N f_M - \partial_M \bar{f}_N) + \Omega_{MNK}\mathcal{H}^K + \delta_{M,i}\delta_{N,\nu}\,\bar{\mathcal{R}}_\nu{}^i - \delta_{N,i}\,\mathcal{R}_M{}^i \\
0 &= (\partial_M f_N - \partial_N f_M) - \Omega_{NKM}\mathcal{H}^K - 2G_{MK}\,\delta_{N,\nu}\,\mathcal{H}_\nu{}^K \\
&\quad -\delta_{M,\mu}\delta_{N,i}\,\bar{\mathcal{P}}_\mu{}^i + \delta_{M,i}\,g_N{}^i \\
0 &= (\partial_M \bar{f}_N - \partial_N \bar{f}_M) + \Omega_{KNM}\mathcal{H}^K + 2G_{MK}\,\delta_{N,\nu}\,\bar{\mathcal{H}}_\nu{}^K \\
&\quad -\delta_{M,\mu}\delta_{N,i}\,\mathcal{P}_\mu{}^i + \delta_{M,i}\,\bar{g}_N{}^i \\
0 &= \partial_K\partial_{[M}\bar{f}_{N]} + R_{KLNM}\mathcal{H}^L + \Omega_{K[M|L}\,\delta_{N],\nu}\,\bar{\mathcal{H}}_\nu{}^L \\
&\quad -\mathcal{P}_{K[M}{}^i\delta_{N],i} - \delta_{K,i}\delta_{[M,\mu}\delta_{N],\nu}\,\bar{\mathcal{R}}_{\mu\nu}{}^i \\
0 &= \partial_K\partial_{[M}f_{N]} - R_{NMKL}\mathcal{H}^L - \Omega_{[M|KL}\delta_{N],\nu}\,\mathcal{H}_\nu{}^L \\
&\quad -\bar{\mathcal{P}}_{K[M}{}^i\delta_{N],i} - \delta_{K,i}\delta_{[M,\mu}\delta_{N],\nu}\,\mathcal{R}_{\mu\nu}{}^i \\
0 &= -\delta_{K,i}\delta_{M,\mu}\,h_\mu{}^K + h_{ik}\delta^{K,k}\,G_{KM} - \delta_{M,j}\,\mathcal{G}_i{}^j \\
0 &= h_{(jk)} \\
0 &= \Omega_{MN}{}^j\,h_{ij} - \delta_{L,i}\delta_{M,\mu}\delta_{N,\nu}\,h_{\mu\nu}{}^L - \delta_{M,\mu}\delta_{N,j}\,\mathcal{Q}_{\mu i}{}^j - \delta_{M,j}\delta_{N,\nu}\,\bar{\mathcal{Q}}_{\nu i}{}^j \\
0 &= \delta_{M,\mu}\delta_{N,i}\,\mathcal{H}_\mu{}^N + \delta_{M,j}\,\bar{\mathcal{Q}}_i{}^j \\
0 &= \delta_{M,\mu}\delta_{N,i}\,\bar{\mathcal{H}}_\mu{}^N + \delta_{M,j}\,\mathcal{Q}_i{}^j \\
0 &= 2G_{K(M}\delta_{N),\nu}\,h_\nu{}^K + \Omega_{K(MN)}\mathcal{H}^K - \delta_{(M,i}\,\mathcal{G}_{N)}{}^i \\
0 &= 2G_{ML}\delta_{N,\nu}\delta_{K,\kappa}\,h_{\nu\kappa}{}^L + \delta_{M,\mu}\,\Omega_{NKL}\,h_\mu{}^L + \mathcal{H}^L\partial_L\Omega_{NKM} \\
&\quad +\delta_{N,\nu}\,\Omega_{LKM}\mathcal{H}_\nu{}^L + \delta_{K,\kappa}\,\Omega_{NLM}\bar{\mathcal{H}}_\kappa{}^L \\
&\quad -\delta_{M,i}\delta_{N,\nu}\delta_{K,\kappa}\,\mathcal{G}_{\nu\kappa}{}^i - \delta_{K,i}\,\mathcal{Q}_{MN}{}^i - \delta_{N,i}\,\bar{\mathcal{Q}}_{MK}{}^i \\
0 &= \tfrac{1}{2}\,\mathcal{H}^R\partial_R R_{NMLK} - \delta_{N,\nu}\,R_{MRLK}\mathcal{H}_\nu{}^R + \delta_{L,\lambda}\,R_{NMRK}\bar{\mathcal{H}}_\lambda{}^R \\
&\quad +\delta_{N,\nu}\delta_{L,\lambda}\,\Omega_{MKR}\,h_{\nu\lambda}{}^R - \delta_{L,i}\,\mathcal{P}_{MNK}{}^i - \delta_{M,i}\,\bar{\mathcal{P}}_{KLN}{}^i
\end{aligned} \tag{6.1.6}
$$

where the last equation has to be antisymmetrized in $M \leftrightarrow N$ and $K \leftrightarrow L$. We will now work out these conditions and remove simultaneously coboundary terms. Since $h_{[ab]}$ can be removed by a coboundary term the equations

$$h_{(ij)} = 0 \qquad \delta_{K,i}\delta_{M,\mu}\,h_\mu{}^K - h_{ik}\delta^{K,k}\,G_{KM} + \delta_{M,j}\,\mathcal{G}_i{}^j = 0$$

require

$$h_{mi} = \mathcal{G}_i{}^j = 0$$

Furthermore

$$
\begin{aligned}
\Omega_{MN}{}^j\,h_{ij} - \delta_{L,i}\delta_{M,\mu}\delta_{N,\nu}\,h_{\mu\nu}{}^L - \delta_{M,\mu}\delta_{N,j}\,\mathcal{Q}_{\mu i}{}^j - \delta_{M,j}\delta_{N,\nu}\,\bar{\mathcal{Q}}_{\nu i}{}^j &= 0 \\
\delta_{M,\mu}\delta_{N,i}\,\mathcal{H}_\mu{}^N + \delta_{M,j}\,\bar{\mathcal{Q}}_i{}^j &= 0 \\
\delta_{M,\mu}\delta_{N,i}\,\bar{\mathcal{H}}_\mu{}^N + \delta_{M,j}\,\mathcal{Q}_i{}^j &= 0
\end{aligned}
$$

require

$$\begin{aligned} h_{\mu\nu i} = \mathcal{Q}_{\mu i}{}^{j} = \bar{\mathcal{Q}}^{j}_{\mu i} &= 0 \\ \mathcal{H}_{\mu i} = \bar{\mathcal{H}}_{\mu i} = \mathcal{Q}_{i}{}^{j} = \bar{\mathcal{Q}}_{i}{}^{j} &= 0 \end{aligned}$$

Again we introduce the following combinations for the f's

$$f_M^+ = f_M + \bar{f}_M \qquad f_M^- = f_M - \bar{f}_M,$$

where f_M^- is determined by

$$(f_M - \bar{f}_M) + 2G_{MN}\,\mathcal{H}^N + \delta_{M,i}\,\mathcal{G}^i = 0.$$

Exploiting the freedom to redefine

$$\begin{aligned} f_M &\to f_M + \partial_M \hat{f} + \delta_{M,i}\hat{g}^i \\ \bar{f}_M &\to \bar{f}_M + \partial_M \hat{f} + \delta_{M,i}\hat{\bar{g}}^i \end{aligned}$$

we can remove $\mathcal{G}^a$ by an appropriate choice of $\hat{g}^a - \hat{\bar{g}}^a$. Still we are left with the freedom to redefine f_M^+. From the third equation of (6.1.6) we obtain by symmetrization and antisymmetrization and the use of the second equation

$$\begin{aligned} \mathcal{L}_{\mathcal{H}} G_{MN} &= \delta_{(M,i}\delta_{N),\nu}\,\bar{\mathcal{R}}_{\nu}{}^{i} - \delta_{(N,i}\,\mathcal{R}_{M)}{}^{i} \\ \mathcal{L}_{\mathcal{H}} B_{MN} &= -\partial_{[M} f''^{+}_{N]} + \delta_{[M,i}\delta_{N],\nu}\,\bar{\mathcal{R}}_{\nu}{}^{i} - \delta_{[N,i}\,\mathcal{R}_{M]}{}^{i}, \end{aligned} \tag{6.1.7}$$

where

$$f''_M = f_M + 2B_{MK}\,\mathcal{H}^K.$$

Symmetrization and antisymmetrization of the fourth and fifth equation of (6.1.6) yields

$$\begin{aligned} \mathcal{L}_{\mathcal{H}} G_{MN} &= 2G_{K(M}\left(\partial_{N)}\mathcal{H}^K - \delta_{N),\nu}\,\mathcal{H}_{\nu}{}^{K}\right) \\ &\quad -\delta_{(M,\mu}\delta_{N),i}\,\bar{\mathcal{P}}_{\mu}{}^{i} + \delta_{(M,i}\,g_{N)}{}^{i}, \\ \mathcal{L}_{\mathcal{H}} B_{MN} &= -\partial_{[M} f''^{+}_{N]} - 2G_{K[M}\left(\partial_{N]}\mathcal{H}^K - \delta_{N],\nu}\,\mathcal{H}_{\nu}{}^{K}\right) \\ &\quad +\delta_{[M,\mu}\delta_{N],i}\,\bar{\mathcal{P}}_{\mu}{}^{i} - \delta_{[M,i}\,g_{N]}{}^{i}, \\ \mathcal{L}_{\mathcal{H}} G_{MN} &= 2G_{K(M}\left(\partial_{N)}\mathcal{H}^K - \delta_{N),\nu}\,\bar{\mathcal{H}}_{\nu}{}^{K}\right) \\ &\quad +\delta_{(M,\mu}\delta_{N),i}\,\mathcal{P}_{\mu}{}^{i} - \delta_{(M,i}\,\bar{g}_{N)}{}^{i}, \\ \mathcal{L}_{\mathcal{H}} B_{MN} &= -\partial_{[M} f''^{+}_{N]} + 2G_{K[M}\left(\partial_{N]}\mathcal{H}^K - \delta_{N],\nu}\,\bar{\mathcal{H}}_{\nu}{}^{K}\right) \\ &\quad +\delta_{[M,\mu}\delta_{N],i}\,\mathcal{P}_{\mu}{}^{i} - \delta_{[M,i}\,\bar{g}_{N]}{}^{i}, \end{aligned} \tag{6.1.8}$$

i.e., they are of the same structure as (6.1.7). We will thus be able to reduce the number of independent coefficient functions by comparing (6.1.7) with

(6.1.8). Furthermore the equations for $C^1_{(1/2,1/2)}$ (6.1.5) give the following identifications

$$\mathcal{H}^M = -\mathcal{K}^M \Rightarrow f'^+ = f''^+$$
$$\mathcal{G}^i = \mathcal{K}^i, \quad \bar{\mathcal{K}}_\nu{}^i = \bar{\mathcal{R}}_\nu{}^i \quad \mathcal{K}_M{}^i = \mathcal{R}_M{}^i$$

To complete the solution we make the following ansatz for $A^1_{(1,1/2)}$ and $A^1_{(1/2,1)}$

$$\begin{aligned}
A^1_{(1,1/2)} &= \bar{\psi}^*_M \mathcal{A}^M_{(0,0)} + F^*_M \mathcal{A}^M_{(1/2,0)} + \phi^*_i \mathcal{A}^i_{(1/2,0)} \\
&\quad + \lambda^*_i \mathcal{B}^i_{(1,0)} + \bar{\lambda}^*_i \mathcal{B}^i_{(1/2,1/2)} + A^*_i \mathcal{E}^i_{(0,1/2)} + \mathcal{D}\lambda^*_i \mathcal{E}^i_{(0,0)} \\
A^1_{(1/2,1)} &= \psi^*_M \bar{\mathcal{A}}^M_{(0,0)} + F^*_M \bar{\mathcal{A}}^M_{(0,1/2)} + \phi^*_i \bar{\mathcal{A}}^i_{(0,1/2)} \\
&\quad + \lambda^*_i \bar{\mathcal{B}}^i_{(1/2,1/2)} + \bar{\lambda}^*_i \bar{\mathcal{B}}^i_{(0,1)} + \bar{A}^*_i \bar{\mathcal{E}}^i_{(1/2,0)} + \bar{\mathcal{D}}\bar{\lambda}^*_i \bar{\mathcal{E}}^i_{(0,0)}
\end{aligned}$$

The coefficient functions contained in the expression above are

$$\begin{aligned}
\mathcal{A}^M_{(0,0)} &= \mathcal{A}^M \\
\mathcal{A}^M_{(1/2,0)} &= \psi^N \mathcal{A}_N{}^M \\
\mathcal{A}^i_{(1/2,0)} &= \psi^N \mathcal{A}_N{}^i \\
\mathcal{B}^i_{(1,0)} &= \mathcal{D}X^M \mathcal{B}_M{}^i + \psi^M \psi^N \mathcal{B}_{MN}{}^i \\
\mathcal{B}^i_{(1/2,1/2)} &= F^M b_M{}^i + \phi^j b_j{}^i + \psi^M \bar{\psi}^N b_{MN}{}^i \\
\mathcal{E}^i_{(0,1/2)} &= \bar{\psi}^M \mathcal{E}_M{}^i \\
\mathcal{E}^i_{(0,0)} &= \mathcal{E}^i \\
\bar{\mathcal{A}}^M_{(0,0)} &= \bar{\mathcal{A}}^M \\
\bar{\mathcal{A}}^M_{(0,1/2)} &= \bar{\psi}^N \bar{\mathcal{A}}_N{}^M \\
\bar{\mathcal{A}}^i_{(0,1/2)} &= \bar{\psi}^N \bar{\mathcal{A}}_N{}^i \\
\bar{\mathcal{B}}^i_{(0,1)} &= \bar{\mathcal{D}}X^M \bar{\mathcal{B}}_M{}^i + \bar{\psi}^M \bar{\psi}^N \bar{\mathcal{B}}_{MN}{}^i \\
\bar{\mathcal{B}}^i_{(1/2,1/2)} &= F^M \bar{b}_M{}^i + \phi^j \bar{b}_j{}^i + \psi^M \bar{\psi}^N \bar{b}_{MN}{}^i \\
\bar{\mathcal{E}}^i_{(1/2,0)} &= \psi^M \bar{\mathcal{E}}_M{}^i \\
\bar{\mathcal{E}}^i_{(0,0)} &= \bar{\mathcal{E}}^i
\end{aligned}$$

Following the procedure for $A^1_{(1,1)}$ we remove trivial parts by considering the Koszul-Tate part of the BRST transformations of

$$\begin{aligned}
A^2_{(1,1/2)} &= F^*_M \bar{\lambda}^*_i \hat{\mathcal{A}}^{iM}_{(0,0)} + \phi^*_i \bar{\lambda}^*_j \hat{\mathcal{A}}^{ji}_{(0,0)} + A^*_i \lambda^*_j \hat{\mathcal{B}}^{ji}_{(0,0)} \\
&\quad + \lambda^*_i \bar{\lambda}^*_j \hat{\mathcal{B}}^{ji}_{(1/2,0)} + \bar{\lambda}^*_i \bar{\lambda}^*_j \hat{\mathcal{E}}^{(ji)}_{(0,1/2)} \\
A^2_{(1/2,1)} &= F^*_M \lambda^*_i \hat{\bar{\mathcal{A}}}^{iM}_{(0,0)} + \phi^*_i \lambda^*_j \hat{\bar{\mathcal{A}}}^{ji}_{(0,0)} + \bar{A}^*_i \bar{\lambda}^*_j \hat{\bar{\mathcal{B}}}^{ji}_{(0,0)} \\
&\quad + \lambda^*_i \bar{\lambda}^*_j \hat{\bar{\mathcal{B}}}^{ji}_{(0,1/2)} + \lambda^*_i \lambda^*_j \hat{\bar{\mathcal{E}}}^{(ji)}_{(1/2,0)}.
\end{aligned}$$

The coefficient functions are again independent of ghosts and antifields and constrained by the conformal weight indicated in the subscript brackets (m,n). Going through the same steps as described in detail for $A^1_{(1,1)}$ and $C^1_{(1/2,1/2)}$ one finds the corresponding equations, which are of the same structure as (6.1.4) and (6.1.4). For $A^1_{(1,1/2)}$ they explicitly read

$$
\begin{aligned}
0 &= B_i^0 + \delta_{M,i}\,\mathcal{A}^M \\
0 &= (f_M - \bar{f}_M) + 2G_{MN}\,\mathcal{A}^N + \delta_{M,i}\,\mathcal{E}^i \\
0 &= (\partial_N f_M - \partial_M \bar{f}_N) + \Omega_{MNK}\mathcal{A}^K \\
&\quad -\delta_{M,i}\delta_{N,\nu}\,\mathcal{E}_\nu{}^i + \delta_{N,i}\,\mathcal{B}_M{}^i \\
0 &= (\partial_M f_N - \partial_N f_M) - \Omega_{NKM}\mathcal{A}^K - 2G_{MK}\,\delta_{N,\nu}\,\mathcal{A}_\nu{}^K \\
&\quad +\delta_{M,i}\,\mathcal{A}_N{}^i + \delta_{N,i}\delta_{M,\mu}\,b_\mu{}^i \\
0 &= \partial_K\partial_{[M}f_{N]} - R_{NMKL}\mathcal{A}^L - \Omega_{[M|KL}\delta_{N],\nu}\,\mathcal{A}_\nu{}^L \\
&\quad +\delta_{[N,i}b_{M]K}{}^i + \delta_{K,i}\delta_{[M,\mu}\delta_{N],\nu}\,\mathcal{B}_{\mu\nu}{}^i \\
0 &= \delta_{M,i}\delta_{N,\nu}\,\mathcal{A}_\nu{}^M + \delta_{N,j}\,b_i{}^j.
\end{aligned}
$$

