



*U. Schweda*

DISSERTATION

# PERTURBATIVE ASPECTS OF NON-LOCAL AND NON-COMMUTATIVE QUANTUM FIELD THEORIES

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Univ.Prof. Dr. Manfred Schweda  
am Institut für Theoretische Physik (E 136)  
der Technischen Universität Wien

eingereicht an der Technischen Universität Wien  
Fakultät für Physik

von

**Dipl.-Ing. Stefan Denk**

Matr. Nr: 9555382  
Leystraße 104/5/28  
A-1200 Wien  
Austria, Europe

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*Stefan Denk*  
*Manfred Schweda*

## Kurzfassung

Das Verständnis fundamentaler Wechselwirkung stützt sich auf das Konzept der Quantenfeldtheorie in Kombination mit Eichfeldern. Sie stellt im Wesentlichen eine Vereinigung aus Spezieller Relativitätstheorie und Quantenmechanik dar und beschreibt so Elementarteilchenprozesse mit erstaunlicher Genauigkeit. Phänomene in Zusammenhang mit der Schwerkraft werden dabei jedoch nicht in zufriedenstellender Art und Weise erfasst. Dies würde eine Vereinheitlichung von Quantentheorie und Allgemeiner Relativitätstheorie, welche eine elegante Beschreibung der Schwerkraft darstellt, erfordern. Ein zentrales Problem stellt folgender Umstand dar: In der Quantenmechanik kann man ein Teilchen im Prinzip innerhalb eines beliebig kleinen Raumes lokalisieren, vorausgesetzt man wendet dafür entsprechend hohe Energien auf. Lässt man nun die Erkenntnisse der Relativitätstheorie einfließen, würde dies bei entsprechender Lokalisierung bedeuten, dass das Teilchen aufgrund der hohen Energiedichte vom Horizont eines schwarzen Loches umgeben wäre. Im Rahmen bisheriger Erkenntnisse würde dies den Erfahrungen widersprechen. So versucht man unter anderem, die beliebige Lokalisierung von Teilchen aus der Quantentheorie auszuschließen. Dies lässt sich beispielsweise dadurch realisieren, dass man die Raum-Zeit-Koordinaten durch hermitesche Operatoren, die nun nicht mehr kommutieren, ersetzt und so zu einer nichtkommutativen Geometrie gelangt. Die entsprechenden nichtkommutativen Quantenfeldtheorien sollen in dieser Doktorarbeit diskutiert werden.

Eine mögliche Beschreibungsform benützt die sogenannte Weyl-Quantisierung. Dabei ist wesentlich, dass in der nichtkommutativen Quantenfeldtheorie lokale Feldprodukte der entsprechenden lokalen Theorie durch nichtlokale Moyal-Weyl-Produkte, die durch einen Deformierungsparameter der Dimension einer Fläche charakterisiert werden, ersetzt werden. Die dadurch eingeführte Nichtlokalität ergibt wiederum schwerwiegende Probleme, beispielsweise mit der Renormierung oder der Unitarität. Letztere Eigenschaft ist verletzt, sobald Nichtlokalitäten in der Zeit auftreten, und sollte durch den Aufbau der Theorie mittels zeitgeordneter Störungstheorie jedoch gewährleistet sein. So werden die Feynman-Regeln im Rahmen der zeitgeordneten Störungstheorie von ihrem bisherigen Geltungsbereich, den lokalen Wechselwirkungen, auf einen sehr allgemeinen Typ nichtlokaler Wechselwirkungen, der in dieser Arbeit festgelegt wird, erweitert. Im Gegensatz zu bisherigen Ergebnissen wird hier der Zusammenhang zu den Feynman-Regeln lokaler Wechselwirkungen sehr deutlich. Im Ortsraum ist neben einer etwas komplizierteren Vorschrift für Vertizes der Propagator der lokalen Theorie durch den hier eingeführten *Contractor* zu ersetzen, welcher eine verallgemeinerte Zeitordnung berücksichtigt. Die Diagrammatik im Impulsraum wird wesentlich einfacher. Bemerkenswert ist hierbei, dass die Funktionen, die den Vertizes zuzuordnen sind, nicht mehr von den vollen off-shell-Impulsen, sondern nur mehr von deren Raum-Komponenten abhängen. Auf sehr allgemeiner Basis werden weiters verschiedene Eigenschaften wie Unitarität oder das Verhalten unter Lorentz-Transformationen, Zeitumkehr oder Parität untersucht. Dabei stellt vor allem die Lorentz-Invarianz ein Problem dar, das für nichtlokale Wechselwirkungen auch im Rahmen der hier vorgenommenen Untersuchungen als kaum lösbar erscheint.

Auf die Problematik der Renormierung wurde im Wesentlichen auf zwei verschiedene Arten eingegangen: Zum einen wurde durch verschmierte Feldoperatoren eine nichtlokale Theorie konstruiert, welche frei von ultravioletten Divergenzen ist. Zum anderen wurde im Falle nichtkommutativer Feldtheorien der Einfluss eines zusätzlichen Wechselwirkungsterms, der

einem harmonischen Oszillator entspricht, untersucht. Dadurch verzichtet man auf Translationsinvarianz. Einfache Rechnungen zeigen dann, dass das UV/IR-Mischungsproblem, welches ein schwerwiegendes Hindernis für Renormierungsprozeduren darstellt, dadurch zumindest gelindert wird, was in Übereinstimmung mit bisherigen Ergebnissen steht.

## Abstract

The understanding of fundamental interactions is based on the concept of quantum field theory in combination with gauge fields. It represents a unification of the theory of special relativity and quantum mechanics. Elementary particle processes are described with amazing precision. But phenomena related to gravity are not included in a satisfactory way. This would require the unification of quantum theory and the general theory of relativity. Thereby, one faces the following problem: In quantum mechanics, particles can in principal be localised within arbitrarily small spaces, provided that one spends sufficiently high energies. Taking the general theory of relativity into account, particles could create black hole horizons for the corresponding localisation. This would contradict our knowledge and experiences. Thus, one tries for example to exclude particle states with arbitrary localisation from quantum theory. This can be realised by replacing the continuous space-time coordinates by non-commuting, Hermitian operators, which leads to non-commutative geometry. The corresponding non-commutative quantum field theories are discussed in this thesis.

One possible description applies the so-called Weyl quantisation. This means that in non-commutative quantum field theory, the local field products of the corresponding local theory are replaced by non-local Moyal-Weyl products, which are characterised by a deformation parameter of the dimension of an area. Thus, one introduces non-localities, which cause serious problems to renormalisation and unitarity, for example. The latter property is violated as soon as there are non-localities in time. Unitarity can now be preserved by strictly constructing the theory on the basis of time-ordered perturbation theory. Within this approach, the validity of the usual Feynman rules is therefore extended to a large class of non-local interactions to be defined in this thesis. The connection to the usual Feynman rules of local quantum field theory becomes clear in contrast to existing results. In coordinate space, the propagator is replaced by the *contractor*, which embeds a more general type of time-ordering. The prescription for vertices becomes also more complicated. The diagrammatic techniques in momentum space are considerably simpler. It is remarkable that the functions associated with vertices do not depend on the full off-shell momenta, but simply on their space components. Properties like unitarity, or the behaviour under Lorentz transformations, time reversal, and parity are investigated on a very general basis. It turns out that especially the requirement for Lorentz invariance represents a serious problem, whose solution seems to be nearly impossible within the framework of the investigations carried out here.

The problem of renormalisation was basically studied in two ways: First, a theory free of ultraviolet divergences was constructed from smeared field operators. Secondly, the influence of an additional, harmonic oscillator term in the action of a non-commutative quantum field theory was investigated. Thus, one drops translation invariance. Simple calculations show that the UV/IR mixing problem, which is a serious obstacle to the renormalisation program, seems to be cured, which is in agreement with existing results.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Non-local Feynman rules</b>	<b>4</b>
2.1	Some basics . . . . .	4
2.2	Non-local interactions . . . . .	9
2.3	Feynman rules for non-local interactions . . . . .	11
2.3.1	Coordinate space rules . . . . .	14
2.3.2	Calculation of the contractor . . . . .	15
2.3.3	Energy-momentum conservation . . . . .	16
2.3.4	Momentum space rules . . . . .	18
2.4	Further contractors . . . . .	20
2.4.1	Contractors of derivative fields . . . . .	20
2.4.2	Contractors of Dirac spinors . . . . .	21
2.4.3	The photon contractor in Coulomb gauge . . . . .	23
2.5	Applications . . . . .	25
2.5.1	Non-commutative interactions . . . . .	25
2.5.2	Non-local interactions of Gaussian type . . . . .	28
2.5.3	Simplified non-local interactions of Gaussian type . . . . .	29
2.6	Summary . . . . .	29
<b>3</b>	<b>Towards UV-finite QFTs</b>	<b>31</b>
3.1	The oscillator representation . . . . .	31
3.2	Smeared field operators . . . . .	33
3.3	Perturbative corrections and power counting . . . . .	38
3.3.1	Power counting for full damping . . . . .	39
3.3.2	Power counting for partial damping . . . . .	42
3.3.3	Explicit calculations in the Euclidean case . . . . .	43
3.3.4	Explicit calculations in the Minkowski case . . . . .	45
3.4	Summary . . . . .	46
<b>4</b>	<b>Special features of non-local perturbation theory</b>	<b>48</b>
4.1	Deformation of the free Hamiltonian . . . . .	49
4.2	Illuminating apparent discrepancies . . . . .	51
4.3	Diagrammatic specialties of non-local interactions . . . . .	55
4.3.1	Amputation of external legs . . . . .	55
4.3.2	Physical $S$ -matrix elements . . . . .	56

