DISSERTATION

Symmetries and Renormalization of Noncommutative Field Theories

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines *Doktors der technischen Wissenschaften* unter der Leitung von

Univ.Prof. Dr. Manfred Schweda
E136
Institut für theoretische Physik

eingereicht an der Technischen Universität Wien
Fakultät für technische Naturwissenschaften und Informatik

von

Dipl.-Ing. Volkmar Putz
9626026
Billrothstraße 45/3/8, A-1190 Wien
Email: putz@hep.itp.tuwien.ac.at

Wien, am 10.12.03

eigenhändige Unterschrift
Volkmar Putz

Deutsche Kurzfassung

Diese Dissertation beschäftigt sich mit nichtkommutativen Feldtheorien. Die beiden Hauptteile befassen sich mit Renormierung, konkret der Formulierung und Lösung des sogenannten UV/IR-mixing Problems. Eingeschoben ist ein Abschnitt über Symmetrien in Zusammenhang mit der Seiberg-Witten Abbildung von Eichfeldtheorien.

Teil I: Kapitel 1 stellt das Problem vor, Kapitel 2 (basierend auf den Arbeiten [1] und [2]) präsentiert zwei einfache Lösungsvorschläge zunächst für die skalare ϕ^4-Theorie, dann für die Vektoreichfeldtheorie.

Intermezzo: Die Kapitel 3 und 4 (nach [3] bzw. [4]) betrachten zwei Aspekte der sogenannten Seiberg-Witten (SW-) Abbildung. Dabei handelt es sich um eine Entwicklung des nichtkommutativen Eichfeldes nach Potenzen des Nichtkommutativitäts-Parameter $\theta_{\mu\nu}$. Kapitel 3 diskutiert die Möglichkeiten einer Konstruktion des Energie-Impuls-Tensors im Rahmen einer SW-entwikkelten Eichfeldtheorie. Kapitel 4 formuliert die SW-Abbildung für den Fall einer nichtkommutativen super Yang-Mills Eichtheorie.

Teil II: Die Kapitel 5, 6 und 7 (nach [5] und [6]) betrachten abermals das UV/IR-mixing Problem, diesmal von wesentlich grundlegenderer Seite. Der Formalismus der Quantenfeldtheorie wird sowohl für die nichtkommutative skalare als auch für die nichtkommutative Vektoreichfeld-Theorie konstruiert. Eine abweichende Definition der Zeitordnung führt zu Unitarität und Renormierbarkeit ohne UV/IR-Probleme, wie anhand der skalaren Feldtheorie gezeigt wird.

To my Parents and my Aunt

Die diese Tat mir auferlegt,
die Götter werden da sein, mir zu helfen.

Hugo von Hofmannsthal

Summary

This thesis deals with noncommutative field theories. The two main parts focus on renormalization, in particular the formulation and solution of the so-called UV/IR-mixing problem. The interjectional "Intermezzo" treats symmetries with respect to the Seiberg-Witten expansion of gauge theories.

Part I: Chapter 1 introduces the problem, chapter 2 (based on the papers [1] and [2]) presents two simple proposals for a solution, first for the scalar ϕ^4-theory, then for the vector gauge field theory.

Intermezzo: Chapters 3 and 4 (after [3] and [4], respectively) consider two aspects of the so-called Seiberg-Witten (SW-) map. This is an expansion of the noncommutative gauge field with respect to the noncommutativity parameter $\theta_{\mu\nu}$. Chapter 3 discusses the possibilities to construct the energy-momentum tensor of a SW-expanded gauge field theory. Chapter 4 formulates the SW-map for a noncommutative super Yang-Mills gauge theory.

Part II: Chapters 5, 6 and 7 (after [5] and [6]) again focus on the UV/IR-mixing problem, now from a more fundamental point of view. The formalism of quantum field theory is constructed for both, noncommutative scalar and vector gauge field theory. A deviating definition of the time ordering restores unitarity and renormalizability without UV/IR-problems, which is demonstrated by means of the scalar field theory.

Contents

First of all I want to thank my supervisor Prof. Manfred Schweda for his dedicated support and encouragement.

Then I have to thank the Max-Planck-Institute for Mathematics in the Sciences in Leipzig, where most of the present work has been done. Special thanks go to Raimar Wulkenhaar who has been my babysitter during my stay in Leipzig.

Finally, I must thank Richard Strauss, whose operas made this work possible.

Part I

Steh mir, Gespenst!

Hugo von Hofmannsthal

Chapter 1

Introduction

1.1 Propaganda

There are two major theories in physics today, Einstein's theory of special and general relativity concerning gravity on one hand and quantum mechanics/quantum field theory (which grew eventually to the standard model) concerning 'the rest of the world' on the other hand. Both theories rely upon standard differential geometry. The standard model is most efficiently described via fibre bundles, and Einstein's theory of gravity uses Riemannian geometry.

Unfortunately, exactly the attempt of combining both theories (GRT and QFT) has shown that the concept of space-time as a differentiable manifold will break down at very small distances. A simple heuristic argument given by [7], [8] combining the principles of general relativity and quantum theory shows that it is impossible to locate a particle with arbitrary accuracy. In short, the energy transfer required to make an increasingly exact observation will eventually create a black hole: A fundamental horizon which makes impossible any measurement below the Planck length. Of course, one could argue that space-time still *is* a manifold. However, history has shown that physics does not use idealised concepts which are proven to be undetectable (*e.g.* the aether which became dispensable after special relativity has been invented). Thus we conclude: Standard differential geometry is not the mathematical framework to describe physics at extremely short distances. Space-time cannot be a manifold.

So, after giving up standard differential geometry—what should replace it? A first, natural thought would be of a lattice. However, the disadvantage of a lattice is that *symmetries e.g.* translational and rotational symmetry in space-time), which are guiding principles in quantum field theory, are lost. Furthermore, there are other conceptual problems, for example with the spin structure and the construction of a differential calculus.

Another, more promising candidate is *noncommutative geometry*, which has been pioneered by Alain Connes, [9] for a review see [10]. Based on an idea by Snyder [11], noncommutative (NC) geometry tries to extend the principles of quantum mechanics to geometry itself: The concepts of operator algebras, Hilbert spaces and functional analysis. Indeed, there exists a class of metric spaces equipped with a differential calculus and a spin structure (to allow for fermions). These mathematical objects are called *spectral triples*, [12]. They are noncommutative geometries consisting of a (space/time-) algebra $\mathcal{A}$, represented on a Hilbert space $\mathcal{H}$, and a Dirac operator $\mathcal{D}$ (which is necessary to describe a spin structure). Moreover, they allow for the definition chirality γ_5 and charge conjugation J, [13].

The great triumph of the spectral triples is that the standard model of modern physics looks much simpler when formulated in the language of spectral triples [13], [14]. The most remarkable features are: the Higgs field appears as a component of a gauge field living on a spectral triple; parity breaking and spontaneous symmetry breaking are enforced by the NC formulation; gravity and gauge fields (Yang-Mills and Higgs) are all 'created' by the the free Dirac operator: The so called spectral action of the Dirac operator $\mathcal{D}$ gives the complete bosonic action of the standard model, the Einstein-Hilbert action (with cosmological constant) and an additional Weyl term in one stroke [15]. Thus the full bosonic content of the standard model is determined by the fermionic content of $\mathcal{D}$.

Of course, despite this remarkable features, there are technical difficulties with spectral triples (such as the restriction to compact spaces with Euclidean signature). In particular, how to perform the quantisation of the spectral action is completely unclear. The simplest way to test whether the standard calculus of quantum field theory extends to spectral triples is to apply it to the most simple examples—deformations of a manifold. This means that we focus our interest on spectral triples depending, let's say, on a set of parameters θ such that for $\theta \to 0$ we recover an ordinary manifold. Then we may expect that all results computed within this spectral triple (we call it θ deformed space, see next chapter) tend for $\theta \to 0$ to the results computed from QFT on a manifold (the ordinary space).

1.2 The Simplest Case

To obtain the most primitive version of a spectral triple, one generalises the usual concepts of quantum mechanics on Euclidean space, which are defined by the following commutation relation [16], [17],

$$[\hat{X}_\mu, \hat{P}_\nu] = i\delta_{\mu\nu}, \quad [\hat{X}_\mu, \hat{X}_\nu] = [\hat{P}_\nu, \hat{P}_\nu] = 0, \qquad (\hbar = 1) \tag{1.1}$$

where $\hat{X}_\nu$ and $\hat{P}_\nu$ are the hermitian position and momentum operators, respectively. Now, in order to get a noncommutative geometry, we assume the following natural generalisation

$$[\hat{X}_\mu, \hat{X}_\nu] = i\theta_{\mu\nu}, \tag{1.2}$$

with the deformation parameter $\theta_{\mu\nu}$ being an antisymmetric constant matrix of dimension $[\text{length}]^2$. Thus the noncommutative space-time becomes blurry, as any two dimensional subspace $\{x_\alpha, x_\beta\}$ is divided into 'plaquettes' of area $\theta_{\alpha\beta}$. Since we want to maintain Lorentz symmetry, the existence of the constant antisymmetric tensor $\theta_{\mu\nu}$ makes necessary a modification of the Lorentz transformation [18].

To construct the perturbative field theory formulation we will use ordinary fields and not operator-valued objects. One has the so-called Moyal-Weyl correspondence defined by

$$\hat{\phi}(\hat{X}) \Longleftrightarrow \phi(x), \tag{1.3}$$

where $\hat{\phi}(\hat{X})$ is an operator valued functional and $\phi(x)$ is the usual scalar field depending on ordinary (commuting) Euclidean coordinates x_μ. The correspondence is given by

$$\begin{aligned} \hat{\phi}(\hat{X}) &= \int \frac{d^4k}{(2\pi)^4} e^{ik\hat{X}} \phi(k), \\ \phi(k) &= \int d^4x e^{-ikx} \phi(x). \end{aligned} \tag{1.4}$$

Here k and x are commutative, real variables. With the help of the Baker-Campbell-Hausdorff-formula one finds (in Euclidean space)

$$\begin{aligned} \hat{\phi}_1(\hat{X})\hat{\phi}_2(\hat{X}) &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{ip\hat{X}} e^{iq\hat{X}} \phi_1(p)\phi_2(q) \\ &= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{i(p+q)\hat{X} - \frac{i}{2}\theta_{\mu\nu}p_\mu q_\nu} \phi_1(p)\phi_2(q). \end{aligned} \tag{1.5}$$

Now one can introduce the Moyal-Weyl (or star-) product,

$$\hat{\phi}_1(\hat{X})\hat{\phi}_2(\hat{X}) \Longleftrightarrow \Big(\phi_1 \star \phi_2\Big)(x), \tag{1.6}$$

which is defined as [16], [19]

$$\Big(\phi_1 \star \phi_2\Big)(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}\right)\phi(x)\phi(y)|_{y=x}\,. \tag{1.7}$$

Note: The derivatives in this exponential form of the $\star$-product are generalised derivatives in the sense of distribution theory, not ordinary derivatives. As such one cannot apply the naïve rules of differential calculus. To make this transparent, write $\phi(x+a)\phi(y) = \exp(a^\mu\partial_\mu^x)\phi(x)\phi(y)$, and hide the exponential of the derivatives in the definition of the product. For example, with respect to time ordering, it would be completely wrong to use the step function $\tau(x^0 - y^0)$ or $\tau(y^0 - x^0)$ for the product $\phi(x+a)\phi(y)$, [20].

Due to this problems, we will employ a slightly different definition of the Moyal product throughout this theses:

$$\Big(\phi_1 \star \phi_2\Big)(x) := \int d^4s \int \frac{d^4l}{(2\pi)^4}\phi_1(x - \frac{1}{2}\tilde{l})\phi_2(x+s)e^{ils}, \quad \tilde{l}_\nu = l_\mu\theta_{\mu\nu}. \tag{1.8}$$

Here the fields need not be analytic functions as in (1.7), but have to vanish rapidly at infinity. Note that x is the commutative coordinate corresponding to $\hat{X}$. The philosophy in deformed field theory is now to realize the noncommutativity property by a mere replacement of all ordinary field products by Moyal products. The advantage of this approach is that one needs not perform calculations explicitly on the noncommutative algebra $\mathcal{A}$ but stays on familiar, commutative space-time.

1.3 Action

With the star-product one is able to define, for example, the noncommutative scalar self-interacting classical action in a four-dimensional Euclidean space,

$$\Gamma^{(0)}[\phi] = \int d^4x\Big(\frac{1}{2}\big(\partial_\mu\phi(x)\partial_\mu\phi(x) + m^2\phi^2(x)\big) + \frac{g^2}{4!}\phi\star\phi\star\phi\star\phi(x)\Big). \tag{1.9}$$

The perturbative properties of such a noncommutative field model are studied in great detail in [21], [22]. Note that the quadratic part of the action (1.9) remains unchanged compared to the commutative theory. The $\star$-product in the interaction term leads to a momentum-dependent phase factor associated with each vertex in a given Feynman diagram.

In perturbation theory, the interplay of these phase factors produces two different types of graphs leading to planar and non-planar contributions, as originally proposed by [16]. These are distinguished by their behaviour with respect to the ultra-violet (UV) region. The planar graphs show the desired effects expected from naive power counting and known from commutative theory. The planar divergent radiative corrections can therefore be discussed in the framework of the usual UV-renormalization procedure. Some of the non-planar diagrams, however, show an ugly nonlocal behaviour. The a priori divergent contributions are regularized by the phase factors that are associated with each crossing of lines in the graph. The rapid oscillations of these phases regulate the integrals and thus suppress any divergence, i. e. an otherwise divergent graph becomes finite with an effective cutoff (at one-loop)

$$\Lambda_{eff} = \frac{1}{\sqrt{\theta_{\mu\nu} p_\nu \theta_{\mu\rho} p_\rho}} \equiv \frac{1}{\sqrt{\tilde{p}^2}}. \tag{1.10}$$

Therefore, the original UV-divergence is replaced by an IR-singularity in the limit of vanishing external momenta p (if one discusses one-loop self-energy corrections) implying $\Lambda_{eff} \to \infty$ (for a rigorous analysis see [23]). This artifact is the so called UV/IR-mixing problem of NCQFT [21].

1.4 The Problem

In order to understand the UV/IR-mixing problem it is useful to show more precisely how it enters the game [24], [21]. The action (1.9) induces the following one-loop Feynman integral describing the first order quantum correction to the two point function

$$\Delta\Sigma = \frac{g^2}{6} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \big(2 + \cos(k\tilde{p})\big), \quad \tilde{p}_\nu = p_\mu \theta_{\mu\nu}, \tag{1.11}$$

where we have used the well-known propagator for the scalar field,

$$\Delta(k) = \frac{1}{k^2 + m^2}, \tag{1.12}$$

and the corresponding Feynman rule for the noncommutative interaction vertex [21]. This integral splits up in a planar contribution (leading to the usual mass renormalization),

$$\Delta\Sigma_p = \frac{g^2}{3}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\,, \tag{1.13}$$

and a non-planar contribution,

$$\Delta\Sigma_{np} = \frac{g^2}{6}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}e^{ik\tilde{p}}\,. \tag{1.14}$$

With the usual techniques using Schwinger parametrisation one gets for the non-planar expression after Gaussian integration

$$\Delta\Sigma_{np} = \frac{g^2}{6}\frac{\pi^2}{(2\pi)^4}\int_0^\infty \frac{d\alpha}{\alpha^2}e^{-\alpha m^2-\frac{\tilde{p}^2}{4\alpha}}\,, \tag{1.15}$$

where $\tilde{p}^2$ acts as regulator.

With $\int_0^\infty \frac{d\alpha}{\alpha^2}\exp(-u\alpha - v/(4\alpha)) = 4\sqrt{(u/v)}K_1(\sqrt{uv})$ for positive real part of u and v we find

$$\Delta\Sigma_{np} = \frac{g^2}{6}\frac{\pi^2}{(2\pi)^4}4\sqrt{\frac{m^2}{\tilde{p}^2}}K_1\left(\sqrt{m^2\tilde{p}^2}\right). \tag{1.16}$$

With the expansion of the modified Bessel function $K_1(x) = \frac{1}{x} + \frac{x}{4}(2\gamma - 1 - 2\ln 2) + \frac{x}{2}\ln x + \mathcal{O}(x^3)$, where $\gamma \simeq 0.577$ is the Euler gamma, we get for small $\tilde{p}^2$

$$\Delta\Sigma_{np} = \frac{g^2}{24\pi^2}\Big(\frac{1}{\tilde{p}^2} + \frac{m^2}{4}\ln(m^2\tilde{p}^2) + \mathcal{O}(1)\Big), \tag{1.17}$$

where $\mathcal{O}(1)$ stands for the terms remaining finite for $\tilde{p}^2 \to 0$. Thus, as suggested above, in the non-planar section of NCQFT the original UV-divergence of the commutative theory has been regularized by the momentum dependent cut-off $\tilde{p}^2$. In the commutative limit $\theta_{\mu\nu} = 0 = \tilde{p}^2$ the divergence reappears. Unfortunately, even the regularized divergence ($\theta_{\mu\nu} \neq 0$) causes troubles: The first term on the right hand side of (1.17) gives rise to severe IR-singularities when inserted into higher order loop integrals $\int d^4p$. This miraculous conservation of misery is exactly the UV/IR-mixing of divergences.

On the other hand, the regularized ($\Lambda \to \infty$) planar term reads

$$\Delta\Sigma_p = \frac{g^2}{12\pi^2}\Big(\Lambda^2 + \frac{m^2}{4}\ln(m^2/\Lambda^2) + \mathcal{O}(1)\Big) = \delta m^2. \qquad (1.18)$$

After performing the ordinary mass renormalization (treating the planar correction) $M^2 = m^2 + \delta m^2$, the effective two point action to first order is

$$\Gamma_2^{(1)} = \qquad (1.19)$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(p)\phi(-p)\Big(p^2 + M^2 + \frac{g^2}{24\pi^2}\big(\frac{1}{\tilde{p}^2} + \frac{M^2}{4}\ln(M^2\tilde{p}^2) + \ldots\big)\Big).$$

Now, in order to eliminate the UV/IR-mixing, one has to handle the horrible nonlocal $1/\tilde{p}^2$ term. At first, we want to present two very simple approaches.

Chapter 2

Redefinition/Resummation

2.1 Redefinition

2.1.1 Field Redefinition in ϕ^4-Theory

The aim of this section is to present a first, simple way of discussing the IR-singularities. In [25], [26] it has been shown that the concept of field redefinition originally proposed in [27], [28], [29] is very useful for the perturbative description of noncommutative $U(1)$ gauge field models. Therefore, it is quite natural to use an appropriate field redefinition also in the present case to analyse the IR-structure of an effective two-point vertex function at $O(g^2)$, at first for the simplest case of a ϕ^4-theory. Similar results have been derived in [24] in the context of Wilsonian Renormalization Group and hard noncommutative loop resummation (see next chapter).

In order to keep trace of the infinities we first introduce an effective cutoff,

$$\frac{1}{\tilde{p}^2} \to \Lambda_{eff}^2 = \frac{1}{\tilde{p}^2 + \frac{1}{\Lambda^2}} \tag{2.1}$$

The one-loop 1PI quadratic effective action (1.19) becomes therefore

$$\Gamma_{eff}^{(2)} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2} \phi(p)\phi(-p) \left(p^2 + M^2 + \frac{g^2}{24\pi^2(\tilde{p}^2 + \frac{1}{\Lambda^2})} \right.$$
$$\left. - \frac{g^2}{96\pi^2} M^2 \ln\left(\frac{1}{M^2(\tilde{p}^2 + \frac{1}{\Lambda^2})} \right) + O(g^4) \right). \tag{2.2}$$

As is explained in [21] the limit $\Lambda \to \infty$ does not commute with the low momentum limit $p \to 0$ (IR-region). Of course, this is again a manifestation of the UV/IR-mixing.

The idea of redefinition is the following: One tries to obtain a "new" effective one-loop two-point vertex function

$$\Gamma'^{(2)}_{eff} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2}\phi(p)\phi(-p)\Bigg(p^2 + M^2 - \frac{g^2}{96\pi^2}M^2 \ln\left(\frac{1}{M^2(\tilde{p}^2 + \frac{1}{\Lambda^2})}\right) + O(g^4)\Bigg) \quad (2.3)$$

(without the problematic term $\frac{1}{\tilde{p}^2+\frac{1}{\Lambda^2}}$) as a result of a field redefinition

$$\phi(p) \to \phi(p) + f(p,\theta,\Lambda)\phi(p). \quad (2.4)$$

A simple calculation shows that a solution $f(p)$ is of the following form

$$f(p) = -\frac{1}{2}\frac{g^2}{24\pi^2}\frac{1}{(p^2+m^2)}\frac{1}{\tilde{p}^2 + \frac{1}{\Lambda^2}}, \quad (2.5)$$

implying the redefinition of the field $\phi(x)$ in position space

$$\phi(x) \to \phi(x) - \frac{1}{2}\frac{g^2}{24\pi^2}\frac{1}{(\Box - m^2)}\frac{1}{\tilde{\partial}^2}\phi(x). \quad (2.6)$$

Therefore, also the action (1.9) must be changed accordingly

$$S' = \int d^4x \left(\frac{1}{2}\left(\partial_\mu\phi\partial_\mu\phi + m^2\phi^2 + \frac{g^2}{24\pi^2}\phi\frac{1}{\tilde{\partial}^2}\phi\right) + \frac{g^2}{4!}\phi\star\phi\star\phi\star\phi + O(g^4)\right). \quad (2.7)$$

It is now straightforward to compute an "IR-regular" quadratic effective action up to the given order in g^2 with this new action, yielding

$$\Gamma'^{(2)}_{eff} = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2}\phi(p)\phi(-p)\left(p^2 + M^2 - \frac{g^2}{24\pi^2}\left(\frac{1}{\tilde{p}^2} - \frac{1}{\tilde{p}^2+\frac{1}{\Lambda^2}}\right) - \frac{g^2}{96\pi^2}M^2 \ln\left(\frac{1}{M^2(\tilde{p}^2 + \frac{1}{\Lambda^2})}\right) + O(g^4)\right). \quad (2.8)$$

In the limit $\Lambda \to \infty$ one arrives at

$$\Gamma'^{(2)}_{eff} = \int \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(p)\phi(-p)\left(p^2 + M^2 - \frac{g^2}{96\pi^2}M^2 \ln\frac{1}{M^2\tilde{p}^2} + O(g^4)\right), \tag{2.9}$$

which does not contain any non-integrable IR-singularities [30].

At this point one has to make the following comments:

- Because of the last term in (2.7) it is clear that the problem of UV/IR-mixing is not solved by this simple field redefinition, since the problems have only been transferred from the 2-point function to higher n-point functions. For example, at order g^4 the field redefinition produces a term proportional to

$$\frac{g^4}{(\Box - m^2)}\frac{1}{\tilde{\partial}^2}\phi^4, \tag{2.10}$$

  which induces new IR-singularities.

- The correction term in the field redefinition (2.6) is of order g^2. Thus, the bare propagator (the free-field case being defined by $g = 0$) remains unchanged:

$$\Delta(p) = \frac{1}{p^2 + m^2}. \tag{2.11}$$

- The field redefinition (2.6) is nonlocal and induces also a nonlocal term in the action (2.7). Such nonlocal field redefinitions are known to arise in non-Abelian gauge field models quantised in the axial gauge, where the redefinition must be compatible with BRST-symmetry [31] (and references therein).

- In order to reproduce the UV/IR-mixing the authors of [21], [22] have interpreted the IR-singularities in the nonplanar one-loop diagrams as tree level exchange of new light degrees of freedom. In our approach there is no need of introducing these degrees of freedom.

- Since the dangerous term $\frac{g^2}{24\pi^2}\frac{1}{\tilde{p}^2+\frac{1}{\Lambda^2}}$ in (2.2) does not depend on the mass (physical or bare mass) the massless case is also well defined implying a field redefinition

$$\phi(x) \to \phi(x) - \frac{1}{2}\frac{g^2}{96\pi^2}\frac{1}{\Box}\frac{1}{\tilde{\partial}^2}\phi(x) \tag{2.12}$$

leading to the following quadratic effective one-loop action

$$\Gamma'^{(2)}_{eff} = \int \frac{d^4p}{(2\pi)^4}\frac{1}{2}\phi(p)\phi(-p)\left(p^2 - \frac{g^2}{24\pi^2}\left(\frac{1}{\tilde{p}^2} - \frac{1}{\tilde{p}^2 + \frac{1}{\Lambda^2}}\right) + O(g^4)\right) \tag{2.13}$$

for finite Λ^2.

2.1.2 Field Redefinition in Gauge Theory

Now we try to sketch the idea of a redefiniton procedure for a gauge field model with BRST-symmetry [32], [33]. We begin with a pure $U(1)$-noncommutative Yang-Mills (NCYM)-theory, which is described in Euclidean space at the classical level by

$$\Gamma^{(0)}_{INV} = \frac{1}{4}\int d^4x F_{\mu\nu} \star F_{\mu\nu}, \tag{2.14}$$

where the field strength $F_{\mu\nu}$ is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]_M, \quad [A, B]_M := A \star B - B \star A. \tag{2.15}$$

The enclosure of fermions can be done in the usual way [34], [35]. Scalar matter fields are treated in [36]. In order to allow a meaningful perturbation theory one has to use the BRST-quantisation procedure [32], [33], which implies the introduction of ghost fields $c, \bar{c}$ and a multiplier field B for the gauge fixing. One has the total action

$$\begin{aligned}\Gamma^{(0)} &= \Gamma^{(0)}_{INV} + \Gamma_{gf} + \Gamma_{matter} = \frac{1}{4}\int d^4x F_{\mu\nu} \star F_{\mu\nu} \\ &+ \int d^4x (gB \star \partial_\mu A_\mu + \frac{\alpha}{2} B \star B - \bar{c} \star \partial_\mu D_\mu c) + \Gamma_{matter},\end{aligned} \tag{2.16}$$

where $D_\mu := \partial_\mu - ig[A_\mu, .]_M$. The corresponding nonlinear, SUSY-like and nilpotent BRST-transformation is given by

$$\begin{gathered} sA_\mu = D_\mu c, \quad sc = ic \star c, \\ s\bar{c} = B, \quad sB = 0, \\ s^2 A_\mu = s^2 c = s^2 \bar{c} = s^2 B = 0 \ . \end{gathered} \tag{2.17}$$

Doing now perturbation theory at the one-loop level (including all contributions coming from the gluon, ghost, fermion etc. fields) one obtains the following problematic contribution to the vacuum polarisation of the photon [34],

$$\Pi_{\mu\nu}(p) = g^2 \beta \, \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} \tag{2.18}$$

(with β some numerical constant of order $(g)^0$), which represents the non-planar one-loop contribution. This is the well-known UV/IR-mixing term for non-supersymmetric NCYM-models. It is singular for $p_\mu \to 0$. Note that (2.18) is, after all, transversal. Its properties regarding gauge covariance will be discussed below. The corresponding term of the effective action is

$$\delta\Gamma = \frac{1}{2} A_\mu(p) \, g^2 \beta \, \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} A_\nu(-p). \tag{2.19}$$

(A discussion of Now, in search for a proper field redefinition we notice that the leading term (order $(g)^0$) of the action is

$$\Gamma = \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} \Big(p_\mu A_\nu(p) - p_\nu A_\mu(p) \Big) \Big(p_\mu A_\nu(-p) - p_\nu A_\mu(-p) \Big) + \mathcal{O}(g^1). \tag{2.20}$$

Now we redefine

$$A_\mu(p) \to A_\mu(p) - \frac{1}{2} g^2 \beta \, \frac{\tilde{p}_\mu \tilde{p}_\nu}{p^2 (\tilde{p}^2)^2} A_\nu(p) \tag{2.21}$$

With this (2.20) becomes (note that $p_\mu \tilde{p}_\mu = 0$)

$$\begin{aligned} \Gamma \;=\; & \frac{1}{4} \int \frac{d^4p}{(2\pi)^4} \Big(p_\mu A_\nu(p) - p_\nu A_\mu(p) \Big) \Big(p_\mu A_\nu(-p) - p_\nu A_\mu(-p) \Big) \\ & -2 A_\mu(p) \, g^2 \beta \, \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} A_\nu(-p) + \mathcal{O}(g^1). \end{aligned} \tag{2.22}$$

The last term cancels exactly (2.19). Of course, this naive picture of a gauge field redefinition suffers from the same problems as for scalar field theory. In particular, it is not clear if the difficulties in higher orders of g can be managed.

2.2 Resummation

Since the method of field redefinitions has the somewhat awkward property of changing the action explicitly, we want to discuss another possibility (avoiding this problem) to cure the UV/IR-mixing. Especially, we investigate the idea of resummation proposed in the literature [24].

2.2.1 The Resummation of ϕ^4-Theory

The recipe to cure the UV/IR-mixing (at one-loop order) via resummation is given by adding and subtracting to the classical action (1.9) the term [24],

$$\frac{g^2}{24\pi^2}\int d^4x \frac{1}{2}\phi(x)\frac{1}{\tilde{\partial}^2}\phi(x), \tag{2.23}$$

implying that one has now the following tree level action,

$$\begin{aligned}\Gamma_R^{(0)}[\phi] &= \int d^4x \Big(\frac{1}{2}\big(\partial_\mu\phi(x)\partial_\mu\phi(x) + m^2\phi^2(x) - \phi(x)\frac{\tilde{c}}{\tilde{\partial}^2}\phi(x) + \phi(x)\frac{\tilde{c}}{\tilde{\partial}^2}\phi(x)\big) \\ &\quad + \frac{g^2}{4!}\phi\star\phi\star\phi\star\phi(x)\Big),\end{aligned} \tag{2.24}$$

with $\tilde{c} = \frac{g^2}{24\pi^2}$. The idea is to treat one of the two cancelling terms as modification of the propagator and the other as a two-point vertex function. Doing loop-expansion, this leads to a mixing of orders in the coupling constant g^2 (which is exactly the desired effect of the resummation procedure), but non-perturbatively the theory remains the same. The process of resummation allows in principle two possibilities for the resummed propagators,

$$\Delta_\pm(k) = \frac{1}{k^2 + m^2 \pm \frac{\tilde{c}}{\tilde{k}^2}}. \tag{2.25}$$

As argued in [24] the negative sign corresponds to unphysical tachyonic poles. Therefore it seems natural that only the positive sign is meaningful. However, aiming at further calculations concerning gauge theory, we want to stay as general as possible.

Now one can compute the one-loop quantum correction with the resummed propagator,

$$
\begin{aligned}
\Delta\Sigma_{\pm} &= \frac{g^2}{6}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2\pm\frac{\tilde{c}}{\tilde{k}^2}}(2+\cos(k\tilde{p})) \\
&= \frac{g^2}{6}\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{k}^2}{k^2\tilde{k}^2+m^2\tilde{k}^2\pm\tilde{c}}(2+e^{ik\tilde{p}}). \qquad (2.26)
\end{aligned}
$$

In the UV-region the integral (2.26) has the same structure as for the non-resummed theory. Thus, we expect a quite similar result, with a quadratically divergent contribution from the planar graph and a finite but nonlocal $\frac{1}{\tilde{p}^2}$-contribution from the non-planar graph. This will be verified by explicit calculation. In order to get a Gaussian integral after Schwinger parametrisation we have to expand (2.26) into partial fractions. Unfortunately, the term $k^2\tilde{k}^2$ causes troubles unless $\tilde{k}^2 \propto k^2$. Since $\theta_{\mu\nu}$ can always be transformed into a block matrix

$$
\theta_{\mu\nu} \doteq \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}, \qquad (2.27)
$$

we see that $\theta_{\mu\nu}$ has only two degrees of freedom. By eliminating one degree of freedom, thus using the choice $\theta_{12} = \theta_{34} =: \theta$, we find $(\theta^2)_{\mu\nu} = \theta^2 \cdot \mathbb{1}_{4\times 4}$. At least for this special choice we can calculate

$$
\begin{aligned}
\Delta\Sigma_{\pm} &= \frac{g^2}{6}\int\frac{d^4k}{(2\pi)^4}\frac{k^2}{(k^2)^2+m^2k^2\pm\frac{\tilde{c}}{\theta^2}}(2+e^{ik\tilde{p}}) \\
&= -\frac{g^2}{6}\int\frac{d^4k}{(2\pi)^4}\frac{k^2}{2u}\Big(\frac{1}{k^2+\frac{m^2}{2}+u}-\frac{1}{k^2+\frac{m^2}{2}-u}\Big)(2+e^{ik\tilde{p}}). \qquad (2.28)
\end{aligned}
$$

Here the (possibly complex) quantity $u = \sqrt{\frac{m^4}{4} \mp \frac{\tilde{c}}{\theta^2}}$. Now we can use formulae (2.40), (2.41) from the section 2.3 (if u is real, we introduce a convergence factor $b \to 0$ by hand) and obtain for the non-planar part

$$
\begin{aligned}
\Delta\Sigma_{np\pm} &= \frac{g^2}{24\pi^2}\frac{1}{2u}\Bigg((\frac{m^2}{2}+u)\sqrt{\frac{\frac{m^2}{2}+u}{\tilde{p}^2}}K_1\left(\sqrt{(\frac{m^2}{2}+u)\tilde{p}^2}\right) \\
&\quad -(\frac{m^2}{2}-u)\sqrt{\frac{\frac{m^2}{2}-u}{\tilde{p}^2}}K_1\left(\sqrt{(\frac{m^2}{2}-u)\tilde{p}^2}\right)\Bigg). \qquad (2.29)
\end{aligned}
$$

For $\tilde{c} = 0 \Rightarrow u = \frac{m^2}{2}$ this yields exactly the result (1.17). With the expansion of the modified Bessel function we find for the ($\tilde{p}^2 \to 0$)-divergent part

$$\begin{aligned} \Delta\Sigma_{np\pm} &= \frac{g^2}{24\pi^2}\Big(\frac{1}{\tilde{p}^2} + \frac{1}{8u}(\frac{m^2}{2} + u)^2 \ln((\frac{m^2}{2} + u)\tilde{p}^2) \\ &\quad -\frac{1}{8u}(\frac{m^2}{2} - u)^2 \ln((\frac{m^2}{2} - u)\tilde{p}^2) + \dots\Big), \end{aligned} \tag{2.30}$$

where the dots denote the terms finite for $\tilde{p}^2 \to 0$. The logarithmic term is harmless, but we have to keep an eye on the $\frac{1}{\tilde{p}^2}$ term. We find that this term is invariant with respect to the choice of sign in (2.25). Therefore it is cancelled by the counterterm in the action only if we choose the positive sign in (2.25), which is—after all—not too surprising a result. Thus the counterterm reads

$$\delta\Gamma = +\int d^4x \phi(x) \frac{\tilde{c}}{\tilde{\partial}^2}\phi(x) \tag{2.31}$$

(note that $\tilde{\partial}^2 \Rightarrow -\tilde{k}^2$). The result for the planar part is obtained by multiplying (2.29) by two and taking the limes $\tilde{p} \to 0$,

$$\begin{aligned} \Delta\Sigma_{p\pm} &= \frac{g^2}{12\pi^2} \lim_{\Lambda\to\infty} \Bigg(\Lambda^2 - \frac{1}{8u}(\frac{m^2}{2} + u)^2 \ln\left(\frac{\Lambda^2}{\frac{m^2}{2} + u}\right) \\ &\quad + \frac{1}{8u}(\frac{m^2}{2} - u)^2 \ln\left(\frac{\Lambda^2}{\frac{m^2}{2} - u}\right) + \dots\Bigg), \end{aligned} \tag{2.32}$$

where again the finite terms are ignored. Thus, also in resummed field theory the planar one-loop correction of the two-point function can be absorbed in an ordinary mass renormalization of the theory.

2.2.2 Resummation in $U(1)$ NCYM-theory

In gauge theory, the corresponding term for resummation would be (see section 2.1.2)

$$\delta\Gamma = \frac{1}{2} g^2 \beta \int \frac{d^4k}{(2\pi)^4} A_\mu(k) A_\nu(-k) \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2}. \tag{2.33}$$

As observed in [36] it is gauge-invariant with respect to an infinitesimal Abelian gauge transformation, $\delta A_\mu = \partial_\mu \lambda$. But this is not the full story, one has to respect also BRST-invariance (2.17). In order to transform (2.33) in a BRST-invariant quantity we first define

$$\tilde{F} := \theta_{\mu\nu} F_{\mu\nu}, \quad \tilde{D}_\mu := \theta_{\mu\nu} D_\nu \,. \tag{2.34}$$

The idea is to replace (2.33) by (written in Euclidean coordinate space)

$$\delta\Gamma_{INV} = +\frac{g^2\beta}{8} \int d^4x \tilde{F} \star \frac{1}{(\tilde{D}^2)^2} \star \tilde{F} \,. \tag{2.35}$$

Of course, $\frac{1}{(\tilde{D}^2)^2}$ is meant as power series in the gauge field A_μ, so we obtain an infinite set of nonlocal vertices (however, to each order in the gauge coupling g only a finite number of these vertices contribute). To lowest order in A_μ (2.33) and (2.35) are identical. Indeed, it is straightforward to show that $\frac{1}{(\tilde{D}^2)^2} \star X$ transforms covariantly if X does (see below). For $X = \tilde{F}$ one has

$$\delta_\lambda \tilde{F} = i[\lambda, \tilde{F}]_M \Longrightarrow \quad \delta_\lambda \Big(\frac{1}{(\tilde{D}^2)^2} \star \tilde{F}\Big) = i \left[\lambda, \frac{1}{(\tilde{D}^2)^2} \star \tilde{F}\right]_M , \tag{2.36}$$

implying that (2.35) is BRST-invariant. In order to get a resummed gauge field model one generalises now the calculation of the last section. The resummed action reads

$$\begin{aligned}\Gamma_R^{(0)} \;&=\; \int d^4x \Big(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + gB \star \partial_\mu A_\nu + \frac{\alpha}{2} B \star B - \bar{c} \star \partial_\mu D_\mu c \\ &\quad + \frac{g^2\beta}{8} \tilde{F} \star \frac{1}{(\tilde{D}^2)^2} \star \tilde{F} - \frac{g^2\beta}{8} \tilde{F} \star \frac{1}{(\tilde{D}^2)^2} \star \tilde{F}\Big) + \Gamma_{matter}. \end{aligned} \tag{2.37}$$

A similar ansatz for the solution of an analogous problem in high temperature QCD can be found in [37]. Taking only the bilinear part of (2.37) one can calculate the resummed U(1)-gauge field propagator as

$$\Delta_{\mu\nu\pm}(k) = -\frac{1}{k^2} \Big(g_{\mu\nu} - (1-\alpha) \frac{k_\mu k_\nu}{k^2} \mp g^2 \beta \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2 (k^2 \pm \frac{g^2\beta}{\tilde{k}^2})} \Big). \tag{2.38}$$

The upper sign corresponds to the inclusion of the positive β-term in (2.37) in the propagator (treating the negative β-term as counterterm). One observes that the new term in the resummed propagator is independent of the gauge parameter α. Moreover, it is transversal due to $k_\mu \tilde{k}_\mu = 0$. The correct sign (we expect the upper one, of course) must be checked by explicit one-loop calculations. Note that the resummation procedure for gauge theories involves also a resummation of the vertices due to the non-bilinear parts in the resummed term in (2.37).

