Dissertation

Filters in Number Theory and Combinatorics

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Kurzfassung

Im Mittelpunkt dieser Arbeit stehen Anwendungen von abstrakten topologischen Methoden in Zahlentheorie und Kombinatorik. Die Stone-Čech Kompaktifizierung βS einer diskreten Halbgruppe S kann als die geeignet topologisierte Menge aller Ultrafilter auf S konstruiert werden. Diese Struktur erlaubt überraschend einfache Beweise der Sätze von Hindman und van der Waerden. Der Satz von Hindman besagt, dass es für jede endliche Färbung der natürlichen Zahlen eine einfärbige Menge A und eine Folge natürlicher Zahlen $(x_n)_{n=1}^{\infty}$ gibt, so dass für jede endliche Teilmenge F der natürlichen Zahlen $\sum_{t\in F} x_t \in A$ gilt. Der Satz von van der Waerden zeigt, dass es stets eine einfärbige Menge B gibt, die beliebig lange arithmetische Folgen enthält.

Kapitel 1: Wir beginnen mit einigen allgemeinen Bemerkungen über Ramseytheoretische Aussagen. Die Ultrafilterkonstruktion der Stone-Čech Kompaktifizierung einer diskreten Halbgruppe S, sowie die grundlegende algebraische Theorie von kompakten Halbgruppen werden besprochen. Dies liefert den notwendigen Hintergrund für die angestrebten kombinatorischen Anwendungen von βS. Als erste Beispiele präsentieren wir kurze Beweise der Sätze von Hindman und van der Waerden. Unser Beweis des Satzes von van der Waerden liefert eine relativ starke Version dieses Theorems. Im letzten Abschnitt dieses Kapitels zeigen wir, dass ähnliche Verallgemeinerungen auch für andere bekannte ramseytheoretische Resultate möglich sind. Die Ergebnisse dieses Kapitels sind zum Teil [BBHSi] entnommen.

Kapitel 2: Am Beginn dieses Kapitels besprechen wir verschiedene Verallgemeinerungen des Satzes von van der Waerden, unter anderem die Sätze von Gallai und von Hales-Jewett, aber auch unterschiedliche multiplikative Versionen des Satzes von van der Waerden. Ein Hauptziel dieses Kapitels ist es zu zeigen, dass es in jeder endlichen Partition der natürlichen Zahlen eine Zelle gibt, welche sowohl additativ als auch multiplikativ hoch organisierte Strukturen enthält. Manche der hierfür entwickelten Methoden sind speziell an die natürlichen Zahlen angepaßt, während andere auf allgemeine Halbgruppen bzw. Ringe anwendbar sind. Um ein Gefühl für die Sätze, die uns beschäftigen, zu vermitteln, bringen wir ein konkretes Beispiel: Sei k eine beliebig große natürliche Zahl. Für jede endliche Färbung der natürlichen

Zahlen gibt es eine einfärbige Menge A und $a, d, r \in A$ sodass

$$\{(a+id)r^j : i, j \in \{0, 1, \ldots, k\}\} \cup \{dr^j : j \in \{0, 1, \ldots, k\}\} \subseteq A.$$

Die Resultate dieses Kapitels wurden in Kooperation mit V. Bergelson, N. Hindman and D. Strauss in [BBHS*i*, BBHS*ii*] erzielt.

Kapitel 3: Fürstenbergs Satz über zentrale Mengen liefert eine natürliche starke gemeinsame Verallgemeinerung der Sätze von Hindman und van der Waerden. Wir beweisen eine mehrdimensionale Variante, die den klassischen Satz von Ramsey beinhaltet. Des weiteren bringen wir verschiedene Verallgemeinerungen des Satzes über zentrale Mengen, die zum Teil Resultate des vorangegangenen Kapitels stark erweitern. Das Material dieses Kapitels entstammt [Bei*ii*] und [BBHS*ii*].

Kapitel 4: Wir verbinden eine Methode Untergruppen von $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ über Filter zu kodieren (vgl. [Wi02]) mit einem Weg abzählbare Untergruppen von $\mathbb{T}$ über Teilfolgen der natürlichen Zahlen zu charakterisieren (vgl. [BDS01]). Das ermöglicht uns das folgende Resultat zu beweisen: Für jede abzählbare Untergruppe G des Torus gibt es eine Folge natürlicher Zahlen $(x_n)_{n=1}^{\infty}$ sodass

$$\alpha \in G \Rightarrow \sum_{n=1}^{\infty} \|x_n \alpha\| < \infty \qquad \alpha \notin G \Rightarrow \limsup_{n \to \infty} \|x_n \alpha\| \geq 1/6.$$

(Hierbei bezeichnet $\|.\|$ den Abstand zur nächsten ganzen Zahl.) Dieses Theorem ist das Hauptresultat von [Bei*i*] und setzt [BDS01, BS03] fort.

Abstract

We are mainly concerned with certain applications of abstract topological methods to combinatorics and number theory. The Stone-Čech Compactification βS of a discrete semigroup S consists of the properly topologized set of ultrafilters on S. These structure provides surprisingly simple proofs of various Ramsey theoretic results. Celebrated examples are the Theorems of Hindman and van der Waerden. Hindman's Theorem states that for any finite colouring of $\mathbb{N}$ there exist a monochrome set $A \subseteq \mathbb{N}$ and a sequence $(x_n)_{n=1}^{\infty}$ such that for all finite non empty sets $F \subseteq \mathbb{N}$, $\sum_{t \in F} x_t \in A$. Van der Waerden's Theorem yields that there exists a monochrome subset B that contains arithmetic progressions of arbitrary finite length.

chapter 1: Some general speculations concerning Ramsey theoretic results are given. We develop the ultrafilter construction of the Stone-Čech Compactification of a discrete semigroup and give a short introduction into the algebraic theory of compact semigroups. This provides the necessary background which is needed for our intended combinatorial applications of βS. As an immediate application short proofs of the Theorems of Hindman and van der Waerden are given. Our proof of van der Waerden's Theorem yields a rather strong version of this Theorem. In the last section we show that a similar strengthening applies to other well known results. The new results of this chapter are in part taken from [BBHS*i*].

chapter 2: We give several generalisations of van der Waerden's Theorem including Gallai's Theorem and the Hales-Jewett Theorem, some emphasis is put on multiplicative versions of van der Waerden's Theorem. A main goal of this chapter is the derivation of results that provide that highly organised structure in an additive as well as in a multiplicative sense is contained in one cell of every finite partition of $\mathbb{N}$. We obtain different methods to achieve these theorems, some apply to general semigroups, while others are limited to the positive integers. To deliver some flavour of the style of the Theorems we are after, some concrete example might be useful: Let $k \in \mathbb{N}$. If $\mathbb{N}$ is finitely coloured, there exist a monochrome set $A \subseteq \mathbb{N}$ and $a, d, r \in A$ such that

$$\{(a + id)r^j : i, j \in \{0, 1, \dots, k\}\} \cup \{dr^j : j \in \{0, 1, \dots, k\}\} \subseteq A.$$

The material in this section is joint work with V. Bergelson, N. Hindman

and D. Strauss in [BBHSi, BBHSii].
chapter 3: Fürstenberg's Central Sets Theorem is a powerful joint generalization of the Theorems of Hindman and van der Waerden. We give an extension of the Central Sets Theorem which also contains Ramsey's classical Theorem in a quite natural way. Other strengthenings of the Central Sets Theorem in the style of the Theorems from section 2 are given. The material of this section is taken from [Beiii] and [BBHSi].
chapter 4: We connect a way of describing subgroups of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ via filters (see [Wi02]) with a method of characterizing countable subgroups of $\mathbb{T}$ via sequences in $\mathbb{N}$. This enables us to prove the following result: If G is a countable subgroup of $\mathbb{T}$ there exists a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that

$$\alpha \in G \Rightarrow \sum_{n=1}^{\infty} \|x_n \alpha\| < \infty \qquad \alpha \notin G \Rightarrow \limsup_{n\to\infty} \|x_n\alpha\| \geq 1/6.$$

(Here $\|.\|$ denotes the distance from nearest integer.) This theorem is taken from [Bei$i$] and continues [BDS01, BS03].

Contents

Chapter 1

An introduction to βS

This chapter is an introduction to ultrafilter lore.
First we describe the structure of the partition theorems we are after and how they are related to "finitistic" Ramsey theorems.
In the second section we introduce the Stone-Čech Compactification βS of a discrete space S via ultrafilters on S. We explain how a semigroup structure on S may be lifted to βS such that βS becomes a compact right topological semigroup.
In the next section we develop the theory of compact right topological semigroups to an extend which is appropriate for our applications.
In section 4 we link combinatorial properties of a subset of a semigroup S to algebraic properties of ultrafilters in βS. As an application short proofs of the Theorems of Hindman and van der Waerden are given. In the last section the new method that is used to prove the latter of these theorems is used to give extensions of other well known Ramsey Theoretic results.

1.1 Some remarks on partition theorems

Sometimes van der Waerden's Theorem is spelled out in the following finitistic formulation:

Theorem 1.1.1 (van der Waerden's Theorem [Wa27]) *Let $r, l \in \mathbb{N}$[1]. There exists $N \in \mathbb{N}$ such that if $\bigcup_{i=1}^{r} A_i = \{1, 2, \ldots, N\}$ there exist $i \in \{1, 2, \ldots, r\}$ and $a, d \in \mathbb{N}$ such that*

$$\{a, a+d, \ldots, a+ld\} \subseteq A_i.$$

In contrast to this our methods usually only yield partition results about infinite sets.

[1] We take $\mathbb{N}$ to be the set of all positive integers and write ω for $\mathbb{N} \cup \{0\}$.

At the first glance Theorem 1.1.1 might seem stronger then the version referring to partitions of $\mathbb{N}$. In fact they are equivalent. To show this in some generality we introduce some notation.
Many of the theorems that we will prove are of the following kind: Let $r \in \mathbb{N}$ and assume that $\bigcup_{i=1}^{r} A_i = \mathbb{N}$. Then there exists some $i \in \{1, 2, \ldots, r\}$ such that A_i contains ..., where ... usually stands for some interesting algebraic structure.
It will be useful to make this more precise:

Definition 1.1.2 *Let S be a set and let $\mathcal{F}$ be a family of subsets of S. $\mathcal{F}$ is called* partition regular *iff for any finite coloring of S there exists some monochrome set that contains a member of $\mathcal{F}$.*

We use Hindman's Theorem as an example to illustrate this notation. (For a set S we will denote the power set of S by $\mathcal{P}(S)$ and use the symbol $\mathcal{P}_f(S)$ for the set of all non empty finite subsets of S.)

Theorem 1.1.3 (Hindman's Theorem [H75]) *Let $r \in \mathbb{N}$ and assume that $\bigcup_{i=1}^{r} A_i = \mathbb{N}$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\sum_{t \in F} x_t \in A$.*

Thus Hindman's Theorem states that the family

$$\mathcal{F} = \{FS((x_n)_{n=1}^{\infty}) : (x_n)_{n=1}^{\infty} \text{ is a sequence in } \mathbb{N}\},$$

where $FS((x_n)_{n=1}^{\infty}) = \{\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ is partition regular.
Similarly van der Waerden's Theorem states that for all $l \in \mathbb{N}$ the family

$$\mathcal{F} = \{\{a, a+d, \ldots, a+ld\} : a, d \in \mathbb{N}\}$$

is partition regular.
Of course we may not hope for a finitistic version of Hindman's Theorem, since its subject are infinite configurations. The following Theorem states that a finitistic version is always true, provided we are interested in finite configurations.

Theorem 1.1.4 *Let S be a set and assume that $\mathcal{F}$ is a partition regular family of finite subsets of S. Let $r \in \mathbb{N}$. Then there exists some finite set $G \subseteq S$ such that if $G = \bigcup_{i=1}^{r} A_i$ there exist some $i \in \{1, 2, \ldots, r\}$ and some $F \in \mathcal{F}$ such that $F \subseteq A_i$.*

proof: Assume for contradiction that for $r \in \mathbb{N}$ the claim is not true. To carry out a compactness argument we switch to the compact space $X = \{1, \ldots, r\}^S$ and consider colorings instead of partitions. For any finite set $G \subseteq S$ let

$$A_G = \{f \in X : \text{No } F \in \mathcal{F}, F \subseteq G \text{ is monochrom with respect to } f\}.$$

By assumption each A_G is not empty and it is routine to check that it is a closed subset of X. For $k \in \mathbb{N}$ and finite sets $G_1, \ldots, G_k \subseteq S$ one has $A_{G_1} \cap \ldots \cap A_{G_k} \supseteq A_{G_1 \cup \ldots \cup G_k}$ and therefore the family $\{A_G : G \subseteq S, |G| < \infty\}$ has the finite intersection property. Thus we may apply compactness to find $f \in \bigcap_{G \subseteq S, |G| < \infty} A_G$. By the partition regularity of $\mathcal{F}$ there exists some $F \in \mathcal{F}$ that is monochrome with respect to f. But this contradicts $f \in A_F$. $\square$

However we want to remark, that our methods never lead to explicit bounds.

1.2 Elementary properties of ultrafilters

Definition 1.2.1 *Let S be a nonempty set. $\mathcal{F} \in \mathcal{P}(S)$ is a* filter *on S iff the following hold:*

(1) $\emptyset \neq \mathcal{F} \neq \mathcal{P}(S)$.

(2) *Whenever $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$ then $B \in \mathcal{F}$.*

(3) *$\mathcal{F}$ is closed under finite intersections: For all $r \in \mathbb{N}$ and $A_1, A_2, \ldots, A_r \in \mathcal{F}$ one has $\bigcap_{i=1}^{r} A_i \in \mathcal{F}$*

An ultrafilter on S is a filter on S which is maximal with respect to inclusion. βS denotes the set of all ultrafilters on S.

Sometimes we will use the notion of a filter limit:

Definition 1.2.2 *Let S be a set, let $\mathcal{F}$ be a filter on S, let x be a point in a topological space X and assume that $f : dom\ f \to X$ is a function with $dom\ f \subseteq S$. We write $\mathcal{F} - \lim_s f(s) = x$ iff for every neighborhood $U \subseteq X$ of x one has $f^{-1}[U] \in \mathcal{F}$.*

For $s \in S$ put $e(s) = \{A \subseteq S : s \in A\}$. Obviously $e(s)$ is an ultrafilter, a so called *principle ultrafilter*. Most of the time we will abuse notation and just write s instead of $e(s)$. (I.e. We consider S to be embedded in βS.)

Theorem 1.2.3 *Let S be a nonempty set. For a Filter p on S the following properties are equivalent.*

(1) *p is an ultrafilter.*

(2) *If $A \cup B = \mathbb{N}$ then $A \in p$ or $B \in p$.*

(3) *Let $r \in \mathbb{N}$ and $\bigcup_{i=1}^{r} A_i \in p$. Then there exists $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$.*

(4) *If $f : dom\ f \subseteq S \to K$ is a function into a compact space satisfying $dom\ f \in p$, $p - \lim_s f(s)$ exists.*

proof:

(1) $\Rightarrow$ (3) Assume that (3) does not hold and pick $A, A_1, A_2, \ldots, A_r$ such that $A = \bigcup_{i=1}^r A_i \in p$ and $A_i \notin p$ for $i \in \{1, 2, \ldots, r\}$. We claim that there exists some $i \in \{1, 2, \ldots, r\}$ such that $A_i \cap B \neq \emptyset$ for all $B \in p$. If not pick for each $i \in \{1, 2, \ldots, n\}$ some B_i such that $A_i \cap B_i = \emptyset$. But then $A \cap \bigcap_{i=1}^r B_i = \emptyset$ which is not possible since p is closed under finite intersections. Thus there is some $i \in \{1, 2, \ldots, r\}$ such that A_i intersects all elements of p non trivially. It is then not difficult see that $\{A_i\} \cup p$ generates a filter on S which is strictly bigger than p.

(3) $\Rightarrow$ (4) For any $A_1, A_2, \ldots, A_r \in p$ we have $\bigcap_{i=1}^r \overline{f[A_i]} \supseteq \overline{f[\bigcap_{i=}^r A_i]} \neq \emptyset$, so by compactness of X, $L = \bigcap_{A \in p} \overline{f[A]} \neq \emptyset$. Assume for contradiction that there exist to distinct points $x, y \in L$. Pick open sets $U, V \subseteq X$ such that $U \cap V = \emptyset$ and $x \in U, y \in V$. Then $f^{-1}[U] \cup f^{-1}[X \setminus U]) = dom\ f \in p$, so $f^{-1}[U] \in p$ or $f^{-1}[X \setminus U] \in p$. It is not hard to see that both cases yield a contradiction. Thus there exists $y \in X$ such that $L = \{y\}$. By using (3) again it is easy to see that $p - \lim_s f(s) = y$.

(4) $\Rightarrow$ (2) Assume that $A \cup B = S$ and that neither A nor B lies in p. Without loss of generality we may further assume that $A \cap B = \emptyset$. Put $f = \chi_A$ such that f is a function from S to the compact space $\{0, 1\}$. Then both $f^{-1}[\{0\}]$ and $f^{-1}[\{1\}]$ do not lie in p, so $p - \lim_s f(s)$ does not exist.

(2) $\Rightarrow$ (1) Assume that p is not maximal. Pick a filter q and a set $A \in q$ such that $q \supseteq p$ and $A \notin p$. Then by (2), $S \setminus A \in p$, so also $S \setminus A \in q$. But then q is not closed under finite intersections.

□

When we apply ultrafilters to show that some family $\mathcal{G}$ is partition regular we usually proceed as follows: First we prove that there exists some ultrafilter p such that each element of p contains a member of $\mathcal{G}$. Then the partition regularity of $\mathcal{G}$ follows immediately: Whenever $A_1 \cup A_2 \cup \ldots \cup A_r = S$ there exists some $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$. But then A_i contains a member of $\mathcal{G}$.
It is interesting that the converse of this principle also holds.

Theorem 1.2.4 *Let $\mathcal{G}$ be a family of subsets of S. Then $\mathcal{G}$ is partition regular iff there exists an ultrafilter p on S such that for every $A \in p$ some element of $\mathcal{G}$ is contained in A.*

proof: Let

$$\mathcal{G}' = \{A \in \mathcal{G} : \text{ Whenever } \bigcup_{i=1}^r A_i = A, \text{ some } A_i \text{ contains an element of } \mathcal{G}.\}$$

Clearly $\mathbb{N} \in \mathcal{G}'$, so $\mathcal{G}'$ is non empty. If C is a chain of filters in $\mathcal{G}'$ then $\bigcup_{\mathcal{F}\in C} \mathcal{F}$ is a filter which again lies in $\mathcal{G}'$ and which is an upper bound of C. By Zorn's pick a filter p which is maximal (with respect to inclusion) among all filters which are contained in $\mathcal{G}'$. Consider $A, B \subseteq S$ satisfying $A \cup B = S$, $A \cap B = \emptyset$. We want to show that $A \in p$ or $B \in p$. First show that for each $C \in \mathcal{G}'$ $A \cap C \in \mathcal{G}'$ or $B \cap C \in \mathcal{G}'$. If not there exist $A_1, A_2, \ldots, A_r$ and $B_1, B_2, \ldots, B_s$ such that $A \cap C = \bigcup_{i=1}^{r} A_i, B \cap C = \bigcup_{i=1}^{s} B_i$ and no A_i or B_j contains an element of $\mathcal{F}$. But $A_1 \cup \ldots \cup A_r \cup B_1 \cup \ldots \cup B_r = C$, a contradiction to the definition of $\mathcal{G}'$. So for each $C \in \mathcal{G}'$ we have $A \cap C \in \mathcal{G}'$ or $B \cap C \in \mathcal{G}'$. Now an argument similar to the one used to show (1) $\Rightarrow$ (3) in the proof of Theorem 1.2.3 yields either $A \in p$ or $B \in p$. $\square$

βS can be made a topological space if we let $\overline{A} = \{p \in \beta S : A \in p\}$ for $A \subseteq S$ and take $\{\overline{A} : A \subseteq S\}$ as a basis for the closed sets. (We have already remarked earlier that we want to regard S as a subset of βS via the embedding $S \to \beta S, s \mapsto e(s)$.)

Theorem 1.2.5 *Let S be a nonempty set. βS is a compact zero dimensional space. For $A \subseteq S$ the set $\overline{A}$ is clopen. In fact all clopen subsets of βS are of this form.*
The function
$$\phi : \mathcal{P}(S) \to \mathit{clopen}(\beta S) \qquad A \mapsto \overline{A}$$
is a homomorphism of Boolean Algebras.
If S is infinite $|\beta S| = 2^{2^{|S|}}$, in particular βS is not metrisable.
If we endow S with the discrete topology, βS is the Stone-Čech Compactification of S: S is dense in βS. If $f : S \to K$ is a (trivially continuous) function from S into a compact space K then the function $\tilde{f} : \beta S \to K$, $f(p) = p - \lim_s f(s)$ is the unique continuous extension of f.

proof: It is an easy exercise using only the basic properties of ultrafilters that the map $A \mapsto \overline{A}$. respects Boolean operations: I.e. for $A_1, A_2, \ldots, A_r \subseteq S$ we have

$$\begin{aligned} \overline{A}_1 \setminus \overline{A}_2 &= \overline{A_1 \setminus A_2} \\ \overline{A}_1 \cup \ldots \cup \overline{A}_r &= \overline{A_1 \cup \ldots \cup A_r} \\ \overline{A}_1 \cap \ldots \cap \overline{A}_r &= \overline{A_1 \cap \ldots \cap A_r} \end{aligned}$$

Let $A \subseteq S$. Since βS is the disjoint union of $\overline{A}$ and $\overline{S \setminus A}$, $\overline{A}$ is indeed clopen and $\{\overline{B} : B \subseteq S\}$ forms also a basis for the open sets. Furthermore we see that βS is Hausdorff and that $\overline{A}$ is in fact the closure of $A \subseteq \beta S$.
To prove compactness, consider a family of closed sets $\{F_t : t \in T\}$ which has the finite intersection property. Without loss of generality we may assume that we are dealing with basic sets, i.e. that $F_t = \overline{A}_t$ for some $A_t \subseteq S$ holds for all $t \in T$. Then $\bigcap_{t\in G} A_t \neq \emptyset$ for every finite set $G \subseteq T$. Thus

$\{A_t : t \in T\}$ generates a filter $\mathcal{F}$ and we may apply Zorn's Lemma to find an ultrafilter $p \supseteq \mathcal{F}$. But then $A_t \in p$ for all $t \in T$ and this is tantamount to $p \in \bigcap_{t \in T} A_t$. So βS is indeed a compact space.
Next we want to show that all clopen sets are of the form $\overline{A}$ for some $A \subseteq S$. Let F be a clopen subset of βS. Since F is open

$$F = \bigcup_{\{A \subseteq S : \overline{A} \subseteq F\}} \overline{A}.$$

Since F is closed finitely many sets of this covering are enough, so there exist $A_1, A_2, \ldots, A_r \subseteq S$ such that

$$F = \overline{A}_1 \cup \ldots \cup \overline{A}_r = \overline{A_1 \cup \ldots \cup A_r},$$

so we are done. In particular $\phi : \mathcal{P}(S) \to \text{clopen}(\beta S)$ is surjective. It is easy to see that ϕ is also $1-1$ so it is in fact an isomorphism.
It is clear that S is dense in βS. If K is a compact space and $f : S \to K$ is a function we may define $\tilde{f}(p) = p - \lim_s f(s)$ for $p \in \beta S$. (By Theorem 1.2.3, (4) this Definition is justified.) By Lemma 1.2.6 below $\tilde{f}$ is a continuous extension of f to βS and since S is dense in βS this extension is unique.
See [HS98], Theorem 3.58 for a proof that $|\beta S| = 2^{2^{|S|}}$ if S is infinite. □

Lemma 1.2.6 *Let S be discrete space, let X be a regular topological space, let $\tilde{f} : \beta S \to X$ be a function and denote the restriction of $\tilde{f}$ to S by f. Then the following are equivalent:*

(1) *$\tilde{f}$ is continuous.*

(2) *For each $p \in \beta S$, $\tilde{f}(p) = p - \lim_s f(s)$.*

(3) *For each $p \in \beta S$ and each neighborhood U of $\tilde{f}(p)$ one has $\{s \in S : f(s) \in U\} \in p$.*

proof: It is clear that (2) and (3) are equivalent.
(1) $\Rightarrow$ (3) : Pick a neighborhood $\overline{A}$ of p such that $\tilde{f}[\overline{A}] \subseteq U$. Then

$$\{s \in S : f(s) \in U\} \supseteq A \in p.$$

(3) $\Rightarrow$ (1) : Pick $p \in \beta S$ and a neighborhood U of $\tilde{f}(p)$. By regularity of X there exists an open neighborhood V of $\tilde{f}(p)$ such that $\overline{V} \subseteq U$. Let

$$A = \{s \in S : f(s) \in \overline{V}\} \in p$$

and pick $q \in \overline{A}$. Assume for contradiction that $\tilde{f}(q) \notin \overline{V}$. By assumption this yields $\{s \in S : f(s) \in X \setminus \overline{V}\} \in q$. But that contradicts $A \in q$ since q is closed under finite intersections. Thus $\tilde{f}[A] \subseteq \overline{V} \subseteq U$. Since p and U where arbitrary, we are done. □

Definition 1.2.7 *Let S be a semigroup. We define an operation $\cdot \cdot \cdot : \beta S \times \beta S \to \beta S$ by*

$$p \cdot q = p - \lim_s (q - \lim_t s \cdot t).$$

We will abuse notation and write pq instead of $p \cdot q$ respectively S instead of $(S, \cdot)$ if there is no danger of confusion.

Definition 1.2.8 *Let S be a semigroup and let $s \in S$. The* left translation *λ_s and the* right translation *ρ_s are the maps*

$$\begin{array}{rcl} \lambda_s : \beta S & \to & \beta S \\ t & \mapsto & st \end{array} \qquad \begin{array}{rcl} \rho_s : \beta S & \to & \beta S \\ t & \mapsto & ts. \end{array}$$

Theorem 1.2.9 *Let S be semigroup. Then $(\beta S, \cdot)$ is a semigroup such that the maps $\lambda_s : \beta S \to \beta S$, $\rho_q : S\beta \to \beta S$ are continuous for $s \in S$ and $q \in \beta S$.*

proof: By Theorem 1.2.5 $\lambda_s : \beta S \to \beta S$ is continuous since it is the continuous extension of $\lambda_s : S \to \beta S$. Similarly $\rho_q : \beta S \to \beta S$ is continuous since it is the continuous extension of $\rho_q : S \to \beta S$.
It remains to show that $\cdot \cdot \cdot : \beta S \times \beta S \to \beta S$ is associative. Let $p, q, r \in \beta S$.

$$\begin{array}{rcl} p \cdot (q \cdot r) & = & \text{(since } \rho_{q \cdot r} \text{ is continuous)} \\ \lim\limits_{s \to p} s \cdot (q \cdot r) & = & \text{(since } \lambda_s \circ \rho_r \text{ is continuous)} \\ \lim\limits_{s \to p} \lim\limits_{t \to q} s \cdot (t \cdot r) & = & \text{(since } \lambda_s \circ \lambda_t \text{ is continuous)} \\ \lim\limits_{s \to p} \lim\limits_{t \to q} \lim\limits_{u \to r} s \cdot (t \cdot u) & = & \text{(since } \cdot \cdot \cdot \text{ is associative)} \\ \lim\limits_{s \to p} \lim\limits_{t \to q} \lim\limits_{u \to r} (s \cdot t) \cdot u & = & \text{(since } \lambda_{s \cdot t} \text{ is continuous)} \\ \lim\limits_{s \to p} \lim\limits_{t \to q} (s \cdot t) \cdot r & = & \text{(since } \rho_r \circ \lambda_s \text{ is continuous)} \\ \lim\limits_{s \to p} (s \cdot q) \cdot r & = & (p \cdot q) \cdot r \text{ (since } \rho_r \circ \rho_q \text{ is continuous).} \end{array}$$

□

Remark 1.2.10 *For $p \in \beta S \setminus S$ the map $\lambda_p : \beta S \to \beta S$, $r \mapsto pr$ is not continuous in general.*

$$\Lambda(\beta S) = \{p \in \beta S : \lambda_p : \beta S \to \beta S \textit{ is continuous}\}$$

is the topological center *of βS. By Theorem 1.2.9, $S \subseteq \Lambda(\beta S)$.*
If S is a commutative semigroup, then for all $s \in S$ and $p \in \beta S$

$$sp = p - \lim_t st = p - \lim_t ts = ps$$

and this shows that S is also contained in the algebraic center of βS

In fact it is not hard to see that the topological center of βS coincides with the algebraic center of βS if S is commutativ ([HS98], Theorem 4.24). In the cases $(S,\cdot) = (\mathbb{N},+)$ and $(S,\cdot) = (\mathbb{N},\cdot)$ we have $\Lambda(\beta S) = S$ ([HS98], Theorem 6.54).

The following notation is an abbreviation that appeals to the readers intuition.

Definition 1.2.11 *Let S be a semigroup, let $s \in S$ and $A \subseteq S$. Then put*

$$\begin{aligned} s^{-1}A &= \lambda_s^{-1}[A] = \{t \in S : st \in A\}, \\ As^{-1} &= \rho_s^{-1}[A] = \{t \in S : ts \in A\}. \end{aligned}$$

If S is a group this coincides with the usual definition of $s^{-1}A$ and As^{-1}. A concrete Characterisation of the multiplication in βS is given in the following Theorem:

Theorem 1.2.12 *Let S be a semigroup, let $s \in S$, let $p, q \in \beta S$ and let $A \subseteq S$. Then*

$$\begin{aligned} A \in sq &\iff s^{-1}A \in q, \\ A \in pq &\iff \{s \in S : s^{-1}A \in q\} \in p. \end{aligned}$$

proof:

$$\begin{array}{llr} & A \in sq \Leftrightarrow sq \in \overline{A} & \\ \Leftrightarrow & \exists B \in q, s\overline{B} \subseteq \overline{A} & (\text{since } \lambda_s \text{ is continuous}) \\ \Leftrightarrow & \exists B \in q, sB \subseteq A & (\text{since } \lambda_s[\overline{B}] = \overline{\lambda_s[B]}) \\ \Leftrightarrow & s^{-1}A \in q & (\text{by the filter properties of } q), \end{array}$$

and

$$\begin{array}{llr} & A \in pq \Leftrightarrow pq \in \overline{A} & \\ \Leftrightarrow & \exists B \in p, \overline{B}q \subseteq \overline{A} & (\text{since } \rho_q \text{ is continuous}) \\ \Leftrightarrow & \exists B \in p, Bq \subseteq \overline{A} & (\text{since } \rho_q[\overline{B}] = \overline{\rho_q[B]}) \\ \Leftrightarrow & \{s : sq \in \overline{A}\} \in p & (\text{by the filter properties of } p) \\ \Leftrightarrow & \{s : s^{-1}A \in q\} \in p. & \end{array}$$

□

It is convenient to know that homomorphism behave like we would expect them to do:

Theorem 1.2.13 *Let S,T be semigroups and let $f : S \to T$ be a homomorphism. Then the continuous extension $\widetilde{f} : \beta S \to \beta T$ is also a homomorphism.*

proof: Let $p, q \in \beta S$. Then

$$\begin{array}{rcccl} \widetilde{f}(pq) & = & \widetilde{f}(p - \lim_s q - \lim_t st) & = & \\ & = & p - \lim_s (q - \lim_t f(st)) & = & \\ & = & p - \lim_s (q - \lim_t f(s) f(t)) & = & \widetilde{f}(p)\widetilde{f}(q). \end{array}$$

□

1.3 The algebraic structure of compact semigroups

We may not expect that a general given semigroup S has interesting algebraic properties, but its Stone-Čech Compactification in fact does. The reason lies in the additional topological structure.

Definition 1.3.1 *S is a* compact right topological semigroup *if S is a compact space such that the map*

$$\begin{array}{rcl} \rho_q : S & \to & S \\ r & \mapsto & rq \end{array}$$

is continuous for all $q \in S$.