4.3.3	The self-energy matrix . . . . .	58
4.4	Symmetries . . . . .	59
4.4.1	Hermiticity . . . . .	59
4.4.2	Unitarity . . . . .	61
4.4.3	Lorentz invariance . . . . .	62
4.4.4	$\mathcal{P}, \mathcal{T}$ . . . . .	65
4.5	A perturbative comparison . . . . .	67
4.6	Summary . . . . .	70
<b>5</b>	<b>Adding a harmonic oscillator</b>	<b>72</b>
5.1	The propagator . . . . .	73
5.1.1	Green functions and kernels . . . . .	73
5.1.2	The free Green function for the dual action . . . . .	74
5.2	Calculation of the tadpole graph . . . . .	75
	<b>Conclusion</b>	<b>81</b>
<b>A</b>	<b>Generalised Wick theorem</b>	<b>83</b>

# Chapter 1

## Introduction

In the last decade, non-local structures were intensively investigated in high energy physics (see [1, 2, 3], for a example, or the detailed reviews [4, 5]). The main focus lies on the construction of a consistent quantum field theory (QFT) on non-commutative (NC) spaces. Such spaces possibly provide a repair of the problems and inconsistencies which arise when combining QFT and the general theory of relativity at the Planck scale ( $\approx 10^{-33}$  cm). Even at the early days of quantum mechanics and QFT, continuous space-time and Lorentz symmetry was considered inappropriate to describe the small scale structure of the universe [6]. However, this discussion was rather based on the interpretation of quantum mechanics than on the failure of geometry at the Planck scale. In [7], this scale was already mentioned in connection with a universal length, but the main focus lied on the introduction of a new length scale of approximately  $10^{-13}$  cm in connection with Fermi's theory of the  $\beta$ -decay. A mathematical formulation of these ideas was first presented in [8], where non-commutative operators were introduced for the coordinates to provide a possible solution of the problems of infinities. The lack of a consistent renormalisation program was the main motivation for this approach. The inconsistencies arising from the combination of QFT and general relativity were then considered in [9], but without referring to non-commutativity. It was in [1], where a motivation for NC spaces was given. There, one dealt with a semi-classical approach to Einstein's general relativity equation, where the energy-momentum tensor corresponds to a localised state of scalar particles. One then admits "only those states whose associated energy-momentum tensor, taken as a source in Einstein's equation, does not generate closed trapped surfaces in the sense of Penrose". In other words, one excludes states which would generate black hole horizons. This suggests uncertainty relations between the coordinates, which are thus assumed to be Hermitian operators  $\hat{x}^\mu$ . The uncertainty relations are realised by the requirement that the coordinate operators  $\hat{x}^\mu$  do not commute anymore. The simplest example is given by the canonical commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} . \quad (1.1)$$

The anti-symmetric deformation parameter  $\theta^{\mu\nu}$  is assumed constant here and has dimension [length]<sup>2</sup>. There are also more complicated cases of non-commutativity treated in literature. First, one can assume the coordinate operators to form a Lie algebra. This was discussed in two approaches: fuzzy spheres [10], and  $\kappa$ -deformation [11, 12, 13]. Furthermore, one can also have quadratic commutation relations [14, 15], where the commutator equals a bilinear combination of coordinate operators.

We will restrict ourselves to the simplest case, the canonical one represented by eq. (1.1).

We will not directly deal with the coordinates as operators, but the Weyl quantisation procedure will be applied [16, 17]. This leads to a QFT on ordinary space-time, but local field products are replaced with the (Weyl-Moyal) star product, to be discussed in more detail in the following chapter. We will refer to this approach as NCQFT in this thesis. The introduction of the star product then generates non-local interactions. In order to carry out perturbative calculations for NCQFT, one can use a set of modified Feynman rules first formulated in [18] and elaborated in great detail in [19] for the one-loop and two-loop approximation of a scalar field model. The non-commutativity of coordinates now manifests a fundamental length scale represented by  $\theta^{\mu\nu}$ , which raised the hope that the usual ultraviolet (UV) divergences could be absent in the case of NCQFT. But as was shown in [18], there are still UV divergences in the so-called planar sector of the theory. Each Feynman diagram of NCQFT can be divided into two parts, its planar, and its non-planar one. The planar parts differ from their commutative counterparts only by a phase factor depending on external momenta. Therefore, the UV behaviour is not at all improved in the planar sector. In the non-planar sector, the phase factors oscillate with respect to the internal momenta rendering certain loop integrals finite. But the wave number (or frequency) is proportional to the external momenta. One therefore obtains the original infinities for the infrared (IR) region, where the external momenta vanish. This entanglement of the internal UV and external IR regions is spoiling the renormalisability of a theory and is known as the UV/IR mixing, which was first recognised in [20, 21, 22] for scalar particles. Similar effects are also known for gauge field models [23, 21, 24, 25].

At this point, one also has to stress that the corresponding deformation parameter  $\theta^{\mu\nu}$  characterising the non-commutativity of space-time must be restricted to the case  $\theta^{0i} = 0$ , when applying the simple Feynman rules [18]. This excludes non-locality in the time direction. The only place, where one encounters time derivatives is the kinetic term of the action and the usual formalism to get Feynman rules is applicable. The case  $\theta^{0i} \neq 0$  is more delicate due to the fact that non-localities are also present in the time direction. This is really an obstacle for the naive application of the perturbation theory à la Feynman, where simply the Feynman propagators of the quantum field models are associated with the Wick contractions. Due to the presence of infinitely many time derivatives, it is difficult to carry out the canonical quantisation in a strict sense [19]. Furthermore, unitarity is violated when applying these simple rules [26, 27]. Since unitarity can be shown quite easily within the framework of time-ordered perturbation theory <sup>1</sup> (TOPT) [29], we will adopt this approach and elaborate it for the case of non-localities also present in the time direction. In contrast to the investigations of other authors [30, 31, 28], a very general type of interactions able to embed many types of non-localities will be treated, and the corresponding Feynman rules will be simple compared to existing results. Most of this thesis concentrates on self-interacting, real, scalar particles. Thus, complications due to constraints are avoided. Emphasis is placed on the non-locality of the interaction, especially the one in the time direction.

This thesis is organised as follows: In chapter 2, a very general type of interactions is defined, and the corresponding Feynman rules are given in coordinate as well as momentum space. These rules are applied to NCQFT, non-local interactions of Gaussian type [32], and a simplified version of the latter ones. Besides a few extensions like to massive spin- $\frac{1}{2}$ -particles or the photon in Coulomb gauge, these results have already been published in

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<sup>1</sup>In [28], the term interaction point time-ordered perturbation theory (IPTOPT) was introduced to emphasise that time-ordering is solely done with respect to the time the interaction depends on. By inspection of the Dyson series, this is clear anyway. We will thus simply keep the abbreviation TOPT.



[33]. The simplified version of non-local interactions of Gaussian type is investigated in more detail in chapter 3, [34]. This approach is motivated by smeared field operators. Several explicit calculations could be carried out due to its simplicity confirming the UV-finiteness of these interactions. In chapter 4, various properties and features of the non-local interactions are investigated. First, we examine the validity of the assumption that the free theory is unaltered. The simple Feynman rules [18] mentioned above can be derived easily from a naive path integral approach (NPIA), where one directly starts with the Lagrangian version. The discrepancy between the NPIA and TOPT is also explained. These two topics are also treated in [35]. The remaining part of this chapter extends special diagrammatic techniques, examines several properties like unitarity and Lorentz invariance, and presents an interesting, new result for the comparison of TOPT and the NPIA. The last chapter addresses the UV/IR mixing problem. It is based on the approach given in [36, 37, 38, 39, 40], which assumes an additional harmonic oscillator term in the action. Thus, one hopes to cure the UV/IR mixing problem at the cost of translation invariance. We carry out calculations within the NPIA, and restrict ourselves to a single tadpole loop.

## Chapter 2

# Non-local Feynman rules

In this chapter, Feynman rules in coordinate as well as momentum space are derived for non-local interactions between scalar particles to be defined in more detail below. Time-ordered perturbation theory is the basis of these calculations. The results are extended to spin- $\frac{1}{2}$ -particles, and photons in Coulomb gauge. Applications to several, specialised types on non-localities are given at the end of the chapter. At the beginning, a few basics of QFT, especially TOPT are repeated to show that this framework in principal also holds for non-local interactions.