The analogous equations for $A^1_{(1/2,1)}$ are

$$
\begin{aligned}
0 &= B_i^0 - \delta_{M,i}\,\bar{\mathcal{A}}^M \\
0 &= (f_M - \bar{f}_M) - 2G_{MN}\,\bar{\mathcal{A}}^N + \delta_{M,i}\,\bar{\mathcal{E}}^i \\
0 &= (\partial_N f_M - \partial_M \bar{f}_N) - \Omega_{MNK}\bar{\mathcal{A}}^K \\
&\quad +\delta_{M,i}\delta_{N,\nu}\,\bar{\mathcal{B}}_\nu{}^i - \delta_{N,i}\,\bar{\mathcal{E}}_M{}^i \\
0 &= (\partial_M \bar{f}_N - \partial_N \bar{f}_M) - \Omega_{KNM}\bar{\mathcal{A}}^K + 2G_{MK}\,\delta_{N,\nu}\,\bar{\mathcal{A}}_\nu{}^K \\
&\quad -\delta_{M,i}\,\bar{\mathcal{A}}_N{}^i + \delta_{M,\mu}\delta_{N,i}\,\bar{b}_\mu i \\
0 &= \partial_K\partial_{[M}\bar{f}_{N]} - R_{KLNM}\bar{\mathcal{A}}^L + \Omega_{K[M|L}\delta_{N],\nu}\,\bar{\mathcal{A}}_n{}^L \\
&\quad -d_{[M,i}\bar{b}_{K|N]}{}^i - \delta_{K,i}\delta_{[M,\mu}\delta_{N],\nu}\,\bar{\mathcal{B}}_{\mu\nu}{}^i \\
0 &= \delta_{M,i}\delta_{N,\nu}\,\bar{\mathcal{A}}_\nu{}^M - \delta_{N,j}\,\bar{b}_i{}^j
\end{aligned}
$$

Comparing the equations above with the the relevant equations for $A^1_{(1,1)}$ leads to the following identifications

$$
\begin{gathered}
\mathcal{H}^M = \mathcal{A}^M = -\bar{\mathcal{A}}^M \quad \mathcal{G}^i = \mathcal{E}^i = \bar{\mathcal{E}}^i \\
\mathcal{R}_M{}^i = -\mathcal{B}_M{}^i = \bar{\mathcal{E}}_M{}^i \quad \mathcal{R}_\mu{}^i = \bar{\mathcal{B}}_\mu{}^i = -\mathcal{E}_\mu{}^i \\
\mathcal{H}_\nu{}^\kappa = \mathcal{A}_\nu^\kappa \quad \bar{\mathcal{H}}_\nu{}^\kappa = \bar{\mathcal{A}}_\nu^\kappa \\
g_N{}^i = \mathcal{A}_N{}^i \quad \bar{g}_N{}^i = -\bar{\mathcal{A}}_N{}^i \quad \mathcal{P}_\mu{}^i = -\bar{b}_\mu{}^i \quad \bar{\mathcal{P}}_\mu{}^i = -b_\mu{}^i \\
\mathcal{R}_{\mu\nu}{}^i = -\mathcal{B}_{\mu\nu}{}^i \quad \bar{\mathcal{R}}_{\mu\nu}{}^i = \bar{\mathcal{B}}_{\mu\nu}{}^i \quad \mathcal{P}_{KM}{}^i = -b_{MK}{}^i \quad \bar{\mathcal{P}}_{KM}{}^i = b_{KM}{}^i
\end{gathered}
$$

Thus the complete result $\omega_1^0 + {}^0\omega_1^1 + {}^1\omega_1^1 + {}^2\omega_1^1$ up to antifield number 1 reads

$$
\begin{aligned}
\omega_1^0 \;=\;& (\eta\mathcal{D}X^M+\bar\eta\bar{\mathcal{D}}X^M)\tfrac{1}{2}\,f_M^+-(\eta\mathcal{D}X^M-\bar\eta\bar{\mathcal{D}}X^M)G_{MN}\mathcal{H}^N\\
&-(\eta\psi^M\psi^N+\bar\eta\bar\psi^M\bar\psi^N)\tfrac{1}{2}\,\partial_N f_M^+\\
&+(\eta\psi^M\psi^N-\bar\eta\bar\psi^M\bar\psi^N)\partial_N(G_{MK}\mathcal{H}^K)\\
&+\tfrac{1}{2}\,(\varepsilon\psi^M+\bar\varepsilon\bar\psi^M)\tfrac{1}{2}\,f_M^+-\tfrac{1}{2}\,(\varepsilon\psi^M-\bar\varepsilon\bar\psi^M)G_{MN}\mathcal{H}^N-C^i\delta_{M,i}\mathcal{H}^M
\end{aligned}
$$

$$
\begin{aligned}
{}^0\omega_1^1 \;=\;& \eta\bar\eta(X_M^*\mathcal{H}^M+\psi_M^*\psi^N\delta_{N,\nu}\delta^{M,\mu}\mathcal{H}_\nu{}^\mu+\bar\psi_M^*\bar\psi^N\delta_{N,\nu}\delta^{M,\mu}\bar{\mathcal{H}}_\nu{}^\mu\\
&+F_M^*(F^N\delta_{N,\nu}\delta^{M,\mu}h_\nu{}^\mu+\psi^N\bar\psi^K\delta_{N,\nu}\delta_{K,\kappa}\delta^{M,\mu}h_{\nu\kappa}{}^\mu)\\
&+\phi_i^*(F^N\mathcal{G}_N{}^i+\psi^N\bar\psi^K\delta_{N,\nu}\delta_{K,\kappa}\mathcal{G}_{\nu\kappa}{}^i)\\
&+\lambda_i^*(\mathcal{D}\bar\psi^M\delta_{M,\mu}\mathcal{P}_\mu{}^i+\mathcal{D}X^M\bar\psi^N\mathcal{P}_{MN}{}^i\\
&\qquad+\psi^M\psi^N\bar\psi^K\mathcal{P}_{MNK}{}^i+F^M\psi^N\mathcal{Q}_{MN}{}^i)\\
&+\bar\lambda_i^*(\bar{\mathcal{D}}\psi^M\delta_{M,\mu}\bar{\mathcal{P}}_m{}^i+\bar{\mathcal{D}}X^M\psi^N\bar{\mathcal{P}}_{MN}{}^i\\
&\qquad+\bar\psi^M\bar\psi^N\psi^K\bar{\mathcal{P}}_{MNK}{}^i+F^M\bar\psi^N\bar{\mathcal{Q}}_{MN}{}^i)\\
&+A_i^*(\bar{\mathcal{D}}X^M\delta_{M,\mu}\bar{\mathcal{R}}_\mu{}^i+\bar\psi^M\bar\psi^N\delta_{M,\mu}\delta_{N,\nu}\bar{\mathcal{R}}_{\mu\nu}{}^i)\\
&+\bar A_i^*(\mathcal{D}X^M\mathcal{R}_M{}^i+\psi^M\psi^N\delta_{M,\mu}\delta_{N,\nu}\mathcal{R}_{\mu\nu}{}^i)\\
&+\bar{\mathcal{D}}\bar\lambda_i^*\psi^M g_M{}^i+\mathcal{D}\lambda_i^*\bar\psi^M\bar g_M{}^i\\
&-\eta C^i A_j^*(\partial^j\delta_{M,i}\mathcal{H}^M)+\bar\eta C^i\bar A_j^*(\partial^j\delta_{M,i}\mathcal{H}^M)
\end{aligned}
$$

$$
\begin{aligned}
{}^1\omega_1^1 \;=\;& \eta\bar\varepsilon\big(\bar\psi_M^*\mathcal{H}^M+F_M^*\psi^N\delta_{N,\nu\delta^{M,\mu}}\mathcal{H}_\nu{}^\mu+\phi_i^*\psi^N g_N{}^i\\
&-\lambda_i^*(\mathcal{D}X^M\mathcal{R}_M{}^i+\psi^M\psi^N\delta_{M,\mu}\delta_{N,\nu}\mathcal{R}_{\mu\nu}{}^i)\\
&-\bar\lambda_i^*(F^M\delta_{M,\mu}\bar{\mathcal{P}}_\mu{}^i+\psi^M\bar\psi^N\mathcal{P}_{NM}{}^i)\\
&-A_i^*\bar\psi^N\delta_{N,\nu}\bar{\mathcal{R}}_\nu{}^i\big)\\
&+\bar\eta\varepsilon\big(-\psi_M^*\mathcal{H}^M+F_M^*\bar\psi^N\delta_{N,\nu}\delta^{M,\mu}\bar{\mathcal{H}}_\nu{}^\mu-\phi_i^*\bar\psi^N\bar g_N{}^i\\
&-\lambda_i^*(F^M\delta_{M,\mu}\mathcal{P}_\mu{}^i-\psi^M\bar\psi^N\bar{\mathcal{P}}_{MN}{}^i)\\
&+\bar\lambda_i^*(\bar{\mathcal{D}}X^M\delta_{M,\mu}\bar{\mathcal{R}}_\mu{}^i+\bar\psi^M\bar\psi^N\delta_{M,\mu}\delta_{N,\nu}\bar{\mathcal{R}}_{\mu\nu}{}^i)\\
&+\bar A_i^*\psi^N\mathcal{R}_N{}^i\big)\\
&+\varepsilon C^i\bar\lambda_j^*(\partial^j\delta_{M,i}\mathcal{H}^M)-\bar\varepsilon C^i\lambda_j^*(\partial^j\delta_{M,i}\mathcal{H}^M)
\end{aligned}
$$

$$
{}^2\omega_1^1 \;=\; \varepsilon\bar\varepsilon\big(-F_M^*\mathcal{H}^M+\phi_i^*\mathcal{G}^i+\lambda_i^*\psi^N\mathcal{R}_N{}^i+\bar\lambda_i^*\bar\psi^N\delta_{N,\nu}\bar{\mathcal{R}}_\nu{}^i\big)
$$

where the coefficient functions have to fulfill the equations (6.1.7) and (6.1.8) and the remaining equations of (6.1.6). In fact it turns out that the number of independent coefficient functions can be reduced, since (6.1.7) and (6.1.8) are of the same structure and the higher order differential equations in (6.1.6) (i.e., the sixth and seventh equation in (6.1.6)) turn out to be derivatives of the generalized Killing equations. We will not work this out for the general case but instead investigate a specific example in the next section.

As a final remark we note that the solution given above is defined only up to redefinitions

$$
\begin{array}{rcl}
f^+_M & \to & f^+_M + \partial_M f + \delta_{M,a} g^a \\
\mathcal{P}_{MN}{}^a & \to & \mathcal{P}_{MN}{}^a - 2\delta_{N,b}\, \hat{g}_M{}^{(ab)} \\
\mathcal{P}_{MNK}{}^a & \to & \mathcal{P}_{MNK}{}^a - 2\delta_{K,b}\, \hat{g}_{[MN]}{}^{(ab)} + \delta_{M,b}\, \hat{\hat{g}}_{NK}{}^{ab} \\
\mathcal{Q}_{MN}{}^a & \to & \mathcal{Q}_{MN}{}^a + \delta_{N,b}\, \hat{\hat{g}}_M{}^{ab} \\
\bar{\mathcal{P}}_{MN}{}^a & \to & \bar{\mathcal{P}}_{MN}{}^a + 2\delta_{N,b}\, \hat{\bar{g}}_M{}^{(ab)} \\
\bar{\mathcal{P}}_{MNK}{}^a & \to & \bar{\mathcal{P}}_{MNK}{}^a + 2\delta_{K,b}\, \hat{\bar{g}}_{[MN]}{}^{(ab)} + \delta_{M,b}\, \hat{\bar{g}}_{KN}{}^{ba} \\
\bar{\mathcal{Q}}_{MN}{}^a & \to & \bar{\mathcal{Q}}_{MN}{}^a - \delta_{N,b}\, \hat{\bar{g}}_M{}^{ba}
\end{array}
$$

which alter the solution by a coboundary.

6.2 Global symmetries

6.2.1 Simplified action

For further discussion we shall assume in the following that the functions D_i coincide with a subset of the fields X^M. We denote this subset by $\{y^i\}$ and the remaining X's by x^μ,

$$\{X^M\} = \{x^\mu, y^i\}, \quad D_i = y^i. \tag{6.2.9}$$

In fact, this assumption is a very mild one because, except at stationary points of $D_i(X)$, (6.2.9) can be achieved by a field redefinition $X^M \to \tilde{X}^M = \tilde{X}^M(X)$ ("coordinate transformation in X-space"), where this redefinition is such that each nonconstant $D_i(X)$ becomes one of the $\tilde{X}$'s. Indeed, constant D_i give only contributions to the Lagrangian which are total derivatives and can thus be neglected, at least classically; nonconstant D_i can be assumed to be independent by a suitable choice of basis for the gauge fields and may thus be taken as $\tilde{X}$'s, at least locally (e.g., if $D_1 = D_2$, the Lagrangian depends only on the combination $A^1_m + A^2_m$ which can be introduced as a new gauge field).

It is now easy to see that the Lagrangian (4.1.14) can actually be simplified by setting the fields $\psi^i, F^i, \lambda^i, \phi^i$ to zero. Indeed, owing to (6.2.9), the classical equations of motion for λ^i and ϕ^i yield $\psi^i = 0$ and $F^i = 0$, respectively. The latter equations are algebraic and can be used in the Lagrangian. Then the Lagrangian does not contain λ^i and ϕ^i anymore and the only remnant of the gauge multiplets are the terms $e\varepsilon^{mn} y^i \partial_m A^i_n$. This reflects that the gauge multiplets carry no dynamical degrees of freedom since the world-sheet is 2-dimensional. Of course, the BRST transformations given in section 3.2 must be adapted in order to provide the gauge symmetries of the simplified Lagrangian: those fields that are eliminated from the action must also be eliminated from the transformations of the remaining fields using the equations of motion of the eliminated fields. This only affects the supersymmetry transformations of y^i and A^i_m. The new supersymmetry transformation of y^i is then simply zero owing to the field equation for λ^i ($\psi^i = 0$). This is not in contradiction with the supersymmetry algebra because the equations of motion for the A^i_m give $\partial_m y^i = 0$ (of course, after eliminating the fields $\psi^i, F^i, \lambda^i, \phi^i$, the supersymmetry algebra holds only on-shell). The latter also shows that the fields y^i carry no dynamical degree of freedom. The new supersymmetry transformation of A^i_m is more complicated and arises from the original one by using the equations of motion for F^i and ψ^i to replace λ^i and ϕ^i, and then setting F^i and ψ^i to zero.

6.2.2 Nontrivial global symmetries

Let us now discuss the nontrivial global symmetries of the action (4.1.14) as obtained from the BRST cohomology in the space of antifield dependent local functionals with ghost number -1. This cohomology feels of course the particular action, for the latter enters the BRST transformations of the antifields through the Euler-Lagrange derivatives of the Lagrangian. We present now the resulting global symmetries for the simplified form of the action arising from the Lagrangian (4.1.14) by eliminating the fields $\psi^i, F^i, \lambda^i, \phi^i$ as described above, assuming (6.2.9). The nontrivial symmetries[3] are generated by the following transformations,

$$
\begin{aligned}
\Delta h_{mn} &= 0\\
\Delta \chi^\alpha_m &= 0\\
\Delta X^M &= \mathcal{H}^M, \quad \mathcal{H}^i = K^i(y), \quad \mathcal{H}^\mu = V^\mu(X)\\
\Delta \psi^\mu_\alpha &= \psi^\nu_\alpha \partial_\nu V^\mu(X)\\
\Delta F^\mu &= F^\nu \partial_\nu V^\mu(X) + \frac{1}{2}\bar{\psi}^\nu \psi^\lambda \partial_\nu \partial_\lambda V^\mu(X)\\
\Delta A^i_m &= b^i_M(X)\partial_m X^M + \varepsilon_m{}^n a^i_M(X)\partial_n X^M - \delta_{jk} A^j_m \partial_i K^k(y)\\
&\quad -\chi_n \gamma_m \gamma^n \gamma_* \psi^\mu a^i_\mu(X) + \tfrac{\mathrm{i}}{2}\,\bar{\psi}^\mu \gamma_m \psi^\nu \partial_{[\nu} b^i_{\mu]}(X)\\
&\quad +\tfrac{\mathrm{i}}{2}\,\bar{\psi}^\mu \gamma_m \gamma_* \psi^\nu \partial_{[\nu} a^i_{\mu]}(X)
\end{aligned}
\tag{6.2.10}
$$

where $\mathcal{H}^M$, a^i_M and b^i_M have to solve the following generalized Killing vector equations,

$$
\begin{aligned}
\mathcal{L}_\mathcal{H} G_{MN} &= -2\delta_{i(M} a^i_{N)}\\
\mathcal{L}_\mathcal{H} B_{MN} &= -2\partial_{[M} p_{N]} - 2\delta_{i[M} b^i_{N]}
\end{aligned}
\tag{6.2.11}
$$

[3]A global symmetry is called trivial in this context when it is equal to a gauge transformation on-shell.

for some functions $p_M(X)$ ($\mathcal{L}_\mathcal{H}$ is the standard Lie derivative along $\mathcal{H}^M$, δ_{iM} is the Kronecker symbol, i.e., $\delta_{iM} = 1$ if $M = i$ and $\delta_{iM} = 0$ otherwise). Note that the p_M do not occur in the Δ-transformations; however, they do contribute to the corresponding Noether currents.