Remark 1.3.2 *The source of the name "right topological semigroup" is of course that the multiplication from the right is continuous. On the other hand the multiplication is continuous in the* left *argument which would indicate to talk about "left topological" semigroups. Furthermore there is no good reason why we shouldn't take λ_q instead of ρ_q to be continuous. So since there are four ways - all of them common in the literature - to handle the subject, there is plenty of space for confusion.*
Our choice (as well as most of our notation) is stimulated by [HS98] which is probably the main source for applications of ultrafilter lore in Ramsey Theory.

If S is a semigroup, βS is a compact right topological semigroup by Theorem 1.2.5 and Theorem 1.2.9.
Ideals respectively left or right ideals are trivial in groups but turn out to be very interesting concepts in compact right topological groups.

Definition 1.3.3 *Let S be a semigroup. We call $I \subseteq S$*

(1) *a* left ideal *if $SI \subseteq I$,*

(2) *a* right ideal *if $IS \subseteq I$,*

(3) *an* ideal *if I is a left ideal as well as a right ideal.*

I is a minimal *left ideal if it doesn't properly contain another left ideal. Minimal right ideals are defined similarly.*

Theorem 1.3.4 *Let S be a compact right topological semigroup. Then S has a minimal left ideal L. All minimal left ideals are closed.*

proof: Let $\mathcal{L}$ be the set of all *closed* left ideal ideals in S. If K is a chain in $\mathcal{L}$ then by compactness of S, $\bigcap_{L\in K} L$ is a lower bound of K in $\mathcal{L}$. By Zorn's Lemma there exist a left ideal L which is minimal in $\mathcal{L}$.
Assume that I is an arbitrary left ideal in S. Pick $s \in I$. Then Ss is a left ideal which is contained in I. Further Ss is closed by continuity of $\rho_s : S \to S$. This shows that every left ideal contains a closed left ideal. Hence L is minimal among all left ideals of S. □

Definition 1.3.5 *Let S be a semigroup. $e \in S$ is called an* idempotent *if $ee = e$. The set of all idempotents of S is denoted by $E(S)$.*

Theorem 1.3.6 *Let S be a compact right topological semigroup. Then there exists an idempotent $e \in S$.*

proof: Let $P = \{M \subseteq S : M$ is closed, non empty and $MM \subseteq M\}$. P is non empty since $S \in P$. Furthermore P is partially ordered by inclusion and every chain K in P has the lower bound $\bigcap_{M\in K} M$. Thus by Zorn's Lemma there exists a minimal element $M_0 \in P$. Let $e \in M_0$ be arbitrary. By continuity of ρ_e, M_0e is closed. Moreover $(M_0e)(M_0e) \subseteq M_0M_0M_0e \subseteq M_0e$ and $M_0e \subseteq M_0$. Thus we have $M_0e = M_0$ by minimality of M_0. In particular $M_1 = \{x \in M_0 : xe = e\}$ is non empty. M_1 is closed and for $x, y \in M_0$, $(xy)e = x(ye) = xe = e$. so M_1 is a closed subgroup of M_0 and again by minimality of M_0 equality holds. This yields $ee = e$. □

Since any minimal left ideal in a compact semigroup is closed it contains an idempotent. In particular any compact semigroup S has a minimal left ideal that contains an idempotent. Later (in particular in Theorem 1.3.17) we will see that this condition guarantees that S has a rich algebraic structure.

Lemma 1.3.7 *Let S be a semigroup and let $e \in E(S)$. Then e is a* right neutral element *in Se, i.e. $xe = e$ for all $x \in Se$, a* left neutral element *in eS and a neutral element in eSe.*

proof: Let $y \in S$. Pick $x \in S$ such that $xe = y$. Then $ye = (xe)e = x(ee) = xe = y$. The rest follows by a left - right switch. □

Theorem 1.3.8 *Let S be a semigroup that has a minimal left ideal L. Pick $s \in S$. Then Ls is also a minimal left ideal. Every minimal left ideal is of the form Ls for some $s \in S$.*
Furthermore every left ideal in S contains a minimal left ideal.

proof: Let $s \in S$ be arbitrary. Then Ls is a left ideal. Let $I \subseteq Ls$ be a left ideal of S and put $J = \{x \in L : xs \in I\}$. For $t \in S$ and $x \in J$ we have $tx \in L$ since L is an ideal and $txs \in I$ since I is a left ideal. Thus $tx \in J$. t and x were arbitrary, so J is a left ideal and by minimality of L we have $J = L$. In particular $I \supseteq Js = Ls$, so $I = Ls$. This shows that Ls is in fact a minimal left ideal.
Next let I be a minimal left ideal of S. Pick $s \in I$. Then Ls is a left ideal that is contained in I so we must have $I = Ls$.
If I is not necessarily minimal then Ls is at least a minimal left ideal that is contained in I. $\square$

Theorem 1.3.9 *Let S be a semigroup and let $e \in E(S)$. Then the following statements are equivalent.*

(1) *Se is a minimal left ideal.*

(2) *eSe is a group.*

(3) *Se is a minimal right ideal.*

proof: (1) $\Rightarrow$ (2): Trivially eSe is closed and by Lemma 1.3.7 e is a neutral element in eSe. Let $x = ese \in S$ be given. Then $Sx \subseteq Se$ is a left ideal of S, so $Sx = Se$. In particular there exists $y \in S$ such that $yx = e$. $eye \in eSe$ and $(eye)x = eyeese = ey(ese) = eyx = ee = e$, thus x has an inverse in eSe.
(2) $\Rightarrow$ (1): Let $L \subseteq Se$ be a left ideal of S and pick $t \in L$. Then $et \in eSe$, so pick $x \in eSe$ such that $x(et) = e$. But then $e = (xe)t \in SL \subseteq L$, so $Se \subseteq L$. Since L was arbitrary Se is a minimal left ideal of S.
Now (2) $\Leftrightarrow$ (3) follows by a left - right switch. $\square$

Corollary 1.3.10 *Let S be a semigroup that has a minimal left ideal that has an idempotent. Then S has a minimal right ideal that has an idempotent.*

proof: Let L be a minimal left ideal that contains an idempotent e. Then $Se = L$ is a minimal left ideal, so by Theorem 1.3.9 eS is a minimal right ideal that contains an idempotent. $\square$

Corollary 1.3.11 *Let S be a semigroup that has a minimal left ideal that contains an idempotent. Then every left ideal has an idempotent.*

proof: Let L be a minimal left ideal that contains an idempotent e (i.e. $L = Se$), let I be an arbitrary left ideal of S and by Lemma 1.3.8 let $x \in S$ such that $I \supseteq Lx$. By Theorem 1.3.9 eSe is a group so let $y = eye$ be the inverse of exe in this group. Then

$$yxyx = (eye)x(eye)x = e(y(exe))yex = eeyex = yx,$$

so yx is an idempotent in I. $\square$

Definition 1.3.12 *Let S be a semigroup. The* kernel *of S is defined by*

$$K(S) = \bigcap\{I : I \textit{ is an ideal of } S\}.$$

Many common semigroups such as $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$ have a trivial kernel. An easy condition assures that the kernel of a semigroup is non trivial, in fact that it is the *smallest ideal* of the semigroup.

Theorem 1.3.13 *Let S be a semigroup that has a minimal left ideal. Then*

$$K(S) = \bigcup\{L : L \textit{ is a minimal left ideal of } S\}.$$

Furthermore this union is disjoint and $K(S)$ is an ideal of S.

proof: Put $K' = \bigcup\{L : L$ is a minimal left ideal of $S\}$. Since the intersection of two left ideals is either empty or a left ideal, the defining union is disjoint.
Let $L \subseteq S$ be a minimal left ideal and let $I \subseteq S$ be an ideal. Since I is a left ideal IL is also left ideal. IL is contained in L, so $IL = L$. Since I is a right ideal $L = IL \subseteq IS \subseteq I$. I and L were arbitrary, so $K(S) \supseteq K'$.
To conclude the proof, we show that K' is an ideal. It is obvious that K' is a left ideal. Thus let $x \in K'$ and $y \in S$. Let L be a minimal left ideal such that $x \in L$. By Lemma 1.3.8 Ly is a minimal left ideal, so $xy \in Ly \subseteq K'$.
$\square$

Theorem 1.3.14 *Let S be a semigroup, let $L \subseteq S$ be a minimal left ideal and let $R \subseteq S$ be a minimal right ideal. Then $RL = R \cap L$ is a group and its neutral element e is an idempotent. Further we have the representations $L = Se, R = eS$ and $RL = eSe$.*

proof: Since L is a left ideal $RL \subseteq L$. Similarly $RL \subseteq R$ and so $RL \subseteq R \cap L$. We have

$$(RL)(RL) = R(LRL) \subseteq RL,$$

so RL is a subsemigroup. For each $s \in L$ we have $Ls = L$ by minimality of L. Analogously $sR = R$ for $s \in R$. This shows

$$sRL = RLs = RL$$

for $s \in RL$. Thus RL has no non trivial left - or right ideal. At this point we need the following well known Lemma:

Lemma 1.3.15 *Assume that S is a semigroup that contains no non trivial left- or right ideal. Then S is a group.*

proof: Let $s, t \in S$. Since $St = S = tS$, there exist $e, r \in S$ such that $et = t$ and $s = tr$ where e only depends on t. We have

$$es = e(tr) = (et)r = tr = s.$$

It follows that e is a left neutral element. Similarly one shows the existence of a right neutral element e' and of course $e = ee' = e'$, so e is a neutral element. As above one sees that for arbitrary $s \in S$ the equations $sx = e, xs = e$ have a solution, so S is indeed a group. $\square$

By the lemma RL is a group. Let e be its neutral element. Again by minimality of L respectively R we have $Se = L$, $eS = R$. Thus

$$RL = eSSe \subseteq eSe \subseteq Re \subseteq RL.$$

This shows $RL = eSe$. Finally we have

$$R \cap L = eS \cap Se \subseteq eSe = RL,$$

so $R \cap L = RL$. $\square$

Corollary 1.3.16 *A semigroup S has a minimal left ideal that contains an idempotent iff it has a minimal left ideal and a minimal right ideal.*

proof: This follows from Corollary 1.3.10 and Theorem 1.3.14. $\square$

Theorem 1.3.17 (Structure Theorem) *Let S be a semigroup that has a minimal left ideal that contains an idempotent.*

(1) *The minimal left ideals are exactly the sets of the form Se for $e \in E(K(S))$. All minimal left ideals are isomorphic.*

(2) *The minimal right ideals are exactly the sets of the form eS for $e \in E(K(S))$. All minimal right ideals are isomorphic.*

(3) *Sets of the form eSe for $e \in E(K(S))$ arise exactly as the intersections $L \cap R = RL$. These sets are the maximal groups in $K(S)$ and all maximal subgroups in $K(S)$ are isomorphic.*

(4) *The kernel of S is the disjoint union of all minimal left ideals of S respectively of all minimal right ideals of S respectively of all maximal groups in $K(S)$:*

$$\begin{aligned} K(S) &= \bigcup \{L : L \text{ is a minimal left ideal of } S\} \\ &= \bigcup \{R : R \text{ is a minimal right ideal of } S\} \\ &= \bigcup \{eSe : e \in E(K(S))\}. \end{aligned}$$

proof: Let $G \subseteq K(S)$ be a group. Then its neutral element e is an idempotent, so $G = eGe \subseteq eSe$. This shows that the maximal groups in $K(S)$ are of the form eSe. Now everything but the isomorphism statements follows directly from other propositions in this section. For a complete proof of the Structure Theorem see for example [HS98], Theorem 1.64. $\square$

Under reasonable assumptions it is easy to determine the smallest ideal of a subsemigroup of a given semigroup:

Theorem 1.3.18 *Let S be a semigroup, let T be a subsemigroup of S, assume that both of them have a minimal left ideal that contains an idempotent and that $T \cap K(S) \neq \emptyset$. Then $T \cap K(S) = K(T)$.*

proof: $T \cap K(S)$ is an ideal of T, so we have to show that it is the smallest one. Let $x \in T \cap K(S)$ be given. Tx is a minimal left ideal of T, so pick an idempotent $e \in Tx$ such that Te is a minimal left ideal of T. Since $x \in K(S)$, Sx is a minimal left ideal of S and $e \in Tx \subseteq Sx$. Thus $Sx = Se$, so $x \in Se$. e is a right neutral element of Se, so $x = xe \in Te$. Since Te is a minimal right ideal of T we are done. $\square$

Theorem 1.3.19 *Let S be a semigroup that has a minimal left ideal that contains an idempotent and let $s \in S$. Then the following statements are equivalent:*

(1) $s \in K(S)$.

(2) *For all* $t \in S$, $s \in Sts$

(3) *For all* $t \in S$, $s \in Sts \cap stS$

proof: (1) $\Rightarrow$ (3): $s \in L$ for some minimal left ideal L. Sts is a left ideal that is contained in L, so $Sts = L$ and in particular $s \in Sts$. Now $s \in stS$ follows by a left right switch.
(3) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1): Pick $t \in K(S)$. Then $s \in Sts \subseteq K(S)$. $\square$

We may define different orders on $E(S)$. Despite they will not coincide in general, they do have the same minimal elements.

Definition 1.3.20 *Let S be a semigroup and let $e, f \in E(S)$. Then*

(1) $e \leq_L f$ *iff* $e = ef$,

(2) $e \leq_R f$ *iff* $e = fe$,

(3) $e \leq f$ *iff* $e = ef = fe$.

The relations $\leq_L, \leq_R, \leq$ are transitive and reflexive. $\leq$ is also antisymmetric.

Theorem 1.3.21 *Let S be a semigroup and let $e \in E(S)$ The following are equivalent:*

(1) *e is minimal with respect to $\leq_L$.*

(2) *e is minimal with respect to $\leq_R$.*

(3) *e is minimal with respect to $\leq$.*

It is important to make precise what we understand by minimality with respect to $\leq_L$ and $\leq_R$: $e \in E(S)$ is minimal with respect to $\leq_L$ iff for $f \in E(S)$, $f \leq_L e$ implies that $e \leq_L f$. (Similarly for $\leq_R$.)
proof: (1) $\Rightarrow$ (3) : Assume that e is minimal with respect to $\leq_L$ and let $f \leq e$ $(ef = fe = f)$. Then $f \leq_L e$ which implies $e \leq_L f$, i.e. $e = ef$, so $e = f$.
(3) $\Rightarrow$ (1) : Assume that e is minimal with respect to $\leq$. Pick $f \in E(S)$ such that $f \leq_L e$ $(f = fe.)$ Let $g = ef$. Then

$$gg = efef = eff = ef = g,$$

so g is an idempotent. Also $g = ef = efe$ which implies

$$ge = efee = efe = g = efe = eefe = eg,$$

such that $g \leq e$. By minimality of e this gives $g = e$, i.e. $ef = e$. Thus $e \leq_L f$ as required.
Now (2) $\Leftrightarrow$ (3) follows by a left-right switch. □

Definition 1.3.22 *Let S be a semigroup. $e \in E(S)$ is called* minimal *iff it is minimal with respect to any of the orders $\leq_L, \leq_R, \leq$.*

Theorem 1.3.23 *Let S be a semigroup that has a minimal left ideal that contains an idempotent and let $e \in E(S)$. Then the following statements are equivalent:*

(1) *e is minimal.*

(2) *$e \in K(S)$.*

proof: (1) $\Rightarrow$ (2): We want to show that Se is a minimal left ideal. Let $L \subseteq Se$ be left ideal of S. Pick by Corollary 1.3.11 an idempotent $f \in L$. Let $x \in L$ such that $f = xe$. Then $fe = xee = xe = f$, so $f \leq_l e$. By minimality of e, $e \leq_L f$. Thus we have $e = ef \in L$ and this implies $L = Se$.
(2) $\Rightarrow$ (1): Se is a minimal left ideal. Let $f \in E(S)$ such that $f \leq_L e$. Since $f = fe \in Se$, $Se = Sf$. Thus f is a right neutral element in Sf, in particular $ef = e$. f was arbitrary, so e is minimal. □

Theorem 1.3.24 *Let S be a semigroup that has a minimal ideal that contains an idempotent and let e be an idempotent in S. There exists a minimal idempotent $f \leq e$.*

proof: Let $L \subseteq Se$ be a minimal left ideal and let $R \subseteq eS$ be a minimal right ideal. Pick $f \in E(L \cap R)$. f is minimal and since e is a right neutral element in Se and a left neutral element in eS we have $f = fe = ef$. $\square$

The proof of the following Theorem is easy, so we skip it.

Theorem 1.3.25 *Let $(S_i)_{i \in I}$ be a family of semigroups. Then $\times_{i \in I} K(S_i) = K(\times_{i \in I} S_i)$.*

Next we want to show how ideals of a semigroup S behave in the Stone-Čech Compactification of S. We will need the following Lemma.

Lemma 1.3.26 *Let S be a compact right topological semigroup, let $A, B \subseteq S$ and assume that for all $s \in A$ the function λ_s is continuous. Then $\overline{A}\,\overline{B} \subseteq \overline{AB}$.*

proof: We use the well known fact that $f[\overline{X}] \subseteq \overline{f[X]}$ for a continuous map $f : T \to T$ and $X \subseteq T$ arbitrary. For arbitrary $C \subseteq T$ we have

$$\overline{A}C = \bigcup_{c \in C} \rho_c[\overline{A}] \subseteq \bigcup_{c \in C} \overline{\rho_c[A]} = \bigcup_{c \in C} \overline{Ac} \subseteq \overline{AC}, \tag{1.1}$$

$$A\overline{B} = \bigcup_{a \in A} \lambda_a[\overline{B}] \subseteq \bigcup_{a \in A} \overline{\lambda_a[B]} = \bigcup_{a \in A} \overline{aB} \subseteq \overline{AB}. \tag{1.2}$$

By taking these inclusions together and specifying $C = \overline{B}$ we get $\overline{A}\,\overline{B} \subseteq \overline{A\overline{B}} \subseteq \overline{\overline{AB}} = \overline{AB}$. $\square$

Corollary 1.3.27 *Let S be a semigroup and assume that $I \subseteq S$ is an ideal of S. Then $\overline{I}$ is an ideal of βS.*

proof: For all $s \in S$, $\lambda_s : \beta S \to \beta S$ is continuous. Thus $\overline{I}\beta S \subseteq \overline{IS} \subseteq \overline{I}$ and $\beta S \overline{I} \subseteq \overline{SI} \subseteq \overline{I}$.
$\square$

We conclude this section with some remarks concerning the connection between compact right topological semigroups and symbolic dynamics:

A *dynamical system* is a tuple $(X, (T_s)_{s \in S})$, where X is a compact space and S is a semigroup that is acting on X via the continuous functions $T_s : X \to X$, $s \in S$. If L is a compact subspace of X we call L a *subsystem* iff $T_s[L] \subseteq L$ for each $s \in S$. Every point $x \in X$ generates a subsystem, namely its *orbit closure* $\overline{\{T_s(x) : s \in S\}}$. X is a *minimal system* iff it has no

proper subsystem. Equivalently the orbit closure of every point is the whole space.
Let T be a compact right topological semigroup. Recall that

$$\Lambda(T) = \{s \in T : \lambda_s : T \to T \text{ is continuous}\}$$

is the topological center of T. Clearly $\Lambda(T)$ is a subsemigroup. If S is a subsemigroup that is contained in the topological center, $(T, (\lambda_s)_{s\in S})$ is a dynamical system. This system has interesting properties if S is dense in T. That situation arises in particular if T is the Stone-Čech Compactification βS of S.

Theorem 1.3.28 *Let T be a compact semigroup, let S be a subsemigroup of $\Lambda(T)$ and assume that S is dense in T. The closed left ideals of T are precisely the subsystems of $(T, (\lambda_s)_{s\in S})$. Moreover L is a minimal subsystem of $(T, (\lambda_s)_{s\in S})$ iff it is a minimal left ideal of T.*

proof: By continuity of $\rho_t : T \to T$ for $t \in T$, $SL \subseteq L$ is equivalent to $\overline{SL} = TL \subseteq L$ for every closed set L. This shows that subsystems correspond to closed left ideals. The rest is clear. □

1.4 Connecting algebraic and combinatorial properties

We shall be concerned with several notions of largeness that originated in the study of topological dynamics and make sense in any semigroup. Four of these, namely *thick*, *syndetic*, *piecewise syndetic* and *IP set* have simple elementary descriptions and we introduce them now. The fifth, *central* is most simply described in terms of the algebraic structure of βS.

Definition 1.4.1 *Let S be a semigroup and let $A \subseteq S$.*

(a) A is thick if and only if whenever $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.

(b) A is syndetic if and only if there exists $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in G} t^{-1}A$.

(c) A is piecewise syndetic if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq \bigcup_{t\in G} t^{-1}A$.

(d) Let $(x_n)_{n=1}^{\infty}$ be a sequence in S. Then

$$FP((x_n)_{n=1}^{\infty}) = \{x_{n_1}x_{n_2}\ldots x_{n_k} : n_1 < n_2 < \ldots < n_k\}.$$

(If we write the semigroup operation additively, we will use the symbol $FS((x_n)_{n=1}^{\infty})$ instead.)

A is an IP set if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in S such that $FP((x_n)_{n=1}^{\infty}) \subseteq A$.

In $(\mathbb{N}, +)$ the notions described in (a), (b) and (c) have a very intuitive meaning: $A \subseteq \mathbb{N}$ is thick iff A contains arbitrarily long intervals, A is syndetic iff it has *bounded gaps* i.e. iff there exists some $d \in \mathbb{N}$ such that $\mathbb{N} \setminus A$ contains no interval of length d. Furthermore A is piecewise syndetic iff there exist a thick set $B \subseteq \mathbb{N}$ and a syndetic set $C \subseteq \mathbb{N}$ such that $A = B \cap C$. (Beware: This does not hold in general semigroups.)
Notice that each of thick and syndetic imply piecewise syndetic and that thick sets are IP sets. It is easy to construct examples in $(\mathbb{N}, +)$ showing that no other implications among these notions is valid in general. If a thick or syndetic set is partitioned into finitely many cells, there will not necessarily be a cell that is still thick respectively syndetic. Piecewise syndetic set and IP sets do have this property. In the first case this has an easy elementary proof, while in the second this was a long open question, solved by N. Hindman in 1975 ([H75]). Later we will see that these results follow easily from the characterisation of these terms via the Stone-Čech Compactification.
Let $(S, \cdot)$ be a semigroup. A family $\mathcal{F} \subseteq \mathcal{P}(S)$ is called *left invariant* iff for all $F \in \mathcal{F}$ and $s \in S$ one has $sF \in \mathcal{F}$. It is called *right invariant* iff for all $F \in \mathcal{F}$ and $s \in S$ one has $Fs \in \mathcal{F}$ and it is called *invariant* iff it is left and right invariant.
The following Lemma gives a hint why piecewise syndetic sets will be interesting for our purposes.

Lemma 1.4.2 *Set S be a semigroup and let $\mathcal{F} \subseteq \mathcal{P}(S)$ be an invariant family of finite sets. Then $\mathcal{F}$ is partition regular iff for any piecewise syndetic set $A \subseteq S$ there exists $F \in \mathcal{F}$ such that $F \subseteq A$.*

proof: In any finite partition of S one cell is piecewise syndetic. This shows that the condition is sufficient.
To prove that it is also necessary, let $\mathcal{F}$ be an invariant family of finite sets. By Theorem 1.1.4 for each $r \in \mathbb{N}$ there exists some finite set $G \subseteq S$ such that for any finite partition of G into r cells one cell must contain an element of $\mathcal{F}$.
Assume that $A \subseteq S$ is piecewise syndetic and let $r \in \mathbb{N}$ and $s_1, \ldots, s_r \in S$ such that $\bigcup_{i=1}^{r} s_i^{-1}A$ is thick. Further let G be a finite subset of S such that for any partition of G into r cells one cell contains an element of $\mathcal{F}$ and let $t \in S$ be such that $Gt \subseteq \bigcup_{i=1}^{r} s_i^{-1}A$. Then $G \subseteq \bigcup_{i=1}^{r} s_i^{-1}At^{-1}$ and thus there exist $i \in \{1, \ldots, r\}$ and $F \in \mathcal{F}$ such that $F \subseteq s_i^{-1}At^{-1}$. Equivalently we have $s_iFt \subseteq A$ and since $s_iFt \in \mathcal{F}$ this suffices to conclude the proof. $\square$

Notice that if S is not commutative, then both multipliers in Lemma 1.4.2 may be required. For example, let S be the free semigroup on the letters a

and b. Then $\mathcal{F} = \{bF : F \in \mathcal{P}_f(S)\}$ and $\mathcal{G} = \{Fb : F \in \mathcal{P}_f(S)\}$ are partition regular, aS and Sa are piecewise syndetic, there do not exist $F \in \mathcal{F}$ and $x \in S$ with $Fx \subseteq aS$, and there do not exist $F \in \mathcal{G}$ and $t \in S$ with $tF \subseteq Sa$. (In fact, aS is syndetic in S.)

Theorem 1.4.3 *Let S be a semigroup and let $A \subseteq S$.*

(1) *A is thick iff there exists a minimal left ideal L of βS such that $\overline{A} \supseteq L$.*

(2) *A is piecewise syndetic iff $\overline{A} \cap K(\beta S) \neq \emptyset$.*

proof:

(1) Assume that A is thick. For all $F \in \mathcal{P}_f(S)$ there exists some $s \in S$ such that $Fs \subseteq A$. Thus the set $T_F = \{p \in \beta S : Fp \subseteq \overline{A}\}$ is non empty. The representation $T_F = \bigcap_{t \in F} \lambda_t^{-1}[\overline{A}]$ shows that T_F is closed. For $F, G \in \mathcal{P}_F(S), T_F \cap T_G = T_{F \cup G}$. This shows that the family $\{T_F : F \in \mathcal{P}_f(S)\}$ has the finite intersection property. By compactness we may pick $p \in \bigcap_{s \in S} T_{\{s\}}$. Then for all $s \in S$, $sp \in \overline{A}$, i.e. $Sp \subseteq \overline{A}$. By continuity of ρ_p we also have $\beta S p \subseteq \overline{A}$. Since βS contains a minimal left ideal we are done.

Let $L \subseteq \overline{A}$ be a minimal left ideal and let $F \in P_f(S)$. Pick $p \in L$. Then $Fp \subseteq L \subseteq \overline{A}$. Thus $\bigcap_{s \in F} s^{-1}A \in p$. In particular this intersection is non empty. For every $t \in \bigcap_{s \in F} s^{-1}A$ we have $Ft \subseteq A$.

(2) By (1), A is piecewise syndetic iff there exists a minimal left ideal L and a set $G \in \mathcal{P}_f(S)$ such that $L \subseteq \bigcup_{s \in G} s^{-1}\overline{A}$.

Now the statement follows from Theorem 1.4.4 by considering the dynamical system $(\beta S, (\lambda_s)_{s \in S})$ and the open set $\overline{A}$.

□

Theorem 1.4.4 *Let $(X, (T_s)_{s \in S})$ be a dynamical system, let L be a minimal subsystem and let $O \subseteq X$ be open. If (and only if) $O \cap L$ is nonempty, there exists a set $F \in \mathcal{P}_f(S)$ such that $L \subseteq \bigcup_{s \in F} T_s^{-1}[O]$.*

proof: Since L is minimal, $\overline{\{T_s(x) : s \in S\}} = L$ for all $x \in L$. In particular there exists some $s \in S$ such that $T_s(x) \in O$. This shows $L \subseteq \bigcup_{s \in S} T_s^{-1}[O]$. Since L is compact, there exists a finite subcovering. □

Theorem 1.4.5 *Let S be semigroup and let $A \subseteq S$. Assume that $L \subseteq \beta S$ is a minimal left ideal that intersects $\overline{A}$ and let $p \in L$. Then the set $\{s \in S : s^{-1}A \in p\}$ is syndetic.*
Let $q \in \beta S$ such that $\{s \in S : s^{-1}A \in q\}$ is syndetic for all $A \in q$. Then $q \in K(\beta S)$.

proof: Pick $G \in \mathcal{P}_f(S)$ such that $L \subseteq \bigcup_{t \in G} t^{-1}\overline{A}$. Let $x \in S$ be arbitrary. Since $xp \in L$ there exists some $t \in G$ such that $xp \in t^{-1}\overline{A}$. Thus $xt \in \{s \in S : sp \in \overline{A}\}$. Since x was arbitrary, we have

$$S = \bigcup_{t \in G} t^{-1}\{s \in S : s^{-1}A \in p\}.$$

Assume now that p is an ultrafilter such that for all $A \in p$ the set $\{s \in S : s^{-1}A \in p\}$ is syndetic. Pick $q \in K(\beta S)$ and assume for contradiction that $p \notin \beta S q p$. Pick $A \in p$ such that $\overline{A} \cap \beta S q p = \emptyset$. Pick $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}\{s \in S : s^{-1}A \in p\}$. Then there exists $t \in G$ such that $q \in t^{-1}\{s \in S : s^{-1}A \in p\}$. Thus

$$\{s \in S : s^{-1}A \in p\} \in tq$$

which is equivalent to $A \in tqp$.
It follows that $p \in \beta S q p \subseteq K(\beta S)$, so we are done. $\square$

Theorem 1.4.6 *Let S be a semigroup and let $A \subseteq S$. A is an IP-set iff there exists an idempotent $e \in \beta S$ such that $A \in e$.*

proof: Assume first that there exists an idempotent $e \in \beta S$ such that $A \in e$. Since $ee = e$ we have $\{s \in S : s^{-1}A \in e\} \in e$. Pick

$$x_1 \in A \cap \{s \in S : s^{-1}A \in e\} \in e.$$

Then $A_1 = A \cap x_1^{-1}A \in e$. Next pick

$$x_2 \in A_1 \cap \{s \in S : s^{-1}A_1 \in e\} \in e.$$

Then $x_2 \in A$ and since $x_2 \in A_1 \subseteq x_1^{-1}A_1$ we have $x_1x_2 \in A$.
Let $A_2 = A_1 \cap x_2^{-1}A_1 \in e$. Pick

$$x_3 \in A_2 \cap \{s \in S : s^{-1}A_2 \in\} \in e.$$

Then since $x_3 \in A_2$, $x_3 \in A_1$ and $x_2x_3 \in A_1$. It follows that

$$x_3, x_1x_3, x_2x_3, x_1x_2x_3 \in A.$$

Next put $A_3 = A_2 \cap x_3^{-1}A_2 \in e. \ \ldots$

By continuing in this fashion we arrive at a sequence $(x_n)_{n=1}^{\infty}$ such that $FP((x_n)_{n=1}^{\infty}) \subseteq A$.