### 2.1 Some basics

First, some conventions on special relativity and units are introduced before we repeat a few basics of scattering theory and QFT. The units are chosen in such a way that the speed of light  $c = 1$  and Planck's constant  $\hbar = 2\pi$  ( $\hbar = 1$ ). The components of four-vectors are labelled from 0 to 3 and are written as

$$x = (x^1, x^2, x^3, x^0)^T \quad (2.1)$$

with 0 representing the time component. Bold symbols denote three-vectors,  $\mathbf{x} = (x^1, x^2, x^3)^T$ . The metric of special relativity is defined by

$$(x_1, x_2, x_3, x_0) \equiv (x^1, x^2, x^3, -x^0) . \quad (2.2)$$

Greek indices usually run from 0 to 3 and Latin indices from 1 to 3. The sum convention is assumed as usual.

The strategy concerning QFT followed here is essentially taken from [29]. Scattering theory will be repeated in detail to demonstrate the validity of the underlying formalism also for non-local interactions. The first thing we start with is the Hamiltonian  $H$ , the generator of time translations. We split it into two parts, the free one,  $H_0$  and the interacting one,  $V$ :

$$H = H_0 + V . \quad (2.3)$$

Here, the free part  $H_0$  is assumed to have the same spectrum as the full Hamiltonian  $H$ . Therefore, the masses appearing in  $H_0$  must represent the physical masses actually measured, which might not be the same as the bare masses. Any differences are then considered to be included in  $V$ . The eigenstates of  $H_0$  we refer to as the free states  $|\alpha\rangle$ :

$$H_0 |\alpha\rangle = E_\alpha |\alpha\rangle . \quad (2.4)$$

The quantum numbers abbreviated by  $\alpha$  represent a set of particle species  $n_i$ , three-momenta  $\mathbf{p}_i$  and spins or helicities  $\sigma_i$ :

$$\alpha = n_1, \mathbf{p}_1, \sigma_1; n_2, \mathbf{p}_2, \sigma_2 \dots \quad (2.5)$$

The energy of such a free state is given by

$$E_\alpha = \sqrt{m_{n_1}^2 + \mathbf{p}_1^2} + \sqrt{m_{n_2}^2 + \mathbf{p}_2^2} \dots \quad (2.6)$$

The vacuum state is written as  $|0\rangle$  and has energy 0. The wave functions are assumed to be already (anti-)symmetrised appropriately for the case that  $\alpha$  includes multiple identical particles. The normalisation is assumed to be

$$\begin{aligned} \langle \alpha | \alpha' \rangle &= \langle n_1, \mathbf{p}_1, \sigma_1; \dots | n'_1, \mathbf{p}'_1, \sigma'_1; \dots \rangle \\ &= \delta_{n_1 n'_1} \delta^3(\mathbf{p}_1 - \mathbf{p}'_1) \delta_{\sigma_1 \sigma'_1} \delta_{n_2 n'_2} \delta^3(\mathbf{p}_2 - \mathbf{p}'_2) \delta_{\sigma_2 \sigma'_2} \dots \pm \text{permutations} \\ &\equiv \delta(\alpha - \alpha') . \end{aligned} \quad (2.7)$$

' $\pm$ permutations' indicate the (anti-)symmetrisation with respect to identical particles. Sums over complete states are abbreviated as

$$\int d\alpha \dots \equiv \sum_{n_1 \sigma_1 n_2 \sigma_2 \dots} \int d^3 p_1 d^3 p_2 \dots \quad (2.8)$$

with the completeness relation (for the free states, for example) written as

$$|\Psi\rangle = \int d\alpha |\alpha\rangle \langle \alpha | \Psi \rangle . \quad (2.9)$$

The free states themselves are not sufficient to describe all physically interesting situations such as scattering experiments. However, they are quite useful even in the latter case. This is because at the beginning of a scattering experiment, the particles are far apart and the interaction is negligible. Therefore, this situation can be identified with a wave packet of free states. At the end of the scattering process, however, the evolved wave function is not a superposition of free particle states anymore such that the several particles are separated far apart. But again, the free particle states come into play, since the experiments are set up to check in which free particle state the system is found. The particles are then far apart and the interaction can be neglected. Summarising, there are two kinds of states, namely in- and out-states,  $|\alpha; \text{in}\rangle$  and  $|\alpha; \text{out}\rangle$ . Both are eigenstates of the full Hamiltonian  $H$  with energy  $E_\alpha$  as given by eq. (2.6). For sufficiently smooth wave packets  $g$ , one then has the condition for the out-states

$$\lim_{t \rightarrow \infty} \int d\alpha e^{-iE_\alpha t} g(\alpha) |\alpha; \text{out}\rangle = \lim_{t \rightarrow \infty} \int d\alpha e^{-iE_\alpha t} g(\alpha) |\alpha\rangle , \quad (2.10)$$

which can be written as

$$\lim_{t \rightarrow \infty} e^{-iHt} \int d\alpha g(\alpha) |\alpha; \text{out}\rangle = \lim_{t \rightarrow \infty} e^{-iH_0 t} \int d\alpha g(\alpha) |\alpha\rangle . \quad (2.11)$$

Similar relations hold for the in-states but with  $t \rightarrow -\infty$ . One can now invert the above relation and the corresponding one for in-states formally:

$$|\alpha; \text{in}\rangle = \Omega(-\infty) |\alpha\rangle , \quad (2.12)$$

$$|\alpha; \text{out}\rangle = \Omega(+\infty) |\alpha\rangle , \quad (2.13)$$

where

$$\Omega(t) \equiv e^{iHt} e^{-iH_0 t} . \quad (2.14)$$

However, it should be kept in mind that  $\Omega(\pm\infty)$  in eqs. (2.12) and (2.13) only makes sense when acting on a smooth superposition of energy eigenstates.

In scattering experiments, one is then interested in the amplitude  $S_{\beta\alpha}$  for the transition of an initial state with particle content  $\alpha$  into the final state with particle content  $\beta$ , which can be calculated by

$$S_{\beta\alpha} = \langle \beta; \text{out} | \alpha; \text{in} \rangle . \quad (2.15)$$

It turns out to be convenient to introduce the  $S$ -operator as follows:

$$S_{\beta\alpha} \equiv \langle \beta | S | \alpha \rangle . \quad (2.16)$$

Note that  $|\alpha\rangle$  and  $|\beta\rangle$  have been defined to be free particle states. The  $S$  operator is then given by

$$S = \Omega(+\infty)^\dagger \Omega(-\infty) = U(+\infty, -\infty) , \quad (2.17)$$

where eqs. (2.15), (2.12) and (2.13) were used and one defines

$$U(t, t') \equiv \Omega(t)^\dagger \Omega(t') = e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'} . \quad (2.18)$$

Before giving a perturbative expansion for  $U$ , it should be mentioned that the  $S$  operator naturally comes into play, when calculating  $S_{\beta\alpha}$  in the Dirac picture at  $t = \pm\infty$ . Here, it is a good point to give the definition of the different kinds of time dependence in the various pictures. So far, the state vectors  $|\Psi\rangle$  have been considered to be time independent. This means that we have been using the *Heisenberg picture* above. A natural kind of time dependence is given by the *Schrödinger picture* with the time dependence of states defined as follows:

$$|\Psi_S(t)\rangle \equiv e^{-iHt} |\Psi\rangle . \quad (2.19)$$

Since this is a unitary transformation, transition amplitudes can also be calculated at arbitrary time  $t$  in the Schrödinger picture  $\langle \Psi | \Phi \rangle = \langle \Psi_S(t) | \Phi_S(t) \rangle$ . The third picture of interest is the *Dirac* or *interaction picture*, giving the time dependence by

$$|\Psi_D(t)\rangle \equiv e^{iH_0 t} |\Psi_S(t)\rangle , \quad (2.20)$$

which again is a unitary transformation allowing transition amplitudes to be calculated at arbitrary time  $t$ ,  $\langle \Psi | \Phi \rangle = \langle \Psi_D(t) | \Phi_D(t) \rangle$ . It is easy to see that the operator  $U$  defined in eq. (2.18) describes the time dependence in the Dirac picture:

$$|\Psi_D(t)\rangle = U(t, t') |\Psi_D(t')\rangle , \quad (2.21)$$

which can be seen simply by using the definitions of the pictures. The matrix elements one is interested in in scattering theory can then also be calculated at arbitrary time, say  $t = +\infty$ , in this picture:

$$S_{\beta\alpha} = \langle (\beta; \text{out})_D(\infty) | (\alpha; \text{in})_D(\infty) \rangle = \langle (\beta; \text{out})_D(\infty) | U(\infty, -\infty) | (\alpha; \text{in})_D(-\infty) \rangle . \quad (2.22)$$

From eqs. (2.12) and (2.13), one then sees

$$|(\alpha; \text{in})_D(-\infty)\rangle = |\alpha\rangle , \quad (2.23)$$

$$|(\beta; \text{out})_D(\infty)\rangle = |\beta\rangle , \quad (2.24)$$

demonstrating the accordance between the definition of the  $S$  operator via the Dirac picture and the definition given in eq. (2.16).