The equations (6.2.11) are actually the same as the equations which also determine the symmetries of bosonic string and D-string actions [38–40, 42], specified for (6.2.9). In absence of gauge fields (no A^i_m, y^i, K^i; $\{\mathcal{H}^M\} \equiv \{V^\mu\}$), they read

$$\mathcal{L}_V G_{\mu\nu} = 0, \quad \mathcal{L}_V B_{\mu\nu} = -2\partial_{[\mu}p_{\nu]} \ . \tag{6.2.12}$$

These equations had been already discussed in [87, 88, 90]. The first equation (6.2.12) is just the standard Killing vector equation for $G_{\mu\nu}$. Hence, the solutions of equations (6.2.12) are those Killing vector fields of $G_{\mu\nu}$ which solve the second equation (6.2.12) (for some p_μ).

The situation changes when gauge fields are present. Then equations (6.2.11) read for $M, N = \mu, \nu$:

$$\begin{aligned} \mathcal{L}_V G_{\mu\nu} &= -K^i\partial_i G_{\mu\nu} \\ \mathcal{L}_V B_{\mu\nu} &= -K^i\partial_i B_{\mu\nu} - 2\partial_{[\mu}p_{\nu]} \end{aligned} \tag{6.2.13}$$

where $\mathcal{L}_V$ is the Lie derivative along the vector field V^M given by $V^i = 0$, $V^\mu = V^\mu(X)$. The remaining equations (6.2.11) just determine the functions a^i_M and b^i_M,

$$\begin{aligned} a^i_\mu = -\mathcal{L}_\mathcal{H} G_{\mu i} \ , \quad a^j_i = -2\mathcal{L}_\mathcal{H} G_{ij} \\ b^i_\mu = \mathcal{L}_\mathcal{H} B_{\mu i} - \partial_i p_\mu \ , \quad b^j_i = 2\mathcal{L}_\mathcal{H} B_{ij} \ . \end{aligned} \tag{6.2.14}$$

Here we have used that p_i and the parts of a^j_i resp. b^j_i which are antisymmetric resp. symmetric in i, j can be set to zero without loss of generality (the corresponding contributions to Δ can be removed by subtracting trivial global symmetries from Δ).

The global symmetries are thus completely determined by equations (6.2.13). Note that these equations reproduce (6.2.12) for $K^i = 0$, except that now $G_{\mu\nu}$ and $B_{\mu\nu}$ depend in general not only on the x^μ but also on the y^i. Hence, in general V^μ and p_μ also depend on the y^i. For the discussion of equations (6.2.13), the y^i may be viewed as parameters of $G_{\mu\nu}$ and $B_{\mu\nu}$. Solutions to equations (6.2.13) with $K^i = 0$ can thus be regarded as solutions to equations (6.2.12) for some $G_{\mu\nu}$ and $B_{\mu\nu}$ involving parameters y^i. The global symmetries with $K^i = 0$ are thus analogous to the symmetries of ordinary superstrings and correspond to isometries of the (parameter-dependent) metric $G_{\mu\nu}$. In contrast, solutions to (6.2.13) with $K^i \neq 0$ have no counterparts among the solutions of (6.2.12). Such solutions may be called "dilatational" solutions, because in special cases they are true dilatations, as we will see in the example below (further examples can be found in [38, 39, 42]).

Finally we note that the solutions to equations (6.2.13) come in infinite families and that, as a consequence, the corresponding commutator algebra of the global symmetries is an infinite dimensional loop-like algebra. This has been observed and discussed already in [38, 39] and is an immediate consequence of the fact that the action depends on the A^i_m only via their field strengths [89]. All members of a family arise from one of its representatives by multiplying the functions $V^\mu(X)$, $K^i(y)$, $p_\mu(X)$ of that representative with an arbitrary function of the y^i. One can directly verify that this makes sense: if $V^\mu(X)$, $K^i(y)$, $p_\mu(X)$ is a solution to equations (6.2.13), then another solution is obtained by simply multiplying V^μ, K^i, p_μ by the same arbitrary function of the y^i. As the y^i are constant on-shell (by the equations of motion for the A^i_m), this infinite dimensionality of the space of global symmetries has no practical importance, i.e., in order to discuss the global symmetries it is sufficient to consider just one representative of each family.

6.3 Example

To illustrate the results presented above, we specify them for a simple class of models, which were treated already in [38, 39] for the purely bosonic case. These models are characterized by Lagrangians containing only one $U(1)$ gauge field A_m and the following choices for the background

$$\begin{aligned} G_{yM} &= 0, \qquad & G_{\mu\nu} &= f(y)\eta_{\mu\nu}, \\ B_{y\mu} &= 0, \qquad & B_{\mu\nu} &= B_{\mu\nu}(y), \end{aligned} \tag{6.3.15}$$

leading to

$$\begin{aligned} e^{-1}L \;=\; & -\tfrac{1}{2}\, h^{mn}\partial_m x^\mu \partial_n x^\nu G_{\mu\nu} + \tfrac{1}{2}\,\varepsilon^{mn}\partial_m x^\mu \partial_n x^\nu B_{\mu\nu} \\ & +\chi_m(\gamma^n\gamma^m)\psi^\nu\partial_n x^\mu G_{\mu\nu} - \tfrac{1}{4}\,\chi_m(\gamma^n\gamma^m C)\chi_n\overline{\psi}^\mu\psi^\nu G_{\mu\nu} \\ & +\tfrac{\mathrm{i}}{2}\,\overline{\psi}^\mu\gamma^m\partial_m\psi^\nu G_{\mu\nu} - \tfrac{\mathrm{i}}{4}\,\overline{\psi}^\nu(\gamma^m\gamma_*)\psi^\mu\partial_m y\,\partial_y B_{\mu\nu} \\ & +\tfrac{1}{2}\,\varepsilon^{mn}(\partial_m A_n - \partial_n A_m)y, \end{aligned} \tag{6.3.16}$$

where the assumption $\{X^M\} = \{x^\mu, y\}$ is taken into account. As shown in [38, 39], in this case the general solution of equations (6.2.11) is (modulo redefinitions corresponding to trivial global symmetries)

$$\begin{aligned} V^\mu &= -\tfrac{1}{2}\, K(y)[\ln f(y)]' x^\mu + r^\mu(y) + r^{[\mu\lambda]}(y)\eta_{\lambda\nu}x^\nu \\ a_\mu &= -V^{\lambda\prime} f(y)\eta_{\mu\lambda}, \quad a_y = 0 \\ b_\mu &= -\tfrac{1}{2}\,(K(y)B'_{\mu\nu})' x^\nu - B'_{\mu\nu}V^\nu, \quad b_y = 0 \\ p_\mu &= K(y)B'_{\mu\nu}x^\nu + 2B_{\mu\nu}V^\nu, \end{aligned} \tag{6.3.17}$$

where a prime denotes differentiation with respect to y:

$$' \equiv \frac{\partial}{\partial y}.$$

$K(y)$, $r^{\mu}(y)$ and $r^{[\mu\lambda]}(y)$ are arbitrary functions of y and correspond to families of dilatations, translations and Lorentz-transformations in target space, respectively. For two reasons the dilatations are special. Firstly, as discussed already above, they have no counterpart among the global symmetries of the ordinary superstring on a flat background. Secondly, they can map solutions to the classical equations of motion with vanishing field strength $\partial_m A_n - \partial_n A_m$ to solution with non-vanishing field strength. This is in sharp contrast to the translations and Lorentz-transformations and most easily seen from $\Delta y = K(y)$, using that the field strength is related to y by the equations of motion through $f'(y) \approx \varepsilon^{mn}\partial_m A_n + \dots$ where $\approx$ is equality on-shell. An analogous reasoning shows that the latter property of 'dilatational symmetries' extends to more complicated backgrounds for which solutions to (6.2.13) with $K^i \neq 0$ exist.

a ter 7

G r l s l i r $g < 4$

7.1 On-shell cohomology

We shall now define and analyse an "on-shell BRST cohomology" $H(\sigma)$ and show that it is isomorphic to its purely bosonic counterpart at ghost numbers < 4, i.e., to the on-shell BRST cohomology of the corresponding bosonic string model. The relevance of $H(\sigma)$ rests on the fact that it is isomorphic to the full local s-cohomology $H(s)$ (in the jet space associated to the fields and antifields), at least at ghost numbers < 4,

$$g < 4: \quad H^g(\sigma) \simeq H^g(s). \tag{7.1.1}$$

This will be proved in section 7.2.

The analysis in this and the next section is general, i.e., it applies to any model with an action (4.1.13) (or, equivalently, (4.1.14)) provided that two rather mild assumptions hold, which are introduced now. The first assumption only simplifies the action a little bit but does not reduce its generality: as we have argued already in [36], one may assume that the functions $D_i(X)$ which occur in the action coincide with a subset of the fields X^M. We denote this subset by $\{y^i\}$ and the remaining X's by x^μ,

$$\{X^M\} = \{x^\mu, y^i\}, \quad D_i(X) \equiv y^i. \tag{7.1.2}$$

For physical applications this "assumption" does not represent any loss of generality because it can always be achieved by a field redefinition ("target space coordinate transformation") $X^M \to \tilde{X}^M = \tilde{X}^M(X)$. The y^i may be interpreted as coordinates of an enlarged target space leading to "frozen extra dimensions" [36]. The second assumption is that $G_{\mu\nu}(x, y)$ is invertible (in contrast, G_{MN} need not be invertible). This is particularly natural in the string theory context, since it allows one to interpret $G_{\mu\nu}$ as a target space metric. It is rather likely that our result holds for even weaker assumptions (but we did not study this question), because the results derived in [39, 40] for bosonic string models do not use the invertibility of $G_{\mu\nu}$.

Let us remark that the isomorphism (7.1.1) is not too surprising, because it is reminiscent of a standard result of local BRST cohomology stating that $H(s)$ is isomorphic to the on-shell cohomology of γ in the space of antifield independent functions, where γ is the part of s with antifield number 0 (see, e.g., section 7.2 of [30]). However, (7.1.1) is not quite the same statement because the definition of σ given below does not take the equations of motion for μ, $\bar{\mu}$, α or $\bar{\alpha}$ into account. Hence, (7.1.1) contains information in addition to the standard result of local BRST cohomology mentioned before: the equations of motion for μ, $\bar{\mu}$, α, $\bar{\alpha}$ are not relevant to the cohomology! This is a useful result as these equations of motion are somewhat unpleasant, because they are not linearizable (the models under study do not fulfill the standard regularity conditions described, e.g., in section 5.1 of [30]).

7.1.1 Definition of σ and $H(\sigma)$

σ is an "on-shell version" of s defined in the space of local functions made of the fields only (but not of any antifields). We work in the 'Beltrami basis' and use the equations of motion obtained by varying the action (4.1.13) with respect to the fields X, ψ, $\bar{\psi}$, $\hat{F}$, $\hat{\phi}$, λ, $\bar{\lambda}$ and A_m. The covariant version of these equations of motion can be obtained from the s-transformations of the corresponding covariant antifields given in appendix C.2 by setting the antifield independent part ('Koszul-Tate part') of these transformations to zero. This gives the following "on-shell equalities" ($\approx$):

$$\hat{F}^i \approx 0 \tag{7.1.3}$$

$$\psi^i \approx 0 \tag{7.1.4}$$

$$\bar{\psi}^i \approx 0 \tag{7.1.5}$$

$$\mathcal{D}y^i \approx 0 \tag{7.1.6}$$

$$\bar{\mathcal{D}}y^i \approx 0 \tag{7.1.7}$$

$$\hat{\phi}^i \approx 2G_{i\mu}\hat{F}^\mu + \psi^\mu\bar{\psi}^\nu\Omega_{\mu\nu i} \tag{7.1.8}$$

$$\begin{aligned}\lambda^i \approx\ & 2G_{i\mu}\mathcal{D}\bar{\psi}^\mu + \mathcal{D}x^\mu\bar{\psi}^\nu\Omega_{\mu\nu i} + \hat{F}^\mu\psi^\nu\Omega_{\nu i\mu} \\ & +\psi^\mu\psi^\nu\bar{\psi}^\rho R_{\nu\mu i\rho}\end{aligned} \tag{7.1.9}$$

$$\begin{aligned}\bar{\lambda}^i \approx\ & -2G_{i\mu}\bar{\mathcal{D}}\psi^\mu - \bar{\mathcal{D}}x^\mu\psi^\nu\Omega_{\nu\mu i} + \hat{F}^\mu\bar{\psi}^\nu\Omega_{i\nu\mu} \\ & +\psi^\mu\bar{\psi}^\nu\bar{\psi}^\rho R_{\mu i\rho\nu}\end{aligned} \tag{7.1.10}$$

$$\hat{F}^\rho \approx -\tfrac{1}{2}\,\psi^\mu\bar{\psi}^\nu\Omega_{\mu\nu}{}^\rho \tag{7.1.11}$$

$$\begin{aligned}\bar{\mathcal{D}}\psi^\mu \approx\ & -\tfrac{1}{2}\,[\bar{\mathcal{D}}x^\nu\psi^\rho\Omega_{\rho\nu}{}^\mu + \tfrac{1}{2}\,\psi^\lambda\bar{\psi}^\sigma\psi^\rho\Omega_{\lambda\sigma}{}^\nu\Omega^\mu{}_{\rho\nu} \\ & +\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma R^\mu{}_{\nu\sigma\rho}]\end{aligned} \tag{7.1.12}$$

$$\begin{aligned}\mathcal{D}\bar{\psi}^\mu \approx\ & \tfrac{1}{2}\,[-\mathcal{D}x^\nu\bar{\psi}^\rho\Omega_{\nu\rho}{}^\mu + \tfrac{1}{2}\,\psi^\lambda\bar{\psi}^\sigma\psi^\rho\Omega_{\lambda\sigma}{}^\nu\Omega_\rho{}^\mu{}_\nu \\ & +\psi^\nu\psi^\rho\bar{\psi}^\sigma R_{\rho\nu\sigma}{}^\mu]\end{aligned} \tag{7.1.13}$$

$$
\begin{aligned}
\mathcal{F}^i &\approx 2G_{i\mu}\mathcal{D}\bar{\mathcal{D}}x^\mu + \mathcal{D}x^\mu\bar{\mathcal{D}}x^\nu\Omega_{\mu\nu i} - \hat{F}^\mu\hat{F}^\nu\Omega_{i\mu\nu} \\
&\quad -\mathcal{D}\bar{\psi}^\mu\bar{\psi}^\nu\Omega_{i\nu\mu} + \psi^\mu\bar{\mathcal{D}}\psi^\nu\Omega_{\mu i\nu} \\
&\quad -\mathcal{D}x^\mu\bar{\psi}^\nu\bar{\psi}^\rho R_{\mu i\rho\nu} - \bar{\mathcal{D}}x^\mu\psi^\nu\psi^\rho R_{\rho\nu\mu i} \\
&\quad -\hat{F}^\mu\psi^\nu\bar{\psi}^\rho\partial_i\Omega_{\nu\rho\mu} - \tfrac{1}{2}\,\psi^\mu\psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma\partial_i R_{\nu\mu\sigma\rho} \qquad (7.1.14)\\
\mathcal{D}\bar{\mathcal{D}}x^\mu &\approx \tfrac{1}{2}\,[-\mathcal{D}x^\nu\bar{\mathcal{D}}x^\rho\Omega_{\nu\rho}{}^\mu + \hat{F}^\nu\hat{F}^\rho\Omega^\mu{}_{\nu\rho} \\
&\quad +\mathcal{D}\bar{\psi}^\nu\bar{\psi}^\rho\Omega^\mu{}_{\rho\nu} - \psi^\nu\bar{\mathcal{D}}\psi^\rho\Omega_\nu{}^\mu{}_\rho \\
&\quad -\mathcal{D}x^\sigma\bar{\psi}^\nu\bar{\psi}^\rho R^\mu{}_{\sigma\rho\nu} + \bar{\mathcal{D}}x^\sigma\psi^\nu\psi^\rho R_{\rho\nu\sigma}{}^\mu \\
&\quad +\hat{F}^\sigma\psi^\nu\bar{\psi}^\rho\partial^\mu\Omega_{\nu\rho\sigma} + \tfrac{1}{2}\,\psi^\lambda\psi^\nu\bar{\psi}^\sigma\bar{\psi}^\rho\partial^\mu R_{\nu\lambda\rho\sigma}] \qquad (7.1.15)
\end{aligned}
$$