On the other hand assume that $(x_n)_{n=1}^{\infty}$ is a sequence in S that satisfies $FP((x_n)_{n=1}^{\infty}) \subseteq A$. For $m \in \mathbb{N}$ put $A_m = FS((x_n)_{n=m}^{\infty})$ and let $B = \bigcap_{m=1}^{\infty} \overline{A_m}$. By compactness of βS, B is non empty. We claim that B is a subsemigroup of βS. Let $p, q \in B$ and pick $C \in pq$ and $m_0 \in \mathbb{N}$. Then

$\{s : s^{-1}C \in q\} \in p$ and $A_{m_0} \in p$, so pick $n_1, n_2, \ldots, n_k$ such that $m_0 \leq n_1 < n_2 < \ldots < n_k$ and

$$\left(\prod_{t=1}^{k} x_{n_t}\right)^{-1} C \in q.$$

$A_{n_k+1} \in q$, so pick $n'_1, n'_2, \ldots, n'_{k'}$ such that $n_k < n'_1 < n'_2 < \ldots < n'_{k'}$ and

$$\prod_{t'=1}^{k'} x_{n'_{t'}} \in \left(\prod_{t=1}^{k} x_{n_t}\right)^{-1} C.$$

This shows that $C \cap A_{m_0} \neq \emptyset$. Since $m_0 \in \mathbb{N}$ was arbitrary we get $\overline{C} \cap B \neq \emptyset$. Since $C \in pq$ was arbitrary, it follows that $pq \in B$, so B is indeed a subsemigroup. Clearly B is closed, so there exists an idempotent $e \in B$. By definition of B, $A \in e$. $\square$

Corollary 1.4.7 (Hindman's Theorem [H75]) (1) *Assume that $\mathbb{N}$ is finitely colored. There exists a monochrome additive IP set.*

(2) *Let S be a semigroup and let A be an IP-set in S. Assume that A is finitely coloured. Then there exists a monochrome IP-set.*

proof: Clearly (1) follows from (2). To prove (2), by Theorem 1.4.6 let $e \in \beta S$ be an idempotent such that $A \in e$. Pick a monochrome set $B \subseteq A$ such that $B \in e$. Then again by Theorem 1.4.6 B is an IP set. $\square$

Definition 1.4.8 *Let S be a semigroup and let $A \subseteq S$. Then A is* central *iff there is a minimal idempotent e of βS such that $A \in e$, i.e. iff $\overline{A} \cap E(K(\beta S)) \neq \emptyset$.*

A central set is in particular a piecewise syndetic IP-set. Given a minimal idempotent p and a finite partition of S, one cell must be a member of p, hence at least one cell of any finite partition of S must be central.
If $B \subseteq S$ is a thick set, $\overline{B}$ contains minimal left ideal and in particular a minimal idempotent. Thus B is central.
If $C \subseteq S$ is piecewise syndetic, there exists some minimal left ideal $L \subseteq \beta S$ such that $L \cap \overline{C} \neq \emptyset$. Pick a minimal idempotent $e \in L$. By Theorem 1.4.5 $\{s \in S : s^{-1}C \in e\}$ is syndetic. Thus there exist in fact many $s \in S$ such that $s^{-1}C$ is central.

Central sets are fundamental to the Ramsey Theoretic applications of the algebra of βS. Central subsets of $\mathbb{Z}$ were introduced by Fürstenberg [F81] using a different but equivalent definition.
Depending on the context it might or might not be appropriate to work with the semigroup $(\omega, +)$ instead of $(\mathbb{N}, +)$. It is convenient to know that they have essentially the same central subsets:

Lemma 1.4.9 *$A \subseteq \omega$ is central in $(\omega, +)$ iff $A \setminus \{0\}$ is central in $(\mathbb{N}, +)$. More generally let $k \in \mathbb{N}$. Then $C \subseteq \omega^l$ is central in $(\omega, +)^k$ iff $C \cap \mathbb{N}^k$ is central in $(\mathbb{N}, +)^k$.*

proof: The crucial fact behind Lemma 1.4.9 is that $\mathbb{N}^k$ and ω^k have the same minimal idempotents:
We will identify $\beta\mathbb{N}^k$ with $\overline{\mathbb{N}}^k \subseteq \beta\omega^k$. $\mathbb{N}^k$ is an ideal of ω^k, thus $\overline{\mathbb{N}}^k$ is an ideal of $\beta\omega^k$ and in particular $K(\beta\omega^k) \subseteq \overline{\mathbb{N}}^k$. So by Lemma 1.3.18

$$K\left(\overline{\mathbb{N}}^k\right) = K\left(\beta\omega^k\right) \cap \overline{\mathbb{N}}^k = K\left(\beta\omega^k\right).$$

In particular e is a minimal idempotent in $\beta\omega^k$ iff it is a minimal idempotent in $\overline{\mathbb{N}}^k$. Hence

$$\overline{C} \cap E\left(K\left(\beta\omega^k\right)\right) \neq \emptyset \Leftrightarrow \overline{C} \cap E\left(K\left(\overline{\mathbb{N}}^k\right)\right) \neq \emptyset \Leftrightarrow \overline{C \cap \mathbb{N}^k} \cap E\left(K\left(\overline{\mathbb{N}}^k\right)\right) \neq \emptyset.$$

□

The following Theorem (which generalizes van der Waerden's Theorem) and its proof are new, but we remark that Theorem 1.4.10 also follows from [BH01]. In the case that $S = (\mathbb{N}, +)$ a very similar Theorem (where central is replaced by piecewise syndetic) was shown by Fürstenberg and Glaser in [FG98].

Theorem 1.4.10 *Let S be a commutative semigroup with identity 1, let $A \subseteq S$ be central and let $k \in \mathbb{N}$. The set $\{(b, r) : b, br, \ldots, br^k \in A\}$ is central in S^2.*

proof: $A \times \{1\}$ is central in $S \times \{1\}$, so pick an idempotent $e \in \overline{S \times \{1\}} \subseteq \beta S^2$ that is minimal in the subgroup $\overline{S \times \{1\}}$ such that $A \times \{1\} \in e$. Pick a minimal idempotent $q \in \beta S^2$ such that $q \leq e$. For $i \in \omega$ put $\theta_i(b, r) = (br^i, 1)$ for $b, r \in S$ and denote its continuous extension $\beta S^2 \to \overline{S \times \{i\}}$ by the same symbol. Note that θ_i is a homomorphism. Thus $\theta_i(q)$ is an idempotent and it follows directly from the definition of $\leq$ that $\theta_i(q) \leq \theta_i(e) = e$. Since e is minimal in $\overline{S \times \{1\}}$, $\theta_i(q) = e$. We have

$$\{(b, r) \in S^2 : br^i \in A\} = \bigcap_{i=0}^{k} \{(b, r) \in S^2 : \theta_i(b, r) \in \overline{A \times \{1\}}\}$$

and by Lemma 1.2.6 each set in the intersection on the right side is contained in q. □

Corollary 1.4.11 *Let $A \subseteq \mathbb{N}$ be central in $(\mathbb{N}, +)$ and let $k \in \mathbb{N}$. The set*

$$\{(a, d) \in \mathbb{N} : a, a + d, \ldots, a + dk \in A\}$$

is central in $(\mathbb{N},+)^2$.
In particular for any finite coloring of $\mathbb{N}$ *there exist* $a,d \in \mathbb{N}$ *and a monochrome set* $A \subseteq \mathbb{N}$ *such that*

$$\{a, a+d, \ldots, a+dk\} \subseteq A.$$

proof: The main statement follows directly by Theorem 1.4.10 applied to the semigroup $(\omega,+)$ if one takes into account that $(\mathbb{N},+)$ and $(\omega,+)$ respectively $(\mathbb{N},+)^2$ and $(\omega,+)^2$ have essentially the same central sets by Lemma 1.4.9. The 'in particular' statement follows immediately. □

1.5 Abundance of partition regular structures in large sets

Often when we establish the existence of a certain structure or some element we find that in fact there are a lot of possible choices. (I.e. "ultrafilter"-many choices)
Our proofs of the Theorems of Hindman and van der Waerden are strong witnesses for this principle.
In this section we somewhat turn the tables in the sense that we use well known partition result to establish that strengthenings that guarantee many of these structures are also valid.
Put

$$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Then van der Waerden's Theorem for arithmetic progressions of length 4 states that for any finite colouring of $\mathbb{N}$ there exist a monochrome set C and a vector $(a,d) \in \mathbb{N}^2$ such that $C(a,d)^T \subseteq C^4$.
Matrix multiplication makes perfect sense when we restrict our attention to commutative semigroups $(S,+)$ that possess an identity 0 and we shall do so.

Definition 1.5.1 *Let* $(S,+)$ *be commutative semigroup with identity* 0 *and let* $u,v \in \mathbb{N}$. *A* $u \times v$ *matrix* C *with entries from* ω *is called image partition regular over* $(S,+)$ *iff whenever* $r \in \mathbb{N}$ *and* $S = \bigcup_{i=1}^{r} A_i$, *there exist* $i \in \{1,2,\ldots,r\}$ *and* $\vec{x} \in (S \setminus \{0\})^v$ *such that all entries of* $C\vec{x}$ *are in* A_i. *(We shall use the custom of denoting the entries of a matrix by the lower case of the same letter whose upper case denotes the matrix, so that the entry in row* i *and column* j *of* C *is denoted by* $c_{i,j}$.*)*

Tightly connected with image partition regular matrices is the notion of a first entries matrix:

Definition 1.5.2 *Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix with entries from ω. C is a first entries matrix iff no row of C is $\vec{0}$ and for all $i, j \in \{1, 2, \ldots, u\}$ and all $k \in \{1, 2, \ldots, v\}$, if $k = \min\{t : c_{i,t} \neq 0\} = \min\{t : c_{j,t} \neq 0\}$, then $c_{i,k} = c_{j,k}$.*
$c \in \omega$ is called a first entry *of C iff it is the value of the first non vanishing entry in some row of C.*

In the formulation of the next Theorem another notion of largeness in semigroups, namely central* will important. A set $B \subseteq S$ is called a central* set if it intersects any central set of S. Central* sets have many interesting properties and we want to mention some of them: Assume that $e \in \beta S$ is a minimal idempotent and that $S \backslash B \in e$. This makes $S \backslash B$ a central set which contradicts the definition of B, so we must have $B \in e$. Thus B is contained in every minimal idempotent in βS, i.e. in symbols $E(K(\beta S)) \subseteq \overline{B}$. It is easy to see that this property characterizes central* sets. Furthermore, it immediately gives that the intersection of finitely many central* sets is a central* set.
Since every thick set contains a central set, B intersects any thick set. For example in $(\mathbb{N}, +)$ or in $(\omega, +)$ this is tantamount to saying that B is syndetic. We want to give an important example of a central* set in $(\omega, +)$: Let $k \in \mathbb{N}$. It is an easy exercise that for any sequence $(x_n)_{n=1}^{\infty}$ of positive integers, $FS((x_n)_{n=1}^{\infty}) \cap k\omega \neq \emptyset$. Thus $k\omega$ intersects all IP sets. All central sets are IP sets, so $k\omega$ intersects in particular all central sets. Thus $k\omega$ is central*. This example is particularly important for the following theorem. It yields that in the case $(S, +) = (\omega, +)$ every first entry c of C has the property that cS is central*.

Theorem 1.5.3 *Let $(S, +)$ be an infinite commutative semigroup with identity 0, let $A \subseteq S$ be central, let $u, v \in \mathbb{N}$ and let $C \in \omega^{u \times v}$ be a first entries matrix. Assume that for each first entry c of C, cS is central*. Then there exists $\vec{x} \in (S \setminus \{0\})^v$ such that $C\vec{x} \in A^u$. In particular C is image partition regular.*

proof: See [HS98], Theorem 15.5 and Corollary 15.6. □

As an example for the usefulness of Theorem 1.5.3 we show how it yields a nice strengthening of van der Waerden's Theorem: Let $(S, +)$ be a commutative semigroup with identity 0.

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

is a first entries matrix. All first entries are 1 and since $1 \cdot S = S$ is clearly central* we may apply Theorem 1.5.3 to the matrix C.

Thus for any central set $A \subseteq S$ there exist $a, d \in S \setminus \{0\}$ such that $\{d, a, a + d, a + 2d, a + 3d\} \subseteq A$. In the same way we see that any central set contains arbitrarily long arithmetic progressions together with the step length of these progressions.
The following theorem shows that concerning the semigroup $(\omega, +)$ there is not much difference between the main statement and the "in particular" statement in Theorem 1.5.3 and that whenever a matrix is image partition regular, there exists a first entries matrix that is responsible for it.

Theorem 1.5.4 *Let $u, v \in \mathbb{N}$ and let $C \in \mathbb{Q}^{u \times v}$. The following are equivalent:*

(1) *C is image partition regular over $(\omega, +)$.*

(2) *There exist $m \in \mathbb{N}$ and $D \in \omega^{u \times m}$ such that given any $\vec{y} \in \mathbb{N}^m$ there exists some $\vec{x} \in \mathbb{N}^v$ such that $C\vec{x} = D\vec{y}$.*

(3) *For each central set $A \subseteq \omega$ exists $\vec{x} \in \mathbb{N}^v$ such that $C\vec{x} \in A^u$.*

proof: [HS98], Theorem 15.24 $\square$

Theorem 1.5.5 *Let $(S, +)$ be a commutative semigroup with identity 0, let $A \subseteq S$ be central, let $u, v \in \mathbb{N}$ and let $C \in \omega^{u \times v}$. Assume that for each central set $B \subseteq S$ there exists $x \in S^v$ such that $C\vec{x} \in B^u$. Then $\{x \in S^v : C\vec{x} \in A^u\}$ is central in S^v.*

proof: Let $e \in \beta S$ be a minimal idempotent such that $A \in e$. Let $\phi : \beta S^v \to (\beta S)^u$ be the continuous extensions of the map that corresponds to the matrix C. Let $M = \{p \in \beta S^v : \phi(p) = (e, \ldots, e) \in (\beta S)^u\}$. For each $B \in e$ there exists $\vec{x}_B \in S^v$ such that $\phi(\vec{x}_B) \in B^u$. By passing to a limit we obtain that M is non empty. Furthermore M is a closed subsemigroup of βS^v. Let q be an idempotent that is minimal in M and let $p \in \beta S^u$ be an idempotent such that $p \leq q$. Observe that $\phi(p) \leq \phi(q) = (e, \ldots, e)$. Since $K((\beta S)^u) = (K(\beta S))^u$, $(e, \ldots, e)$ is minimal in $(\beta S)^u$. Thus by minimality of $(e, \ldots, e)$, $\phi(p) = (e, \ldots, e)$. Thus $p \in M$ and so $p = q$. This shows that q is minimal in βS^u. By Lemma 1.2.6 continuity of ϕ implies $\{\vec{x} \in S^u : \phi(\vec{x}) \in \overline{A}^v\} \in q$. Thus

$$\{\vec{x} \in S^u : C\vec{x} \in A^v\} = S^u \cap \phi^{-1}[\overline{A^v}] \in q.$$

$\square$

Together with Theorem 1.5.3 Theorem 1.5.5 yields a nice generalisation of Theorem 1.4.10:

Theorem 1.5.6 *Let $(S,+)$ be a commutative semigroup with identity 0, let $A \subseteq S$ be central and let $k \in \mathbb{N}$. The set*

$$\{(a,d) : a, a+d, \ldots, a+dk, d \in A\}$$

is central in $(S,+)^2$.

proof: In the case $k = 3$ use the matrix C defined above to show that every central set $B \subseteq S$ contains a configuration of the type $\{a, a+d, a+2d, a+3d, d\}$. Then the claim follows directly by Theorem 1.5.3. If k is bigger (or smaller) then 3, one has to add (or remove) some rows of the matrix C first. □

Another way to characterize sets of the form $\{d, a, a+d, a+2d, a+3d\}$ is to ask for the elements of the kernel of the matrix

$$D = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

This approach leads to the following definition:

Definition 1.5.7 *Let $u, v \in \mathbb{N}$. A matrix $C \in \mathbb{Z}^{u \times v}$ is called* kernel partition regular *iff for any finite colouring of* $\mathbb{N}$ *there exist a monochrome set A and $\vec{x} \in A^v$ such that $C\vec{x} = 0$.*

Rado's Theorem [R33] gives an explicit method (which we will not describe here) to determine whether a given matrix C is kernel partition regular. We want to remark that kernel partition regular matrices are also linked to central sets: By [HS98], Theorem 15.16 a matrix $C \in \mathbb{Z}^{u \times v}$ is kernel partition regular iff for each central set $A \subseteq \mathbb{N}$ there exists $\vec{x} \in A^v$ such that $C\vec{x} = 0$.

Theorem 1.5.8 *Let $u, v \in \mathbb{N}$ and let $C \in \mathbb{Z}^{u \times v}$ be kernel partition regular. Assume that A is central in $(\omega, +)$. Then $\{x \in A^v : C\vec{x} = 0\}$ is central in the semigroup $\{x \in \omega^v : C\vec{x} = 0\}$.*

proof: Let $e \in \beta S$ be a minimal idempotent such that $A \in e$. Let $\phi : \beta S^v \to (\beta S)^u$ be the continuous extension of the map that corresponds to the matrix C and for $i \in \{1, 2, \ldots, v\}$ denote by $\pi_i : S^v \to S$ the projection onto the i-th coordinate respectively its continuous extension. Let

$$M = \{p \in \beta S^v : \phi(p) = 0, \pi_1(p) = \pi_2(p) = \ldots = \pi_v(p) = e\}.$$

For each $B \in e$ there exists $\vec{x}_B \in B^v$ such that $\phi(\vec{x}_B) = 0$. By passing to a limit we obtain that M is non empty. Furthermore M is a closed subsemigroup of βS^v. Let q be an idempotent that is minimal in M. Let $E = \{p \in \beta S^v : \phi(p) = 0\}$. E is a closed subsemigroup that contains M.

Let $p \in E$ be an idempotent such that $p \leq q$. Then for $i \in \{1, 2, \ldots, v\}$, $\pi_i(p) \leq \pi_i(q) = e$ and by minimality of e, $\pi_i(p) = e$. Thus $p \in M$ and this shows that q is minimal in $E = \overline{\{x \in S^v : C\vec{x} = 0\}}$. So

$$\{x \in A^v : C\vec{x} = 0\} = \{x \in S^v : C\vec{x} = 0\} \cap \bigcap_{i=1}^{v} \pi_i^{-1}[\overline{A}] \in q.$$

□

Note that we may not achieve that $\{x \in A^v : C\vec{x} = 0\}$ is central in $\mathbb{N}^v$: The set $\{(d, a, a + d, a + 2d, a + 3d) : a, d \in \mathbb{N}\}$ is not even piecewise syndetic in $\mathbb{N}^5$.

Let $k \in \mathbb{N}$. From [BH01], Theorem 3.7 it follows that if $A \subseteq \mathbb{N}$ is 'large' then $\{(a, d) \in \mathbb{N}^2 : \{a, a+d, \ldots, a+kd\} \in A\}$ is 'large' in $\mathbb{N}^2$, where 'large' is any of the notions piecewise syndetic, central, central*, PS*, IP*. We want to investigate this question for the set $\{(a, d) \in \omega^2 : \{a, a + d, \ldots, a + kd, d\} \in A\}$ or more generally for the set $\{\vec{x} \in S^u : C\vec{x} \in A\}$, where $(S, +)$ is a commutative semigroup with identity 0 and $C \in \omega^{u \times v}$ is a matrix which satisfies reasonable conditions.
We have already defined the notion of a central* set. IP* sets and PS* sets are defined similarly: $B \subseteq S$ is an IP* set iff B intersects any IP set in S, $C \subseteq S$ is a PS* set iff it intersects any piecewise syndetic set in S. As with central* sets, it is easy to see that IP* sets are characterized by the property that their closures contain $E(\beta S)$ and that C is a PS* set iff $\overline{C} \supseteq K(\beta S)$. Thus IP* sets and PS* set are closed under finite intersections. Furthermore if B is an IP* set and C is a PS* set then $\overline{B \cap C} \supseteq E(\beta S) \cap K(\beta S) \supseteq E(K(\beta S))$, so $B \cap C$ is central*. We have already seen that in ω for each $k \in \mathbb{N}$, $k\omega$ is central*. The proof given there shows in fact that $k\omega$ is even IP*.

$2\mathbb{N} + 1$ is syndetic, but doesn't contain any configurations of the form $\{a, d, a + d\}, a, d \in \mathbb{N}$. Thus 'syndetic' and 'piecewise syndetic' are not promising notions of largeness in the sense of the following theorem.

Theorem 1.5.9 *Let $(S, +)$ be a commutative semigroup with identity 0. Let $u, v \in \mathbb{N}$, let $A \subseteq S$ and let $C \in \omega^{u \times v}$ be any matrix.*

(1) *If A is IP* in S then $\{\vec{x} \in S^v : C\vec{x} \in A^u\}$ is IP* in S^u.*

(2) *Let 'large' be any of the terms 'central', 'central*', 'PS*' and let C be a first entries matrix such that for each first entry c of C, cS is central*. If A is large in S then $\{\vec{x} \in S^v : C\vec{x} \in A^u\}$ is large in S^u.*

proof: Denote by $\phi : \beta S^v \to (\beta S)^u$ the continuous extension of the map induced by C.

(1) Since ϕ is a homomorphism, it maps idempotents to idempotents. Let $e \in \beta S^v$ be an arbitrary idempotent. Then $\overline{A}^u$ is a neighborhood of $\phi(e)$. By continuity (we are again using the characterisation given in Lemma 1.2.6) of ϕ,

$$\{\vec{x} \in S^v : C\vec{x} \in A^u\} = \{\vec{x} \in S^v : \phi(\vec{x}) \in \overline{A}^u\} \in e$$

so we are done.

(2) For 'large'='central' (which is the most difficult part) this follows Theorem 1.5.3 and Theorem 1.5.5.

$\phi[\beta S^v]$ is a compact subgroup of $(\beta S)^u$ and $\phi[K(\beta S^v)] = K(\phi[\beta S^v])$. It follows from Theorem 1.5.3 and the proof of Theorem 1.5.5 that for each minimal idempotent $e \in \beta S$ there exists a minimal idempotent $q \in \beta S^v$ such that $\phi(q) = (e, \ldots, e) \in K((\beta S))^u$. It follows that

$$\phi[\beta S^v] \cap K((\beta S)^u) = \phi[\beta S^v] \cap (K(\beta S))^u \neq \emptyset.$$

Thus by Theorem 1.3.18

$$\phi[K(\beta S^v)] = \phi[\beta S^v] \cap (K(\beta S))^u.$$

Assume now that A is a PS* set and pick $p \in K(\beta S^v)$. Then $\overline{A}^u$ is a neighborhood of $\phi(p)$. By continuity of ϕ,

$$\{\vec{x} \in S^v : C\vec{x} \in A^u\} = \{\vec{x} \in S^v : \phi(\vec{x}) \in \overline{A}^u\} \in p.$$

Since p was arbitrary, $\{\vec{x} \in S^v : C\vec{x} \in A^u\}$ is indeed PS*.

The proof in the case 'large'='central*' is similar.

□

Chapter 2

Geoarithmetic Progressions

In section 1 we give some motivation for the intended results. In particular we review some results of V. Bergelson from [Ber$i$]. This seems to be the first paper that focuses on so called geoarithmetic progressions in large subsets of $\mathbb{N}$ and was a starting point for our research.
In the next section we discuss various versions of van der Waerden's Theorem, observe some interactions between them and derive some immediate corollaries in the direction of geoarithmetic progressions. Sections 3 and 4 are dedicated to more sophisticated ways of deriving joint extensions of known partition result. While section 3 concentrates on methods that are applicable to general semigroups, the theorems of the last section are more specialized for applications in $\mathbb{N}$.
To deliver some feeling why ultrafilters might be useful to link different partition results, we want to state a very simple result in this direction:

Theorem 2.0.1 *Let S be a semigroup, let $\mathcal{F}$ and $\mathcal{G}$ be partition regular families of subset of S and assume that $\mathcal{F}$ or $\mathcal{G}$ consists of finite sets. Then $\{FG : F \in \mathcal{F}, G \in \mathcal{G}\}$ is partition regular.*

proof: Without loss of generality we may assume that $\mathcal{F}$ consists of finite sets. Let $p, q \in \beta S$ be such that every element of q contains a member of $\mathcal{F}$ and that every element of p contains a member of $\mathcal{G}$. Assume that $A_1, \ldots, A_r$ is a partition of S. Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in pq$. Then $B = \{s : s^{-1}A_i \in p\} \in q$. Pick $F \in \mathcal{F}$ such that $F \subseteq B$. Since F is finite $\bigcap_{t \in F} t^{-1}A_i \in q$, thus we may pick $G \in \mathcal{G}$ such that $G \subseteq \bigcap_{t \in F} t^{-1}A_i$. Equivalently $FG \subseteq A_i$. $\square$

2.1 Some motivations

Van der Waerden's Theorem says that whenever the set $\mathbb{N}$ of positive integers is divided into finitely many classes, one of these classes contains arbitrar-

ily long arithmetic progressions. The corresponding statement about geometric progressions is easily seen to be equivalent via the homomorphisms $b : (\mathbb{N}, +) \to (\mathbb{N}, \cdot)$ and $\ell : (\mathbb{N} \setminus \{1\}, \cdot) \to (\mathbb{N}, +)$ where $b(n) = 2^n$ and $\ell(n)$ is the length of the prime factorization of n.
In 1975, Szemerédi [Sz75], showed that any set with positive upper asymptotic density contains arbitrarily long arithmetic progressions. (An ergodic theoretic proof of Szemerédi's Theorem can be found in [F81].)
In the last chapter we have seen that an appropriate "topological" notion of largeness behind van der Waerden's Theorem is piecewise syndeticy.
The density analogon is the notion of positive upper Banach density which we will use in $(\mathbb{N}, +)$ as well as in $(\mathbb{N}, \cdot)$:

Definition 2.1.1 *Let $A \subseteq \mathbb{N}$, let $(p_n)_{n=1}^{\infty}$ be the sequence of primes in their natural order and let $F_n = \{\prod_{i=1}^{n} p_i^{\alpha_i}$: for each $i \in \{1, 2, \ldots, n\}$, $\alpha_i \in \{0, 1, \ldots, n\}\}$ for $n \in \mathbb{N}$. The* upper additive Banach density $d^*(A)$ *and the* upper multiplicative Banach density $d_m^*(A)$ *are defined by*

$$d^*(A) = \sup\left\{\limsup_{n\to\infty} \frac{|A \cap \{a_n, \ldots, b_n\}|}{b_n - a_n + 1} : \lim_{n\to\infty} b_n - a_n = \infty\right\}, \quad (2.1)$$

$$d_m^*(A) = \sup\left\{\limsup_{n\to\infty} \frac{|A \cap r_n F_n|}{|F_n|} : (r_n)_{n=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}\right\}. \quad (2.2)$$

Clearly any additively or multiplicatively piecewise syndetic set has positive upper additive respectively upper multiplicative Banach density.
Via a compactness argument, similar to the one given in the proof of Theorem 1.1.4 it is not hard to see that Szemerédi's Theorem implies that any set with positive upper additive Banach density contains arbitrarily long arithmetic progressions.
It has recently been shown ([Ber*i*], Theorem 1.3) that any set having positive multiplicative upper Banach density must contain substantial combined additive and multiplicative structure. That is, geoarithmetic progressions of arbitrary large order, that is, sets of the following form:

Definition 2.1.2 *A geoarithmetic progression (of order k) is a set of the form $\{r^j(a + id) : i, j \in \{0, 1, \ldots, k\}\}$ where $a, d, k \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$.*

Another simply stated result from [Ber*i*] is that any multiplicatively large set contains geometric progressions in which the common ratios form an arithmetic progression, that is a set of the form $\{b(a + id)^j : i, j \in \{0, 1, \ldots, k\}\}$.
The apparatus used in modern[1] density Ramsey Theory is that of Ergodic Theory in the sense of recurrence theorems in measure preserving systems. This parallels the important role of symbolic dynamics and the Stone-Čech

[1]I.e. in density Ramsey Theory after the publication of Fürstenberg's seminal Ergodic Theoretic proof of Szeméredi's Theorem in [F77].

Compactification of discrete semigroups in the case of partition Ramsey Theory.
For any partition result one can easily formulate the corresponding density statement, and it is neither absurd to hope that it is valid. Call a family $\mathcal{F}$ of subsets of $\mathbb{N}$ density regular iff any set of positive upper additive Banach density contains a member of $\mathcal{F}$. For quite a while it was an open question if every invariant partition regular family of finite subsets of $\mathbb{N}$ must in fact be density regular. (This problem was settled negatively by Kriz [K87] in 1987)
As a matter of fact, density statements are much harder to prove than partition results. (Van der Waerden's Theorem was shown in 1932 while Szeméredi's Theorem appeared in 1975.) It is quite an exceptional situation that density theorems serve as motivation to search for (desirable simpler) proofs of the derived partition statements that avoid the huge machinery behind the density statements and yield stronger implications.
We want to give some example of what combined additive and multiplicative structures can be guaranteed to lie in one cell of a finite partition of $\mathbb{N}$: It was shown in 1975 [H79] that there exist sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ such that $FS((x_n)_{n=1}^{\infty}) \cup FP((y_n)_{n=1}^{\infty})$ is contained in one cell, and in 1988 in [BH88] that one cell must in addition contain arbitrarily long arithmetic and geometric progressions.
To get an idea of the new results we want to establish, consider the following result, which is a consequence of [Ber*i*], Theorem 3.13.

Theorem 2.1.3 *Let $m, k \in \mathbb{N}$. For each $i \in \{0,1,\ldots,k\}$ let $(x_{i,t})_{t=1}^{\infty}$ and $(y_{i,t})_{t=1}^{\infty}$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1,2,\ldots,m\}$, $F, G \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\left\{ b\left(a + \sum_{t\in F} x_{i,t}\right) \cdot \left(\prod_{t\in G} y_{j,t}\right) : i,j \in \{0,1,\ldots,k\} \right\} \subseteq A_s.$$

Notice that a particular consequence of Theorem 2.1.3 is that one cell of each finite partition of $\mathbb{N}$ must contain arbitrarily long geoarithmetic progressions. Further, the common ratio r can be taken from $FP((y_n)_{n=1}^{\infty})$ for any prescribed $(y_n)_{n=1}^{\infty}$ and the additive increment d can be guaranteed to be a multiple of some member of $FS((x_n)_{n=1}^{\infty})$ for any prescribed $(x_n)_{n=1}^{\infty}$. To see this, for $i \in \{1,2,\ldots,k\}$ and $t \in \mathbb{N}$, let $x_{i,t} = ix_t$ and $y_{i,t} = (y_t)^i$. Given F and G as guaranteed by Theorem 2.1.3, let $d = b \cdot \sum_{t\in F} x_t$ and $r = \prod_{t\in G} y_t$.
We show in Theorem 2.2.9 that one may take $F = G$ in Theorem 2.1.3 and in Corollary 2.4.12 that one may eliminate b. We show also that one may not simultaneously take $F = G$ and eliminate b.
The following consequence of Corollary 2.4.6 (or alternatively of Corollary 2.4.12(c) or Corollary 2.3.9) says that one can always get the additive in-

crement of a geoarithmetic progression as the initial term of a geometric progression in one cell of a finite partition of $\mathbb{N}$.

Theorem 2.1.4 *Let $m \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $a, d \in A_s$ and $r \in A_s \setminus \{1\}$ such that*

$$\{r^j(a + id) : i, j \in \{0, 1, \ldots, k\}\} \cup \{dr^j : j \in \{0, 1, \ldots, k\}\} \subseteq A_s.$$

2.2 Different versions of van der Waerden's Theorem

Some experts (e.g. Vitaly Bergelson) think that the Hales-Jewett Theorem [HJ63] is the "right" version van der Waerden's Theorem. We need to introduce some notation to state it: Consider some alphabet (i.e. non empty set) Λ and let v be a "variable" that is not contained in Λ. Let W be the free semigroup of all words over the alphabet $\Lambda \cup \{v\}$, denote the subsemigroup of all words that consist of letters of Λ by W_0 and write W_1 for the subsemigroup of all words in which v occurs. The elements of W_1 are called *variable words.*
For each $a \in \Lambda$, define $\theta_a : W \to W_0$ by

$$\theta_a(w)(t) = \begin{cases} w(t) & \text{if } w(t) \in \Lambda \\ a & \text{if } w(t) = v\,. \end{cases}$$

for $w \in W$ and $t \in dom\ (w)$. That is, $\theta_a(w)$ is the result of replacing each occurrence of v in w by a. Notice that θ_a is the identity on W_0 hence this also holds for its continuous extension on βW_0.