A perturbative expansion of  $U(t, t')$  can be obtained by differentiating it with respect to  $t$  and solving the corresponding differential equation

$$i \frac{d}{dt} U(t, t') = V(t) U(t, t') \quad (2.25)$$

with

$$V(t) \equiv e^{iH_0 t} V e^{-iH_0 t} \quad (2.26)$$

and boundary condition  $U(t', t') = 1$ . The above differential equation is rewritten in terms of an integral equation by integrating it:

$$U(t, t') = 1 - i \int_{t'}^t d\tau V(\tau) U(\tau, t') . \quad (2.27)$$

This relation is then applied iteratively for  $U$  starting with  $U_0 = 1$  yielding the desired perturbative expansion. Setting  $t = +\infty$  and  $t' = -\infty$ , one obtains the well known *Dyson series*

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n T\{V(t_1) \dots V(t_n)\} \quad (2.28)$$

for the  $S$ -operator, where  $T$  is the time-ordering operator defined by

$$\begin{aligned} T\{O(t)\} &\equiv O(t) , \\ T\{O_1(t_1) \dots O_n(t_n)\} &\equiv \sum_{P \in S^n} \theta(t_{P_1} - t_{P_2}) \theta(t_{P_2} - t_{P_3}) \dots O_{P_1}(t_{P_1}) O_{P_2}(t_{P_2}) \dots \end{aligned}$$

Here,  $\theta(t)$  is the time-ordering step function  $\theta(t) = 1$  for  $t > 0$  and  $\theta(t) = 0$  for  $t < 0$  (not to be confused with the non-commutativity parameter  $\theta$  of NCQFT),  $S^n$  is the group of permutations of the integers  $\{1, \dots, n\}$ .  $P_i$  denotes the integer  $P(i)$ , the integer  $i$  is mapped to by the permutation  $P$  (see also [41] for more details on permutations).

Details of the structure of the interaction  $V$  have not been used so far. Of course, the convergence of the power series will depend on  $V$ , but we will not concern ourselves with this problem. The *Gell-Mann-Low* formula is another important relation of QFT. Again, its proof does not directly involve details of  $V$  (see [29]), and we will assume its validity here:

$$\begin{aligned} \langle 0 | T\{o_{1H}(x_1) \dots o_{rH}(x_r)\} | 0 \rangle &= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_m \times \\ &\quad \langle 0 | T\{o_1(x_1) \dots o_r(x_r) V(t_1) \dots V(t_m)\} | 0 \rangle . \end{aligned} \quad (2.29)$$

The subscript  $H$  indicates that the operators are given in the Heisenberg picture:

$$o_{iH}(\mathbf{x}, t) \equiv e^{iHt} o_i(\mathbf{x}, 0) e^{-iHt} , \quad (2.30)$$

and no subscript automatically refers to the interaction picture:

$$o_i(\mathbf{x}, t) \equiv e^{iH_0 t} o_i(\mathbf{x}, 0) e^{-iH_0 t} . \quad (2.31)$$

Sometimes, it is helpful to emphasise that a certain operator is in the interaction or Dirac picture, and the subscripts  $I$  or  $D$  will be written explicitly.

The interaction is rewritten in terms of a density  $\mathcal{H}$ :

$$V(t) = \int d^3x \mathcal{H}(\mathbf{x}, t) . \quad (2.32)$$

Together with the time integrals of eq. (2.28), one now has four-integrals in the perturbative expansion which is the first step towards Poincaré-invariance. The cluster decomposition principle is then satisfied by rewriting  $\mathcal{H}$  in terms of *creation* and *annihilation* operators  $a^\dagger(n, \mathbf{k}, \sigma)$  and  $a(n, \mathbf{k}, \sigma)$  in an appropriate way. Since these operators are to be used to compose  $V(t)$ , whose time dependence is governed by the free Hamiltonian (see eq. (2.26),  $a^\dagger$  is defined by acting on free states:

$$a^\dagger(n, \mathbf{k}, \sigma) |n_1, \mathbf{k}_1, \sigma_1; n_2, \mathbf{k}_2, \sigma_2; \dots\rangle \equiv |n, \mathbf{k}, \sigma; n_1, \mathbf{k}_1, \sigma_1; n_2, \mathbf{k}_2, \sigma_2; \dots\rangle . \quad (2.33)$$

This relation fully defines  $a^\dagger$  as well as  $a$ . The following (anti-)commutation relations are of special importance for later considerations:

$$\left[ a(n', \mathbf{k}', \sigma'), a^\dagger(n, \mathbf{k}, \sigma) \right]_{\mp} = \delta_{n'n} \delta^3(\mathbf{k}' - \mathbf{k}) \delta_{\sigma'\sigma} , \quad (2.34)$$

$$\left[ a(n', \mathbf{k}', \sigma'), a(n, \mathbf{k}, \sigma) \right]_{\mp} = \left[ a^\dagger(n', \mathbf{k}', \sigma'), a^\dagger(n, \mathbf{k}, \sigma) \right]_{\mp} = 0 , \quad (2.35)$$

with

$$[A, B]_{\mp} \equiv AB \mp BA .$$

The upper sign refers to bosonic operators (*commutator*), and the lower one to fermionic ones (*anti-commutator*). The commutator is sometimes also simply written as  $[A, B] \equiv [A, B]_-$ , and the anti-commutator as  $\{A, B\} \equiv [A, B]_+$ . The relation

$$a(n, \mathbf{p}, \sigma) |0\rangle = 0 \quad (2.36)$$

is also needed later.

The interaction itself is not directly constructed from annihilation and creation operators, but of some kind of Fourier transforms called *field operators*. For the moment, we will only introduce the field operators for neutral, scalar particles, since this model will be studied in the following section. These particles have some finite mass  $m$ , do not carry electrical charge and have spin  $s = 0$ . Since the model we will study only contains one kind of particles, we can leave out the particle label. The spin quantum number can be left out due to  $s = 0$ . Thus, the commutation relations of the annihilation and creation operators read

$$\left[ a(\mathbf{p}), a^\dagger(\mathbf{q}) \right] = \delta^3(\mathbf{p} - \mathbf{q}) , \quad (2.37)$$

with all other commutators vanishing. The aforementioned field operator is then given by

$$\phi(x) = \phi^+(x) + \phi^-(x) , \quad (2.38)$$

where

$$\phi^+(x) = (2\pi)^{-\frac{3}{2}} \int d^3p \frac{e^{ip^+x}}{\sqrt{2\omega_p}} a(\mathbf{p}) , \quad (2.39)$$

$$\phi^-(x) = (2\pi)^{-\frac{3}{2}} \int d^3p \frac{e^{-ip^+x}}{\sqrt{2\omega_p}} a^\dagger(\mathbf{p}) . \quad (2.40)$$

We have defined  $\omega_p \equiv \sqrt{p^2 + m^2}$  and  $q^\sigma \equiv (\mathbf{q}, \sigma \omega_q)^T$  for  $\sigma = \pm 1$ . The reason for constructing the field operator in this way is to achieve Lorentz invariance and causality in local field theories. Field operators for other types of particles will only be given when needed explicitly. Let us now focus on QFTs for scalar particles.

## 2.2 Non-local interactions

In order to deal with the algebra in eq. (1.1), one introduces the so-called (Weyl-Moyal) star product defined in [18, 19]:

$$(f * g)(x) \equiv e^{\frac{i}{2} \theta^{\mu\nu} \partial_\mu^\zeta \partial_\nu^\eta} f(x + \zeta) g(x + \eta) \Big|_{\zeta=\eta=0} . \quad (2.41)$$

In eq. (2.41),  $x^\mu$  is an element of the usual commutative space-time and  $f(x)$ ,  $g(x)$  stand for the usual operators of the commutative theory like the scalar field  $\phi(x)$  for example. The product of eq. (2.41) is now to be used instead of the local products of fields present in the usual field theory. The self interaction density of a real, scalar  $\phi^4$ -theory is then modified in the following way:

$$\mathcal{H}(z) = \frac{\kappa}{4!} \phi(z)^4 \rightarrow \frac{\kappa}{4!} \phi(z) * \phi(z) * \phi(z) * \phi(z) . \quad (2.42)$$

This is related to the interaction  $V(z^0)$  in the interaction picture by  $V(z^0) = \int d^3z \mathcal{H}(z)$ . For  $\theta^{0i} = 0$ , it has been described in [18, 19] how to derive Feynman rules and the outcome showed that the usual causal Feynman propagator can be used. However, the interaction vertices are modified by certain phase factors. The existence of such phase factors leads to planar and non-planar contributions. But furthermore, it was pointed out in [19] that the case  $\theta^{0i} \neq 0$  is difficult to handle due to the fact that the Lagrangian containing star products of fields consequently also depends on infinitely many time derivatives acting on fields. Thus, it may be doubted that the Lagrangian formalism can be applied for  $\theta^{0i} \neq 0$  in the usual, traditional way.