where indices μ of Ω, R, ∂ have been raised with the inverse of $G_{\mu\nu}(x,y)$, and ψ^i, $\bar{\psi}^i$ and $\hat{F}^i$ belong to the same supersymmetry multiplet as y^i (the auxiliary fields $\hat{F}^i$ should not be confused with the supercovariant field strengths $\mathcal{F}^i$ of the gauge fields). Note that the right hand sides of (7.1.8), (7.1.9), (7.1.10), (7.1.14) and (7.1.15) still contain $\hat{F}^\mu$, $\bar{\mathcal{D}}\psi^\mu$ or $\mathcal{D}\bar{\psi}^\mu$, which are to be substituted for by the expressions given in (7.1.11), (7.1.12) and (7.1.13), respectively. Furthermore, in (7.1.14) one has to substitute the expression resulting from (7.1.15) for $\mathcal{D}\bar{\mathcal{D}}x^\mu$. Using Eqs. (7.1.3) through (7.1.15) and their $\mathcal{D}$ and $\bar{\mathcal{D}}$ derivatives, we eliminate all variables on the left hand sides of these equations and all the covariant derivatives of these variables. Furthermore, we use these equations to define the σ-transformations of the remaining field variables from their s-transformations. For instance, one gets

$$
\begin{aligned}
\sigma y^i &= 0 \qquad (7.1.16)\\
\sigma x^\mu &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})x^\mu + \varepsilon\psi^\mu + \bar{\varepsilon}\bar{\psi}^\mu \qquad (7.1.17)\\
\sigma\psi^\mu &= \eta\mathcal{D}\psi^\mu - \tfrac{1}{2}\,\bar{\eta}[\bar{\mathcal{D}}x^\nu\psi^\rho\Omega_{\rho\nu}{}^\mu \\
&\quad +\tfrac{1}{2}\,\psi^\lambda\bar{\psi}^\sigma\bar{\psi}^\rho\Omega_{\lambda\sigma}{}^\nu\Omega^\mu{}_{\rho\nu} + \psi^\nu\bar{\psi}^\rho\bar{\psi}^\sigma R^\mu{}_{\nu\sigma\rho}] \\
&\quad +\tfrac{1}{2}\,\partial\eta\psi^\mu + \varepsilon\mathcal{D}x^\mu + \tfrac{1}{2}\,\bar{\varepsilon}\psi^\rho\bar{\psi}^\nu\Omega_{\rho\nu}{}^\mu. \qquad (7.1.18)
\end{aligned}
$$

The σ-transformations of η, $\bar{\eta}$, ε, $\bar{\varepsilon}$, μ, $\bar{\mu}$, α, $\bar{\alpha}$ coincide with their s-transformations. The cohomology $H(\sigma)$ is the cohomology of σ in the space of local functions of the variables $\{u^\ell, v^\ell, W^A\}$, where the u's and v's are the same as in sections 3.3 and 4, while the W's are given by

$$
\begin{aligned}
\{W^A\} &= \{y^i, x^\mu, \mathcal{D}^k x^\mu, \bar{\mathcal{D}}^k x^\mu, \mathcal{D}^r\psi^\mu, \bar{\mathcal{D}}^r\bar{\psi}^\mu, \partial^r\eta, \bar{\partial}^r\bar{\eta} \\
&\quad \partial^r\varepsilon, \bar{\partial}^r\bar{\varepsilon}, C^i : k = 1,2,\dots\ ,\ r = 0,1,\dots\}. \qquad (7.1.19)
\end{aligned}
$$

$H(\sigma)$ is well-defined because σ squares to zero,

$$\sigma^2 = 0. \qquad (7.1.20)$$

This holds because the (covariant) equations of motion of the fields X, ψ, $\bar{\psi}$, $\hat{F}$, $\hat{\phi}$, λ, $\bar{\lambda}$, A_m and their covariant derivatives transform into each other under diffeomorphisms and supersymmetry transformations but not into the equations of motion of μ, $\bar{\mu}$, α or $\bar{\alpha}$ [as can be read off from the s-transformations of the antifields X^*, ψ^*, $\bar{\psi}^*$, $\hat{F}^*$, $\hat{\phi}^*$, λ^*, $\bar{\lambda}^*$ and A_m^* in appendix C.2].

7.1.2 Relation to $H(\sigma, \mathcal{W})$

σ acts on the variables $\{u^\ell, v^\ell, W^A\}$ according to $\sigma u^\ell = v^\ell$, $\sigma W^A = r^A(W)$. Furthermore, analogously to (4.0.2) one has

$$\Big\{\sigma\,,\,\frac{\partial}{\partial(\partial\eta)}\Big\}W^A = L_0 W^A\ ,\ \Big\{\sigma\,,\,\frac{\partial}{\partial(\bar{\partial}\bar{\eta})}\Big\}W^A = \bar{L}_0 W^A\ , \qquad (7.1.21)$$

i.e., in the space of local functions of the W's the derivatives with respect to $\partial\eta$ and $\bar{\partial}\bar{\eta}$ are contracting homotopies for L_0 and $\bar{L}_0$, respectively. Hence, the same standard arguments, which were used already in section 4 yield that $H(\sigma)$ is given by $H_{\mathrm{dR}}(GL^+(2)) \otimes H(\sigma, \mathcal{W})$, where $H_{\mathrm{dR}}(GL^+(2))$ reflects the nontrivial de Rham cohomology of the zweibein manifold (see theorem 5.1 of [79]), while $H(\sigma, \mathcal{W})$ is the σ-cohomology in the space of local functions with vanishing conformal weights made solely of the variables (7.1.19),

$$\begin{aligned} H(\sigma) &= H_{\mathrm{dR}}(GL^+(2)) \otimes H(\sigma, \mathcal{W}), \\ \mathcal{W} &= \{\omega : \omega = \omega(W),\ (L_0\omega, \bar{L}_0\omega) = (0,0)\}. \end{aligned} \qquad (7.1.22)$$

The factor $H_{\mathrm{dR}}(GL^+(2))$ is irrelevant for the following discussion because it just reflects $\det e_m^a \neq 0$ and makes no difference between superstring and bosonic string models.

7.1.3 Decomposition of σ

To study $H(\sigma, \mathcal{W})$ we decompose σ into pieces of definite degree in the supersymmetry ghosts and the fermions[1]. The corresponding counting operator is denoted by N,

$$N = N_\varepsilon + N_{\bar{\varepsilon}} + N_\psi + N_{\bar{\psi}} \qquad (7.1.23)$$

with N_ε and $N_{\bar{\varepsilon}}$ as in (4.0.4) and

$$N_\psi = \sum_{r\geq 0} (\mathcal{D}^r\psi^\mu)\,\frac{\partial}{\partial(\mathcal{D}^r\psi^\mu)}\ ,\quad N_{\bar{\psi}} = \sum_{r\geq 0} (\bar{\mathcal{D}}^r\bar{\psi}^\mu)\,\frac{\partial}{\partial(\bar{\mathcal{D}}^r\bar{\psi}^\mu)}\ .$$

Using the formulae given above, it is easy to verify that σ decomposes into pieces with even N-degree,

$$\sigma = \sum_{n\geq 0} \sigma_{2n}\quad ,\quad [N, \sigma_{2n}] = 2n\,\sigma_{2n} \qquad (7.1.24)$$

[1]We are referring here to the variables (7.1.19) themselves, and not to the fermions that are implicitly contained in these variables through covariant derivatives.

where, on each variable (7.1.19), only finitely many σ_{2n} are non-vanishing. For instance, (7.1.18) yields

$$\begin{aligned}
\sigma_0\psi^\mu &= \eta\mathcal{D}\psi^\mu - \tfrac{1}{2}\,\bar\eta\bar{\mathcal{D}}x^\nu\psi^\rho\Omega_{\rho\nu}{}^\mu + \tfrac{1}{2}\,\partial\eta\psi^\mu + \varepsilon\mathcal{D}x^\mu \\
\sigma_2\psi^\mu &= -\tfrac{1}{4}\,\bar\eta\psi^\lambda\bar\psi^\sigma\bar\psi^\rho\Omega_{\lambda\sigma}{}^\nu\Omega^\mu{}_{\rho\nu} - \tfrac{1}{2}\,\bar\eta\psi^\nu\bar\psi^\rho\bar\psi^\sigma R^\mu{}_{\nu\sigma\rho} + \tfrac{1}{2}\,\bar\varepsilon\psi^\rho\bar\psi^\nu\Omega_{\rho\nu}{}^\mu \\
\sigma_{2n}\psi^\mu &= 0 \quad \text{for} \quad n > 1.
\end{aligned}$$

7.1.4 Decomposition of σ_0

We shall prove the asserted result by an inspection of the cohomology of σ_0. To that end we decompose σ_0 according to the supersymmetry ghosts. That decomposition has only two pieces owing to the very definition of σ_0 and N,

$$\sigma_0 = \sigma_{0,0} + \sigma_{0,1}\ , \quad [N_\varepsilon + N_{\bar\varepsilon}\,, \sigma_{0,0}] = 0\ , \quad [N_\varepsilon + N_{\bar\varepsilon}\,, \sigma_{0,1}] = \sigma_{0,1}\ . \tag{7.1.25}$$

$\sigma_{0,1}$ acts notrivially only on the fermions and their derivatives $\mathcal{D}^r\psi^\mu$ and $\bar{\mathcal{D}}^r\bar\psi^\mu$ with $r = 0, 1, \dots$. One easily verifies by induction that $\sigma_{0,1}$ has the following simple structure

$$\begin{aligned}
\sigma_{0,1}\mathcal{D}^r\psi^\mu &= \sum_{k=0}^{r} \binom{r}{k}\, \partial^k\varepsilon\,\mathcal{D}^{r+1-k}x^\mu \\
\sigma_{0,1}\bar{\mathcal{D}}^r\bar\psi^\mu &= \sum_{k=0}^{r} \binom{r}{k}\, \bar\partial^k\bar\varepsilon\,\bar{\mathcal{D}}^{r+1-k}x^\mu
\end{aligned} \tag{7.1.26}$$

7.1.5 $H(\sigma_0, \mathcal{W})$ at ghost numbers < 5

The cocycle condition of $H(\sigma_0, \mathcal{W})$ reads

$$\sigma_0\omega = 0, \quad \omega \in \mathcal{W}. \tag{7.1.27}$$

We analyse (7.1.27) using (7.1.25). To that end we decompose ω according to the number of supersymmetry ghosts,

$$\omega = \sum_{k=\underline{k}}^{\overline{k}} \omega_k\ , \quad (N_\varepsilon + N_{\bar\varepsilon})\omega_k = k\omega_k\ . \tag{7.1.28}$$

Note that $\overline{k}$ is finite, $\overline{k} \leq \mathrm{gh}\,(\omega)$. Hence, the cocycle condition (7.1.27) decomposes into

$$\sigma_{0,1}\omega_{\overline{k}} = 0, \quad \sigma_{0,0}\omega_{\overline{k}} + \sigma_{0,1}\omega_{\overline{k}-1} = 0, \quad \dots \quad , \quad \sigma_{0,0}\omega_{\underline{k}} = 0. \tag{7.1.29}$$

We can neglect contributions $\sigma_{0,1}\hat\omega_{\overline{k}-1}$ to $\omega_{\overline{k}}$ because such contributions can be removed by subtracting $\sigma_0\hat\omega_{\overline{k}-1}$ from ω. Hence, $\omega_{\overline{k}}$ can be assumed to be a nontrivial representative of $H(\sigma_{0,1}, \mathcal{W})$. That cohomology is computed in appendix A.1 and yields

$$\begin{aligned}\omega_{\overline{k}} &= h(y,x,C,[\varepsilon,\eta],[\bar\varepsilon,\bar\eta]) + \eta\mathcal{D}x^\mu h_\mu(y,x,\partial\eta,C,[\bar\varepsilon,\bar\eta]) \\ &\quad +\bar\eta\bar{\mathcal{D}}x^\mu \bar h_\mu(y,x,\bar\partial\bar\eta,C,[\varepsilon,\eta]) \\ &\quad +\eta\bar\eta\mathcal{D}x^\mu\bar{\mathcal{D}}x^\nu h_{\mu\nu}(y,x,\partial\eta,\bar\partial\bar\eta,C)\end{aligned} \qquad (7.1.30)$$

where $\sigma_{0,1}$-exact pieces have been neglected, and $[\varepsilon,\eta]$ and $[\bar\varepsilon,\bar\eta]$ denote dependence on the variables $\partial^r\varepsilon, \partial^r\eta$ and $\bar\partial^r\bar\varepsilon, \bar\partial^r\bar\eta$ $(r = 0,1,\dots)$, respectively. The result (7.1.30) holds for all ghost numbers and shows in particular that $\omega_{\overline{k}}$ can be assumed not to depend on the fermions $(\mathcal{D}^r\psi^\mu,\ \bar{\mathcal{D}}^r\bar\psi^\mu)$ at all. We now insert this result in the second equation (7.1.29), which requires that $\sigma_{0,0}\omega_{\overline{k}}$ be $\sigma_{0,1}$-exact. At ghost numbers < 5 this requirement kills completely the dependence of $\omega_{\overline{k}}$ on the supersymmetry ghosts as we show in appendix A.2. The result for these ghost numbers is thus that, modulo σ_0-exact pieces, the solutions to (7.1.27) neither depend on the fermions nor on the supersymmetry ghosts,

$$\begin{aligned}\mathrm{gh}\,(\omega) < 5:\ \omega &= \sigma_0\hat\omega + h(y,x,C,[\eta],[\bar\eta]) \\ &\quad +\eta\mathcal{D}x^\mu h_\mu(y,x,\partial\eta,C,[\bar\eta]) \\ &\quad +\bar\eta\bar{\mathcal{D}}x^\mu\bar h_\mu(y,x,\bar\partial\bar\eta,C,[\eta]) \\ &\quad +\eta\bar\eta\mathcal{D}x^\mu\bar{\mathcal{D}}x^\nu h_{\mu\nu}(y,x,\partial\eta,\bar\partial\bar\eta,C).\end{aligned} \qquad (7.1.31)$$

Furthermore, (7.1.25) and (7.1.26) show that a function which neither depends on the fermions nor on the supersymmetry ghosts is σ_0-exact if and only if it is the σ_0-transformation of a function which does not depend on these variables either. Combining this with (7.1.31) one concludes

$$g < 5: \quad H^g(\sigma_0, \mathcal{W}) \simeq H^g(\sigma_0, \mathcal{W}_0), \qquad (7.1.32)$$

where $\mathcal{W}_0$ is the subspace of $\mathcal{W}$ containing the functions with vanishing N-eigenvalues,

$$\mathcal{W}_0 = \{\omega \in \mathcal{W} : N\omega = 0\}.$$

This subspace can be made very explicit. The only variables (7.1.19) with negative conformal weights on which a function $\omega \in \mathcal{W}_0$ can depend are the undifferentiated ghosts η and $\bar\eta$ [note: the only other variables (7.1.19) with negative conformal weights are the undifferentiated supersymmetry ghosts, but they cannot occur in $\omega \in \mathcal{W}_0$ by the very definition of $\mathcal{W}_0$]. Since η and $\bar\eta$ are anticommuting variables and have conformal weights $(-1,0)$ and $(0,-1)$, respectively, each of them can occur at most once in a monomial contributing to $\omega \in \mathcal{W}_0$. Hence, functions in $\mathcal{W}_0$ can only depend on those w's with conformal weights ≤ 1 (as higher weights cannot be compensated for by variables with negative weights), and a variable with L_0-weight ($\bar{L}_0$-weight) 1 must necessarily occur together with η ($\bar{\eta}$). This yields

$$\omega \in \mathcal{W}_0 \;\Leftrightarrow\; \omega = f(y, x, C, \partial\eta, \bar{\partial}\bar{\eta}, \eta\mathcal{D}x^{\mu}, \bar{\eta}\bar{\mathcal{D}}x^{\mu}, \eta\partial^2\eta, \bar{\eta}\bar{\partial}^2\bar{\eta}). \qquad (7.1.33)$$

Note that $H(\sigma_0, \mathcal{W}_0)$ is nothing but the on-shell cohomology $H(\sigma, \mathcal{W})$ of the corresponding bosonic string model, since elements of $\mathcal{W}_0$ neither depend on the fermions nor on the supersymmetry ghosts, and since σ_0 reduces in $\mathcal{W}_0$ to $\sigma_{0,0}$, which encodes only the diffeomorphism transformations but not the supersymmetry transformations.