2.3 Supplement

In order to keep the above considerations free of long and unilluminating technical aspects, we present the calculations necessary for the resummed two-point one-loop graph now.

First we need the complex Euclidean Gauss integral (α real)

$$\int d^4k e^{\pm i\alpha k^2 + ik\tilde{p}} = \int d^4k e^{\pm i\alpha(k \pm \frac{\tilde{p}}{2\alpha})^2 \mp \frac{i\tilde{p}^2}{4\alpha}} = \\ = \lim_{\varepsilon\to 0} \int d^4k' e^{-(\varepsilon \mp i\alpha)k'^2 \mp \frac{i\tilde{p}^2}{4\alpha}} = -\frac{\pi^2}{\alpha^2} e^{\mp \frac{i\tilde{p}^2}{4\alpha}}. \tag{2.39}$$

With this we get for real $b > 0$, a real,

$$\int d^4k \frac{k^2}{k^2 + a \pm ib} e^{ik\tilde{p}} = \\ = \int d^4k (\pm i) \int_0^\infty d\alpha (\pm i \frac{\partial}{\partial\alpha} + a \pm ib) e^{\pm i\alpha(k^2 + a \pm ib) + ik\tilde{p}} \\ = \pm i \int_0^\infty d\alpha (\pm i \frac{\partial}{\partial\alpha} + a \pm ib) \big(-\frac{\pi^2}{\alpha^2} e^{\mp \frac{i\tilde{p}^2}{4\alpha} \pm i\alpha(a \pm ib)} \big) \\ = -(\pm i)\pi^2 (a \pm ib) \lim_{\varepsilon\to 0} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\frac{\varepsilon \pm i\tilde{p}^2}{4\alpha} - (b \mp ia)\alpha}. \tag{2.40}$$

Here we use $\int_0^\infty \frac{d\alpha}{\alpha^2} \exp(-u\alpha - v/(4\alpha)) = 4(\sqrt{u}/\sqrt{v}) K_1(\sqrt{uv})$ for positive real part of u and v and find

$$= -(\sqrt{\pm i})^2 4\pi^2 (a \pm ib) \frac{\sqrt{b \mp ia}}{\sqrt{\pm i\tilde{p}^2}} K_1 \big(\sqrt{\pm i\tilde{p}^2 (b \mp ia)} \big) \\ = -4\pi^2 (a \pm ib) \sqrt{\frac{a \pm ib}{\tilde{p}^2}} K_1 \big(\sqrt{(a \pm ib)\tilde{p}^2} \big). \tag{2.41}$$

Of course one has to be careful with respect to the sign of the roots, thus one should check the result (2.41) for special cases (e. g. $a > 0,\ b \to 0$) via ordinary (non-complex) Schwinger parametrisation.

Second we want to give a formal proof that $D^{-1}F$ is a covariant quantity if F is covariant (D^{-1} is the inverse of the covariant derivative D). In order to stay as simple as possible, we perform the calculation on ordinary space first. We have

$$\begin{aligned}\delta_\lambda F &= i[\lambda, F], \qquad (2.42)\\ \delta_\lambda DF &= (\delta_\lambda D)F + D(\delta_\lambda F) = (\delta_\lambda D)F + i[(D\lambda), F] + i[\lambda, (DF)]\\ &\stackrel{!}{=} i[\lambda, DF].\end{aligned}$$

From this we conclude $\delta_\lambda D\ . = -i[(D\lambda), .]$. With $(\delta_\lambda D^{-1}\ .)$ $= -D^{-1}(\delta_\lambda D)D^{-1}\ .$ we find

$$\begin{aligned}\delta_\lambda D^{-1}F &= (\delta_\lambda D^{-1})F + D^{-1}\delta_\lambda F = -D^{-1}(\delta_\lambda D)D^{-1}F + iD^{-1}[\lambda, F]\\ &= iD^{-1}[(D\lambda), D^{-1}F] + iD^{-1}[\lambda, DD^{-1}F]\\ &= iD^{-1}[(D\lambda), D^{-1}F] + iD^{-1}D[\lambda, D^{-1}F] - iD^{-1}[(D\lambda), D^{-1}F]\\ &= i[\lambda, D^{-1}F]. \qquad (2.43)\end{aligned}$$

Now on Moyal deformed space we notice that also there D acts as derivation. Thus the proof carries on to $D^{-1} \star F$. □

2.4 Conclusion

In our first try we have demonstrated that the (quadratic) IR-singularities appearing in the 2-point function of noncommutative ϕ^4-theory may be shifted to higher n-point functions via a field redefinition. One could speculate if this method, initiating an infinite chain of field redefinitions, could in fact be used to totally remove the IR-singularities.

In our second attempt we tried a resummation procedure. The results seemed to be very promising. Unfortunately, whereas the complexity of the calculations is manageable at 1-loop order, higher orders appear to be inexecutable especially in gauge theory, where also the vertices have to be resummed.

On the other hand, there were hints that the full theory still *is* renormalizable. Thus we decided not to pursue loop calculations any further but to use more rigorous techniques. In concrete, we tried to establish Polchinski's renormalization group approach [38] on noncommutative field theory. Unfortunately, the result was a disaster. Due to completely different types of nonplanarities appearing at any loop order (see [23]), the elegant inductive proof of renormalizability of commutative ϕ^4-theory presented by Polchinsky did not carry on to the noncommutative ϕ^4-theory. It was not even possible to show that the theory is *not* renormalizable. This uncomfortable sort of stalemate lead to a severe crisis in our research program.

The solution came—as always—from a completely unexpected direction. It will be presented in the third part of this thesis. But before (in order to hold up the thrill a bit...) we want to focus on a *completely different* aspect of noncommutative field theories, *i.e.* we want to present some nice results concerning the so called Seiberg-Witten map.

Proudly presents: The Intermezzo...

Intermezzo:
The Seiberg-Witten Map

Ist das die Beute, die du mir schlägst?

Hugo von Hofmannsthal

Premises

In [47] Seiberg and Witten argued that there is an equivalence map between ordinary Yang-Mills theory and its noncommutative analogue (see also [50], [27]). This so called *Seiberg-Witten map* was originally defined through the gauge equivalence condition

$$\hat{A}_\mu(A,\theta) + \hat{\delta}_\lambda \hat{A}_\mu(A,\theta) = \hat{A}_\mu(A + \delta_\lambda A, \theta), \tag{2.44}$$

introducing a possibly θ-dependent noncommutative gauge field $\hat{A}$ distinguished from the ordinary (we avoid the word 'commutative' since it could still be Lie-algebra valued) gauge field A. Indeed, to first order in θ the map takes the form (we will use Minkowskian signature throughout the 'Intermezzo')

$$\hat{A}_\mu(A,\theta) = A_\mu - \frac{1}{4}\theta^{\rho\sigma} A_\rho(\partial_\sigma A_\mu + F_{\sigma\mu}) + \mathcal{O}(\theta^2). \tag{2.45}$$

The Seiberg-Witten map ensures the gauge equivalence between an ordinary gauge field and its noncommutative counterpart. It implies that the noncommutative gauge field $\hat{A}_\mu$ (and also $\hat{F}_{\mu\nu}$) can be expanded in a series in the deformation parameter $\theta^{\mu\nu}$ of the noncommutative space-time geometry, with coefficients depending on the ordinary gauge field.

In the first chapter of the 'Intermezzo' we want to discuss the translation invariance of the SW-expansion of the noncommutative $U(N)$-Yang Mills (NCYM-)theory. In the second chapter we will focus on the thrilling search of a Seiberg-Witten map for the noncommutative super Yang-Mills field.

Chapter 3

EM-Tensor in NC Gauge Field Models

3.1 Introduction

In the paper [39] the energy-momentum tensor on noncommutative spaces was analysed and it was found that the dilation symmetry is broken due to the presence of the deformation parameter $\theta^{\mu\nu}$ characterising the noncommutative geometry (1.1).

The existence of a constant, fixed antisymmetric tensor field $\theta^{\mu\nu}$ clearly also breaks the Lorentz symmetry [40], [41] if $\theta^{\mu\nu}$ does not have a tensorial transformation behaviour with respect to Lorentz transformations. This situation resembles in some sense the axial gauge in gauge field models. There the presence of the constant, fixed gauge 'direction' n^μ breaks the Lorentz invariance, too [31].

In particular, the occurence of $\theta^{\mu\nu}$ in noncommutative quantum field models induces that the corresponding energy-momentum tensor needs neither be symmetric (for massless models), nor traceless.

The aim of this chapter is the investigation of the construction of the energy-momentum tensor in massless and commutative gauge field models and their noncommutative counterparts, in order to work out the different aspects of the stress tensor for both cases.

Generally, the usual Noether procedure for the construction of the canonical energy-momentum tensor in the worst case needs an improvement procedure and the Belinfante trick [42], [43], [44] in order to get a symmetric and traceless stress tensor. However, due to the idea of Jackiw [44], [45], [46], there is a more direct method to get the correct stress tensor by combining the Noether procedure translations with field dependent gauge transformations.

3.2 Energy-Momentum Tensor in Ordinary Yang-Mills Theory

In order to demonstrate the various possible constructions (canonical form, Belinfante procedure, construction modulo a gauge transformation) of the energy-momentum tensor [39], [42], [43], [44], [45], let us start with a commutative Yang-Mills model, where the gauge field is matrix-valued, $A_\mu = A_\mu^a X^a$, X^a being the corresponding generators of the gauge group $U(N)$ with $[X^a, X^b] = d^{ab} \cdot \mathbb{1} + f^{abc} X^c$.

The corresponding infinitesimal gauge transformation is given by

$$\delta_\lambda A_\mu = \partial_\mu \lambda - ig[A_\mu, \lambda] =: D_\mu \lambda, \tag{3.1}$$

implying that the non-Abelian field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{3.2}$$

transforms covariantly,

$$\delta_\lambda F_{\mu\nu} = i[\lambda, F_{\mu\nu}]. \tag{3.3}$$

In the following we set $g = 1$. Therefore, the gauge invariant non-Abelian action at the classical level is given by

$$\Gamma_{inv}[A] = -\frac{1}{4} \int d^4x \, tr\left(F_{\mu\nu} F^{\mu\nu}\right) =: -\frac{1}{4} \int d^4x \, tr F^2. \tag{3.4}$$

The equation of motion for the gauge field is

$$\frac{\delta}{\delta A_\nu} \Gamma_{inv}[A] = D_\rho F^{\rho\nu} = 0. \tag{3.5}$$

The symmetry transformation (3.1) may be expressed by a functional differential operator, the global Ward-identity (WI)-operator

$$W^G(\lambda) = \int d^4x \, tr D_\mu \lambda(x) \frac{\delta}{\delta A_\mu(x)}. \tag{3.6}$$

Gauge invariance is stated through $W^G(\lambda)\Gamma_{inv} = 0$. This implies the following local identity for the gauge symmetry,

$$\frac{\delta}{\delta\lambda(x)} W^G(\lambda)\Gamma_{inv} = 0. \tag{3.7}$$

We calculate this explicitly,

$$\begin{aligned}
\frac{\delta}{\delta\lambda} W_\lambda \Gamma &= \frac{\delta}{\delta\lambda} \int d^4x\, tr(D_\mu \lambda \frac{\delta\Gamma}{\delta A_\mu}) \\
&= -D_\mu D_\rho F^{\rho\mu} = -D_\mu(\partial_\rho F^{\rho\mu} - i[A_\rho, F^{\rho\mu}]) \\
&= [A_\mu, [A_\rho, F^{\rho\mu}]] + i[\partial_\mu A_\rho, F^{\rho\mu}] \\
&= i\partial_\mu [A_\rho, F^{\rho\mu}] - i[A_\rho, \underbrace{D_\mu F^{\rho\mu}}_{=0\ e.o.m.}] = 0.
\end{aligned}$$

This defines the locally conserved current for the gauge symmetry,

$$j^\mu_G = -i[A_\rho, F^{\rho\mu}], \quad -\partial_\mu j^\mu_G = i\partial_\mu [A_\rho, F^{\rho\mu}] = 0. \tag{3.8}$$

Of course, by direct computation and by use of the equation of motion one easily verifies (3.8).

Now we want to discuss the infinitesimal translation described by the following global WI-operator,

$$W^T_\mu = \int d^4x\, tr \partial_\mu A_\nu(x) \frac{\delta}{\delta A_\nu(x)}. \tag{3.9}$$

For convenience, we first calculate (for arbitrary quantities M)

$$\begin{aligned}
-\frac{1}{4}\int d^4x\, tr M(x) \frac{\delta(F_{\rho\sigma}F^{\rho\sigma})}{\delta A_\nu} &= \frac{1}{2}\int d^4x\, tr\Big(M(x)\partial_\rho F^{\rho\nu} - M(x)\partial_\sigma F^{\rho\sigma} \\
&\quad + iM(x)[A_\sigma, F^{\nu\sigma}] - iM(x)[A_\rho, F^{\rho\nu}]\Big) \\
&= \int d^4x\, tr\big(M(x) D_\rho F^{\rho\nu}\big).
\end{aligned} \tag{3.10}$$

By applying the WI-operator (3.9) to the gauge invariant action one gets (as usual) the canonical energy-momentum tensor due to translational invariance,

$$
\begin{aligned}
W^T_\mu \Gamma_{inv} &= \int d^4x\, tr\Big(\partial_\mu A_\nu(\partial_\rho F^{\rho\nu} - i[A_\rho, F^{\rho\nu}])\Big) \\
&= \int d^4x\, tr\Big(\partial_\mu A_\nu(\partial_\rho F^{\rho\sigma}) + \partial_\mu(-\frac{1}{4}F^2) + \partial_\mu\partial_\rho A_\nu F^{\rho\nu}\Big) \\
&= \int d^4x\, tr\Big(\partial^\rho(\frac{1}{2}\{F_{\rho\nu}, \partial_\mu A^\nu\} - \frac{1}{4}g_{\rho\mu}F^2)\Big) \\
&= -\int d^4x\, \partial^\rho T^c_{\rho\mu} = 0.
\end{aligned}
\tag{3.11}
$$

Thus, the canonical energy-momentum tensor is defined as

$$
T^c_{\rho\mu} := -tr\big(\frac{1}{2}\{F_{\rho\nu}, \partial_\mu A^\nu\} - \frac{1}{4}g_{\rho\mu}F^2\big). \tag{3.12}
$$

It is simple to show that $T^c_{\rho\mu}$ is locally conserved by using the equation of motion.

However, $T^c_{\rho\mu}$ is not gauge invariant, not traceless and not symmetric in (ρ, μ). In order to obtain a symmetric stress tensor one has two possibilities [42], [43]. Here we follow the method proposed originally by R. Jackiw [45] in using an alternative representation for infinitesimal translations. Modulo a field dependent gauge transformation a possible description of translations is given by

$$
W^F_\mu = \int d^4x\, tr F_{\mu\nu}(x)\frac{\delta}{\delta A_\nu(x)}, \tag{3.13}
$$

leading to (with the help of $F_{\rho\nu}D^\rho F^{\mu\nu} = \frac{1}{2}F_{\rho\nu}D^\mu F^{\rho\nu}$ following from the Bianchi identity)

$$
\begin{aligned}
W^F_\mu \Gamma_{inv} &= \int d^4x\, tr\Big(F_{\mu\nu}(\partial_\rho F^{\rho\nu} - i[A_\rho, F^{\rho\nu}])\Big) \\
&= \int d^4x\, tr\Big(\partial^\rho(\frac{1}{2}\{F_{\rho\nu}, F_\mu{}^\nu\}) - F_{\rho\nu}\partial^\rho F_\mu{}^\nu + iF_{\rho\nu}[A^\rho, F_\mu{}^\nu]\Big) \\
&= \int d^4x\, tr\Big(\partial^\rho(\frac{1}{2}\{F_{\rho\nu}, F_\mu{}^\nu\}) - \frac{1}{4}D_\mu F^2\Big) \\
&= \int d^4x\, tr\Big(\partial^\rho(\frac{1}{2}\{F_{\rho\nu}, F_\mu{}^\nu\} - \frac{1}{4}g_{\rho\mu}F^2)\Big) \\
&= -\int d^4x\, \partial^\rho T^s_{\rho\mu} = 0,
\end{aligned}
\tag{3.14}
$$

where $T^s_{\rho\mu}$ is gauge invariant, symmetric and traceless,

$$T^s_{\rho\mu} := -tr\big(\frac{1}{2}\{F_{\rho\nu}, F_\mu{}^\nu\} - \frac{1}{4}g_{\rho\mu}F^2\big). \tag{3.15}$$

One observes that the Jackiw construction unifies the Belinfante and improvement procedure.

Using the splitting

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = \partial_\mu A_\nu - D_\nu A_\mu, \tag{3.16}$$

one gets for the canonical tensor

$$T^c_{\rho\mu} = -tr\big(\frac{1}{2}\{F_{\rho\nu}, F_\mu{}^\nu + D^\nu A_\mu\} - \frac{1}{4}g_{\rho\mu}F^2\big), \tag{3.17}$$

implying that the difference between the canonical tensor and the symmetric one becomes

$$T^c_{\rho\mu} - T^s_{\rho\mu} = -\frac{1}{2}tr\{F_{\rho\nu}, D^\nu A_\mu\}. \tag{3.18}$$

Due to the fact that the WI-operator of the translation is represented by

$$W^T_\mu = W^F_\mu + W^G_\mu = \int d^4x\, tr\Big(F_{\mu\nu}(x)\frac{\delta}{\delta A_\nu(x)} + D_\nu A_\mu(x)\frac{\delta}{\delta A_\nu(x)}\Big), \tag{3.19}$$

the field dependent gauge transformation corresponds to the difference $T^c_{\rho\mu} - T^s_{\rho\mu}$,

$$\begin{aligned} -W^G_\mu\Gamma_{inv} &= -\int d^4x\, tr\Big(D_\nu A_\mu(x)\frac{\delta\Gamma_{inv}}{\delta A_\nu(x)}\Big) \\ &= -\int d^4x\, tr\partial_\rho\big(\frac{1}{2}\{F_{\rho\nu}, D^\nu A_\mu\}\big). \end{aligned} \tag{3.20}$$

This is easily checked by explicit calculation with the use of partial integration,

$$\begin{aligned} -W^G_\mu\Gamma_{inv} &= -\int d^4x\, tr\big(D_\nu A_\mu D_\rho F^{\rho\nu}\big) \\ &= -\int d^4x\, tr\Big(\partial^\rho\big(\frac{1}{2}\{F_{\rho\nu}, D^\nu A_\mu\}\big) - D_\rho D_\nu A_\mu F^{\rho\nu}\Big). \end{aligned} \tag{3.21}$$

With the antisymmetry of $F^{\rho\nu}$ the second term is easily shown to vanish. This is very similar to the construction of the stress tensor of the Maxwell theory [49].

Another interesting comment has to be made. If we omit the tr symbol in the definition of the energy momentum tensor $T^s_{\rho\mu}$,

$$\tilde{T}^s_{\rho\mu} := -\left(\frac{1}{2}\{F_{\rho\nu}, F_\mu{}^\nu\} - \frac{1}{4}g_{\rho\mu}F^2\right), \tag{3.22}$$

we get an object which is (due to the equation of motion and the Bianchi identity) *covariantly* conserved (proof see below),

$$D^\rho \tilde{T}^s_{\rho\mu} = 0. \tag{3.23}$$

In discussing the noncommutative counterpart we will find that a similar 'covariant conservation' is also valid there. With (3.23) one finds

$$\partial^\rho \tilde{T}^s_{\rho\mu} = i[A^\rho, \tilde{T}^s_{\rho\mu}], \tag{3.24}$$

which is consistent with

$$W^F_\mu \Gamma_{inv} = -\int d^4x\, tr(\partial^\rho \tilde{T}^s_{\rho\mu}) = 0. \tag{3.25}$$

3.3 Energy-Momentum Tensor in Noncommutative YM-Theory

It is now straightforward to discuss also the noncommutative structure in the spirit of the considerations done in the previous section. In noncommutative gauge field models one has to replace all field products by $\star$-products [16] and one introduces the noncommutative matrix valued gauge field $\hat{A}$ as defined in (2.44). The corresponding gauge invariant action is therefore

$$\hat{\Gamma}_{inv}[\hat{A}] = -\frac{1}{4}\int d^4x\, tr\left(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}\right), \tag{3.26}$$

with the noncommutative field strength

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star. \tag{3.27}$$

Here the Moyal commutator is given by

$$[\hat{A}_\mu, \hat{A}_\nu]_\star := \hat{A}_\mu \star \hat{A}_\nu - \hat{A}_\nu \star \hat{A}_\mu, \tag{3.28}$$

using the $\star$-product,

$$A(x) \star B(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial^\xi_\mu \partial^\eta_\nu} A(x+\xi)B(x+\eta)|_{\xi=\eta=0}. \tag{3.29}$$

The infinitesimal gauge transformation is defined as

$$\delta_{\hat{\lambda}} \hat{A}_\mu = \partial_\mu \hat{\lambda} - i[\hat{A}_\mu, \hat{\lambda}]_\star =: \hat{D}_\mu \star \hat{\lambda}, \tag{3.30}$$

where $\hat{\lambda}$ is the noncommutative counterpart of λ of equation (3.1) The equation of motion for the gauge field then is

$$\frac{\delta}{\delta \hat{A}_\nu} \hat{\Gamma}_{inv}[\hat{A}] = \hat{D}_\rho \star \hat{F}^{\rho\nu} = 0, \tag{3.31}$$

and for the locally conserved gauge current we get

$$\hat{j}^\mu_G = -i[\hat{A}_\rho, \hat{F}^{\rho\mu}]_\star. \tag{3.32}$$

At the level of noncommutative gauge field models one can perform the same steps as in the previous section. With

$$\begin{aligned} \hat{W}^T_\mu \hat{\Gamma}_{inv} &= \frac{1}{2}\int d^4x\, tr\Big(\partial_\mu \hat{A}_\nu(x) \star \frac{\delta\hat{\Gamma}_{inv}}{\delta \hat{A}_\nu(x)} + \frac{\delta\hat{\Gamma}_{inv}}{\delta \hat{A}_\nu(x)} \star \partial_\mu \hat{A}_\nu(x)\Big) \\ &= \int d^4x\, tr\Big(\partial^\rho(\frac{1}{2}\{\hat{F}_{\rho\nu}, \partial_\mu \hat{A}^\nu\}_\star - \frac{1}{4} g_{\rho\mu} \hat{F}_{\alpha\beta} \star \hat{F}^{\alpha\beta})\Big) \end{aligned} \tag{3.33}$$

we find

$$\hat{T}^c_{\rho\mu} = -(\frac{1}{2}\{\hat{F}_{\rho\nu}, \partial_\mu \hat{A}^\nu\}_\star - \frac{1}{4} g_{\rho\mu} \hat{F}_{\alpha\beta} \star \hat{F}^{\alpha\beta}). \tag{3.34}$$

Analogously we have

$$\begin{aligned} \hat{W}^F_\mu \hat{\Gamma}_{inv} &= \frac{1}{2}\int d^4x\, tr\Big(\hat{F}_{\mu\nu}(x) \star \frac{\delta\hat{\Gamma}_{inv}}{\delta \hat{A}_\nu(x)} + \frac{\delta\hat{\Gamma}_{inv}}{\delta \hat{A}_\nu(x)} \star \hat{F}_{\mu\nu}(x)\Big) \\ &= \int d^4x\, tr\Big(\partial^\rho(\frac{1}{2}\{\hat{F}_{\rho\nu}, \hat{F}_\mu{}^\nu\}_\star - \frac{1}{4} g_{\rho\mu} \hat{F}_{\alpha\beta} \star \hat{F}^{\alpha\beta})\Big) \end{aligned} \tag{3.35}$$

and

$$\hat{T}^{s}_{\rho\mu} := -\big(\frac{1}{2}\{\hat{F}_{\rho\nu}, \hat{F}_{\mu}{}^{\nu}\}_{\star} - \frac{1}{4}g_{\rho\mu}\hat{F}_{\alpha\beta} \star \hat{F}^{\alpha\beta}\big). \tag{3.36}$$

Here $\{\ ,\ \}_{\star}$ represents the Moyal anti-commutator in the sense of (3.28). Note that in order to define 'local' quantities $\int d^4x\, tr$—the integration over space-time and the trace over the colour indices—cannot be separated in noncommutative geometry. After all, (3.36) is symmetric, traceless and transforms covariantly with respect to (3.30), [44]. It is also 'locally' *covariantly* conserved (see (3.23)),

$$\hat{D}^{\rho} \star \hat{T}^{s}_{\rho\mu} = 0. \tag{3.37}$$

This is shown with the help of the equation of motion ($\hat{D}^{\rho} \star \hat{F}_{\rho\mu} = 0$) and $\{\hat{F}_{\rho\nu}, \hat{D}^{\rho} \star \hat{F}_{\mu}{}^{\nu}\}_{\star} = \frac{1}{2}\{\hat{F}_{\rho\nu}, \hat{D}_{\mu} \star \hat{F}^{\rho\nu}\}_{\star}$,

$$\hat{D}^{\rho} \star \hat{T}^{s}_{\rho\mu} = -\frac{1}{2}\{\hat{F}_{\rho\nu}, \hat{D}^{\rho} \star \hat{F}_{\mu}{}^{\nu}\}_{\star} + \frac{1}{4}g_{\rho\mu}\{\hat{D}^{\rho} \star \hat{F}_{\alpha\beta}, \hat{F}^{\alpha\beta}\}_{\star} = 0. \tag{3.38}$$

We find that the energy-momentum tensors are not locally conserved[1],

$$\partial^{\rho}\hat{T}^{s}_{\rho\mu} \neq 0 \neq \partial^{\rho}\hat{T}^{c}_{\rho\mu}, \tag{3.39}$$

which is already known from the works [39], [48].

3.4 Energy-Momentum Tensor via Seiberg-Witten Map

Now we want to go a step further and use the Seiberg-Witten expansion of A_{μ}. The starting point is equation (3.26).

$$\hat{\Gamma}_{inv}[\hat{A}] = -\frac{1}{4}\int d^4x\, tr\big(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}\big). \tag{3.40}$$

The SW-map to lowest order in $\theta^{\mu\nu}$ for the noncommutative gauge field is

$$\hat{A}_{\mu} = A_{\mu} - \frac{1}{4}\theta^{\rho\sigma}\{A_{\rho}, \partial_{\sigma}A_{\mu} + F_{\sigma\mu}\}, \tag{3.41}$$

[1]Even a 'formal' definition of the tensors including the *tr*-symbol is useless, since the trace of a Moyal commutator is *not* vanishing locally.

implying the following field strength expansion [47]

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4}\theta^{\rho\sigma}\big(2\{F_{\mu\rho}, F_{\nu\sigma}\} - \{A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu}\}\big). \tag{3.42}$$

For the ordinary (commutative, Lie-algebra valued) field A_μ and field strength $F_{\mu\nu}$ the corresponding gauge transformations are given by (3.1) and (3.3), respectively.

Expanding the $\star$-product in (3.26) we have [27], [51]

$$\begin{aligned}\Gamma^\theta_{inv}[A] &= \int d^4x\, tr\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}(F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} - \frac{1}{4}F_{\alpha\beta}F^2)\Big) + \mathcal{O}(\theta^2)\\ &= \int d^4x\, tr\mathcal{L}^\theta_{inv} + \mathcal{O}(\theta^2).\end{aligned} \tag{3.43}$$

The corresponding equation of motion is[2]

$$\begin{aligned}-D_\rho\Pi^{\rho\nu} &= D_\rho\big(F^{\rho\nu} - \frac{1}{4}(\theta^{\rho\nu}F_{\alpha\beta}F^{\alpha\beta} + \{F^{\rho\nu}, \theta^{\alpha\beta}F_{\alpha\beta}\})\\ &\quad + \frac{1}{2}(\{\theta^{\nu\beta}F_{\alpha\beta}, F^{\rho\alpha}\} - \{\theta^{\rho\beta}F_{\alpha\beta}, F^{\nu\alpha}\} + \{F^\rho{}_\alpha, \theta^{\alpha\beta}F^\nu{}_\beta\})\big)\\ &= 0.\end{aligned} \tag{3.44}$$

The quantity $\Pi^{\rho\nu}$ is antisymmetric,

$$\Pi^{\rho\nu} = -\Pi^{\nu\rho} = \frac{\partial\mathcal{L}^\theta_{inv}}{\partial(\partial_\rho A_\nu)}, \tag{3.45}$$

and $\Pi^{0\rho}$ is the canonical momentum. The analogous calculations as in section 1 give now

$$\begin{aligned}W^T_\mu\Gamma^\theta_{inv} &:= \int d^4x\, tr\partial_\mu A_\nu(x)\frac{\delta\Gamma^\theta_{inv}}{\delta A_\nu(x)}\\ &= \int d^4x\, tr\big(\partial^\rho(\frac{1}{2}\{-\Pi_{\rho\nu}, \partial_\mu A^\nu\} + g_{\rho\mu}\mathcal{L}^\theta_{inv})\big)\\ &= -\int d^4x\, \partial^\rho T^{c,\theta}_{\rho\mu} = 0.\end{aligned} \tag{3.46}$$

[2]The long but straightforward computation analogous to the previous sections is not given explicitly

Thus, the canonical energy-momentum tensor becomes

$$T^{c,\theta}_{\rho\mu} := tr\big(\frac{1}{2}\{\Pi_{\rho\nu}, \partial_\mu A^\nu\} - g_{\rho\mu}\mathcal{L}^\theta_{inv}\big). \tag{3.47}$$

Similarly, one gets

$$\begin{aligned} W^F_\mu \Gamma^\theta_{inv} &= \int d^4x\, tr F_{\mu\nu}(x) \frac{\delta\Gamma^\theta_{inv}}{\delta A_\nu(x)} \\ &= \int d^4x\, tr\Big(\partial^\rho\big(\frac{1}{2}\{-\Pi_{\rho\nu}, F_\mu{}^\nu\} + g_{\rho\mu}\mathcal{L}^\theta_{inv}\big)\Big) \\ &= -\int d^4x\, \partial^\rho T^{s,\theta}_{\rho\mu} = 0, \end{aligned} \tag{3.48}$$

implying the following definition,

$$T^{s,\theta}_{\rho\mu} := tr\big(\frac{1}{2}\{\Pi_{\rho\nu}, F_\mu{}^\nu\} - g_{\rho\mu}\mathcal{L}^\theta_{inv}\big). \tag{3.49}$$

Both currents (3.47) and (3.49) are locally conserved,

$$\partial^\rho T^{c,\theta}_{\rho\mu} = \partial^\rho T^{s,\theta}_{\rho\mu} = 0, \tag{3.50}$$

and they are related by a Belinfante like procedure

$$\begin{aligned} T^{s,\theta}_{\rho\mu} &= T^{c,\theta}_{\rho\mu} + tr\big(D^\nu(A_\mu \Pi_{\rho\nu})\big) \\ &= T^{c,\theta}_{\rho\mu} + \partial^\nu \chi_{[\nu\rho]\mu}. \end{aligned} \tag{3.51}$$

One observes that both versions of the energy-momentum tensor, (3.47) and (3.49), are neither symmetric nor traceless. This is due to the fact that the Lorentz invariance and the dilation symmetry are no longer maintained [46]. However, one has to stress that $T^{s,\theta}_{\rho\mu}$ is invariant with respect to infinitesimal gauge transformations (3.1).

3.5 The $U(1)$-Case: θ-Deformed Maxwell Theory

The simplest, but still interesting, case of a θ-expanded gauge theory is the $U(1)$-NCYM, the θ-deformed Maxwell theory (without sources). One just replaces in the expressions derived in the previous section the matrix-valued $U(N)$ gauge field $A^a X^a$ by the ordinary photon field. Omitting the trace symbols we get

$$\begin{aligned}\Gamma^{\theta}_{inv}[A] &= \int d^4x \Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\theta^{\alpha\beta}(F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu} - \frac{1}{4}F_{\alpha\beta}F^2)\Big) + \mathcal{O}(\theta^2) \\ &= \int d^4x\, \mathcal{L}^{\theta}_{inv} + \mathcal{O}(\theta^2). \end{aligned} \tag{3.52}$$

Again, with the canonical momentum,

$$\begin{aligned}\Pi^{\rho\nu} &= \frac{\partial \mathcal{L}^{\theta}_{inv}}{\partial(\partial_\rho A_\nu)} = -F^{\rho\nu} + \frac{1}{4}(\theta^{\rho\nu}F_{\alpha\beta}F^{\alpha\beta}) + \frac{1}{2}F^{\rho\nu}\theta^{\alpha\beta}F_{\alpha\beta} \\ &\quad -(\theta^{\nu\beta}F_{\alpha\beta}F^{\rho\alpha} - \theta^{\rho\beta}F_{\alpha\beta}F^{\nu\alpha}) - F^{\rho}{}_{\alpha}\theta^{\alpha\beta}F^{\nu}{}_{\beta}, \end{aligned} \tag{3.53}$$

we find the equation of motion [48],

$$\partial_\rho \Pi^{\rho\nu} = 0. \tag{3.54}$$

The stress tensors read

$$\begin{aligned} T^{c,\theta}_{\rho\mu} &= \Pi_{\rho\nu}\partial_\mu A^\nu - g_{\rho\mu}\mathcal{L}^{\theta}_{inv}, \\ T^{s,\theta}_{\rho\mu} &= \Pi_{\rho\nu}F_{\mu}{}^{\nu} - g_{\rho\mu}\mathcal{L}^{\theta}_{inv}. \end{aligned} \tag{3.55}$$

Explicitly we have for $T^{s,\theta}_{\rho\mu}$

$$\begin{aligned} T^{s,\theta}_{\rho\mu} &= -g_{\rho\mu}\mathcal{L}^{\theta}_{inv} - F_{\mu\nu}F_{\rho}{}^{\nu}(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}) + \frac{1}{4}F_{\mu\nu}\theta_{\rho}{}^{\nu}F^2 \\ &\quad + F_{\mu\nu}\theta_{\rho\beta}F_{\alpha}{}^{\beta}F^{\nu\alpha} - (F_{\mu\alpha}F_{\rho\nu} + F_{\rho\alpha}F_{\mu\nu})F^{\alpha}{}_{\beta}\theta^{\nu\beta}. \end{aligned} \tag{3.56}$$

The latter equation confirms Kruglov's result [48]. One has to stress that $T^{s,\theta}_{\rho\mu}$ is *not* symmetric and *not* traceless. In order to expand the tensor (3.36) for the $U(1)$-case to first order in $\theta^{\mu\nu}$ one needs (remember $\hat{A} \star \hat{B} = \hat{A}\hat{B} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\hat{A}\partial_\beta\hat{B}$)

$$\begin{aligned} \hat{F}_{\rho\nu} &= F_{\rho\nu} + \theta^{\alpha\beta}(F_{\rho\alpha}F_{\nu\beta} - A_\alpha\partial_\beta F_{\rho\nu}) \\ \frac{1}{2}\{\hat{F}_{\rho\nu}, \hat{F}_{\mu}{}^{\nu}\}_\star &= F_{\rho\nu}F_{\mu}{}^{\nu} + \underbrace{\frac{i}{4}\theta^{\alpha\beta}(\partial_\alpha F_{\rho\nu}\partial_\beta F_{\mu}{}^{\nu} + (\alpha \leftrightarrow \beta))}_{=0} + F_{\rho\nu}\theta^{\alpha\beta}F_{\mu\alpha}F^{\nu}{}_{\beta} \\ &\quad + F_{\mu}{}^{\nu}\theta^{\alpha\beta}F_{\rho\alpha}F_{\nu\beta} - F_{\rho\nu}\theta^{\alpha\beta}A_\alpha\partial_\beta F_{\mu}{}^{\nu} - F_{\mu}{}^{\nu}\theta^{\alpha\beta}A_\alpha\partial_\beta F_{\rho\nu} \\ &= F_{\rho\nu}F_{\mu}{}^{\nu}(1 - \frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}) + (F_{\rho\alpha}\theta^{\nu\beta}F_{\mu\nu}F^{\alpha}{}_{\beta} + F_{\mu\alpha}\theta^{\nu\beta}F_{\rho\nu}F^{\alpha}{}_{\beta}) \\ &\quad -\theta^{\alpha\beta}\partial_\beta(A_\alpha F_{\rho\nu}F_{\mu}{}^{\nu}). \end{aligned}$$

Thus we get

$$\begin{aligned}\hat{T}^{s}_{\rho\mu}|_{\mathcal{O}(\theta)} &= -g_{\rho\mu}\mathcal{L}^{\theta}_{inv} - F_{\mu\nu}F_{\rho}{}^{\nu}(1-\frac{1}{2}\theta^{\alpha\beta}F_{\alpha\beta}) \\ &\quad -(F_{\mu\alpha}F_{\rho\nu}+F_{\rho\alpha}F_{\mu\nu})F^{\alpha}{}_{\beta}\theta^{\nu\beta}+\theta^{\alpha\beta}\partial_{\beta}(A_{\alpha}F_{\rho\nu}F_{\mu}{}^{\nu}) \\ &\neq T^{s,\theta}_{\rho\mu}. \end{aligned} \tag{3.57}$$

We observe that (ignoring the total derivative) the non-symmetric parts of $T^{s,\theta}_{\rho\mu}$ do not appear in the expansion of $\hat{T}^{s}_{\rho\mu}$. Moreover, these are exactly the terms where $\theta_{\rho}{}^{\nu}$ carries a free index ρ. For $T^{c,\theta}_{\rho\mu}$ and $\hat{T}^{c}_{\rho\mu}$ we get the analogous result.