An alternative approach was already proposed in [1] and analysed explicitly in [31, 28]. The calculations carried out there are based on the Gell-Mann-Low formula (2.29) applied to field operators  $\phi$

$$\langle 0 | T \{ \phi_H(x_1) \dots \phi_H(x_n) \} | 0 \rangle = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \dots \int_{-\infty}^{\infty} dt_m \times \\ \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) V(t_1) \dots V(t_m) \} | 0 \rangle . \quad (2.43)$$

Following the derivation of this formula (see [29], for example), it is clear that time-ordering is to be done with respect to  $x_1^0, \dots, x_n^0$  and  $t_1, \dots, t_m$ , called *time stamps* from now on. In order to apply eq. (2.43) to the interaction given in eq. (2.42), it is helpful to rewrite  $\mathcal{H}$  in the form of integral representations [28]

$$\mathcal{H}(z) = \frac{\kappa}{4!} \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{i l_i s_i} \right) \times \\ \phi(z - \frac{1}{2} \tilde{l}_1) \phi(z + s_1 - \frac{1}{2} \tilde{l}_2) \phi(z + s_1 + s_2 - \frac{1}{2} \tilde{l}_3) \phi(z + s_1 + s_2 + s_3) , \quad (2.44)$$

where  $\tilde{a}^\nu \equiv a_\mu \theta^{\mu\nu}$ . Time-ordering only involves  $z^0$  and not any other time components occurring in the field operators in eq. (2.44). The advantages of this representation are that the non-locality can be nicely seen and that one does not have to care about derivatives, especially time derivatives, any more as it is the case when explicitly using the star product in the form of eq. (2.41). Setting  $\theta = 0$ , the usual, local  $\phi^4$ -theory is recovered.

Now, the question is, how to do perturbation theory with interactions like in eq. (2.44)? The case of  $\theta_{0i} \neq 0$  was already treated in [31, 28] and rules how to do calculations were also given. But as was also pointed out, combinatorics was not explicitly treated. Furthermore, it is hard to see the connection between the rules given in [31, 28] and ordinary Feynman rules, which should come out for the case  $\theta = 0$ . Therefore, we develop graphical rules in the fashion of Feynman rules for perturbation theory of non-local interactions of scalar particles of the type

$$V(t) = \int d\lambda v(\lambda, t) \phi(g_1(\lambda, t)) \dots \phi(g_k(\lambda, t)) \quad (2.45)$$

in the following section.  $\lambda = (\lambda_1, \dots, \lambda_e)$  denotes a set of  $e$  real parameters,  $v$  is a  $\mathbb{C}$ -function and the  $g_i$ 's map  $(\lambda, t)$  into a four vector.  $d\lambda$  simply abbreviates  $d\lambda_1 \dots d\lambda_e$ . By non-local, we usually address products of fields as the interaction given in eq. (2.45), which also involve configurations where the arguments of fields are not identical. A big part of the considerations carried out in this thesis are applicable to interactions as specified in eq. (2.45). This treatment is much more general than the local case, which is already well known. It should be noted that eq. (2.45) also covers the local case with

$$\begin{aligned} \lambda &= (\mathbf{z}) , \\ v(\lambda, t) &= \text{const} , \\ g_i(\lambda, t) &= (\mathbf{z}, z^0)^T , \end{aligned}$$

representing  $V(z^0) \propto \int d^3z \phi(z)^k$ . Thus, non-local is sometimes also to be understood as a generalisation of the local case not necessarily excluding it.

Concerning eq. (2.44), eq. (2.45) is applicable the following way:

$$\begin{aligned} k &= 4 , \\ \lambda &= (\mathbf{z}, l_1, l_2, l_3, s_1, s_2, s_3) , \\ v(\lambda, t) &= \kappa/4! (2\pi)^{-12} e^{i(l_1 s_1 + l_2 s_2 + l_3 s_3)} , \\ g_1(\lambda, t) &= z - \tilde{l}_1/2 , \\ g_2(\lambda, t) &= z + s_1 - \tilde{l}_2/2 , \\ g_3(\lambda, t) &= z + s_1 + s_2 - \tilde{l}_3/2 , \\ g_4(\lambda, t) &= z + s_1 + s_2 + s_3 , \end{aligned}$$

with  $t = z^0$ . The inclusion of a larger class of non-local interactions like the one studied in [32], for example, was the main motivation for attacking the problem in such a general way:

$$\begin{aligned} V(z^0) &= \kappa c_k \int d^3z \int d^4a_1 \dots d^4a_k : \phi(z + \zeta a_1) \dots \phi(z + \zeta a_k) : \times \\ &\quad \exp \left\{ -\frac{1}{2} \sum_{j=1}^k (a_j^\mu)^2 \right\} \delta^4 \left( \frac{1}{k} \sum_{j=1}^k a_j \right) , \end{aligned} \quad (2.46)$$



with  $\kappa, c_k, \zeta \in \mathbb{R}$ .  $\kappa$  is the coupling constant and  $c_k$  denotes some normalisation constant depending on  $k$ .  $\zeta$  has a physical dimension of a length and should be very small (maybe in the range of the Planck length). We have explicitly introduced this parameter, since the limit  $\zeta \rightarrow 0$  represents the corresponding local theory. We will refer to interactions of this type as *non-local interactions of Gaussian type* from now on. Besides the usual normal ordering indicated by the colons,  $V(t)$  can be put into the form of eq. (2.45):

$$\begin{aligned}\underline{\lambda} &= (\mathbf{z}, a_1, \dots, a_k), \\ v(\underline{\lambda}, t) &= \kappa c_k \exp \left\{ -\frac{1}{2} \sum_{j=1}^k (a_j^\mu)^2 \right\} \delta^4 \left( \frac{1}{k} \sum_{j=1}^k a_j \right), \\ g_j(\underline{\lambda}, t) &= z + \zeta a_j,\end{aligned}$$

with  $j = 1, \dots, k$  and  $z^0 = t$ . The effect of normal ordering will be discussed below.

Before applying the Gell-Mann-Low formula (2.43) to these interactions, it must be mentioned that  $V(t)$  should have a time dependence according to the interaction picture:

$$V(t) = \exp(iH_0 t) V(0) \exp(-iH_0 t). \quad (2.47)$$

Choosing  $\phi$  in the interaction picture, this relation can be easily satisfied by

$$v(\underline{\lambda}, t) = v(\underline{\lambda}, 0), \quad (2.48)$$

$$g_i(\underline{\lambda}, t)^T = (g_i(\underline{\lambda}, 0)^1, g_i(\underline{\lambda}, 0)^2, g_i(\underline{\lambda}, 0)^3, g_i(\underline{\lambda}, 0)^0 + t). \quad (2.49)$$

The interactions of eqs. (2.44) and (2.46) obey these conditions.

## 2.3 Feynman rules for non-local interactions

In this section, diagrammatic rules will be given for calculating

$$G_m^n(x_1, \dots, x_n) \equiv \frac{(-i)^m}{m!} \int dt_{n+1} \dots dt_N \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) V(t_{n+1}) \dots V(t_N) \} | 0 \rangle, \quad (2.50)$$

with non-local interactions as given in eq. (2.45).  $n$  is the number of external points, and  $m$  denotes the order of interactions.  $N = n + m$ , and the time stamps  $t_1, \dots, t_m$  have been renamed by  $t_{n+1}, \dots, t_{n+m}$ . In order to understand the general case which is discussed in appendix A, we consider the simplest non-local interaction of the following form in a first step :

$$V(t) = \int d\underline{\lambda} v(\underline{\lambda}, t) \phi(g_1(\underline{\lambda}, t)) \phi(g_2(\underline{\lambda}, t)).$$

Such an interaction could be responsible for “mass-renormalisation” if one investigates the connected piece of a “physical” two-point function

$$G_1^2(x, y) = \int dt \int d\underline{\lambda} v(\underline{\lambda}, t) \langle 0 | T \{ \phi(x) \phi(y) \phi(g_1(\underline{\lambda}, t)) \phi(g_2(\underline{\lambda}, t)) \} | 0 \rangle. \quad (2.51)$$

As pointed out in [30, 31, 28, 42, 35], one emphasises here that the time-ordering involves only the times  $x^0, y^0$  and the time stamp of the interaction part due to the unitarity problem

which is discussed in [27, 42]. Thus, time ordering is not done with respect to  $g_i(\underline{\lambda}, t)^0$ , but simply  $t$ .