7.1.6 $H(\sigma)$ at ghost numbers < 4

We shall now show that $H(\sigma, \mathcal{W})$ is at ghost numbers < 4 isomorphic to $H(\sigma_0, \mathcal{W}_0)$,

$$g < 4: \quad H^g(\sigma, \mathcal{W}) \simeq H^g(\sigma_0, \mathcal{W}_0). \qquad (7.1.34)$$

Because of (7.1.22) this implies that $H(\sigma)$ is isomorphic to its counterpart in the corresponding bosonic string model (recall that the factor $H_{\mathrm{dR}}(GL^+(2))$ is present in the case of bosonic strings as well, and that $H^g(\sigma_0, \mathcal{W}_0)$ is the on-shell cohomology of the bosonic string model). To derive (7.1.34), we consider the cocycle condition of $H(\sigma, \mathcal{W})$,

$$\sigma\omega = 0, \quad \omega \in \mathcal{W}. \qquad (7.1.35)$$

We decompose ω into pieces with definite degree in the supersymmetry ghosts and fermions,

$$\omega = \sum_{n=\underline{n}}^{\overline{n}} \omega_n, \quad N\omega_n = n\,\omega_n, \qquad (7.1.36)$$

with N as in (7.1.23) [actually there are only even values of n in this decomposition because ω has vanishing conformal weights]. The cocycle condition (7.1.35) implies in particular

$$\sigma_0\omega_{\underline{n}} = 0, \qquad (7.1.37)$$

where we used the decomposition (7.1.24) of σ. Hence, every cocycle ω of $H^g(\sigma, \mathcal{W})$ contains a coycle $\omega_{\underline{n}}$ of $H^g(\sigma_0, \mathcal{W})$. Our result (7.1.32) on $H^g(\sigma_0, \mathcal{W})$ implies that this relation between representatives of $H^g(\sigma, \mathcal{W})$ and $H^g(\sigma_0, \mathcal{W})$ gives rise to a one-to-one correspondence between the cohomology classes of $H^g(\sigma, \mathcal{W})$ and $H^g(\sigma_0, \mathcal{W}_0)$ for $g < 4$ and thus to (7.1.34). The arguments are standard and essentially the following:

(i) When $g < 5$, $\omega_{\underline{n}}$ can be assumed to be nontrivial in $H^g(\sigma_0, \mathcal{W})$ and represents thus a class of $H^g(\sigma_0, \mathcal{W}_0)$. Indeed, assume it were trivial, i.e.,

$\omega_{\underline{n}} = \sigma_0 \hat{\omega}_{\underline{n}}$ for some $\hat{\omega}_{\underline{n}} \in \mathcal{W}$. In that case we can remove $\omega_{\underline{n}}$ from ω by subtracting $\sigma\hat{\omega}_{\underline{n}}$. $\omega' := \omega - \sigma\hat{\omega}_{\underline{n}} \in \mathcal{W}$ is cohomologically equivalent to ω and its decomposition (7.1.36) starts at some degree $\underline{n}' > \underline{n}$ unless it vanishes (which implies already $\omega = \sigma\hat{\omega}_{\underline{n}}$). The cocycle condition for ω' implies $\sigma_0\omega'_{\underline{n}'} = 0$ and thus $\omega'_{\underline{n}'} = \sigma_0\hat{\omega}'_{\underline{n}'}$ for some $\hat{\omega}'_{\underline{n}'} \in \mathcal{W}$ as a consequence of (7.1.32) (owing to $\underline{n}' > \underline{n} \geq 0$). Repeating the arguments, one concludes that ω is σ-exact, $\omega = \sigma(\hat{\omega}_{\underline{n}} + \hat{\omega}'_{\underline{n}'} + \dots)$ [it is guaranteed that the procedure terminates, i.e., that the sum $\hat{\omega}_{\underline{n}} + \hat{\omega}'_{\underline{n}'} + \dots$ is finite and thus local, because the number of supersymmetry ghosts is bounded by the ghost number and thus the number of fermions is bounded too because ω has vanishing conformal weights].

(ii) When $g < 4$, every nontrivial cocycle ω_0 of $H^g(\sigma_0, \mathcal{W}_0)$ can be completed to a nontrivial cocycle ω of $H^g(\sigma, \mathcal{W})$. Indeed suppose we had constructed $\omega_n \in \mathcal{W}$, $n = 0, \dots, m$ with ghost number g such that $\omega^{(m)} :=$ ${}^{m}_{n=0}\,\omega_n$ fulfills $\sigma\omega^{(m)} = {}_{n\geq m+1} R_n$ with $NR_n = nR_n$ [for $m = 0$ this is implied by $\sigma_0\omega_0 = 0$ which holds because ω_0 is a σ_0-cocycle by assumption]. $\sigma^2 = 0$ implies $\sigma\ {}_{n\geq m+1} R_n = 0$ and thus $\sigma_0 R_{m+1} = 0$ at lowest N-degree. Note that R_{m+1} is in $\mathcal{W}$ (owing to $\sigma\mathcal{W} \subset \mathcal{W}$) and that it has ghost number $g + 1 < 5$ because $\omega^{(m)}$ has ghost number $g < 4$. (7.1.32) guarantees thus that there is some $\omega_{m+1} \in \mathcal{W}$ such that $R_{m+1} = -\sigma_0\omega_{m+1}$, which implies that $\omega^{(m+1)} := \omega^{(m)} + \omega_{m+1}$ fulfills $\sigma\omega^{(m+1)} = {}_{n\geq m+2} R'_n$. By induction this implies that every solution to (7.1.37) with ghost number < 4 can indeed be completed to a solution of (7.1.35) [the locality of ω holds by the same arguments as above]. If ω_0 is trivial in $H^g(\sigma_0, \mathcal{W}_0)$, then its completion ω is trivial in $H^g(\sigma, \mathcal{W})$ by arguments used in (i). Conversely, the triviality of ω in $H^g(\sigma, \mathcal{W})$ ($\omega = \sigma\eta$) implies obviously the triviality of ω_0 in $H^g(\sigma_0, \mathcal{W}_0)$ ($\omega_0 = \sigma_0\eta_0$) because there are no negative N-degrees.

7.2 elation to the cohomology of bosonic strings

We shall now derive (7.1.1) and the announced isomorphism between the s-cohomologies of a superstring and the corresponding bosonic string model. Both results can be traced to the existence of variables $\{\tilde{u}^{\bar{\ell}}, \tilde{v}^{\bar{\ell}}, \tilde{W}^{\bar{A}}\}$ on which s takes a form very similar to σ on the variables $\{u^\ell, v^\ell, w^A\}$ used in section 7.1. In the 'Beltrami basis' the set of $\tilde{u}$'s consists of: (i) $\tilde{u}$'s with ghost number 0 which coincide with the u^ℓ; (ii) $\tilde{u}$'s with ghost number -1 given by the superconformal antifields X^*_M, ψ^*_M, $\bar{\psi}^*_M$, F^*_M, ϕ^*_i, λ^*_i $\bar{\lambda}^*_i$, A^*_i (recall that we have dropped the hats on these antifields) and all covariant derivatives of these antifields plus the $\bar{A}^*_i$ and all their $\bar{\mathcal{D}}$-derivatives ($\bar{\mathcal{D}}^r \bar{A}^*_i$, $r = 0, 1, \dots)^2$; (iii) $\tilde{u}$'s with ghost number -2 given by the antifields of the

[2] The $\mathcal{D}^k\bar{\mathcal{D}}^r\bar{A}^*_i$ with $k > 0$ do not count among the u's because the antifield independent parts of $s\mathcal{D}^k\bar{\mathcal{D}}^r\bar{A}^*_i$ and $-s\mathcal{D}^{k-1}\bar{\mathcal{D}}^{r+1}A^*_i$ are equal (both are given by $\mathcal{D}^k\bar{\mathcal{D}}^{r+1}y^i$). Rather, they are substituted for by the v's corresponding to the $\mathcal{D}^{k-1}\bar{\mathcal{D}}^r C^*_i$ ($k > 0$) owing to

ghosts, i.e., by η^*, $\bar{\eta}^*$, ε^*, $\bar{\varepsilon}^*$, C_i^* and all their derivatives. It can be readily checked that a complete set of new local jet coordinates in the Beltrami basis is given by $\{\tilde{u}^{\tilde{\ell}}, \tilde{v}^{\tilde{\ell}}, \tilde{W}^{\tilde{A}}_{(0)}\}$ with $\tilde{v}^{\tilde{\ell}} = s\tilde{u}^{\tilde{\ell}}$ and

$$\begin{aligned}\{\tilde{W}^{\tilde{A}}_{(0)}\} = \{ & y^i, x^\mu, \mathcal{D}^k x^\mu, \bar{\mathcal{D}}^k x^\mu, \mathcal{D}^r \psi^\mu, \bar{\mathcal{D}}^r \bar{\psi}^\mu, \partial^r \eta, \bar{\partial}^r \bar{\eta}, \partial^r \varepsilon, \bar{\partial}^r \bar{\varepsilon}, C^i, \\ & \partial^r \mu^*, \bar{\partial}^r \bar{\mu}^*, \partial^r \alpha^*, \bar{\partial}^r \bar{\alpha}^* : k = 1, 2, \ldots\ , \ r = 0, 1, \ldots\}. \end{aligned} \tag{7.2.38}$$

Note that $\{\tilde{W}^{\tilde{A}}_{(0)}\}$ does not only contain the W^A listed in (7.1.19), but in addition the variables $\partial^r \mu^*$, $\bar{\partial}^r \bar{\mu}^*$, $\partial^r \alpha^*$, $\bar{\partial}^r \bar{\alpha}^*$. The latter occur here because their s-transformations contain no linear parts and can therefore not be used as $\tilde{v}$'s[3]. The $\tilde{W}^{\tilde{A}}_{(0)}$ fulfull

$$s\tilde{W}^{\tilde{A}}_{(0)} = r^{\tilde{A}}(\tilde{W}_{(0)}) + O(1) \tag{7.2.39}$$

where $O(1)$ collects terms which are at least linear in the $\tilde{u}$'s and $\tilde{v}$'s. As shown in [94], (7.2.39) implies the existence of variables $\tilde{W}^{\tilde{A}} = \tilde{W}^{\tilde{A}}_{(0)} + O(1)$ which fulfill

$$s\tilde{W}^{\tilde{A}} = r^{\tilde{A}}(\tilde{W}) \tag{7.2.40}$$

with the *same* functions $r^{\tilde{A}}$ as in (7.2.39). Furthermore the algorithm described in [94] for the construction of the $\tilde{W}^{\tilde{A}}$ results in local expressions when applied in the present case. This can be shown by means of arguments similar to those used within the discussion of the examples in [94][4].

(7.2.40) implies that the s-transformations of those $\tilde{W}$'s which correspond to the variables (7.1.19) can be obtained from the σ-transformations of the latter variables simply by substituting there $\tilde{W}$'s for the corresponding W's. For instance, this gives

$$\begin{aligned} sy^{i\prime} &= 0, & (7.2.41)\\ sx^{\mu\prime} &= \eta(\mathcal{D}x^\mu)' + \bar{\eta}(\bar{\mathcal{D}}x^\mu)' + \varepsilon\psi^{\mu\prime} + \bar{\varepsilon}\bar{\psi}^{\mu\prime} & (7.2.42)\end{aligned}$$

where here and in the following a prime on a variable indicates a $\tilde{W}$-variable[5]. For instance, $y^{i\prime}$ is the $\tilde{W}$-variable corresponding to y^i and explicitly given

$s\mathcal{D}^{k-1}\bar{\mathcal{D}}^r C_i^* = -\mathcal{D}^k \bar{\mathcal{D}}^r \bar{A}_i^* + \ldots$.

[3]The other derivatives of the antifields μ^*, $\bar{\mu}^*$, α^*, $\bar{\alpha}^*$, such as the $\bar{\partial}^k \partial^r \mu^*$ $(k > 0)$, do not occur among the $\tilde{W}_{(0)}$'s because they are substituted for by the $\tilde{v}$'s corresponding to η^*, $\bar{\eta}^*$, ε^*, $\bar{\varepsilon}^*$ and their derivatives (e.g., one has $s\eta^* = -\bar{\partial}\mu^* + \ldots$).

[4]In the present case, the suitable 'degrees' to be used in these arguments are the conformal weights and the ghost number. Using these degrees one can prove that the algorithm produces local (though not necessarily polynomial) expressions: the resulting $\tilde{W}$'s can depend nonpolynomially on the x^μ, y^i and on the two particular combinations $\varepsilon\bar{\lambda}_i^*$ and $\bar{\varepsilon}\lambda_i^*$ but they are necessarily polynomials in all variables which contain derivatives of fields or antifields.

[5]The construction of the $\tilde{W}$'s implies $(\partial^r\eta)' = \partial^r\eta$, $(\bar{\partial}^r\bar{\eta})' = \bar{\partial}^r\bar{\eta}$, $(\partial^r\varepsilon)' = \partial^r\varepsilon$ and $(\bar{\partial}^r\bar{\varepsilon})' = \bar{\partial}^r\bar{\varepsilon}$ because the s-transformation of these ghost variables do not contain any $\tilde{u}$'s or $\tilde{v}$'s. This has been used in (7.2.42).

by

$$y^{i\prime} = y^i + \varepsilon \bar{\lambda}_i^* - \bar{\varepsilon}\lambda_i^* - \eta A_i^* + \bar{\eta}\bar{A}_i^* + \eta\bar{\eta}C_i^* \ . \tag{7.2.43}$$

This very close relation between s on the $\tilde{W}$-variables and σ on the variables (7.1.19) would immediately imply $H(s) \simeq H(\sigma)$ if the $\tilde{W}$-variables $(\partial^r\mu^*)'$, $(\bar{\partial}^r\bar{\mu}^*)'$, $(\partial^r\alpha^*)'$, $(\bar{\partial}^r\bar{\alpha}^*)'$ were not present. Nevertheless the asserted isomorphism (7.1.1) holds because the conformal weights of the latter variables are too high so that they cannot contribute nontrivially to $H^g(s)$ for $g < 4$. To show this we analyse $H(s)$ along the same lines as $H(\sigma)$ in section 7.1.

The first step of that analysis gives

$$\begin{aligned} H(s) &\simeq H_{\mathrm{dR}}(GL^+(2)) \otimes H(s, \tilde{\mathcal{W}}), \\ \tilde{\mathcal{W}} &= \{\omega : \omega = \omega(w),\ (L_0\omega, \bar{L}_0\omega) = (0,0)\}. \end{aligned} \tag{7.2.44}$$

This result is analogous to (7.1.22) and expresses that the zweibein gives the only nontrivial cohomology in the subspace of $\tilde{u}$'s and $\tilde{v}$'s and that there is a contracting homotopy for L_0 and $\bar{L}_0$ because (7.2.40) implies

$$\Big\{ s\,,\, \frac{\partial}{\partial(\partial\eta)} \Big\} \tilde{W}^{\tilde{A}} = L_0 \tilde{W}^{\tilde{A}} \ , \ \Big\{ s\,,\, \frac{\partial}{\partial(\bar{\partial}\bar{\eta})} \Big\} \tilde{W}^{\tilde{A}} = \bar{L}_0 \tilde{W}^{\tilde{A}} \ .$$

The conformal weights of $\alpha^{*\prime}$, $\bar{\alpha}^{*\prime}$, $\mu^{*\prime}$ and $\bar{\mu}^{*\prime}$ are $(3/2, 0)$, $(0, 3/2)$, $(2, 0)$ and $(0, 2)$, respectively.