Thus we find that the calculation of the energy-momentum tensor does not commute with the Seiberg-Witten expansion of fields and Moyal products.

3.6 Conclusion

For noncommutative gauge field models we have studied (at the classical level) the construction of the various energy-momentum tensors in order to describe translation invariance of different noncommutative gauge field theories. Due to the presence of the deformation parameter $\theta^{\mu\nu}$ (as a constant, antisymmetric, fixed tensor) Lorentz and dilation invariance are manifestly broken, entailing that the corresponding stress tensors are not symmetric and not traceless. The obtained results may be the basis for the discussion of broken Lorentz and dilation symmetry.

Chapter 4

Seiberg-Witten Map for NCSYM-Theory

4.1 Introduction

The Seiberg-Witten equivalence between commutative and noncommutative gauge fields can be traced back [18] to a deeper discussion of Lorentz transformations [53]: In presence of θ one has to distinguish between 'observer Lorentz transformations', which transform θ as a second order Lorentz tensor, and 'particle Lorentz transformations', which leave θ invariant. It turns out that observer Lorentz transformations are symmetries of the theory, whereas particle Lorentz symmetry is broken. Being (in principle) an observable, the breaking of particle Lorentz symmetry must be gauge-invariant [18]. This is not automatically the case and demands a covariant redefinition of the splitting of the observer Lorentz transformation into particle Lorentz transformation plus θ-transformation, which is governed by the Seiberg-Witten differential equations.

This chapter is an extension of [18] to the components of a noncommutative super vector field in the Wess-Zumino gauge. We derive the Seiberg-Witten differential equations of super Yang-Mills theory via a covariant splitting of the observer Lorentz transformations into particle Lorentz transformations and a remainder, using the splitting for the gauge field derived in [18] as the starting point. The Seiberg-Witten differential equations lead to a θ-expansion of the noncommutative super Yang-Mills action in terms of fields living on commutative space-time. This θ-expanded action is automatically invariant under commutative gauge transformations and commutative Lorentz transformations. It is however not invariant under commutative supersymmetry transformations. Instead, the θ-expansion of the noncommutative supersymmetry transformation yields a symmetry transformation of the θ-expanded action which extends the usual supersymmetry transformations by terms of order $n \geq 1$ in θ. This result implies that the Seiberg-Witten map for super Yang-Mills theory cannot be expressed in terms of superfields.

4.2 The NCSYM Action and its Symmetries

The most compact way to formulate supersymmetric theories is to use the superfield formalism. The various fields of super Yang-Mills theory can be regarded as components of the real superfield

$$\begin{aligned}\hat{\phi} &= -2\hat{C} - 2\hat{\chi}^a\theta_a - 2\bar{\theta}_{\dot{a}}\hat{\bar{\chi}}^{\dot{a}} - \theta^a\theta_a\hat{M} - \bar{\theta}_{\dot{a}}\bar{\theta}^{\dot{a}}\hat{\bar{M}} \\ &\quad -2\theta^a\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\hat{A}_\mu - \bar{\theta}_{\dot{a}}\theta^a\theta_a\hat{\bar{\lambda}}^{\dot{a}} - \theta^a\bar{\theta}_{\dot{a}}\bar{\theta}^{\dot{a}}\hat{\lambda}_a - \frac{1}{2}\theta^a\theta_a\bar{\theta}_{\dot{a}}\bar{\theta}^{\dot{a}}\hat{D}\,. \end{aligned} \tag{4.1}$$

The anti-commuting variables $\theta^a, \bar{\theta}^{\dot{a}}$ should not be confused with the non-commutativity parameter $\theta^{\mu\nu}$. The Wess-Zumino gauge consists in setting the components $\hat{C}, \hat{\chi}^a, \hat{\bar{\chi}}^{\dot{a}}, \hat{M}, \hat{\bar{M}}$ equal to zero. One has $\hat{\phi} \star \hat{\phi} \star \hat{\phi} = 0$ in this gauge. For details about the superfield formalism we refer to [56]. The gauge transformation is

$$\delta\hat{\phi} = i(\hat{\Lambda} - \hat{\bar{\Lambda}}) + \frac{i}{2}[\hat{\phi}, \hat{\Lambda} + \hat{\bar{\Lambda}}] + \frac{i}{12}[\hat{\phi}[\hat{\phi}, \hat{\Lambda} - \hat{\bar{\Lambda}}]] + 0, \tag{4.2}$$

due to the Wess-Zumino gauge. Here we have

$$\begin{aligned}\hat{\Lambda} &= (1 - i\theta\sigma^\mu\bar{\theta}\partial_\mu - \frac{1}{2}\theta^2\bar{\theta}^2\partial^2)(-2\hat{a} - 2\theta\hat{\rho} - 2\theta^2\hat{f}) \\ &= -2\hat{a} - 2\theta\hat{\rho} - 2\theta^2\hat{f} + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\hat{a} + 2i\theta\sigma^\mu\bar{\theta}\partial_\mu(\theta\hat{\rho}) + \theta^2\bar{\theta}^2\partial^2\hat{a}, \\ \hat{\bar{\Lambda}} &= -2\hat{\bar{a}} - 2\bar{\theta}\hat{\bar{\rho}} - 2\bar{\theta}^2\hat{\bar{f}} - 2i\theta\sigma^\mu\bar{\theta}\partial_\mu\hat{\bar{a}} - 2i\theta\sigma^\mu\bar{\theta}\partial_\mu(\bar{\theta}\hat{\bar{\rho}}) + \theta^2\bar{\theta}^2\partial^2\hat{\bar{a}}. \end{aligned} \tag{4.3}$$

In order to maintain the Wess-Zumino gauge we need $\hat{\rho} = \hat{\bar{\rho}} = \hat{f} = \hat{\bar{f}} = 0$. Defining $\hat{a} = \frac{\hat{\omega}}{2} = \hat{\bar{a}}$ we find for $\delta\hat{\phi}$ (note that $\theta^\alpha\theta_\beta = \frac{1}{2}\delta^\alpha_\beta\theta^2$ and $\bar{\theta}_{\dot{\alpha}}\theta^{\dot{\beta}} = \frac{1}{2}\delta^{\dot{\beta}}_{\dot{\alpha}}\bar{\theta}^2$)

$$\begin{aligned}
\delta\hat{\phi}|_{\theta\bar{\theta}} &= -2\theta\sigma^\mu\bar{\theta}\partial_\mu(\hat{a}+\hat{\bar{a}}) - i[-2\theta\sigma^\mu\bar{\theta}\hat{A}_\mu, \hat{a}+\hat{\bar{a}}] \\
\Rightarrow \delta\hat{A}_\mu &= \partial_\mu\hat{\omega} - i[\hat{A}_\mu, \hat{\omega}], \\
\delta\hat{\phi}|_{\bar{\theta}\theta^2,\theta\bar{\theta}^2} &= +i[\bar{\theta}_{\dot{a}}\theta^2\hat{\bar{\lambda}}^{\dot{a}} + \theta^a\bar{\theta}^2\hat{\lambda}_a, \hat{a}+\hat{\bar{a}}] \\
\Rightarrow \delta\hat{\lambda}_a &= -i[\hat{\lambda}_a, \hat{\omega}] \\
\Rightarrow \delta\hat{\bar{\lambda}}_{\dot{a}} &= -i[\hat{\bar{\lambda}}_{\dot{a}}, \hat{\omega}], \\
\delta\hat{\phi}|_{\theta^2\bar{\theta}^2} &= -i[-\frac{1}{2}\theta^2\bar{\theta}^2\hat{D}, \hat{a}+\hat{\bar{a}}] \\
\Rightarrow \delta\hat{D} &= -i[\hat{D}, \hat{\omega}].
\end{aligned} \tag{4.4}$$

The construction of the supersymmetric $\mathcal{N}$=1 Yang-Mills action is done in [56]. In the Wess-Zumino gauge its (noncommutative) component formulation reads

$$\Gamma = \int d^4x\, tr\Big(-\frac{1}{4}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} + \frac{i}{4}\hat{\lambda}^a\sigma^\mu_{a\dot{a}}\hat{D}_\mu\hat{\bar{\lambda}}^{\dot{a}} + \frac{1}{8}\hat{D}^2\Big)\,, \tag{4.5}$$

where

$$\begin{aligned}
\hat{F}_{\mu\nu} &:= \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]_\star\,, \\
\hat{D}_\mu\hat{\bar{\lambda}}^{\dot{a}} &:= \partial_\mu\hat{\bar{\lambda}}^{\dot{a}} - i[\hat{A}_\mu, \hat{\bar{\lambda}}^{\dot{a}}]_\star\,.
\end{aligned} \tag{4.6}$$

Some useful properties of objects carrying spinor indices $a, \dot{a} \in \{1,2\}$ are listed at the end of the chapter. The $\star$-(anti)commutators of matrix-valued Schwartz class functions f, g are defined by

$$[f,g]_\star = g\star f - f\star g\,, \qquad \{f,g\}_\star = g\star f + f\star g\,, \tag{4.7}$$

where the $\star$-product is defined by

$$(f\star g)(x) = \int d^4y \int \frac{d^4k}{(2\pi)^4}\, f(x+\tfrac{1}{2}\theta\cdot k)\, g(x+y)\, e^{ik\cdot y}\,, \tag{4.8}$$

with $(\theta\cdot k)^\mu := \theta^{\mu\nu}k_\nu$, $k\cdot y := k_\mu y^\mu$ and $\theta^{\mu\nu} = -\theta^{\nu\mu} \in M_4(\mathbb{R})$. We consider $\theta^{\mu\nu}$ as the components of a translation invariant tensor field. The action (4.5) is invariant under gauge transformations

$$W^G_{\hat{\omega}} = \int d^4x\, tr\Big(\hat{D}_\mu\hat{\omega}\,\frac{\delta}{\delta\hat{A}_\mu} - i[\hat{\bar{\lambda}}^{\dot{a}},\hat{\omega}]_\star\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}} - i[\hat{\lambda}^a,\hat{\omega}]_\star\frac{\delta}{\delta\hat{\lambda}^a} - i[\hat{D},\hat{\omega}]_\star\frac{\delta}{\delta\hat{D}}\Big)\,, \tag{4.9}$$

observer transformations (translation, Lorentz rotation and dilatation)

$$W^T_\tau = \int d^4x\, tr\Big(\partial_\tau\hat{A}_\mu\,\frac{\delta}{\delta\hat{A}_\mu} + \partial_\tau\hat{\lambda}^a\frac{\delta}{\delta\hat{\lambda}^a} + \partial_\tau\hat{\bar{\lambda}}^{\dot{a}}\,\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}} + \partial_\tau\hat{D}\frac{\delta}{\delta\hat{D}}\Big)\,, \tag{4.10}$$

$$\begin{aligned}
W^R_{\alpha\beta} := \int d^4x\, tr\bigg(&\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat{A}_\mu\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat{A}_\mu\}_\star + g_{\mu\alpha}\hat{A}_\beta - g_{\mu\beta}\hat{A}_\alpha\Big)\frac{\delta}{\delta\hat{A}_\mu}\\
&+\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat{\lambda}^a\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat{\lambda}^a\}_\star + \frac{i}{2}\hat{\lambda}^b\sigma_{\alpha\beta b}{}^a\Big)\frac{\delta}{\delta\hat{\lambda}^a}\\
&+\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat{\bar{\lambda}}^{\dot{a}}\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat{\bar{\lambda}}^{\dot{a}}\}_\star - \frac{i}{2}\bar{\sigma}^{\dot{a}}_{\alpha\beta\dot{b}}\hat{\bar{\lambda}}^{\dot{b}}\Big)\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}}\\
&+\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat{D}\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat{D}\}_\star\Big)\frac{\delta}{\delta\hat{D}}\bigg)\\
+\Big(\delta^\mu_\alpha\theta_\beta{}^\nu - \delta^\mu_\beta\theta_\alpha{}^\nu + \delta^\nu_\alpha\theta^\mu{}_\beta - \delta^\nu_\beta\theta^\mu{}_\alpha\Big)&\frac{\partial}{\partial\theta^{\mu\nu}}\,,
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
W^D := \int d^4x\, tr\bigg(&\Big(\frac{1}{2}\{x^\delta,\partial_\delta\hat{A}_\mu\}_\star + \hat{A}_\mu\Big)\,\frac{\delta}{\delta\hat{A}_\mu} + \Big(2\hat{D} + \frac{1}{2}\{x^\delta,\partial_\delta\hat{D}\}_\star\Big)\,\frac{\delta}{\delta\hat{D}}\\
&+\Big(\frac{3}{2}\hat{\lambda}^a + \frac{1}{2}\{x^\delta,\partial_\delta\hat{\lambda}^a\}_\star\Big)\frac{\delta}{\delta\hat{\lambda}^a} + \Big(\frac{3}{2}\hat{\bar{\lambda}}^{\dot{a}} + \frac{1}{2}\{x^\delta,\partial_\delta\hat{\bar{\lambda}}^{\dot{a}}\}_\star\Big)\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}}\bigg)\\
-2\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\,,&
\end{aligned} \tag{4.12}$$

and supersymmetry transformations [54]

$$\begin{aligned}
W^S_a &= \int d^4x\, tr\Big(\frac{1}{2}\sigma_{\mu a\dot{a}}\hat{\bar{\lambda}}^{\dot{a}}\,\frac{\delta}{\delta\hat{A}_\mu} + \big(\delta^b_a\hat{D} + \sigma^{\mu\nu}_a{}^b\hat{F}_{\mu\nu}\big)\frac{\delta}{\delta\hat{\lambda}^b} - i\sigma^\mu_{a\dot{a}}\hat{D}_\mu\hat{\bar{\lambda}}^{\dot{a}}\,\frac{\delta}{\delta\hat{D}}\Big)\,,\\
W^{\bar{S}}_{\dot{a}} &= \int d^4x\, tr\Big(\frac{1}{2}\hat{\lambda}^a\sigma_{\mu a\dot{a}}\frac{\delta}{\delta\hat{A}_\mu} + \big(\delta^{\dot{b}}_{\dot{a}}\hat{D} - \bar{\sigma}^{\mu\nu\dot{b}}{}_{\dot{a}}\hat{F}_{\mu\nu}\big)\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{b}}} - i\hat{D}_\mu\hat{\lambda}^a\sigma^\mu_{a\dot{a}}\frac{\delta}{\delta\hat{D}}\Big)\,.
\end{aligned} \tag{4.13}$$

The partial derivative with respect to $\theta^{\mu\nu}$ has the property

$$\frac{\partial(\hat{U}\star\hat{V})}{\partial\theta^{\mu\nu}} = \frac{\partial\hat{U}}{\partial\theta^{\mu\nu}}\star\hat{V} + \hat{U}\star\frac{\partial\hat{V}}{\partial\theta^{\mu\nu}} + \frac{i}{2}(\partial_\mu\hat{U})\star(\partial_\nu\hat{V})\,, \tag{4.14}$$

where the fields $\hat{A}_\mu, \hat{\lambda}^a, \hat{\bar{\lambda}}^{\dot{a}}, \hat{D}$ must be assumed to be independent of θ.

4.3 Seiberg-Witten Differential Equations

As in (non-supersymmetric) noncommutative Yang-Mills theory [18] we derive the Seiberg-Witten differential equations via a splitting of the observer Lorentz transformation $W^R_{\alpha\beta}$ into the covariant particle Lorentz transformation $\tilde{W}^R_{\hat{\phi};\alpha\beta}$ and a remaining piece $\tilde{W}^R_{\theta;\alpha\beta}$ involving the Seiberg-Witten differential equation:

$$W^R_{\alpha\beta} \equiv \tilde{W}^R_{\hat{\phi};\alpha\beta} + \tilde{W}^R_{\theta;\alpha\beta} , \tag{4.15}$$

$$\tilde{W}^R_{\hat{\phi};\alpha\beta}(\theta^{\mu\nu}) = 0 , \tag{4.16}$$

$$[\tilde{W}^R_{\hat{\phi};\alpha\beta}, W^G_{\hat{\omega}}] = W^G_{\hat{\omega}'_{\alpha\beta}} , \qquad [\tilde{W}^R_{\theta;\alpha\beta}, W^G_{\hat{\omega}}] = W^G_{\hat{\omega}''_{\alpha\beta}} . \tag{4.17}$$

The motivation for this ansatz is the following. The commutator of an *observer Lorentz rotation* (4.11) with a gauge transformation (4.9) is again a gauge transformation,

$$[W^R_{\alpha\beta}, W^G_{\hat{\omega}}] = W^G_{\hat{\omega}_{\alpha\beta}} , \tag{4.18}$$

for some infinitesimal gauge parameter $\hat{\omega}_{\alpha\beta}[\hat{\omega}]$. A *particle Lorentz transformation* is defined as the part of an observer Lorentz transformation which does not transform the field $\theta^{\mu\nu}$, see (4.16). However, one should require that a particle Lorentz transformation transforms a gauge-invariant quantity into another gauge-invariant quantity, otherwise the particle Lorentz transformation cannot be considered as well-defined [18]. It is sufficient to demand (4.17) in order to achieve this property. To find the sought for splitting we first apply the ansatz of [18] for the Yang-Mills field $\hat{A}_\mu$:

$$\tilde{W}^R_{\hat{\phi};\alpha\beta}\hat{A}_\mu = \hat{D}_\mu\hat{\chi}_{\alpha\beta} + \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{F}_{\beta\mu}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{F}_{\alpha\mu}\}_\star - W^R_{\alpha\beta}(\theta^{\rho\sigma})\hat{\Omega}_{\rho\sigma\mu}\Big) , \tag{4.19}$$

where $\hat{X}^\mu = x^\mu + \theta^{\mu\nu}\hat{A}_\nu$ are the covariant coordinates [50] and $\hat{\Omega}_{\rho\sigma\mu}$ is a polynomial in covariant quantities such as $\theta^{\alpha\beta}, \hat{F}_{\kappa\lambda}, \hat{D}_{\mu_1}\dots\hat{D}_{\mu_n}\hat{F}_{\kappa\lambda}$, antisymmetric in ρ, σ, of power-counting dimension 3, and expresses the freedom in the splitting. In the following we set $\hat{\Omega}_{\rho\sigma\mu} = 0$. The parameter $\hat{\chi}_{\alpha\beta}$ is unchanged and given by [18]

$$\hat{\chi}_{\alpha\beta} = \frac{1}{4}\{2x_\alpha + \theta_\alpha{}^\rho \hat{A}_\rho, \hat{A}_\beta\}_\star - \frac{1}{4}\{2x_\beta + \theta_\beta{}^\rho \hat{A}_\rho, \hat{A}_\alpha\}_\star \,. \tag{4.20}$$

Comparing (4.19) with the $\hat{A}_\mu$-part of (4.11) and extending this covariantisation to the remaining fields $\hat{\lambda}^a, \hat{\bar{\lambda}}^{\dot{a}}, \hat{D}$ we obtain from (4.11)

$$\begin{aligned}
&\tilde{W}^R_{\phi;\alpha\beta} = \\
&= W^G_{\hat{\chi}_{\alpha\beta}} + \int d^4x\, tr\Bigg(\Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{F}_{\beta\mu}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{F}_{\alpha\mu}\}_\star\Big)\frac{\delta}{\delta \hat{A}_\mu} \\
&+ \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{D}_\beta \hat{\lambda}^a\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{D}_\alpha \hat{\lambda}^a\}_\star + \frac{i}{2}\hat{\lambda}^b \sigma_{\alpha\beta b}{}^a\Big)\frac{\delta}{\delta \hat{\lambda}^a} \\
&+ \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{D}_\beta \hat{\bar{\lambda}}^{\dot{a}}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{D}_\alpha \hat{\bar{\lambda}}^{\dot{a}}\}_\star - \frac{i}{2}\bar{\sigma}^{\dot{a}}_{\alpha\beta\dot{b}} \hat{\bar{\lambda}}^{\dot{b}}\Big)\frac{\delta}{\delta \hat{\bar{\lambda}}^{\dot{a}}} \\
&+ \Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{D}_\beta \hat{D}\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{D}_\alpha \hat{D}\}_\star\Big)\frac{\delta}{\delta \hat{D}}\Bigg) \,,
\end{aligned} \tag{4.21}$$

Now it is straightforward to evaluate

$$\tilde{W}^R_{\theta;\alpha\beta} = W^R_{\alpha\beta} - \tilde{W}^R_{\phi;\alpha\beta} = W^R_{\alpha\beta}(\theta^{\rho\sigma})\frac{d}{d\theta^{\rho\sigma}} \,, \tag{4.22}$$

with

$$\frac{d}{d\theta^{\rho\sigma}} = \frac{\partial}{\partial\theta^{\rho\sigma}} + \int d^4x\, tr\left(\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}}\frac{\delta}{\delta \hat{A}_\mu} + \frac{d\hat{\lambda}^a}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{\lambda}^a} + \frac{d\hat{\bar{\lambda}}^{\dot{a}}}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{\bar{\lambda}}^{\dot{a}}} + \frac{d\hat{D}}{d\theta^{\rho\sigma}}\frac{\delta}{\delta\hat{D}}\right) \,, \tag{4.23}$$

which yields the Seiberg-Witten differential equations (the proof will be given later using the superspace formalism)

$$\begin{aligned}
\frac{d\hat{A}_\mu}{d\theta^{\rho\sigma}} &= -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{A}_\mu + \hat{F}_{\sigma\mu}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{A}_\mu + \hat{F}_{\rho\mu}\big\}_\star \,, \\
\frac{d\hat{\lambda}^a}{d\theta^{\rho\sigma}} &= -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{\lambda}^a + \hat{D}_\sigma \hat{\lambda}^a\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{\lambda}^a + \hat{D}_\rho \hat{\lambda}^a\big\}_\star \,, \\
\frac{d\hat{\bar{\lambda}}^{\dot{a}}}{d\theta^{\rho\sigma}} &= -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{\bar{\lambda}}^{\dot{a}} + \hat{D}_\sigma \hat{\bar{\lambda}}^{\dot{a}}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{\bar{\lambda}}^{\dot{a}} + \hat{D}_\rho \hat{\bar{\lambda}}^{\dot{a}}\big\}_\star \,, \\
\frac{d\hat{D}}{d\theta^{\rho\sigma}} &= -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma \hat{D} + \hat{D}_\sigma \hat{D}\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho \hat{D} + \hat{D}_\rho \hat{D}\big\}_\star \,.
\end{aligned} \tag{4.24}$$

The first of these equations was first found in [47].

4.4 θ-Expansion of the Action

The differential equations (4.24) are now taken as the starting point for a θ-expansion of the action,

$$\Gamma^{(n)} := \sum_{j=0}^{n} \frac{1}{j!}\theta^{\rho_1\sigma_1}\ldots\theta^{\rho_j\sigma_j}\Big(\frac{d^j\Gamma}{d\theta^{\rho_1\sigma_1}\ldots d\theta^{\rho_j\sigma_j}}\Big)_{\theta=0}. \tag{4.25}$$

It follows from the the second identity in (4.17) that the θ-expansion (4.25) of the action (4.5) is invariant under commutative gauge transformations. One also checks the identity

$$\Big[W^T_\tau,\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\Big] = \Big[W^R_{\alpha\beta},\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\Big] = \Big[W^D,\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\Big] = 0 \tag{4.26}$$

for super Yang-Mills theory, which means that the θ-expansion of the fields leads to a commutative action invariant under commutative rotations and translations and with commutative dilatational symmetry. The θ-expansion of (4.5) yields an action which is ***not invariant*** under commutative supersymmetry transformations. Indeed, if we compute the commutators

$$\begin{aligned}
&[\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}, W^S_a]\hat{A}_\mu = \\
&= -\frac{1}{8}\theta^{\rho\sigma}\sigma_{\sigma\, a\dot{a}}\{\hat{A}_\rho,\hat{D}_\mu\hat{\bar{\lambda}}^{\dot{a}}\}_\star + \frac{1}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{\bar{\lambda}}^{\dot{a}},\partial_\sigma\hat{A}_\mu + \hat{F}_{\sigma\mu}\}_\star \\
&= \frac{1}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\hat{D}_\mu\{\hat{A}_\sigma,\hat{\bar{\lambda}}^{\dot{a}}\}_\star + \frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{F}_{\sigma\mu},\hat{\bar{\lambda}}^{\dot{a}}\}_\star\,,
\end{aligned} \tag{4.27}$$

$$\begin{aligned}
&[\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}, W^S_a]\hat{\lambda}^b = \\
&= \theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}(\delta^b_a\hat{D} + \sigma^{\mu\nu\, b}_a\hat{F}_{\mu\nu}) + \frac{1}{4}\theta^{\rho\sigma}W^S_a\{\hat{A}_\rho,\partial_\sigma\hat{\lambda}^b + \hat{D}_\sigma\hat{\lambda}^b\}_\star \\
&= \frac{1}{2}\theta^{\rho\sigma}\sigma^{\mu\nu\, b}_a\{\hat{F}_{\mu\rho},\hat{F}_{\nu\sigma}\}_\star + \frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},\hat{D}_\sigma\hat{\lambda}^b]_\star \\
&+ \frac{i}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},[\hat{A}_\sigma,\hat{\lambda}^b]_\star]_\star + \frac{i}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{A}_\sigma,\{\hat{\bar{\lambda}}^{\dot{a}},\hat{\lambda}^b\}_\star\}_\star \\
&= \frac{1}{2}\theta^{\rho\sigma}\sigma^{\mu\nu\, b}_a\{\hat{F}_{\mu\rho},\hat{F}_{\nu\sigma}\}_\star + \frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},\hat{D}_\sigma\hat{\lambda}^b]_\star
\end{aligned}$$

$$+\frac{i}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{\lambda}^{b},\{\hat{A}_{\sigma},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}\}_{\star}\,, \tag{4.28}$$

$$\begin{aligned}
&[\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}},W^{S}_{a}]\hat{\bar{\lambda}}^{\dot{b}} = \\
&= \frac{1}{4}\theta^{\rho\sigma}W^{S}_{a}\{\hat{A}_{\rho},\partial_{\sigma}\hat{\bar{\lambda}}^{\dot{b}}+\hat{D}_{\sigma}\hat{\bar{\lambda}}^{\dot{b}}\}_{\star} \\
&= \frac{1}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},\partial_{\sigma}\hat{\bar{\lambda}}^{\dot{b}}+\hat{D}_{\sigma}\hat{\bar{\lambda}}^{\dot{b}}]_{\star}+\frac{i}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{A}_{\sigma},\{\hat{\bar{\lambda}}^{\dot{a}},\hat{\bar{\lambda}}^{\dot{b}}\}_{\star}\}_{\star} \\
&= \frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},\hat{D}_{\sigma}\hat{\bar{\lambda}}^{\dot{b}}]_{\star}+\frac{i}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{\bar{\lambda}}^{\dot{b}},\{\hat{A}_{\sigma},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}\}_{\star}\,,
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
&[\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}},W^{S}_{a}]\hat{D} = \\
&= -i\theta^{\rho\sigma}\frac{d}{d\theta^{\rho\sigma}}\sigma^{\mu}_{a\dot{a}}\hat{D}_{\mu}\hat{\bar{\lambda}}^{\dot{a}}+\frac{1}{4}\theta^{\rho\sigma}W^{S}_{a}\{\hat{A}_{\rho},\partial_{\sigma}\hat{D}+\hat{D}_{\sigma}\hat{D}\}_{\star} \\
&= \frac{i}{4}\theta^{\rho\sigma}\sigma^{\mu}_{a\dot{a}}\hat{D}_{\mu}\{\hat{A}_{\rho},\partial_{\sigma}\hat{\bar{\lambda}}^{\dot{a}}+\hat{D}_{\sigma}\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}+\frac{1}{4}\theta^{\rho\sigma}\sigma^{\mu}_{a\dot{a}}[\{\hat{A}_{\rho},\partial_{\sigma}\hat{A}_{\mu}+\hat{F}_{\sigma\mu}\},\hat{\bar{\lambda}}^{\dot{a}}]_{\star} \\
&- \frac{i}{2}\theta^{\rho\sigma}\sigma^{\mu}_{a\dot{a}}\{\partial_{\rho}\hat{A}_{\mu},\partial_{\sigma}\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}+\frac{1}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{\bar{\lambda}}^{\dot{a}},\partial_{\sigma}\hat{D}+\hat{D}_{\sigma}\hat{D}\}_{\star} \\
&- \frac{i}{4}\theta^{\rho\sigma}\sigma^{\mu}_{a\dot{a}}\{\hat{A}_{\rho},\partial_{\sigma}\hat{D}_{\mu}\hat{\bar{\lambda}}^{\dot{a}}+\hat{D}_{\sigma}\hat{D}_{\mu}\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}-\frac{i}{8}\theta^{\rho\sigma}\sigma_{\sigma\, a\dot{a}}\{\hat{A}_{\rho},[\hat{\bar{\lambda}}^{\dot{a}},\hat{D}]_{\star}\}_{\star} \\
&= \frac{i}{2}\theta^{\rho\sigma}\sigma^{\mu}_{a\dot{a}}\{\hat{F}_{\sigma\mu},\hat{D}_{\rho}\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}+\frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}\{\hat{\bar{\lambda}}^{\dot{a}},\hat{D}_{\sigma}\hat{D}\}_{\star} \\
&- \frac{i}{8}\theta^{\rho\sigma}\sigma_{\rho\, a\dot{a}}[\hat{D},\{\hat{A}_{\sigma},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}]_{\star}\,,
\end{aligned} \tag{4.30}$$

we find that the commutator of a SUSY transformation and a θ-differentiation is given by[1] (with $\hat{\epsilon}=\frac{1}{16}\sigma_{\rho\, a\dot{a}}\{\hat{A}_{\sigma},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}-\frac{1}{16}\sigma_{\sigma\, a\dot{a}}\{\hat{A}_{\rho},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}$)

$$\begin{aligned}
&\Big[\frac{d}{d\theta^{\rho\sigma}},W^{S}_{a}\Big] = \\
&= \tilde{W}^{G}_{\hat{\epsilon}}+\int d^{4}x\, tr\Big(\Big(\frac{1}{8}\sigma_{\rho\, a\dot{a}}\{\hat{F}_{\sigma\mu},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}-\frac{1}{8}\sigma_{\sigma\, a\dot{a}}\{\hat{F}_{\rho\mu},\hat{\bar{\lambda}}^{\dot{a}}\}_{\star}\Big)\frac{\delta}{\delta\hat{A}_{\mu}} \\
&\quad +\Big(\frac{1}{4}\sigma^{\mu\nu\, b}_{a}\{\hat{F}_{\mu\rho},\hat{F}_{\nu\sigma}\}_{\star}+\frac{1}{8}\sigma_{\rho\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},\hat{D}_{\sigma}\hat{\lambda}^{b}]_{\star}-\frac{1}{8}\sigma_{\sigma\, a\dot{a}}[\hat{\bar{\lambda}}^{\dot{a}},\hat{D}_{\rho}\hat{\lambda}^{b}]_{\star}\Big)\frac{\delta}{\delta\hat{\lambda}^{b}}
\end{aligned}$$

[1] There is of course a freedom in the differential equations (4.24) given by the Ω-terms in (4.19) and similarly for the other fields. This freedom is not sufficient to obtain a vanishing right hand side of (4.31).

$$+\Big(\frac{1}{8}\sigma_{\rho a\dot a}[\hat{\bar\lambda}^{\dot a},\hat D_\sigma\hat{\bar\lambda}^{\dot b}]_\star-\frac{1}{8}\sigma_{\sigma a\dot a}[\hat{\bar\lambda}^{\dot a},\hat D_\rho\hat{\bar\lambda}^{\dot b}]_\star\Big)\frac{\delta}{\delta\hat{\bar\lambda}^{\dot b}}$$
$$+\Big(\frac{i}{4}\sigma^\mu_{a\dot a}\{\hat F_{\sigma\mu},\hat D_\rho\hat{\bar\lambda}^{\dot a}\}_\star-\frac{i}{4}\sigma^\mu_{a\dot a}\{\hat F_{\rho\mu},\hat D_\sigma\hat{\bar\lambda}^{\dot a}\}_\star$$
$$+\frac{1}{8}\sigma_{\rho a\dot a}\{\hat{\bar\lambda}^{\dot a},\hat D_\sigma\hat D\}_\star-\frac{1}{8}\sigma_{\sigma a\dot a}\{\hat{\bar\lambda}^{\dot a},\hat D_\rho\hat D\}_\star\Big)\frac{\delta}{\delta\hat D}\Big)\,, \tag{4.31}$$

where the gauge transformation with respect to a fermionic parameter $\tilde\omega$ is defined by

$$\tilde W^G_{\tilde\omega}=\int d^4x\; tr\Big(\hat D_\mu\hat\omega\,\frac{\delta}{\delta\hat A_\mu}+i\{\hat{\bar\lambda}^{\dot a},\tilde\omega\}_\star\frac{\delta}{\delta\hat{\bar\lambda}^{\dot a}}+i\{\hat\lambda^a,\tilde\omega\}_\star\frac{\delta}{\delta\hat\lambda^a}-i[\hat D,\tilde\omega]_\star\frac{\delta}{\delta\hat D}\Big)\,. \tag{4.32}$$

The action (4.5) is invariant under the transformation (4.32). It follows now from (4.25) that the θ-expansion of (4.5) is invariant under the transformation

$$W^{S,comm}_a$$
$$=\big(W^S_a\big)_{\theta=0}+\sum_{n=1}^{\infty}\frac{1}{n!}\theta^{\rho_1\sigma_1}\cdots\theta^{\rho_n\sigma_n}\Big(\Big[\frac{d}{d\theta^{\rho_1\sigma_1}},\Big[\cdots\Big[\frac{d}{d\theta^{\rho_n\sigma_n}},W^S_a\Big]\cdots\Big]\Big]\Big)_{\theta=0}\,, \tag{4.33}$$

which due to $[\frac{d}{d\theta},W^S_a]\neq 0$ is different from the commutative supersymmetry transformation $\big(W^S_a\big)_{\theta=0}$. The first terms of (4.33) read

$$W^{S,comm}_a=$$
$$\int d^4x\; tr\Big(\Big(\frac{1}{2}\sigma_{\mu a\dot a}\bar\lambda^{\dot a}+\frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho a\dot a}\{F_{\sigma\mu},\bar\lambda^{\dot a}\}\Big)\frac{\delta}{\delta A_\mu}+\Big(\frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho a\dot a}[\bar\lambda^{\dot a},D_\sigma\bar\lambda^{\dot b}]\Big)\frac{\delta}{\delta\bar\lambda^{\dot b}}$$
$$+\Big(\delta^b_aD+\sigma^{\mu\nu\,b}_a F_{\mu\nu}+\frac{1}{2}\theta^{\rho\sigma}\sigma^{\mu\nu\,b}_a\{F_{\mu\rho},F_{\nu\sigma}\}+\frac{1}{2}\theta^{\rho\sigma}\sigma_{\rho a\dot a}\{F_{\sigma\mu},\bar\lambda^{\dot a}\}\Big)\frac{\delta}{\delta\lambda^b}$$
$$+\Big(-i\sigma^\mu_{a\dot a}D_\mu\bar\lambda^{\dot a}+\frac{i}{2}\theta^{\rho\sigma}\sigma^\mu_{a\dot a}\{F_{\sigma\mu},D_\rho\bar\lambda^{\dot a}\}+\frac{1}{4}\theta^{\rho\sigma}\sigma_{\rho a\dot a}\{\bar\lambda^{\dot a},D_\sigma D\}\Big)\frac{\delta}{\delta D}\Big)$$
$$+\mathcal{O}(\theta^2)\,. \tag{4.34}$$

Similar formulae exist for the anti-supersymmetry transformation $W^{\bar S}_{\dot a}$. At order $n=0$ in θ the expansion of (4.5) is obviously the standard super Yang-Mills action

$$\Gamma^{(0)}=\int d^4x\; tr\Big(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{i}{4}\lambda^a\sigma^\mu_{a\dot a}D_\mu\bar\lambda^{\dot a}+\frac{1}{8}D^2\Big)\,, \tag{4.35}$$

where $\phi = \hat{\phi}|_{\theta=0}$ for $\phi \in \{A_\mu, \lambda^a, \bar{\lambda}^{\dot{a}}, D\}$. At first order in θ one finds

$$\begin{aligned}\Gamma^{(1)} &= \Gamma^{(0)}\\ &-\frac{1}{2}\int d^4x\; tr\Big(\theta^{\rho\sigma}F_{\rho\sigma}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{i}{8}\bar{\lambda}^{\dot{a}}\bar{\sigma}^{\mu}_{\dot{a}a}D_\mu\lambda^a+\frac{i}{8}\lambda^a\sigma^{\mu}_{a\dot{a}}D_\mu\bar{\lambda}^{\dot{a}}+\frac{1}{8}D^2\Big)\\ &+\theta^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}F^{\mu\nu}+\frac{1}{4}\theta^{\rho\sigma}F_{\mu\rho}\big(i\bar{\lambda}^{\dot{a}}\bar{\sigma}^{\mu}_{\dot{a}a}D_\sigma\lambda^a+i\lambda^a\sigma^{\mu}_{a\dot{a}}D_\sigma\bar{\lambda}^{\dot{a}}\big)\Big)\,. \qquad (4.36)\end{aligned}$$

The θ-expanded action (4.36) could be further analysed, for instance with respect to new decay channels of supersymmetric particles—in a similar manner as investigations of models without supersymmetry, see *e.g.* [55].