Theorem 2.2.1 (Hales-Jewett Theorem [HJ63]) *Let Λ be a finite alphabet and let $W_0 = \bigcup_{i=1}^{m} A_i$. Then there exist $i \in \{1, 2, \ldots, m\}$ and a variable word $w \in W_1$ such that*

$$\{\theta_a(w) : a \in \Lambda\} \subseteq A_i.$$

proof: The proof is similar to our proof of van der Waerden's Theorem: Let $e \in \beta W_0$ be a minimal idempotent and let $A \in e$. We want to show that $\{w \in W_1 : \theta_a(w) \in A \text{ for all } a \in \Lambda\}$ is central in W.
For $a \in \Lambda$ we denote the continuous extension of θ_a to βW by the same symbol and remark that $\theta_a : \beta W \to \beta W_0$ is also a homomorphism. Put $M = \{p \in \beta W : \theta_a(p) = e \text{ for all } a \in \Lambda\}$. Since each θ_a is the identity function on W_0 and hence on $\beta W_0 = \overline{W}_0$, we have $e \in M$. M is a closed subgroup of βW, so pick a minimal idempotent $q \in M$ such that $q \leq e$.
We claim that q is minimal in βW. Pick an idempotent $p \in \beta W$ such that $p \leq q$. Then $\theta_a(p) \leq \theta_a(q) = e$ for all $a \in \Lambda$. By minimality of e in $\overline{W}_0$,

$\theta_a(p) = e$ and this shows that $p \in M$. So q is in fact minimal. W_1 is an ideal in W, so $\overline{W}_1$ is an ideal of βW and hence $W_1 \in q$. By continuity of θ_a we have $\theta_a^{-1}[A] \in q$ for each $a \in \Lambda$ and this yields

$$\{w \in W_1 : \theta_a(w) \in A \text{ for all } a \in \Lambda\} = \bigcap_{a \in \Lambda} \{w \in W_1 : \theta_a^{-1}(w_1) \in \overline{A}\} \in q.$$

□

(In [BL99] a very strong "polynomial" extension of the Hales-Jewett Theorem is established.)

Applications which we will use later are the following theorems. These results are well known among afficianados.

Corollary 2.2.2 *Let $(S, \cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, let $k \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, k\}$ let $(y_{i,n})_{i=1}^{\infty}$ be a sequence in S. There exist $F \in \mathcal{P}_f(\mathbb{N})$ and $b \in S$ such that*

$$\{b\} \cup \left\{ b \prod_{t \in F} y_{i,t} : i \in \{1, \ldots, k\} \right\} \subseteq A.$$

proof: By the virtue of Lemma 1.4.2 it is sufficient to show that the family $\{\{b\} \cup \{b \prod_{t \in F} y_{i,t} : i \in \{1, \ldots, k\}\} : b \in S, F \in \mathcal{P}_f(\mathbb{N})\}$ is partition regular. Let $\Lambda = \{0, 1, \ldots, k\}$ and let W_0 be the free semigroup on the alphabet Λ. Let $b_0 \in S$ be an arbitrary, fixed element. Given a word $w = l_1 l_2 \cdots l_n$ of length n in S, define

$$f(w) = b_0 \prod_{t \in \{1,2,\ldots,n\}, l_t \neq 0} y_{l_t,t}$$

if there exists some $t \in \{1, 2, \ldots, n\}$ such that $l_t \neq 0$ and $f(w) = b_0$ otherwise.
Consider a partition $\{A_1, A_2, \ldots, A_m\}$ of S. Then $W_0 = \bigcup_{s=1}^{m} f^{-1}[A_s]$ so pick $s \in \{1, 2, \ldots, m\}$ and a variable word $w = l_1 l_2 \cdots l_n$ (with each $l_t \in \Lambda \cup \{v\}$) such that $\{\theta_i(w) : i \in \Lambda\} \subseteq f^{-1}[A_s]$.
Let $F = \{t \in \{1, 2, \ldots, n\} : l_t = v\}$, let $G = \{1, 2, \ldots, n\} \setminus F$ and let $b = f(w(0))$. Then $b \prod_{t \in F} y_{i,t} = f(w(i))$ for $i \in \{1, 2, \ldots, k\}$ and thus $\{b\} \cup \{b \prod_{t \in F} y_{i,t} : i \in \{1, \ldots, k\}\} \subseteq A_s$. □

Theorem 2.2.3 *Let $(S, \cdot)$ be a commutative semigroup, let A be a piecewise syndetic subset of S, let B be an IP set in S and let $k \in \mathbb{N}$. There exist $b \in S$ and $r \in B$ such that*

$$\{b, br, br^2, \ldots, br^k\} \subseteq A.$$

If A is central we may in particular take $A = B$, such that

$$\{r, b, br, br^2, \ldots, br^k\} \subseteq A.$$

proof: Let $(x_n)_{n=1}^{\infty}$ be a sequence in S such that $FP((x_n)_{n=1}^{\infty}) \subseteq B$. For $i \in \{1, 2, \ldots, k\}$ and $n \in \mathbb{N}$, let $y_{i,n} = (x_n)^i$. Pick b and F as guaranteed by Theorem 2.2.2 and let $r = \prod_{t \in F} x_t$.
Any central set is a piecewise syndetic IP set and thus the in particular statement follows. □

Another simple consequence of Theorem 2.2.2 is a nice multidimensional version of van der Waerden's Theorem, namely Gallai's Theorem.
Before we state and prove this theorem, we want to remark which multidimensional version of van der Waerden's Theorem we may achieve just with the use of the one dimensional version: Let $k \in \mathbb{N}$ and put

$$\mathcal{F} = \{\{(a, 0), (a + c, 0), \ldots, (a + kc, 0)\} : a, c \in \mathbb{N}\},$$

$$\mathcal{G} = \{\{(0, b), (0, b + d), \ldots, (0, b + kd)\} : b, d \in \mathbb{N}\}.$$

By van der Waerden's Theorem $\mathcal{F}$ and $\mathcal{G}$ are partition regular families of finite subsets ω^2.
Thus by Theorem 2.0.1 for every finite colouring there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F + G$ is contained in a monochrome subset A. I.e. we may find $a, b, c, d \in \mathbb{N}$ such that for all $i, j \in \{1, 2, \ldots, k\}$, $(a + ic, b + id) \in A$.
Gallai's Theorem strengthens this by allowing to take $c = d$.

Theorem 2.2.4 (Gallai's Theorem) [2] *Let $k, l, r \in \mathbb{N}$ and assume that $\bigcup_{i=1}^{r} A_i = \mathbb{N}^k$. Then there exist $s \in \{1, 2, \ldots, r\}$, $a \in \mathbb{N}^k$ and $d \in \mathbb{N}$ such that*

$$\{a + d(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \{0, 1, \ldots, l\}\} \subseteq A_s.$$

proof: For $x_1, \ldots, x_k \in \{0, 1, \ldots, l\}$ and $n \in \mathbb{N}$ put $y_{(x_1,\ldots,x_k),n} = (x_1, \ldots, x_k)$. Pick $s \in \{1, 2, \ldots, r\}$ such that A_s is piecewise syndetic and by Theorem 2.2.2 $a \in A_s$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{t \in F} y_{(x_1,\ldots,x_k),t} \in A$ for all $x_1, \ldots, x_k \in \{0, 1, \ldots, l\}$. Put $d = |F|$. Since $a + \sum_{t \in F} y_{(x_1,\ldots,x_k),t} = a + d(x_1, \ldots, x_k)$ we are done. □

We want to suggest another version of a multiplicative van der Waerden type Theorem that is nicely connected with Gallai's Theorem:

Theorem 2.2.5 [3] *For any $k \in \mathbb{N}$ and any finite colouring of $\mathbb{N}$ there exist $b, r \in \mathbb{N}$ and a monochrome set $A \subseteq \mathbb{N}$ such that*

$$\{b, b2^r, b3^r, \ldots, bk^r\} \subseteq A.$$

[2]This Theorem was never published by its author. Sometimes it is referred to it as Grünwald's Theorem, Grünwald being the original name of the author. During the period surrounding World War II Grünwald changed his name to Gallei, presumably for fear of persecution by the national socialists.

[3]A density version of this Theorem can be found in [Ber*i*].

Theorem 2.2.5 is easily seen to imply the partition regularity of geometric progressions: Take $k_2 = 2^k$. Then

$$\{bi^r : i \in \{1, \dots, k_2\}\} \supseteq \{b(2^i)^r : i \in \{0, \dots, k\}\} = \{b(2^r)^i : i \in \{0, \dots, k\}\}.$$

Above we mentioned that the partition regularity of geometric progressions is equivalent to van der Waerden's Theorem. This is paralleled by the following statement.

Theorem 2.2.6 *Theorem 2.2.5 is 'equivalent' to Gallai's Theorem.*

proof: Let us first prove Theorem 2.2.5 with the help of Gallai's Theorem: Consider a covering $A_1, \dots, A_m$ of $\mathbb{N}$, the map

$$\phi : \mathbb{N}^k \to \mathbb{N} \qquad (x_1, \dots, x_k) \mapsto \prod_{i=1}^{k} i^{x_i}$$

and the induced covering $B_1 = \phi^{-1}[A_1], \dots, B_k = \phi^{-1}[A_k]$ of $\mathbb{N}^k$. By Gallai's Theorem there exist $a_1, \dots, a_k, r \in \mathbb{N}$ and $s \in \{1, \dots, m\}$ such that

$$\{(a_1 + r, a_2, \dots, a_k), \dots, (a_1, a_2 + r, \dots, a_k), \dots, (a_1, \dots, a_k + r)\} \subseteq B_s.$$

If we write $b = \prod_{i=1}^{k} i^{a_i}$, this yields that $bi^r \in A_s$ for $i \in \{1, \dots, k\}$.
To see that Gallai's Theorem follows from Theorem 2.2.5 let $p_1, p_2, p_3, \dots$ be an enumeration of the primes and for $i, N \in \mathbb{N}$ let

$$\psi_i(N) = \max\{x \in \omega : p_i^x | N\}.$$

Consider a covering $B_1, \dots, B_m$ of ω^k, the map $\psi = (\psi_1, \dots, \psi_k) : \mathbb{N} \to \omega^k$ and the induced covering $A_1 = \psi^{-1}[B_1], \dots, A_k = \psi^{-1}[B_k]$ of $\mathbb{N}$. Let $l \in \mathbb{N}$ and $n \geq \left(\prod_{i=1}^{k} p_i\right)^l$. By Theorem 2.2.5 there exist $s \in \{1, ..., m\}$ and $b, r \in \mathbb{N}$ such that $bi^r \in A_s$ for $i \in \{1, \dots, n\}$. Let $a_1 = \psi_1(b), \dots, a_k = \psi_k(b)$. Then $(a_1, \dots, a_k) + r(x_1, \dots, x_k) \in B_s$ for $x_1, \dots, x_k \in \{0, \dots, l\}$. $\square$

Gallai's Theorem provides an easy way to see that in one cell of a finite partition of $\mathbb{N}$ there exist geoarithmetic progressions of arbitrarily high order. In fact it can do even more for us: If a sufficiently large geoarithmetic progression is partitioned into finitely many cells, one cell must contain a large geoarithmetic progression itself:

Theorem 2.2.7 *Let $k, r \in \mathbb{N}$. There exists $K \in \mathbb{N}$ such that if a geoarithmetic progression of order K is partitioned into r cells, then one cell must contain a geoarithmetic progression of order k.*

proof: By Theorem 1.1.4 there exists $K \in \mathbb{N}$ such that whenever $\{0,1,\ldots,K\}^2$ is coloured with r colours, there exist a monochrome set $A \subseteq \{0,1,\ldots,K\}^2$ and $a \in \mathbb{N}^2, d \in \mathbb{N}$ such that $a + d(x_1, x_2) \in A$ for all $x_1, x_2 \in \{0,1,\ldots,k\}$. Let $\overline{a}, \overline{d} \in \mathbb{N}$ and $\overline{r} \in \mathbb{N} \setminus \{1\}$ and consider a partition $A_1, A_2, \ldots, A_r$ of

$$\{(\overline{a} + i\overline{d})\overline{r}^j : i,j \in \{0,1,\ldots,K\}\}.$$

Put $\phi(i,j) = (\overline{a} + i\overline{d})\overline{r}^j$ for $(i,j) \in \{0,1,\ldots,K\}$. Then there exist $s \in \{1,2,\ldots,r\}$ and $a_1, a_2, d \in \mathbb{N}$ such that $(a_1,a_2) + d(x_1,x_2) \in \phi^{-1}[A_s]$ for all $x_1, x_2 \in \{0,1,\ldots,k\}$. Equivalently

$$(\overline{a} + (a_1 + di)\overline{d})\overline{r}^{a_2+dj} = ((\overline{a} + a_1\overline{d}\overline{r}^{a_2}) + i(d\overline{r}^{a_2}\overline{d}))(\overline{r}^d)^j \in A_s$$

for all $i,j \in \{0,1,\ldots,k\}$. $\square$

Now, as we promised in the introductory section of this chapter, we turn our attention to an extension of the following result from [Ber*i*].

Theorem 2.2.8 *Let $m, k \in \mathbb{N}$. For each $i \in \{0,1,\ldots,k\}$ let $(x_{i,t})_{t=1}^\infty$ and $(y_{i,t})_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^m A_s$. Then there exist $s \in \{1,2,\ldots,m\}$, $F, G \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\left\{ b\left(a + \sum_{t\in F} x_{i,t}\right) \cdot \left(\prod_{t\in G} y_{j,t}\right) : i,j \in \{0,1,\ldots,k\} \right\} \subseteq A_s.$$

proof: By [Ber*i*], Theorem 3.13, every set A with $d_m^*(A) > 0$ contains such a configuration and for some s, $d_m^*(A_s) > 0$. $\square$

We shall show in Theorem 2.2.9 that one may take $F = G$ in Theorem 2.2.8 and in Corollary 2.4.12(a) that the multiplier b may be eliminated. We show in Corollary 2.4.16, however, that one cannot simultaneously take $F = G$ and eliminate b.

Theorem 2.2.9 *Let $m, k \in \mathbb{N}$. For each $i \in \{0,1,\ldots,k\}$ let $(x_{i,t})_{t=1}^\infty$ and $(y_{i,t})_{t=1}^\infty$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^m A_s$. Then there exist $s \in \{1,2,\ldots,m\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}&\{ba\} \cup \{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0,1,\ldots,k\}\} \cup \\ &\{ba\textstyle\prod_{t\in F} y_{j,t} : j \in \{0,1,\ldots,k\}\} \cup \\ &\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\} \subseteq A_s\,.\end{aligned}$$

proof: Let $x_{k+1,t} = 0$ and $y_{k+1,t} = 1$ for all t. Let $A_0 = \{0\}$. Let $\Lambda = \{0,1,\ldots,k+1\} \times \{0,1,\ldots,k+1\}$ and let W_0 be the free semigroup on the alphabet Λ. Given a word $w = l_1 l_2 \cdots l_n$ of length n in W_0, define

$$f(w) = \left(1 + \sum_{t=1}^n x_{\pi_1(l_t),t}\right) \prod_{t=1}^n y_{\pi_2(l_t),t}.$$

(It is necessary to add 1 to assure that the range of f lies in $\mathbb{N}$.) Then $W_0 = \bigcup_{s=0}^{m} f^{-1}[A_s]$ so pick $s \in \{0,1,\ldots,m\}$ and a variable word $w = l_1 l_2 \cdots l_n$ (with each $l_t \in \Lambda \cup \{v\}$) such that $\{w(c) : c \in \Lambda\} \subseteq f^{-1}[A_s]$. Notice that $s \neq 0$.
Let $F = \{t \in \{1,2,\ldots,n\} : l_t = v\}$ and let $G = \{1,2,\ldots,n\} \setminus F$. Let $a = 1 + \sum_{t\in G} x_{\pi_1(l_t),t}$ and let $b = \prod_{t\in G} y_{\pi_2(l_t),t}$. Then given $i,j \in \{0,1,\ldots,k+1\}$, $f(w(i,j)) = (a + \sum_{t\in F} x_{i,t}) \cdot b \cdot \prod_{t\in F} y_{j,t}$. □

Corollary 2.2.10 *Let $k \in \mathbb{N}$. For each $i \in \{0,1,\ldots,k\}$ let $(x_{i,t})_{t=1}^{\infty}$ and $(y_{i,t})_{t=1}^{\infty}$ be sequences in $\mathbb{N}$ and let A be piecewise syndetic in $(\mathbb{N},\cdot)$. Then there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}&\{ba\} \cup \{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0,1,\ldots,k\}\} \cup \\ &\{ba \textstyle\prod_{t\in F} y_{j,t} : j \in \{0,1,\ldots,k\}\} \cup \\ &\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\} \subseteq A.\end{aligned}$$

proof: By Theorem 2.2.9 the collection of sets H of the form

$$\begin{aligned}H \;=\; &\{ba\} \cup \{b(a + \textstyle\sum_{t\in F} x_{i,t}) : i \in \{0,1,\ldots,k\}\} \cup \\ &\{ba \textstyle\prod_{t\in F} y_{j,t} : j \in \{0,1,\ldots,k\}\} \cup \\ &\{b(a + \textstyle\sum_{t\in F} x_{i,t}) \cdot (\prod_{t\in F} y_{j,t}) : i,j \in \{0,1,\ldots,k\}\}\end{aligned}$$

is partition regular, so by Lemma 1.4.2 there is some $t \in \mathbb{N}$ and some such H with $tH \subseteq A$. Replacing b by tb yields the desired conclusion. □

2.3 Combining results in general semigroups

Our first result in this direction (Theorem 2.3.2) replaces r in a geometric progression by members of any invariant partition regular family of finite sets.
For that result, one needs to add a multiplier b because one can certainly not expect to find a set of the form $\{r, r^2\}$ for $r > 1$ in one cell of an arbitrary finite partition of $\mathbb{N}$; one may assign the members of $\mathbb{N} \setminus \{x^2 : x \in \mathbb{N} \setminus \{1\}\}$ to A_1 or A_2 at will, and then assign x^2 to the cell that x is not in, x^4 to the cell x^2 is not in, and so on.
We start with tuning up our version of van der Waerden's Theorem 1.4.10:

Theorem 2.3.1 *Let S be a commutative semigroup, let $A \subseteq S$ be central and let $k \in \mathbb{N}$. The set*

$$\{(b,r) \in S^2 : b, br, \ldots, br^k, r \in A\}$$

is central in S^2.

It is possible to show Theorem 2.3.1 by employing Theorem 1.5.3 (which we didn't prove) together with 1.5.5 and playing a little bit with adjoined identities in case S doesn't have one. To be self contained we prove the theorem directly.
proof: Pick a minimal idempotent $e \in \beta S$ such that $A \in e$. Put $\theta_i(b,r) = br^i$ and $\phi(b,r) = r$ for $i \in \{0,1,\ldots,k\}$ and $b,r \in S$ and denote the continuous extensions $\beta S^2 \to \beta S$ of these functions by the same symbols. Note that these functions are homomorphisms. Let

$$M = \{p \in \beta S : \theta_0(p) = \theta_1(p) = \ldots = \theta_k(p) = \phi(p) = e\}.$$

For each $B \in e$ there exist $b_B, r_B \in \mathbb{N}$ such that

$$\theta_0(b_B,r_B), \theta_1(b_B,r_B), \ldots, \theta_k(b_B,r_B), \phi(b_B,r_B) \in B$$

by Theorem 2.2.3. By passing to a limit point, we see that M is not empty. In fact M is a compact subsemigroup, so pick a minimal idempotent $q \in M$. We claim that q is minimal in βS^2: Let $p \leq q$ be minimal in βS^2. Then $\phi(p) \leq \phi(q) = e$. Since e is minimal we have $\phi(p) = e$. Similarly $\theta_i(p) = e$ for $i \in \{0,1,\ldots,k\}$. Thus $p \in M$ and so by minimality of q in M we have $p = q$. Since

$$\{(b,r) \in S^2 : b, br, \ldots, br^s, r \in A\} = S^2 \cap \bigcap_{i=0}^{r} \theta_i^{-1}[\overline{A}] \cap \phi^{-1}[\overline{A}] \in q$$

we are done. □

Corollary 2.3.2 *Let S be a commutative semigroup, let $\mathcal{F}$ be a partition regular invariant family of finite subsets of $\mathbb{N}$, let $k \in \mathbb{N}$ and let A be central. Then there exist $b \in \mathbb{N}$ and $F \in \mathcal{F}$ such that*

$$F \cup \{by^j : j \in \{0,1,\ldots,k\} \text{ and } y \in F\} \subseteq A.$$

proof: By Theorem 2.3.1 the set $M = \{(b,r) \in S^2 : r, b, br, \ldots, br^k \in A\}$ is central. The family $\mathcal{F}' = \{\{x\} \times F : F \in \mathcal{F} \text{ and } x \in S\}$ is invariant and partition regular in S^2, thus by Lemma 1.4.2 there exist $b \in S$ and $F \in \mathcal{F}$ such that $\{b\} \times F \subseteq M$. Equivalently $F \cup \{by^j : y \in F, j \in \{1,2,\ldots,k\}\} \subseteq A$.
□

Corollary 2.3.3 *Let $k \in \mathbb{N}$ and let A be central in $(\mathbb{N},\cdot)$. Then there exist $a, b, d \in A$ such that*

$$\{a, a+d, \ldots, a+dk\} \cup \{b, bd, \ldots, bd^r\} \cup$$

$$\{b(a+id)^j : i,j \in \{0,1,\ldots,k\}\} \subseteq A.$$

proof: The family $\mathcal{F} = \{\{a, a+d, \ldots, a+dk, d\} : a, d \in \mathbb{N}\}$ is invariant under multiplication in $(\mathbb{N}, \cdot)$. Via Theorem 2.2.3 applied to $(\mathbb{N}, +)$ one sees that $\mathcal{F}$ is partition regular. So Corollary 2.3.2 applies. $\square$

The following corollary is also a consequence of [Ber*i*], Theorem 3.15. (Our conclusion is a little bit stronger in the sense that we may also take $bd^j \in A$.)

Corollary 2.3.4 *Let $k \in \mathbb{N}$ and let A be piecewise syndetic in $(\mathbb{N}, \cdot)$. Then there exist $a, b, d \in \mathbb{N}$ such that*

$$\{b(a+id)^j, bd^j : i, j \in \{0, 1, \ldots, k\}\} \subseteq A.$$

proof: The required structure is a multiplicatively invariant portion of Corollary 2.3.3, so Lemma 1.4.2 applies. $\square$

Corollary 2.3.3 naturally extends to commutative rings. First we need the following Lemma.

Lemma 2.3.5 *Let $(S, +, \cdot)$ be a commutative ring[4] and let $G \subseteq S$ be a finite set. For any finite colouring of S there exist a monochrome set $A \subseteq S$ and $a, d \in A$ such that*

$$\{a + dg : g \in G\} \subseteq A.$$

proof: Let $A \subseteq S$ be an additively central set. Let $(y_n)_{n=1}^{\infty}$ be a sequence in S such that $FS((y_n)_{n=1}^{\infty}) \subseteq A$. Corollary 2.2.2 applied to the sequences $(gx_n)_{n \in \mathbb{N}}, g \in G$ yields that there exist $a \in A$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{t \in F} gx_t \in A$ for all $g \in G$. If we put $d = \sum_{t \in F} x_t$, the statement follows. $\square$

Corollary 2.3.6 *Let $(S, +, \cdot)$ be a commutative ring, let $G \subseteq S$ be a finite set and let A be central in $(S, \cdot)$. Then there exist $a, b, d \in A$ such that*

$$\{a + dg : g \in G\} \cup \{ba^j, bd^j : j \in \{0, 1, \ldots, k\}\} \cup$$
$$\{b(a + dg)^j : g \in G, j \in \{0, 1, \ldots, k\}\} \subseteq A.$$

proof: The family $\mathcal{F} = \{\{a + dg : d \in G\} \cup \{a, d\} : a, d \in S\}$ is invariant under multiplication in $(S, \cdot)$. By Lemma 2.3.5 $\mathcal{F}$ is partition regular. So we may apply Corollary 2.3.2. $\square$

We proceed with another Theorem that allows to intertwine different partition regular structures.

Theorem 2.3.7 *Let $(S, \cdot)$ be a semigroup, let $\mathcal{F}$ be a set of subsets of S with the property that each central subset of S contains a member of $\mathcal{F}$, let $\mathcal{G}$ be a partition regular family of finite subsets of S, and let A be a central subset of S. Then there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $t \in S$ such that $F \cup tGF \subseteq A$.*

---

[4] We call $(S, +, \cdot)$ a commutative ring if $(S, +)$ and $(S, \cdot)$ are commutative semigroups and $\cdot$ distributes over $+$.

proof: Pick a minimal idempotent p of βS with $A \in p$. Then by Theorem 1.4.5 $\{s \in S : s^{-1}A \in p\}$ is syndetic so pick $H \in \mathcal{P}_f(S)$ such that

$$S = \bigcup_{t \in H} t^{-1}\{s \in S : s^{-1}A \in p\}.$$

Pick $G \in \mathcal{G}$ and $t \in H$ such that $G \subseteq t^{-1}\{s \in S : s^{-1}A \in p\}$. Then for each $s \in G$, $(ts)^{-1}A \in p$ so $A \cap \bigcap_{s \in G}(ts)^{-1}A \in p$. Pick $F \in \mathcal{F}$ such that $F \subseteq A \cap \bigcap_{s \in G}(ts)^{-1}A$. $\square$

We remark that the following Corollary could also be derived by Theorem 2.3.13 below.

Corollary 2.3.8 *Let $(S, \cdot)$ be a semigroup, let $\mathcal{F}$ and $\mathcal{G}$ be partition regular families of finite subsets of S. Assume that $\mathcal{F}$ is invariant and let A be a piecewise syndetic subset of S. Then there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $t \in S$ such that $tGF \subseteq A$. If S is commutative, then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $GF \subseteq A$.*

proof: Note that by Lemma 1.4.2 $\mathcal{F}$ has the property that every piecewise syndetic subset of S contains a member of $\mathcal{F}$. In particular every central subset of S contains a member of $\mathcal{F}$. Pick by Theorem 1.4.5 some $x \in S$ such that $x^{-1}A$ is central. Pick by Theorem 2.3.7 some $F \in \mathcal{F}$, $G \in \mathcal{G}$, and $t \in S$ such that $F \cup tGF \subseteq x^{-1}A$. Then $(xt)GF \subseteq A$. $\square$

The following corollary extends a portion of Theorem 2.1.4. Notice that any central set is a piecewise syndetic IP set.

Corollary 2.3.9 *Let A be a piecewise syndetic IP set in $(\mathbb{N}, \cdot)$ with $1 \notin A$ and let $k \in \mathbb{N}$. Then there exist $a, r, d \in A$ such that*

$$\{r^j(a + id) : i, j \in \{0, 1, \ldots, k\}\} \cup \{dr^j : j \in \{0, 1, \ldots, k\}\} \subseteq A.$$

proof: Let $\mathcal{F} = \{br^j : j \in \{0, 1, \ldots, k\}\} : b \in \mathbb{N}$ and $r \in A\}$ and let

$$\mathcal{G} = \{\{d\} \cup \{a + id : i \in \{0, 1, \ldots, k\}\} : a, d \in \mathbb{N}\}.$$

By Theorem 2.2.3, $\mathcal{F}$ and $\mathcal{G}$ are partition regular. And trivially if $F \in \mathcal{F}$ and $t \in \mathbb{N}$, then $tF \in \mathcal{F}$. Pick by Corollary 2.3.8 some $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $GF \subseteq A$. Pick $b \in \mathbb{N}$ and $r \in A$ such that $F = \{br^j : j \in \{0, 1, \ldots, k\}\}$ and pick $a_1, d_1 \in \mathbb{N}$ such that $G = \{d_1\} \cup \{a_1 + id_1 : i \in \{0, 1, \ldots, k\}\}$. Let $a = a_1 b$ and $d = d_1 b$. $\square$

Again we see that Corollary 2.3.9 extends to commutative rings.

Theorem 2.3.10 *Let $(S, +, \cdot)$ be a commutative ring, let A be a piecewise syndetic IP set in $(S, \cdot)$ and let $G \in \mathcal{P}_f(S)$. Then there exist $a, r, d \in A$ such that*

$$\{r^j(a + dg) : g \in G, j \in \{0, 1, \ldots, k\}\} \cup \{ar^j, dr^j : j \in \{0, 1, \ldots, k\}\} \subseteq A.$$

proof: The proof is almost identical to the one of Corollary 2.3.9. Virtually the only difference is that one has to replace $\mathcal{G}$ by the family

$$\mathcal{G}' = \{\{a + dg : g \in G\} \cup \{a, d\} : a, d \in S\}$$

which is partition regular by Lemma 2.3.5. □

We see that we can turn the tables somewhat, translating geometric progressions by arithmetic progressions. (Since addition does not distribute over multiplication, we end up with the four variables a, d, b, and r, rather than just the three of Corollary 2.3.9.)

Corollary 2.3.11 *Let A be a piecewise syndetic IP set in $(\mathbb{N}, +)$ and let $k \in \mathbb{N}$. Then there exist $d \in A$, $a, b \in \mathbb{N}$, and $r \in \mathbb{N} \setminus \{1\}$ such that*

$$\{a + id + br^j : i, j \in \{0, 1, \dots, k\}\} \cup \{a + id + r : i \in \{0, 1, \dots, k\}\} \subseteq A\,.$$

proof: Let $\mathcal{F} = \{\{a + id : i \in \{0, 1, \dots, k\}\} : a \in \mathbb{N}$ and $d \in A\{$ and let

$$\mathcal{G} = \{\{r\} \cup \{br^j : j \in \{0, 1, \dots, k\}\} : b \in \mathbb{N} \text{ and } r \in \mathbb{N} \setminus \{1\}\}\,.$$

Exactly as in the proof of Corollary 2.3.9, $\mathcal{F}$ and $\mathcal{G}$ are partition regular and if $F \in \mathcal{F}$ and $t \in \mathbb{N}$, then $t + F \in \mathcal{F}$. Pick by Corollary 2.3.8 $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $G + F \subseteq A$. Pick $b \in \mathbb{N}$ and $r \in \mathbb{N} \setminus \{1\}$ such that $G = \{r\} \cup \{br^j : j \in \{0, 1, \dots, k\}\}$. Pick $a \in \mathbb{N}$ and $d \in A$ such that $F = \{a + id : i \in \{0, 1, \dots, k\}\}$. □

We do not know whether we can require that any of a, b, or r be in A in Corollary 2.3.11.
We will prove now another Theorem that allows to combine different partition regular families. We need the following refinement of Lemma 1.4.2 that may be of some interest in its own right:

Lemma 2.3.12 *Let S be a semigroup and let $\mathcal{F}$ be a partition regular family of subsets of S. Then*

$$P = \{p \in \beta S : \textit{Every } A \in p \textit{ contains a member of } \mathcal{F}\}$$

is a nonempty closed subset of βS.

(1) *If $\mathcal{F}$ is left invariant then P is a left ideal of βS.*

(2) *If $\mathcal{F}$ is right invariant and all members of $\mathcal{F}$ are finite then P is a right ideal of βS.*

proof: By Theorem 1.2.4 P is non empty and it is not hard to see that P is closed.
To prove (1) assume that $\mathcal{F}$ is left invariant, let $p \in P$ and let $s \in S$. Pick $A \in sp$. Then $s^{-1}A \in p$ and by assumption there exists some $F \in \mathcal{F}$ such

that $F \subseteq s^{-1}A$. Equivalently $sF \subseteq A$. Since A was arbitrary $sp \in P$. Thus we have $SP \subseteq P$. By continuity of $\rho_p : \beta S \to \beta S$ for $p \in \beta S$, P is a left ideal.
Next assume that $\mathcal{F}$ consists of finite sets and is right invariant. Let $p \in P, q \in \beta S$ and $A \in pq$. We have $B = \{s : s^{-1}A \in q\} \in p$. By definition of P there exists some $F \in \mathcal{F}$ such that $F \subseteq B$, thus $s^{-1}A \in q$ for $s \in F$. In particular we may pick $t \in \bigcap_{s\in F} s^{-1}A$. So $Ft \subseteq A$. Since A was arbitrary, we are done. □

Let $\mathcal{F}$ be a partition regular invariant family of finite sets. In the light of Theorem 1.4.3 Lemma 1.4.2 states that any ultrafilter p in the smallest ideal of βS guarantees that its members contain sets of $\mathcal{F}$.
By Lemma 2.3.12 the set of all ultrafilters with this property is an ideal (which necessarily contains the smallest ideal).