$H(s, \tilde{\mathcal{W}})$ can be analysed by means of a decomposition of s analogous to the σ-decomposition in (7.1.24), using a counting operator N' for all those $\tilde{W}$'s which have half-integer conformal weights,

$$N' = N_\varepsilon + N_{\bar{\varepsilon}} + N_{\psi'} + N_{\bar{\psi}'} + N_{\alpha^{*\prime}} + N_{\bar{\alpha}^{*\prime}} \ .$$

The decomposition of s reads

$$s = \sum_{n\geq 0} s_{2n} \quad , \quad [N', s_{2n}] = 2n\, s_{2n} \ .$$

Next we examine the s_0-cohomology. Analogously to (7.1.25) one has

$$s_0 = s_{0,0} + s_{0,1} \ , \quad [N_\varepsilon + N_{\bar{\varepsilon}}\,, s_{0,0}] = 0 \ , \quad [N_\varepsilon + N_{\bar{\varepsilon}}\,, s_{0,1}] = s_{0,1} \ .$$

We now determine the cohomology of $s_{0,1}$ along the lines of the investigation of the $\sigma_{0,1}$-cohomology in appendix A.1 by inspecting the part of $s_{0,1}$ which contains the undifferentiated ghost ε. That part is the analog of $\sigma_{0,1,1}$ in (A.1.2) and takes the form $\varepsilon\, \hat{G}'_{-1/2}$. $\hat{G}'_{-1/2}$ acts nontrivially only on the ψ', $\alpha^{*\prime}$ and their (covariant) derivatives according to

$$\hat{G}'_{-1/2}(\mathcal{D}^r\psi^\mu)' = (\mathcal{D}^{r+1}x^\mu)' \quad , \quad \hat{G}'_{-1/2}(\partial^r\alpha^*)' = -(\partial^r\mu^*)'$$

We define a contracting homotopy B' which is analogous to the contracting homotopy B in appendix A.1,

$$B' = \sum_{r\geq 0} \Big[(\mathcal{D}^r\psi^\mu)' \frac{\partial}{\partial(\mathcal{D}^{r+1}x^\mu)'} - (\partial^r\alpha^*)' \frac{\partial}{\partial(\partial^r\mu^*)'}\Big].$$

Using B' one proves that the functions f'_r with $r > 0$ which are analogous to the functions f_r in appendix A.1 can be assumed not to depend on the variables $(\mathcal{D}^r\psi^\mu)'$, $(\mathcal{D}^{r+1}x^\mu)'$, $(\partial^r\alpha^*)'$ or $(\partial^r\mu^*)'$.[6] In the case $r = 0$ one gets that f'_0 does not depend on $(\partial^r\alpha^*)'$ or $(\partial^r\mu^*)'$, simply because the conformal weights of these variables are too large [cf. the arguments in the text after (A.1.9)]. This implies the analog of equation (A.1.11), with functions f'_r and g'_μ which may still depend on $(\bar{\mathcal{D}}^r\bar{\psi}^\mu)'$, $(\bar{\mathcal{D}}^{r+1}x^\mu)'$, $(\bar\partial^r\bar\alpha^*)'$ or $(\bar\partial^r\bar\mu^*)'$. The dependence on these variables can be analysed analogously, using a contracting homotopy $\bar{B}'$ for these variables, along the lines of the remaining analysis in appendix A.1. One finally obtains the following result for $H(s_{0,1},\tilde{\mathcal{W}})$:

$$\begin{aligned}
&s_{0,1}\omega = 0, \quad \omega\in\tilde{\mathcal{W}} \quad \Rightarrow \\
&\omega = h(y', x', C', [\varepsilon,\eta], [\bar\varepsilon,\bar\eta]) \\
&\qquad + \eta(\mathcal{D}x^\mu)' h_\mu(y', x', \partial\eta, C', [\bar\varepsilon,\bar\eta]) \\
&\qquad + \bar\eta(\bar{\mathcal{D}}x^\mu)' \bar{h}_\mu(y', x', \bar\partial\bar\eta, C', [\varepsilon,\eta]) \\
&\qquad + \eta\bar\eta(\mathcal{D}x^\mu)'(\bar{\mathcal{D}}x^\nu)' h_{\mu\nu}(y', x', \partial\eta, \bar\partial\bar\eta, C') \\
&\qquad + s_{0,1}\hat\omega(w), \ \hat\omega\in\tilde{\mathcal{W}}.
\end{aligned} \tag{7.2.45}$$

Hence, $H(s_{0,1},\tilde{\mathcal{W}})$ is completely isomorphic to $H(\sigma_{0,1},\mathcal{W})$ (for all ghost numbers). In particular, the representatives do not depend on $(\partial^r\alpha^*)'$, $(\bar\partial^r\bar\alpha^*)'$, $(\partial^r\mu^*)'$ or $(\bar\partial^r\bar\mu^*)'$ [recall that the reason is that the conformal weights of these variables are too high; if, for instance, $\mu^{*\prime}$ had conformal weights $(1,0)$ instead of $(2,0)$ it had contributed to (7.2.45) analogously to $(\mathcal{D}x^\mu)'$]. This implies the results announced above: arguments which are completely analogous to those used to derive first (7.1.31) and then (7.1.34) lead to

$$g < 4: \quad H^g(s,\tilde{\mathcal{W}}) \simeq H^g(s_0,\tilde{\mathcal{W}}_0), \quad \tilde{\mathcal{W}}_0 = \{\omega\in\tilde{\mathcal{W}} : N'\omega = 0\}. \tag{7.2.46}$$

Analogously to (7.1.33), the elements of $\tilde{\mathcal{W}}_0$ can only depend on those w's

[6] For this argument it is important that there is a finite maximal value $\bar{r}$ of r. In the case of the σ-cohomology, r was bounded from above by the ghost number but now the ghost number alone does not give a bound because there are variables with negative ghost numbers, the $(\partial^r\alpha^*)'$, $(\bar\partial^r\bar\alpha^*)'$, $(\partial^r\mu^*)'$ and $(\bar\partial^r\bar\mu^*)'$. Nevertheless there is a bound because $\omega(\tilde{W})$ does not only have fixed ghost number but also vanishing conformal weights. Indeed, it is easy to show that this forbids arbitrarily large powers of ε because the $(\partial^r\alpha^*)'$ and $(\partial^r\mu^*)'$ have ghost number -1 and conformal weights $\geq 3/2$.

with conformal weights ≤ 1, i.e.,

$$\omega' \in \tilde{\mathcal{W}}_0 \;\Leftrightarrow\; \omega' = f(y', x', C', \partial\eta, \bar{\partial}\bar{\eta}, \eta(\mathcal{D}x^\mu)', \bar{\eta}(\bar{\mathcal{D}}x^\mu)', \eta\partial^2\eta, \bar{\eta}\bar{\partial}^2\bar{\eta}). \tag{7.2.47}$$

Because of (7.2.40), s_0 takes exactly the same form in $\tilde{\mathcal{W}}_0$ as σ_0 in $\mathcal{W}_0$. This implies (for all ghost numbers)

$$H(s_0, \tilde{\mathcal{W}}_0) \simeq H(\sigma_0, \mathcal{W}_0). \tag{7.2.48}$$

Because of (7.2.46) and (7.1.34) (as well as (7.2.44) and (7.1.22)) this yields (7.1.1). (7.2.46) establishes also the equivalence between the cohomologies of the superstring and the corresponding bosonic string at ghost numbers < 4 because $H_{\mathrm{dR}}(GL^+(2)) \otimes H(s_0, \tilde{\mathcal{W}}_0)$ is nothing but the s-cohomology of the bosonic string.

endix

l l i s

A.1 Cohomology of $\sigma_{0,1}$ in $\mathcal{W}$

In this appendix we compute $H(\sigma_{0,1}, \mathcal{W})$ where $\sigma_{0,1}$ is given in (7.1.26). The cocycle condition reads

$$\sigma_{0,1}\omega = 0, \quad \omega \in \mathcal{W}. \tag{A.1.1}$$

We decompose this equation into pieces with definite degree in the undifferentiated supersymmetry ghosts ε. $\sigma_{0,1}$ decomposes into two pieces, $\sigma_{0,1,0}$ and $\sigma_{0,1,1}$, where $\sigma_{0,1,0}$ does not change the degree in the undifferentiated ε, whereas $\sigma_{0,1,1}$ increases this degree by one unit. $\sigma_{0,1,1}$ reads

$$\sigma_{0,1,1} = \varepsilon\, \hat{G}_{-1/2}\ , \quad \hat{G}_{-1/2} = \sum_{r\geq 0} (\mathcal{D}^{r+1}x^\mu)\, \frac{\partial}{\partial(\mathcal{D}^r\psi^\mu)}\ . \tag{A.1.2}$$

ω can be assumed to have fixed ghost number and is thus a polynomial in the undifferentiated ε,

$$\omega = \sum_{r=\underline{r}}^{\overline{r}} \varepsilon^r f_r\ , \tag{A.1.3}$$

where f_r can depend on all variables (7.1.19) except for the undifferentiated ε. At highest degree in the undifferentiated ε, (A.1.1) implies $\sigma_{0,1,1}(\varepsilon^{\overline{r}} f_{\overline{r}}) = 0$ and thus

$$\hat{G}_{-1/2} f_{\overline{r}} = 0. \tag{A.1.4}$$

We analyse this condition by means of the contracting homotopy

$$B = \sum_{r\geq 0} (\mathcal{D}^r\psi^\mu)\, \frac{\partial}{\partial(\mathcal{D}^{r+1}x^\mu)}\ .$$

The anticommutator of B and $\hat{G}_{-1/2}$ is the counting operator for all variables $\mathcal{D}^r\psi^\mu$ and $\mathcal{D}^{r+1}x^\mu$ $(r = 0, 1, \dots)$,

$$\{B, \hat{G}_{-1/2}\} = \sum_{r\geq 0} \Big[(\mathcal{D}^r\psi^\mu)\, \frac{\partial}{\partial(\mathcal{D}^r\psi^\mu)} + (\mathcal{D}^{r+1}x^\mu)\, \frac{\partial}{\partial(\mathcal{D}^{r+1}x^\mu)} \Big].$$

Hence, (A.1.4) implies by standard arguments that $f_{\overline{r}}$ is $\hat{G}_{-1/2}$-exact up to a function that does not depend on the $\mathcal{D}^r\psi^\mu$ or $\mathcal{D}^{r+1}x^\mu$,

$$f_{\overline{r}} = \hat{G}_{-1/2}\, g_{\overline{r}} + h_{\overline{r}}(y, x, C, [\bar{\mathcal{D}}x, \bar{\psi}], [\partial\varepsilon, \eta], [\bar{\varepsilon}, \bar{\eta}]) \tag{A.1.5}$$

where $g_{\overline{r}}$ is a function that can depend on all variables (7.1.19) except for the undifferentiated ε, $[\bar{\mathcal{D}}x, \bar{\psi}]$ denotes collectively the variables $\bar{\mathcal{D}}^{r+1}x^\mu, \bar{\mathcal{D}}^r\bar{\psi}^\mu$, and $[\partial\varepsilon, \eta]$ and $[\bar{\varepsilon}, \bar{\eta}]$ denote collectively the variables $\partial^{r+1}\varepsilon, \partial^r\eta$ and $\bar{\partial}^r\bar{\varepsilon}, \bar{\partial}^r\bar{\eta}$, respectively ($r = 0, 1, \dots$ in all cases). We shall first study the case $\overline{r} > 0$ [the case $\overline{r} = 0$ will be included automatically below]. (A.1.5) implies

$$\begin{aligned} \overline{r} > 0: \quad \omega \;&=\; \sigma_{0,1}(\varepsilon^{\overline{r}-1}g_{\overline{r}}) + \varepsilon^{\overline{r}-1}f'_{\overline{r}-1} + \sum_{r=\underline{r}}^{\overline{r}-2} \varepsilon^r f_r \\ &\quad + \varepsilon^{\overline{r}}h_{\overline{r}}(y, x, C, [\bar{\mathcal{D}}x, \bar{\psi}], [\partial\varepsilon, \eta], [\bar{\varepsilon}, \bar{\eta}]) \end{aligned} \tag{A.1.6}$$

where

$$f'_{\overline{r}-1} = f_{\overline{r}-1} - \sigma_{0,1,0}\, g_{\overline{r}}\ .$$

The exact piece $\sigma_{0,1}(\varepsilon^{\overline{r}-1}g_{\overline{r}})$ on the right hand side of (A.1.6) will be neglected in the following, i.e., actually we shall examine $\omega' := \omega - \sigma_{0,1}(\varepsilon^{\overline{r}-1}g_{\overline{r}})$ in the following. However, for notational convenience, we shall drop the primes (of ω' and $f'_{\overline{r}-1}$) and consider now

$$\overline{r} > 0: \quad \omega = \sum_{r=\underline{r}}^{\overline{r}-1} \varepsilon^r f_r + \varepsilon^{\overline{r}}h_{\overline{r}}(y, x, C, [\bar{\mathcal{D}}x, \bar{\psi}], [\partial\varepsilon, \eta], [\bar{\varepsilon}, \bar{\eta}]) \tag{A.1.7}$$

We have thus learned that, if $\overline{r} > 0$, the piece in ω with highest degree in the undifferentiated ε can be assumed not to depend on any of the variables $\mathcal{D}^r\psi^\mu$ or $\mathcal{D}^{r+1}x^\mu$ $(r = 0, 1, \dots)$. As a consquence, the $\sigma_{0,1}$-transformation of that piece does not depend on these variables either and $\sigma_{0,1}\omega = 0$, with ω as in (A.1.7), implies

$$\hat{G}_{-1/2}f_{\overline{r}-1} = 0. \tag{A.1.8}$$

We can now analyse (A.1.8) in the same way as (A.1.4) and repeat the arguments until we reach an equation

$$\hat{G}_{-1/2}f_0 = 0 \tag{A.1.9}$$

where f_0 is a function with conformal weights $(0,0)$ which does not depend the undifferentiated ε [note that f_r has conformal weights $(r/2,0)$ because $\varepsilon^r f_r$ has conformal weights $(0,0)$; if $\overline{r}$ had been zero, we had arrived at (A.1.9) immediately]. The only way in which f_0 can depend nontrivially on the variables $\mathcal{D}^r\psi^\mu$ or $\mathcal{D}^{r+1}x^\mu$ $(r=0,1,\dots)$ is through terms of the form $\eta\psi^\mu\psi^\nu f_{\mu\nu}(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}])$, $\eta\partial\varepsilon\psi^\mu f_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}])$, or $\eta\mathcal{D}x^\mu g_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}])$ [recall that the only variables (7.1.19) with negative L_0-weights are the undifferentiated η and ε and that η is an anticommuting variable]. (A.1.9) implies $f_{\mu\nu}(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}])=0$ and $f_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}])=0$. We conclude

$$\begin{aligned} f_0 \;=\; & \eta\mathcal{D}x^\mu g_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}]) \\ & +h_0(y,x,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\partial\varepsilon,\eta],[\bar{\varepsilon},\bar{\eta}]) \end{aligned} \tag{A.1.10}$$

We thus get the following intermediate result: without loss of generality we can assume

$$\begin{aligned} \omega \;=\; & \sum_r \varepsilon^r h_r(y,x,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\partial\varepsilon,\eta],[\bar{\varepsilon},\bar{\eta}]) \\ & +\eta\mathcal{D}x^\mu g_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}]). \end{aligned} \tag{A.1.11}$$

The only part of $\sigma_{0,1}$ which is active on such an ω is the part

$$\hat{\sigma}_{0,1}=\sum_{r\geq 0}\sum_{k=0}^{r}\binom{r}{k}(\bar{\partial}^k\bar{\varepsilon}\,\bar{\mathcal{D}}^{r+1-k}x^\mu)\,\frac{\partial}{\partial(\bar{\mathcal{D}}^r\bar{\psi}^\mu)}\;.$$

Note that $\hat{\sigma}_{0,1}$ touches only the dependence on the variables $\bar{\mathcal{D}}^r\bar{\psi}^\mu$, $\bar{\mathcal{D}}^{r+1}x^\mu$ and $\bar{\partial}^r\bar{\varepsilon}$ $(r=0,1,\dots)$ and treats all other variables as contants. Hence, for ω as in (A.1.11), $\sigma_{0,1}\omega=0$ implies

$$\begin{aligned} \hat{\sigma}_{0,1}h_r(y,x,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\partial\varepsilon,\eta],[\bar{\varepsilon},\bar{\eta}])=0 \quad \forall r, \\ \hat{\sigma}_{0,1}g_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}])=0. \end{aligned} \tag{A.1.12}$$

These equations are decomposed into pieces with definite degree in the undifferentiated $\bar{\varepsilon}$ and then analysed using the contracting homotopy

$$\bar{B}=\sum_{r\geq 0}(\bar{\mathcal{D}}^r\bar{\psi}^\mu)\,\frac{\partial}{\partial(\bar{\mathcal{D}}^{r+1}x^\mu)}\;.$$

By means of arguments analogous to those that have led to (A.1.11) we conclude that we can assume, without loss of generality,

$$\begin{aligned} h_r(y,x,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\partial\varepsilon,\eta],[\bar{\varepsilon},\bar{\eta}]) = & \sum_q \bar{\varepsilon}^q h_{r,q}(y,x,C,[\partial\varepsilon,\eta],[\bar{\partial}\bar{\varepsilon},\bar{\eta}]) \\ & +\bar{\eta}\bar{\mathcal{D}}x^\mu g_{r,\mu}(y,x,\bar{\partial}\bar{\eta},C,[\partial\varepsilon,\eta]), \\ g_\mu(y,x,\partial\eta,C,[\bar{\mathcal{D}}x,\bar{\psi}],[\bar{\varepsilon},\bar{\eta}]) = & \sum_q \bar{\varepsilon}^q h_{\mu,q}(y,x,\partial\eta,C,[\bar{\partial}\bar{\varepsilon},\bar{\eta}]) \\ & +\bar{\eta}\bar{\mathcal{D}}x^\nu g_{\mu,\nu}(y,x,C,\partial\eta,\bar{\partial}\bar{\eta}). \end{aligned} \tag{A.1.13}$$

Since the $h_{r,q}$, $g_{r,\mu}$, $h_{\mu,q}$ and $g_{\mu,\nu}$ do not depend on the fermions, they are $\sigma_{0,1}$-invariant. We have thus proved that (A.1.1) implies

$$\begin{aligned}\omega &= h(y,x,C,[\varepsilon,\eta],[\bar\varepsilon,\bar\eta]) \\ &\quad +\eta\mathcal{D}x^\mu h_\mu(y,x,\partial\eta,C,[\bar\varepsilon,\bar\eta])+\bar\eta\bar{\mathcal{D}}x^\mu\bar h_\mu(y,x,\bar\partial\bar\eta,C,[\varepsilon,\eta]) \\ &\quad +\eta\bar\eta\mathcal{D}x^\mu\bar{\mathcal{D}}x^\nu h_{\mu\nu}(y,x,\partial\eta,\bar\partial\bar\eta,C)+\sigma_{0,1}\hat\omega \end{aligned} \tag{A.1.14}$$

where the functions on the right hand side (h, $\eta\mathcal{D}x^\mu h_\mu$, ... , $\hat\omega$) are elements of $\mathcal{W}$. Note also that the sum on the right hand side is direct: no nonvanishing function $h+\eta\mathcal{D}x^\mu h_\mu+\bar\eta\bar{\mathcal{D}}x^\mu\bar h_\mu+\eta\bar\eta\mathcal{D}x^\mu\bar{\mathcal{D}}x^\nu h_{\mu\nu}$ is $\sigma_{0,1}$-exact because the various terms either do not contain variables $\mathcal{D}^{r+1}x^\mu$ or $\bar{\mathcal{D}}^{r+1}x^\mu$ at all, or they contain $\mathcal{D}x^\mu$ but no ε, or $\bar{\mathcal{D}}x^\mu$ but no $\bar\varepsilon$. Hence, our result characterizes $H(\sigma_{0,1},\mathcal{W})$ completely.