4.5 Remarks on the Superspace Formalism

Now we want to return to the superspace formalism. We notice that due to $[\frac{d}{d\theta}, W^S_a] \neq 0$, see (4.31), a Seiberg-Witten map in terms of superfields cannot exist. All one can do is to write the previous formulae in a more compact form, in which the super vector field is understood to be in Wess-Zumino gauge. The gauge transformations and observer Lorentz transformations can be written in the compact form

$$W^G_{\hat{\omega}} = \int d^4x\; tr\Big(\Big(-2\theta^a\sigma^{\mu}_{a\dot{a}}\bar{\theta}^{\dot{a}}\partial_\mu\hat{\omega} - i[\hat{\phi},\hat{\omega}]_\star\Big)\frac{\delta}{\delta\hat{\phi}}\Big)\,, \qquad (4.37)$$

$$W^T_\tau := \int d^4x\; tr\Big(\partial_\tau\hat{\phi}\,\frac{\delta}{\delta\hat{\phi}}\Big), \qquad (4.38)$$

$$\begin{aligned}W^R_{\alpha\beta} := &\int d^4x\; tr\Big(\Big(\frac{1}{2}\{x_\alpha,\partial_\beta\hat{\phi}\}_\star - \frac{1}{2}\{x_\beta,\partial_\alpha\hat{\phi}\}_\star + \Sigma_{\alpha\beta}\hat{\phi}\Big)\frac{\delta}{\delta\hat{\phi}}\Big)\\ &+\Big(\delta^\mu_\alpha\theta_\beta{}^\nu - \delta^\mu_\beta\theta_\alpha{}^\nu + \delta^\nu_\alpha\theta^\mu{}_\beta - \delta^\nu_\beta\theta^\mu{}_\alpha\Big)\frac{\partial}{\partial\theta^{\mu\nu}}\,, \qquad (4.39)\end{aligned}$$

$$W^D = \int d^4x\; tr\Big(\frac{1}{2}\{x^\delta,\partial_\delta\hat{\phi}\}_\star\frac{\delta}{\delta\hat{\phi}}\Big) - 2\theta^{\mu\nu}\frac{\partial}{\partial\theta^{\mu\nu}}\,. \qquad (4.40)$$

Here $\Sigma_{\alpha\beta} = -\frac{i}{2}\theta^a\sigma_{\alpha\beta\,a}{}^b\frac{\partial}{\partial\theta^b} + \frac{i}{2}\bar{\theta}_{\dot{a}}\bar{\sigma}^{\dot{a}}_{\alpha\beta\,\dot{b}}\frac{\partial}{\partial\bar{\theta}_{\dot{b}}}$ is the spin operator for the superfield.

The covariant particle Lorentz rotation reads

$$\begin{aligned}\tilde{W}^R_{\hat{\phi};\alpha\beta} &:= W^G_{\hat{\chi}_{\alpha\beta}}\\ &+ \int d^4x \, tr\Big(\Big(\frac{1}{2}\{\hat{X}_\alpha, \hat{F}_\beta\}_\star - \frac{1}{2}\{\hat{X}_\beta, \hat{F}_\alpha\}_\star + \Sigma_{\alpha\beta}(\hat{\phi} + 2\theta^a\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\hat{A}_\mu)\Big)\frac{\delta}{\delta\hat{\phi}}\Big),\end{aligned} \tag{4.41}$$

where $\hat{\chi}_{\alpha\beta}$ is given by (4.20) and

$$\hat{F}_\sigma := \partial_\sigma\hat{\phi} + 2\theta^a\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\partial_\mu\hat{A}_\sigma - i[\hat{A}_\sigma, \hat{\phi}]_\star\,. \tag{4.42}$$

This object, resembling the usual field strength tensor $F_{\mu\nu}$, transforms covariantly under supergauge transformations (4.37),

$$\begin{aligned} W^G_{\hat{\omega}}(\partial_\sigma\hat{\phi}) &= -2\theta\sigma\bar{\theta}\partial\partial_\sigma\hat{\omega} - i[\partial_\sigma\hat{\phi}, \hat{\omega}]_\star - i[\hat{\phi}, \partial_\sigma\hat{\omega}]_\star,\\ W^G_{\hat{\omega}}(2\theta\sigma\bar{\theta}\partial\hat{A}_\sigma) &= 2\theta\sigma\bar{\theta}\partial\partial_\sigma\hat{\omega} - 2i[\theta\sigma\bar{\theta}\partial\hat{A}_\sigma, \hat{\omega}]_\star - 2i[\hat{A}_\sigma, \theta\sigma\bar{\theta}\partial\hat{\omega}]_\star,\\ W^G_{\hat{\omega}}(-i[\hat{A}_\sigma, \hat{\phi}]_\star) &= -i[\partial_\sigma\hat{\omega} - i[\hat{A}_\sigma, \hat{\omega}]_\star, \hat{\phi}]_\star - i[\hat{A}_\sigma, -2\theta\sigma\bar{\theta}\partial, \hat{\omega} - i[\hat{\phi}, \hat{\omega}]_\star]_\star.\end{aligned}$$

Using the Jacobi identity this yields

$$W^G_{\hat{\omega}}(\hat{F}_\sigma) = -i[\hat{F}_\sigma, \hat{\omega}]_\star. \tag{4.43}$$

The calculation of the Seiberg-Witten expansion is straightforward. We need

$$\begin{aligned}(W^R_{\alpha\beta} - \tilde{W}^R_{\hat{\phi};\alpha\beta})\hat{\phi} &= \tilde{W}^R_{\theta,\alpha\beta}\hat{\phi} = W^R_{\alpha\beta}(\theta^{\rho\sigma})\frac{d}{d\theta^{\rho\sigma}}\hat{\phi}\\ &= -2\Big(\frac{d\phi}{d\theta^{\beta\rho}}\theta_\alpha{}^\rho - (\alpha \leftrightarrow \beta)\Big).\end{aligned} \tag{4.44}$$

With the use of

$$\Sigma_{\alpha\beta}(2\theta^a\sigma^\mu_{a\dot{a}}\bar{\theta}^{\dot{a}}\hat{A}_\mu) = 2\theta^a\sigma_{\alpha,a\dot{a}}\bar{\theta}^{\dot{a}}\hat{A}_\beta - (\alpha \leftrightarrow \beta) \tag{4.45}$$

we find

$$
\begin{aligned}
&(W^R_{\alpha\beta} - \tilde{W}^R_{\hat{\phi};\alpha\beta})\hat{\phi} = \\
&= \frac{1}{2}\{x_\alpha, \partial_\beta\hat{\phi}\}_\star - \frac{1}{2}\{\hat{X}_\alpha, \hat{F}_\beta\}_\star - 2\theta\sigma_\alpha\bar{\theta}\hat{A}_\beta + 2\theta\sigma\bar{\theta}\partial(\frac{1}{4}\{2x_\alpha + \theta_\alpha{}^\rho\hat{A}_\rho, \hat{A}_\beta\}_\star) \\
&\quad + i[\hat{\phi}, \frac{1}{4}\{2x_\alpha + \theta_\alpha{}^\rho\hat{A}_\rho, \hat{A}_\beta\}_\star]_\star - (\alpha \leftrightarrow \beta) \\
&= -\frac{1}{2}\{x_\alpha, 2\theta\sigma\bar{\theta}\partial\hat{A}_\beta - i[\hat{A}_\beta, \hat{\phi}]_\star\}_\star \\
&\quad - \frac{1}{2}\{\theta_\alpha{}^\rho\hat{A}_\rho, \partial_\beta\hat{\phi} + 2\theta\sigma\bar{\theta}\partial\hat{A}_\beta - i[\hat{A}_\beta, \hat{\phi}]_\star\}_\star \\
&\quad - 2\theta\sigma_\alpha\bar{\theta}\hat{A}_\beta + \frac{1}{2}\theta\sigma\bar{\theta}\partial\{2x_\alpha + \theta_\alpha{}^\rho\hat{A}_\rho, \hat{A}_\beta\}_\star + \frac{i}{4}[\hat{\phi}, \{2x_\alpha + \theta_\alpha{}^\rho\hat{A}_\rho, \hat{A}_\beta\}_\star]_\star \\
&\quad - (\alpha \leftrightarrow \beta) \\
&= -\frac{1}{2}\{x_\alpha, 2\theta\sigma\bar{\theta}\partial\hat{A}_\beta\}_\star + \frac{i}{2}\{x_\alpha, i[\hat{A}_\beta, \hat{\phi}]_\star\}_\star - \frac{1}{2}\{\theta_\alpha{}^\rho\hat{A}_\rho, \partial_\beta\hat{\phi} + 2\theta\sigma\bar{\theta}\partial\hat{A}_\beta\}_\star \\
&\quad + \frac{i}{2}\{\theta_\alpha{}^\rho\hat{A}_\rho, [\hat{A}_\beta, \hat{\phi}]_\star\}_\star - 2\theta\sigma_\alpha\bar{\theta}\hat{A}_\beta + 2\theta\sigma_\alpha\bar{\theta}\hat{A}_\beta + \{\frac{1}{2}\theta\sigma\bar{\theta}\partial(\theta_\alpha{}^\rho\hat{A}_\rho), \hat{A}_\beta\}_\star \\
&\quad + \{2x_\alpha + \theta_\alpha{}^\rho\hat{A}_\rho, \frac{1}{2}\theta\sigma\bar{\theta}\partial\hat{A}_\beta\}_\star + \frac{i}{2}[\hat{\phi}, \{x_\alpha, \hat{A}_\beta\}_\star]_\star + \frac{i}{4}[\hat{\phi}, \{\theta_\alpha{}^\rho\hat{A}_\rho, \hat{A}_\beta\}_\star]_\star \\
&\quad - (\alpha \leftrightarrow \beta).
\end{aligned}
\tag{4.46}
$$

After cancellation and with the help of the Jacobi identity we find (note that $[x_\alpha, \hat{\phi}]_\star = i\theta_\alpha{}^\rho\partial_\rho\hat{\phi}$)

$$
\begin{aligned}
&\Big(W^R_{\alpha\beta} - \tilde{W}^R_{\hat{\phi};\alpha\beta}\Big)\hat{\phi} = \\
&= \theta_\alpha{}^\rho\Big(\frac{2}{4}\{\hat{A}_\beta, \partial_\rho\hat{\phi}\}_\star + \frac{1}{4}\{\hat{A}_\beta, 2\theta\sigma\bar{\theta}\partial\hat{A}_\rho\}_\star - \frac{i}{4}\{\hat{A}_\beta, [\hat{A}_\rho, \hat{\phi}]_\star\}_\star - (\beta \leftrightarrow \rho)\Big) \\
&\quad -\Big(\alpha \leftrightarrow \beta\Big) \\
&= \theta_\alpha{}^\rho\Big(\frac{1}{4}\{\hat{A}_\beta, \partial_\rho\hat{\phi} + \hat{F}_\rho\}_\star - (\beta \leftrightarrow \rho)\Big) - \Big(\alpha \leftrightarrow \beta\Big) \\
&= -\Big(2\frac{d\theta\phi}{d\theta^{\beta\rho}}\theta_\alpha{}^\rho - (\alpha \leftrightarrow \beta)\Big).
\end{aligned}
$$

Thus we get the final result

$$
\frac{d\hat{\phi}}{d\theta^{\rho\sigma}} = -\frac{1}{8}\big\{\hat{A}_\rho, \partial_\sigma\hat{\phi} + \hat{F}_\sigma\big\}_\star + \frac{1}{8}\big\{\hat{A}_\sigma, \partial_\rho\hat{\phi} + \hat{F}_\rho\big\}_\star . \tag{4.47}
$$

Written in components this leads exactly to the formulae (4.24).

4.6 Conclusion

Following the ideas of [18, 57] we have derived the Seiberg-Witten map for noncommutative super Yang-Mills theory in Wess-Zumino gauge via the splitting of the observer Lorentz transformation into a covariant particle Lorentz transformation and a remainder, which directly leads to the Seiberg-Witten differential equations. We have also computed the θ-expansion of the noncommutative super Yang-Mills action, up to first order in θ. The θ-expanded action is invariant under a transformation which *differs* from the commutative supersymmetry transformations by terms of order $n \geq 1$ in θ. For this reason the Seiberg-Witten map cannot be expressed in terms of superfields.

4.7 Useful Formulae

Spinor indices $a, \dot{a} \in \{1, 2\}$ are shifted by the antisymmetric metric $\varepsilon^{ab} = -\varepsilon^{ba}, \varepsilon^{\dot{a}\dot{b}} = -\varepsilon^{\dot{b}\dot{a}}$ according to

$$\chi_a = \varepsilon_{ab}\chi^b \,, \qquad\qquad \bar{\chi}^{\dot{a}} = \varepsilon^{\dot{a}\dot{b}}\bar{\chi}_{\dot{b}} \,. \tag{4.48}$$

Note that spinors are anti-commuting,

$$\chi^a\eta_a = -\chi_a\eta^a = \eta^a\chi_a = -\eta_a\chi^a \,, \qquad \bar{\chi}_{\dot{a}}\bar{\eta}^{\dot{a}} = -\bar{\chi}^{\dot{a}}\bar{\eta}_{\dot{a}} = \bar{\eta}_{\dot{a}}\bar{\chi}^{\dot{a}} = -\bar{\chi}^{\dot{a}}\bar{\eta}_{\dot{a}} \,. \tag{4.49}$$

The 2×2 σ-matrices are given by

$$\sigma^\mu_{a\dot{a}} = \left(1, \vec{\sigma}\right)_{a\dot{a}} \,, \qquad \bar{\sigma}^{\mu\,\dot{a}a} = \left(1, -\vec{\sigma}\right)^{\dot{a}a} \,, \qquad \sigma^\mu_{a\dot{a}} = \bar{\sigma}^\mu_{\dot{a}a} \,, \tag{4.50}$$

where $\vec{\sigma}$ denotes the three Pauli matrices. The σ-matrices satisfy

$$\begin{aligned}
\sigma^\mu_{a\dot{a}}\bar{\sigma}^{\nu\,\dot{a}b} &= g^{\mu\nu}\delta_a^{\ b} - i\sigma^{\mu\nu\,b}_{a} \,, \\
\bar{\sigma}^{\mu\,\dot{a}a}\sigma^\nu_{a\dot{b}} &= g^{\mu\nu}\delta^{\dot{a}}_{\ \dot{b}} - i\bar{\sigma}^{\mu\nu\,\dot{a}}{}_{\dot{b}} \,, \\
\sigma^\mu_{a\dot{a}}\bar{\sigma}^{\nu\,\dot{a}b}\sigma^\rho_{b\dot{b}} &= g^{\mu\nu}\sigma^\rho_{a\dot{b}} + g^{\nu\rho}\sigma^\mu_{a\dot{b}} - g^{\rho\mu}\sigma^\nu_{a\dot{b}} - i\epsilon^{\mu\nu\rho\lambda}\sigma_{\lambda\,a\dot{b}} \,, \\
\bar{\sigma}^{\mu\,\dot{a}a}\sigma^\nu_{a\dot{b}}\bar{\sigma}^{\rho\,\dot{b}b} &= g^{\mu\nu}\bar{\sigma}^{\rho\,\dot{a}b} + g^{\nu\rho}\bar{\sigma}^{\mu\,\dot{a}b} - g^{\rho\mu}\bar{\sigma}^{\nu\,\dot{a}b} + i\epsilon^{\mu\nu\rho\lambda}\bar{\sigma}^{\dot{a}b}_{\lambda} \,, \\
\sigma^\mu_{a\dot{a}}\sigma_{\mu\,b\dot{b}} &= 2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}} \,,
\end{aligned} \tag{4.51}$$

with $\sigma^{\mu\nu\,b}_{a} = -\sigma^{\nu\mu\,b}_{a}$ and $\bar{\sigma}^{\mu\nu\,\dot{a}}{}_{\dot{b}} = -\bar{\sigma}^{\nu\mu\,\dot{a}}{}_{\dot{b}}$.

Part II

Aus unsern Taten steigt ein Gericht...

Hugo von Hofmannsthal

Chapter 5

Back to the Roots

5.1 Introduction

Now we want to return to the problem of UV/IR mixing presented in Part I of this thesis. However, from now on we will use Minkowskian signature of space-time. Apparently, the origin of this inconvenient feature of noncommutative field theories is too fundamental to be curable by a mere application of mathematical techniques. One has to focus on the underlying physics. Indeed, this will eventually lead to the solution of another problem: the violation of unitarity for $\theta_{0i} \neq 0$.

It was pointed out in [58] that in the Minkowskian (non-degenerate) case the Wick rotation of Euclidean Green's function does *not* give a meaningful result, first of all because unitarity would be lost [59]. The reason is that the Osterwalder-Schrader theorem [60] does not hold. Already in [61] there was given a proposal for a correct quantisation of field theories on space/time noncommutative geometries: Starting with interaction Hamiltonians on a Fock space in the Dirac picture (with free fields)

$$H_I(t) = \int_{x^0=t} d^3x \ :(\phi_0 \star \phi_0 \star \cdots \star \phi_0)(x): \tag{5.1}$$

(and averaging over the noncommutativity parameter), where the dots denote the normal ordering of fields, the contributions to the scattering amplitudes were defined as the Dyson series

$$G_n^{con}(x_1, \ldots, x_k) :=$$
$$\frac{(-i)^n}{n!} \int dt_1 \ldots dt_n \left\langle 0 \middle| T\phi(x_1) \ldots \phi(x_k) H_I(t_1) \cdots H_I(t_n) \middle| 0 \right\rangle_{(0)}^{con}, \qquad (5.2)$$

where T denotes the time-ordering with respect to $\{x_1^0, \ldots, x_k^0, t_1, \ldots, t_n\}$ and $|0\rangle$ the vacuum state. Unitarity is preserved.

In [58] there was added a second proposal, the iterative solution of the (interacting) field equation (Yang-Feldman approach), which has the advantage of being manifestly covariant. Unitarity is preserved as well.

A third approach, the direct application of the Gell-Mann–Low formula for Green's functions,

$$G_n^{con}(x_1, \ldots, x_k) :=$$
$$\frac{i^n}{n!} \int d^4z_1 \ldots d^4z_n \left\langle 0 \middle| T\phi(x_1) \ldots \phi(x_k) \mathcal{L}_I(z_1) \cdots \mathcal{L}_I(z_n) \middle| 0 \right\rangle_{(0)}^{con} \qquad (5.3)$$

where $\mathcal{L}_I$ is the interaction Lagrangian, was elaborated in [62]. The superscript ^{con} means projection onto the connected part. Unitarity was investigated in [63]. That approach was called "time-ordered perturbation theory" in [62], a name which we find ambiguous. The time-ordering in [62] is considered for external vertices and *interaction points* only, and not with respect to the *actual time-order of the fields* in the interaction Lagrangian. Thus it is an *interaction-point time-ordering* (IPTO), it is explicitly accusal and to distinguish from a true *causal time-ordering*. A detailed explanation will be given below.

After all, these results motivated ourselves to have a closer look at the very basics of quantum field theory. So long, deformed field theory has been pursued in a somewhat ambivalent way (see Part I): The Moyal product has been used in the interaction part of the Lagrangian (remember that the bilinear term remained unchanged). The validity of the usual Lagrangian formalism and ordinary Feynman rules has ever since been *assumed* but never been proven. Since it is completely unclear if this formalisms still hold on noncommutative space, we will now construct the whole perturbative quantum field theory on deformed space (following [64]), beginning with the Schrödinger equation on commutative space...

5.2 Pictures

5.2.1 Schrödinger Picture

We start with the Schrödinger equation (with $\hbar = 1$),

$$i\frac{\partial}{\partial t}|Z_S(t)\rangle = H_S|Z_S(t)\rangle, \tag{5.4}$$

where $H_S = H_{0S} + H_{IS} =$ is the Hamilton operator consisting of a free part H_{0S} and an interaction part H_{IS}, and $|Z_S(t)\rangle$ is a time dependent Schrödinger state. As long as H_S is time independent, we have the simple solution

$$|Z_S(t)\rangle = e^{-iH_S(t-t_0)}|Z_S(t_0)\rangle, \tag{5.5}$$

with some initial state $|Z_S(t_0)\rangle$.

Now the question is: What is the particular feature of a specific model described by eq. (5.4)? The answer is simple: Different models are distinguished by different Hamiltonians. In particular, the states are defined as solutions of eq. (5.4) with a particular Hamiltonian.

In the Schrödinger picture (which is denoted by the index S) we have the notion of a time independent Hamiltonian, which generates time dependent states. Physics is described via matrix elements of operators with these states. Those operators are assumed time independent and (if we are lucky) known, so the interesting thing is to get the correct states.

5.2.2 Heisenberg Picture

But if we have a closer look at the matrix elements of some time independent operator A_S,

$$\langle A\rangle = \langle Z_S(t)|A_S|Z_S(t)\rangle = \langle Z_S(t_0)|e^{+iH_S(t-t_0)}A_S e^{-iH_S(t-t_0)}|Z_S(t_0)\rangle, \tag{5.6}$$

we could also argue that we have time independent states $|Z_H\rangle := |Z_S(t_0)\rangle$ and time dependent *Heisenberg* operators A_H (let $t_0 = 0$),

$$A_H(t) := e^{+iH_S t}A_S e^{-iH_S t}. \tag{5.7}$$

This is the Heisenberg picture. Instead of fixing the operators and searching for time dependent states, we keep the states fixed and put our interest in time evoluting operators.

Instead of the Schrödinger equation (5.4) for the states, we now have the Heisenberg equation for the Heisenberg operators, obtained from differentiating (5.7) with respect to the time (note that $A_S \neq A_S(t)$),

$$-i\frac{\partial}{\partial t}A_H(t) = [H_H, A_H(t)]. \tag{5.8}$$

Here, $H_H = H_S$ is still time independent. The Heisenberg equation looks indeed very similar to the Schrödinger equation.

5.2.3 Dirac (Interaction) Picture

Somehow in between is the Dirac picture, where states and operators have a time evolution. The *free* part of the Hamiltonian is used to describe the time evolution of the operators, whereas the *interaction part* will describe the time evolution of the states and is treated like other operators. The states in the interaction picture are defined as

$$|Z_D(t)\rangle := e^{iH_{0S}t}|Z_S(t)\rangle \ . \tag{5.9}$$

From $\langle Z_S|A_S|Z_S\rangle = \langle Z_D|A_D|Z_D\rangle$ we conclude

$$A_D(t) := e^{+iH_{0S}t}A_S e^{-iH_{0S}t} \ . \tag{5.10}$$

With

$$\begin{aligned} H_{0D} = H_{0S}, \quad H_{ID}(t) &:= e^{+iH_{0S}t}H_{ID}e^{-iH_{0S}t}, \\ [H_{0S}, \exp(\pm iH_{0S}t)] &= 0 \end{aligned} \tag{5.11}$$

we find the two evolution equations (from 5.9 and (5.4), (5.9), respectively)

$$\begin{aligned} -i\frac{\partial}{\partial t}A_D(t) &= [H_{0D}, A_D(t)], \\ i\frac{\partial}{\partial t}|Z_D(t)\rangle &= H_{ID}(t)|Z_D(t)\rangle. \end{aligned} \tag{5.12}$$

Note that the first equation is only true for $A_S \neq A_S(t)$. We see that the time evolution of the operators is defined by the *free* Hamiltonian, so that $A_D(t)$ is simply a solution of the free theory. The time evolution of the states, on the other hand, depends only on the *interaction* Hamiltonian. Note: Since the free Hamiltonians are the same in all pictures, we define $H_{0S} = H_{0H} = H_{0D} = H_0$ from now on.

5.3 Lagrange and Hamilton Formalisms for the Scalar Field

5.3.1 Commutative Space

The free scalar field is described by the equation of motion

$$(\Box + m^2)\phi = 0. \tag{5.13}$$

This equation of motion can be obtained by a field variation of the action,

$$\begin{aligned} &W = \int dt L_0 = \int d^4x \mathcal{L}_0, \quad \mathcal{L}_0 = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2, \\ &\delta W = 0 \Rightarrow (\Box + m^2)\phi = 0. \end{aligned} \tag{5.14}$$

Another possibility is the description via the Hamiltonian $H = \int d^3x \mathcal{H}$,

$$\Pi := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}, \quad \mathcal{H} := \dot{\phi}\Pi - \mathcal{L}. \tag{5.15}$$

An explicit calculation yields

$$H_0 = \int d^3x \frac{1}{2}(\dot{\phi}^2 + (\vec{\partial}\phi)^2 + m^2\phi^2) \geq 0. \tag{5.16}$$

H_0 is interpreted as the total energy of the free field system. Energy conservation $\frac{d}{dt}H_0 = \frac{d}{dt}\int d^3x \mathcal{H}_0 = 0$ is obtained by use of partial integration and the equation of motion.

Note: ϕ and Π correspond to x (the current elongation) and p (the momentum) of the harmonic oscillator, whereas the space coordinates $\vec{x}$ could be thought of as 'labels' of the infinitely many harmonic oscillators hanging around in space. Only time is always time.

5.3.2 Noncommutative Space

Now we have two possibilities in defining the Lagrangian, leading to the same equation of motion.

$$\mathcal{L}_0 = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 \quad \text{or} \quad \mathcal{L}_0 = \frac{1}{2}(\partial^\mu\phi) \star (\partial_\mu\phi) - \frac{1}{2}m^2\phi \star \phi. \tag{5.17}$$

Of course, on noncommutative space the second possibility seems to be somewhat more natural. Fortunately, integration $\int d^4x$ leads to the same action W for both Lagrangians and the difference is thus not problematic. However, in defining the Hamiltonian the difference seems to be crucial. In order to see what consequences the insertion of the Moyal products into the Lagrangian has we use the second definition. $\Pi = \dot{\phi}$ remains unchanged. Now we have

$$\begin{aligned} H_0 &= \int d^3x \mathcal{H}_0 = \int d^3x (\dot{\phi} \star \Pi - \mathcal{L}_0) \\ &= \int d^3x \frac{1}{2} (\dot{\phi} \star \dot{\phi} + (\vec{\partial}\phi) \star (\vec{\partial}\phi) + \frac{1}{2} m^2 \phi \star \phi). \end{aligned} \tag{5.18}$$

Note: Since we have only $\int d^3x$ we must not drop the stars here! Now we find

$$\frac{dH}{dt} = \frac{1}{2} \int d^3x (\dot{\phi} \star (\ddot{\phi} - \Delta\phi + m^2\phi) + (\ddot{\phi} - \Delta\phi + m^2\phi) \star \dot{\phi})|_{e.o.m.} = 0. \tag{5.19}$$

A modification of the equation of motion is not necessary to guarantee energy conservation.

Since we know now that the free field equation is not modified on noncommutative space, we can use the well known solutions $\phi(x) = \phi^+(x) + \phi^-(x)$,

$$\begin{aligned} \phi^-(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}}\, a^-(\vec{k})\, e^{-ix_\mu k^{+\mu}}, \\ \phi^+(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}}\, a^+(\vec{k})\, e^{+ix_\mu k^{+\mu}}. \end{aligned} \tag{5.20}$$

Here we have $k^+ = (\omega_k, \vec{k})$, $\omega_k = \sqrt{\vec{k}^2 + m^2}$.

5.4 Quantum Field Theory

Quantisation is now performed in the following way. We declare the so long classical fields (5.20) being *operators* in the interaction picture, obeying $[\Pi(t,\vec{x}), \Phi(t,\vec{x}')] = -i\delta(\vec{x} - \vec{x}'))$. This leads to a natural interpretation of $a^+(\vec{k})$ and $a^-(\vec{k})$ as creation and annihilation operators, respectively,

$$[a^-(\vec{k}), a^+(\vec{k}')] = \delta^3(\vec{k} - \vec{k}'). \tag{5.21}$$

With this we find

$$[\phi^-(x), \phi^+(y)] = D^+(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-i(x-y)k^+}. \tag{5.22}$$

Inserting (5.20) into the expression (5.18) for the free Hamiltonian we have

$$\begin{aligned}
H_0 &= \int d^3x \frac{1}{2}(\dot\phi \star \dot\phi + (\vec\partial\phi) \star (\vec\partial\phi) + \frac{1}{2}m^2\phi \star \phi) \\
&= \int d^3x \frac{1}{2}\frac{1}{(2\pi)^3} \int \frac{d^3k d^3k'}{\sqrt{4\omega\omega'}} \times \Big(\\
&-\omega\omega'(a^+(\vec k)e^{+ik^+x} - a^-(\vec k)e^{-ik^+x}) \star (a^+(\vec k')e^{+ik'^+x} - a^-(\vec k')e^{-ik'^+x}) \\
&-(\vec k\vec k')(a^+(\vec k)e^{+ik^+x} - a^-(\vec k)e^{-ik^+x}) \star (a^+(\vec k')e^{+ik'^+x} - a^-(\vec k')e^{-ik'^+x}) \\
&+m^2(a^+(\vec k)e^{+ik^+x} + a^-(\vec k)e^{-ik^+x}) \star (a^+(\vec k')e^{+ik'^+x} + a^-(\vec k')e^{-ik'^+x})\Big).
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{1}{(2\pi)^3}\int d^3x e^{ik_1x} \star e^{ik_2x} = \frac{1}{(2\pi)^3}\int d^3x e^{i(k_1+k_2)x} e^{-\frac{i}{2}k_1\theta k_2} \\
&= \delta^{(3)}(\vec k_1 + \vec k_2) e^{i(k_1^0+k_2^0)x_0} e^{-\frac{i}{2}(k_1^0k_2^i - k_1^ik_2^0)\theta_{0i}}.
\end{aligned}$$

Collecting the factors of a^+a^+, a^-a^-, a^+a^-, a^-a^+ we find (with three momentum conservation and use of the equation of motion) that the a^+a^+, a^-a^- terms vanish and the a^+a^-, a^-a^+ terms have a total factor of $2\omega^2$ without any phase factor. Thus we finally get

$$H_0 = \int d^3k\, \omega\, \frac{1}{2}(a^+(\vec k)a^-(\vec k) + a^-(\vec k)a^+(\vec k)). \tag{5.23}$$

This is exactly the same expression we would have found for the commutative theory (which can be seen most easily, since it does not depend on θ anymore). Thus, the insertion of the Moyal products does not change the free Hamiltonian at all. In particular, remind that we have $\frac{d}{dt}H_0 = 0$ (which is essential for the construction of our perturbation theory) due to the equation of motion.

One checks explicitly with (5.21) that the quantised Dirac operators of the field (5.20) fulfil the evolution equation, which is most simply done by use of H_0 as presented in eq. (5.23) (note that H_0 does not depend on x^μ),

$$\begin{aligned}[H_0, \phi^\pm(x)] &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \frac{1}{2} \int d^3k' \omega'_k \\ &\quad \times \underbrace{[a^+(\vec{k}')a^-(\vec{k}') + a^-(\vec{k}')a^+(\vec{k}'), a^\pm(\vec{k})]}_{\pm 2a^\pm(\vec{k}')\delta^3(\vec{k}-\vec{k}')} e^{\pm i x_\mu k^{\pm,\mu}} \\ &= \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} (\pm 2a^\pm(\vec{k}))\omega_k e^{\pm i x_\mu k^{\pm,\mu}} = -i\frac{\partial}{\partial t}\phi^\pm(x). \end{aligned} \tag{5.24}$$

Next we want to introduce some interaction Hamiltonian (written already in terms of Dirac fields), for example

$$H_D = H_0 + H_{ID}, \qquad H_{ID} = -\frac{g}{4!} \int d^3x (\phi_0 \star \phi_0 \star \phi_0 \star \phi_0)(x),, \tag{5.25}$$

where $H_{ID} = H_{ID}(t)$ is time dependent, as defined above. The evolution equation for the Dirac states is

$$i\frac{\partial}{\partial t}|Z_D(t)\rangle = H_{ID}|Z_D(t)\rangle. \tag{5.26}$$

Since H_{ID} depends explicitly on time we have to solve this iteratively, which leads to the well known result

$$|Z_D(t)\rangle = \underbrace{Te^{-i\int_{t_0}^t H_{ID}(t')dt'}}_{U(t,t_0)} |Z_D(t_0)\rangle, \tag{5.27}$$

with T being the usual time ordering operator acting on the (formal) time argument t' of H_{ID} (see the next subsection). This defines the time evolution operator $U(t, t_0)$. In order to describe scattering processes, we need the S-operator defined by

$$S = U(\infty, -\infty) = Te^{+i\int dt \int d^3x \mathcal{L}_{ID}(t,\vec{x})}, \tag{5.28}$$

where $\mathcal{L}_{ID}(x) = -\mathcal{H}_{ID}(x)$ is the interaction Lagrangian. Again, T acts on the time argument of $\int d^3x \mathcal{L}_{ID}(t, \vec{x})$.

S-matrix elements are thus given by

$$S_{fi} := \langle f|S|i\rangle, \tag{5.29}$$

where $|i\rangle$ and $\langle f|$ are the incoming and outgoing states, respectively, q.e.d.

5.5 A Conceptual Note

On a true noncommutative, four dimensional algebra the "integration" (i.e. trace operation) over a submanifold $\int d^3x$ is tedious. In Moyal deformed QFT, which just replaces ordinary products by Moyal products, this problem does not occur. Here by "noncommutative $\mathbb{R}^4$" one understands the *algebra* $\mathbb{R}^4_\theta$ of Schwartz class functions on ordinary four-dimensional space, equipped with the multiplication rule

$$(f \star g)(x) = \int d^4s \int \frac{d^4l}{(2\pi)^4}\, f(x - \tfrac{1}{2}\tilde{l})\, g(x+s)\, e^{ils}\,, \quad \tilde{l}^\nu := l_\mu \theta^{\mu\nu}\,. \tag{5.30}$$

The product (5.30) characterised by a real skew-symmetric translation invariant tensor $\theta^{\mu\nu} = -\theta^{\nu\mu}$ of dimension [length]^2 is associative and noncommutative, it is a *non-local* product on rapidly decreasing functions.

We consider a scalar field theory on $\mathbb{R}^4_\theta$ given by the classical action

$$\Gamma = \int d^4z \Big(\frac{1}{2} g^{\mu\nu} (\partial_\mu\phi \star \partial_\nu\phi)(z) - \frac{1}{2} m^2 (\phi\star\phi)(z) + \frac{g}{4!}(\phi\star\phi\star\phi\star\phi)(z)\Big)\,, \tag{5.31}$$

with $\phi \in \mathbb{R}^4_\theta$.

Since the Moyal product is highly nonlocal, the time dependence of H_{ID} is somewhat problematic. This can be seen most easily by the integral representation of the Moyal product,

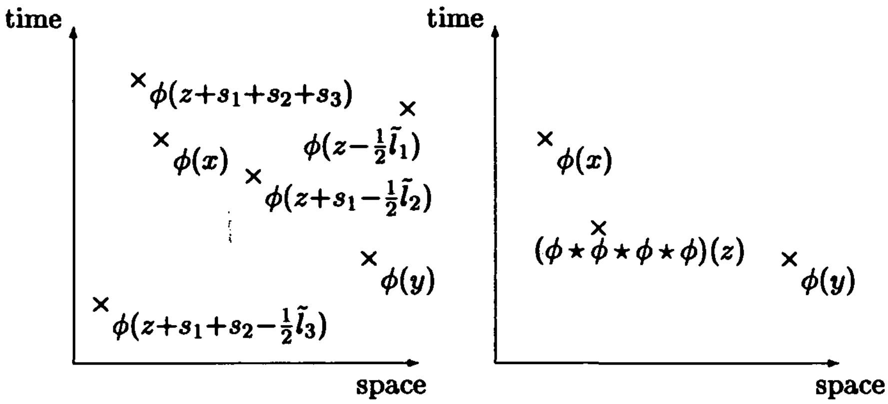
$$\begin{aligned}(\phi\star\phi\star\phi\star\phi)(x) &= \int \prod_{i=1}^{3} \Big(d^4s_i \frac{d^4l_i}{(2\pi)^4} e^{il_is_i}\Big) \qquad (5.32)\\ &\times \phi(z - \frac{1}{2}\tilde{l}_1)\phi(z+s_1-\frac{1}{2}\tilde{l}_2)\phi(z+s_1+s_2-\frac{1}{2}\tilde{l}_3)\phi(z+s_1+s_2+s_3),\end{aligned}$$

$\tilde{l}^\nu := l_\mu\theta^{\mu\nu}$, which shows that the time arguments of f and g are not quite fixed. For further calculations, the x^0 component of the left side (the so-called 'time stamp') is assumed to be 'the time coordinate' of a given field product, e.g. of H_{ID}. At least, this definition guarantees formal unitarity of the S-matrix.