Theorem 2.3.13 *Let S be a semigroup, let $\mathcal{F}, \mathcal{G}$ be partition regular families of subsets of S, assume that $\mathcal{F}$ is left invariant, that $\mathcal{G}$ is right invariant and that $\mathcal{F}$ or $\mathcal{G}$ consists of finite sets. Then the family*

$$\{F \cup G \cup GF : F \in \mathcal{F}, G \in \mathcal{G}\}$$

is partition regular.

proof: Without loss of generality we may assume that $\mathcal{G}$ consists of finite sets. Let

$$P_{\mathcal{F}} = \{p \in \beta S : \text{Every } A \in p \text{ contains a member of } \mathcal{F}\},$$

$$P_{\mathcal{G}} = \{p \in \beta S : \text{Every } A \in p \text{ contains a member of } \mathcal{G}\}.$$

By Lemma 2.3.12 $P_{\mathcal{F}}$ is a left ideal of βS and $P_{\mathcal{G}}$ is a right ideal of βS. Pick a minimal left ideal $L \subseteq P_{\mathcal{F}}$, a minimal right ideal $R \subseteq P_{\mathcal{G}}$ and an idempotent $e \in L \cap R$. Let $A \in e$. Then $A^* = \{s \in A : s^{-1}A \in e\} = A \cap \{s \in S : s^{-1}A \in e\} \in e$. Thus we may pick $G \in \mathcal{G}$ such that $G \subseteq A^*$. Then $B = A \cap \bigcap_{s\in G} s^{-1}A \in e$, so pick $F \in \mathcal{F}$ such that $F \subseteq B$. Clearly $F \cup G \subseteq A$. Further $F \subseteq \bigcap_{s\in G} s^{-1}A$ and this is equivalent to $FG \subseteq A$. □

Corollary 2.3.14 *Let $m, k \in \mathbb{N}$ and assume that $\bigcup_{i=1}^m A_i = \mathbb{N}$. There exist $i \in \{1, 2, \ldots, m\}$ and $a, b, d, r \in A_i$ such that*

$$\{a + di, br^j, (a + di)br^j : i, j \in \{0, 1, \ldots, k\}\} \subseteq A_i.$$

proof: Let $\mathcal{F} = \{\{a, a+d, \ldots, a+dk\} : a, d \in \mathbb{N}\}$ and let $\mathcal{G} = \{\{b, br, \ldots, br^k\} : b, r \in \mathbb{N}\}$. $\mathcal{F}$ and $\mathcal{G}$ are invariant under multiplication and partition regular, so by Theorem 2.3.13 there exist $F \in \mathcal{F}$, $G \in \mathcal{G}$ and a cell A_i such that $F \cup G \cup FG \subseteq A_i$. $\square$

As it was mentioned in the previous section, geoarithmetic progressions are strongly partition regular. We show now that configurations of the sort produced by Corollary 2.3.14 are not strongly partition regular.

Theorem 2.3.15 *There is a set $C \subseteq \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exist $b, a, d \in \mathbb{N}$ and $r \in \mathbb{N}\setminus\{1\}$ such that $\{br^n(a+td) : n, t \in \{0,1,\ldots,k\}\} \cup \{br^n : n \in \{0,1,\ldots,k\}\} \cup \{a+td : t \in \{0,1,\ldots,k\}\} \subseteq C$ and there exist sets A_1 and A_2 such that $C = A_1 \cup A_2$ and there do not exist $i \in \{1,2\}$, $c, a, d \in \mathbb{N}$, and $s \in \mathbb{N}\setminus\{1\}$ such that $\{cs, cs^2, cs(a+d), cs^2(a+d), cs(a+2d)\} \subseteq A_i$.*

proof: Let $r_1 = 5$. Inductively choose a prime $r_{k+1} > (r_k(2k+1))^2$. For each $k \in \mathbb{N}$, let $B_k = \{{r_k}^n x : n \in \{1,2,\ldots,k+1\}$ and $x \in \{k+1, k+2, \ldots, 2k+1\}\}$ and let $B = \bigcup_{k=1}^{\infty} B_k$.

Lemma 2.3.16 *If $a, d \in \mathbb{N}$ and $\{a+d, a+2d\} \subseteq B$, then there exist $k \in \mathbb{N}$ and $n \in \{1,2,\ldots,k+1\}$ such that $\{a+d, a+2d\} \subseteq \{{r_k}^n x : x \in \{k+1, k+2, \ldots, 2k+1\}\}$.*

proof: Pick $k \in \mathbb{N}$, $n \in \{1,2,\ldots,k+1\}$, and $x \in \{k+1, k+2, \ldots, 2k+1\}$ such that $a+d = {r_k}^n x$. Then $a+2d < 2(a+d) = 2{r_k}^n x$. Also $2{r_k}^n x < {r_k}^{n+1}(k+1)$ and $2{r_k}^n x < r_{k+1}(k+2)$. The first member of B larger than ${r_k}^n(2k+1)$ is ${r_k}^{n+1}(k+1)$ (if $n \leq k$) or $r_{k+1}(k+2)$ (if $n = k+1$). Thus there is some $y \in \{x+1, x+2, \ldots, k+1\}$ such that $a + 2d = {r_k}^n y$. $\square$

Lemma 2.3.17 *If $c \in \mathbb{N}$, $s \in \mathbb{N}\setminus\{1\}$, and $\{cs, cs^2\} \subseteq B$, then there exist $k \in \mathbb{N}$, $n \in \{0,1,\ldots,k\}$, $t \in \{1,2,\ldots,k+1-n\}$, and $y \in \{k+1, k+2, \ldots, 2k+1\}$ such that $c = {r_k}^n y$ and $s = {r_k}^t$.*

proof: Pick $k \leq m$, $\delta \in \{1,2,\ldots,k+1\}$, $\nu \in \{1,2,\ldots,m+1\}$, $y \in \{k+1, k+2, \ldots, 2k+1\}$, and $z \in \{m+1, m+2, \ldots, 2m+1\}$ such that $cs = {r_k}^\delta y$, and $cs^2 = {r_m}^\nu z$.

Now $s \leq {r_k}^\delta y \leq {r_k}^{k+1}(2k+1)$ and $s = \dfrac{{r_m}^\nu z}{{r_k}^\delta y} > \dfrac{r_m}{{r_k}^{k+1}(2k+1)}$ so

$$r_m < \left({r_k}^{k+1}(2k+1)\right)^2 < r_{k+1}$$

and so $m \leq k$ and thus $m = k$. Therefore $s = {r_k}^{\nu-\delta}\dfrac{z}{y}$. Since r_k is a prime which does not divide y, we must have that y divides z and therefore that $y = z$. Let $t = \nu - \delta$. Since $c{r_k}^{\nu-\delta} = cs = {r_k}^\delta y$ we have $c = {r_k}^{2\delta-\nu} y$. Let $n = 2\delta - \nu$. Since $c = {r_k}^n y$ and $s = {r_k}^t$ we have that $n \geq 0$ and $t \geq 1$. Since $n + t = \delta$ we have that $n + t \leq k+1$. $\square$

To complete the proof of the theorem, let $A_1 = B$, let $A_2 = \{{r_k}^n : k \in \mathbb{N}$ and $n \in \{1,2,\ldots,k+1\}\}$, and let $C = A_1 \cup A_2$. Given $k \in \mathbb{N}$, let

$a = r_k(k+1)$ and let $d = b = r = r_k$. Then for $t, n \in \{0, 1, \ldots, k-1\}$ one has $br^n = r_k{}^{n+1} \in A_2$, $a + td = r_k(k+t+1) \in A_1$, and $br^n(a+td) = r_k{}^{n+2}(k+t+1) \in A_1$.
It is trivial that A_2 does not contain $\{cs(a+d), cs(a+2d)\}$ as the latter element is less than twice the former. Suppose we have some $c, a, d \in \mathbb{N}$ and some $s \in \mathbb{N} \setminus \{1\}$ such that

$$\{cs, cs^2, cs(a+d), cs^2(a+d), cs(a+2d)\} \subseteq A_1 .$$

Pick by Lemma 2.3.17 some $k \in \mathbb{N}$, $n \in \{0, 1, \ldots, k\}$, $t \in \{1, 2, \ldots, k+1-n\}$, and $y \in \{k+1, k+2, \ldots, 2k+1\}$ such that $c = r_k{}^n y$ and $s = r_k{}^t$. Again invoking Lemma 2.3.17, pick some $k' \in \mathbb{N}$, $m \in \{0, 1, \ldots, k'\}$, $t' \in \{1, 2, \ldots, k'+1-m\}$, and $z \in \{k'+1, k'+2, \ldots, 2k'+1\}$ such that $c(a+d) = r_{k'}{}^m z$ and $s = r_{k'}{}^{t'}$.
Since $r_{k'}{}^{t'} = s = r_k{}^t$ we have $k = k'$ and $t = t'$. Pick by Lemma 2.3.16 $k'' \in \mathbb{N}$ and $\nu \in \{1, 2, \ldots, k''+1\}$ such that

$$\{cs(a+d), cs(a+2d)\} \subseteq \{r_{k''}{}^{\nu} w : w \in \{k''+1, k''+2, \ldots, 2k''+1\}\} .$$

Since $cs(a+d) = r_k{}^{t+m} z$ we have $k'' = k$ and $\nu = t+m$. Since $cs = r_k{}^{t+n} y$ we have $a + d = r_k{}^{m-n}\frac{z}{y}$. Since r_k is a prime which does not divide y we have that y divides z so $y = z$ and thus $a + d = r_k{}^{m-n}$.
Pick $w \in \{k+1, k+2, \ldots, 2k+1\}$ such that $cs(a+2d) = r_k{}^{t+m} w$. Then $a + 2d = r_k{}^{m-n}\frac{w}{y}$ so w divides y and thus $a + 2d = r_k{}^{m-n}$. Therefore $d = 0$, a contradiction. □

2.4 Combining additive and multiplicative structure in $\mathbb{N}$

As mentioned in the introductory section for any finite colouring of $\mathbb{N}$ there exists a monochrome set $A \subseteq \mathbb{N}$ that contains arbitrarily long arithmetic progressions, arbitrarily long geometric progressions, and is an additive as well as a multiplicative IP set. This statement follows from the fact that there exists an ultrafilter $q \in \beta\mathbb{N}$ such that every $A \in q$ is additively and multiplicatively central. Since several theorems in this section are based on this fact (or refinements thereof) we will prove this in some detail.
Remember that for $A \subseteq \mathbb{N}$ the upper density of A is given by $\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1,2,\ldots,n\}|}{n}$.

Definition 2.4.1

$$\mathbb{D} = \left\{ p \in \beta\mathbb{N} : \begin{array}{ll} 1) & p \textit{ is a minimal idempotent in } (\beta\mathbb{N}, +), \\ 2) & \overline{d}(A) > 0 \textit{ for all } A \in p. \end{array} \right\}$$

Theorem 2.4.2 *$\overline{\mathbb{D}}$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.*

The following Lemma is crucial for the proof of Theorem 2.4.2:

Lemma 2.4.3 (1) *Let e be an idempotent in $(\beta\mathbb{N}, +)$ and let $n \in \mathbb{N}$. Then ne is also an idempotent in $(\beta\mathbb{N}, +)$.*

(2) *Let $p \in K((\beta\mathbb{N}, +))$ and let $n \in \mathbb{N}$. Then $np \in K((\beta\mathbb{N}, +))$.*

(3) *Let p be an ultrafilter such that every element of p has positive upper density and let $n \in \mathbb{N}$. Then every element of np has positive upper density.*

proof:

(1) Let $A \in ne$. Then $n^{-1}A \in e$. Since e is an idempotent $\{s : -s + n^{-1}A \in e\} \in e$. This implies $n\{s : -s + n^{-1}A \in e\} \in ne$. Let $t \in n\{s : -s + n^{-1}A \in e\}$. Then there exists some $s \in \mathbb{N}$ such that $t = ns$ and $-s + n^{-1}A \in e$. Also $n(-s + n^{-1}A) = -ns + n \cdot n^{-1}A \in ne$. Since $n \cdot n^{-1}A \subseteq A$ this implies $-t + A \in ne$. t was arbitrary, so

$$\{t : -t + A \in ne\} \supseteq n\{s : -s + n^{-1}A \in e\} \in ne.$$

Equivalently $A \in ne + ne$. $A \in ne$ was arbitrary, so we have $ne \subseteq ne + ne$. Since we are dealing with ultrafilters this implies $ne = ne + ne$.

(2) Let $A \in np$. Then $n^{-1}A \in p$ so $\{s : -s + n^{-1}A \in p\}$ is syndetic by Theorem 1.4.5. Clearly this implies that $n\{s : -s + n^{-1}A \in p\}$ is also syndetic and as above we have

$$n\{s : -s + n^{-1}A \in p\} \subseteq \{t : -t + A \in np\}.$$

Since $A \in p$ was arbitrary we may apply Theorem 1.4.5 once more to get $p \in K((\beta\mathbb{N}, +))$.

(3) The family of all sets with positive upper density is invariant under multiplication, so by Lemma 2.3.12 $\mathbb{D}_0 = \{p \in \beta\mathbb{N} : \overline{d}(A) > 0 \text{ for all } A \in p\}$ is a left ideal of $(\beta\mathbb{N}, \cdot)$.

□

We are now ready to give the proof of Theorem 2.4.2:
proof: Similar to the proof of the lemma the family of all sets with positive upper density is invariant under addition, so by Lemma 2.3.12 $\mathbb{D}_0 = \{p \in \beta\mathbb{N} : \overline{d}(A) > 0 \text{ for all } A \in p\}$ is a left ideal of $(\beta\mathbb{N}, +)$. Thus $\mathbb{D} = \mathbb{D}_0 \cap E(K((\beta\mathbb{N}, +))) \neq \emptyset$.

By Lemma 2.4.3 we have $\mathbb{N}\mathbb{D} \subseteq \mathbb{D}$. By continuity of $\lambda_n^{(m)} : \beta\mathbb{N} \to \beta\mathbb{N}$ for $n \in \mathbb{N}$ this implies $\mathbb{N}\overline{\mathbb{D}} \subseteq \overline{\mathbb{D}}$. Then by continuity of the multiplication from the right this gives $\beta\mathbb{N}\overline{\mathbb{D}} \subseteq \overline{\mathbb{D}}$. □

Theorem 2.4.2 has quite nice combinatorial applications: Since $\overline{\mathbb{D}}$ is a left ideal of $(\beta\mathbb{N}, \cdot)$ it contains multiplicative minimal idempotents. Such ultrafilters deserve a special name:

Definition 2.4.4 *Let $q \in \overline{\mathbb{D}}$ be a multiplicative minimal idempotent. Then q is called a* combinatorially rich ultrafilter.[5]

If q is a combinatorially rich ultrafilter, each $A \in q$ is additively and multiplicatively central. In particular for any finite colouring of $\mathbb{N}$ there exists some monochrome set which has this property.

We have just seen that $\overline{K((\beta\mathbb{N},+))} \cap K((\beta\mathbb{N},\cdot))$ is quite large.
It has been known for quite a while that $K((\beta\mathbb{N},+)) \cap K((\beta\mathbb{N},\cdot)) = \emptyset$. Also it is not possible to find ultrafilters which are additively and multiplicatively idempotent. (Actually the only solution of the equation $e + e = ee$ in $\beta\mathbb{N}$ is $e = 2$.) It has been a long open question if $K((\beta\mathbb{N},+)) \cap \overline{K((\beta\mathbb{N},\cdot))} \neq \emptyset$. Having combinatorial applications in mind, it would have been quite interesting to find additive idempotents in $\overline{K((\beta\mathbb{N},\cdot))}$. Only recently both problems were answered negatively in a rather strong way. In [St*i*] Dona Strauss shows that

$$(\beta\mathbb{N} \setminus \mathbb{N} + \beta\mathbb{N} \setminus \mathbb{N}) \cap \overline{K((\beta\mathbb{N},\cdot))} = \emptyset.$$

Since $(\beta\mathbb{N} \setminus \mathbb{N} + \beta\mathbb{N} \setminus \mathbb{N})$ contains $K((\beta\mathbb{N},+))$ as well as all additive idempotents of $\beta\mathbb{N}$ this answers both questions simultaneously.
We turn now to applications of combinatorially large ultrafilters.

Theorem 2.4.5 *Let $\mathcal{F}$ and $\mathcal{G}$ be families of subsets of $\mathbb{N}$ such that each set in a combinatorially large ultrafilter contains a member of $\mathcal{F}$ and a member of $\mathcal{G}$. Assume that $\mathcal{F}$ or $\mathcal{G}$ consists of finite sets. Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r A_i$, there exists $i \in \{1, 2, \ldots, r\}$ such that $d_m^*(A_i) > 0, \overline{d}(A_i) > 0$ and that there exist $B \in \mathcal{F}$ and $C \in \mathcal{G}$ such that*

$$B \cup C \cup B \cdot C \subseteq A_i.$$

proof: Without loss of generality we may assume that $\mathcal{G}$ is finite. Pick a combinatorially rich ultrafilter q. Choose $i \in \{1, 2, \ldots, r\}$ such that $A_i \in q$. The claimed density properties follow immediately since q is combinatorially rich. Since $q = q \cdot q$, $\{x \in A_i : x^{-1}A_i \in q\} \in q$. Pick $B \in \mathcal{G}$ such that

[5]This notion was invented by V. Bergelson and N. Hindman

$B \subseteq \{x \in A_i : x^{-1}A_i \in q\}$. Since B is finite, $A_i \cap \bigcap_{x \in B} x^{-1}A_i \in q$. Pick $C \in \mathcal{F}$ such that $C \subseteq A_i \cap \bigcap_{x \in B} x^{-1}A_i$. $\square$

Theorem 2.4.5 is in part an enhancement of Theorem 2.3.13. To prove Corollary 2.3.14 we applied Theorem 2.3.13 to the multiplicatively invariant families $\{\{a, a+d, \ldots, a+kd\} : a, d \in \mathbb{N}\}$ and $\{\{b, br, \ldots, br^k\} : b, r \in \mathbb{N}\}$. Theorem 2.4.5 is also applicable to the families $\{\{d, a, a+d, \ldots, a+kd\} : a, d \in \mathbb{N}\}$ and $\{\{r, b, br, \ldots, br^k\} : b, r \in \mathbb{N}\}$ where the latter is neither multiplicatively nor additively invariant. We use this in the following Corollary.

Corollary 2.4.6 *Let $m, k \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^m A_i$. Then there exist $i \in \{1, 2, \ldots, m\}$, $a, d \in A_i$, and $r \in A_i \setminus \{1\}$ such that $\overline{d}(A_i) > 0$, $d_m^*(A_i) > 0$, and*

$$\{r^s(a + td) : s, t \in \{0, 1, \ldots, k\}\} \cup \{dr^s : s \in \{0, 1, \ldots, k\}\} \subseteq A_i.$$

proof: Let $\mathcal{G} = \{\{br^s : s \in \{0, 1, \ldots, k\}\} \cup \{r\} : b, r \in \mathbb{N}\}$ and let

$$\mathcal{H} = \{\{a + td : t \in \{0, 1, \ldots, k\}\} \cup \{d\} : a, d \in \mathbb{N}\}.$$

By applying Theorem 2.2.3 to $(\mathbb{N}, \cdot)$ respectively to $(\mathbb{N}, +)$ one achieves that every multiplicatively central set contains a member of $\mathcal{G}$ and that every additively central set contains a member of $\mathcal{H}$. Thus we may apply Theorem 2.4.5. By assigning 1 to its own cell one may ensure that $r \neq 1$. Put $a_1 = ab$ and $d_1 = db$. Then for some $i \in \{1, 2, \ldots, m\}$, $\{a_1\} \cup \{d_1\} \cup \{r^s(a_1 + td_1) : s, t \in \{0, 1, \ldots, k\}\} \cup \{d_1 r^s : s \in \{0, 1, \ldots, k\}\} \subseteq A_i$ $\square$

Lemma 2.4.7 *Let S be a commutative semigroup. Let L be a minimal left ideal of βS. Let $\mathcal{F}$ be a family of finite subsets of S such that the family*

$$\{bF : F \in \mathcal{F}, b \in S\}$$

is partition regular. Let $A \subseteq S$ such that $\overline{A} \cap L \neq \emptyset$. Then there exists $F \in \mathcal{F}$ such that

$$L \cap \bigcap_{y \in F} \overline{y^{-1}A} \neq \emptyset.$$

proof: Pick $v \in \overline{A} \cap L$. Pick a minimal right ideal R of βS such that $v \in R$ and pick an idempotent $u \in R$. Then $v = uv$ so

$$B = \{x \in S : x^{-1}A \in v\} \in u\,.$$

In particular B is central so pick by Lemma 2.2.2, some $b \in S$ and $F \in \mathcal{F}$ such that $bF \subseteq B$. So for each $y \in F$, $(by)^{-1}A \in v$. Equivalently for each $y \in F$, $y^{-1}A \in bv$. Since $bv \in L$, we are done. $\square$

If S has an identity the following version of Lemma 2.4.7 follows directly from the original statement. If it doesn't, at least its proof is almost identical to the one of 2.4.7, so we skip it.

Lemma 2.4.8 *Let S be a commutative semigroup. Let L be a minimal left ideal of βS. Let $\mathcal{F}$ be a family of finite subsets of S such that the family*

$$\{\{b\} \cup bF : F \in \mathcal{F}, b \in S\}$$

is partition regular. Let $A \subseteq S$ such that $\overline{A} \cap L \neq \emptyset$. Then there exists $F \in \mathcal{F}$ such that

$$\overline{A} \cap L \cap \bigcap_{y \in F} \overline{y^{-1}A} \neq \emptyset.$$

Corollary 2.4.9 *Let S be a semigroup. Let L be a minimal left ideal of βS and let $A \subseteq S$ such that $\overline{A} \cap L \neq \emptyset$. Let $k \in \mathbb{N}$. Then $R = \{r \in S : \overline{A} \cap \bigcap_{i=1}^{k} \overline{(r^k)^{-1}A} \cap L \neq \emptyset\}$ intersects any IP set. In particular if $S = (\mathbb{N}, +)$, R is syndetic.*

Before we prove the corollary, we want to recall from chapter 1, section 5 that a set which intersects any IP set is called an IP* set. The family of all IP* sets is closed under finite intersections. In ω respectively in $\mathbb{N}$ the set $k\omega$ respectively $k\mathbb{N}$ is an IP* set for each $k \in \mathbb{N}$. Furthermore IP* sets in ω and in $\mathbb{N}$ are syndetic.
After this lengthy remark we will append the proof of Corollary 2.4.9:

proof: Let B be an IP set. By Corollary 2.2.2 the family

$$\mathcal{G} = \{\{b, br, \ldots, br^k\} : b \in \mathbb{N}, r \in B\}$$

is partition regular. Thus by Lemma 2.4.8 there exists $r \in B$ such that $\overline{A} \cap \bigcap_{i=1}^{k} \overline{(r^i)^{-1}A} \cap L \neq \emptyset$. $\square$

In attempting to derive results about geoarithmetic progressions, the approach that one might try first after a little experience in deriving Ramsey Theoretic consequences of the algebra of $\beta\mathbb{N}$ would be to choose an appropriate idempotent q in $(\beta\mathbb{N}, \cdot)$ and show that if $A \in q$, then there is some r, preferably in A, such that $\bigcap_{s=0}^{k} (r^s)^{-1}A \in q$. The corresponding statement would lead to nicer versions of Lemma 2.4.7 (respectively Lemma 2.4.8 and Corollary 2.4.9). We show now that such an approach is doomed to failure.

Theorem 2.4.10

(a) For all $q \in \beta\mathbb{N}$, there exists a partition $\{A_0, A_1\}$ of $\mathbb{N}$ such that for all $i \in \{0, 1\}$ and all $x \in \mathbb{N}$, $(-x + A_i) \cap (-2x + A_i) \notin q$. In particular there exists $A \in q$ such that for all $x \in \mathbb{N}$, either $-x + A \notin q$ or $-2x + A \notin q$.

(b) There does not exist $q \in \beta\mathbb{N}$ such that for each $A \in q$ there is some $r \in \mathbb{N} \setminus \{1\}$ with $r^{-1}A \in q$ and $(r^2)^{-1}A \in q$.

proof: (a) Let $q \in \beta\mathbb{N}$. Then $q + \beta\mathbb{N}$ is a right ideal of $(\beta\mathbb{N}, +)$ so there is an additive idempotent in $q + \beta\mathbb{N}$. Pick $r \in \beta\mathbb{N}$ such that $q + r$ is an idempotent in $(\beta\mathbb{N}, +)$. Then $q + r \in \bigcap_{n=1}^{\infty} \overline{\mathbb{N}2^n}$ by [HS98], Lemma 6.6.

Define $f : \mathbb{N} \to \omega = \mathbb{N} \cup \{0\}$ by $f(n) = \min F$ where $F \in \mathcal{P}_f(\omega)$ and $n = \sum_{t \in F} 2^t$. Then f has a continuous extension $\widetilde{f} : \beta\mathbb{N} \to \beta\omega$. For $i \in \{0,1\}$ let $A_i = \{x \in \mathbb{N} : (2\mathbb{N} - i) \in \widetilde{f}(x+r)\}$.
Let $i \in \{0,1\}$ and let $x \in \mathbb{N}$ and suppose that $(-x + A_i) \cap (-2x + A_i) \in q$. Pick $j, k \in \omega$ such that $x = 2^j(2k+1)$. Denote addition of z on the left in $\beta\mathbb{N}$ by λ_z and addition of z on the right by ρ_z. Then $\widetilde{f} \circ \lambda_x$ is constantly equal to $f(x)$ and $\widetilde{f} \circ \lambda_{2x}$ is constantly equal to $f(x) + 1$ on $\mathbb{N}2^{j+2}$, which is a member of $q + r$. So $\widetilde{f}(x + q + r) = f(x)$ and $\widetilde{f}(2x + q + r) = f(x) + 1$. Therefore $\widetilde{f} \circ \lambda_x \circ \rho_r(q) = f(x)$ and $\widetilde{f} \circ \lambda_{2x} \circ \rho_r(q) = f(x) + 1$ so

$$\{y \in \mathbb{N} : \widetilde{f}(x + y + r) = f(x) \text{ and } \widetilde{f}(2x + y + r) = f(x) + 1\} \in q$$

so pick $y \in (-x + A_i) \cap (-2x + A_i)$ such that $\widetilde{f}(x + y + r) = f(x)$ and $\widetilde{f}(2x + y + r) = f(x) + 1$.
Since $x + y \in A_i$, we have that $2\mathbb{N} - i \in \widetilde{f}(x+y+r) = f(x)$ so $f(x) + i \in 2\mathbb{N}$. (Recall that we are identifying points of $\mathbb{N}$ with the principle ultrafilters they generate.) Since $2x + y \in A_i$, we have that $2\mathbb{N} - i \in \widetilde{f}(2x+y+r) = f(x) + 1$ so $f(x) + i + 1 \in 2\mathbb{N}$, a contradiction.
(b) For $x \in \mathbb{N}\backslash\{1\}$, let $\ell(x)$ be the number of terms in the prime factorization of x. Then ℓ is a homomorphism from $(\mathbb{N} \setminus \{1\}, \cdot)$ onto $(\mathbb{N}, +)$ and so its continuous extension $\widetilde{\ell} : (\beta\mathbb{N} \setminus \{1\}, \cdot) \to (\beta\mathbb{N}, +)$ is also a homomorphism. □

We will see in Corollary 2.4.12 that Lemma 2.4.7 is still very useful for our purposes.
We know that $\overline{\mathbb{D}}$ is a left ideal of $(\beta\mathbb{N}, \cdot)$. Pick a minimal left ideal L that is contained in $\mathbb{D}$ and a combinatorially rich ultrafilter $p \in L$. In Theorem 2.4.11 and in Corollary 2.4.12 let $A \subseteq \mathbb{N}$ be such that $\overline{A} \in p$. Since in any finite partition $\{A_1, \ldots, A_m\}$ there is one cell A_i such that $A_i \in p$ the partition versions of these Theorems are also valid.

Theorem 2.4.11 *Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N}$ such that the family $\{bF : F \in \mathcal{F}, b \in \mathbb{N}\}$ is partition regular and let $\mathcal{G}$ be a family of subsets of $\mathbb{N}$ such that any set which is central in $(\mathbb{N}, +)$ contains a member of $\mathcal{G}$. Then there exist $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that*

$$\overline{d}\left(\bigcap_{y \in F} y^{-1}A\right) > 0, d^*_m\left(\bigcap_{y \in F} y^{-1}A\right) > 0 \text{ and } FG \subseteq A.$$

proof: Pick, by Lemma 2.4.7, $F \in \mathcal{F}$ such that $L \cap \overline{\bigcap_{y \in F} y^{-1}A} \neq \emptyset$. Since $L \subseteq K(\beta\mathbb{N}, \cdot)$, $d^*_m(\bigcap_{y \in F} y^{-1}A) > 0$. Since $L \subseteq \overline{\mathbb{D}}$, pick $q \in \mathbb{D}$ such that q is a minimal idempotent of $(\beta\mathbb{N}, +)$ and $\bigcap_{y \in F} y^{-1}A \in q$. Then this set is central in $(\mathbb{N}, +)$ so pick $G \in \mathcal{G}$ such that $G \subseteq \bigcap_{y \in F} y^{-1}A$. Since $q \in \mathbb{D}$, $\overline{d}(\bigcap_{y \in F} y^{-1}A) > 0$. □

Corollary 2.4.12 *Let $k \in \mathbb{N}$.*

(a) For each $i \in \{1,2,\dots,k\}$ let $(x_{i,t})_{t=1}^{\infty}$ and $(y_{i,t})_{t=1}^{\infty}$ be sequences in $\mathbb{N}$. Then there exist $H_m, H_a \in \mathcal{P}_f(\mathbb{N})$ and $a \in A$ such that

$$\left\{a + \sum_{t\in H_a} x_{i,t} : i \in \{1,2,\dots,k\}\right\} \cup \left\{a \cdot \prod_{t\in H_m} y_{j,t} : i \in \{1,2,\dots,k\}\right\} \cup$$
$$\left\{\left(a + \sum_{t\in H_a} x_{i,t}\right) \cdot \prod_{t\in H_m} y_{j,t} : i,j \in \{1,2,\dots,k\}\right\} \subseteq A.$$

(b) There exist $a, r, d \in A$ such that $r > 1$ and

$$\{(a+id)r^j : i,j \in \{0,1,\dots,k\}\} \cup \{dr^j : j \in \{0,1,\dots,k\}\} \subseteq A.$$

(c) There exist $a, r, d \in A$ such that $r > 1$ and

$$\{(a+id)j^r : i \in \{0,1,\dots,k\}, j \in \{1,2,\dots,k\}\} \cup$$
$$\{dj^r : j \in \{1,2,\dots,k\}\} \subseteq A.$$

proof: 1 is not contained in any minimal left ideal of $(\mathbb{N},\cdot)$. Thus by considering $A \setminus \{1\}$ instead of A we may assume that $1 \notin A$. Let

$\mathcal{F}_1 = \{\{1\} \cup \{\prod_{t\in H} y_{i,t} : i \in \{1,2,\dots,k\}\} : H \in \mathcal{P}_f(\mathbb{N})\}$,
$\mathcal{G}_1 = \{\{a\} \cup \{a + \sum_{t\in H} x_{i,t} : i\imath\{1,2,\dots,k\} : H \in \mathcal{P}_f(\mathbb{N}), a \in \mathbb{N}\}$,
$\mathcal{F}_2 = \{\{r^i : i \in \{0,1,\dots,k\}\} : r \in A\}$
$\mathcal{G}_2 = \{\{d\} \cup \{a + di : i \in \{0,1,\dots,k\}\} : a,d \in \mathbb{N}\}$

and put $\mathcal{F}_i' = \{bF : b \in \mathbb{N}, F \in \mathcal{F}_i\}$ for $i \in \{1,2\}$. By applying Corollary 2.2.2 respectively Corollary 2.2.3 to the semigroup $(\mathbb{N},\cdot)$ we see that the families $\mathcal{F}_1$ and $\mathcal{F}_2$ are partition regular. Similarly by Corollary 2.2.2 respectively Corollary 2.2.3 applied to the semigroup $(\mathbb{N},+)$, every subset of $\mathbb{N}$ that is central in $(\mathbb{N},+)$ contains a member of $\mathcal{G}_1$ and a member of $\mathcal{G}_2$. Thus we get (a) by applying Theorem 2.4.11 to $\mathcal{F}_1$ and $\mathcal{G}_1$ and (b) by applying Theorem 2.4.11 to $\mathcal{F}_2$ and $\mathcal{G}_2$.
We will prove (c) by using Theorem 2.4.11 with $\mathcal{F}_1$ and $\mathcal{G}_2$, where we define the sequences $(y_{i,n})_{n=1}^{\infty}$, $i \in \{1,2,\dots,k\}$ appropriately. A is central in $(\mathbb{N},\cdot)$, so choose a sequence $(r_n)_{n=1}^{\infty}$ such that $FS((r_n)_{n=1}^{\infty}) \subseteq A$. Using this put $y_{i,n} = i^{r_n}$ for $i \in \{1,2,\dots,k\}$ and $n \in \mathbb{N}$. By Theorem 2.4.11 we find $a, d \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $G = \{d\} \cup \{a + id : i \in \{0,1,\dots,k\}\}$ and $F = \{\prod_{t\in H} y_{j,t} : j \in \{1,2,\dots,k\}\}$ satisfy the conclusion of Theorem 2.4.11 Let $r = \sum_{t\in H} r_t \in A$. Then for $j \in \{1,2,\dots,k\}$, $\prod_{t\in H} y_{j,t} = \prod_{t\in H} j^{r_t} = j^r$. Thus we see that (c) is valid. □

We now turn our attention to showing that one cannot simultaneously let $F = G$ and eliminate the multiplier b in Theorem 2.2.8.