A.2 Derivation of (7.1.31)

We shall show that (7.1.30) implies (7.1.31). The proof is a case-by-case study for $g=0,\dots,4$. Since $\omega_{\bar k}$ does not depend on the fermions and has vanishing conformal weights, it can be assumed to contain only terms with even N_ε-degree and even $N_{\bar\varepsilon}$-degree. Hence, it does not depend on the supersymmetry ghosts if $g=0$ or $g=1$ which gives (7.1.31) in these cases. If $2\le g\le 4$ the assertion follows from

$$\sigma_{0,0}\omega_{\bar k}+\sigma_{0,1}\omega_{\bar k-1}=0, \tag{A.2.15}$$

which is the second equation in (7.1.29).

<u>$g=2$:</u> Only $\omega_{\bar k=2}$ can depend on the supersymmetry ghosts. One has

$$\omega_{\bar k=2}=\varepsilon\partial\varepsilon a(X)+\bar\varepsilon\bar\partial\bar\varepsilon\bar a(X)$$

where $a(X)$ and $\bar a(X)$ are functions of the undifferentiated x^μ and y^i. $\sigma_{0,0}\omega_{\bar 2}$ contains for instance $\eta(\partial\varepsilon)^2a(X)$ and $\bar\eta(\bar\partial\bar\varepsilon)^2\bar a(X)$ because $\sigma_{0,0}\varepsilon$ and $\sigma_{0,0}\bar\varepsilon$ contain $\eta\partial\varepsilon$ and $\bar\eta\bar\partial\bar\varepsilon$, respectively. If $a\neq 0$ or $\bar a\neq 0$, these terms are not $\sigma_{0,1}$-exact because they do not contain derivatives of an x^μ. We conclude that $a=0$ and $\bar a=0$ and thus that (7.1.31) holds for $g=2$.

<u>$g=3$:</u> Again, only $\omega_{\bar k=2}$ can depend on the supersymmetry ghosts. The terms in $\omega_{\bar k=2}$ depending on ε or its derivatives are

$$\begin{aligned}&\eta\varepsilon\partial^2\varepsilon a(X)+\varepsilon\partial\varepsilon\partial\eta b(X)+\varepsilon\partial\varepsilon\bar\partial\bar\eta c(X)+\varepsilon\partial\varepsilon C^i d_i(X) \\ &\quad +\bar\eta\bar{\mathcal{D}}x^\mu\varepsilon\partial\varepsilon e_\mu(X)+\eta(\partial\varepsilon)^2 f(X)+\partial^2\eta\varepsilon^2 g(X).\end{aligned} \tag{A.2.16}$$

In addition there are analogous terms with $\bar\varepsilon$ or its derivatives. A straightforward calculation shows that (A.2.15) imposes

$$b=0,\quad c=0,\quad d_i=0,\quad e_\mu=\partial_\mu a,\quad f=a,\quad g=-\tfrac{1}{2}\,a \tag{A.2.17}$$

where $a = a(X)$ is an arbitary function of the y^i and x^μ. Using (A.2.17) in (A.2.16), the latter becomes

$$\begin{aligned}&[\eta\varepsilon\partial^2\varepsilon + \bar{\eta}\bar{\mathcal{D}}x^\mu\varepsilon\partial\varepsilon\partial_\mu + \eta(\partial\varepsilon)^2 - \tfrac{1}{2}\,\partial^2\eta\varepsilon^2]a(X)\\ &\quad = \sigma_0[\varepsilon\partial\varepsilon a(X)] + \sigma_{0,1}[\eta\partial\varepsilon\psi^\mu\partial_\mu a(X)].\end{aligned} \tag{A.2.18}$$

This shows that all terms containing ε or its derivatives can be removed from $\omega_{\overline{k}=2}$ by the redefinition $\omega' = \omega - \sigma_0[\varepsilon\partial\varepsilon a(X) + \eta\partial\varepsilon\psi^\mu\partial_\mu a(X)]$. Similarly one can remove all terms containing $\bar{\varepsilon}$ or its derivatives. Hence, without loss of generality one can assume $\omega_{\overline{k}=2} = 0$ which implies (7.1.31) for $g = 3$.

<u>$g = 4$:</u> Now $\omega_{\overline{k}=4}$ and ω_2 can depend on the supersymmetry ghosts. One has

$$\begin{aligned}\omega_{\overline{k}=4} &= \varepsilon^3\partial^2\varepsilon a(X) + \varepsilon^2(\partial\varepsilon)^2 b(X) + \bar{\varepsilon}^3\bar{\partial}^2\bar{\varepsilon}\bar{a}(X)\\ &\quad +\bar{\varepsilon}^2(\bar{\partial}\bar{\varepsilon})^2\bar{b}(X) + \varepsilon\partial\varepsilon\bar{\varepsilon}\bar{\partial}\bar{\varepsilon}c(X).\end{aligned}$$

The fact that $\sigma_{0,0}\partial^2\varepsilon$ contains $-(1/2)\varepsilon\partial^3\eta$ implies $a = 0$. Analogously one concludes $\bar{a} = 0$. The fact that $\sigma_{0,0}\partial\varepsilon$ and $\sigma_{0,0}\bar{\partial}\bar{\varepsilon}$ contain $\eta\partial^2\varepsilon$ and $\bar{\eta}\bar{\partial}^2\bar{\varepsilon}$, respectively, implies $b = 0$, $\bar{b} = 0$ and $c = 0$.

ω_2 is of the form $P^A(\text{ghosts}, \mathcal{D}x^\mu, \bar{\mathcal{D}}x^\mu)a_A(X)$ where the P^A either depend on ε and its derivatives, or on $\bar{\varepsilon}$ and its derivatives. The complete list of polynomials P^A depending on ε and its derivatives is

$$\begin{gathered}\eta\partial\eta\varepsilon\partial^2\varepsilon, \eta\partial\eta(\partial\varepsilon)^2, \partial^2\eta\partial\eta\varepsilon^2, \eta\partial^2\eta\varepsilon\partial\varepsilon, \eta\partial^3\eta\varepsilon^2,\\ \bar{\eta}\bar{\mathcal{D}}x^\mu\eta\varepsilon\partial^2\varepsilon, \bar{\eta}\bar{\mathcal{D}}x^\mu\eta(\partial\varepsilon)^2, \bar{\eta}\bar{\mathcal{D}}x^\mu\partial\eta\varepsilon\partial\varepsilon, \bar{\eta}\bar{\mathcal{D}}x^\mu\partial^2\eta\varepsilon^2,\\ \eta\bar{\partial}\bar{\eta}\varepsilon\partial^2\varepsilon, \eta\bar{\partial}\bar{\eta}(\partial\varepsilon)^2, \bar{\eta}\bar{\partial}^2\bar{\eta}\varepsilon\partial\varepsilon, \partial\eta\bar{\partial}\bar{\eta}\varepsilon\partial\varepsilon, \partial^2\eta\bar{\partial}\bar{\eta}\varepsilon^2, \bar{\eta}\bar{\mathcal{D}}x^\mu\bar{\partial}\bar{\eta}\varepsilon\partial\varepsilon,\\ \eta C^i\varepsilon\partial^2\varepsilon, \eta C^i(\partial\varepsilon)^2, \partial\eta C^i\varepsilon\partial\varepsilon, \partial^2\eta C^i\varepsilon^2, \bar{\partial}\bar{\eta}C^i\varepsilon\partial\varepsilon, \bar{\eta}\bar{\mathcal{D}}x^\mu C^i\varepsilon\partial\varepsilon, C^iC^j\varepsilon\partial\varepsilon,\end{gathered}$$

Starting with the terms

$$\varepsilon\partial^2\varepsilon\eta\partial\eta A_1(X) + (\partial\varepsilon)^2\eta\partial\eta B_1(X) + \varepsilon^2\partial\eta\partial^2\eta E_2(X) \tag{A.2.19}$$

one finds that (A.2.15) implies $A_1(X) = B_1(X) = 2E_2(X)$. Considering the terms

$$\begin{gathered}\varepsilon\partial\varepsilon\eta\partial^2\eta B_5(X) + \varepsilon^2\eta\partial^3\eta E_1(X)\\ +\varepsilon\partial^2\varepsilon\eta\bar{\eta}\bar{\mathcal{D}}x^\mu A_{4,\mu}(X) + (\partial\varepsilon)^2\eta\bar{\eta}\bar{\mathcal{D}}x^\mu B_{4,\mu}(X)\\ +\varepsilon\partial\varepsilon\partial\eta\bar{\eta}\bar{\mathcal{D}}x^\mu C_{4,\mu}(X) + \varepsilon^2\partial^2\eta\bar{\eta}\bar{\mathcal{D}}x^\mu E_{6,\mu}(X),\end{gathered} \tag{A.2.20}$$

one observes that the σ_0 transformation of these terms neither contain $\bar{\partial}^k\bar{\eta}$ or $\bar{\partial}^k\bar{\varepsilon}$ terms nor $U(1)$ ghosts. Thus they have to fulfill (A.2.15) separately and one obtains

$$\begin{aligned}C_{4,\mu}(X) &= -\partial_\mu A_1(X)\\ B_{4,\mu}(X) &= -\partial_\mu B_5(X) + \partial_\mu A_1(X) - 2E_{6,\mu}(X)\\ A_{4,\mu}(X) &= -2\partial_\mu E_1(X) - 2E_{6,\mu}(X).\end{aligned}$$

Eliminating the coefficients one finds that (A.2.19) + (A.2.20) can be expressed by

$$\begin{aligned}&\sigma_0\big(\eta(\partial\varepsilon)^2(B_5(X)-A_1(X))+\eta\varepsilon\partial^2\varepsilon E_1(X)\\&\quad+\varepsilon\partial\varepsilon\partial\eta A_1(X)-2\varepsilon\partial\varepsilon\bar\eta\bar{\mathcal{D}}x^\mu E_{6,\mu}(X)\big)\\&+\sigma_{0,1}\big(-\eta\partial\eta\partial\varepsilon\psi^\mu\partial_\mu A_1(X)-2\bar\eta\eta\partial\varepsilon\bar{\mathcal{D}}x^\mu\psi^\nu\partial_\nu E_{6,\mu}\\&\quad-\bar\eta\eta\partial\varepsilon\bar{\mathcal{D}}x^\rho\psi^\nu\Omega_{\rho\nu}{}^\mu E_{6,\mu}\big),\end{aligned}\tag{A.2.21}$$

where we have used the on-shell equality (7.1.15). Next we consider the terms involving derivatives of $\bar\eta$

$$\begin{aligned}&\varepsilon\partial^2\varepsilon\eta\bar\partial\bar\eta A_2(X)+(\partial\varepsilon)^2\eta\bar\partial\bar\eta B_2(X)+\varepsilon\partial\varepsilon\bar\eta\bar\partial^2\bar\eta B_6(X)\\&+\varepsilon\partial\varepsilon\partial\eta\bar\partial\bar\eta B_7(X)+\varepsilon^2\partial^2\eta\bar\partial\bar\eta E_3(X)+\varepsilon\partial\varepsilon\bar\partial\bar\eta\bar\eta\bar{\mathcal{D}}x^\mu C_{5,\mu}(X),\end{aligned}\tag{A.2.22}$$

which implies via (A.2.15)

$$\begin{aligned}B_7(X)=0,\quad A_2(X)=B_6(X)=B_2(X)=-2E_3(X),\\C_{5,\mu}(X)=-\partial_\mu A_2(X).\end{aligned}\tag{A.2.23}$$

Thus (A.2.22) can be written as

$$\sigma_0\left(\varepsilon\partial\varepsilon\bar\partial\bar\eta A_2(X)\right)-\sigma_{0,1}\left(\partial\varepsilon\bar\partial\bar\eta\eta\psi^\mu\partial_\mu A_2(X)\right)\tag{A.2.24}$$

and thus be removed from ω_2. In the last step we consider contributions containing $U(1)$ ghosts, i.e.

$$\begin{aligned}&\varepsilon\partial^2\varepsilon\eta C^iA_{3,i}(X)+(\partial\varepsilon)^2\eta C^iB_{3,i}(X)+\varepsilon\partial\varepsilon\partial\eta C^iB_{8,i}(X)\\&\quad+\varepsilon^2\partial^2\eta C^iE_{4,i}(X)+\varepsilon\partial\varepsilon\bar\eta C^iB_{9,i}(X)\\&\quad+\varepsilon\partial\varepsilon C^i\bar\eta\bar{\mathcal{D}}x^\mu C_{6,\mu i}(X)+\varepsilon\partial\varepsilon C^iC^jB_{10,ij}(X).\end{aligned}\tag{A.2.25}$$

(A.2.15) imposes $B_{10,ij}(X)=B_{9,i}(X)=B_{8,i}(X)=0$. Furthermore we derive the conditions

$$A_{3,i}(X)=B_{3,i}(X)=-2E_{4,i}(X)\quad C_{6,\mu i}(X)=-\partial_\mu A_{3,i}(X).\tag{A.2.26}$$

Using the on-shell equality (7.1.14), (A.2.25) can be written as

$$\begin{aligned}&\sigma_0\left(\varepsilon\partial\varepsilon C^iA_{3,i}(X)\right)+\sigma_{0,1}\big(\partial\varepsilon C^i\eta\psi^\mu\partial_\mu A_{3,i}(X)\\&\quad-\partial\varepsilon\eta\bar\eta\psi^\mu\bar{\mathcal{D}}x^\nu(\Omega_{\mu\nu i}-\Omega_{\mu\nu}{}^\lambda G_{\lambda i})A_{3,i}(X)\big).\end{aligned}\tag{A.2.27}$$

Hence, as in the case $g=3$ one finds that (A.2.15) implies $\omega_2=\sigma_0(\dots)+\sigma_{0,1}(\dots)$ which implies (7.1.31) for $g=4$.

endix B

l sis i i i i i s

In this appendix we summarize briefly the investigation of the Bianchi identities for two-dimensional supergravity coupled to Maxwell theory. The starting point is the structure equation

$$[\mathcal{D}_A, \mathcal{D}_B\} = -T_{AB}{}^C \mathcal{D}_C - R_{AB}\delta_L - F_{AB}{}^i \delta_i, \tag{B.0.1}$$

where $[\cdot,\cdot\}$ denotes the graded commutator, $\{\mathcal{D}_A\} = \{\mathcal{D}_a, \mathcal{D}_\alpha\}$ contains the covariant derivatives $\mathcal{D}_a$ and covariant supersymmetry transformations $\mathcal{D}_\alpha$, $\delta_L = (1/2)\varepsilon^{ab} l_{ab}$ is the Lorentz generator and δ_i are the $U(1)$ generators (represented trivially in our case). The "torsions" $T_{AB}{}^C$, "curvatures" R_{AB} and "field strengths" $F_{AB}{}^i$ are generically field dependent and determined from the Bianchi identities implied by (B.0.1). Using the constraints (3.2.6) and (3.2.7) one obtains for the torsions

$$\begin{aligned} T_{\alpha\beta}{}^a &= 2\mathrm{i}(\gamma^a C)_{\alpha\beta} \\ T_{a\beta}{}^\alpha &= \tfrac{1}{4}\, S(\gamma_a)_\beta{}^\alpha \\ T_{ab}{}^\alpha &= \tfrac{\mathrm{i}}{4}\, \varepsilon_{ab}(C\gamma_*)^{\alpha\beta}\mathcal{D}_\beta S, \end{aligned} \tag{B.0.2}$$

where S is the auxiliary scalar field of the gravitational multiplet. For the curvatures one obtains

$$\begin{aligned} R_{\alpha\beta} &= \mathrm{i}S(\gamma_* C)_{\alpha\beta} \\ R_{a\alpha} &= \tfrac{\mathrm{i}}{2}\,(\gamma_a\gamma_*)_\alpha{}^\beta \mathcal{D}_\beta S \\ R_{ab} &= \tfrac{1}{4}\,\varepsilon_{ab}(S^2 + \mathcal{D}^2 S), \end{aligned} \tag{B.0.3}$$

and the field strengths are given by

$$\begin{aligned} F_{\alpha\beta}{}^i &= 2\mathrm{i}(\gamma_* C)_{\alpha\beta}\phi^i \\ F_{a\alpha}{}^i &= (\gamma_a)_\alpha{}^\beta \lambda^i_\beta\ . \end{aligned} \tag{B.0.4}$$