Let us now investigate in more detail the difference between total and interaction-point time ordering. We consider the simplest case of the two-point function at first order in g. The Gell-Mann Low-formula implies

$$G(x,y) = \frac{g}{4!}\int d^4z \Big\langle 0 \Big| T\big(\phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big)\Big| 0\Big\rangle_{(0)}\,. \tag{5.33}$$

(We put the missing factor i directly into the formula for the element of the S-matrix.) In the same manner as on commutative space-time, the integration over the interaction point is performed *after* taking the time-ordered product. Since the $\star$-product for $\theta^{0i} \neq 0$ is non-local in time, one has to say clearly what one understands under time-ordering. Let us discuss this for the geometrical situation relevant for (5.33):



The arangement of fields for the left graph corresponds to the following non-vanishing contribution to the true time-ordering of (5.33):

$$
\begin{aligned}
G(x,y) &= \\
&= \int d^4z \int \prod_{i=1}^{3} \Big(d^4 s_i \frac{d^4 l_i}{(2\pi)^4}\, e^{i l_i s_i} \Big) \tau(s_1^0+s_2^0+s_3^0+\tfrac{1}{2}\tilde{l}_1^0)\tau(z^0-\tfrac{1}{2}\tilde{l}_1^0-x^0) \\
&\times \tau(x^0-z^0-s_1^0+\tfrac{1}{2}\tilde{l}_2^0)\tau(z^0+s_1^0-\tfrac{1}{2}\tilde{l}_2^0-y^0)\tau(y^0-z^0-s_1^0-s_2^0+\tfrac{1}{2}\tilde{l}_3^0) \\
&\times \Big\langle 0 \Big| \phi(z+s_1+s_2+s_3)\phi(z-\tfrac{1}{2}\tilde{l}_1)\phi(x)\phi(z+s_1-\tfrac{1}{2}\tilde{l}_2)\phi(y)\phi(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3) \Big| 0 \Big\rangle_{(0)}.
\end{aligned}
\tag{5.34}
$$

Here, $\tau(t)$ denotes the step function $\tau(t) = 1$ for $t > 0$ and $\tau(t) = 0$ for $t < 0$. There are $6! = 720$ different contributions to (5.33) when interpreting the time-ordering in the Gell-Mann–Low formula as the name suggests. The time-ordering guarantees that only causal processes contribute to the S-matrix. Positive energy solutions propagate forward in time and negative energy solutions backward.

In contrast to this *true* time ordering, we now have *interaction point* time ordering (right graph), which is defined with respect to the *interaction* *point*:

$$G'(x,y) = \int d^4z \int \prod_{i=1}^{3} \left(d^4s_i \frac{d^4l_i}{(2\pi)^4}\, e^{il_is_i} \right) \tau(x^0-z^0)\tau(z^0-y^0)$$
$$\times \left\langle 0 \middle| \phi(x)\phi(z-\tfrac{1}{2}\tilde{l}_1)\phi(z+s_1-\tfrac{1}{2}\tilde{l}_2)\phi(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3)\phi(z+s_1+s_2+s_3)\phi(y) \middle| 0 \right\rangle_{(0)}. \tag{5.35}$$

There are now only $3! = 6$ different contributions of this type. Since the individual fields are now (in most of the cases) at the wrong place with respect to the time-order, the noncommutative version (5.35) of the Gell-Mann–Low formula violates causality. Now both energy solutions propagate in any direction of time. After all, contributions (5.2) to the Dyson series are precisely ordered with respect to the time stamp of the interaction Hamiltonians. It does not matter how the time-dependence of the interaction Hamiltonian is produced from the time-dependence of the constituents.

5.6 Gauge Field Theory

5.6.1 Gauge Fixed Lagrangian

The gauge fixed Lagrangian for a pure $U(1)$ gauge field model reads (see [50], where this has been derived very elegant by use of covariant coordinates as presented in chapter 5)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu) \star (\partial^\nu A_\nu). \tag{5.36}$$

We define

$$F_{\mu\nu} := \underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}_{f_{\mu\nu}} - ig[A_\mu, A_\nu]_\star. \tag{5.37}$$

The free part of the Lagrangian thus reads

$$\mathcal{L}_0 = -\frac{1}{4} f_{\mu\nu} \star f^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu) \star (\partial^\nu A_\nu). \tag{5.38}$$

The free field equation reads

$$\Box A_\mu - (1-\frac{1}{\alpha})\partial_\mu(\partial^\nu A_\nu) = 0. \tag{5.39}$$

For the free field momenta we find

$$\Pi^i = f^{i0} = +f_{0i}, \qquad \Pi^0 = -\frac{1}{\alpha}(\partial^\mu A_\mu). \tag{5.40}$$

Now we define

$$\begin{aligned}
H_0 &= \int d^3x \mathcal{H}_0 = \int d^3x (\frac{1}{2}\{\dot{A}_\mu, \Pi_\mu\}_\star - \mathcal{L}_0) \\
&= \int d^3x \frac{1}{2}\{\partial_0 A_i, f_{0i}\}_\star + \underbrace{\frac{1}{2}\partial^\mu A^\nu \star f_{\mu\nu}}_{-\frac{1}{4}\{\partial_0 A_i, f_{0i}\}_\star - \frac{1}{4}\{\partial_i A_0, f_{i0}\}_\star + \frac{1}{2}\partial_i A_j \star f_{ij}} \\
&\quad -\frac{1}{2\alpha}\{\partial_0 A_0, \partial^\mu A_\mu\}_\star + \frac{1}{2\alpha}(\partial^\mu A_\mu)\star(\partial^\nu A_\nu) \\
&= \int d^3x \frac{1}{4}\{(\partial_0 A_i + \partial_i A_0), f_{0i}\}_\star + \underbrace{\frac{1}{2}\partial_i A_j \star f_{ij}}_{\frac{1}{2}(\partial_i A_j \star \partial_i A_j - \partial_i A_i \star \partial_j A_j)} \\
&\quad -\frac{1}{2\alpha}(\partial_0 A_0)\star(\partial_0 A_0) + \frac{1}{2\alpha}(\partial_i A_i)\star(\partial_j A_j) + \dot{A}_0 \star \dot{A}_0 - \dot{A}_0 \star \dot{A}_0 \\
&= \frac{1}{2}\int d^3x \Big(\dot{A}_i \star \dot{A}_i - \dot{A}_0 \star \dot{A}_0 + \partial_j A_i \star \partial_j A_i - \partial_j A_0 \star \partial_j A_0 \\
&\quad +(\frac{\alpha-1}{\alpha})(\dot{A}_0 \star \dot{A}_0 - (\partial_i A_i)\star(\partial_j A_j))\Big).
\end{aligned} \tag{5.41}$$

We check explicitly the time independence of the Hamiltonian,

$$\begin{aligned}
\dot{H}_0 &= \frac{1}{2}\int d^3x \frac{1}{2}\{\dot{A}_i, \ddot{A}_i - \vec{\partial}^2 A_i + (\frac{\alpha-1}{\alpha})(-\partial_i\partial^\mu A_\mu + \underbrace{\partial_i\partial^0 A_0}))\}_\star \\
&\quad -\frac{1}{2}\{\dot{A}_0, \ddot{A}_0 - \vec{\partial}^2 A_0 + (\frac{\alpha-1}{\alpha})\partial_0\partial^0 A_0\}_\star.
\end{aligned} \tag{5.42}$$

The first line (without the underbraced term) is zero due to the equation of motion for A_i. After partial integration of the underbraced term with respect to $\partial_i = -\partial^i$ it combines with the second line to the equation of motion for A_0. Thus we see that $\frac{d}{dt}H_0 = 0$.

For quantisation we rewrite H_0 in a convenient form

$$H_0 = \frac{1}{2}\int d^3x\Big(- \partial_\mu A^\nu \star \partial_\mu A_\nu + (\frac{\alpha-1}{2\alpha})\{\partial^\mu A_\mu, \partial_\nu A_\nu\}_\star\Big). \qquad (5.43)$$

We make the ansatz

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_k}}\Big(a^+_\mu(\vec{k})e^{+ikx} + a^-_\mu(\vec{k})e^{-ikx}\Big), \qquad (5.44)$$

where $k_0 = \omega_k(\vec{k}) > 0$ is a (not yet specified) function of $|\vec{k}|$. Inserting this into the expression for H_0 we find

$$\begin{aligned}
H_0 = \frac{1}{2}&\int \frac{d^3k}{\sqrt{2\omega_k}}\int \frac{d^3q}{\sqrt{2\omega_q}}\\
\Big(&e^{it(\omega_k+\omega_q)-\frac{i}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta(\vec{k}+\vec{q})\big(k_\mu q_\mu a^{\nu+}(\vec{k})a^+_\nu(\vec{q})\\
&+(\frac{\alpha-1}{2\alpha})(-k^\mu q_\nu a^+_\mu(\vec{k})a^+_\nu(\vec{q}) - k_\nu q^\mu a^+_\nu(\vec{k})a^+_\mu(\vec{q}))\big)\\
&+e^{-it(\omega_k+\omega_q)-\frac{i}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta(\vec{k}+\vec{q})\big(k_\mu q_\mu a^{\nu-}(\vec{k})a^-_\nu(\vec{q})\\
&+(\frac{\alpha-1}{2\alpha})(-k^\mu q_\nu a^-_\mu(\vec{k})a^-_\nu(\vec{q}) - k_\nu q^\mu a^-_\nu(\vec{k})a^-_\mu(\vec{q}))\big)\\
&+e^{it(\omega_k-\omega_q)+\frac{i}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta(\vec{k}-\vec{q})\big(- k_\mu q_\mu a^{\nu+}(\vec{k})a^-_\nu(\vec{q})\\
&+(\frac{\alpha-1}{2\alpha})(+k^\mu q_\nu a^+_\mu(\vec{k})a^-_\nu(\vec{q}) + k^\mu q_\nu a^-_\nu(\vec{k})a^+_\mu(\vec{q}))\big)\\
&+e^{-it(\omega_k-\omega_q)+\frac{i}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta(\vec{k}-\vec{q})\big(- k_\mu q_\mu a^{\nu-}(\vec{k})a^+_\nu(\vec{q})\\
&+(\frac{\alpha-1}{2\alpha})(+k^\mu q_\nu a^-_\mu(\vec{k})a^+_\nu(\vec{q}) + k^\mu q_\nu a^+_\nu(\vec{k})a^-_\mu(\vec{q}))\big)\Big).
\end{aligned}$$

Using now the delta functions we find

$$\begin{aligned}
H_0 = \frac{1}{2}&\int \frac{d^3k}{\sqrt{2\omega_k}}\int \frac{d^3q}{\sqrt{2\omega_q}}\\
\Big(&e^{it(\omega_k+\omega_q)-\frac{i}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta(\vec{k}+\vec{q})\big(\frac{1}{2}(k^2 a^{\nu+}(\vec{k})a^+_\nu(\vec{q}) + q^2 a^+_\nu(\vec{k})a^{\nu+}(\vec{q}))\\
&+(\frac{\alpha-1}{2\alpha})(-k^\mu k^\nu a^+_\mu(\vec{k})a^+_\nu(\vec{q}) - q^\nu q^\mu a^+_\nu(\vec{k})a^+_\mu(\vec{q}))\big)\\
&+e^{-it(\omega_k+\omega_q)-\frac{i}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta(\vec{k}+\vec{q})\big(\frac{1}{2}(k^2 a^{\nu-}(\vec{k})a^-_\nu(\vec{q}) + q^2 a^-_\nu(\vec{k})a^{\nu-}(\vec{q})
\end{aligned}$$

$$
\begin{aligned}
&+(\frac{\alpha-1}{2\alpha})(-k^\mu k^\nu a^-_\mu(\vec{k})a^-_\nu(\vec{q})-q^\nu q^\mu a^-_\nu(\vec{k})a^-_\mu(\vec{q}))) \\
&+\delta(\vec{k}-\vec{q})\big(-k_\mu k_\mu a^{\nu+}(\vec{k})a^-_\nu(\vec{k}) \\
&+(\frac{\alpha-1}{2\alpha})(+k^\mu k_\nu a^+_\mu(\vec{k})a^-_\nu(\vec{k})+k^\mu k_\nu a^-_\nu(\vec{k})a^+_\mu(\vec{k}))) \\
&+\delta(\vec{k}-\vec{q})\big(-k_\mu k_\mu a^{\nu-}(\vec{k})a^+_\nu(\vec{k}) \\
&+(\frac{\alpha-1}{2\alpha})(+k^\mu k_\nu a^-_\mu(\vec{k})a^+_\nu(\vec{k})+k^\mu k_\nu a^+_\nu(\vec{k})a^-_\mu(\vec{k}))))\Big).
\end{aligned}
$$

With the equation of motion (5.39) expressed in terms of $a^\pm_\mu$,

$$k^2 a^\pm_\mu(\vec{k})-(\frac{\alpha-1}{\alpha})k_\mu(k^\nu a^\pm_\nu(\vec{k}))=0, \tag{5.45}$$

the terms with the non zero exponentials vanish. The remaining terms simplify considerably with the help of the equation of motion. So we get

$$
\begin{aligned}
H_0 &= \frac{1}{2}\int\frac{d^3k}{2\omega_k}\big(-k_\mu k_\mu a^{\nu+}(\vec{k})a^-_\nu(\vec{k})+k^2a^+_\nu(\vec{k})a^-_\nu(\vec{k}) \\
&\qquad -k_\mu k_\mu a^{\nu-}(\vec{k})a^+_\nu(\vec{k})+k^2a^-_\nu(\vec{k})a^+_\nu(\vec{k})) \\
&= \int\frac{d^3k}{2\omega_k}\big(-\vec{k}^2(a^+_0a^-_0+a^-_0a^+_0)+\omega^2_k(a^+_ia^-_i+a^-_ia^+_i)\big).
\end{aligned} \tag{5.46}
$$

Quantisation can now be performed in the usual way by imposing the commutator relation

$$[a^-_\rho(\vec{k}),a^+_\mu(\vec{k}\,')]=-g_{\rho\mu}\delta^{(3)}(\vec{k}-\vec{k}\,')\ . \tag{5.47}$$

5.6.2 BRST-Symmetry

The free part of the BRST-expanded Lagrangian reads

$$L_0=\int d^3x\Big(-\frac{1}{4}F_{\mu\nu}\star F^{\mu\nu}+B\star_s(\partial^\mu A_\mu+\frac{\alpha}{2}B)+\partial^\mu\bar{c}\star_s\partial_\mu c\Big). \tag{5.48}$$

Note: For shortness we have defined the symmetrized $\star_s$-product,

$$A\star_s B=\frac{1}{2}(A\star B\pm B\star A), \tag{5.49}$$

where the sign is positive for usual fields and negative for Grassmann valued fields. The use of this $\star_s$-product is crucial! The equations of motion read

$$\begin{aligned}\frac{\partial \mathcal{L}_0}{\partial A^\mu} &= \Box A_\mu - \partial_\mu(\partial^\nu A_\nu) - \partial_\mu B = 0 \\ \frac{\partial \mathcal{L}_0}{\partial B} &= \partial^\mu A_\mu + \alpha B = 0.\end{aligned} \tag{5.50}$$

We postpone the treatment of the ghost sector, which in the free theory decouples from the photon sector anyway. In order to construct the Hamiltonian we have

$$\begin{aligned}&\frac{\partial \mathcal{L}_0}{\partial \dot{A}^i} =: \Pi_i = f_{i0}, \\ &\frac{\partial \mathcal{L}_0}{\partial \dot{A}^0} =: \Pi_0 = B, \qquad \frac{\partial \mathcal{L}_0}{\partial \dot{B}} =: \Pi_B = 0.\end{aligned} \tag{5.51}$$

The latter two equations are primary constraints. Since their Poisson bracket is not weakly zero,

$$\{\phi_1(\vec{x}), \phi_2(\vec{x}')\}_{PB} = \{\Pi_0(\vec{x}) - B(\vec{x}), \Pi_B(\vec{x}')\}_{PB} = -\delta^3(\vec{x} - \vec{x}'), \tag{5.52}$$

they are second class constraints. The total Hamiltonian reads (with use of $\dot{A}_i \star_s \Pi^i = (\partial_i A_0 - \Pi_i) \star_s \Pi^i$ and partial integration)

$$\begin{aligned}H_T &= \int d^3x \Big(\dot{A}_\mu \star_s \Pi^\mu + \dot{B} \star_s \Pi_B - \mathcal{L}_0 + \lambda_1' \star_s \phi_1 + \lambda_2' \star_s \phi_2 \Big) \\ &= \int d^3x \Big(\underbrace{(\lambda_1' + \dot{A}_0)}_{\lambda_1} \star_s (\Pi^0 - B) + \underbrace{(\lambda_2' + \dot{B})}_{\lambda_2} \star_s \Pi_B - A_0 \star_s \partial_i \Pi^i \\ &\qquad - B \star_s \partial_i A^i - \frac{\alpha}{2} B \star B + \frac{1}{2} \Pi^i \star \Pi^i + \frac{1}{4} F^{ij} \star F^{ij} \Big).\end{aligned} \tag{5.53}$$

Since the constraints should be preserved in time, we find conditions for λ_i,

$$\begin{aligned}\{H_T, \phi_1\}_{PB} &= \lambda_2 - \partial_i \Pi^i = 0, \\ \{H_T, \phi_2\}_{PB} &= -\lambda_1 - \partial_i A^i - \alpha B = 0.\end{aligned} \tag{5.54}$$

According to Dirac [65], for quantisation the second class constraints are imposed as strong operator equations. This is only possible after elimination of the unphysical degrees of freedom corresponding to the second class constraints. Clearly, these degrees of freedom are simply B, Π_B. So, with $\Pi_B = 0$ and $B = \Pi^0$ we get the new, quantizable Hamiltonian

$$H' = \int d^3x \Big(- A_0 \star_s \partial_i \Pi^i - \Pi^0 \star_s \partial_i A^i \\ - \frac{\alpha}{2} \Pi^0 \star \Pi^0 + \frac{1}{2} \Pi^i \star \Pi^i + \frac{1}{4} F^{ij} \star F^{ij} \Big). \tag{5.55}$$

With use of the Hamiltonian equations of motion for the fields,

$$\dot{A}_0 = \frac{\delta H'}{\delta \Pi^0} = -\partial_i A^i - \alpha \Pi^0, \qquad \dot{A}_i = \frac{\delta H'}{\delta \Pi^i} = \partial_i A^0 - \Pi_i, \tag{5.56}$$

we may express the field momenta by the fields and their time derivative. Inserting this yields exactly the Hamiltonian (5.41) we have found for the gauge fixed theory:

$$\begin{aligned} H' = \int d^3x \Big(& (-A^0 \star_s \partial^i \dot{A}^i + A^0 \star_s \partial^i \partial^i A^0) \\ & +(\frac{1}{\alpha} \partial_i A^i \star \partial_j A^j + \frac{1}{\alpha} \dot{A}^0 \star_s \partial_j A^j) \\ & +(-\frac{1}{2\alpha} \partial_i A^i \star \partial_j A^j - \frac{1}{2\alpha} \dot{A}^0 \star \dot{A}^0 - \frac{2}{2\alpha} \partial_i A^i \star_s \dot{A}^0) \\ & +(\frac{1}{2} \dot{A}^i \star \dot{A}^i + \frac{1}{2} \partial^i A^0 \star \partial^i A^0 - \dot{A}^i \star_s \partial^i A^0) \\ & +(\frac{1}{2} \partial^i A^j \star \partial^i A^j - \frac{1}{2} \partial^i A^j \star \partial^j A^i) \Big) \\ = \frac{1}{2} \int d^3x \Big(& \dot{A}_i \star \dot{A}_i - \dot{A}_0 \star \dot{A}_0 + \partial_j A_i \star \partial_j A_i - \partial_j A_0 \star \partial_j A_0 \\ & +(\frac{\alpha - 1}{\alpha})(\dot{A}_0 \star \dot{A}_0 - (\partial_i A_i) \star (\partial_j A_j)) \Big). \end{aligned} \tag{5.57}$$

Note: The elimination of the B-field does not spoil our considerations with respect to the construction of perturbation theory, since the B field has no interaction vertex.

Now for c, $\bar{c}$ the situation is very simple,

$$\Gamma_{\phi\pi} = \int d^3x \partial^\mu \bar{c} \star_s \partial_\mu c. \tag{5.58}$$

The equations of motion and the momenta are

$$\begin{aligned} \frac{\delta\Gamma_{\phi\pi}}{\delta\bar{c}} &= -\Box c = 0, & \frac{\delta\Gamma_{\phi\pi}}{\delta c} &= \Box\bar{c} = 0, \\ \frac{\delta\Gamma_{\phi\pi}}{\delta\dot{\bar{c}}} &= \Pi_{\bar{c}} = \dot{c}, & \frac{\delta\Gamma_{\phi\pi}}{\delta\dot{c}} &= \Pi_c = -\dot{\bar{c}}. \end{aligned} \tag{5.59}$$

There are no constraints. For the Hamiltonian we have

$$\begin{aligned} H_{\phi\pi} &= \int d^3x(\dot{\bar{c}}\star_s \Pi_{\bar{c}} + \dot{c}\star_s \Pi_c + \Pi_c \star_s \Pi_{\bar{c}} + \partial_i\bar{c}\star_s \partial_i c) \\ &= \int d^3x(\dot{\bar{c}}\star_s \dot{c} - \dot{c}\star_s \dot{\bar{c}} - \dot{\bar{c}}\star_s \dot{c} + \partial_i\bar{c}\star_s \partial_i c) \\ &= \int d^3x(\dot{\bar{c}}\star_s \dot{c} + \partial_i\bar{c}\star_s \partial_i c). \end{aligned} \tag{5.60}$$

We check time independence of $H_{\phi\pi}$,

$$\dot{H}_{\phi\pi} = \int d^3x\Big((\ddot{\bar{c}} - \partial_i\partial_i\bar{c})\star_s \dot{c} + \dot{\bar{c}}\star_s (\ddot{c} - \partial_i\partial_i c)\Big) = 0. \tag{5.61}$$

Using the following ansatz for $c, \bar{c}$,

$$\begin{aligned} c(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_k}}\Big(c^+(\vec{k})e^{ikx} + c^-(\vec{k})e^{-ikx}\Big), \\ \bar{c}(x) &= \frac{1}{(2\pi)^{3/2}}\int \frac{d^3k}{\sqrt{2\omega_k}}\Big(\bar{c}^+(\vec{k})e^{ikx} - \bar{c}^-(\vec{k})e^{-ikx}\Big) \end{aligned} \tag{5.62}$$

(note that $\bar{c}(x)$ is here imaginary), and the Poisson bracket for Grassmann fields, $\{c(\vec{x}), \Pi_c(\vec{x}')\} = \{\bar{c}(\vec{x}), \Pi_{\bar{c}}(\vec{x}')\} = -\delta^3(\vec{x} - \vec{x}')$, we find

$$\{\bar{c}^+(\vec{k}), c^-(\vec{k}')\} = \{\bar{c}^-(\vec{k}), c^+(\vec{k}')\} = -i\delta^3(\vec{k} - \vec{k}'). \tag{5.63}$$

For $H_{\phi\pi}$ we find with the help of the equations of motion (with $k_0 = \omega_k = \vec{k}^2$, so that $k_\mu k_\mu = 2\omega_k^2$)

$$\begin{aligned} H_{\phi\pi} &= \int d^3x\partial_\mu\bar{c}(x)\star_s \partial_\mu c(x) \\ &= \int \frac{d^3x}{(2\pi)^3}\int \frac{d^3k\, d^3k'}{2\sqrt{\omega_k\omega_{k'}}}\Big((+ik_\mu\bar{c}^+(\vec{k})e^{ikx} + ik_\mu\bar{c}^-(\vec{k})e^{-ikx}) \\ &\qquad \star_s(+ik'_\mu c^+(\vec{k}')e^{ik'x} - ik'_\mu c^-(\vec{k}')e^{-ik'x})\Big) \\ &= \int d^3k\omega_k\Big(\bar{c}^+(\vec{k})c^-(\vec{k}) + c^+(\vec{k})\bar{c}^-(\vec{k})\Big). \end{aligned} \tag{5.64}$$

We find that noncommutativity does not spoil the free theory. Quantisation is done by the replacement of the Poisson brackets by commutators,

$$\{\,,\,\}_{PB} \Rightarrow -i[\,,\,]\,, \tag{5.65}$$

which again leads to the well known commutator relations between annihilation and creation operators of fields.

5.7 Conclusion

We have constructed perturbation theory on noncommutative space from the beginning. The most important result is equation (5.27): The time ordering operator T must not act on the true time argument of the fields in the exponential but only on the time stamp of the Moyal product. With this prescription unitarity is maintained even in the case $\theta_{0i} \neq 0$. Indeed, we are sure that the naive application of the Lagrangian formalism comes along with a time ordering also of the fields within the Moyal product and is thus wrong.

Furthermore we have shown that the free Hamiltonian of ϕ^4- and gauge field theories are not changed by noncommutativity. Especially, H_0 remains time independent, which is essential for the construction of a perturbation theory. Note however that in gauge field theory $\mathcal{H}_I \neq -\mathcal{L}_I$ due to the time derivatives in the 3-interaction vertex of the gauge field theory.

In the next chapter we will now perform the first simple calculations in ϕ^4-Interaction Point Time Ordered Perturbation Theory (IPTOPT).

Chapter 6

IPTOPT

6.1 The One-Loop Two-Point Function in "Interaction Point Time-Ordered Perturbation Theory"

Since the calculation of the sum of terms (5.35) is (at least) by a factor of 120 simpler than the calculation of the sum of terms (5.34), we are happy to use (5.35). In order to distinguish from the *true* time ordering, we call it "interaction point time-ordered perturbation theory", and use the symbol T_I instead of the true causal time-ordering T. The calculation can be shortened considerably when starting directly from the Feynman rule (6.27) derived in section 6.2. But without computing at least one example one has little understanding for the starting point (6.23) of the general derivation.

Note: In order to look for new aspects we explicitly set $\theta_{0i} \neq 0$, which is exactly the case excluded in most papers on noncommutative field theory (due to the unitarity problem).

With these remarks now we are prepared to calculate the entire contribution to the one-loop two-point function in noncommutative ϕ^4 theory reads

$$
\begin{aligned}
G(x,y) &= \frac{g}{4!}\int d^4z \left\langle 0 \middle| T_I\big(\phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big) \middle| 0 \right\rangle_{(0)} \\
&= \frac{g}{4!}\int d^4z \Big(\tau(x^0-y^0)\tau(y^0-z^0)\left\langle 0 \middle| \phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z) \middle| 0 \right\rangle_{(0)} \\
&\qquad + \tau(x^0-z^0)\tau(z^0-y^0)\left\langle 0 \middle| \phi(x)\big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(y) \middle| 0 \right\rangle_{(0)}
\end{aligned}
$$

$$
\begin{aligned}
&+\tau(y^0-x^0)\tau(x^0-z^0)\,\big\langle 0\big|\phi(y)\phi(x)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big|0\big\rangle_{(0)}\\
&+\tau(y^0-z^0)\tau(z^0-x^0)\,\big\langle 0\big|\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(x)\big|0\big\rangle_{(0)}\\
&+\tau(z^0-x^0)\tau(x^0-y^0)\,\big\langle 0\big|\big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(x)\phi(y)\big|0\big\rangle_{(0)}\\
&+\tau(z^0-y^0)\tau(y^0-x^0)\,\big\langle 0\big|\big(\phi\star\phi\star\phi\star\phi\big)(z)\phi(y)\phi(x)\big|0\big\rangle_{(0)}\Big)\,,
\end{aligned}
\tag{6.1}
$$

with the $\star$-product given in (5.32). We follow the usual strategy to obtain in the end the amputated on-shell momentum-space one-loop two-point function. We insert (5.32) into (6.1) and split each field (at given position x) $\phi(x)=\phi^+(x)+\phi^-(x)$ into negative and positive frequency parts, which have the property

$$
\phi^-(x)\big|0\big\rangle=0\,,\qquad\qquad \big\langle 0\big|\phi^+(x)=0\,. \tag{6.2}
$$

It is convenient now to commute the ϕ^- to the right and the ϕ^+ to the left, using the commutation rule

$$
[\phi^-(x_1),\phi^+(x_2)]=D^+(x_1-x_2)\,, \tag{6.3}
$$

where $D^+(x_1-x_2)$ is the positive frequency propagator (as defined above)

$$
D^+(x_1-x_2)=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\,e^{-ik^+(x_1-x_2)}\,,\qquad \omega_k=\sqrt{\vec{k}^2+m^2}\,, \tag{6.4}
$$

and $k^+_\mu=(+\omega_k,-\vec{k})$ the positive energy on-shell four-momentum. A lengthy but completely standard computation yields

$$
G(x,y)=G^{con}(x,y)+G^{discon}(x,y)\,, \tag{6.5}
$$

where we have defined

$$
\begin{aligned}
&G^{discon}(x,y)=\\
&=\frac{g}{4!}\int d^4z\int\prod_{i=1}^{3}\Big(d^4s_i\frac{d^4l_i}{(2\pi)^4}\,e^{il_is_i}\Big)\Big\{\Big(\tau(x^0-y^0)\tau(y^0-z^0)D^+(x-y)\\
&\quad+\tau(x^0-z^0)\tau(z^0-y^0)D^+(x-y)+\tau(z^0-x^0)\tau(x^0-y^0)D(x-y)\Big)\\
&\quad+(x\leftrightarrow y)\Big\}\Big(D^+(-\tfrac{1}{2}\tilde{l}_2-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2-s_3)
\end{aligned}
$$

$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(-\tfrac{1}{2}\tilde{l}_2-s_2-s_3)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(-\tfrac{1}{2}\tilde{l}_3-s_3)\Big) \ , \qquad (6.6)$$

$$G^{con}(x,y) =$$
$$= \frac{g}{4!}\int d^4z \int \prod_{i=1}^{3}\left(d^4s_i \frac{d^4l_i}{(2\pi)^4}\, e^{il_is_i}\right)\Bigg\{\Bigg(\tau(x^0-y^0)\tau(y^0-z^0)$$
$$\times \Bigg\{\Big(D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2-s_3)D^+(x-z-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(y-z-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(x-z-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(y-z-s_1-s_2-s_3)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(x-z-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(y-z-s_1-s_2-s_3)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_2-s_2-s_3)D^+(x-z+\tfrac{1}{2}\tilde{l}_1)D^+(y-z-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_2-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(x-z+\tfrac{1}{2}\tilde{l}_1)D^+(y-z-s_1-s_2-s_3)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_3-s_3)D^+(x-z+\tfrac{1}{2}\tilde{l}_1)D^+(y-z-s_1-\tfrac{1}{2}\tilde{l}_2)\Big)$$
$$+ (x \leftrightarrow y)\Bigg\}$$
$$+ \tau(x^0-z^0)\tau(z^0-y^0)$$
$$\times \Big\{D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2-s_3)D^+(x-z-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(z+s_1-\tfrac{1}{2}\tilde{l}_2-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_2-s_2-s_3)D^+(x-z-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(z-\tfrac{1}{2}\tilde{l}_1-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(x-z-s_1-s_2-s_3)D^+(z+s_1-\tfrac{1}{2}\tilde{l}_2-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_2-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(x-z-s_1-s_2-s_3)D^+(z-\tfrac{1}{2}\tilde{l}_1-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2-s_3)D^+(x-z-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_3-s_3)D^+(x-z-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(z-\tfrac{1}{2}\tilde{l}_1-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(x-z-s_1-s_2-s_3)D^+(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(x-z-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(z+s_1+s_2+s_3-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_1-s_1+\tfrac{1}{2}\tilde{l}_2)D^+(x-z-s_1-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(z+s_1+s_2+s_3-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_2-s_2-s_3)D^+(x-z+\tfrac{1}{2}\tilde{l}_1)D^+(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_3-s_3)D^+(x-z+\tfrac{1}{2}\tilde{l}_1)D^+(z+s_1-\tfrac{1}{2}\tilde{l}_2-y)$$
$$+ D^+(-\tfrac{1}{2}\tilde{l}_2-s_2+\tfrac{1}{2}\tilde{l}_3)D^+(x-z+\tfrac{1}{2}\tilde{l}_1)D^+(z+s_1+s_2+s_3-y)\Big\}$$
$$+ \tau(z^0-x^0)\tau(x^0-y^0)$$

$$
\begin{aligned}
&\times \Big\{ \Big(D^+(-\tfrac{1}{2}\tilde{l}_1 - s_1 - s_2 - s_3)\, D^+(z+s_1-\tfrac{1}{2}\tilde{l}_2 - x) D^+(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3 - y) \\
&\quad + D^+(-\tfrac{1}{2}\tilde{l}_2 - s_2 - s_3)\, D^+(z-\tfrac{1}{2}\tilde{l}_1 - x) D^+(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3 - y) \\
&\quad + D^+(-s_3 - \tilde{l}_3)\, D^+(z-\tfrac{1}{2}\tilde{l}_1 - x) D^+(z+s_1-\tfrac{1}{2}\tilde{l}_2 - y) \\
&\quad + D^+(-\tfrac{1}{2}\tilde{l}_1 - s_1 - s_2 + \tfrac{1}{2}\tilde{l}_3) D^+(z+s_1-\tfrac{1}{2}\tilde{l}_2 - x) D^+(z+s_1+s_2+s_3-y) \\
&\quad + D^+(-\tfrac{1}{2}\tilde{l}_2 - s_2 + \tfrac{1}{2}\tilde{l}_3) D^+(z-\tfrac{1}{2}\tilde{l}_1 - x) D^+(z+s_1+s_2+s_3-y) \\
&\quad + D^+(-\tfrac{1}{2}\tilde{l}_1 - s_1 + \tfrac{1}{2}\tilde{l}_2) D^+(z+s_1+s_2-\tfrac{1}{2}\tilde{l}_3 - x) D^+(z+s_1+s_2+s_3-y) \Big) \\
&+ \big(x \leftrightarrow y\big) \Big\} \Big) + \big(x \leftrightarrow y\big) \Big\} . \qquad (6.7)
\end{aligned}
$$

We have to take the connected part $G^{con}(x,y)$ only. Inserting (6.4) we can perform the s_i-integrations, which result in δ-distributions in l_i, so that the l_i integration can be performed as well. The result has a remarkably compact form:

$$
\begin{aligned}
G^{con}(x,y) = \frac{g}{12} \int d^4z \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \int \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} \cos(\tfrac{1}{2}k_1^+ \tilde{k}_2^+) \\
\times \Big(\tau(x^0-y^0)\tau(y^0-z^0) e^{-ik_1^+(x-z)} e^{-ik_2^+(y-z)} \mathcal{I}^{++}(k_1^+,k_2^+) \\
+ \tau(y^0-x^0)\tau(x^0-z^0) e^{-ik_1^+(x-z)} e^{-ik_2^+(y-z)} \mathcal{I}^{++}(k_1^+,k_2^+) \\
+ \tau(x^0-z^0)\tau(z^0-y^0) e^{-ik_1^+(x-z)} e^{-ik_2^+(z-y)} \mathcal{I}^{+-}(k_1^+,k_2^+) \\
+ \tau(y^0-z^0)\tau(z^0-x^0) e^{-ik_1^+(z-x)} e^{-ik_2^+(y-z)} \mathcal{I}^{-+}(k_1^+,k_2^+) \\
+ \tau(z^0-x^0)\tau(x^0-y^0) e^{-ik_1^+(z-x)} e^{-ik_2^+(z-y)} \mathcal{I}^{--}(k_1^+,k_2^+) \\
+ \tau(z^0-y^0)\tau(y^0-x^0) e^{-ik_1^+(z-x)} e^{-ik_2^+(z-y)} \mathcal{I}^{--}(k_1^+,k_2^+) \Big) ,
\end{aligned}
\qquad (6.8)
$$

where $(\tilde{k}^+)^\nu \equiv (k^+)_\mu \theta^{\mu\nu}$ and

$$
\begin{aligned}
\mathcal{I}^{\kappa\lambda}(k_1^+,k_2^+) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \big(3 + e^{i\kappa k_1^+ \tilde{k}^+ + i\lambda k_2^+ \tilde{k}^+} + e^{i\kappa k_1^+ \tilde{k}^+} + e^{i\lambda k_2^+ \tilde{k}^+}\big) , \\
(\kappa,\lambda = \pm 1) . \qquad (6.9)
\end{aligned}
$$

Next we pass to the Fourier transformed Green's function

$$
G^{con}(p,q) = \int d^4x\, d^4y\, e^{ipx+iqy}\, G^{con}(x,y) . \qquad (6.10)
$$

We insert the identity (use the residue theorem)

$$\tau(x^0 - y^0) = \lim_{\delta\to 0} \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{e^{-it(x^0-y^0)}}{t + i\delta} \tag{6.11}$$

and perform the integrations over x, y, z. The result is a host of δ-distributions, which allow us to integrate over $\vec{k}_1, \vec{k}_2, t_1, t_2$:

$$\begin{aligned}
&G^{con}(p,q) \\
&= \lim_{\delta_1,\delta_2\to 0} \frac{g}{12}\left(\frac{i}{2\pi}\right)^2 \int d^4x\, d^4y\, d^4z \int_{-\infty}^{\infty} \frac{dt_1}{t_1 + i\delta_1} \int_{-\infty}^{\infty} \frac{dt_2}{t_2 + i\delta_2} \\
&\qquad \times \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \int \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} \cos(\tfrac{1}{2}k_1^+ \tilde{k}_2^+) \\
&\times \Big(e^{i\{x^0(p_0-t_1-\omega_{k_1})+y^0(q_0+t_1-t_2-\omega_{k_2})+z^0(t_2+\omega_{k_1}+\omega_{k_2})+\vec{x}(\vec{k}_1-\vec{p})+\vec{y}(\vec{k}_2-\vec{q})-\vec{z}(\vec{k}_1+\vec{k}_2)\}} \\
&\qquad\qquad \times \mathcal{I}^{++}(k_1^+, k_2^+) \\
&\quad + e^{i\{x^0(p_0+t_1-t_2-\omega_{k_1})+y^0(q_0-t_1-\omega_{k_2})+z^0(t_2+\omega_{k_1}+\omega_{k_2})+\vec{x}(\vec{k}_1-\vec{p})+\vec{y}(\vec{k}_2-\vec{q})-\vec{z}(\vec{k}_1+\vec{k}_2)\}} \\
&\qquad\qquad \times \mathcal{I}^{++}(k_1^+, k_2^+) \\
&\quad + e^{i\{x^0(p_0-t_1-\omega_{k_1})+y^0(q_0+t_2+\omega_{k_2})+z^0(t_1-t_2+\omega_{k_1}-\omega_{k_2})+\vec{x}(\vec{k}_1-\vec{p})-\vec{y}(\vec{k}_2+\vec{q})+\vec{z}(\vec{k}_2-\vec{k}_1)\}} \\
&\qquad\qquad \times \mathcal{I}^{+-}(k_1^+, k_2^+) \\
&\quad + e^{i\{x^0(p_0+t_2+\omega_{k_1})+y^0(q_0-t_1-\omega_{k_2})+z^0(t_1-t_2-\omega_{k_1}+\omega_{k_2})-\vec{x}(\vec{k}_1+\vec{p})+\vec{y}(\vec{k}_2-\vec{q})+\vec{z}(\vec{k}_1-\vec{k}_2)\}} \\
&\qquad\qquad \times \mathcal{I}^{-+}(k_1^+, k_2^+) \\
&\quad + e^{i\{x^0(p_0+t_1-t_2+\omega_{k_1})+y^0(q_0+t_2+\omega_{k_2})-z^0(t_1+\omega_{k_1}+\omega_{k_2})-\vec{x}(\vec{k}_1+\vec{p})-\vec{y}(\vec{k}_2+\vec{q})+\vec{z}(\vec{k}_1+\vec{k}_2)\}} \\
&\qquad\qquad \times \mathcal{I}^{--}(k_1^+, k_2^+) \\
&\quad + e^{i\{x^0(p_0+t_2+\omega_{k_1})+y^0(q_0+t_1-t_2+\omega_{k_2})-z^0(t_1+\omega_{k_1}+\omega_{k_2})-\vec{x}(\vec{k}_1+\vec{p})-\vec{y}(\vec{k}_2+\vec{q})+\vec{z}(\vec{k}_1+\vec{k}_2)\}} \\
&\qquad\qquad \times \mathcal{I}^{--}(k_1^+, k_2^+)\Big) \\
&= \lim_{\delta_1,\delta_2\to 0} \frac{g}{12}(2\pi)^4 \delta(p+q) \\
&\times \Big(\frac{1}{p_0-\omega_p+i\delta_1} \frac{1}{\omega_p+\omega_q-i\delta_2} \frac{\cos(\frac{1}{2}p^+\tilde{q}^+)}{4\omega_p\omega_q} \mathcal{I}^{++}(p^+, q^+) \\
&\quad + \frac{1}{q_0-\omega_q+i\delta_1} \frac{1}{\omega_p+\omega_q-i\delta_2} \frac{\cos(\frac{1}{2}p^+\tilde{q}^+)}{4\omega_p\omega_q} \mathcal{I}^{++}(p^+, q^+) \\
&\quad + \frac{1}{p_0-\omega_p+i\delta_1} \frac{1}{q_0+\omega_q-i\delta_2} \frac{\cos(\frac{1}{2}p^+(-\tilde{q})^+)}{4\omega_p\omega_q} \mathcal{I}^{+-}(p^+, (-q)^+)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{q_0-\omega_q+i\delta_1}\,\frac{1}{p_0+\omega_p-i\delta_2}\,\frac{\cos(\frac{1}{2}(-p)^+\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}^{-+}((-p)^+,q^+)\\
&+\frac{1}{\omega_p+\omega_q-i\delta_1}\,\frac{1}{-q_0-\omega_q+i\delta_2}\,\frac{\cos(\frac{1}{2}(-p)^+(-\tilde{q})^+)}{4\omega_p\omega_q}\mathcal{I}^{--}((-p)^+,(-q)^+)\\
&+\frac{1}{\omega_p+\omega_q-i\delta_1}\,\frac{1}{-p_0-\omega_p+i\delta_2}\,\frac{\cos(\frac{1}{2}(-p)^+(-\tilde{q})^+)}{4\omega_p\omega_q}\mathcal{I}^{--}((-p)^+,(-q)^+)\Big)\,.
\end{aligned}
\tag{6.12}
$$

Note the appearance of $\delta(p+q)$ implementing conservation of the four momentum (translation invariance). We have used $\omega_{\pm k}=\omega_k$.