The following theorem is of interest in its own right. Recall from Corollary 2.2.3 that when $\mathbb{N}$ is finitely colored, one can find arbitrarily long monochrome arithmetic progressions with increments chosen from any IP set. This theorem tells us that at least relatively thin sequences cannot replace IP sets.

Theorem 2.4.13 *Let $(d_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $3d_n \leq d_{n+1}$. There exists a partition $\{A_0, A_1, A_2, A_3\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0,1,2,3\}$ and $a, k \in \mathbb{N}$ with $\{a, a+d_k\} \subseteq A_s$.*

proof: For $\alpha \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ we denote by $\|\alpha\|$ the distance to the nearest integer. We will not distinguish strictly between equivalence classes and their representatives in $[0,1)$.

Lemma 2.4.14 *There exists $\alpha \in \mathbb{T}$ such that $\|\alpha d_n\| \geq 1/4$ for each $n \in \mathbb{N}$.*

proof: For each $n \in \mathbb{N}$ put $R_n = \{\alpha \in \mathbb{T} : \|\alpha d_n\| \geq 1/4\}$. Each R_n consists of intervals of length $1/2d_n$ which are separated by gaps of the same length. Since $d_{n+1} \geq 3d_n$ every interval of R_n is 3 times longer than an interval or a gap of R_{n+1}. Thus any interval of R_n contains an interval of R_{n+1}. This shows that for each $N \in \mathbb{N}$, $\bigcap_{n=1}^{N} R_n \neq \emptyset$. By compactness of $\mathbb{T}$ there exists $\alpha \in \bigcap_{n=1}^{\infty} R_n$ and this proves the Lemma. $\square$

Let $\alpha \in \mathbb{T}$ such that $d_n\alpha \in (1/4, 3/4)$ for each $n \in \mathbb{N}$. For $i \in \{0,1,2,3\}$ put $A_i = \{m \in \mathbb{N} : m\alpha \in [i/4, (i+1)/4)\}$. Then for any $a, k \in \mathbb{N}$ $\alpha(a + d_k) = \alpha a + \beta$ for some $\beta \in (1/4, 3/4)$ and thus αa and $\alpha(a+d_k)$ must not lie in the same quarter of $[0,1)$. Equivalently there exists no $s \in \{0,1,2,3\}$ such that $\{a, a+d_k\} \subseteq A_s$. $\square$

We remark that Lemma 2.4.14 is well known. Under the much weaker assumption, that the growth rate of the sequence $(d_n)_{n=1}^{\infty}$ is bounded from below by some $q > 1$ B. de Mathan [Ma80] and A. Pollington [P79] show that there exists some $\alpha \in \mathbb{T}$ such that $\{\alpha n : n \in \mathbb{N}\}$ is not dense in $\mathbb{T}$. In order to give a self contained proof we have chosen to go with the weaker statement. The loss is that we have to make an additional step to show that any growth rate $q > 1$ is sufficient to avoid monochrome arithmetic progressions with some d_k as increment.

Corollary 2.4.15 *Let $q \in \mathbb{R}$ with $q > 1$ and assume that $(d_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $qd_n \leq d_{n+1}$. There exists a finite partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ such that there do not exist $s \in \{1, 2, \ldots, r\}$ and $a, k \in \mathbb{N}$ with $\{a, a+d_k\} \subseteq A_s$.*

proof: Pick $m \in \mathbb{N}$ such that $q^m \geq 3$. For $t \in \{0, 1, \ldots, m-1\}$ and $n \in \mathbb{N}$, let $c_{t,n} = d_{nm-t}$. Given $t \in \{0, 1, \ldots, m\}$ one has that $3c_{t,n} \leq c_{t,n+1}$ for each n so pick by Theorem 2.4.13 some $\{B_{t,0}, B_{t,1}, B_{t,2}, B_{t,3}\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0,1,2,3\}$ and $a, k \in \mathbb{N}$ with $\{a, a+c_{t,k}\} \subseteq B_{t,i}$. Let $r = 4^m$

and define a partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ with the property that x and y lie in the same cell of the partition if and only if $x \in B_{t,i} \Leftrightarrow y \in B_{t,i}$ for each $t \in \{0, 1, \ldots, m-1\}$ and each $i \in \{0, 1, 2, 3\}$. $\square$

Corollary 2.4.16 *There exist sequences $(x_{i,n})_{n=1}^{\infty}$, $i \in \{0,1\}$, $(y_n)_{n=1}^{\infty}$ and a partition $\{A_0, A_1, A_2, A_3\}$ of $\mathbb{N}$ such that there do not exist $s \in \{0,1,2,3\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a \in \mathbb{N}$ with $\{(a + \sum_{n\in F} x_{i,n}) \prod_{n\in F} y_n : i \in \{0,1\}\} \subseteq A_s$.*

proof: For each $n \in \mathbb{N}$ let $x_{0,n} = 0$, $x_{1,n} = 1$ and $y_{i,n} = 3$. For each $n \in \mathbb{N}$, let $d_n = n3^n$. Pick A_0, A_1, A_2, A_3 as guaranteed by Theorem 2.4.13. Suppose one has $F \in \mathcal{P}_f(\mathbb{N})$ and $a \in \mathbb{N}$ with $\{(a + \sum_{n\in F} x_{i,n}) \prod_{n\in F} y_n : i \in \{0,1\}\} \subseteq A_s$. Let $n = |F|$. Then for $i \in \{0,1\}$, $\sum_{n\in F} x_{i,n} \prod_{n\in F} y_n = id_n$, a contradiction. $\square$

We have already shown that one cannot eliminate the multiplier b from Theorem 2.2.9. We show now that this multiplier cannot be eliminated from Corollary 2.2.10 even is 'piecewise syndetic' is replaced by 'thick'. (Recall that thick sets in any semigroup are also piecewise syndetic, in fact central.)

Theorem 2.4.17 *There exists a set A which is thick in $(\mathbb{N}, \cdot)$ and a sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{N}$ with the property that there do not exist $a \in \mathbb{N}$ and $d \in FS((x_n)_{n=1}^{\infty})$ with $\{a, a+d\} \subseteq A$.*

proof: Let $A = \bigcup_{n=1}^{\infty} \{(3n)!, 2(3n)!, \ldots, n(3n)!\}$ and for each n, let $x_n = (3n+1)!$. Observe that A is thick in $(\mathbb{N}, \cdot)$. Let $a \in A$ and let $d \in FS((x_n)_{n=1}^{\infty})$. We shall show that $a+d \notin A$. Pick $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$ such that $a = k(3n)!$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $d = \sum_{t\in F} x_t$ and let $m = \max F$. Then $(3m+1)! \leq d < (3m+2)!$.
If $m < n$ we have $k(3n)! < a+d < (k+1)(3n)!$ so $a+d \notin A$. If $m \geq n$, then $a < (3m+1)!$ so $(3m+1)! < a+d < (3m+3)!$ and thus $a+d \notin A$. $\square$

It was shown in [Ber*i*], Theorem 1.3 that the fact that a subset A of $\mathbb{N}$ satisfies $d_m^*(A) > 0$ is enough to guarantee that A contains arbitrarily large geoarithmetic progressions. However, by considering the set $A = \{x \in \mathbb{N} :$ the number of terms in the prime factorization of x is odd$\}$, one sees that the fact that $\overline{d_m}(A) > 0$ is not enough to guarantee geoarithmetic progressions together with the common ratio r, nor together with both b and a.

During this chapter we presented numerous examples of partition regular structures which are highly organised in a multiplicative as well as in an additive sense. In fact several Theorems state that any set which is multiplicatively large in an appropriate sense contains multiplicative *and additive* structure. In contrast to this additively large set usually contain very little multiplicative structure. For example there exists an additively thick set $A \subseteq \mathbb{N}$ with $\overline{d}(A) = 3/4$ which does not contain a three term geometric progression ([BBHS*ii*]).

We conclude this chapter with an example of a Theorem that guarantees that at least some multiplicative structure can be found in an additively large set:

Theorem 2.4.18 *Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$, let $a \in \mathbb{Z}$ and put*

$$B = a + FS((x_n)_{n=1}^{\infty}) \cap \mathbb{N}.$$

There exists a sequence $(y_n)_{n=1}^{\infty}$ in $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $\prod_{k=1}^{n} y_k \in B$ and that $y_{n+1} \equiv 1(mod\ \prod_{k=1}^{n} y_k)$. In particular the elements of $(y_n)_{n=1}^{\infty}$ are pairwise relative prime.

proof: We claim that for each $s \in B$ there exists $t \in \mathbb{N}$ such that $t \equiv 1(\bmod s)$ and $st \in B$: To see this pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $s = a + \sum_{t \in F} y_t$ and let $k = \max F + 1$. Pick $w \in FS((x_n)_{n=k}^{\infty}) \cap \mathbb{N}s^2$ and let $u = w/s$. Note that $s|u$. Let $t = 1 + u$. Then

$$st = s + su = s + w = a + \sum_{k \in F} x_k + w \in B.$$

The claim being established it is easy to inductively construct the sequence $(y_n)_{n=1}^{\infty}$: Start with any $y_1 \in B$. If $y_1, y_2, \ldots, y_n$ have already been constructed put $s = \prod_{k=1}^{n} y_k$ and determine t as described above. Finally let $y_{n+1} = t$. □

Chapter 3

Variations on the Central Sets Theorem

This chapter is devoted to the Central Sets Theorem and several generalizations of it. The Central Sets Theorem for the semigroup $(\mathbb{Z}, +)$ is due to Fürstenberg [F81], Proposition 8.21. See [HS98], Part III for a strengthening that applies to general commutative semigroups and numerous combinatorial applications of the Central Sets Theorem. To provide some flavour of the Theorem, we give an easy stated consequence of it:

Theorem 3.0.1 *Let $r \in \mathbb{N}$ and assume that $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $s \in \{1, 2, \ldots, r\}$ and sequences $(a_n)_{n=0}^{\infty}, (d_n)_{n=0}^{\infty}$ in $\mathbb{N}$ such that for each $H \in \mathcal{P}_f(\omega)$*

$$\sum_{t \in H} \{a_t, a_t + d_t, \ldots, a_t + td_t\} \subseteq A_s.$$

We see that the Central Sets Theorem naturally extends and intertwines the Theorems of Hindman and van der Waerden: Not only that finite sums from some sequence and arbitrarily long arithmetic progressions are contained in one cell of a finite partition. In fact all finite sums from members of the arithmetic progressions are provided to be in this cell.
In section 3.1 we state and prove the Central Sets Theorem. Some connections with methods of chapter 2 are pointed out and easy extensions along the lines of the Central Sets Theorem are given.
Section 3.2 is devoted to a common generalization of Ramsey's Theorem and the Central Sets Theorem. In the course of the proof we also derive a strong version of the Milliken-Taylor Theorem.
In section 3.3 we review the notion of a *partial semigroup* introduced in [BBH94]. This enables us to prove strong generalizations of the Central Sets Theorem as well as of some theorems of the previous chapter.

3.1 An introduction to the Central Sets Theorem

To prove Hindman's Theorem one inductively constructs a sequence $(x_n)_{n=1}^{\infty}$ such that $FS((x_n)_{n=1}^{\infty})$ is contained in some set that is large in an appropriate sense. The key idea behind the proof of the Central Sets Theorem is the observation that in the ultrafilter proof of Hindman's Theorem, whenever some x_n is chosen, there is actually a huge amount of possible choices.
In order to give these comments a rigorous meaning we formalize the notion of a tree:

Definition 3.1.1 *Let S be a set and put $S^{<\omega} = \bigcup_{n=0}^{\infty} S^{\{0,\dots,n-1\}}$. A non empty set $T \subseteq S^{<\omega}$ is a* tree *in S iff for all $f \in S^{<\omega}$, $g \in T$ such that $dom\ f \subseteq dom\ g$, $g_{\restriction dom\ f} = f$ one has $f \in T$. For $f \in S^{<\omega}$ we put*

$$T(f) = \{s \in S : f^\frown s \in T\}.$$

We will identify the function $f \in S^{\{0,\dots,n-1\}}$ with the finite sequence $(f(0),\dots, f(n-1))$. For $s \in S$ put $f^\frown s = (f(0),\dots,f(n-1),s)$.
If S is a semigroup, we put

$$FP(T) = \left\{ \prod_{t\in F} f(t) : f \in T, f \neq \emptyset, F \subseteq dom\ f \right\}.$$

Lemma 3.1.2 *Let S be a semigroup, let e be an idempotent in βS and assume that $A \subseteq e$. There exists a tree T in S such that:*

(1) *For all $f \in T$, $T(f) \in e$.*

(2) *$FP(T) \subseteq A$.*

proof: We will inductively construct a sequence of trees $(T_n)_{n=0}^{\infty}$, satisfying $T_n = \{f_{\restriction\{0,\dots,n-1\}} : f \in T_{n+1}\}$ for each $n \in \omega$, such that

(1) If $f \in T_n$ satisfies $dom\ f \subseteq \{0,\dots,n-2\}$ then $T_n(f) \in e$.

(2) For all $f \in T_n$, for all $H \subseteq dom\ f$, $\prod_{t\in H} f(t) \in A$.

(3) For all $f \in T_n$, for all $H \subseteq dom\ f$, $\left(\prod_{t\in H} f(t)\right)^{-1} A \in e$.

To start with this construction we put $T_0 = \{\emptyset\}$, such that T_0 is a tree in S which trivially satisfies (1) - (3).
Assume that $T_0,\dots,T_n$ are trees satisfying $T_j = \{f_{\restriction\{0,\dots,j-1\}} : f \in T_{j+1}\}$ for $j < n$, such that (1) - (3) hold. For $f \in T_n$, $dom\ f = \{0,\dots,n-1\}$ we put

$$E_f = A \cap \bigcap_{H\subseteq dom\ f} \left(\prod_{t\in H} f(t)\right)^{-1} A.$$

Using this define $T_{n+1}(f) = E_f \cap \{s \in S : s^{-1}E_f \in e\}$. Finally we put

$$T_{n+1} = T_n \cup \{f^\frown s : f \in T_n, dom\ f = \{0, \ldots, n-1\}, s \in T_{n+1}(f)\}.$$

Obviously T_{n+1} is a tree such that $T_n = \{f_{\restriction\{0,\ldots,n-1\}} : f \in T_{n+1}\}$.
We need to check that (1) - (3) are satisfied. Let $f \in T_n$ with $dom\ f = \{0, \ldots, n-1\}$ be fixed.
E_f is defined as an intersection of finitely many sets, all of which are contained in e by the hypothesis of the induction. e is idempotent, so for any set $B \subseteq S$ one has $B \in e = ee \Leftrightarrow \{s : s^{-1}B \in e\} \in e$. Applying this to the set E_f we see that $T_{n+1}(f) \in e$. So (1) is satisfied.
To prove (2) and (3) let $s \in T(f), \tilde{f} = f^\frown s$ and $H \subseteq \{1, 2, \ldots, n\}, H \neq \emptyset$. If $\max H < n$, the claim follows trivially from the hypothesis of the induction. If not, (2) follows since $s \in E_f$ and (3) follows since $s^{-1}E_f \in e$.
The construction of the $T_n, n \in \omega$ being complete, we put $T = \bigcup_{n=0}^{\infty} T_n$. Via properties (1) and (2) of $T_n, n \in \omega$ one sees that T is as claimed. $\square$

Theorem 3.1.3 (Central Sets Theorem) *Let S be a commutative semigroup, let $A \subseteq S$ be central and let $g : \omega \to \omega$ be an arbitrary function. For each $l \in \omega$, let $(y_{l,n})_{n=0}^{\infty}$ be a sequence in S. There exist a sequence $(a_n)_{n=0}^{\infty}$ in S and a sequence $H_0 < H_1 < \ldots$ in $\mathcal{P}_f(\omega)$ such that for each sequence $(i_n)_{n=0}^{\infty}$ in ω satisfying $i_n \leq g(n)$ for all $n \in \omega$*

$$FP\left(\left(a_n + \sum_{t \in H_n} y_{i_n,t}\right)_{n=0}^{\infty}\right) \subseteq A.$$

proof: Fix a minimal idempotent $e \in \beta S \setminus S$. Denote by Φ the set of all sequences $(i_n)_{n=0}^{\infty}$ satisfying $i_n \leq g(n)$ for all $n \in \omega$. Let $T \subseteq S^{<\omega}$ be as provided by Lemma 3.1.2. We will inductively construct sequences $(a_n)_{n=0}^{\infty}$ in S and $H_0 < H_1 < \ldots$ in $\mathcal{P}_f(\omega)$ such that for all $n \in \omega$ and all sequences $(i_n)_{n=0}^{\infty} \in \Phi$

$$\left(a_0 + \textstyle\sum_{t \in H_0} y_{i_0,t}, \ldots, a_n + \sum_{t \in H_n} y_{i_n,t}\right) \in T. \tag{3.1}$$

By the properties of T this is sufficient to proof the Theorem.
Assume that $a_0, \ldots, a_{n-1} \in S$ und $H_0 < \ldots < H_{n-1} \in \mathcal{P}_f(\omega)$ have already been constructed such that (3.2) is true for all $(i_n)_{n=0}^{\infty} \in \Phi$. We have

$$G_n = \bigcap_{(i_n)_{n=0}^{\infty} \in \Phi} T\left(\left(a_0 + \sum_{t \in H_0} y_{i_0,t}, \ldots, a_{n-1} + \sum_{t \in H_{n-1}} y_{i_{n-1},t}\right)\right) \in e.$$

Let $m = \max H_{n-1}$. By applying Theorem 2.2.2 to the set G_n and the sequences $(y_{0,k})_{k=m+1^\infty}, \ldots, (y_{g(n),k})_{k=m+1}^{\infty}$ we find $a_n \in S$ und $H_n \in \mathcal{P}_f(\omega)$, $H_n > H_{n-1}$ such that $a_n + \sum_{t \in H_n} y_{0,t}, \ldots, a_n + \sum_{t \in H_n} y_{g(n),t} \in G_n$.

Thus for all $(i_n)_{n=0}^{\infty} \in \Phi$, $\left(a_0 + \sum_{t \in H_0} y_{i_0,k}, \ldots, a_n + \sum_{t \in H_n} y_{i_n,k}\right) \in T$, as we wanted to show. $\square$

We will shortly describe how Theorem 3.0.1 follows from Theorem 3.1.3: Take $S = (\mathbb{N}, +)$ and put $g(n) = n$ for $n \in \omega$. Further specify $y_{l,n} = l$ for $l, n \in \omega$. One cell A_s of the partition is central, so let $(a_n)_{n=0}^{\infty}$ and $(H_n)_{n=0}^{\infty}$ be as supplied by Theorem 3.1.3. Now Theorem 3.0.1 follows if we put $d_n = |H_n|$ for $n \in \omega$.
In almost the same fashion as the Central Sets Theorem, we may derive the following corollary from Lemma 3.1.2:

Corollary 3.1.4 *Let S be a semigroup, let e be an idempotent of βS, let $A \in e$ and assume that for each $n \in \mathbb{N}$, $\mathcal{F}_n$ is a family of finite subsets of S such that each set in e contains some member of $\mathcal{F}_n$. Then there exists a sequence $(F_n)_{n=1}^{\infty}$ where each $F_n \in \mathcal{F}_n$ such that for any $G \in \mathcal{P}_f(\mathbb{N})$*

$$\sum_{t \in G} F_t \subseteq A.$$

We want to remark that Corollary 3.1.4 vastly extends Theorem 2.3.13 and Theorem 2.4.5 if one takes e to be the ultrafilter that is employed in the proof of Theorem 2.3.13 respectively Theorem 2.4.5 and assumes that $\mathcal{F}$ and $\mathcal{G}$ consist of finite sets. These theorems state that configurations of the type $\{x, y, yx\}$ where x varies over some member of $\mathcal{F}$ and y varies over some member of $\mathcal{G}$ are contained in a central set A. Under similar assumptions Corollary 3.1.4 provides that all finite sums from an infinite sequence are contained in a large set.

Combining Theorem 2.4.5 with Corollary 3.1.4 yields the following strong partition result:

Theorem 3.1.5 *Let for each $n \in \mathbb{N}$, $\mathcal{F}_n$ and $\mathcal{G}_n$ be families of finite subsets of $\mathbb{N}$ such that each combinatorially large ultrafilter contains a member of $\mathcal{F}_n$ and a member of $\mathcal{G}_n$. Whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exists $i \in \{1, 2, \ldots, r\}$ such that $d_m^*(A_i) > 0, \overline{d}(A_i) > 0$ and that there exist sequences $(B_n)_{n=1}^{\infty}$ and $(C_n)_{n=1}^{\infty}$ satisfying $B_n \in \mathcal{F}_n$ and $C_n \in \mathcal{G}_n$ such that for each $H \in \mathcal{P}_f(\mathbb{N})$*

$$\prod_{t \in H} B_t \cup C_t \cup B_t C_t \subseteq A_i.$$

By Lemma 2.3.12 we know that $P = \{p \in \beta\mathbb{N} :$ Every element of p contains arbitrarily long geometric progressions$\}$ is a closed ideal of $(\beta\mathbb{N}, \cdot)$ and thus contains $\overline{K((\beta\mathbb{N}, \cdot))}$. (It is not hard to see that this inclusion is proper.) In the last chapter we shortly discussed that $\overline{K((\beta\mathbb{N}, \cdot))}$ doesn't contain additive idempotents. In the following Theorem we see what nice consequences we could derive from the existence of an additive idempotent in P

Theorem 3.1.6 *Assume that there exists some $e \in P$ such that $e + e = e$. Then whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $(F_n)_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, F_n is a length n geometric progression and for every $H \in \mathcal{P}_f(\mathbb{N})$, one has*

$$\sum_{n \in H} F_n \subseteq A_i.$$

proof: Define $\mathcal{F}_n$ to be the set of all n-term geometric progressions and apply Corollary 3.1.4 . □

3.2 A multidimensional Central Sets Theorem

Ramsey's Theorem (at least the version we are concerned with) states that whenever the complete graph of an infinite set S is finitely coloured there exists an infinite set $T \subseteq S$ such that the complete graph of the set T is monochrome.
For a formal treatment we set up some notation:

Definition 3.2.1 *Let S be an infinite set and let $k \in \mathbb{N}$. Then $[S]^k$ is the set of all subsets of S that consist of exactly k elements.*

Ramsey's Theorem is not restricted to colorings of the complete graph of S which may be identified with the set of all unordered pairs $[S]^2$, it works equally well with $[S]^k$ for arbitrary $k \in \mathbb{N}$:
Assume that $\bigcup_{i=1}^{r} A_i = [S]^k$. There exist $i \in \{1, 2, \ldots, r\}$ and an infinite set $T \subseteq S$ such that $[T]^k \subseteq [S]^k$. For $k = 1$ this is just the pigeonhole principle. For $k' > k$ Ramsey's Theorem for colorings of $[S]^{k'}$ implies Ramsey's Theorem for colorings of $[S]^k$:
Without loss of generality we may assume that we are working with a countable set, so let $S = \mathbb{N}$. Let $A_1, \ldots, A_r$ be a partition of $[\mathbb{N}]^k$. Define a partition of $[\mathbb{N}]^{k'}$ via

$$B_i = \{F \in [\mathbb{N}]^{k'} : \text{The set of the } k \text{ smallest elements of } F \text{ lies in } A_i\}$$

for $i \in \{1, 2, \ldots, r\}$. If $T \subseteq \mathbb{N}$ has the property that $[T]^{k'} \subseteq B_i$ then $[T]^k \subseteq A_i$.

K. Milliken and A. Taylor ([Mi75, T76]) found a quite natural common extension of the Theorems of Hindman and Ramsey. The following seems to be the appropriate generalization of the notion of finite sums from a sequence:

Definition 3.2.2 *For a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{N}$ and $k \geq 1$ put*

$$FS[(x_n)_{n=1}^\infty]^k_< = \left\{ \left\{ \sum_{t\in H_1} x_t, \ldots, \sum_{t\in H_k} x_t \right\} : H_1 < \ldots < H_k \in \mathcal{P}_f(\mathbb{N}) \right\},$$

where we write $H < H'$ for $H, H' \in \mathcal{P}_f(\mathbb{N})$ iff $\max H < \min H'$. We will use the same terminology with FS replaced by FP in abstract semigroups.

Using this, the Milliken-Taylor Theorem may be stated as follows:

Theorem 3.2.3 (Milliken-Taylor Theorem [Mi75, T76]) *Let $k, r \in \mathbb{N}$. If $[\mathbb{N}]^k = \bigcup_{i=1}^r A_i$ then there exist $i \in \{1, \ldots, r\}$ and a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{N}$ such that*

$$FS[(x_n)_{n=1}^\infty]^k_< \subseteq A_i.$$

From the Graph Theoretic point of view, the Milliken Taylor Theorem generalizes Hindman's Theorem in the same way as Ramsey's Theorem outranges the pigeonhole principle.

The main goal of this section is to prove a similar multidimensional extension that applies to the Central Sets Theorem instead of Hindman's Theorem. For sake of readability we will first state a quite special case of the theorem we are after. (More precisely this is a multidimensional version of Theorem 3.0.1 in the Introduction.)

Theorem 3.2.4 *Let $r, k \in \mathbb{N}$ and let $\bigcup_{i=1}^r A_i = [\mathbb{N}]^k$. There exist sequences $(a_n)_{n=1}^\infty, (d_n)_{n=1}^\infty$ in $\mathbb{N}$ such that for each sequence $(x_n)_{n=1}^\infty$ where $x_n \in \{a_n, a_n + d_n, \ldots, a_n + nd_n\}$*

$$[FS((x_n)_{n=1}^\infty)]^k_< \subseteq A_i.$$

If S is an infinite set an arbitrary non principal ultrafilter $p \in \beta S$ may be used to give a proof of Ramsey's Theorem. (This proof is by now classical. See [CN74] p.39 for a discussion of its origins.) It's an idea of V. Bergelson and N. Hindman that in the case $S = \mathbb{N}$, something might be gained by using an ultrafilter with special algebraic properties. Via this approach in [BH89] a short proof of the Milliken-Taylor Theorem is given and a very strong simultaneous generalization of Ramsey's Theorem and numerous single-dimensional Ramsey-type Theorems (including van der Waerden's Theorem) is obtained. Our Theorems result from a variation on their idea. The following lemma is the basic tool in the ultrafilter proof of Ramsey's theorem:

Lemma 3.2.5 *Let S be a set, let $p \in \beta S \setminus S$, let $k, r \in \mathbb{N}$, and let $[S]^k = \bigcup_{i=1}^r A_i$. For each $i \in \{1, \ldots, r\}$, each $t \in \{1, \ldots, k\}$ and each $E \in [S]^{t-1}$, define $B_t(E, i)$ by downward induction on t:*

(1) For $E \in [S]^{k-1}, B_k(E,i) = \{y \in S \setminus E : E \cup \{y\} \in A_i\}$*.*

(2) For $1 \leq t < k$ *and* $E \in [S]^{t-1}$,

$$B_t(E,i) = \{y \in S \setminus E : B_{t+1}(E \cup \{y\}, i) \in p\}.$$

Then there exists some $i \in \{1, \ldots, r\}$ *such that* $B_1(\emptyset, i) \in p$.

proof: For each $E \in [S]^{k-1}$ one has $S = E \cup \bigcup_{i=1}^r B_k(E,i)$, so there exists $i \in \{1, \ldots, r\}$ such that $B_k(E,i) \in p$. Next let $E \in [S]^{k-2}$ and $y \in S \setminus E$. Then there exists $i \in \{1, \ldots, r\}$ such that $B_k(E \cup \{y\}, i) \in p$. Thus $S = E \cup \bigcup_{i=1}^r B_{k-1}(E,i)$.
After iterating this argument $k-1$ times we achieve $S = \emptyset \cup \bigcup_{i=1}^r B_1(\emptyset, i)$ which clearly proves the statement. $\square$

The following lemma extends Lemma 3.1.2. We will use it in quite a similar way.

Lemma 3.2.6 *Let* S *be a semigroup such that there exists an idempotent* $e \in \beta S \setminus S$, *let* $k, r \in \mathbb{N}$ *and let* $[S]^k = \bigcup_{i=1}^r A_i$. *Then there exist* $i \in \{1, \ldots, r\}$ *and a tree* $T \subseteq S^{<\omega}$ *such that for all* $f \in T$ *and* $H_1 < \ldots < H_k \subseteq$ *dom* $f, H_i \in \mathcal{P}_f(\omega)$ *one has:*

(1) $T(f) \in e$.

(2) $\left\{\prod_{t \in H_1} f(t), \ldots, \prod_{t \in H_k} f(t)\right\} \in A_i$.

proof: Let $i \in \{1, \ldots, r\}$ such that $B_1(\emptyset, i) \in e$. (We use the notation of Lemma 3.2.5.) We will inductively construct a sequence of trees $(T_n)_{n=0}^{\infty}$, satisfying $T_n = \{f_{\restriction \{0, \ldots, n-1\}} : f \in T_{n+1}\}$ for each $n \in \omega$, such that

(1) If $f \in T_n$ satisfies *dom* $f \subseteq \{0, \ldots, n-2\}$ then $T_n(f) \in e$.

(2) For all $f \in T_n$, for all $k' \in \{1, \ldots, k\}$, for all $H_1 < \ldots < H_{k'} \subseteq$ *dom* f, $H_i \in \mathcal{P}_f(\omega)$ one has

$$\prod_{t \in H_{k'}} f(t) \in B_{k'}\left(\left\{\prod_{t \in H_1} f(t), \ldots, \prod_{t \in H_{k'-1}} f(t)\right\}, i\right)^{1}.$$

(3) For all $f \in T_n$, for all $k' \in \{1, \ldots, k\}$, for all $H_1 < \ldots < H_{k'} \subseteq$ *dom* f, $H_i \in \mathcal{P}_f(\omega)$ one has

$$\left(\prod_{t \in H_{k'}} f(t)\right)^{-1} B_{k'}\left(\left\{\prod_{t \in H_1} f(t), \ldots, \prod_{t \in H_{k'-1}} f(t)\right\}, i\right) \in e.$$

[1] For $k' = 1$ this is meant to be $B_1(\emptyset, i)$.

To start with this construction we put $T_0 = \{\emptyset\}$, such that T_0 is a tree in S which trivially satisfies (1) - (3).