Abbreviating  $g_1(\underline{\lambda}, t) = z + a$  and  $g_2(\underline{\lambda}, t) = z + b$ , with  $t$  only occurring in  $z^0 = t$ , eq. (2.49) is automatically satisfied. We are now able to evaluate eq. (2.51) in terms of  $\phi^+(x)$  and  $\phi^-(x)$  according to eq. (2.38). The free field commutator

$$\Delta^+(x, y) \equiv [\phi^+(x), \phi^-(y)] \quad (2.52)$$

is a  $\mathbb{C}$ -valued distribution. Using this relation, annihilation operators  $\phi^+$  are easily pushed to the right of creation operators  $\phi^-$ . The connected part of the vacuum expectation value can now be written as

$$\begin{aligned} \langle 0 | T \{ \phi(x) \phi(y) \phi(z+a) \phi(z+b) \} | 0 \rangle_0^{con} = \\ \theta(x^0 - y^0) \theta(y^0 - z^0) [\Delta^+(x, z+a) \Delta^+(y, z+b) + \Delta^+(x, z+b) \Delta^+(y, z+a)] \\ + \theta(y^0 - x^0) \theta(x^0 - z^0) [\Delta^+(x, z+a) \Delta^+(y, z+b) + \Delta^+(x, z+b) \Delta^+(y, z+a)] \\ + \theta(x^0 - z^0) \theta(z^0 - y^0) [\Delta^+(x, z+a) \Delta^+(z+b, y) + \Delta^+(x, z+b) \Delta^+(z+a, y)] \\ + \theta(y^0 - z^0) \theta(z^0 - x^0) [\Delta^+(z+a, x) \Delta^+(y, z+b) + \Delta^+(z+b, x) \Delta^+(y, z+a)] \\ + \theta(z^0 - x^0) \theta(x^0 - y^0) [\Delta^+(z+a, x) \Delta^+(z+b, y) + \Delta^+(z+b, x) \Delta^+(z+a, y)] \\ + \theta(z^0 - y^0) \theta(y^0 - x^0) [\Delta^+(z+a, x) \Delta^+(z+b, y) + \Delta^+(z+b, x) \Delta^+(z+a, y)] . \end{aligned} \quad (2.53)$$

The first (second) summands appearing in square brackets above will be referred to as the uncrossed (crossed) terms. The crossed terms can be obtained from the uncrossed ones by simply exchanging  $x \leftrightarrow y$  or  $a \leftrightarrow b$ . At this step, one gets rid of all operators, and only  $\mathbb{C}$ -valued distributions are left. However, even for this simple case, the result is quite lengthy and familiar structures like Feynman rules cannot be recognised easily. Adding a few more fields to the vacuum expectation value would just blow up the problem dramatically, when using the same algorithm. Of course, computer algebra systems might be helpful to carry out such lengthy tasks. At the beginning of the investigations, a corresponding software package was developed applying the following steps for vacuum expectation values:

- Decompose each field operator into its annihilation and creation part.
- Move annihilation fields to the right and creation parts to the left by applying the corresponding commutation relations.
- Drop terms where annihilation operators act on the vacuum (or the adjoint happens). Thus, one arrives at terms only involving commutators of annihilation and creation fields which are simply  $\mathbb{C}$ -valued distributions.

The manual application of this scheme is a lot of work. For the cases of interest, this is a rather trivial task for a computer. But it turns out that the results become extremely lengthy. For  $k$  fields, one gets

$$\frac{k!}{2^{k/2}(k/2)!}$$

summands each involving  $k/2$  commutators. One is interested in the 4-point function of  $\phi^4$  theory at one-loop level, for example, which involves  $k = 12$  fields. This means one obtains 10395 terms! Dropping unconnected terms and terms involving tadpoles, only 1728 terms remain. But unfortunately, this is not the end of the story since one has to consider all

possible time-orderings. For the described case, one has 6 time stamps (4 corresponding to the external fields and 2 corresponding to the interactions) which means that the vacuum expectation values, each yielding 1728 terms, have to be considered  $6! = 720$  times. Of course, one can reduce the amount of work by only considering special configurations among the external fields and generating the others by exchanging external coordinates. But still, one is confronted with extremely lengthy results, which are best used as wallpaper patterns.

By the way, the combinatorial problems would be the same for local theories if one directly used the Gell-Mann-Low formula. But in this case, one can apply the compact set of Feynman rules instead, where all the 1728 terms are described by one diagram including all possible time-orderings. The key object of these rules is the propagator which can be written as

$$-i\Delta_F(x, x') = \theta(x^0 - x'^0)\Delta^+(x, x') + \theta(x'^0 - x^0)\Delta^+(x', x) . \quad (2.54)$$

The left overs of time-ordering are present through the  $\theta$ -functions. The field arguments occur in the commutators  $\Delta^+$ . I realized that for NCQFT, time-ordering is done with respect to the time stamps and not necessarily the time components of field arguments as in local QFT. Thus, I guessed <sup>1</sup> that instead of the propagator the following object might be relevant for perturbation theory:

$$-i\Delta(x, t; x', t') \equiv \theta(t - t')\Delta^+(x, x') + \theta(t' - t)\Delta^+(x', x) \quad (2.55)$$

with  $x, t$  and  $x', t'$  denoting pairs of field arguments and associated time stamps, respectively. This thing is given the name *contractor*. The name is created in analogy to the usual Wick contractions of commutative local perturbation theory in the sense of Feynman. The semi-colon just visualises the connection between four vectors  $x$  or  $x'$  and time stamps  $t$  and  $t'$ , respectively. For  $t = x^0$  and  $t' = x'^0$ , which means locality in time (including local field theory), it reduces to the usual Feynman propagator  $\Delta_F$ . To make clear how and if at all the contractor is relevant in perturbation theory, we return to the example discussed above. Let us pick the uncrossed terms containing  $\Delta^+(y, z + b)$  out of eq. (2.53):

$$\Delta^+(y, z + b) \left[ \theta(x^0 - y^0)\theta(y^0 - z^0)\Delta^+(x, z + a) + \theta(y^0 - x^0)\theta(x^0 - z^0)\Delta^+(x, z + a) + \theta(y^0 - z^0)\theta(z^0 - x^0)\Delta^+(z + a, x) \right] .$$

For  $(x^0 \neq y^0 \neq t)$  <sup>2</sup>, one can now apply a trick: multiply the first term in square brackets with  $\theta(x^0 - z^0)$ , the second with  $\theta(y^0 - z^0)$ , the third with  $\theta(y^0 - x^0)$ , and add within the square brackets  $\theta(y^0 - z^0)\theta(z^0 - x^0)\theta(x^0 - y^0)\Delta^+(z + a, x)$ , which is 0. Thus, one does not alter the result and gets

$$\theta(y^0 - z^0)\Delta^+(y, z + b) \left\{ \theta(x^0 - z^0)\Delta^+(x, z + a) + \theta(z^0 - x^0)\Delta^+(z + a, x) \right\} .$$

The expression in curly brackets can clearly be rewritten by the contractor introduced above:

$$\theta(x^0 - z^0)\Delta^+(x, z + a) + \theta(z^0 - x^0)\Delta^+(z + a, x) = -i\Delta(x, x^0; z + a, z^0) . \quad (2.56)$$

Treating the remaining terms of eq. (2.53) in the same way,  $G_1^2$  can now be rewritten as

$$G_1^2(x, y) = - \int dt \int d\lambda v(\lambda, t) \times \left[ \Delta(x, x^0; g_1(\lambda, t), t) \Delta(y, y^0; g_2(\lambda, t), t) + \Delta(x, x^0; g_2(\lambda, t), t) \Delta(y, y^0; g_1(\lambda, t), t) \right] . \quad (2.57)$$

<sup>1</sup>One year later, a similar guess [43] based on [33] lead to the reproduction of the results already given in [33].

<sup>2</sup>The configurations where some time stamps are the same can be neglected.

This example will be discussed further in section 2.5. Before, the diagrammatic rules for the general case will be given.

### 2.3.1 Coordinate space rules

Again, we refer to appendix A, where the general case has been treated including beside the connected parts also tadpole contributions and disconnected diagrams. In order to calculate  $G_m^n(x_1, \dots, x_n)$ , one has to apply the following diagrammatic algorithm:

- Draw  $n$  points and label them with the external coordinates  $x_1, \dots, x_n$ . Their time stamps are  $x_1^0, \dots, x_n^0$ , respectively.
- Draw  $m$  circles and label them with the parameters  $\underline{\lambda}_1, \dots, \underline{\lambda}_m$  and time stamps  $t_{n+1}, \dots, t_{n+m}$ .
- Draw  $k$  points into each circle and label them with  $g_1, \dots, g_k$ .
- For each possibility of connecting two points pairwise by a line, so that each point is connected to exactly one line, draw a diagram with points and circles as given above.
- For each line connecting two points with coordinates and time stamp  $x, t$  and  $x', t'$ , respectively, write down a contractor

$$-i\Delta(x, t; x', t') ,$$

if the points do not belong to the same circle. If they belong to the same circle, write down

$$\text{either } \Delta^+(x, x') \text{ or } \Delta^+(x', x) ,$$

depending on whether  $\phi(x)$  stands left of  $\phi(x')$  within the interaction  $V(t)$  or vice versa. External points already carry the coordinates as label and the corresponding time stamp is simply the 0th component of that label. For points within circles, the time stamp  $t$  is simply the time stamp of the circle. The coordinate  $x$  of such a point is given by the label  $\underline{\lambda}$  and  $t$  of the circle and the label  $g_j$  of the point as  $x = g_j(\underline{\lambda}, t)$ .

- For each circle labelled with  $\underline{\lambda}_i$  and  $t_{n+i}$ , perform an integration according to

$$(-i) \int dt_{n+i} d\underline{\lambda}_i v(\underline{\lambda}_i, t_{n+i}) .$$

- Sum up the contributions of all diagrams.