We amputate the external legs by multiplying (6.12) by the inverse propagators $-i(p_0^2-\omega_p^2)$ and $-i(q_0^2-\omega_q^2)$. Using $(\pm k)^+=\pm k^\pm$, in particular the identity

$$
\begin{aligned}
&\mathcal{I}^{\pm\pm}((\pm p)^+,(\pm q)^+)=\\
&=\int\frac{d^3k}{(2\pi)^3 2\omega_k}\left(3+e^{ip^\pm\tilde{k}^+ + iq^\pm\tilde{k}^+}+e^{ip^\pm\tilde{k}^+}+e^{iq^\pm\tilde{k}^+}\right)\equiv\mathcal{I}(p^\pm,q^\pm)\,,
\end{aligned}
\tag{6.13}
$$

defining now the 1PI-vertex functional Γ we obtain

$$
\begin{aligned}
&(2\pi)^4\delta(p+q)\Gamma(p,q)=\\
&=-(p_0^2-\omega_p^2)(q_0^2-\omega_q^2)G(p,q)\\
&=-\lim_{\delta_1,\delta_2\to 0}\frac{g}{12}(2\pi)^4\delta(p+q)\,(p_0^2-\omega_p^2)(q_0^2-\omega_q^2)\\
&\times\Big(\frac{1}{p_0-\omega_p+i\delta_1}\,\frac{1}{\omega_p+\omega_q-i\delta_2}\,\frac{\cos(\frac{1}{2}p^+\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^+)\\
&+\frac{1}{q_0-\omega_q+i\delta_1}\,\frac{1}{\omega_p+\omega_q-i\delta_2}\,\frac{\cos(\frac{1}{2}p^+\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^+)\\
&+\frac{1}{p_0-\omega_p+i\delta_1}\,\frac{1}{q_0+\omega_q-i\delta_2}\,\frac{\cos(\frac{1}{2}p^+\tilde{q}^-)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^-)\\
&+\frac{1}{q_0-\omega_q+i\delta_1}\,\frac{1}{p_0+\omega_p-i\delta_2}\,\frac{\cos(\frac{1}{2}p^-\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^+)\\
&+\frac{1}{\omega_p+\omega_q-i\delta_1}\,\frac{1}{-q_0-\omega_q+i\delta_2}\,\frac{\cos(\frac{1}{2}p^-\tilde{q}^-)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^-)\\
&+\frac{1}{\omega_p+\omega_q-i\delta_1}\,\frac{1}{-p_0-\omega_p+i\delta_2}\,\frac{\cos(\frac{1}{2}p^-\tilde{q}^-)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^-)\Big)\,.
\end{aligned}
\tag{6.14}
$$

Taking on-shell external momenta $p_0 = \omega_p$ and $q_0 = -\omega_q$ there survives a single term (the third one):

$$\Gamma(p^+, q^-) = \lim_{p_0 \to \omega_p,\, q_0 \to -\omega_q} \Gamma(p,q) = \frac{g}{12} \cos(\tfrac{1}{2} p^+ \tilde{q}^-) \mathcal{I}(p^+, q^-)$$
$$= \frac{g}{12} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(4 + 2\cos(k^+ \tilde{p}^+) \Big) . \tag{6.15}$$

In the last line we have used momentum conservation $p^+ = -q^-$ and the skew-symmetry of θ. The remaining integral over $\vec{k}$ consists of a planar θ-independent part and a non-planar θ-dependent part (the cosine). The planar part coincides (up to a factor $\frac{2}{3}$) with the commutative result, it is divergent and to be renormalized as usual by multiplicative renormalization (or better completely removed by normal ordering).

To compute the non-planar part, first note that

$$\cos(k^+ \tilde{p}^+) = \cos\big(\omega_k \tilde{p}_0 - \vec{k}\vec{\tilde{p}}\big) = \cos\big(\omega_k \tilde{p}_0\big) \cos(\vec{k}\vec{\tilde{p}}) + \sin\big(\omega_k \tilde{p}_0\big) \sin(\vec{k}\vec{\tilde{p}}) , \tag{6.16}$$

where $\tilde{p}_0 := (\tilde{p}^+)_0$ and $\vec{\tilde{p}} = \overrightarrow{\tilde{p}^+}$. The uneven sine-term will drop under the integral. Using the residue theorem we have

$$\frac{e^{i\omega_k \tilde{p}_0}}{2\omega_k} = \begin{cases} \lim\limits_{\epsilon \to 0} \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k + i\epsilon)(k_0 - \omega_k - i\epsilon)} & \text{for } \tilde{p}_0 > 0 , \\ \lim\limits_{\epsilon \to 0} \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} dk_0 \, \frac{-e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k - i\epsilon)(k_0 - \omega_k + i\epsilon)} & \text{for } \tilde{p}_0 < 0 , \end{cases} \tag{6.17}$$

$$\frac{e^{-i\omega_k \tilde{p}_0}}{2\omega_k} = \begin{cases} \lim\limits_{\epsilon \to 0} \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} dk_0 \, \frac{-e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k - i\epsilon)(k_0 - \omega_k + i\epsilon)} & \text{for } \tilde{p}_0 > 0 , \\ \lim\limits_{\epsilon \to 0} \dfrac{1}{2\pi i} \displaystyle\int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 \tilde{p}_0}}{(k_0 + \omega_k + i\epsilon)(k_0 - \omega_k - i\epsilon)} & \text{for } \tilde{p}_0 < 0 . \end{cases} \tag{6.18}$$

Inserting (6.16), (6.17) and (6.18) into (6.15) we obtain for the non-planar graph

$$\Gamma_{non-planar}(p^+, q^-) \equiv \frac{g}{6} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \cos(k^+ \tilde{p}^+)$$
$$= \lim_{\epsilon \to 0} \frac{g}{6} \int \frac{d^4k}{(2\pi)^4} \Re\Big(\frac{i}{k_0^2 - (\vec{k}^2 + m^2) + i\epsilon} \Big) e^{-ik\tilde{p}^+} , \quad (6.19)$$

independent of the sign of $\tilde{p}_0$. The result (6.19) can obviously be obtained by Feynman rules, with the prescription that in non-planar tadpoles the propagator to use is *the real part of the Feynman propagator.* That real part is arithmetic mean of causal and acausal propagators. The observed acausality is no surprise, because according to (5.35) the interaction time-ordering T_I explicitly violates causality. As we shall see in section 6.2, the just given Feynman rule is true for tadpole lines only.

Apart from taking the real part, the evaluation of (6.19) coincides with the computation in the "naïve" Feynman graph approach. Let us nevertheless repeat the steps. We employ Zimmermann's ϵ-trick

$$\frac{1}{k^2 - m^2 + i\epsilon} \mapsto \frac{1}{k_0^2 + \omega_k^2(i\epsilon - 1)} = \frac{\epsilon' - i}{(\epsilon' - i)k_0^2 + \omega_k^2(\epsilon - \epsilon' + i + i\epsilon\epsilon')} , \quad (6.20)$$

the denominator of which has for $\epsilon' < \epsilon$ a positive real part, which allows us to introduce a Schwinger parameter (remember $p_0 = \omega_p$):

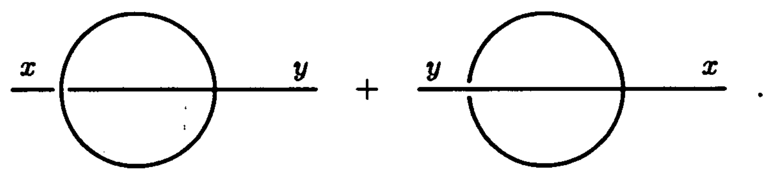
$$\Gamma_{non-planar}(p^+, q^-) =$$
$$= \Re\Big(\lim_{\epsilon \to 0, \epsilon' < \epsilon} \frac{ig}{6}$$
$$\times \int \frac{d^4k}{(2\pi)^4} \int_0^\infty d\alpha\, (\epsilon' - i) e^{-\alpha\{(\epsilon' - i)k_0^2 + (\vec{k}^2 + m^2)(\epsilon - \epsilon' + i + i\epsilon\epsilon')\} - ik_0\tilde{p}_0 + i\vec{k}\vec{\tilde{p}}} \Big)$$
$$= \Re\Big(\lim_{\epsilon \to 0, \epsilon' < \epsilon} \frac{ig}{6(4\pi)^2} \frac{(\epsilon' - i)^{\frac{1}{2}}}{(\epsilon - \epsilon' + i + i\epsilon\epsilon')^{\frac{3}{2}}}$$
$$\times \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-\frac{\tilde{p}_0^2}{4\alpha(\epsilon' - i)} - \frac{\vec{\tilde{p}}^2}{4\alpha(\epsilon - \epsilon' + i + i\epsilon\epsilon')} - \alpha m^2(\epsilon - \epsilon' + i + i\epsilon\epsilon')} \Big)$$
$$= \Re\Big(\lim_{\epsilon \to 0} \frac{2ig}{3(4\pi)^2} \frac{1}{(i\epsilon - 1)^{\frac{3}{2}}} \sqrt{\frac{m^2(i\epsilon - 1)}{\tilde{p}_0^2 + \frac{\vec{\tilde{p}}^2}{(i\epsilon - 1)}}} K_1\Big(\sqrt{m^2(\vec{\tilde{p}}^2 + (i\epsilon - 1)\tilde{p}_0^2)} \Big) \Big)$$
$$= -\Re\Big(\frac{2g}{3(4\pi)^2} \sqrt{-\frac{m^2}{\tilde{p}^2}} K_1\big(\sqrt{-\tilde{p}^2 m^2}\big) \Big) . \quad (6.21)$$

We have used $\int_0^\infty \frac{d\alpha}{\alpha^2}\exp(-u\alpha - v/(4\alpha)) = 4\sqrt{(u/v)}K_1(\sqrt{uv})$ for $\Re u > 0$ and $\Re v > 0$.

In the particular case where the external momentum p is put on-shell, we have

$$-\tilde{p}^2 = \vec{\tilde{p}}^2 - \tilde{p}_0^2 = (\theta_{i0}\sqrt{\vec{p}^2 + m^2} + \theta_{ij}p^j)^2 - (\theta_{0j}p^j)^2 \geq 0\,, \tag{6.22}$$

because $\tilde{p}^\mu$ has to be space-like or null as a vector which is orthogonal to the time-like vector p^μ. Thus, the projection onto the real part in (6.21) is superfluous, and (6.21) agrees exactly with the naïve Feynman rule computation of the sum of graphs



However, if these graphs appear as subgraphs in a bigger graph, one could expect (but it is not clear) that the momentum p may be the off-shell momentum through a propagator, and the projection to the real part makes a difference. This will be discussed in the next chapter.

6.2 The General Case

The graph we have computed (for off-shell external momenta!) is very often made responsible for the so-called UV/IR mixing. In fact the situation is more complex, as it is very well described in [23]. The ultimate goal must be to derive the power-counting theorem for interaction point time-ordered perturbation theory (for noncommutative space and time). In a first step one has to derive graphical rules to assign an integral to a given graph.

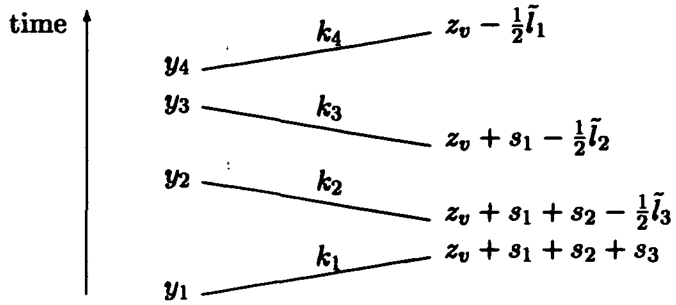
Let us therefore consider the momentum integral for a general Feynman graph for a noncommutative ϕ^4 theory. A given connected contribution to the E-point function at order V in the coupling constant has after performing the Wick contractions, insertion of the D^+ according to (6.4), integration over s_i and l_i appearing in (5.32) and insertion of step functions (6.11) the form

$$
\begin{aligned}
&G(x_1,\ldots,x_E) = \\
&\lim_{\epsilon\to 0}\int\prod_{v=1}^{V}\frac{g\,d^4z_v}{4!}\int\prod_{e=1}^{E}\frac{d^3p_e}{(2\pi)^3 2\omega_{p_e}}\int\prod_{i=1}^{I}\frac{d^3k_i}{(2\pi)^3 2\omega_{k_i}}\int\prod_{s=1}^{E+V-1}\frac{i\,dt_s}{(2\pi)(t_s+i\epsilon)} \\
&\times\exp\Big(-i\sum_{v=1}^{V}\sum_{s=1}^{E+V-1}T_{vs}z_v^0t_s - i\sum_{e=1}^{E}\sum_{s=1}^{E+V-1}T_{es}x_e^0t_s\Big) \\
&\times\exp\Big(-i\sum_{v=1}^{V}z_v\Big(\sum_{i=1}^{I}J_{vi}k_i^+ + \sum_{e=1}^{E}J_{ve}p_e^+\Big)\Big)\exp\Big(-i\sum_{e=1}^{E}\sigma_e p_e^+ x_e\Big) \\
&\times\exp\Big(i\theta^{\mu\nu}\Big(\sum_{i,j=1}^{I}I_{ij}k_{i,\mu}^+k_{j,\nu}^+ + \sum_{i=1}^{I}\sum_{e=1}^{E}I_{ie}k_{i,\mu}^+p_{e,\nu}^+ + \sum_{e,f=1}^{E}I_{ef}p_{e,\mu}^+p_{f,\nu}^+\Big)\Big)\,.
\end{aligned}
\tag{6.23}
$$

There are $E{+}V{-}1$ step functions according to the time differences of the E external points x_e and the V interaction points z_v. For each s there are two non-vanishing T_{*s}, where these two indices $*$ are either two indices e, one index e and one index v, or two indices v. The T_{*s} for which the vertex $*$ (z_v or x_e) is later equals $+1$, the other one -1. This gives the second line in (6.23). An external point x_e is linked via the external line with momentum p_e to exactly one vertex z_v, i.e. for given e there is a single non-vanishing J_{ve}. For our ϕ^4 theory there are $I = 2V - \frac{1}{2}E$ internal lines (E is even) with momentum k_i which link a vertex z_v to another vertex $z_{v'}$. Thus, if $v \neq v'$ (no tadpoles) for given i there are two non-vanishing J_{vi}, whereas for $v = v'$ we have $J_{vi}k_i^+ \equiv 0$. We orient the internal and external lines forward in time. Then, the incidence matrices J_{vi}, J_{ve} equal -1 if the line leaves v and $+1$ if the line arrives at v. Similarly, $\sigma_e = -1$ if the line e leaves x_e and $\sigma_e = +1$ if the line e arrives at x_e. The matrices I_{ij}, I_{ie}, I_{ef} are the intersection matrices [16, 23], which instead of the Euclidean rosette construction are in IPTO obtained as follows: According to the definition (5.32) of the $\star$-product, write at each vertex v the four fields in (5.32) as a time-sequence where $z_v{-}\frac{1}{2}\tilde{l}_1$ is the latest point and $z_v{+}s_1{+}s_2{+}s_3$ the earliest point[1], irrespective of the *actual time-order* of these four points. Connect

[1]By the way, this defines the time-orientation of tadpole lines.

these points with vertices y_1, y_2, y_3, v_4 according to the following picture:



The phase factor produced by the s_n and l_n variables is then given by

$$\begin{aligned}
&\int \prod_{n=1}^{3} \Big(d^4 s_n \frac{d^4 l_n}{(2\pi)^4} \exp(i s_n l_n) \Big) \exp \Big(- i k_1^+ (s_1 + s_2 + s_3) J_{v1} \\
&\qquad - i k_2^+ (s_1 + s_2 - \tfrac{1}{2}\tilde{l}_3) J_{v2} - i k_3^+ (s_1 - \tfrac{1}{2}\tilde{l}_2) J_{v3} - i k_4^+ (-\tfrac{1}{2}\tilde{l}_1) J_{v4} \Big) \\
&= \exp \Big(\frac{i}{2} \theta^{\mu\nu} \sum_{j=2}^{4} \sum_{i=1}^{j-1} k_{i,\mu}^+ J_{vi} k_{j,\nu}^+ J_{vj} \Big) \equiv \exp \Big(\frac{i}{2} \theta^{\mu\nu} \sum_{i,j=1}^{4} \tau_{ij}^v k_{i,\mu}^+ J_{vi} k_{j,\nu}^+ J_{vj} \Big) .
\end{aligned} \tag{6.24}$$

We have to define $\tau_{ij}^v = +1$ if the line i is connected to an "earlier" field ϕ in the vertex v than the line j, otherwise $\tau_{ij}^v = 0$. Summing over all vertices and distinguishing external and internal lines, we are led to the following identification in (6.23):

$$I_{ij} = \frac{1}{2} \sum_{v=1}^{V} \tau_{ij}^v J_{vi} J_{vj} , \quad I_{ie} = \frac{1}{2} \sum_{v=1}^{V} \big(\tau_{ie}^v - \tau_{ei}^v \big) J_{vi} J_{ve} , \quad I_{ef} = \frac{1}{2} \sum_{v=1}^{V} \tau_{ef}^v J_{ve} J_{vf} . \tag{6.25}$$

Once more we notice the enormous computational advantage of using the $\star$-product in the form (5.30).

We perform the Fourier transformation $\int \prod_{e=1}^{E} \big(d^4 x_e \exp(i q_e x_e) \big)$ of (6.23) to external momentum variables q as well as the z_v integrations:

$$G(q_1,\ldots,q_E) =$$

$$\lim_{\epsilon\to 0}\frac{g^V}{(4!)^V}\prod_{e=1}^{E}\frac{1}{2\omega_{q_e}}\int\prod_{i=1}^{I}\frac{d^3k_i}{(2\pi)^3 2\omega_{k_i}}\int\prod_{s=1}^{E+V-1}\frac{i\,dt_s}{(2\pi)(t_s+i\epsilon)}$$
$$\times\prod_{v=1}^{V}(2\pi)^3\delta^3\Big(\sum_{i=1}^{I}J_{vi}\vec{k}_i+\sum_{e=1}^{E}J_{ve}\sigma_e\vec{q}_e\Big)\prod_{e=1}^{E}(2\pi)\delta\Big(q_e^0-\sigma_e\omega_{q_e}-\sum_{s=1}^{E+V-1}T_{es}t_s\Big)$$
$$\times\prod_{v=1}^{V}(2\pi)\delta\Big(\sum_{i=1}^{I}J_{vi}\omega_{k_i}+\sum_{e=1}^{E}J_{ve}\omega_{q_e}+\sum_{s=1}^{E+V-1}T_{vs}t_s\Big)$$
$$\times\exp\Big(i\theta^{\mu\nu}\Big(\sum_{i,j=1}^{I}I_{ij}k^+_{i,\mu}k^+_{j,\nu}+\sum_{i=1}^{I}\sum_{e=1}^{E}I_{ie}\sigma_e k^+_{i,\mu}q^{\sigma_e}_{e,\nu}+\sum_{e,f=1}^{E}I_{ef}\sigma_e\sigma_f q^{\sigma_e}_{e,\mu}q^{\sigma_f}_{f,\nu}\Big)\Big)\,. \tag{6.26}$$

The vectors $\vec{q}_e$ are always outgoing from internal vertices. There are now $E{+}V$ time-component δ-functions involving the $E{+}V{-}1$ integration variables t_s, after integration over which there is one remaining δ-function for the energy conservation $\delta(\sum_{e=1}^{E}q_e^0)$. We multiply (6.26) by the inverse propagators $\prod_{e=1}^{E}(-i)((q_e^0)^2-\omega_{q_e}^2)$, remove $(2\pi)^4\delta^4(\sum_{e=1}^{E}q_e)$ by convention and put $q_e^0=\sigma_e\omega_{q_e}$. There is a non-vanishing contribution only if the external vertices x_e are either before or after the internal vertices z_i. Defining a time-order of vertices $v'<v$ if $z^0_{v'}<z^0_v$ we finally get (an illuminating example will be given below)

$$\Gamma(q_1^{\sigma_1},\ldots,q_E^{\sigma_E}) =$$

$$\lim_{\epsilon\to 0}\frac{g^V}{(4!)^V}\int\prod_{i=1}^{I}\frac{d^3k_i}{(2\pi)^3 2\omega_{k_i}}\prod_{v=1}^{V-1}\frac{i(2\pi)^3\delta^3\Big(\sum_{i=1}^{I}J_{vi}\vec{k}_i+\sum_{e=1}^{E}J_{ve}\sigma_e\vec{q}_e\Big)}{\sum_{v'\leq v}\Big(\sum_{i=1}^{I}J_{v'i}\omega_{k_i}+\sum_{e=1}^{E}J_{v'e}\omega_{q_e}\Big)+i\epsilon}$$
$$\times\exp\Big(i\theta^{\mu\nu}\Big(\sum_{i,j=1}^{I}I_{ij}k^+_{i,\mu}k^+_{j,\nu}+\sum_{i=1}^{I}\sum_{e=1}^{E}I_{ie}\sigma_e k^+_{i,\mu}q^{\sigma_e}_{e,\nu}+\sum_{e,f=1}^{E}I_{ef}\sigma_e\sigma_f q^{\sigma_e}_{e,\mu}q^{\sigma_f}_{f,\nu}\Big)\Big)\,. \tag{6.27}$$

The vertex which is missing in the product over v is the latest one. There remain $I{-}V{+}1{=}L$ momentum integrations to perform, where L is the number of loops. The integral (6.27) corresponds to a particular graph with E

external and V internal vertices which all have different dates. The internal vertices are composed of four different points according to the four fields building the vertex, with the time-interval within a vertex smaller than the time-distance to the neighboured vertices. Any external vertex is a single point which is either later or earlier than all points in internal vertices. A graph is the connection of each two of these $4V+E$ points by a line which is oriented forward in time, such that at each point we find exactly one end of a line. We assign to this graph the integral (6.27) according to the incidence matrices, which also enter in (6.25). Finally, one has to sum over all different graphs. Note that a given graph does not have any symmetry because the four points in the vertices have clearly distinguished dates. The Feynman rule (6.27) is easily generalised to other than ϕ^4 theories. Eq. (6.27) is the analytic expression of the Feynman rules listed in [62], apart from a disagreement in the symmetry factor.

We now see that the graph we have computed was very special. Because of $V=1$ the denominator in (6.27) was absent so that the integration over the propagator momentum k_1 was identical to the naïve Feynman graph computation. This remains true for all tadpole lines i, because for them $J_{vi}k_i^+ = 0$ for all v. For internal lines connecting points in different vertices we need new techniques to perform the integrations.

6.3 Example

Since the step from (6.26) to (6.27) is somewhat complicated, we want to give a simple example involving two external lines e, f and two vertices a, b, ordered in the following way:

time ↑

$\bullet\ x_f$

t_3

$\bullet\ b$

t_2

$\bullet\ a$

t_1

$\bullet\ x_e$

Note: Nothing is said about the inner lines between the vertices, only the time ordering of the points is necessary. For this case (6.26) reads ($V = 2,\ E = 2,$)

$$G(q_e, q_f) = \lim_{\epsilon\to 0} \frac{g^V}{(4!)^V} \int \prod_{i=1}^{I} \frac{d^3k_i}{(2\pi)^3 2\omega_{k_i}} \times G'(k,q)$$
$$\times \exp\Big(i\theta^{\mu\nu}\Big(\sum_{i,j=1}^{I} I_{ij} k^+_{i,\mu} k^+_{j,\nu} + \sum_{i=1}^{I}\sum_{e=1}^{E} I_{ie}\sigma_e k^+_{i,\mu} q^{\sigma_e}_{e,\nu} + \sum_{e,f=1}^{E} I_{ef}\sigma_e\sigma_f q^{\sigma_e}_{e,\mu} q^{\sigma_f}_{f,\nu}\Big)\Big) . \tag{6.28}$$

Here we have defined

$$G'(k,q) = \frac{1}{2\omega_{q_e} 2\omega_{q_f}} \int \frac{i\,dt_1\, i\,dt_2\, i\,dt_3}{(2\pi)^3 (t_1+i\epsilon)(t_2+i\epsilon)(t_3+i\epsilon)}$$
$$\times \prod_{v=a,b} (2\pi)^3 \delta^3\Big(\sum_{i=1}^{I} J_{vi}\vec{k}_i + \sum_{d=e,f} J_{vd}\sigma_d \vec{q}_d\Big) 2\pi\delta\Big(q_e^0 - \sigma_e\omega_{q_e} + t_1\Big)$$
$$\times 2\pi\delta\Big(q_f^0 - \sigma_f\omega_{q_f} - t_3\Big) 2\pi\delta\Big(\sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d} + t_1 - t_2\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{bi}\omega_{k_i} + \sum_{d=e,f} J_{bd}\omega_{q_d} + t_2 - t_3\Big) . \tag{6.29}$$

Now we perform the integration $\int dt_1$ and multiply with the inverse propagator $-i(q_e^0 - \sigma_e\omega_{q_e})(q_e^0 + \sigma_e\omega_{q_e})$,

$$G'(k,q) =$$
$$\frac{-i(q_e^0 - \sigma_e\omega_{q_e})(q_e^0 + \sigma_e\omega_{q_e}) i2\pi}{2\omega_{q_e} 2\omega_{q_f} 2\pi(-q_e^0 + \sigma_e\omega_{q_e})} \int \frac{i\,dt_2\, i\,dt_3}{(2\pi)^2 (t_2+i\epsilon)(t_3+i\epsilon)}$$
$$\times \prod_{v=a,b} (2\pi)^3 \delta^3\Big(\sum_{i=1}^{I} J_{vi}\vec{k}_i + \sum_{d=e,f} J_{vd}\sigma_d \vec{q}_d\Big) 2\pi\delta\Big(q_f^0 - \sigma_f\omega_{q_f} - t_3\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d} - q_e^0 + \sigma_e\omega_{q_e} - t_2\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{bi}\omega_{k_i} + \sum_{d=e,f} J_{bd}\omega_{q_d} + t_2 - t_3\Big) . \tag{6.30}$$

After cancellation we set $q_e^0 = \sigma_e \omega_{q_e}$ and use $\sigma_e = -1$. This leads to

$$G'(k,q) = \frac{1}{2\omega_{q_f}} \int \frac{i\,dt_2\, i\,dt_3}{(2\pi)^2(t_2+i\epsilon)(t_3+i\epsilon)}$$
$$\times \prod_{v=a,b} (2\pi)^3\delta^3\Big(\sum_{i=1}^{I} J_{vi}\vec{k}_i + \sum_{d=e,f} J_{vd}\sigma_d\vec{q}_d\Big) 2\pi\delta\Big(q_f^0 - \sigma_f\omega_{q_f} - t_3\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d} - t_2\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{bi}\omega_{k_i} + \sum_{d=e,f} J_{bd}\omega_{q_d} + t_2 - t_3\Big)\,. \tag{6.31}$$

Now the integration $\int dt_2$ yields

$$G'(k,q) = \frac{1}{2\omega_{q_f}} \int \frac{i\,dt_3}{(2\pi)(t_3+i\epsilon)} \frac{i}{\sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d}}$$
$$\times \prod_{v=a,b} (2\pi)^3\delta^3\Big(\sum_{i=1}^{I} J_{vi}\vec{k}_i + \sum_{d=e,f} J_{vd}\sigma_d\vec{q}_d\Big) 2\pi\delta\Big(q_f^0 - \sigma_f\omega_f - t_3\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{bi}\omega_{k_i} + \sum_{d=e,f} J_{bd}\omega_{q_d} + \sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d} - t_3\Big). \tag{6.32}$$

Doing the integration $\int dt_3$ and multiplying with the inverse propagator $-i(q_f^0 - \sigma_f\omega_{q_f})(q_f^0 + \sigma_f\omega_{q_f})$ yields (after cancellation and with $\sigma_f = +1$)

$$G'(k,q) = \prod_{v=a,b} (2\pi)^3\delta^3\Big(\sum_{i=1}^{I} J_{vi}\vec{k}_i + \sum_{d=e,f} J_{vd}\sigma_d\vec{q}_d\Big)$$
$$\times 2\pi\delta\Big(\sum_{i=1}^{I} J_{bi}\omega_{k_i} + \sum_{d=e,f} J_{bd}\omega_{q_d} + \sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d}\Big)$$
$$\times \frac{i}{\sum_{i=1}^{I} J_{ai}\omega_{k_i} + \sum_{d=e,f} J_{ad}\omega_{q_d}}. \tag{6.33}$$

Now one can rewrite the δ-functions in order to split off total conservation of the external four-momenta, $(2\pi)^4\delta^4(q_e + q_f)$. There remains a δ-function for three momentum conservation at each vertex but one. We are free to choose this one to be the latest vertex of a given graph.

6.4 Summary

As a warm-up for the general treatment we have computed the one-loop two-point function for a ϕ^4 theory on noncommutative space and time in the framework of "interaction point time-ordered perturbation theory". The calculation is based on free fields (on the mass shell). Our final result (for that graph) agrees with a Feynman graph computation, provided that *one assigns to the internal line the real part of the Feynman propagator.* This can be understood as the inclusion of acausal processes in the S-matrix, because IPTO explicitly violates causality. We think that the true time-ordering of the $\star$-product (5.34) would produce the naïve Feynman rules involving the standard causal Feynman propagator in non-planar graphs. This approach, however, was shown to violate unitarity of the S-matrix. We have thus given up (nano-) causality in order to achieve unitarity in noncommutative field theories.

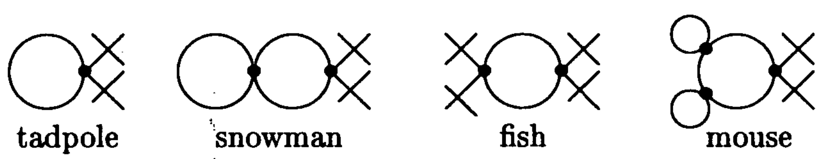
Next we have derived the Feynman rules (6.27) for general Green's functions. Power-counting tells us that (6.27) is expected to diverge if there are subgraphs with $E \leq 4$ external lines. If there are non-planar divergent graphs, it is not possible to absorb the divergences by local (hence planar) counterterms as usual. One has therefore to analyse whether the oscillating phases render the power-counting divergent integral finite. This requires to develop techniques for the computation of (6.27) in analogy to the treatment of the Euclidean case in [23]. Of urgent interest are the evaluations of the two-loop two-point function and the one-loop four-point function.

Chapter 7

Feynman Rules and UV/IR-Mixing

7.1 Examples

Now we want to use the noncommutative version of the time ordered expression for Green functions [66], eq. (6.27) to obtain explicit results for the Fourier transformed (FT), amputated on-shell two-point one-loop amplitude $\Gamma^{(2,1)}$ (tadpole, fig. 7.1), two-point two-loop amplitude $\Gamma^{(2,2)}$ (snowman, fig. 7.2), and four-point one-loop amplitude $\Gamma^{(4,1)}$ (fish, fig. 7.3).



7.1.1 Two-Point One-Loop Tadpole

To see how eq. (6.27) works, we first want to review the on-shell one-loop correction to the two-point function. One typical contribution to this diagram is given in figure 7.1.

With eq. (6.27) a general contribution reads

$$\Gamma^{(2,1)} = \frac{g}{4!} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \cdot 1 \cdot \exp(i\theta^{\mu\nu}\phi_{\mu\nu}), \tag{7.1}$$

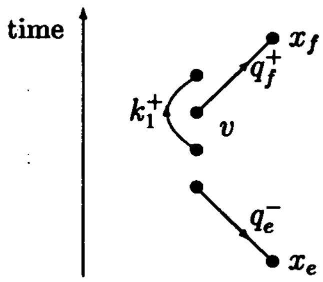


Figure 7.1: The contribution $(e, \bar{1}, f, 1)$

where $\Phi_{\mu\nu}$ is the phase depending on the special configuration of lines at the vertex. Since there is only one inner line, the I_{ij} term in (6.27) is vanishing. For the I_{ie} term we have to look at all possible configurations of lines at the 4-field vertex v. We have

$$I_{ij} = \frac{1}{2}\sum_v \tau_{ij}^v J_{vi} J_{vj}, \quad I_{ie} = \frac{1}{2}\sum_v (\tau_{ie}^v - \tau_{ei}^v) J_{vi} J_{ve}, \quad I_{ef} = \frac{1}{2}\sum_v \tau_{ef}^v J_{ve} J_{vf}. \tag{7.2}$$

The sum is over all vertices in a particular graph. $\tau_{ie}^v = +1$ if the line i is connected to an "earlier" field ϕ in the vertex v than the line e, otherwise $\tau_{ij}^v = 0$. We have $\sigma_e = -1$, $J_{ve} = +1$, $\sigma_f = +1$, $J_{vf} = -1$. For the inner line we have to distinguish between the one leaving (we denote this by $i = \bar{1}$) and the one arriving ($i = 1$). Then $J_{v\bar{1}} = -1$ and $J_{v1} = +1$. Note that the inner line is by definition oriented forward in time and $k_1 \equiv k_{\bar{1}}$. We write the time-ordering configuration at the vertex as an array, the contribution in figure 7.1 is labelled $(e, \bar{1}, f, 1)$. Then we find, for the I_{ie} and the I_{ef} terms:

$$\sum_{i=1,\bar{1}} k_i^+ I_{ie}(-q_e^-) + \sum_{i=1,\bar{1}} k_i^+ I_{if}(+q_f^+) + \sum_{e',f'=e,f} I_{e'f'}(\sigma_{e'} q_{e'}^{\sigma_{e'}})(\sigma_{f'} q_{f'}^{\sigma_{f'}}) =$$

$$\begin{aligned}
&(e, f, \bar{1}, 1) : 0 + 0 + \frac{1}{2} q_e^- q_f^+ &\qquad& (f, e, \bar{1}, 1) : 0 + 0 + \frac{1}{2} q_f^+ q_e^- \\
&(\bar{1}, 1, e, f) : 0 + 0 + \frac{1}{2} q_e^- q_f^+ && (\bar{1}, 1, f, e) : 0 + 0 + \frac{1}{2} q_f^+ q_e^- \\
&(e, \bar{1}, 1, f) : 0 + 0 + \frac{1}{2} q_e^- q_f^+ && (f, \bar{1}, 1, e) : 0 + 0 + \frac{1}{2} q_f^+ q_e^- \\
&(\bar{1}, e, f, 1) : -k_1^+(-q_e^-) + k_1^+ q_f^+ + \frac{1}{2} q_e^- q_f^+ &&
\end{aligned}$$

$$(\bar{1}, f, e, 1) : -k_1^+(-q_e^-) + k_1^+ q_f^+ + \frac{1}{2} q_f^+ q_e^-$$

$$(e, \bar{1}, f, 1) : +k_1^+ q_f^+ + \frac{1}{2} q_e^- q_f^+ \qquad (\bar{1}, f, 1, e) : +k_1^+ q_f^+ + \frac{1}{2} q_f^+ q_e^-$$

$$(\bar{1}, e, 1, f) : -k_1^+(-q_e^-) + \frac{1}{2} q_e^- q_f^+ \qquad (f, \bar{1}, e, 1) : -k_1^+(-q_e^-) + \frac{1}{2} q_f^+ q_e^-.$$

Thus for the sum over all possible phase factors we obtain

$$\sum_\phi \exp(i\theta^{\mu\nu}\phi_{\mu\nu}) = 2\cos\left(\frac{1}{2}\theta^{\mu\nu} q_{e,\mu}^- q_{f,\nu}^+\right) \times \left(3e^0 + e^{i\theta^{\mu\nu} k_{1,\mu}^+ q_{e,\nu}^-} + e^{i\theta^{\mu\nu} k_{1,\mu}^+ q_{f,\nu}^+} + e^{i\theta^{\mu\nu}(k_{1,\mu}^+ q_{e,\nu}^- + k_{1,\mu}^+ q_{f,\nu}^+)}\right). \tag{7.3}$$

Inserting this into (7.1) and with $q_f^+ = -q_e^-$ we find for the total Γ

$$\Gamma_{tot}^{(2,1)} = \frac{g}{12} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(4 + 2\cos(\theta^{\mu\nu} k_\mu^+ q_{f,\nu}^+)\right). \tag{7.4}$$

This result agrees with the corresponding result (6.15), where the same amplitude was obtained by explicitly commuting out the free field operators.