The supersymmetry transformations of λ^i_β and F^i_{ab} turn out to be

$$\begin{array}{rcl}\mathcal{D}_\alpha\lambda^i_\beta & = & \mathrm{i}(\gamma^a\gamma_* C)_{\alpha\beta}\mathcal{D}_a\phi^i + \frac{\mathrm{i}}{2}\,(\gamma_* C)_{\alpha\beta}\varepsilon^{ab}F^i_{ba} + \frac{\mathrm{i}}{2}\,(\gamma_* C)_{\alpha\beta}S\phi^i \\ \mathcal{D}_\alpha F^i_{ab} & = & -(\gamma_b\mathcal{D}_a\lambda^i)_\alpha + (\gamma_a\mathcal{D}_b\lambda^i)_\alpha + \frac{1}{2}\,\varepsilon_{ab}\mathcal{D}_\alpha S\phi^i \\ & & +\frac{1}{2}\,\varepsilon_{ab}S(\gamma_*)_\alpha{}^\delta\lambda^i_\delta. \end{array} \tag{B.0.5}$$

Introducing the corresponding connection 1-forms and proceeding along the lines of [82] one identifies the covariant derivatives $\mathcal{D}_a$ in terms of partial derivatives and connections, and the curvatures, field strengths and torsions with two lower Lorentz indices in terms of the connections and the other field strengths. Owing to the constraint $T_{ab}{}^c = 0$ this yields the expression (3.2.3) for the spin connection. Furthermore one obtains

$$F_{ab}{}^i = E_a{}^n E_b{}^m(\partial_n A^i_m - \partial_m A^i_n - (\chi_m\gamma_n\lambda^i) + (\chi_n\gamma_m\lambda^i) - 2\mathrm{i}(\chi_m\gamma_* C\chi_n)\phi^i)$$

and the expression for $T_{ab}{}^\alpha$ can be used to express the supersymmetry transformation of the auxiliary field S as

$$\mathcal{D}_\alpha S = 4\mathrm{i}(\gamma_* C)_{\alpha\beta}\varepsilon^{nm}\nabla_m\chi_n{}^\beta - \mathrm{i}(\gamma^m C)_{\alpha\beta}\chi_m{}^\beta S.$$

The full BRST transformations (3.2.2), (3.2.4) and (3.2.5) are then obtained by adding the Weyl transformations by hand and imposing $s^2 = 0$ on all fields. To achieve this in an off-shell setting, one introduces the super-Weyl symmetry on the gravitino and the gaugino and the local shift symmetry of the auxiliary field S.

endix

r s r i s

C.1 B ST transformations of superconformal tensor fields

This appendix collects the BRST transformations of the superconformal tensor fields and corresponding ghost variables derived in section 3.3. The transformations of the undifferentiated fields read

$$
\begin{aligned}
s\eta &= \eta\partial\eta - \varepsilon\varepsilon \\
s\bar\eta &= \bar\eta\bar\partial\bar\eta - \bar\varepsilon\bar\varepsilon \\
s\varepsilon &= \eta\partial\varepsilon - \tfrac{1}{2}\,\varepsilon\partial\eta \\
s\bar\varepsilon &= \bar\eta\bar\partial\bar\varepsilon - \tfrac{1}{2}\,\bar\varepsilon\bar\partial\bar\eta \\
sC^i &= \eta\bar\eta\mathcal{F}^i + \eta\bar\varepsilon\lambda^i + \bar\eta\varepsilon\bar\lambda^i + \varepsilon\bar\varepsilon\hat\phi^i \\
sX^M &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})X^M + \varepsilon\psi^M + \bar\varepsilon\bar\psi^M \\
s\psi^M &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\psi^M + \tfrac{1}{2}\,\partial\eta\psi^M + \varepsilon\mathcal{D}X^M - \bar\varepsilon\hat F^M \\
s\bar\psi^M &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\bar\psi^M + \tfrac{1}{2}\,\bar\partial\bar\eta\bar\psi^M + \bar\varepsilon\bar{\mathcal{D}}X^M + \varepsilon\hat F^M \\
s\hat F^M &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\hat F^M + \tfrac{1}{2}\,(\partial\eta + \bar\partial\bar\eta)\hat F^M + \varepsilon\mathcal{D}\bar\psi^M - \bar\varepsilon\bar{\mathcal{D}}\psi^M \\
s\hat\phi^i &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\hat\phi^i + \tfrac{1}{2}\,(\partial\eta + \bar\partial\bar\eta)\hat\phi^i + \varepsilon\lambda^i + \bar\varepsilon\bar\lambda^i \\
s\lambda^i &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\lambda^i + (\partial\eta + \tfrac{1}{2}\,\bar\partial\bar\eta)\lambda^i + \varepsilon\mathcal{D}\hat\phi^i + \bar\varepsilon\mathcal{F}^i + \partial\varepsilon\hat\phi^i \\
s\bar\lambda^i &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\bar\lambda^i + (\tfrac{1}{2}\,\partial\eta + \bar\partial\bar\eta)\bar\lambda^i + \bar\varepsilon\bar{\mathcal{D}}\hat\phi^i - \varepsilon\mathcal{F}^i + \bar\partial\bar\varepsilon\hat\phi^i \\
s\mathcal{F}^i &= (\eta\mathcal{D} + \bar\eta\bar{\mathcal{D}})\mathcal{F}^i + (\partial\eta + \bar\partial\bar\eta)\mathcal{F}^i - \varepsilon\mathcal{D}\bar\lambda^i + \bar\varepsilon\bar{\mathcal{D}}\lambda^i - \partial\varepsilon\bar\lambda^i + \bar\partial\bar\varepsilon\lambda^i
\end{aligned}
$$

The s-transformations of covariant $\mathcal{D}$ or $\bar{\mathcal{D}}$ derivatives (of first or higher order) of a field are obtained by applying $\mathcal{D}$'s and/or $\bar{\mathcal{D}}$'s to the transformations given above, using the rules $\mathcal{D}\eta = \partial\eta$, $\mathcal{D}\bar\eta = 0$, $\mathcal{D}\varepsilon = \partial\varepsilon$, $\mathcal{D}\bar\varepsilon = 0$ etc, as well as $[\mathcal{D}, \bar{\mathcal{D}}] = 0$. E.g., one gets

$$
\begin{aligned}
s\mathcal{D}X^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\mathcal{D}X^M + \partial\eta\mathcal{D}X^M + \varepsilon\mathcal{D}\psi^M + \bar{\varepsilon}\mathcal{D}\bar{\psi}^M + \partial\varepsilon\psi^M \\
s\bar{\mathcal{D}}X^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\bar{\mathcal{D}}X^M + \bar{\partial}\bar{\eta}\bar{\mathcal{D}}X^M + \varepsilon\bar{\mathcal{D}}\psi^M + \bar{\varepsilon}\bar{\mathcal{D}}\bar{\psi}^M + \bar{\partial}\bar{\varepsilon}\bar{\psi}^M \\
s\mathcal{D}\bar{\mathcal{D}}X^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\mathcal{D}\bar{\mathcal{D}}X^M + (\partial\eta + \bar{\partial}\bar{\eta})\mathcal{D}\bar{\mathcal{D}}X^M \\
&\quad + \varepsilon\mathcal{D}\bar{\mathcal{D}}\psi^M + \bar{\varepsilon}\mathcal{D}\bar{\mathcal{D}}\bar{\psi}^M + \partial\varepsilon\bar{\mathcal{D}}\psi^M + \bar{\partial}\bar{\varepsilon}\mathcal{D}\bar{\psi}^M \\
s\mathcal{D}\psi^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\mathcal{D}\psi^M + \tfrac{3}{2}\,\partial\eta\mathcal{D}\psi^M + \tfrac{1}{2}\,\partial^2\eta\psi^M \\
&\quad + \varepsilon\mathcal{D}^2X^M + \partial\varepsilon\mathcal{D}X^M - \bar{\varepsilon}\mathcal{D}\hat{F}^M \\
s\bar{\mathcal{D}}\bar{\psi}^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\bar{\mathcal{D}}\bar{\psi}^M + \tfrac{3}{2}\,\bar{\partial}\bar{\eta}\bar{\mathcal{D}}\bar{\psi}^M + \tfrac{1}{2}\,\bar{\partial}^2\bar{\eta}\bar{\psi}^M \\
&\quad + \bar{\varepsilon}\bar{\mathcal{D}}^2X^M + \bar{\partial}\bar{\varepsilon}\bar{\mathcal{D}}X^M + \varepsilon\bar{\mathcal{D}}\hat{F}^M \\
s\bar{\mathcal{D}}\psi^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\bar{\mathcal{D}}\psi^M + \tfrac{1}{2}\,\partial\eta\bar{\mathcal{D}}\psi^M + \bar{\partial}\bar{\eta}\bar{\mathcal{D}}\psi^M \\
&\quad + \varepsilon\mathcal{D}\bar{\mathcal{D}}X^M - \bar{\partial}\bar{\varepsilon}\hat{F}^M - \bar{\varepsilon}\bar{\mathcal{D}}\hat{F}^M \\
s\mathcal{D}\bar{\psi}^M &= (\eta\mathcal{D} + \bar{\eta}\bar{\mathcal{D}})\mathcal{D}\bar{\psi}^M + \partial\eta\mathcal{D}\bar{\psi}^M + \tfrac{1}{2}\,\bar{\partial}\bar{\eta}\mathcal{D}\bar{\psi}^M \\
&\quad + \bar{\varepsilon}\mathcal{D}\bar{\mathcal{D}}X^M + \partial\varepsilon\hat{F}^M + \varepsilon\mathcal{D}\hat{F}^M
\end{aligned}
$$

C.2 B ST transformations of superconformal antifields

In this appendix we present the full s transformations of the superconformal antifields associated with the matter and gauge multiplets, using the following notation:

$$
\begin{aligned}
G_{MN} &:= H_{(MN)}(X) \\
D_i &:= D_i(X) \\
\Omega_{KNM} &:= \partial_K H_{MN}(X) - \partial_M H_{KN}(X) + \partial_N H_{KM}(X) \\
&= 2\Gamma_{KNM} - H_{KNM} \quad (H_{KNM} = 3\partial_{[K}H_{NM]}) \\
R_{KLMN} &:= \partial_M\partial_{[K}H_{L]N}(X) - \partial_N\partial_{[K}H_{L]M}(X) \\
&= \tfrac{1}{2}\,(\partial_K\Omega_{LMN} - \partial_L\Omega_{KMN}) = \tfrac{1}{2}\,(\partial_M\Omega_{KNL} - \partial_N\Omega_{KML}).
\end{aligned}
$$

Ω_{KNM} and R_{KLMN} enjoy the following properties:

$$
\begin{gathered}
\Omega_{KMN} + \Omega_{KNM} = \Omega_{MKN} + \Omega_{NKM} = 2\partial_K G_{MN} \\
R_{KLMN} = -R_{LKMN} = -R_{KLNM}\ , \quad \partial_{[J}R_{KL]MN} = 0.
\end{gathered}
$$

The full BRST transformations of the undifferentiated superconformal matter antifields are

$$\begin{aligned}
sF^*_M &= -\hat{\phi}^i\partial_M D_i + 2G_{MN}\hat{F}^N + \psi^K\bar{\psi}^N\Omega_{KNM} \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})F^*_M + \tfrac{1}{2}\,(\partial\eta+\bar{\partial}\bar{\eta})F^*_M - \varepsilon\bar{\psi}^*_M + \bar{\varepsilon}\psi^*_M \\
s\psi^*_M &= \bar{\lambda}^i\partial_M D_i + \bar{\psi}^N\hat{\phi}^i\partial_N\partial_M D_i + 2G_{MN}\bar{\mathcal{D}}\psi^N \\
&\quad +\bar{\mathcal{D}}X^N\psi^K\Omega_{KNM} - \hat{F}^N\bar{\psi}^K\Omega_{MKN} - \psi^K\bar{\psi}^N\bar{\psi}^L R_{KMLN} \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\psi^*_M + (\tfrac{1}{2}\,\partial\eta+\bar{\partial}\bar{\eta})\psi^*_M + \varepsilon X^*_M + \bar{\varepsilon}\bar{\mathcal{D}}F^*_M + \bar{\partial}\bar{\varepsilon}F^*_M \\
s\bar{\psi}^*_M &= -\lambda^i\partial_M D_i - \psi^N\hat{\phi}^i\partial_N\partial_M D_i + 2G_{MN}\mathcal{D}\bar{\psi}^N \\
&\quad +\mathcal{D}X^N\bar{\psi}^K\Omega_{NKM} + \hat{F}^N\psi^K\Omega_{KMN} + \psi^K\psi^L\bar{\psi}^N R_{LKMN} \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\bar{\psi}^*_M + (\partial\eta+\tfrac{1}{2}\,\bar{\partial}\bar{\eta})\bar{\psi}^*_M + \bar{\varepsilon}X^*_M - \varepsilon\mathcal{D}F^*_M - \partial\varepsilon F^*_M \\
sX^*_M &= -2G_{MN}\mathcal{D}\bar{\mathcal{D}}X^N - \mathcal{D}X^K\bar{\mathcal{D}}X^L\Omega_{KLM} + \hat{F}^K\hat{F}^L\Omega_{MKL} \\
&\quad +\mathcal{D}\bar{\psi}^K\bar{\psi}^L\Omega_{MLK} - \psi^K\bar{\mathcal{D}}\psi^L\Omega_{KML} \\
&\quad +\mathcal{D}X^N\bar{\psi}^K\bar{\psi}^L R_{NMLK} + \bar{\mathcal{D}}X^N\psi^K\psi^L R_{LKNM} \\
&\quad +\hat{F}^N\psi^K\bar{\psi}^L\partial_M\Omega_{KLN} + \tfrac{1}{2}\,\psi^R\psi^K\bar{\psi}^N\bar{\psi}^L\partial_M R_{KRLN} \\
&\quad +\mathcal{F}^i\partial_M D_i - (\psi^N\bar{\lambda}^i - \bar{\psi}^N\lambda^i + \hat{F}^N\hat{\phi}^i + \psi^N\bar{\psi}^K\hat{\phi}^i\partial_K)\partial_N\partial_M D_i \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})X^*_M + (\partial\eta+\bar{\partial}\bar{\eta})X^*_M \\
&\quad +\varepsilon\mathcal{D}\psi^*_M + \bar{\varepsilon}\bar{\mathcal{D}}\bar{\psi}^*_M + \partial\varepsilon\psi^*_M + \bar{\partial}\bar{\varepsilon}\bar{\psi}^*_M
\end{aligned}$$

The s transformation of the superconformal antifields for the gauge multiplet read

$$\begin{aligned}
s\lambda^*_i &= \bar{\psi}^M\partial_M D_i \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\lambda^*_i + \tfrac{1}{2}\,\bar{\partial}\bar{\eta}\lambda^*_i + \varepsilon\phi^*_i - \bar{\varepsilon}\bar{A}^*_i \\
s\bar{\lambda}^*_i &= -\psi^M\partial_M D_i \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\bar{\lambda}^*_i + \tfrac{1}{2}\,\partial\eta\bar{\lambda}^*_i + \bar{\varepsilon}\phi^*_i - \varepsilon A^*_i \\
s\phi^*_i &= -\hat{F}^M\partial_M D_i - \psi^M\bar{\psi}^N\partial_N\partial_M D_i \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\phi^*_i + \tfrac{1}{2}\,(\partial\eta+\bar{\partial}\bar{\eta})\phi^*_i \\
&\quad +\varepsilon\mathcal{D}\lambda^*_i + \bar{\varepsilon}\bar{\mathcal{D}}\bar{\lambda}^*_i - \varepsilon\bar{\varepsilon}C^*_i \\
sA^*_i &= -\mathcal{D}X^M\partial_M D_i \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})A^*_i + \partial\eta A^*_i \\
&\quad +\bar{\varepsilon}\mathcal{D}\lambda^*_i - \varepsilon\mathcal{D}\bar{\lambda}^*_i - \partial\varepsilon\bar{\lambda}^*_i - \bar{\varepsilon}\varepsilon C^*_i \\
s\bar{A}^*_i &= \bar{\mathcal{D}}X^M\partial_M D_i \\
&\quad +(\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})\bar{A}^*_i + \bar{\partial}\bar{\eta}\bar{A}^*_i \\
&\quad +\varepsilon\bar{\mathcal{D}}\bar{\lambda}^*_i - \bar{\varepsilon}\bar{\mathcal{D}}\lambda^*_i - \bar{\partial}\bar{\varepsilon}\lambda^*_i - \varepsilon\varepsilon C^*_i \\
sC^*_i &= -\mathcal{D}\bar{A}^*_i - \bar{\mathcal{D}}A^*_i + (\eta\mathcal{D}+\bar{\eta}\bar{\mathcal{D}})C^*_i + (\partial\eta+\bar{\partial}\bar{\eta})C^*_i
\end{aligned}$$

The BRST transformations of covariant derivatives of the covariant antifields (such as $s\mathcal{D}X^*_M$) are obtained from the above formulae by means of the rules described in appendix C.1.

i li r

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