The rules given above are considerably more complicated than the usual Feynman rules of the corresponding local field theory. But using this diagrammatic formalism, eq. (A.21) is much more comfortable to handle than the usual algorithm of commuting out all creation and annihilation fields. It should also be mentioned that for each diagram with  $m$  identical vertices, there are  $m!$  diagrams which only differ by a rearrangement of the vertices. This fact has already been implemented in the rules to cancel the factor  $1/m!$  of the Gell-Mann-Low formula (2.43). Consequently, one must include diagrams differing by a rearrangement of vertices exactly once.

Further simplifications might be possible when permuting the labels  $g_i$  of a given circle among each other. But this will in general depend on the interaction itself and will be studied below for a more special type of interactions. Here it is a good point to comment on the prescription for calculating tadpole contributions. The rules given above mean that for a tadpole loop between points with coordinates  $x$  and  $x'$ , one has to include either  $\Delta^+(x, x')$  or  $\Delta^+(x', x)$ , depending on the definition of the interaction. If the time stamp of the circle is  $t$ , the contractor for these coordinates would give

$$-i\Delta(x, t; x', t) = \theta(0) [\Delta^+(x, x') + \Delta^+(x', x)]$$

which clearly is not what one needs for the general case. But in a local field theory,  $\Delta^+(x, x)$  is to be taken for tadpole loops. Thus, by defining  $\theta(0) \equiv 1/2$ , one does not have to treat tadpole contributions exceptionally, but can also use the contractor  $-i\Delta(x, t; x, t) = \Delta^+(x, x)$  in local field theories.

### 2.3.2 Calculation of the contractor

In order to evaluate the contractor, one uses eqs. (2.37), (2.39), and (2.40) to get

$$\Delta^+(x, x') = [\phi^+(x), \phi^-(x')] = (2\pi)^{-3} \int \frac{d^3p}{2\omega_p} e^{ip^+(x-x')} . \quad (2.58)$$

Expressing the  $\theta$ -function as

$$\theta(t) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} ds \frac{e^{-ist}}{s + i\epsilon} , \quad (2.59)$$

one obtains

$$\begin{aligned} \theta(t-t')\Delta^+(x, x') &= \\ \frac{i}{(2\pi)^4} \int \frac{d^3p ds}{2\omega_p} e^{ip^+(x-x') - is(x^0 - x'^0)} \frac{e^{is(x^0 - t - (x'^0 - t'))}}{s + i\epsilon} &= \\ \frac{i}{(2\pi)^4} \int \frac{d^4p}{2\omega_p} e^{ip(x-x') + ip^0(x^0 - t - (x'^0 - t'))} \frac{e^{-i\omega_p(x^0 - t - (x'^0 - t'))}}{p^0 - \omega_p + i\epsilon} . \end{aligned}$$

The last line was obtained by the transformation  $s \equiv p^0 - \omega_p$ . The other term of eq. (2.55) can be obtained by simply exchanging  $(x, t)$  and  $(x', t')$ , and additionally one transforms  $p \rightarrow -p$ . The obtained expression only differs in the last fraction which is

$$\frac{e^{i\omega_p(x^0 - t - (x'^0 - t'))}}{-p^0 - \omega_p + i\epsilon} .$$

Thus, one finally gets

$$\begin{aligned} \Delta(x, t; x', t') &= \frac{1}{(2\pi)^4} \int d^4q \frac{e^{iq(x-x') + iq^0(x^0 - t - (x'^0 - t'))}}{q^2 + m^2 - i\epsilon} \\ &\quad \left[ \cos(\omega_q(x^0 - t - (x'^0 - t'))) - \frac{iq^0}{\omega_q} \sin(\omega_q(x^0 - t - (x'^0 - t'))) \right] . \end{aligned} \quad (2.60)$$

For theories, where the time stamps are always identical with the time components of field arguments, say  $t = x^0$  and  $t' = x'^0$ , this reduces to the usual Feynman propagator

$$\Delta_F(x, x') = \frac{1}{(2\pi)^4} \int d^4q \frac{e^{iq(x-x')}}{q^2 + m^2 - i\epsilon} .$$

### 2.3.3 Energy-momentum conservation

Now we present the calculation of diagrams as shown in fig. 2.1, for example, using translation invariant interactions. Fig. 2.1 consists of one interaction vertex containing  $k$  points labelled  $g_i$  with  $i = 1, \dots, k$ . These points all carry the same time stamp. The  $k$  "external" points are labelled  $z_i + a_i$  all carrying different time stamps  $z_i^0$  ( $i = 1, \dots, k$ ). Actually, a typical external point would carry time stamp  $z_i^0 + a_i^0$ . But the intension of this calculation is to work out Feynman rules in more detail. The "external" points are thus kept more general in order to be also usable as internal points of larger diagrams containing fig. 2.1. The very general interaction defined by eq. (2.45) is now specialised in order to satisfy translation invariance:

$$\underline{\lambda} = (\underline{z}, \underline{\mu}) , \quad (2.61)$$

$$v(\underline{\lambda}, t) = w(\underline{\mu}) , \quad (2.62)$$

$$g_i(\underline{\lambda}, t) = z + h_i(\underline{\mu}) , \quad (2.63)$$

where  $z^0 \equiv t$ .  $\underline{\mu} = (\mu_1, \dots, \mu_{e-3})$  denotes a set of  $e - 3$  real parameters,  $w$  is a  $C$ -function and the  $h_i$ 's map  $\underline{\mu}$  into a four vector.  $d\underline{\mu}$  simply abbreviates  $d\mu_1 \dots d\mu_{e-3}$ . The very general interaction defined by eq. (2.45) is thus specialised slightly:

$$V(z^0) = \int d^3 z \int d\underline{\mu} w(\underline{\mu}) \phi(z + h_1(\underline{\mu})) \dots \phi(z + h_k(\underline{\mu})) . \quad (2.64)$$

This way, translation invariance of the model is manifested:

$$e^{-i\epsilon P_{(0)}} V(z^0) e^{i\epsilon P_{(0)}} = V(z^0 + \epsilon^0) , \quad (2.65)$$

where  $P_{(0)}$  denotes the free four-momentum operator generating translations in space-time.  $\epsilon$  represents the constant translation in coordinate space. This way, eq. (2.26) is automatically satisfied.

Concerning the non-commutative interaction given by eq. (2.44), the above specialisation means

$$\begin{aligned} k &= 4 , \\ \underline{\mu} &= (l_1, l_2, l_3, s_1, s_2, s_3) , \\ w(\underline{\mu}) &= \kappa / (4! (2\pi)^4) \exp(i(l_1 s_1 + l_2 s_2 + l_3 s_3)) , \\ h_1(\underline{\mu}, t) &= -\tilde{l}_1/2 , \\ h_2(\underline{\mu}, t) &= s_1 - \tilde{l}_2/2 , \\ h_3(\underline{\mu}, t) &= s_1 + s_2 - \tilde{l}_3/2 , \\ h_4(\underline{\mu}, t) &= s_1 + s_2 + s_3 , \\ \int d\underline{\mu} &= \int d^3 z \prod_{i=1}^3 \int d^4 s_i \int d^4 l_i . \end{aligned}$$

Now, we are able to write down the contribution of the diagram shown in fig. 2.1:

$$\begin{aligned} G_I(z_1 + a_1, z_1^0; \dots; z_k + a_k, z_k^0) &\equiv -i \int d\underline{\mu} d^4 z w(\underline{\mu}) \times \\ &(-i\Delta(z_1 + a_1, z_1^0; z + h_1(\underline{\mu}), z^0)) \dots (-i\Delta(z_k + a_k, z_k^0; z + h_k(\underline{\mu}), z^0)) , \end{aligned} \quad (2.66)$$

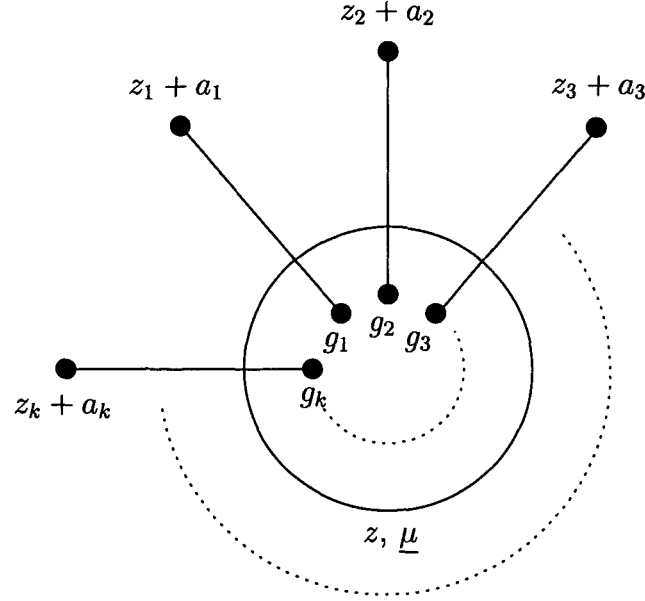


Figure 2.1: Vertex of an interaction with  $k$  fields. The points out of the circle are considered to be either external or part of other vertices. Their time stamps are  $z_1^0, \dots, z_k^0$ . The time stamp of the circle is  $z_0$ .