7.1.2 Two-Loop Snowman

For the two-loop snowman, in addition to the inner configuration of the lines at the vertices, we have to respect the two possibilities of time ordering of the vertices, see figure 7.2.

With $V = 2$, $E = 2$, $I = 3$, eq. (6.27) reads for the left graph, where the vertex v is before the vertex w:

$$\Gamma^{(2,2)} = \frac{g^2}{(4!)^2} \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9 8\omega_1\omega_2\omega_3} \frac{i(2\pi)^3\delta^3(-\vec{k}_2 - \vec{k}_3 - \vec{q}_e - \vec{q}_f)}{-\omega_2 - \omega_3 + \omega_e - \omega_f + i\varepsilon} \exp(i\theta^{\mu\nu}\phi_{\mu\nu}). \tag{7.5}$$

We have $J_{v2} = J_{v3} = +1$, $J_{w2} = J_{w3} = -1$, $\sigma_e = +1$, $\sigma_f = -1$. We obtain a non-trivial I_{ij} term from the vertex v. For example, the phase of the vertex v in the left graph is

$$(2, \bar{1}, 3, 1)_v : -k_1^+ k_3^+ + \frac{1}{2} k_2^+ k_3^+, \tag{7.6}$$

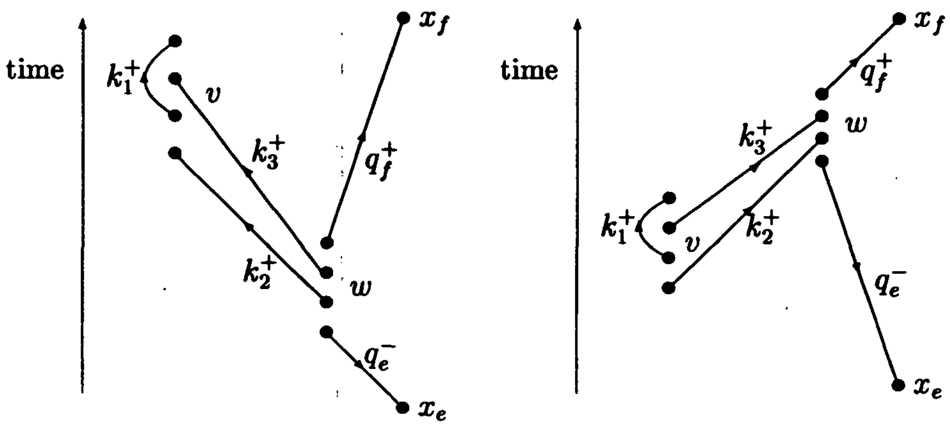


Figure 7.2: $(2,\bar{1},3,1)\times(e,\bar{2},\bar{3},f)$ $(e,\bar{2},\bar{3},f)\times(2,\bar{1},3,1)$

and is similar for the other 11 contributions. For the vertex w the I_{ij} and I_{ie} terms are non-zero. Again, we present only one contribution (note that $-q_e^- = +q_f^+$, owing to momentum conservation):

$$
\begin{aligned}
(e,2,3,f)_w: \quad & \frac{1}{2}k_2^+(-q_e^-)+\frac{1}{2}k_3^+(-q_e^-)+\frac{1}{2}k_2^+q_f^+ + \frac{1}{2}k_3^+q_f^+ + \frac{1}{2}k_2^+k_3^+ \\
= \quad & k_2^+q_f^+ + k_3^+q_f^+ + \frac{1}{2}k_2^+k_3^+. \qquad (7.7)
\end{aligned}
$$

Collecting the other 23 terms would be fairly edifying for a computer. Summing up all contributions, using again $q_e^- = -q_f^+$, integrating out $\vec{k}_3$ and setting $\varepsilon = 0$ yields

$$
\begin{aligned}
\Gamma_{left}^{(2,2)} = & -\frac{ig^2}{(4!)^2}\int\frac{d^3k_2}{(2\pi)^3 8\omega_2^3}\int\frac{d^3k_1}{(2\pi)^3 2\omega_1}2\cos\left(\frac{1}{2}k_2^+\tilde{k}_2^-\right) \\
& \times\left(3+e^{-i\theta^{\mu\nu}k_{1,\mu}^+k_{2,\nu}^+}+e^{+i\theta^{\mu\nu}k_{1,\mu}^+k_{2,\nu}^-}+e^{-i\theta^{\mu\nu}(k_{1,\mu}^+k_{2,\nu}^+-k_{1,\mu}^+k_{2,\nu}^-)}\right) \\
& \times 2\cos\left(\frac{1}{2}k_2^+\tilde{k}_2^-\right)\Big(6+2\cos(\theta^{\mu\nu}k_{2,\mu}^+q_{f,\nu}^+) \\
& +2\cos(\theta^{\mu\nu}k_{2,\mu}^-q_{f,\nu}^+)+2\cos(\theta^{\mu\nu}(k_{2,\mu}^+-k_{2,\mu}^-)q_{f,\nu}^+)\Big). \qquad (7.8)
\end{aligned}
$$

The first two lines of the integral kernel are exactly eq. (7.4), with the obvious replacements (note the correct signs coming from the σ's and J's) $-q_e^- \to +k_2^+$ and $-q_f^+ \to +k_3^+ \to -k_2^-$. For $\Gamma_{right}^{(2,2)}$ we find the same expression with

$k^+_{2,3} \to -k^+_{2,3}$, because of the reversed sign of J_{ve}, etc. This yields exactly the complex-conjugated expression, so with the help of $4\cos^2(\frac{x}{2}) = 2 + 2\cos(x)$ we get

$$\Gamma^{(2,2)}_{tot} = -\frac{ig^2}{(4!)^2}\int \frac{d^3k_2}{(2\pi)^3 8\omega_2^3}(2 + 2\cos(k_2^+\tilde{k}_2^-)) \tag{7.9}$$
$$\times\Big(6 + 2\cos(k_2^+\tilde{q}_f^+) + 2\cos(k_2^-\tilde{q}_f^+) + 2\cos((k_2^+ - k_2^-)\tilde{q}_f^+)\Big)$$
$$\times\int \frac{d^3k_1}{(2\pi)^3 2\omega_1}\Big(6 + 2\cos(k_1^+\tilde{k}_2^+) + 2\cos(k_1^+\tilde{k}_2^-) + 2\cos(k_1^+(\tilde{k}_2^+ - \tilde{k}_2^-))\Big).$$

Note the extra i due to the slightly unusual definition of the S-matrix used in the previous chapter.

7.1.3 Four-Point One-Loop Correction

Finally, for the one-loop correction to the t-channel four-point function we have the contributions of figure 7.3.

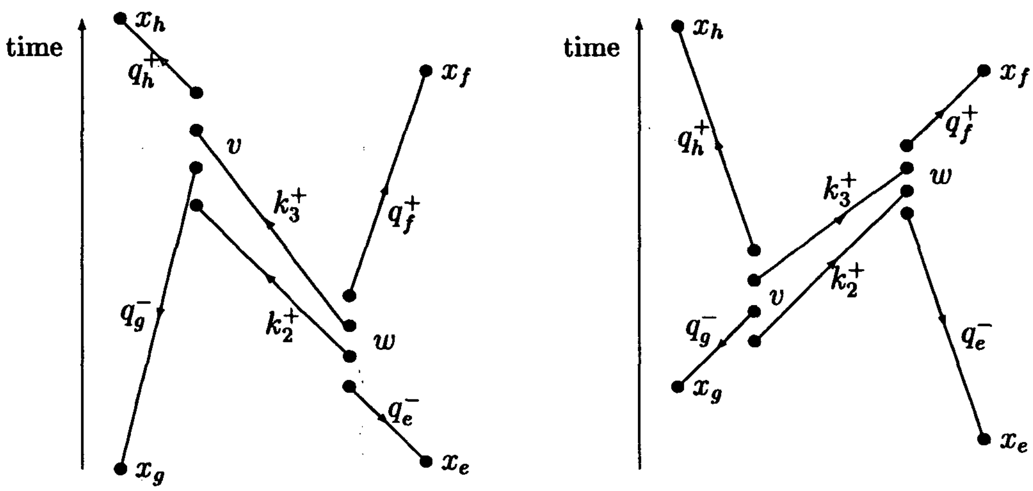


Figure 7.3: Two contributions to $\Gamma^{(4,1)}$

Without going into detail with respect to the phase, we can prove the IR finiteness of the sum of these contributions:

$$
\begin{aligned}
\Gamma^{(4,1)} = \frac{g^2}{(4!)^2} \int \frac{d^3k_2}{(2\pi)^3 2\omega_2} \int \frac{d^3k_3}{(2\pi)^3 2\omega_3} \\
\times \Big(\frac{i(2\pi)^3 \delta^3(-\vec{k}_2 - \vec{k}_3 - \vec{q}_e - \vec{q}_f)}{-\omega_2 - \omega_3 + \omega_e - \omega_f + i\varepsilon} \\
\times \Psi(-q_e^-, -q_f^+, -k_2^+, -k_3^+) \Psi(-q_g^-, -q_h^+, k_2^+, k_3^+) \\
+ \frac{i(2\pi)^3 \delta^3(-\vec{k}_2 - \vec{k}_3 - \vec{q}_g - \vec{q}_h)}{-\omega_2 - \omega_3 + \omega_g - \omega_h + i\varepsilon} \\
\times \Psi(-q_e^-, -q_f^+, +k_2^+, +k_3^+) \Psi(-q_g^-, -q_h^+, -k_2^+, -k_3^+) \Big).
\end{aligned}
\tag{7.10}
$$

Here the phase Ψ will be defined in section 7.2.5. With conservation of the global 4-momentum $\delta^4(q_e^- + q_f^+ + q_g^- + q_h^+)$, we have $\omega_g - \omega_h = -(\omega_e - \omega_f)$ in the denominator of the second term. Before integrating out k_3 we let $\vec{k}_2 \to -\vec{k}_2, \vec{k}_3 \to -\vec{k}_3$ in the second term, so that $\vec{k}_3 = -\vec{k}_2 - \vec{q}_e - \vec{q}_f$ in both terms. Thus we find

$$
\begin{aligned}
\Gamma^{(4,1)} = -\frac{g^2}{(4!)^2} \int \frac{d^3k_2}{(2\pi)^3 2\omega_2} \frac{1}{(2\pi)^3 2\omega_3} i(2\pi)^3 \\
\times \Big(\frac{\Psi(-q_e^-, -q_f^+, -k_2^+, -k_3^+) \Psi(-q_g^-, -q_h^+, k_2^+, k_3^+)}{\omega_2 + \omega_3 - (\omega_e - \omega_f) - i\varepsilon} \\
+ \frac{\Psi(-q_e^-, -q_f^+, -k_2^-, -k_3^-) \Psi(-q_g^-, -q_h^+, k_2^-, k_3^-)}{\omega_2 + \omega_3 + (\omega_e - \omega_f) - i\varepsilon} \Big) \Big|_{\vec{k}_3 = -(\vec{k}_2 + \vec{q}_e + \vec{q}_f)}.
\end{aligned}
\tag{7.11}
$$

We find that the denominators are strictly positive,

$$
\begin{aligned}
&|(\omega_2 + \omega_3)|^2 - |(\omega_e - \omega_f)|^2 = \\
&\vec{k}_2^2 + m^2 + (\vec{k}_2 + \vec{q}_e + \vec{q}_f)^2 + m^2 + 2\omega_2\omega_3 \\
&\qquad -\vec{q}_e^2 - m^2 - \vec{q}_f^2 - m^2 + 2\omega_e\omega_f = \\
&2\Big(\vec{k}_2(\vec{k}_2 + \vec{q}_e + \vec{q}_f) + \vec{q}_e\vec{q}_f + \omega_2\omega_3 + \omega_e\omega_f\Big) \Big|_{\vec{k}_3 = -(\vec{k}_2 + \vec{q}_e + \vec{q}_f)} > 0 \\
&\qquad (|\vec{p} \cdot \vec{q}| < \omega_p\omega_q, \ m > 0).
\end{aligned}
$$

Thus, no new kinematic IR divergence occurs with respect to the commutative case, although the usual cancellations could not take place because of the different phases. Hence we made sure that no novel problems arise from this quarter.

7.2 The Feynman Rules for IPTOPT

To obtain the set of diagrammatic rules for our model we have to answer three questions:

1. What is the vertex?
2. What is the propagator?
3. How to construct graphs?

The first of these we postpone to section 7.2.5, while the other two are tackled by retracing our steps to the explicit result for the tadpole obtained in the previous chapter.

7.2.1 The Full Noncommutative Propagator

We start our search for the Feynman(-like) rules of noncommutative IPTOPT at the explicit expression for the two-point one-loop tadpole $G^{(2,1)}$.

Repeating the notation from the previous chapter (recall that $p^{\pm} := (\pm\omega_p, \vec{p})$, $\omega_p := \sqrt{\vec{p}^2 + m^2}$, $\tilde{p}^{\nu} := p_{\mu}\theta^{\mu\nu}$):

$$\begin{aligned}
\mathcal{I}^{\pm\pm}((\pm p)^+, (\pm q)^+) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(3 + e^{ip^{\pm}\tilde{k}^{+} + iq^{\pm}\tilde{k}^{+}} + e^{ip^{\pm}\tilde{k}^{+}} + e^{iq^{\pm}\tilde{k}^{+}}\right) \\
&\equiv \mathcal{I}(p^{\pm}, q^{\pm}) \,,
\end{aligned} \tag{7.12}$$

we retrace one step and give the unamputated FT Green function

$$\begin{aligned}
G^{(2,1)}(p,q) = -\lim_{\delta_1,\delta_2\to 0} \frac{g}{12}(2\pi)^4\delta(p+q) \\
\times \Big(\frac{1}{p_0-\omega_p+i\delta_1} \frac{1}{\omega_p+\omega_q-i\delta_2} \frac{\cos(\frac{1}{2}p^+\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^+) \\
+ \frac{1}{q_0-\omega_q+i\delta_1} \frac{1}{\omega_p+\omega_q-i\delta_2} \frac{\cos(\frac{1}{2}p^+\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^+) \\
+ \frac{1}{p_0-\omega_p+i\delta_1} \frac{1}{q_0+\omega_q-i\delta_2} \frac{\cos(\frac{1}{2}p^+\tilde{q}^-)}{4\omega_p\omega_q}\mathcal{I}(p^+,q^-) \\
+ \frac{1}{q_0-\omega_q+i\delta_1} \frac{1}{p_0+\omega_p-i\delta_2} \frac{\cos(\frac{1}{2}p^-\tilde{q}^+)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^+)
\end{aligned}$$

$$+\frac{1}{\omega_p+\omega_q-i\delta_1}\frac{1}{-q_0-\omega_q+i\delta_2}\frac{\cos(\frac{1}{2}p^-\tilde{q}^-)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^-)$$
$$+\frac{1}{\omega_p+\omega_q-i\delta_1}\frac{1}{-p_0-\omega_p+i\delta_2}\frac{\cos(\frac{1}{2}p^-\tilde{q}^-)}{4\omega_p\omega_q}\mathcal{I}(p^-,q^-)\Big) . \qquad (7.13)$$

Making use of local energy–momentum conservation $p^0 = -q^0$ and $\omega_p = +\omega_q$, and of the relation $q^\pm = -p^\mp$, we eliminate q and contract eq. (7.13) to

$$G^{(2,1)}(p) = -\lim_{\varepsilon\to 0}\frac{g}{12(2\omega_p)^2}(2\pi)^4\delta(p+q)$$
$$\times\Big(\frac{1}{p_0-\omega_p+i\varepsilon}\,\frac{1}{p_0+\omega_p-i\varepsilon}\,\cos\left(\frac{1}{2}p^+\tilde{p}^-\right)\left(\mathcal{I}(p^+,-p^-)+\mathcal{I}(p^-,-p^+)\right)$$
$$-\frac{1}{p_0-\omega_p+i\varepsilon}\,\frac{1}{p_0-\omega_p+i\varepsilon}\,\cos\left(\frac{1}{2}p^+\tilde{p}^+\right)\,\mathcal{I}(p^+,-p^+)$$
$$-\frac{1}{p_0+\omega_p-i\varepsilon}\,\frac{1}{p_0+\omega_p-i\varepsilon}\,\cos\left(\frac{1}{2}p^-\tilde{p}^-\right)\,\mathcal{I}(p^-,-p^-)\Big). \qquad (7.14)$$

This can easily be written as the sum over two signs:

$$\begin{aligned} G^{(2,1)}(p) &= \frac{g}{12}(2\pi)^4\delta(p+q)\sum_{\sigma}^{+1,-1}\sum_{\sigma'}^{+1,-1}\cos(p^\sigma\tilde{p}^{\sigma'})\mathcal{I}(p^\sigma,-p^{\sigma'}) \\ &\quad \frac{1}{2\omega_p}\frac{1}{\sigma p_0-\omega_p+i\varepsilon}\;\frac{1}{2\omega_p}\frac{1}{\sigma' p_0-\omega_p+i\varepsilon}. \end{aligned} \qquad (7.15)$$

7.2.2 The TO Propagator

Equation (7.15) lets us read off the answers to both our questions. Since we have not performed any amputation yet, two propagators must be included in the above expression. We easily identify the TO propagator as

$$i\Delta^{TO} := \frac{\delta_{\sigma,-\sigma'}}{2\omega_p}\frac{i}{\sigma p^0-\omega_p+i\varepsilon}. \qquad (7.16)$$

The $\delta_{\sigma,-\sigma'}$ was included to guarantee TO-diagrammatic consistency: every directed TO line that leaves one vertex (σ) has to arrive at another one (σ'). (The correctness of this addition will become evident in the following examples.)

Note that the same result is also obtained in [68], where the TO propagator is called "contractor".

The global TO of the Note that the same result is also obtained in [68], where the TO propagator is called "contractor".

The global TO of the vertices is another necessary issue to be encoded in Δ^{TO}: every line has to leave its earlier vertex and arrive at its later vertex, and this must be consistently so for all lines of the diagram. This property is taken care of by the sign of the pole prescription. As illustrated in the amplitudes (re)calculated in section 7.3, only products of TO propagators in TO consistent graphs (if $A < B$ and $C < A$ then $C < B$) will contribute. All others (e.g. $A < B$ and $C < A$ but $B < C$) will have their poles bundled in the same complex half-plane and hence vanish upon integrating over p^0.

7.2.3 Building Graphs

In addition to providing us with a propagator, eq. (7.15) also tells us how to construct graphs: multiply together all the building blocks for a graph of given topology — lines, vertices, subgraphs — which all depend on the entering or leaving ($\sigma_i = \pm 1$) of the lines running into them. Then sum over all signs. The propagators take care of the correct connection of all parts of the diagram, especially causal consistency: if vertex A is later than vertex B and B is later than C, than A is also later than C.

Even at this point we may already calculate the two-point zero-loop function, the usual covariant propagator,

$$\begin{aligned} i\Delta_F &= \sum_{\sigma}^{+1,-1} i\Delta^{TO}(\sigma) = \sum_{\sigma}^{+1,-1} \frac{1}{2\omega} \frac{i}{\sigma p_0 - \omega + i\varepsilon} \\ &= \frac{i}{2\omega}\Big(\frac{1}{+p_0 - \omega + i\varepsilon} + \frac{1}{-p_0 - \omega + i\varepsilon}\Big) = \frac{i}{p_0^2 - \omega^2 + i\varepsilon}. \end{aligned} \tag{7.17}$$

7.2.4 Complete One-Loop Integrals

To complete our discussion of $G^{(2,1)}$, and for further use in section 7.3.4, we evaluate the $\mathcal{I}$'s occurring in eq. (7.15).

Abbreviating the (cut-off-regularized) divergent part of the planar term by $\mathcal{Q} = \Lambda^2 + \frac{m^2}{2}\ln(\frac{m^2}{\Lambda^2})$, we give $\mathcal{I}(p^+, -p^+)$, which was already calculated in the previous chapter, eq. (31):

$$\mathcal{I}(p^+, -p^+) = \frac{2}{(2\pi)^2}\Big(\mathcal{Q} - \sqrt{-\frac{m^2}{\tilde{p}_+^2}}K_1\Big(\sqrt{-m^2\tilde{p}_+^2}\Big)\Big). \tag{7.18}$$

Analogously we find

$$\mathcal{I}(p^-, -p^-) = \frac{2}{(2\pi)^2}\Big(\mathcal{Q} - \sqrt{-\frac{m^2}{\tilde{p}_-^2}}K_1\Big(\sqrt{-m^2\tilde{p}_-^2}\Big)\Big). \tag{7.19}$$

Calculating the sum of the remaining integrals still has to be done. Adding the integrands gives

$$\begin{aligned}
&\mathcal{I}(p^+, -p^-) + \mathcal{I}(p^-, -p^+) = \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\big(6 + e^{-ik^+\tilde{p}^+ + ik^+\tilde{p}^-} + e^{-ik^+\tilde{p}^+} + e^{+ik^+\tilde{p}^-} \\
&\qquad + e^{+ik^+\tilde{p}^+ - ik^+\tilde{p}^-} + e^{+ik^+\tilde{p}^+} + e^{-ik^+\tilde{p}^-}\big) \qquad (7.20)\\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} 2\big(3 + \cos(k^+\tilde{p}^+) + \cos(k^+\tilde{p}^-) + \cos(k^+(\tilde{p}^+ - \tilde{p}^-))\big).
\end{aligned}$$

The first and second cosine terms are just the ones yielding the non-planar parts of eqs. (7.18) and (7.19). The third one has to be dealt with explicitly. With $(\tilde{p}^+ - \tilde{p}^-)_\mu = 2\Theta_{0\mu}\omega$ and $\theta_{00} = 0$ we can choose a coordinate system with the z-axis parallel to the 3-vector θ_{0i}. Thus integrating out the angles yields

$$\frac{2}{(2\pi)^2|\Theta_{0i}|\omega}\int_0^\infty dk\frac{|\vec{k}|}{\omega_k}\sin(2|\vec{k}||\Theta_{0i}|\omega). \tag{7.21}$$

This we evaluate as

$$= \frac{m}{(2\pi)^2|\Theta_{0i}|\omega}K_1\big(2m|\Theta_{0i}|\omega\big). \tag{7.22}$$

Hence we have

$$\begin{aligned}
\mathcal{I}(p^+, -p^-) + \mathcal{I}(p^-, -p^+) &= \frac{2}{(2\pi)^2}\Big(\frac{3}{2}\mathcal{Q} - 2\frac{m}{|\tilde{p}^+|}K_1(m|\tilde{p}^+|) \\
&-2\frac{m}{|\tilde{p}^-|}K_1(m|\tilde{p}^-|) + \frac{m}{|\Theta_{0i}|\omega}K_1\big(2m|\Theta_{0i}|\omega\big)\Big).
\end{aligned} \tag{7.23}$$

For further use (see eq. (7.39)), we finally present another result. Iff $\mathcal{I}(p^+, -p^-)$ occurs under an integral over d^3p together with functions $f(\vec{p})$ invariant under $\vec{p} \to -\vec{p}$, we have:

$$
\begin{aligned}
&\int d^3p f(\vec{p}) \mathcal{I}(p^+, -p^-) = \\
&\int d^3p f(\vec{p}) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(3 + e^{ip^+\tilde{k}^+} + e^{-ip^-\tilde{k}^+} + e^{i(p^+-p^-)\tilde{k}^+}\right) \\
&= \int d^3p f(\vec{p}) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \Big(3 + i\sin(p^+\tilde{k}^+) - i\sin(p^-\tilde{k}^+) \\
&+ i\sin((p^+ - p^-)\tilde{k}^+) + \cos(k^+\tilde{p}^+) + \cos(k^+\tilde{p}^-) + \cos(k^+(\tilde{p}^+ - \tilde{p}^-))\Big).
\end{aligned}
$$

$\sin((p^+ - p^-)\tilde{k}^+) = \sin(2\omega_p \theta_{0i} k_i)$ is antisymmetric in $\vec{k}$ and thus this term vanishes. The remaining two sine-terms can be added up (let $\vec{p} \to -\vec{p}$ in the second term) to $2i\sin(k^+\tilde{p}^+)$. Now we have

$$
\begin{aligned}
&\int d^3p f(\vec{p}) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sin(k^+\tilde{p}^+) \\
&= \int d^3p f(\vec{p}) \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sin(\omega_k \tilde{p}^0)\cos(\vec{k}\vec{\tilde{p}}) + \underbrace{\cos(\omega_k \tilde{p}^0)\sin(\vec{k}\vec{\tilde{p}})}_{=0 \text{ (antisymmetric in } \vec{k})}.
\end{aligned}
$$

Choosing $\vec{\tilde{p}}$ parallel to the z-axis we further evaluate

$$
\begin{aligned}
&= 2\pi \int d^3p f(\vec{p}) \int_0^\infty |\vec{k}|^2 \frac{d|\vec{k}|}{(2\pi)^3 2\omega_k} \int_{-1}^1 d(\cos\vartheta) \sin(\omega_k \tilde{p}^0) \cos\left(\cos\vartheta |\vec{k}||\vec{\tilde{p}}|\right) \\
&= \lim_{\epsilon \to 0} \int d^3p \frac{f(\vec{p})}{|\vec{\tilde{p}}|} \int_0^\infty \frac{d|\vec{k}||\vec{k}|}{(2\pi)^2 2\omega_k} \frac{1}{2i} \left(e^{i\omega_k(\tilde{p}^0 + i\epsilon)} - e^{-i\omega_k(\tilde{p}^0 - i\epsilon)}\right) 2\sin(|\vec{k}||\vec{\tilde{p}}|) \\
&= 0.
\end{aligned}
$$

Here we have expressed one sine by (regularized) exponentials. The last equation follows from Gradstein [67] 3.961 ($\gamma = m$, $\sqrt{\gamma^2 + x^2} = \omega_k$):

$$
\int_0^\infty e^{-\beta\sqrt{\gamma^2+x^2}} \sin ax \frac{x dx}{\sqrt{\gamma^2 + x^2}} = \frac{a\gamma}{\sqrt{a^2 + \beta^2}} K_1(\gamma\sqrt{a^2 + \beta^2}),
$$

for positive real parts of γ, β and a. Therefore all the sine terms are shown to vanish. The cosine integrals have been calculated above and thus we finally find for even $f(\vec{p})$

$$\int d^3p f(\vec{p}) \mathcal{I}(p^+, -p^-) = \int d^3p f(\vec{p}) \frac{1}{(2\pi)^2}$$
$$\times \Big(\frac{3}{2}\mathcal{Q} - \frac{2m}{|\tilde{p}^+|} K_1(m|\tilde{p}^+|) - \frac{2m}{|\tilde{p}^-|} K_1(m|\tilde{p}^-|) + \frac{m}{\omega_p |\Theta_{0i}|} K_1(2m\omega_p |\Theta_{0i}|)\Big). \tag{7.24}$$

With $\mathcal{I}(p^-, -p^+)$ we find an analogous result.

7.2.5 The Vertex

To answer our first question we straightforwardly peruse eq. (6.27) for no internal lines and four external ones with general causalities (σ's). Summing over all possible inner (nano-) TO of the vertex, we proceed as in section 7.1 and find ($\tilde{p}^\nu := p_\mu \theta^{\mu\nu}$)

$$\Gamma^{(4,0)}(p_1^{\sigma_1}, p_2^{\sigma_2}, p_3^{\sigma_3}, p_4^{\sigma_4}) := \frac{g}{4!} \Psi(-p_1^{\sigma_1}, -p_2^{\sigma_2}, -p_3^{\sigma_3}, -p_4^{\sigma_4}) =$$
$$= \frac{g}{3}\Big(\cos\Big(\frac{1}{2} p_1^{\sigma_1} \tilde{p}_2^{\sigma_2}\Big) \cos\Big(\frac{1}{2} p_3^{\sigma_3} \tilde{p}_4^{\sigma_4}\Big) \cos\Big(\frac{1}{2}(p_1^{\sigma_1} + p_2^{\sigma_2})(\tilde{p}_3^{\sigma_3} + \tilde{p}_4^{\sigma_4})\Big)$$
$$+ (2) \leftrightarrow (3) + (2) \leftrightarrow (4)\Big). \tag{7.25}$$

Note that here all the momenta are defined outgoing of the vertex. With the symmetry of the cosine we explicitly check the invariance of (7.25) with respect to any permutation of the momenta.

Unfortunately, the tadpole has to be treated separately. From eq. (6.27) it follows that the tadpole line has to be oriented forward in time. Thus only $\frac{24!}{2}$ nano-configurations at the vertex contribute. We find for the phase factor of a 1-loop tadpole (defining $p_2^{\sigma_2}$, $p_3^{\sigma_3}$ outgoing, loop momentum p_1^+)

$$\frac{g}{4!} \exp\Big(i\theta_{\mu\nu} \sum_{a,b=1}^{3} \tau_{ab}^{v} p_a^{\sigma_a} p_b^{\sigma_b}\Big) =: \frac{g}{4!} \Phi(p_1^+; -p_2^{\sigma_2}, -p_3^{\sigma_3})$$
$$= \frac{g}{12}\Big(3 + e^{ip_1^+ \tilde{p}_2^{\sigma_2}} + e^{ip_1^+ \tilde{p}_3^{\sigma_3}} + e^{ip_1^+ (\tilde{p}_2^{\sigma_2} + \tilde{p}_3^{\sigma_3})}\Big) \cos\Big(\frac{1}{2} p_2^{\sigma_2} \tilde{p}_3^{\sigma_3}\Big). \tag{7.26}$$

7.2.6 Summary of Diagrammatics

To calculate a Fourier transformed, amputated amplitude, use the following rules:

1. An amputated external line carries the momentum $q_e^{\sigma_e}$; $\sigma_e = +1$ if the line is directed into the future, $\sigma_e = -1$ if it runs into the past:

$$q_e^{\sigma_e} = (\sigma_e \sqrt{\vec{q}^2 + m^2}, \vec{q})^T. \tag{7.27}$$

2. For a general, non-tadpolic vertex write a factor

$$\begin{aligned} & \frac{g}{4!}\Psi(-p_1^{\sigma_1}, -p_2^{\sigma_2}, -p_3^{\sigma_3}, -p_4^{\sigma_4}) = \\ = \; & \frac{g}{3}\Big(\cos\left(\frac{1}{2} p_1^{\sigma_1} \tilde{p}_2^{\sigma_2}\right) \cos\left(\frac{1}{2} p_3^{\sigma_3} \tilde{p}_4^{\sigma_4}\right) \cos\left(\frac{1}{2}(p_1^{\sigma_1} + p_2^{\sigma_2})(\tilde{p}_3^{\sigma_3} + \tilde{p}_4^{\sigma_4})\right) \\ + \; & (2) \leftrightarrow (3) + (2) \leftrightarrow (4)\Big), \end{aligned} \tag{7.28}$$

   where all momenta are oriented outwards from the vertex.

3. For a tadpolic vertex (with loop momentum p_1^+), write a factor

$$\begin{aligned} & \frac{g}{4!}\Phi(p_1^+; -p_2^{\sigma_2}, -p_3^{\sigma_3}) = \\ = \; & \frac{g}{12}\left(3 + e^{ip_1^+ \tilde{p}_2^{\sigma_2}} + e^{ip_1^+ \tilde{p}_3^{\sigma_3}} + e^{ip_1^+(\tilde{p}_2^{\sigma_2} + \tilde{p}_3^{\sigma_3})}\right) \cos\left(\frac{1}{2} p_2^{\sigma_2} \tilde{p}_3^{\sigma_3}\right), \end{aligned} \tag{7.29}$$

   where p_2, p_3 are oriented outwards from the vertex.

4. For an inner line, write the propagator

$$i\Delta^{TO} = \frac{i}{2\omega} \frac{\delta_{\sigma,-\sigma'}}{\sigma p^0 - \omega_p + i\varepsilon}. \tag{7.30}$$

5. Sum over all σ's of the internal lines in order to include all possible contributions with respect to the time ordering of the inner vertices.

6. Integrate over all loop momenta (including tadpole momenta).

Remember that 4-momentum conservation is valid at all vertices and along all lines.

7.3 Examples for the Application of the Feynman Rules for NC-IPTOPT

In order to both illustrate the applicability and demonstrate the validity of the new-found FR (and since a motivation was given for them, rather than a derivation), we employ them in the recalculation of the diagrams of section 7.1.

In addition we will finally be able to calculate the "mouse"-diagram $\Gamma^{(2,3)}$, which was one of the main motivations for the development of this diagrammatics.

7.3.1 The Diagrammatic Tadpole

Once again we turn toward the tadpole, obtained by explicitly commuting out the free-field operators in the previous chapter and by use of (6.27).

Simplifying eq. (7.26) by using 4-momentum conservation $p_2^\mu =: q^\mu = -p_3^\mu$, setting the external momenta on-shell $\sigma_2 = +1, \sigma_3 = -1$, and defining $p_1^+ =: k^+$, $p_\mu \Theta^{\mu\nu} =: \tilde{p}$ we find for the vertex factor

$$\frac{g}{4!}\Phi(k^+; -q^+, q^+) = \frac{g}{6}\big(2 + \cos(k^+\tilde{q}^+)\big). \tag{7.31}$$

Note that the σ of the looped line does not occur. Multiplying with the propagator eq. (7.16), summing over σ, σ' and integrating over phase space then yields the FT NC tadpole amplitude, which is well known by now:

$$\begin{aligned}
\Gamma^{(2,1)} &= \int \frac{d^4k}{(2\pi)^4}\frac{g}{6}(2 + \cos(k^+\tilde{q}^+))\sum_{\sigma,\sigma'}^{\pm 1}\frac{\delta_{\sigma,-\sigma'}}{2\omega}\frac{i}{\sigma k^0 - \omega_k + i\varepsilon} \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{g}{6}(2 + \cos(k^+\tilde{q}^+))\frac{i}{2\omega_k}\Big(\frac{1}{k^0 - \omega_k + i\varepsilon} - \frac{1}{k^0 + \omega_k - i\varepsilon}\Big) \\
&= \frac{g}{6}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}(2 + \cos(k^+\tilde{q}^+)).
\end{aligned} \tag{7.32}$$

The actual k^0-integration can be performed directly for both terms separately, heeding non-vanishing semi-circles at infinity. Alternatively they can be brought over a common denominator, resulting in the usual Feynman propagator.

7.3.2 The Diagrammatic Snowman

To further strengthen our confidence in Δ^{TO} and the vertices of eqs. (7.25) and (7.26), we demonstrate how to utilise them to evaluate the snowman of section 7.1.2.