$\vdots$

Assume that $T_0, \ldots, T_n$ are trees satisfying $T_j = \{f_{\restriction\{0,\ldots,j-1\}} : f \in T_{j+1}\}$ for $j < n$, such that (1) - (3) hold. For $f \in T_n$, *dom* $f = \{0, \ldots, n-1\}$ we put

$$E_f = \bigcap_{\substack{H_1 < \ldots < H_{k'}, \\ \max H_{k'} < n, k' \leq k-1}} B_{k'+1}\left(\left\{\prod_{t\in H_1} f(t), \ldots, \prod_{t\in H_{k'}} f(t)\right\}, i\right)$$

$$\cap \bigcap_{\substack{H_1 < \ldots < H_{k'}, \\ \max H_{k'} < n, k' \leq k}} \left(\prod_{t\in H_{k'}} f(t)\right)^{-1} B_{k'}\left(\left\{\prod_{t\in H_1} f(t), \ldots, \prod_{t\in H_{k'-1}} f(t)\right\}, i\right).$$

Using this we define $T_{n+1}(f) = E_f \cap \{s \in S : s^{-1}E_f \in e\}$. Finally we put

$$T_{n+1} = T_n \cup \{f^\frown s : f \in T_n, dom\ f = \{0, \ldots, n-1\}, s \in T_{n+1}(f)\}.$$

Obviously T_{n+1} is a tree such that $T_n = \{f_{\restriction\{0,\ldots,n-1\}} : f \in T_{n+1}\}$. We need to check that (1) - (3) are satisfied. Let $f \in T_n$ with *dom* $f = \{0, \ldots, n-1\}$ be fixed.
E_f is defined as an intersection of finitely many sets, all of which are contained in e by the hypothesis of the induction. e is idempotent, so for any set $A \subseteq S$ one has $A \in e = ee \Leftrightarrow \{s : s^{-1}A \in e\} \in e$. Applying this to the set E_f we see that $T_{n+1}(f) \in e$. So (1) is satisfied.
To prove (2) and (3) let $s \in T(f), \tilde{f} = f^\frown s$ and $H_1 < \ldots < H_{k'} \subseteq \{0, \ldots, n\}$ for some $k' \leq k$ be fixed. If $\max H_{k'} < n$, the claim follows trivially from the hypothesis of the induction, so we assume $n \in H_{k'}$. We distinguish the cases (A) in which $H_{k'} = \{n\}$ and (B) in that $H_{k'} = \overline{H}_{k'} \cup \{n\}$ for some non empty set $\overline{H}_{k'} \subseteq \{0, \ldots, n-1\}$.

(2A) This is clear by the definition of E_f.

(2B) One has

$$s \in E_f \subseteq \left(\prod\nolimits_{t\in\overline{H}_{k'}} f(t)\right)^{-1} B_{k'}\left(\left\{\prod\nolimits_{t\in H_1} f(t), \ldots, \prod\nolimits_{t\in H_{k'-1}} f(t)\right\}, i\right),$$

which implies the claim.

(3A) We have $e \ni s^{-1}E_f \subseteq s^{-1}B_{k'}\left(\left\{\prod_{t\in H_0} f(t), \ldots, \prod_{t\in H_{k'-1}} f(t)\right\}, i\right)$.

(3B) $e \ni s^{-1}E_f \subseteq s^{-1}\left(\prod_{t\in \overline{H}_{k'}} f(t)\right)^{-1} B_{k'}\left(\left\{\prod_{t\in H_0} f(t),\ldots,\prod_{t\in H_{k'-1}} f(t)\right\}, i\right) = \left(\prod_{t\in H_{k'}} \tilde{f}(t)\right)^{-1} B_{k'}\left(\left\{\prod_{t\in H_0} f(t),\ldots,\prod_{t\in H_{k'-1}} f(t)\right\}, i\right)$, so we are done.

$\vdots$

Finally we put $T = \bigcup_{n=0}^{\infty} T_n$ such that for all $f \in T$ the following holds:

(1) $T(f) \in e$.

(2) For all $H_1, \ldots, H_k \in \mathcal{P}_f(\omega)$ satisfying $H_1 < \ldots < H_k \subseteq dom\ f$ one has $\prod_{t\in H_k} f(t) \in B_k\left(\left\{\prod_{t\in H_1} f(t),\ldots,\prod_{t\in H_{k-1}} f(t)\right\}, i\right)$ which implies $\left\{\prod_{t\in H_1} f(t),\ldots,\prod_{t\in H_k} f(t)\right\} \in A_i$.

□

From this Lemma one may directly derive the following strong version of the Milliken-Taylor Theorem:

Corollary 3.2.7 *Let $k, r \in \mathbb{N}$, let S be a semigroup, let $(x_n)_{n=0}^{\infty}$ be a sequence in S and let $[S]^k = \bigcup_{i=1}^{r} A_i$. Assume that for every idempotent $s \in S$ there exists some $m \in \mathbb{N}$ such that $s \notin FP((x_n)_{n=m}^{\infty})$. Then there exist $i \in \{1,\ldots,r\}$ and a sequence $H_0 < H_1 < \ldots$ in $\mathcal{P}_f(\omega)$ such that*

$$\left[FP\left(\left(\prod_{t\in H_n} x_t\right)_{n=0}^{\infty}\right)\right]_<^k \subseteq A_i.$$

proof: By Lemma 1.4.6 there exists an idempotent $e \in \beta S$, such that for all $m \geq 0$, $FP((x_n)_{n=m}^{\infty}) \in e$ and by our assumption we have $e \in \beta S \setminus S$. Let $i \in \{1,\ldots,r\}$ and $T \subseteq S^{<\omega}$ be as provided by Lemma 3.2.6. We have $T(\emptyset) \cap FS((x_n)_{n=0}^{\infty}) \in e$. In particular this set is not empty, so we may choose $H_0 \in \mathcal{P}_f$ such that $\prod_{t\in H_0} x_t \in T(\emptyset)$. Let $m_0 = \max H_0$. As before $T\left(\left(\sum_{t\in H_0} x_t\right)\right) \cap FS\left((x_n)_{n=m_0+1}^{\infty}\right) \in e$, so we find $H_1 > H_0, H_1 \in \mathcal{P}_f(\omega)$ such that $\prod_{t\in H_1} x_t \in T\left(\left(\prod_{t\in H_0} x_t\right)\right)$. By continuing in this fashion we achieve a sequence with the required properties. □

(Corollary 3.2.7 differs from [HS98], Corollary 18.9 only in that there our restriction on the idempotents contained in $FP((x_n)_{n=m}^{\infty})$ is omitted. But in fact some additional assumption is required to guarantee that $FP((x_n)_{n=0}^{\infty})$ does not degenerate. Consider for example $S = (\omega, +)$ and $(x_n)_{n=0}^{\infty} = (0, 0, \ldots)$. In this case $[FS((x)_n)]_<^k = \{\{0\}\} \not\subseteq [\mathbb{N}]^k$ for $k \geq 2$.)

Next we state and prove the main Theorem of this section: (We remark that the proof of Theorem 3.2.8 is virtually identical to the proof of Theorem 3.1.3, one only has to replace Lemma 3.1.2 by Lemma 3.2.6.)

Theorem 3.2.8 *Let S be a commutative semigroup and assume that there exists a non principal minimal idempotent in βS. Let $g : \omega \to \omega$ be an arbitrary function. For each $l \in \omega$, let $(y_{l,n})_{n=0}^{\infty}$ be a sequence in S. Let $k, r \geq 1$ and let $[S]^k = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, \dots, k\}$, a sequence $(a_n)_{n=0}^{\infty}$ in S and a sequence $H_0 < H_1 < \dots$ in $\mathcal{P}_f(\omega)$ such that for each sequence $(i_n)_{n=0}^{\infty}$ satisfying $i_n \leq g(n)$ for $n \in \omega$,*

$$\left[FP\left(\left(a_n + \sum_{t \in H_n} y_{i_n,t} \right)_{n=0}^{\infty} \right) \right]_{<}^{k} \subseteq A_i.$$

proof: Fix a minimal idempotent $e \in \beta S \setminus S$. Let $i \in \{1, \dots, r\}$ and $T \subseteq S^{<\omega}$ be as provided by lemma 3.2.6. Denote by Φ the set of all sequences $(i_n)_{n=0}^{\infty}$ satisfying $i_n \leq g(n)$ for all $n \in \omega$. We will inductively construct sequences $(a_n)_{n=0}^{\infty}$ in S and $H_0 < H_1 < \dots$ in $\mathcal{P}_f(\omega)$ such that for all $n \in \omega$ and all sequences $(i_n)_{n=0}^{\infty} \in \Phi$

$$\left(a_0 + \textstyle\sum_{t \in H_0} y_{i_0,t}, \dots, a_n + \sum_{t \in H_n} y_{i_n,t}\right) \in T. \tag{3.2}$$

By the properties of T this is sufficient to proof the Theorem.
Assume that $a_0, \dots, a_{n-1} \in S$ und $H_0 < \dots < H_{n-1} \in \mathcal{P}_f(\omega)$ have already been constructed such that (3.2) is true for all $(i_n)_{n=0}^{\infty} \in \Phi$. We have

$$G_n = \bigcap_{(i_n)_{n=0}^{\infty} \in \Phi} T\left(\left(a_0 + \sum_{t \in H_0} y_{i_0,t}, \dots, a_{n-1} + \sum_{t \in H_{n-1}} y_{i_{n-1},t} \right) \right) \in e.$$

Let $m = \max H_{n-1}$. By applying Theorem 2.2.2 to the set G_n and the sequences $(y_{0,k})_{k>m}, \dots, (y_{g(n),k})_{k>m}$ we find $a_n \in S$ and $H_n \in \mathcal{P}_f(\omega)$, $H_n > H_{n-1}$ such that $a_n + \sum_{t \in H_n} y_{0,t}, \dots, a_n + \sum_{t \in H_n} y_{g(n),t} \in G_n$.
Thus for all $(i_n)_{n=0}^{\infty} \in \Phi$, $\left(a_0 + \sum_{t \in H_0} y_{g(0),k}, \dots, a_n + \sum_{t \in H_n} y_{g(n),k}\right) \in T$, as we wanted to show. □

The case $k = 1$ of Theorem 3.2.8 is exactly the Central Sets Theorem. Theorem 3.2.4 follows from Theorem 3.2.8 similarly as Theorem 3.0.1 follows from the Central Sets Theorem.
For $k = 1$ the somewhat odd assumption that βS should contain a non principal minimal idempotent is not needed. In general this condition will be satisfied if S is *weakly (left) cancellative*, i.e. if for all $u, v \in S$ the set $\{s \in S : us = v\}$ is finite and S itself is infinite (see [HS98], Theorem 4.3.7). In particular the conclusion of Theorem 3.2.8 holds in the semigroups $(\mathbb{N}, +), (\mathbb{N}, \cdot), (\mathcal{P}_f, \cup)$.

3.3 Applications of located words

Theorem 3.3.7 is the major result of this section. It has several earlier theorems as immediate corollaries. In particular, it implies a stronger version of Theorem 2.2.9 and significantly strengthens the Central Sets Theorem. To establish this theorem we shall use the notion of *partial semigroup* introduced in [BBH94].

Definition 3.3.1 *(a) A partial semigroup is a set S together with an operation $\cdot$ that maps a subset of $S \times S$ into S and satisfies the associative law $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ in the sense that if either side is defined, then so is the other and they are equal.*

(b) Given a partial semigroup $(S, \cdot)$ and $x \in S$, $\phi(x) = \{y \in S : x \cdot y$ is defined$\}$.

(c) Given a partial semigroup $(S, \cdot)$, $x \in S$ and $A \subseteq S$, $x^{-1}A = \{y \in \phi(x) : x \cdot y \in A\}$.

(d) A partial semigroup $(S, \cdot)$ is adequate if and only if for each $F \in \mathcal{P}_f(S)$, $\bigcap_{x \in F} \phi(x) \neq \emptyset$.

(e) Given an adequate partial semigroup $(S, \cdot)$, $\delta S = \bigcap_{x \in S} \overline{\phi(x)}$.

Before we derive some elementary properties of partial semigroups, we shall describe the partial semigroup we are mainly interested in: Let Λ be an alphabet (i.e. some non empty set) and let v be a "variable" that is not contained in Λ. A *located word* over Λ is a function w from a finite subset *dom* w of $\mathbb{N}$ to Λ. Let L_0 be the set of located words over Λ and let L_1 be the set of located variable words over Λ, that is the set of words over $\Lambda \cup \{v\}$ in which v occurs. Let $L = L_0 \cup L_1$. Given $u, w \in L$, if $\max(dom\ u) < \min(dom\ w)$, then define $u \cdot w$ by $dom\ (u \cdot w) = dom\ u \cup dom\ w$ and for $t \in dom\ (u \cdot w)$,

$$(u \cdot w)(t) = \begin{cases} u(t) & \text{if } t \in dom\ u \\ w(t) & \text{if } t \in dom\ w\,. \end{cases}$$

(Or - somewhat more concise - $uw = u \cup w$.) With this operation L, L_0, and L_1 are adequate partial semigroups.

It would be easy to define structures similar to L_0 that carry a semigroup structure. For example as in chapter 2 we could consider words over Λ (instead of located words) and use concatenation as the semigroup operation. Another possibility would be to consider the semigroup $(\mathcal{P}_f(\mathbb{N} \times \Lambda), \cup)$. We will shortly explain why these semigroups are not well suited for our purposes:

Let $(T, \cdot)$ be a semigroup and for each $\lambda \in \Lambda$ let $(x_{\lambda,n})_{n=1}^{\infty}$ be a sequence in T. Put

$$f : L_0 \to T \qquad w \mapsto \prod_{t \in dom\ w} x_{w(t),t}.$$

f has the nice "homomorphism property" that whenever uw is defined, we have $f(uw) = f(u) \cdot f(w)$. Of course we can not expect that the proposals above allow similar "homomorphisms".

Lemma 3.3.2 *Let $(S, \cdot)$ be an adequate partial semigroup and for $p \in \beta S$, $q \in \delta S$ define $p \cdot q = \{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\}$. The map $\rho_q : \beta S \to \delta S$ is continuous. With the relative topology inherited from βS, $(\delta S, \cdot)$ is a compact right topological semigroup.*

proof: It follows directly from Definition 3.3.1 that $\delta S = \bigcap_{s \in S} \overline{\phi(s)}$ is a non empty compact subset of βS if S is adequate. For each $s \in S$ exists a unique continuous extension of $\lambda_s : \phi(s) \to \beta S$ to $\beta\phi(s)$. Rigorously $\beta\phi(s)$ does not coincide with $\overline{\phi(s)} \subseteq \beta S$, but since ultrafilters on S that contain $\phi(s)$ naturally correspond to ultrafilters on $\phi(s)$ we shall not bother with this problem. So for all $s \in S$, the continuous extension of λ_s (which will again be denoted by λ_s) is a continuous function that is defined on δS. It is easy to check that for $p \in \delta S$, $\lambda_s(p) = sp = \{A \subseteq S : s^{-1}A \in p\}$.
By considering the continuous extensions of the functions ρ_q for $q \in \delta S$, we may assume that pq is defined for $p \in \beta S$ and $q \in \delta S$ and as in Theorem 1.2.12 one easily checks that $A \in pq \Leftrightarrow \{s \in S : s^{-1}A \in q\} \in p$.
Pick $s \in S$ and $t \in \phi(s)$. We have

$$\begin{aligned} t^{-1}\phi(s) &= \{x \in \phi(t) : tx \in \phi(s)\} && (3.3) \\ &= \{x \in \phi(t) : s(tx) \text{ is defined}\} && (3.4) \\ &= \{x \in \phi(t) : (st)x \text{ is defined}\} && (3.5) \\ &= \phi(t) \cap \phi(st). && (3.6) \end{aligned}$$

Thus for $q \in \delta S$, $t^{-1}\phi(s) = \phi(t) \cap \phi(st) \in q$. Equivalently we have $tq \in \overline{\phi(s)}$. By continuity $\overline{\phi(s)}q \subseteq \overline{\phi(s)}$. In particular $pq \in \overline{\phi(s)}$ for each $p \in \delta S$. Since s was arbitrary, it follows that $pq \in \delta S$. Thus $\delta S \delta S \subseteq \delta S$.
Associativity follows similarly as in 1.2.9. □

L, L_0, and L_1 are adequate partial semigroups so by Lemma 3.3.2 δL, δL_0, and δL_1, are compact right topological semigroups. Also δL_1 is an ideal of δL. (The verification of this latter statement is an easy exercise and a good chance for the reader to see whether she has grasped the definition of the operation.) Notice that for $j \in \{1, 2\}$ and $p \in \beta L_j$, one has that $p \in \delta L_j$ if and only if for each $n \in \mathbb{N}$, $\{w \in L_j : \min(dom\ w) > n\} \in p$.
Lemma 3.1.2 extends to adequate partial semigroups without difficulties:

Lemma 3.3.3 *Let S be an adequate partial semigroup, let e be an idempotent in δS and assume that $A \subseteq e$. There exists a tree T in S such that for all $f \in T$:*

(1) $T(f) \in e$.

(2) *For all non empty $H \subseteq dom\ f$, $\prod_{t \in H} f(t)$ is defined and lies in A.*

proof: The proof is very similar to the one of Lemma 3.1.2, so we skip it. □

Lemma 3.3.4 *Let $(S, \cdot)$ and $(T, *)$ be adequate partial semigroups and let $f : S \to T$ have the property that for all $x \in S$ and all $y \in \phi_S(x)$, $f(y) \in \phi_T(f(x))$ and $f(x \cdot y) = f(x) * f(y)$ and assume that f is onto. Let $\tilde{f} : \beta S \to \beta T$ be the continuous extension of f. Then the restriction of $\tilde{f}$ to δS is a homomorphism from $(\delta S, \cdot)$ to $(\delta T, *)$.*

proof: Let $p \in \delta S$. We want show that $\tilde{f}(p) \in \delta T$. Let $b \in T$ be arbitrary. Choose $a \in S$ such that $f(a) = b$. Then $\overline{\phi_S(a)} \in p$. We have $\phi_S(a) \subseteq f^{-1}[\phi_T(b)]$, so $f^{-1}[\phi_T(b)] \in p$. Thus $\tilde{f}(p) \in \overline{\phi_T(b)}$. Since b was arbitrary it follows that $\tilde{f}(p) \in \delta T$. To show that $\tilde{f} \restriction \delta S$ is a homomorphism proceed as in the proof of Theorem 1.2.13. □

In the following we will denote the continuous extension of a function by the same letter as the function itself.
For each $a \in L$, define $\theta_a : L \to L_0$ as follows. For $w \in S$, let $dom\ \theta_a(w) = dom\ w$ and for $t \in dom\ w$, let

$$\theta_a(w)(t) = \begin{cases} w(t) & \text{if } w(t) \in L \\ a & \text{if } w(t) = v\,. \end{cases}$$

That is, $\theta_a(w)$ is the result of replacing each occurrence of v in w by a. Notice that θ_a is the identity on L_0 hence this also holds for its continuous extension on βL_0.

Lemma 3.3.5 (1) *Let $q \in \delta L_0$ be a minimal idempotent. There exists a minimal idempotent $r \in \delta L_1$ such that $\theta_a(r) = q$ for all $a \in \Lambda$.*

(2) *Let $A \subseteq L_0$ be central and let $a_1, a_2, \ldots, a_k \in \Lambda$. There exists a central set $B \subseteq L_1$ such that $\theta_{a_i}[B] \subseteq A$ for all $i \in \{1, 2, \ldots, k\}$.*

proof:

(1) Pick a minimal idempotent $r \leq q$ in L. Since δL_1 is an ideal of δL we have $r \in \delta L_1$. Let $a \in \Lambda$ be arbitrary. $\theta_a(r) \leq \theta_a(q) \leq q$. Since q is minimal in L_0, $\theta_a(r) = q$.

(2) Pick a minimal idempotent $q \in \delta L_0$ such that $A \in q$ and let r be as provided by (1). By continuity we have

$$B = L_1 \cap \bigcap_{i=1}^{k} \{w \in L : \theta_{a_i}(w) \in \overline{A}\} \in r.$$

□

The following theorems apply equally well to finitely many semigroups instead of just E_1, E_2. We just have chosen to go with two semigroups since this simplifies the notation, while nothing essential has to be changed to treat arbitrarily many semigroups.

Lemma 3.3.6 *Let E_1, E_2 be countable semigroups with identities e_1, e_2, for $j \in \mathbb{N}$ let $(x_{j,n})_{n=1}^{\infty}$ be a sequence in E_1 and let $(y_{j,n})_{n=1}^{\infty}$ be a sequence in E_2. Assume further that $x_{1,n} = e_1$ for $n \in \mathbb{N}$, that every element of E_1 appears infinitely often in $(x_{2,n})_{n=1}^{\infty}$ and that the analogous statement holds for $(y_{i,n})_{n=1}^{\infty}$ for $i \in \{1,2\}$. Let q_i be a minimal idempotent in E_i for $i \in \{1,2\}$. Let $\Lambda = \mathbb{N}^2$ and define*

$$g_1 : L_0 \to E_1 \qquad w \mapsto \prod_{t \in dom\ w} x_{\pi_1(w(t)),t} \tag{3.7}$$

$$g_2 : L_0 \to E_2 \qquad w \mapsto \prod_{t \in dom\ w} y_{\pi_2(w(t)),t}, \tag{3.8}$$

where $\pi_1, \pi_2 : \mathbb{N}^2 \to \mathbb{N}$ are the projections onto the first respectively the second coordinate. Then there exists a minimal idempotent $q \in \delta L_0$ such that $g_1(q) = q_1$ and $g_2(q) = q_2$.

proof: We claim that if $b_i \in E_i$ for each $i \in \{1,2\}$ and if $n \in \mathbb{N}$, there exists $w \in S_0$ such that $g_i(w) = b_i$ for every $i \in \{1,2\}$ and $\min(dom\ w) > n$. To see this, observe that we can choose n_1, n_2 in $\mathbb{N}$ such that $n < n_1 < n_2$ and $x_{2,n_1} = b_1, y_{2,n_2} = b_2$. We can then define w by putting $dom\ w = \{n_1, n_2\}$, $w(n_1) = (2,1), w(n_2) = (1,2)$.
In particular each $g_i : L_0 \to E_i$ is surjective and so, by Lemma 3.3.4, the restriction of g_i to δL_0 is a homomorphism to $\delta E_i = \beta E_i$.
Given $(X_1, X_2, n) \in q_1 \times q_2 \times \mathbb{N}$ we choose $w(X_1, X_2, n) \in S_0$ such that $\min(dom\ w(X_1, X_2, n)) > n$ and $g_i(w(X_1, X_2, n)) \in X_i$ for each $i \in \{1,2\}$. We give $p_1 \times p_2 \times \mathbb{N}$ a directed set ordering by stating that $(X_1, X_2, n) \prec (X_1', X_2', n')$ iff $X_i' \subseteq X_i$ for each $i \in \{1,2\}$ and $n < n'$. If x is any limit point of the net $(w(X_1, X_2, n))$ in βL_0, we have $x \in \delta L_0$ and $g_i(x) = p_i$ for every $i \in \{1,2\}$. (That $x \in \delta L_0$ follows from the fact that $\min(dom\ w(X_1, X_2, n)) > n$. To see that $g_i(x) = q_i$, let $A \in q_i$ and suppose $g_i(x) \notin \overline{A}$. Pick $B \in x$ such that $g_i[\overline{B}] \cap \overline{A} = \emptyset$. Let $X_i = A$ and for $j \neq i$ let $X_j = E_j$. Pick $(X_1', X_2', n') \succ (X_1, X_2, 1)$ such that $w(X_1', X_2', n') \in \overline{B}$. But $g_i(w(X_1', X_2', n')) \in X_i' \subseteq X_i = A$, a contradiction.)
Let $C = \{x \in \delta L_0 : g_i(x) = q_i \text{ for } i \in \{1,2\}\}$. We have just seen that C is nonempty, and so it is a compact subsemigroup of δL_0. Let q be a minimal idempotent in C. Then q is minimal in δL_0, because if q' is any idempotent of δL_0 satisfying $q' \leq q$, we have $g_i(q') \leq g_i(q) = q_i$ for $i \in \{1,2\}$. This

implies that $g_i(q') = q_i$. So $q' \in C$ and thus $q' = q$. $\square$

The following theorem is a rather strong generalisation of the Central Sets Theorem.

Theorem 3.3.7 *Let E_1, E_2 be commutative semigroups with identities e_1, e_2, let $C_1 \subseteq E_1$ and $C_2 \subseteq E_2$ be central sets. For $j \in \mathbb{N}$ let $(x_{j,n})_{n=1}^{\infty}$ be a sequence in E_1 and let $(y_{j,n})_{n=1}^{\infty}$ be a sequence in E_2.*
Let X be a nonempty set that is finitely coloured and let $\phi : E_1 \times E_2 \to X$ be an arbitrary function.
Let $h : \omega \to \mathbb{N}$ be a function which is growing arbitrarily fast.
There exist sequences $(a_n)_{n=0}^{\infty}$ in E_1, $(b_n)_{n=0}^{\infty}$ in E_2 and $H_0 < H_1 < \ldots$ in $\mathcal{P}_f(\mathbb{N})$ and a monochrome set $M \subseteq X$ such that for all $F \in \mathcal{P}_f(\omega)$ and all $h_1, h_2 \leq h$

$$\prod_{n\in F} a_n \prod_{t\in H_n} x_{h_1(n),t} \in C_1, \qquad \prod_{n\in F} b_n \prod_{t\in H_n} y_{h_2(n),t} \in C_2$$

$$\phi\left(\prod_{n\in F} a_n \prod_{t\in H_n} x_{h_1(n),t}, \prod_{n\in F} b_n \prod_{t\in H_n} y_{h_2(n),t}\right) \subseteq M.$$

proof: By eventually switching to the subgroups that are generated by the sequences $(x_{j,n})_{n=1}^{\infty}, (y_{j,n})_{n=1}^{\infty}, j \in \mathbb{N}$, we may assume that G_1 and G_2 are countable.
Further we shall assume that the sequences $(x_{i,n})_{n=1}^{\infty}, (y_{i,n})_{n=1}^{\infty}, i \in \{1,2\}$ satisfy the hypothesis of Lemma 3.3.6. (Perhaps we have to add some new sequences and replace h by $h+2$.)
For $i \in \{1,2\}$ pick a minimal idempotent $q_i \in \beta E_i$ such that $C_i \in q_i$.
We will again work with L_0, L_1 where $\Lambda = \mathbb{N}^2$. Define g_1, g_2 as in Lemma 3.3.6 and pick a minimal idempotent $q \in \delta L_0$ such that $g_1(q) = q_1, g_2(q) = q_2$. Via the map

$$\psi : L_0 \to X \qquad w \mapsto \phi(g_1(w), g_2(w))$$

the colouring of X induces a colouring of L_0. Let K be a monochrome set that is contained in q. Then $A = g_1^{-1}[C_1] \cap g_2^{-1}[C_2] \cap K \in q$.
By Lemma 3.3.3 there is a tree T in L_0 such that for all $f \in T$ the following holds:

(1) $T(f) \in q$.

(2) Let $dom\ f = \{0, 1, \ldots, l\}$. Then $\max f(0) < \min f(1), \max f(1) < \min f(2), \ldots, \max f(l-1) < \min f(l)$ and for all $F \subseteq \{0, 1, \ldots, l\}$, $\bigcup_{t\in F} f(t) \in A$.

Assume that w is a variable word with $w^{-1}[\{v\}] = H$. Then for $i, j \in \mathbb{N}$, $(g_1(\theta_{(i,j)}(w)), g_2(\theta_{(i,j)}(w)))$ is of the form

$$\left(a \prod_{t \in H} x_{i,t}, b \prod_{t \in H} y_{j,t}\right),$$

where $a \in E_1, b \in E_2$ do no depend on (i, j).
By the properties of T and A it is sufficient to construct a sequence $(w_n)_{n=0}^{\infty}$ in L_1 such that for all $h_1, h_2 \leq h$ and each $l \in \omega$

$$(\theta_{(h_1(0),h_2(0))}(w_0), \ldots, \theta_{(h_1(l),h_2(l))}(w_l)) \in T.$$

Assume that this was already done for $n < l$. We show how to construct w_l:

$$D = \bigcap_{h_1,h_2 \leq h} T\left(\theta_{(h_1(0),h_2(0))}(w_0), \ldots, \theta_{(h_1(l-1),h_2(l-1))}(w_{l-1})\right) \in q.$$

By Lemma 3.3.5 we may pick $w_l \in L_1$ such that $\theta_{(i,j)}(w_l) \in D$ for all $i, j \in \{1, 2, \ldots, h(l)\}$. □

We show now how to derive a simple strengthening of Theorem 2.2.9 from Theorem 3.3.7.

Corollary 3.3.8 *Let $m, k \in \mathbb{N}$. Let C_1 be central in $(\mathbb{N}, +)$ and let C_2 be central in $(\mathbb{N}, \cdot)$. For each $i \in \{1, 2, \ldots, k\}$ let $(x_{i,t})_{t=1}^{\infty}$ and $(y_{i,t})_{t=1}^{\infty}$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\begin{aligned}
&\{ba\} \cup \{b(a + \textstyle\sum_{t \in F} x_{i,t}) : i \in \{1, 2, \ldots, k\}\} \cup \\
&\{ba \cdot \textstyle\prod_{t \in F} y_{j,t} : j \in \{1, 2, \ldots, k\}\} \cup \\
&\{b(a + \textstyle\sum_{t \in F} x_{i,t}) \cdot (\textstyle\prod_{t \in F} y_{j,t}) : i, j \in \{1, 2, \ldots, k\}\} \subseteq A_s\,, \\
&\{a\} \cup \{a + \textstyle\sum_{t \in F} x_{i,t} : i \in \{1, 2, \ldots, k\}\} \subseteq C_1\,, \text{ and} \\
&\{b\} \cup \{b \cdot \textstyle\prod_{t \in F} y_{j,t} : j \in \{1, 2, \ldots, k\}\} \subseteq C_2\,.
\end{aligned}$$

proof: Let $E_1 = (\omega, +)$ and let $E_2 = (\mathbb{N}, \cdot)$. Define $\phi : E_1 \times E_2 \to \omega$ by $\psi(a, b) = ab$. For $t \in \mathbb{N}$ let $x_{k+1,t} = 0$ and $y_{k+1,t} = 1$. (For $j > k + 1$ we do not care what $x_{j,t}$ and $y_{j,t}$ are.) Let $h \equiv k + 1$.
By Lemma refcentralegal, C_1 is central in E_1. Pick $(H_n)_{n=0}^{\infty}$, $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, and M as guaranteed by Theorem 3.3.7. Pick $s \in \{1, 2, \ldots, m\}$ such that $M \subseteq A_s$. Let $a = a_0$, let $b = b_0$ and let $F = H_0$. □

The following is a nice strengthening of Lemma 3.3.6 that applies if the semigroups are the same. This time we derive the statement for arbitrarily many semigroups simultaneously.