To obtain the amputated, FT snowman amplitude, we multiply the terms for the two vertices with each other and with one Δ^{TO} for the head-loop and two for the body-loop. Using 4-momentum conservation $k_3^\mu = -k_2^\mu, k_3^\pm = -k_2^\mp$, summing over $\sigma_1^v, \sigma_1^w, \sigma_2^v, \sigma_2^w, \sigma_3^v, \sigma_3^w = \pm 1$ and integrating over the two loop-momenta k_1^μ, k_2^μ, we find

$$
\begin{aligned}
\Gamma^{(2,2)} = \frac{g}{4!}\int \frac{d^4k_1}{(2\pi)^4}
&\sum_{\sigma_2^v,\sigma_2^w,\sigma_3^v,\sigma_3^w}^{+,-} \overbrace{\frac{g}{4!}\int\frac{d^4k_2}{(2\pi)^4}\sum_{\sigma_1,\sigma_1'}\frac{i}{2\omega_1}\frac{\delta_{\sigma_1,-\sigma_1'}}{\sigma_1 k_1^0-\omega_1+i\varepsilon}\Phi^v(k_1^+;-k_2^{\sigma_2^v},+k_2^{-\sigma_3^v})}^{\Gamma^{(2,1)}(k_2;\sigma_2^v,\sigma_3^v)} \\
&\times\Psi^w(-q_e^-,-q_f^+,+k_2^{-\sigma_2^w},-k_2^{+\sigma_3^w}) \\
&\times\frac{i}{2\omega_2}\frac{i}{2\omega_2}\,\frac{\delta_{\sigma_2^v,-\sigma_2^w}}{\sigma_2^v k_2^0-\omega_2+i\varepsilon}\,\frac{\delta_{\sigma_3^v,-\sigma_3^w}}{-\sigma_3^v k_2^0-\omega_2+i\varepsilon} \\
= \frac{ig^2}{(4!)^2}\int&\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4(2\omega_2)^2(2\pi)^4} \\
&\sum_{\sigma_2^v,\sigma_3^v}^{+,-}\Phi^v(k_1^+;-k_2^{\sigma_2^v},+k_3^{-\sigma_3^v})\Psi^w(-q_e^-,-q_f^+,+k_2^{\sigma_2^v},-k_2^{-\sigma_3^v}) \\
\times&\frac{1}{(k_1^0)^2-\omega_1^2+i\varepsilon}\,\frac{1}{\sigma_2^v k_2^0-\omega_2+i\varepsilon}\,\frac{1}{\sigma_3^v k_2^0+\omega_2-i\varepsilon} \\
= \frac{g^2}{(4!)^2}\int&\frac{d^3k_1}{2\omega_1(2\pi)^3}\frac{d^3k_2}{(2\omega_2)^2(2\pi)^3}\frac{dk_2^0}{2\pi} \\
&\Big(\Phi^v(k_1^+;-k_2^+,+k_2^-)\Psi^w(-q_e^-,-q_f^+,+k_2^+,-k_2^-) \\
&\qquad\times\frac{1}{+k_2^0-\omega_2+i\varepsilon}\,\frac{1}{+k_2^0+\omega_2-i\varepsilon} \\
&+\Phi^v(k_1^+;-k_2^+,+k_2^+)\Psi^w(-q_e^-,-q_f^+,+k_2^+,-k_2^+) \\
&\qquad\times\frac{1}{+k_2^0-\omega_2+i\varepsilon}\,\frac{1}{-k_2^0+\omega_2-i\varepsilon}
\end{aligned}
$$

$$
\begin{aligned}
&+\Phi^v(k_1^+;-k_2^-,+k_2^-)\Psi^w(-q_e^-,-q_f^+,+k_2^-,-k_2^-) \\
&\qquad\times\ \frac{1}{-k_2^0-\omega_2+i\varepsilon}\,\frac{1}{+k_2^0+\omega_2-i\varepsilon} \\
&+\Phi^v(k_1^+;-k_2^-,+k_2^+)\Psi^w(-q_e^-,-q_f^+,+k_2^-,-k_2^+) \\
&\qquad\times\ \frac{1}{-k_2^0-\omega_2+i\varepsilon}\,\frac{1}{-k_2^0+\omega_2-i\varepsilon}\Big).
\end{aligned}
\tag{7.33}
$$

In the last step we integrated over k_1^0 as in section 7.3.1 and expanded the sums over σ_2^v, σ_3^a.

Performing the k_2^0 integration reveals how Δ^{TO} selects the correct σ-signs: the poles in the second and the third term are double poles, both lying on top of each other in the same complex half-plane. Hence we may close the contour in the other half without enclosing any residuum, yielding a vanishing integral (mark that the auxiliary semi-circle is harmless, contrary to the tadpole case).

In the first and the fourth term the poles lie in opposite halves and yield upon integration $2\pi i/(2\omega_2)$. Hence we find

$$
\begin{aligned}
\Gamma^{(2,2)} &= -\frac{ig^2}{(4!)^2}\int \frac{d^3k_1}{2\omega_1(2\pi)^3}\,\frac{d^3k_2}{(2\omega_2)^3(2\pi)^3} \\
&\quad \Big(\Phi(k_1^+;-k_2^+,+k_2^-)\Psi(-q_e^-,-q_f^+,+k_2^+,-k_2^-)+ \\
&\quad \Phi(k_1^+;-k_2^-,+k_2^+)\Psi(-q_e^-,-q_f^+,+k_2^-,-k_2^+)\Big).
\end{aligned}
\tag{7.34}
$$

Evaluation of the phases Φ and Ψ, using momentum conservation $q_e^- = -q_f^+$ and doing some trivial but tedious trigonometry, yields

$$
\begin{aligned}
&\Phi(k_1^+;-k_2^+,+k_2^-) = \\
&\qquad 2\cos(\tfrac{1}{2}k_2^+\tilde{k}_2^-)\big(3+e^{-ik_1^+\tilde{k}_2^+}+e^{+ik_1^+\tilde{k}_2^-}+e^{-ik_1^+(\tilde{k}_2^+-\tilde{k}_2^-)}\big) \\
&\Phi(k_1^+;-k_2^-,+k_2^+) = \\
&\qquad 2\cos(\tfrac{1}{2}k_2^+\tilde{k}_2^-)\big(3+e^{+ik_1^+\tilde{k}_2^+}+e^{-ik_1^+\tilde{k}_2^-}+e^{+ik_1^+(\tilde{k}_2^+-\tilde{k}_2^-)}\big) \\
&\Psi(-q_e^-,-q_f^+,+k_2^+,-k_2^-) = 4\cos(\tfrac{1}{2}k_2^+\tilde{k}_2^-)\big(3+\cos(k_2^+\tilde{q}_f^+)+ \\
&\qquad\qquad \cos(k_2^-\tilde{q}_f^+)+\cos((k_2^+-k_2^-)\tilde{q}_f^+)\big) \\
&\Psi(-q_e^-,-q_f^+,+k_2^-,-k_2^+) = \Psi(-q_e^-,-q_f^+,-k_2^+,+k_2^-).
\end{aligned}
$$

Inserting this into eq. (7.34) we find exactly the same result as eq. (7.9), as obtained by the TO procedure in section 7.1.2.

7.3.3 The Diagrammatic Fish

To demonstrate that our diagrammatic rules also work in a non-tadpolic context, we recalculate the t-channel four-point one-loop fish graph evaluated in section 7.1.3. As above we restrict ourselves to the t-channel.

Using the same notation as in fig. 7.3, we fix the external on-shell momenta as above: $\sigma_e = -1, \sigma_f = +1, \sigma_g = -1, \sigma_h = +1$. 4-momentum conservation yields

$$\vec{k}_3 = -\vec{k}_2 - \vec{q}_e - \vec{q}_f, \qquad k_3^0 = -k_2^0 + \omega_e - \omega_f \quad . \tag{7.35}$$

$\Gamma^{(4,1)}$ is then given as

$$\begin{aligned}
\Gamma^{(4,1)} &= \frac{g^2}{(4!)^2}\int\frac{d^4k_2}{(2\pi)^4}\sum_{\sigma_2^w,\sigma_2^v,\sigma_3^w,\sigma_3^v}^{+1,-1}\frac{i}{2\omega_2}\frac{i}{2\omega_3} \\
&\quad \Psi^w(-q_e^-,-q_f^+,-k_2^{\sigma_2^w},-k_3^{\sigma_3^w})\Psi^v(-q_g^-,-q_h^+,+k_2^{-\sigma_2^v},+k_3^{-\sigma_3^v}) \\
&\quad \frac{\delta_{\sigma_2^w,-\sigma_2^v}}{\sigma_2^w k_2^0-\omega_2+i\varepsilon}\;\frac{\delta_{\sigma_3^w,-\sigma_3^v}}{-\sigma_3^w(k_2^0+\omega_e-\omega_f)-\omega_3+i\varepsilon} \\
&= \frac{g^2}{(4!)^2}\int\frac{d^4k_2}{(2\pi)^4}\sum_{\sigma_2^w,\sigma_3^w}^{+1,-1}\frac{1}{2\omega_2 2\omega_3} \\
&\quad \Psi^w(-q_e^-,-q_f^+,-k_2^{\sigma_2^w},-k_3^{\sigma_3^w})\Psi^v(-q_g^-,-q_h^+,+k_2^{+\sigma_2^w},+k_3^{+\sigma_3^w}) \\
&\quad \frac{1}{\sigma_2^w k_2^0-\omega_2+i\varepsilon}\;\frac{1}{\sigma_3^w k_2^0+\sigma_3^w\omega_e-\sigma_3^w\omega_f+\omega_3-i\varepsilon} \\
&= \frac{g^2}{(4!)^2}\int\frac{d^4k_2}{(2\pi)^4}\frac{1}{2\omega_2}\frac{1}{2\omega_3} \\
&\quad \Big(\frac{\Psi^w(-q_e^-,-q_f^+,-k_2^+,-k_3^+)}{k_2^0-\omega_2+i\varepsilon}\,\frac{\Psi^v(-q_g^-,-q_h^+,+k_2^+,+k_3^+)}{k_2^0+\omega_e-\omega_f+\omega_3-i\varepsilon} \\
&+ \frac{\Psi^w(-q_e^-,-q_e^+,-k_2^+,-k_3^-)}{+k_2^0-\omega_2+i\varepsilon}\,\frac{\Psi^v(-q_g^-,-q_h^+,+k_2^+,+k_3^-)}{-k_2^0-\omega_e+\omega_f+\omega_3-i\varepsilon} \\
&+ \frac{\Psi^w(-q_e^-,-q_f^+,-k_2^-,-k_3^+)}{-k_2^0-\omega_2+i\varepsilon}\,\frac{\Psi^v(-q_g^-,-q_h^+,+k_2^-,+k_3^+)}{+k_2^0+\omega_e-\omega_f+\omega_3-i\varepsilon}
\end{aligned}$$

$$+ \quad \frac{\Psi^w(-q_e^-,-q_f^+,-k_2^-,-k_3^-)}{-k_2^0-\omega_2+i\varepsilon}\frac{\Psi^v(-q_g^-,-q_h^+,+k_2^-,+k_3^-)}{-k_2^0-\omega_e+\omega_f+\omega_3-i\varepsilon}\Bigg). \quad (7.36)$$

Inspecting the complex k_2^0 plane of the four terms we see that the poles of the second and the third term lie on the same half-plane and hence yield vanishing integrals. We thus find (for shortness we retain k_3, ω_3, but of course eqs. (7.35) still apply)

$$\begin{aligned}\Gamma^{(4,1)} &= \frac{-ig^2}{(4!)^2}\int\frac{d^3k_2}{(2\pi)^3}\frac{1}{2\omega_2}\frac{1}{2\omega_3} \qquad (7.37)\\ &\Bigg(\frac{\Psi^w(-q_e^-,-q_f^+,-k_2^+,-k_3^+)\Psi^v(-q_g^-,-q_h^+,+k_2^+,+k_3^+)}{\omega_2+\omega_3+\omega_e-\omega_f-i\varepsilon}\\ &+\frac{\Psi^w(-q_e^-,-q_f^+,-k_2^-,-k_3^-)\Psi^v(-q_g^-,-q_h^+,+k_2^-,+k_3^-)}{\omega_2+\omega_3-(\omega_e-\omega_f)-i\varepsilon}\Bigg),\end{aligned}$$

which is identical to eq. (7.11).

7.3.4 The Diagrammatic Mouse - where the UV/IR Mixing Should Occur

Confident in our new tools, we embark on calculating the two-point three-loop amplitude of "mouse-like morphology", figure 7.4.

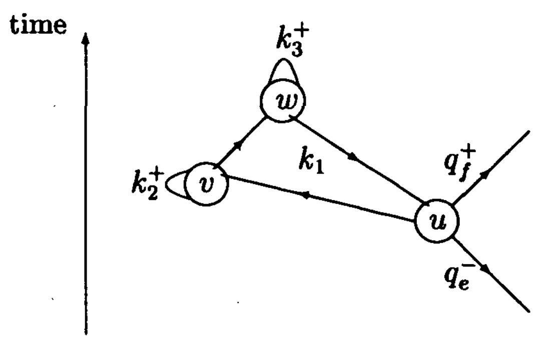


Figure 7.4: The macro-contribution uvw

This amplitude is of great interest since in usual noncommutative QFT it is the simplest graph that becomes undefined due to the notorious UV/IR-mixing problem: the two tadpoles inserted into the third each bring a $1/\tilde{k}_1^2$, introducing a non-integrable IR-singularity into the remaining, otherwise UV-finite, loop-integral – usually...

Beginning as in the previous sections (and skipping the steps that are now familiar), the amplitude of interest is written as

$$\begin{aligned}
&\Gamma^{(2,3)} = \\
&\frac{ig^3}{(4!)^3} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{d^4k_3}{(2\pi)^4} \frac{1}{(2\omega_1)^3} \frac{1}{2\omega_2} \frac{1}{2\omega_3} \\
&\sum_{\sigma_2,\sigma_3}^{+1,-1} \frac{1}{\sigma_2 k_2^0 - \omega_2 + i\varepsilon} \frac{1}{\sigma_3 k_3^0 - \omega_3 + i\varepsilon} \\
&\sum_{\sigma_u,\sigma_v,\sigma_w}^{+1,-1} \Phi^v(k_2^+; k_1^{\sigma_u}, -k_1^{\sigma_v}) \Phi^w(k_3^+; k_1^{\sigma_v}, -k_1^{\sigma_w}) \Psi^u(q_f^+, -q_f^+, k_1^{\sigma_w}, -k_1^{\sigma_u}) \\
&\frac{1}{\sigma_u k_1^0 - \omega_1 + i\varepsilon} \frac{1}{\sigma_v k_1^0 - \omega_1 + i\varepsilon} \frac{1}{\sigma_w k_1^0 - \omega_1 + i\varepsilon}.
\end{aligned} \tag{7.38}$$

Two of the eight possible combinations of $\sigma_u = \pm, \sigma_v = \pm, \sigma_w = \pm$ result in the coincidence of all three poles on the same half of the complex plane, and they thus vanish under k_1^0-integration. The remaining six summands yield

$$\begin{aligned}
&\Gamma^{(2,3)} = \\
&\frac{ig^3}{(4!)^3} \Big(-i \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{(2\omega_1)^5} \Big) \Big(-i \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{2\omega_2} \Big) \Big(-i \int \frac{d^3k_3}{(2\pi)^3} \frac{1}{2\omega_3} \Big) \\
&\Big(\Phi^v(k_2^+; k_1^+, -k_1^+) \Phi^w(k_3^+, k_1^+, -k_1^-) \Psi^u(q_f^+, -q_f^+, k_1^-, -k_1^+) \\
&+ \Phi^v(k_2^+; k_1^+, -k_1^-) \Phi^w(k_3^+, k_1^-, -k_1^+) \Psi^u(q_f^+, -q_f^+, k_1^+, -k_1^+) \\
&+ \Phi^v(k_2^+; k_1^-, -k_1^+) \Phi^w(k_3^+, k_1^+, -k_1^+) \Psi^u(q_f^+, -q_f^+, k_1^+, -k_1^-) \\
&+ \Phi^v(k_2^+; k_1^+, -k_1^-) \Phi^w(k_3^+, k_1^-, -k_1^-) \Psi^u(q_f^+, -q_f^+, k_1^-, -k_1^+) \\
&+ \Phi^v(k_2^+; k_1^-, -k_1^+) \Phi^w(k_3^+, k_1^+, -k_1^-) \Psi^u(q_f^+, -q_f^+, k_1^-, -k_1^-) \\
&+ \Phi^v(k_2^+; k_1^-, -k_1^-) \Phi^w(k_3^+, k_1^-, -k_1^+) \Psi^u(q_f^+, -q_f^+, k_1^+, -k_1^-) \Big).
\end{aligned} \tag{7.39}$$

Here the six terms correspond to the six possible macro-time orderings of the vertices: $uvw, wuv, vwu, uwv, vuw, wvu$, respectively.

7.4 No UV/IR Mixing in IPTOPT

The most interesting feature of IPTOPT is the apparent absence of the UV/IR-mixing problem. This can be seen in the amplitudes calculated so far by explicitly performing the loop integrations in the result of the previous section 7.3.4. No divergence will be fed down via the phases to the next loop.

7.4.1 Explicit Result for the UV/IR Divergence-free Mouse

To evaluate eq. (7.39) explicitly, we start by integrating over k_2 and k_3. These integrals yield, apart from a possible overall cosine in k_1, exactly the $\mathcal{I}$'s from section 7.2.1 and the previous chapter:

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Phi(k^+;k_1^\sigma,-k_1^{\sigma'}) \\
=\ &\int \frac{d^3k}{(2\pi)^3}\frac{2}{2\omega_k}\cos\left(\frac{1}{2}k_1^\sigma\tilde{k}_1^{\sigma'}\right)\left(3+e^{-ik^+\tilde{k}_1^\sigma}+e^{ik^+\tilde{k}_1^{\sigma'}}+e^{-ik^+(\tilde{k}_1^\sigma-\tilde{k}_1^{\sigma'})}\right) \\
=\ &2\cos\left(\frac{1}{2}k_1^\sigma\tilde{k}_1^{\sigma'}\right)\mathcal{I}(k_1^\sigma,-k_1^{\sigma'}).
\end{aligned} \tag{7.40}$$

Since these were already evaluated in eq. (7.18)–(7.24), determining the result for all but the last loop integration is a mere task of compilation. Using the same abbreviations as above we find

$$\begin{aligned}
&\Gamma^{(2,3)} = \\
&-\frac{g^3}{4(3!)^3}\frac{1}{(2\pi)^4}\int\frac{d^3k_1}{(2\pi)^3}\frac{1}{(2\omega_{k_1})^5}\left[1+\cos(k_1^+\tilde{k}_1^-)\right] \\
&\times\left[\frac{3}{2}\mathcal{Q}-\frac{2m}{|\tilde{k}_1^+|}K_1(m|\tilde{k}_1^+|)-\frac{2m}{|\tilde{k}_1^-|}K_1(m|\tilde{k}_1^-|)+\frac{m}{\omega_{k_1}|\Theta_{0i}|}K_1(2m\omega_{k_1}|\Theta_{0i}|)\right] \\
&\times\Big[\left(4+\cos(k_1^+\tilde{q}_f^+)+\cos(k_1^-\tilde{q}_f^+)\right) \\
&\times\left(\frac{3}{2}\mathcal{Q}-\frac{2m}{|\tilde{k}_1^+|}K_1(m|\tilde{k}_1^+|)-\frac{2m}{|\tilde{k}_1^-|}K_1(m|\tilde{k}_1^-|)+\frac{m}{\omega_{k_1}|\Theta_{0i}|}K_1(2m\omega_{k_1}|\Theta_{0i}|)\right) \\
&+\ \left(3+\cos(k_1^+\tilde{q}_f^+)+\cos(k_1^-\tilde{q}_f^+)+\cos((k_1^+-k_1^-)\tilde{q}_f^+)\right) \\
&\times\left(2\mathcal{Q}-\frac{m}{|\tilde{k}_1^+|}K_1(m|\tilde{k}_1^+|)-\frac{m}{|\tilde{k}_1^-|}K_1(m|\tilde{k}_1^-|)\right)\Big]
\end{aligned} \tag{7.41}$$

where $K_1(x)$ is the modified Bessel function. So far in noncommutative QFT this expression contained an IR divergence: poles in k^2 of 2nd order. This is not the case here as

$$\lim_{\vec{k}\to\vec{0}}(\tilde{k}_\mu^\pm)^2 = -m^2\Theta_{0i}^2 < 0. \tag{7.42}$$

This limit will be discussed in more detail in the next section.

Where does this first instance of the absence of the notorious UV/IR-mixing problem stem from? It is due to the appearance of on-shell 4-momenta in the noncommutative phases: since, because of the mass, the 0-component remains non-vanishing for all values of the three-momentum, no pole can appear.

7.4.2 Argument for the General Absence of UV/IR Mixing

Encouraged by the above explicit result we give an argument for the absence of this problem to all orders — for all $\Gamma^{(n,l)}$ — in (IPTO) perturbation theory in a more general way (although we refrain from writing "proof").

To arrive at this conclusion we remember the n-th order k-point Green functions given by the noncommutative Gell-Mann–Low formula

$$\begin{aligned} &G_n(x_1,\dots,x_k) \\ &= \frac{i^n}{n!}\int d^4z_1\dots d^4z_n \langle 0|T_I\phi(x_1)\dots\phi(x_k)\mathcal{L}_I(z_1)\dots\mathcal{L}_I(z_n)|0\rangle_{(0)}^{con}, \end{aligned} \tag{7.43}$$

where T_I denotes the interaction point time ordering and $\mathcal{L}_I(z)$ is the interaction part of the Lagrangian, $\frac{g}{4!}(\phi\star\phi\star\phi\star\phi)$ for noncommutative ϕ^4-theory.

Note that all fields occurring in eq. (7.43) are *free fields*, their Fourier transforms are on-shell quantities, the 0-component of the four-vector being $\omega(\vec{k})$:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3}\Big(\tilde{\phi}(\vec{k})e^{-ik^+x} + \tilde{\phi}^\dagger(\vec{k})e^{+ik^+x}\Big). \tag{7.44}$$

Evaluating the $\star$-product between these FT free fields hence produces phase factors containing *on-shell* momenta $k_\mu^\pm$ only. This remains true after integrating out some (or all) of the loop-momenta occurring later in the evaluation. At no point of the further calculations (evaluating TO, FT, amputation, ...) will this property be changed.

Why does this novel feature of IPTOPT prohibit the occurrence of the usual UV/IR problem? First note that for time-like (on-shell) four vectors k^μ we find $\tilde{k}^\mu$ to be space-like

$$\Theta_{\mu\nu}k^\mu k^\nu = 0 = (k^\mu\Theta_{\mu\nu})k^\nu := \tilde{k}^\mu k_\mu \tag{7.45}$$

and hence

$$\tilde{k}^\mu\tilde{k}_\mu < 0 \quad \forall\tilde{k}^\mu \neq 0^\mu \quad , \quad \tilde{k}^\mu\tilde{k}_\mu = 0 \leftrightarrow \tilde{k}^\mu = 0^\mu. \tag{7.46}$$

The case $\tilde{k}^\mu = 0^\mu$ is only possible for massive theories eff $\Theta_{\mu\nu}$ is of less then full rank, which is excluded in IPTOPT since we demand $\Theta_{i0} \neq 0$: if $\Theta_{\mu\nu}$ were of less than full rank, one could always transform it into $\Theta'_{\mu\nu}$, with $\Theta'_{i0} = 0$, which we excluded by definition.

Hence we find

$$\tilde{k}^2 = \tilde{k}^\mu\tilde{k}_\mu < 0 \quad \forall\vec{k}. \tag{7.47}$$

As the usual (i.e. the one found in the literature) UV/IR problem always occurs in the form of a $1/\tilde{k}^2$ pole, which, for off-shell k^μ and $\tilde{k}^\mu$, introduces a possible new singularity at 0, we see that IPTOPT is free from this (type of) problem: zero is never reached by an on-shell $\tilde{k}^2$.

The mathematical reason for $\tilde{k}^2$ being *not* off-shell is possibly the following: We have no time ordering θ-function for the nano-configuration of fields at a vertex. Thus, there is no parameter t from the integral form

$$\theta(x^0 - y^0) = \lim_{\delta\to 0}\frac{i}{2\pi}\int_{-\infty}^{\infty} dt\frac{e^{-it(x^0-y^0)}}{t+i\delta} \; . \tag{7.48}$$

Since it is exactly this parameter t which would become the zeroth (off-shell) component of the internal loop momentum, the absence of t keeps the loop momentum on-shell.

It is at this point that our argument degrades from being a proof, since it excludes the appearance of this particular form of mixing only. But in what other guises it still has to be excluded we are not able to discuss yet.

7.4.3 A Short Note on *PT*

As a short side-remark we would like to draw your attention to the behaviour of the amplitudes calculated above under P and/or T acting on the external momenta:

$$P: \quad \vec{q} \to -\vec{q}, \qquad T: \sigma \to -\sigma. \tag{7.49}$$

Hence we find that

$$P(q^{\pm}) = -q^{\mp}, \qquad T(q^{\pm}) = q^{\mp}, \tag{7.50}$$

which do not leave the amplitudes calculated above invariant when only one of P, T acts on them. However, under the combined action of PT:

$$PT(q^{\pm}) = P(q^{\mp}) = -q^{\pm}, \tag{7.51}$$

the amplitudes remain unchanged, since the external momenta occur in cosine only.

Invariance under PT, however, is a direct consequence of the unitarity of the S-matrix and the existence of free states; see [69] and references therein.

7.5 Outlook

The first steps were taken in the program of IPTOPT: Feynman rules were stated and demonstrated to yield the same results as the TO amplitude. In a sense, IPTOPT developed into interaction-point diagrammatics. Although note must be taken that these FR are rather conjectured than truly derived, since (Minkowskian) canonical instead of (Euclidean) PI quantisation was employed. A more general and rigorous method for obtaining them has recently been found in [68].

Also a strong motivation for further work utilising this approach was discovered: the possibility of the general absence of the UV/IR problem. Although a strong argument in favour of this feature was given, a true proof is still missing and certainly highly desirable. In principle two routes to this end are imaginable: either continuing in IPTOPT, investigating eq. (6.27) for the possibility of an inductive proof; or by making use of the diagrammatics proposed in this work. The second approach could also yield important insights into how to pursue the great question of renormalizability and renormalization of noncommutative QFT.

Further work may deepen our understanding of the intricate connections between nano-causality, unitarity, UV/IR mixing (i.e. its absence), CPT invariance and renormalization. Moreover, possible phenomenological implications of IPTOPT will be of great interest [70]. Anyway, with noncommutative

QFT a tool to a better understanding of commutative QFT is available, illustrating by similarities and differences the fundamental features of the two sets of theories.

Die überschwenglich guten Götter sind's
die das gegeben haben.

Hugo von Hofmannsthal

Bibliography

[1] J. M. Grimstrup, H. Grosse, L. Popp, V. Putz, M. Schweda, M. Wickenhauser and R. Wulkenhaar, arXiv:hep-th/0202093.

[2] F. Aigner, M. Hillbrand, J. Knapp, G. Milovanovic, V. Putz, R. Schofbeck and M. Schweda, "Technical remarks and comments on the UV/IR-mixing problem of a noncommutative scalar quantum field theory," arXiv:hep-th/0302038.

[3] J. M. Grimstrup, B. Kloibock, L. Popp, V. Putz, M. Schweda and M. Wickenhauser, "The energy-momentum tensor in noncommutative gauge field models," arXiv:hep-th/0210288, accepted for publication by Int. J. Mod. Phys. A.

[4] V. Putz and R. Wulkenhaar, "Seiberg-Witten map for noncommutative super Yang-Mills theory," Int. J. Mod. Phys. A **18** (2003) 3325 [arXiv:hep-th/0205094].

[5] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda and R. Wulkenhaar, "Space/time noncommutative field theories and causality," Eur. Phys. J. C **29** (2003) 133 [arXiv:hep-th/0209253].

[6] P. Fischer and V. Putz, "No UV / IR mixing in unitary space-time noncommutative field theory," arXiv:hep-th/0306099, accepted for publication by Eur. Phys. J. C.

[7] S. Doplicher, K. Fredenhagen and J. E. Roberts, Phys. Lett. B **331** (1994) 39.

[8] S. Doplicher, K. Fredenhagen and J. E. Roberts, "The Quantum structure of space-time at the Planck scale and quantum fields," Commun. Math. Phys. **172** (1995) 187 [arXiv:hep-th/0303037].

[9] A. Connes, "Noncommutative Geometry,"Academic Press, San Diego (1994).

[10] A. Connes, "A short survey of noncommutative geometry," J. Math. Phys. **41** (2000) 3832 [arXiv:hep-th/0003006].

[11] H. S. Snyder, "Quantized Space-Time," Phys. Rev. **71** (1947) 38.

[12] A. Connes, "Noncommutative Geometry And Reality," J. Math. Phys. **36** (1995) 6194.

[13] A. Connes, "Gravity coupled with matter and the foundation of non-commutative geometry," Commun. Math. Phys. **182** (1996) 155 [arXiv:hep-th/9603053].

[14] L. Carminati, B. Iochum and T. Schucker, "Noncommutative Yang-Mills and noncommutative relativity: A bridge over trouble water," Eur. Phys. J. C **8** (1999) 697 [arXiv:hep-th/9706105].

[15] A. H. Chamseddine and A. Connes, "The spectral action principle," Commun. Math. Phys. **186** (1997) 731 [arXiv:hep-th/9606001].

[16] T. Filk, "Divergencies In A Field Theory On Quantum Space," Phys. Lett. B **376** (1996) 53.

[17] A. Micu and M. M. Sheikh Jabbari, "Noncommutative phi**4 theory at two loops," JHEP **0101** (2001) 025 [arXiv:hep-th/0008057].

[18] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative Lorentz symmetry and the origin of the Seiberg-Witten map," Eur. Phys. J. C **24** (2002) 165 [arXiv:hep-th/0108045].

[19] J. M. Gracia-Bondia and J. C. Varilly, "Algebras Of Distributions Suitable For Phase Space Quantum Mechanics. 1," J. Math. Phys. **29** (1988) 869.

[20] R. Wulkenhaar, "Quantum field theories on noncommutative $\mathbb{R}^4$ versus θ-expanded quantum field theories," arXiv:hep-th/0206018.

[21] S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[22] M. Van Raamsdonk and N. Seiberg, "Comments on noncommutative perturbative dynamics," JHEP **0003** (2000) 035 [arXiv:hep-th/0002186].

[23] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," JHEP **0103** (2001) 001 [arXiv:hep-th/0008090].

[24] L. Griguolo and M. Pietroni, "Wilsonian renormalization group and the non-commutative IR/UV connection," JHEP **0105** (2001) 032 [arXiv:hep-th/0104217].

[25] A. Bichl, J. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, "Renormalization of the noncommutative photon selfenergy to all orders via Seiberg-Witten map," JHEP **0106** (2001) 013 [arXiv:hep-th/0104097].

[26] I. Frühwirth, J. M. Grimstrup, Z. Morsli, L. Popp, M. Schweda, "The role of the field redefinition in noncommutative Maxwell theory," arXiv:hep-th/0202092.

[27] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda and R. Wulkenhaar, "Perturbative analysis of the Seiberg-Witten map," arXiv:hep-th/0102044.

[28] F. Bastianelli, "BRST Symmetry From A Change Of Variables And The Gauged WZNW Models," Nucl. Phys. B **361** (1991) 555.

[29] J. Alfaro and P. H. Damgaard, "BRST symmetry of field redefinitions," Annals Phys. **220** (1992) 188.

[30] H. Grosse, T. Krajewski and R. Wulkenhaar, "Renormalization of noncommutative Yang-Mills theories: A simple example," arXiv:hep-th/0001182.

[31] A. Boresch, S. Emery, O. Moritsch, M. Schweda, T. Sommer, H. Zerrouki, "Applications of Noncovariant Gauges in the Algebraic Renormalization Procedure," *Singapore, Singapore: World Scientific (1998) 150 p.*

[32] C. Becchi, A. Rouet and R. Stora, "Renormalization Of Gauge Theories," Annals Phys. **98** (1976) 287.

[33] C.V. Tyutin, "Gauge Invariance in Field Theory and in Statistical Mechanics," preprint P.N. Lebedev Physical Inst. USSR Acad. Sci., FIAN # 39, (1975), unpublished.

[34] M. Hayakawa, "Perturbative analysis on infrared and ultraviolet aspects of noncommutative QED on R**4," arXiv:hep-th/9912167.

[35] A. Matusis, L. Susskind and N. Toumbas, "The IR/UV connection in the non-commutative gauge theories," JHEP **0012** (2000) 002 [arXiv:hep-th/0002075].

[36] M. Van Raamsdonk, "The meaning of infrared singularities in non-commutative gauge theories," JHEP **0111** (2001) 006 [arXiv:hep-th/0110093].

[37] M. Kreuzer, A. Rebhan and H. Schulz, 'Resummed Loop Expansion For High Temperature QCD," Phys. Lett. B **244** (1990) 58.

[38] J. Polchinski, "Renormalization And Effective Lagrangians," Nucl. Phys. B **231** (1984) 269.

[39] A. Gerhold, J. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, "The energy-momentum tensor on noncommutative spaces: Some pedagogical comments," arXiv:hep-th/0012112.

[40] A. Iorio and T. Sykora, "On the space-time symmetries of non-commutative gauge theories," Int. J. Mod. Phys. A **17** (2002) 2369 [arXiv:hep-th/0111049].

[41] S. M. Carroll, J. A. Harvey, V. A. Kostelecky, C. D. Lane and T. Okamoto, "Noncommutative field theory and Lorentz violation," Phys. Rev. Lett. **87** (2001) 141601 [arXiv:hep-th/0105082].

[42]

[42] C. G. Callan, S. R. Coleman and R. Jackiw, "A New Improved Energy - Momentum Tensor," Annals Phys. **59** (1970) 42.

[43] S. R. Coleman and R. Jackiw, "Why Dilatation Generators Do Not Generate Dilatations?," Annals Phys. **67** (1971) 552.

[44] M. Abou-Zeid and H. Dorn, "Comments on the energy-momentum tensor in non-commutative field theories," Phys. Lett. B **514** (2001) 183 [arXiv:hep-th/0104244].

[45] R. Jackiw, "Gauge Covariant Conformal Transformations," Phys. Rev. Lett. **41** (1978) 1635.

[46] A. A. Bichl *et al.*, "Non-commutative U(1) super-Yang-Mills theory: Perturbative self-energy corrections," arXiv:hep-th/0203141.

[47] N. Seiberg and E. Witten, "String theory and noncommutative geometry," JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[48] S. I. Kruglov, "Maxwell's theory on non-commutative spaces and quaternions," Annales Fond. Broglie **27** (2002) 343 [arXiv:hep-th/0110059].

[49] J. D. Jackson, "klassische Elektrodynamik", Walter de Gruyter - Berlin - New York 1981

[50] J. Madore, S. Schraml, P. Schupp and J. Wess, "Gauge theory on noncommutative spaces," Eur. Phys. J. C **16** (2000) 161 [arXiv:hep-th/0001203].

[51] B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, "Construction of non-Abelian gauge theories on noncommutative spaces," Eur. Phys. J. C **21** (2001) 383 [arXiv:hep-th/0104153].

[52] I. Chepelev and R. Roiban, "Convergence theorem for non-commutative Feynman graphs and renormalization," JHEP **0103** (2001) 001 [arXiv:hep-th/0008090].

[53] D. Colladay and V. A. Kostelecky, "Lorentz-violating extension of the standard model," Phys. Rev. D **58** (1998) 116002 [hep-ph/9809521].

[54] O. Piguet, "Supersymmetry, supercurrent, and scale invariance," arXiv:hep-th/9611003.

[55] W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, "The $Z \to \gamma\gamma$, gg decays in the noncommutative standard model," arXiv:hep-ph/0202121.

[56] O. Piguet and K. Sibold, "Renormalized Supersymmetry. The Perturbation Theory Of $N = 1$ Supersymmetric Theories In Flat Space-Time," *Boston, USA: Birkhaeuser (1986) 346 P. (Progress In Physics, 12).*

[57] J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda and R. Wulkenhaar, "Noncommutative spin-1/2 representations," arXiv:hep-th/0110231.

[58] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, "On the unitarity problem in space/time noncommutative theories," Phys. Lett. B **533** (2002) 178 [arXiv:hep-th/0201222].

[59] J. Gomis and T. Mehen, "Space-time noncommutative field theories and unitarity," Nucl. Phys. B **591** (2000) 265 [arXiv:hep-th/0005129].

[60] K. Osterwalder and R. Schrader, "Axioms For Euclidean Green's Functions," Commun. Math. Phys. **31** (1973) 83.

[61] S. Doplicher, K. Fredenhagen and J. E. Roberts, "The Quantum Structure Of Space-Time At The Planck Scale And Quantum Fields," Commun. Math. Phys. **172** (1995) 187.

[62] Y. Liao and K. Sibold, "Time-ordered perturbation theory on noncommutative spacetime: Basic rules," arXiv:hep-th/0205269.

[63] Y. Liao and K. Sibold, "Time-ordered perturbation theory on noncommutative spacetime. II. Unitarity," arXiv:hep-th/0206011.

[64] S. Weinberg, "The Quantum Theory of Fields", Cambridge University Press (1996).

[65] P. A. M. Dirac, "Lectures on quantum mechanics," Yeshiva University, New York (1964).

[66] See e.g. eq. (9.72) of G. Sterman, "An Introduction to Quantum Field Theory," Cambridge Univ. Press, 1993.

[67] I. S. Gradstein, I. M. Ryshik, "Tables," Verlag Harri Deutsch (1981).

[68] S. Denk and M. Schweda, "Time ordered perturbation theory for non-local interactions; applications to NCQFT," TUW-03-19 [arXiv:hep-th/0306101].

[69] L. Álvarez-Gaumé and M. A. Vázquez-Mozo, "General properties of non-commutative field theories," CERN-TH/2003-100, HU-EP-03/19, [arXiv:hep-th/0305093].

[70] Y. Liao and C. Dehne, "Some phenomenological consequences of the time-ordered perturbation theory of QED on non-commutative space-time," [arXiv:hep-ph/0211425].

Curriculum Vitae

Dipl.-Ing. Volkmar Putz

Persönliche Angaben

> **Geburtsdatum:** 23. Oktober 1976
> **Geburtsort:** Bad Ischl, Österreich
> **Sprachen:** Deutsch (Muttersprache), Englisch (fließend), Französisch (lesend)
> **Eltern:** Ingrid Putz, geb. Mayrhofer, AHS-Lehrerin
> Helmar Putz, Innenausbaumeister

Volksschule, Bad Goisern, 1983 - 1987

Realistisches Gymnasium, Bad Ischl, 1987 - 1995
Matura mit ausgezeichnetem Erfolg

Präsenzdienst, Schwarzenbergkaserne Salzburg, 1995 - 1996

Technische Universität Wien, 1996-2001
Diplom-Ingenieur der technischen Physik
Diplom (mit Auszeichnung) in theoretischer Physik
Betreuer: Univ.Prof. Manfred Schweda
Diplomarbeit: *Noncommutative Supersymmetric Yang-Mills Theory*

Technische Universität Wien, Max-Planck-Institut für Mathematik in den Wissenschaften Leipzig, 2001-2003
Doktoratsstudium in technischer Physik
Betreuer: Univ.Prof. Manfred Schweda
Doktorarbeit: *Symmetries and Renormalization of Noncommutative Field Theories*

Technische Universität Wien, 1998-2003
Unterrichtstätigkeit am Institut für theoretische Physik