Lemma 3.3.9 *Let $k \in \mathbb{N}$ and let E be a countable commutative semigroup with identity e, for $j \in \mathbb{N}$ and $i \in \{1, 2, \ldots, k\}$ let $(x_{j,n}^{(i)})_{n=1}^{\infty}$ be a sequence* *in E. Assume further that $x_{1,n}^{(i)} = e$ for $n \in \mathbb{N}$ and that every element of E appears infinitely often in $(x_{2,n}^{(i)})_{n=1}^{\infty}$ for each $i \in \{1,2,\ldots,k\}$. Let p be a minimal idempotent in E, let $\Lambda = \mathbb{N}^k$ and define*

$$g_i : L_0 \to E \qquad w \mapsto \prod_{t \in dom\ w} x_{\pi_i(w(t)),t}^{(i)}$$

for $i \in \{1,2,\ldots,k\}$. ($\pi_i : \mathbb{N}^k \to \mathbb{N}$ denotes the projection onto the i-th coordinate). Further put $\gamma_F(w) = \prod_{i \in F} g_i(w)$ for $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$. Then there exists a minimal idempotent $q \in \delta L_0$ such that for each $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F(q) = p$.

proof: As in the proof of Theorem 3.3.6 we see that given any $b_1, b_2, \ldots, b_k \in E$ there is some $w \in L_0$ such that $g_i(w) = b_i$ for each $i \in \{1,2,\ldots,k\}$. In particular each γ_F is a surjective homomorphism so by Lemma 3.3.4 the restriction of γ_F to δL_0 is a homomorphism to βE.
We claim that for any $B \in p$ and any $n \in \mathbb{N}$ there exists $w_{B,n} \in L_0$ such that for all $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F(w_{B,n}) \in B$ and that $\min(dom\ w_{B,n}) > n$. To see this pick $b_1, b_2, \ldots, b_k$ such that $FP((b_t)_{t=1}^k) \subseteq B$, which one may do because p is an idempotent. Pick $w_{B,n}$ such that $g_i(w_{B,n}) = b_i$ and that $\min(dom\ w_{B,n} > n)$ for each $i \in \{1,2,\ldots,k\}$.
Direct $\mathcal{D} = \{(B,n) : B \in p \text{ and } n \in \mathbb{N}\}$ by $(B,n) \prec (B',n')$ if and only if $B' \subseteq B$ and $n < n'$. Let u be a limit point of the net $(w_{B,n})_{(B,n)\in\mathcal{D}}$ in βL_0. We see as in the proof of Lemma 3.3.6 that $u \in \delta L_0$ and $\gamma_F(u) = p$ for all $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$. Let

$$J = \{w \in \delta S_0 : \gamma_F(w) = p \text{ for all } F \in \mathcal{P}_f(\{1,2,\ldots,k\})\}.$$

Then J is a compact subsemigroup of δL_0 since each γ_F is a continuous homomorphism. Pick a minimal idempotent q of J. Given any idempotent $q' \in \delta L_0$ such that $q' \le q$, for each $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$, $\gamma_F(q') \le \gamma_F(q) = p$ so $\gamma_F(q') = p$. Thus $q' \in J$ and so $q' = q$. That is, q is minimal in δL_0. □

Theorem 3.3.10 *Let $k \in \mathbb{N}$, let E be a countable commutative semigroup with identity e and let C be a central subset of E.*

(1) *Let $m \in \mathbb{N}$. For all $j \in \{1,2,\ldots,k\}, i \in \{1,2,\ldots,m\}$ let $(x_{i,n}^{(j)})_{n=1}^{\infty}$ be a sequence in E. There exist $G \in \mathcal{P}_f(\mathbb{N})$ and $b_1, \ldots, b_k \in E$ such that for all $F \in \mathcal{P}_f(\{1,2,\ldots,k\})$ and each $f : \{1,2,\ldots,k\} \to \{1,2,\ldots,m\}$ one has*
$$\prod_{j\in F} b_j \prod_{t \in G} x_{f(j),t}^{(j)} \in C.$$

(2) *Let $R_1, R_2, \ldots, R_k$ be IP sets in E. There exist $r_j \in R_j$ and $b_j \in E$ for each $j \in \{1,2,\ldots,k\}$ such that whenever $f : \{1,2,\ldots,k\} \to$*

$\{1,2,\ldots,k\}$, $h:\{1,2,\ldots,k\}\to\{0,1,\ldots,k\}$ and $F\in\mathcal{P}_f(\{1,2,\ldots,k\})$, one has

$$\prod_{j\in F} b_j\cdot\left(r_{f(j)}\right)^{h(j)}\in C.$$

proof:

(1) By eventually switching to the subsemigroups that are generated by the sequences $(x^{(j)}_{i,n})_{n=1}^{\infty}$ we may assume that G is countable.

Further we shall assume that the sequences $(x^{(j)}_{i,n})_{n=1}^{\infty}$ $i\in\{1,2\},j\in\{1,2,\ldots,k\}$ satisfy the hypothesis of Lemma 3.3.9. (Perhaps we have to add some new sequences and replace m by $m+2$.)

Pick a minimal idempotent $p\in\beta E$ such that $C\in p$.

We will work with L_0,L_1 where $\Lambda=\mathbb{N}^k$. Define $g_i, i\in\{1,2,\ldots,k\},\gamma_F$, $F\in\mathcal{P}_f(\{1,2,\ldots,k\})$ as in Lemma 3.3.6 and pick a minimal idempotent $q\in\delta L_0$ such that $\gamma_F(q)=p$ for all $F\in\mathcal{P}_f(\{1,2,\ldots,k\})$.

Put $A=\bigcap_{F\in\mathcal{P}_f(\{1,2,\ldots,k\})}\gamma_F^{-1}[C]\in q$

By Lemma 3.3.5 we may pick a vairable word $w\in L_1$ such that $\theta_{(i_1,\ldots,i_k)}(w)\in A$ for all $i_1,\ldots,i_k\in\{1,2,\ldots,m\}$. Let $H=w^{-1}[\{v\}]$. For $i_1,\ldots,i_k\in\mathbb{N}$, $(g_1(\theta_{(i_1,\ldots,i_k)}(w)),\ldots,g_k(\theta_{(i_1,\ldots,i_k)})(w))$ is of the form

$$\left(b_1\prod_{t\in H}x^{(1)}_{i_1,t},\ldots,b_k\prod_{t\in H}x^{(k)}_{i_k,t}\right),$$

where $b_1,\ldots,b_k\in E$ are independent of $i_1,\ldots,i_k$, and so we are done.

(2) For each $j\in\{1,2,\ldots,k\}$ let $(z_{j,n})_{n=1}^{\infty}$ be a sequence in E such that $FP((z_{j,n})_{n=1}^{\infty})\subseteq R_j$. Put $m=k^2$ and for $i,j,l\in\{1,2,\ldots,k\},n\in\mathbb{N}$ put $x^{(j)}_{k(i-1)+l,n}=z^l_{i,n}$. (I.e.: The choice of $(x^{(j)}_{i,n})_{n=1}^{\infty}$ is independent of j, the number $i\in\{1,2,\ldots,k\}$ of the desired IP set and the desired power $l\in\{1,2,\ldots,k\}$ are coded in the number $k(i-1)+l\in\{1,2,\ldots,k^2\}=\{1,2,\ldots,m\}$.) Pick $b_1,\ldots,b_n$ and G as guaranteed by (1). Put $r_i=\prod_{t\in G}z_{i,t}$ for $i\in\{1,2,\ldots,k\}$. Then $r^{g(i)}_{f(i)}=\prod_{t\in G}x^{(j)}_{k(f(i)-1)+g(i),n}$ so the statement follows.

□

Chapter 4

Characterising Sequences

This chapter is entirely devoted to the proof of one major Theorem. András Biró and Vera Sós [BS03] prove that for any subgroup G of $\mathbb{T}$ generated freely by finitely many generators there is a sequence $A \subseteq \mathbb{N}$ such that for all $\beta \in \mathbb{T}$ we have ($\|.\|$ denotes the distance to the nearest integer)

$$\beta \in G \Rightarrow \sum_{n \in A} \|n\beta\| < \infty, \qquad \beta \notin G \Rightarrow \limsup_{n \in A, n \to \infty} \|n\beta\| > 0.$$

We extend this result to arbitrary countable subgroups of $\mathbb{T}$. We also show that not only the sum of norms but the sum of arbitrary small powers of these norms can be kept small. Our proof combines ideas from the above article with new methods, involving a filter characterization of subgroups of $\mathbb{T}$.

4.1 Introduction

We study certain subgroups of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and methods to describe them by sequences of positive integers. By $\|.\|$ we denote the distance to the nearest integer. It is easily seen that for any sequence $A \subseteq \mathbb{N}$ the set $\{\beta \in \mathbb{T} : \lim_{n \in A, n \to \infty} \|n\beta\| = 0\}$ is a subgroup of $\mathbb{T}$. It seems natural to ask which subgroups arise in this way. In [BDS01] A. Biró, J.-M. Deshouillers and V. T. Sós show that for any countable group $G < \mathbb{T}$ there is some $A \subseteq \mathbb{N}$ that characterizes G in the above sense.
Another way to connect subsets of $\mathbb{N}$ and $\mathbb{T}$ is to consider the set $\{\beta \in \mathbb{T} : \sum_{n \in A} \|n\beta\| < \infty\}$ which again is a subgroup of $\mathbb{T}$. Following a question of P. Liardet, A. Biró and V. T. Sós show in [BS03] that if $1, \alpha_1, \ldots, \alpha_t \in \mathbb{T}$ are linearly independent over the rationals there is a sequence $A \subseteq \mathbb{N}$, that characterizes $\langle \alpha_1, \ldots, \alpha_t \rangle$ simultaneously in both ways. Such a sequence is called a 'strong characterizing sequence' of $\langle \alpha_1, \ldots, \alpha_t \rangle$. Our aim is to find strong characterizing sequences for arbitrary countable subgroups of $\mathbb{T}$. The main result is:

Theorem 4.1.1 *Let $G = \{\alpha_t : t \in \mathbb{N}\}$ be a subgroup of $\mathbb{T}$. Then there exists a sequence $A \subseteq \mathbb{N}$, such that for all $\beta \in \mathbb{T}$*

$$\beta \in G \Rightarrow \forall r > 0 \sum_{n \in A} \|n\beta\|^r < \infty, \qquad \beta \notin G \Rightarrow \limsup_{n \in A, n \to \infty} \|n\beta\| \geq 1/6.$$

4.2 Connecting two methods

In our proof we use the following reformulation of Theorem 1 in [Wi02]:

Proposition 4.2.1 *Let G be an arbitrary subgroup of $\mathbb{T}$. Then there is a filter $\mathcal{F}$ on $\mathbb{N}$ that characterizes G in the sense that for all $\beta \in \mathbb{T}$*

$$\beta \in G \iff \mathcal{F} - \lim_n \|n\beta\| = 0.$$

We remind the reader that here '$\mathcal{F} - \lim_n \|n\beta\| = 0$' means that for all $\varepsilon > 0$ one has $\{n \in \mathbb{N} : \|n\beta\| \leq \varepsilon\} \in \mathcal{F}$. The filter-convergence defined in this way is more general than ordinary convergence: For a sequence $A \subseteq \mathbb{N}$ let $\mathcal{F}(A)$ be the filter consisting of all sets containing $\{k \in A : k \geq n\}$ for some $n \in \mathbb{N}$. Then we have for all $\beta \in \mathbb{T}$

$$\lim_{n \in A, n \to \infty} \|n\beta\| = 0 \iff \mathcal{F}(A) - \lim_n \|n\beta\| = 0.$$

The following notation will be useful: Given $\alpha_1, \dots, \alpha_t \in \mathbb{T}$, $\varepsilon > 0$ and $N \in \mathbb{N}$ the corresponding infinite respectively finite Bohr sets are defined by

$$\begin{aligned} H_\varepsilon(\alpha_1, \dots, \alpha_t) &= \{n \in \mathbb{N} : \|n\alpha_1\|, \dots, \|n\alpha_t\| \leq \varepsilon\}, \\ H_{N,\varepsilon}(\alpha_1, \dots, \alpha_t) &= \{n \leq N : \|n\alpha_1\|, \dots, \|n\alpha_t\| \leq \varepsilon\}. \end{aligned}$$

Using the finite intersection property of filters, one sees that $\mathcal{F} - \lim_n \|n\beta\| = 0$ for all elements β of some given $G < \mathbb{T}$ implies that for all $\alpha_1, \dots, \alpha_t \in G$ and $\varepsilon > 0$ $H_\varepsilon(\alpha_1, \dots, \alpha_t) \in \mathcal{F}$. For each subgroup $G < \mathbb{T}$ there is a canonical (i.e. smallest) candidate for a filter that characterizes G, namely the filter $\mathcal{F}_G$ which consists of all sets containing a set $H_\varepsilon(\alpha_1, \dots, \alpha_t)$ ($\varepsilon > 0, t \in \mathbb{N}, \alpha_1, \dots, \alpha_t \in G$).

To illustrate the connections between the number theoretic approach in [BDS01] respectively [BS03] and the more abstract point of view in [Wi02] we show that the result on the characterization of countable subgroups by sequences of positive integers in [BDS01] implies Proposition 4.2.1:

proof: Let $G < \mathbb{T}$ be an arbitrary subgroup and let $\mathcal{F}_G$ be the filter described above. By definition of $\mathcal{F}_G$ we have $\mathcal{F}_G - \lim_n \|n\beta\| = 0$ for each $\beta \in G$. Now assume $\mathcal{F}_G - \lim_n \|n\beta\| = 0$ for some $\beta \in \mathbb{T}$. For $k \in \mathbb{N}$ let $M_k = H_{1/k}(\beta) \in \mathcal{F}_G$. According to the construction of $\mathcal{F}_G$, there are sequences $t_1 < t_2 < \dots$ ($t_k \in \mathbb{N}$), $(\alpha_t)_{t=1}^\infty$ ($\alpha_t \in G$) and $\varepsilon_1 > \varepsilon_2 > \dots$ ($\varepsilon_k > 0$) such that $M_k \supseteq H_{\varepsilon_k}(\alpha_1, \dots, \alpha_{t_k})$ for all $k \in \mathbb{N}$.

By the result of A. Biró, J.-M. Deshouillers and V. T. Sós there is a sequence $A \subseteq \mathbb{N}$, such that

$$\{\beta \in \mathbb{T} : \lim_{n \in A, n \to \infty} \|n\beta\| = 0\} = \langle \alpha_t : t \in \mathbb{N} \rangle.$$

In particular we have $\lim_{n \in A, n \to \infty} \|n\alpha_t\| = 0$ for all $t \in \mathbb{N}$. Thus for fixed $m \in \mathbb{N}$ we can find $n_m \in \mathbb{N}$ satisfying $\|n\alpha_t\| \le \varepsilon_m$ for all $n \in A, n \ge n_m$ and for all $t \le t_m$. This implies

$$\{n \in A : n > n_m\} \subseteq H_{\varepsilon_m}(\alpha_1, \dots, \alpha_{t_m}) \subseteq M_m,$$

i.e. for all $n \in A, n \ge n_m$ we have $\|n\beta\| \le 1/m$. Since $m \in \mathbb{N}$ was arbitrary this yields $\lim_{n \in A, n \to \infty} \|n\beta\| = 0$ and, as A is a characterizing sequence, $\beta \in \langle \alpha_t : t \in \mathbb{N} \rangle < G$. □

4.3 Ideas of the proof

The rest of this chapter focuses on the proof of Theorem 4.1.1. The proof splits in several lemmas. Before we state and prove them rigorously, we want to give a short sketch of the strategy of the proof and the informal meaning of the individual lemmas:
Lemma 4.4.3 shows how the countable group G may be represented as the limes inferior of certain open subsets V_t of $\mathbb{T}$. These sets may by seen as approximations of G.
Lemma 4.4.2 shows that the behaviour of the values $\|n\beta\|$, where n runs in an appropriate finite Bohr set, may decide whether β lies in an approximation V_t of G. Part (1) of the Lemma uses Theorem 4.2.1, while part (2) follows easily by a compactness argument similar to the reasoning in [BDS01].
The methods developed so far are powerful enough to prove the existence of sequences that characterize countable groups in the sense of [BDS01]. To provide a strong characterizing sequence we use Lemma 4.4.5 to replace a Bohr set H by a somewhat thinner set S that contains the same amount of information but allows in addition to keep the sum $\sum_{n \in S} \|n\alpha\|^r$ ($\alpha \in G, r > 0$) under control. The proof of Lemma 4.4.5 is based on Lemma 4.4.4, a deep result on the structure of Bohr sets due to A. Biró and V. T. Sós ([BS03]).

4.4 Preparations

The following technical facts will be needed later. The proof is elementary, so we skip it.

Lemma 4.4.1 *Let $\alpha, \beta \in \mathbb{T}$ and $n \in \mathbb{N}$.*

(1) Assume $\|\alpha\|, \|2\alpha\|, \dots, \|n\alpha\| \le d < 1/3$. Then $\|\alpha\| \le d/n$.

(2) Assume $\|\beta + 2^0\alpha\|, \|\beta + 2^1\alpha\|, \ldots, \|\beta + 2^n\alpha\| \leq d < 1/6$. *Then* $\|\alpha\| \leq d/2^{n-2}$.

Given $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ and $M \in \mathbb{N}$ we define

$$\langle \alpha_1, \ldots, \alpha_t \rangle_M = \{k_1\alpha_1 + \ldots + k_t\alpha_t : |k_1|, \ldots, |k_t| \leq M\}.$$

We further define $\|\beta S\| = \sup\{\|n\beta\| : n \in S\}$ for $\beta \in \mathbb{T}$ and $S \subseteq \mathbb{N}$.

Lemma 4.4.2 *Let* $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ *and* $\varepsilon > 0$.

(1) There exists some positive integer M *such that*

$$\|\beta H_\varepsilon(\alpha_1, \ldots, \alpha_t)\| \leq 1/6 \;\Rightarrow\; \beta \in \langle \alpha_1, \ldots, \alpha_t \rangle_M.$$

(2) If $V \supseteq \langle \alpha_1, \ldots, \alpha_t \rangle_M$ *is an open subset of* $\mathbb{T}$, *there exists some positive integer* N *such that*

$$\|\beta H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t)\| \leq 1/6 \;\Rightarrow\; \beta \in V.$$

proof: Throughout the proof we suppress mentioning $\alpha_1, \ldots, \alpha_t$ while notating Bohr sets.

(1) Suppose β satisfies $\|\beta H_\varepsilon\| \leq 1/6$. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ that characterizes $\langle \alpha_1, \ldots, \alpha_t \rangle$ and let $m \in \mathbb{N}$ be fixed. Of course we have $H_{\varepsilon/m} \in \mathcal{F}$. For $n \in H_{\varepsilon/m}$ and $k \leq m$ we have $kn \in H_\varepsilon$ and in particular $\|k \cdot n\beta\| \leq 1/6$. Since $1/6 < 1/3$ this implies $\|n\beta\| \leq \frac{1}{6m}$ by Lemma 4.4.1. Thus we have $\|\beta H_{\varepsilon/m}\| \leq \frac{1}{6m}$ and since m was arbitrary we get $\mathcal{F} - \lim_n \|n\beta\| = 0$. $\mathcal{F}$ was assumed to characterize $\langle \alpha_1, \ldots, \alpha_t \rangle$ thus we have $\beta \in \langle \alpha_1, \ldots, \alpha_t \rangle$.
It remains to show that $\{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\}$ is finite. The torsion subgroup of $\langle \alpha_1, \ldots, \alpha_t \rangle$ is finite and cyclic, let its order be $q \in \mathbb{N}$. Then $q\langle \alpha_1, \ldots, \alpha_t \rangle$ is torsion free, hence we find some $\gamma_1, \ldots, \gamma_n \in \mathbb{T}$, such that $q\langle \alpha_1, \ldots, \alpha_t \rangle$ is freely generated by $q\gamma_1, \ldots, q\gamma_n$. We have $\langle \alpha_1, \ldots, \alpha_t \rangle = \langle \gamma_1, \ldots, \gamma_n, 1/q \rangle$ and there are uniquely determined $k_{ij} \in \mathbb{Z}$ $(i \leq t, j \leq n)$ and $k_i \in \{0, \ldots, q-1\}$ $(i \leq t)$, such that

$$\alpha_i = \sum_{j=1}^{n} k_{ij}\gamma_j + k_i/q \;\; (i \leq t).$$

Thus we can find some $\delta > 0$, such that for all $m \in q\mathbb{N}$

$$\|m\gamma_1\|, \ldots, \|m\gamma_n\| \leq \delta \;\Rightarrow \|m\alpha_1\|, \ldots, \|m\alpha_t\| \leq \varepsilon.$$

For each β satisfying $\|\beta H_\varepsilon\| \leq 1/6$ there are uniquely determined $k_j \in \mathbb{Z}$ $(j \leq n)$ and $k \in \{0, \ldots, q-1\}$ such that $\beta = \sum_{j=1}^{n} k_j\gamma_j + k/q$.

If the k_j ($j \leq n$) don't vanish simultaneously, Kronecker's theorem assures that we can find $m \in q\mathbb{N}$, such that

$$\begin{array}{lccccc} \forall j \leq n & \frac{1}{6\sum_{i=1}^n |k_i|} & < & \mathrm{sign}(k_j) m\gamma_j & < & \frac{5}{6\sum_{i=1}^n |k_i|} & \bmod 1 \\ \implies & \frac{1}{6} & < & m\sum_{i=1}^n k_i\gamma_i & < & \frac{5}{6} & \bmod 1, \end{array}$$

i.e. $\|m\beta\| > 1/6$. Thus $\frac{5}{6\sum_{i=1}^n |k_i|} > \delta$. This shows that there are only finitely many choices for the k_i ($i \leq n$). Thus $\{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\}$ is also finite and we can find some $M \in \mathbb{N}$, such that $\{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\} \subseteq \langle \alpha_1, \ldots, \alpha_t \rangle_M$.

(2) Let M be as in (1). Then $\langle \alpha_1, \ldots, \alpha_t \rangle_M \subseteq V$ implies

$$\emptyset = V^c \cap \{\beta \in \mathbb{T} : \|\beta H_\varepsilon\| \leq 1/6\} = V^c \cap \bigcap_{n \in H_\varepsilon} \{\beta \in \mathbb{T} : \|n\beta\| \leq 1/6\}.$$

Since $\mathbb{T}$ is compact and all of the above sets are closed, the intersection of finitely many of these sets must be empty, i.e. we can find some $N \in \mathbb{N}$ such that $V^c \cap \bigcap_{n \in H_{N,\varepsilon}} \{\beta \in \mathbb{T} : \|n\beta\| \leq 1/6\} = \emptyset$. Obviously this N is as required.

□

Lemma 4.4.3 *Let $G = \{\alpha_t : t \in \mathbb{N}\}$ be a subgroup of $\mathbb{T}$ and let $(M_t)_{t=1}^\infty$ be a sequence of positive integers. There exists a sequence $(V_t)_{t=1}^\infty$ of open subsets of T such that*

(i) $V_t \supseteq \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}$ ($t \in \mathbb{N}$),

(ii) $\bigcup_{k=1}^\infty \bigcap_{t=k}^\infty V_t = \liminf_{t\to\infty} V_t = G$.

proof: We may assume that $(M_t)_{t=1}^\infty$ is increasing. We choose a sequence $(\delta_t)_{t=1}^\infty$ of positive numbers that decreases to 0 and satisfies for all $t \in \mathbb{N}$

(1) $2\delta_t < \min\{\|\alpha - \alpha'\| : \alpha, \alpha' \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}, \alpha \neq \alpha'\}$,

(2) $\delta_t + \delta_{t+1} < \min\left\{\|\alpha - \alpha'\| : \begin{array}{l} \alpha \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}, \\ \alpha' \in \langle \alpha_1, \ldots, \alpha_{t+1} \rangle_{M_{t+1}} \setminus \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t} \end{array}\right\}$.

Using this, we define

$$V_t = \{\beta \in \mathbb{T} : \exists \alpha \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t} \ \|\alpha - \beta\| < \delta_t\}.$$

We obviously have $\liminf_{t\to\infty} V_t \supseteq G$. To show the reverse inclusion, assume $\beta \in \liminf_{t\to\infty} V_t$, i.e. $\beta \in V_t$ for all $t \geq t_0$ for some $t_0 \in \mathbb{N}$. By definition of the V_t for all $t \geq t_0$ there is some $\gamma_t \in \langle \alpha_1, \ldots, \alpha_t \rangle_{M_t}$ satisfying $\|\beta - \gamma_t\| < \delta_t$

and (1) shows that this γ_t is uniquely determined. Further $\gamma_t \neq \gamma_{t+1}$ for some $t \geq t_0$ would contradict (2), thus we have $\gamma_{t_0} = \gamma_{t_0+1} = \gamma_{t_0+2} = \cdots$. In particular this shows $\|\beta - \gamma_{t_0}\| = \|\beta - \gamma_t\| < \delta_t \to 0$, hence $\beta = \gamma_{t_0} \in G$. $\square$

From Lemma 1 in [BS03] one gets:

Lemma 4.4.4 *Let $t \in \mathbb{N}$. There exists some constant $C_1 = C_1(t)$, such that for all $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$, positive $\varepsilon \leq 1/C_1$ and positive integers N there are suitable nonzero integers $n_1, \ldots, n_R$ and positive integers $K_1, \ldots, K_R$, $R \leq C_1$ satisfying*

(a) $\sum_{i=1}^{R} K_i \|n_i \alpha_j\| \leq C_1 \cdot \varepsilon \quad (1 \leq j \leq t)$

(b) $\sum_{i=1}^{R} K_i |n_i| \leq C_1 \cdot N$,

(c) $H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t) \subseteq \left\{ \sum_{i=1}^{R} k_i n_i : 1 \leq k_i \leq K_i \right\}$.

Lemma 4.4.5 *Let $t \in \mathbb{N}$. There exists some constant $C_2 = C_2(t)$, such that for all $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$, positive $\varepsilon \leq 1/C_1(t)$, positive $r \leq 1$ and positive integers N and U there is a suitable nonempty finite set S of integers satisfying*

(i) $U < \min S$,

(ii) for all $j \leq t$ *we have* $\sum_{n \in S} \|n\alpha_j\|^r \leq C_2 \cdot \frac{\varepsilon^r}{2^r - 1}$,

(iii) for all $\beta \in \mathbb{T}$ *we have* $\min\{1/6, \|\beta H_{N,\varepsilon}(\alpha_1, \ldots, \alpha_t)\|\} \leq \|\beta S\|$.

proof: Let $\alpha_1, \ldots, \alpha_t \in \mathbb{T}$ and C_1, R, K_i, n_i $(i \leq R)$ as given by Lemma 4.4.4. Let $m > U$ be an integer satisfying

$$\|m\alpha_j\|^r \leq \frac{\varepsilon^r}{\lg_2(8 \cdot C_1^2 \cdot N)}$$

for all $j \leq t$ and let

$$S = \{m + 2^l |n_i| : 2^l \leq 8 \cdot K_i \cdot R\}.$$

Clearly S satisfies (i).
For each $j \leq t$ we have

$$\sum_{n \in S} \|n\alpha_j\|^r \leq \mathrm{card}(S) \cdot \|m\alpha_j\|^r + \sum_{n \in S} \|(n-m)\alpha_j\|^r.$$

To find an upper bound for the first term, we observe that $K_i \leq C_1 \cdot N$ implies $\mathrm{card}(S) \leq R \lg_2(8 \cdot C_1 \cdot N \cdot R)$. Thus

$$\mathrm{card}(S) \cdot \|m\alpha_j\|^r \leq R \lg_2(8 \cdot C_1 \cdot N \cdot R) \cdot \frac{\varepsilon^r}{\lg_2(8 \cdot C_1^2 \cdot N)} \leq C_1 \cdot \varepsilon^r.$$

The second term can be estimated by

$$\sum_{i=1}^{R}\left(\sum_{l=0}^{\lfloor \lg_2(8\cdot K_i\cdot R)\rfloor} 2^l \|n_i\alpha_j\|\right)^r \le \sum_{i=1}^{R}\frac{(2^r)^{\lg_2(8K_iR)+1}-1}{2^r-1}\|n_i\alpha\|^r$$
$$< \frac{16^r R^r}{2^r-1}\sum_{i=1}^{R} K_i^r \|n_i\alpha\|^r.$$

For any $a_1,\dots,a_R$ we have $\frac{1}{R}\sum_{i=1}^{R} a_i^r \le \left(\frac{1}{R}\sum_{i=1}^{R} a_i\right)^r$ by Jensen's inequality. This yields

$$\sum_{n\in S}\|(n-m)\alpha_j\|^r \le \frac{16^r R}{2^r-1}\left(\sum_{i=1}^{R} K_i\|n_i\alpha_j\|\right)^r \le \frac{16^r C_1}{2^r-1}\left(C_1\varepsilon\right)^r.$$

Thus S will satisfy (ii) if we let $C_2 = C_1 + 16\cdot C_1^2$.
Finally let $\beta\in\mathbb{T}$ and $d = \|\beta S\|$. We may assume $d < 1/6$. Thus by Lemma 4.4.1 for all $i \le R$

$$\|m\beta + 2^l |n_i|\beta\| \le d \ (l \le \lg_2(8\cdot K_i\cdot R))$$

implies

$$\|n_i\beta\| \le \frac{d}{2^{\lfloor lg_2(8\cdot K_i\cdot R)\rfloor - 2}} \le \frac{d}{K_i\cdot R}.$$

By Lemma 4.4.4 each $n\in H_{N,\varepsilon}(\alpha_1,\dots,\alpha_t)$ has a representation $n=\sum_{i=1}^{R} k_i n_i$ for some integers $k_i, (1\le i\le R)$ satisfying $1\le k_i\le K_i$. Using this representation we get

$$\|n\beta\| = \left\|\sum_{i=1}^{R} k_i n_i\beta\right\| \le \sum_{i=1}^{R} K_i\|n_i\beta\| \le \sum_{i=1}^{R} K_i\frac{d}{K_i\cdot R} = d.$$

Thus S satisfies (iii). □

4.5 Proof of the Theorem

Finally we are able to give the proof of Theorem 4.1.1. Let $(\varepsilon_t)_{t=1}^{\infty}$ be a sequence of positive numbers, satisfying $\varepsilon_t < 1/C_1(t)$ and $\sum_{t=1}^{\infty} C_2(t)\frac{\varepsilon_t^{1/t}}{2^{1/t}-1} < \infty$. Combining Lemma 4.4.2 and Lemma 4.4.3 we find a sequence $(N_t)_{t=1}^{\infty}$ of positive integers and a sequence $(V_t)_{t=1}^{\infty}$ of open subsets of $\mathbb{T}$, such that:

(1) For all $\beta\in\mathbb{T}$ and for all $t\in\mathbb{N}$ $\|\beta H_{N_t,\varepsilon_t}(\alpha_1,\dots,\alpha_t)\| \le 1/6 \ \Rightarrow\ \beta\in V_t$.

(2) $\bigcup_{k=1}^{\infty}\bigcap_{t=k}^{\infty} V_t = G$.

Using Lemma 4.4.5 we find some sequence $(S_t)_{t=1}^{\infty}$ of subsets of $\mathbb{N}$ such that for all $t \in \mathbb{T}$

(i) $\max S_t < \min S_{t+1}$,

(ii) $\sum_{n \in S_t} \|n\alpha_j\|^{1/t} \leq C_2(t) \cdot \frac{\varepsilon_t^{1/t}}{2^{1/t}-1}$ $(j \leq t)$,

(iii) for all $\beta \in \mathbb{T}$ $\min\{1/6, \|\beta H_{N_t,\varepsilon_t}(\alpha_1, \dots, \alpha_t)\|\} \leq \|\beta S_t\|$.

By defining $A = \bigcup_{t=1}^{\infty} S_t$ we will in fact get a strong characterizing sequence of G as stated in Theorem 4.1.1:
Assume $\beta \in G$ and $r > 0$. Then $\beta = \alpha_{t_0}$ for some $t_0 \in \mathbb{N}$. If we let $m > \max\{t_0, 1/r\}$, we have

$$\sum_{n \in A, n \geq \min S_m} \|n\beta\|^r \leq \sum_{t \geq m} \sum_{n \in S_t} \|n\alpha_{t_0}\|^{1/t} \leq \sum_{t \geq t_0} C_2(t) \frac{\varepsilon_t^{1/t}}{2^{1/t} - 1} < \infty.$$

Finally, assume $\beta \notin G$. There exists a sequence $t_1 < t_2 < \dots$ of positive integers such that $\beta \notin V_{t_k}$ $(k \in \mathbb{N})$. So for each $k \in \mathbb{N}$ we have $\|\beta H_{\varepsilon_{t_k}, N_{t_k}}(\alpha_1, \dots, \alpha_{t_k})\| > 1/6$ and thus can find some $n_k \in S_{t_k}$ satisfying $\|\beta n_k\| \geq 1/6$. This shows $\limsup_{n \in A, n \to \infty} \|n\beta\| \geq 1/6$.

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