where  $I$  stands for the identity permutation of  $S^k$ .  $G_I$  is only one certain contribution. On the whole, there are  $k!$  terms similar to  $G_I$  but only differing in the way the external points are connected to the internal ones. Denoting an arbitrary permutation by  $Q \in S^k$ , one can write these terms as

$$G_Q(z_1 + a_1, z_1^0; \dots; z_k + a_k, z_k^0) \equiv -i \int d\mu d^4z w(\mu) \times \quad (2.67)$$

$$(-i\Delta(z_1 + a_1, z_1^0; z + h_{Q_1}(\mu), z^0) \dots (-i\Delta(z_k + a_k, z_k^0; z + h_{Q_k}(\mu), z^0) .$$

$G_Q$  gives the vertex where each point  $z_i + a_i$  is attached to  $g_{Q_i}(\underline{\lambda}, t) = z + h_{Q_i}(\underline{\mu})$ . The total contribution of these terms is given by

$$G(z_1 + a_1, z_1^0; \dots; z_k + a_k, z_k^0) \equiv \sum_{Q \in S^k} G_Q(z_1 + a_1, z_1^0; \dots; z_k + a_k, z_k^0) . \quad (2.68)$$

All contractors connecting external and internal points are of the following form:

$$\Delta(z + a, z^0; z' + a', z'^0) = \frac{1}{(2\pi)^4} \int d^4q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} \sum_{\sigma \in \{1, -1\}} \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} e^{iq^\sigma(a-a')} . \quad (2.69)$$

It is remarkable that  $z$  and  $z'$  only occur in the first exponential. This expresses translation invariance for time-ordered perturbation theory of non-local interactions also involving time.

Integrating over  $z$  immediately yields energy-momentum conservation and one gets

$$\begin{aligned}
& G(z_1 + a_1, z_1^0; \dots; z_k + a_k, z_k^0) \\
&= \frac{(-i)^{k+1}}{(2\pi)^{4k}} \sum_{Q \in S^k} \int d\mu \, w(\mu) \int d^4 q_1 \dots d^4 q_k (2\pi)^4 \delta^4(q_1 + \dots + q_k) \times \\
& \prod_{i=1}^k \left( \frac{e^{iq_i z_i}}{q_i^2 + m^2 - i\epsilon} \sum_{\sigma_i} \frac{\omega_{\mathbf{q}_i} + \sigma_i q_i^0}{2\omega_{\mathbf{q}_i}} e^{iq_i^{\sigma_i} (a_i - h_{Q_i})} \right) \\
&= \frac{(-i)^{k+1}}{(2\pi)^{4k}} \int d^4 q_1 \dots d^4 q_k (2\pi)^4 \delta^4(q_1 + \dots + q_k) \prod_{i=1}^k \left( \frac{e^{iq_i z_i}}{q_i^2 + m^2 - i\epsilon} \right) \times \quad (2.70) \\
& \sum_{\sigma_1, \dots, \sigma_k} \prod_{i=1}^k \left( \frac{\omega_{\mathbf{q}_i} + \sigma_i q_i^0}{2\omega_{\mathbf{q}_i}} e^{iq_i^{\sigma_i} a_i} \right) \chi_k(q_1^{\sigma_1}, \dots, q_k^{\sigma_k}),
\end{aligned}$$

with

$$\chi_k(q_1^{\sigma_1}, \dots, q_k^{\sigma_k}) \equiv \int d\mu \, w(\mu) \sum_{Q \in S^k} \exp \left( -i \sum_{i=1}^k q_{Q_i}^{\sigma_i} h_i(\mu) \right). \quad (2.71)$$

Of course, the calculated vertex is much more complicated than what one knows from local field theories. But the positive aspect is given by the fact that the details of the non-locality only occur in the factor  $\chi_k$ . For the case of NCQFT, it contains the phase factors leading to planar and non-planar contributions.  $\chi_k$  can be said to be the central quantity of time-ordered perturbation theory for non-local interactions obeying translation invariance. We will refer to it as the *vertex factor* from now on.

Corresponding to this result, it is now straight forward to extract the momentum space Feynman rules for non-local interactions.

### 2.3.4 Momentum space rules

Usually, one is interested in the Fourier transform of eq. (2.43). Correspondingly, one has to evaluate <sup>3</sup>

$$\begin{aligned}
G^{con, nt}(p_1, \dots, p_n) &\equiv \prod_{i=1}^n \left( \int d^4 x_i e^{ip_i x_i} \right) \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \} | 0 \rangle_H^{con, nt} \\
&= (2\pi)^4 \delta^4 \left( \sum_{i=1}^n p_i \right) \prod_{i=1}^{n-1} \left( \int d^4 z_i e^{ip_i z_i} \right) \langle 0 | T \{ \phi(z_1) \dots \phi(z_{n-1}) \phi(0) \} | 0 \rangle_H^{con, nt} \\
&\equiv (2\pi)^4 \delta^4 \left( \sum_{i=1}^n p_i \right) G_{trunc}^{con, nt}(p_1, \dots, p_{n-1}). \quad (2.72)
\end{aligned}$$

This task can be simplified using the rules of the last section. The  $\delta$ -function in eq. (2.72) represents the total momentum conservation under the assumption that all momenta are flowing into the graph. The main result of the calculation of the last section is the fact that summing up diagrams, which only differ by the way how the contractors are attached to

<sup>3</sup>The superscript *con, nt* indicates that we restrict ourselves to connected diagrams without tadpoles. The discussion of tadpoles will be treated as an example in section 2.5.



points within fixed circles, can easily be done by including the factor  $\chi_k$  of eq. (2.71) for each vertex. But, using  $\chi_k$ , one has to take care of two things:

- When calculating  $\sum_Q G_Q$ , it was assumed that all momenta are flowing into the vertex. Thus, one has to insert  $-(p^\sigma)$  into  $\chi_k$  if  $p$  is flowing out of the vertex.
- Furthermore, for certain diagrams, one would get double counting. Thus one has to include a symmetry factor  $1/S$  (see also [31]). It must be taken the same as for common Feynman diagrams.

Before presenting the final rules, we will briefly repeat topological considerations. A general graph is characterised by a certain number of internal lines  $I$  and by the number  $V$  of interaction vertices of a given type. The number of independent loops are denoted by  $L$ , which is

$$L = I - V + 1 . \quad (2.73)$$

With the above momentum assignment that all momenta are incoming, we are now able to state the Feynman rules for the calculation of expression (2.72):

1. Draw all possible momentum space Feynman diagrams having  $E = n$  external lines.
2. Carefully label each line with four momentum including its flow and make use of the conservation of four momentum at each vertex. Due to definition (2.72), the external lines are labelled with momenta  $p_1, \dots, p_n$  with the convention that the  $p_i$ 's are incoming. Also assign a variable  $\sigma_i$  to each line.
3. For each line, one has to include a factor

$$\frac{-i}{q^2 + m^2 - i\epsilon} \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} , \quad (2.74)$$

where  $q$  and  $\sigma$  now represent the labels of the corresponding line. This factor is referred to as the *pole factor* from now on.

4. For each vertex, write down the vertex factor

$$-i \chi_k(\dots)$$

with the rule to insert  $\pm(q_i^{\sigma_i}) = (\pm \mathbf{q}_i, \pm \sigma_i \omega_{\mathbf{q}_i})^T$  into  $\chi_k$  for each line (at the vertex) labelled  $q_i, \sigma_i$ ; the “+” sign for momenta flowing into the vertex and “−” otherwise. Due to the symmetry of  $\chi_k$  concerning permutations of arguments, the order of arguments is not relevant.

5. Include the symmetry factor

$$1/S .$$

6. Assure momentum conservation by a factor

$$(2\pi)^4 \delta^4(p_1 + \dots + p_n) .$$

7. Integrate over the  $L$  independent loop momenta, which are not fixed by energy-momentum conservation and multiply by  $(2\pi)^{-4L}$ . Sum over all  $\sigma$ 's.

8. Sum up all diagrams in the usual sense.

Now, one is ready to apply the result of the rather general treatment of time-ordered perturbation theory valid for a large class of non-local interactions to special cases. But before, let us generalise these results to other types of fields.

## 2.4 Further contractors

We have seen above that one has to replace the propagator of local QFT with the contractor as soon as non-localities in time are involved. The extension to other types of fields is now straight forward. Before dealing with the physically more interesting cases, the contractor for derivatives of scalar fields will be given.

### 2.4.1 Contractors of derivative fields

First, we are interested in the result of contracting a scalar field  $\phi(z' + a')$  with its derivative  $\partial_\mu \phi(z + a)$ , having time stamps  $z'^0, z^0$ , respectively. This means, we want to calculate

$$\begin{aligned} -i\Delta_\mu(z + a, z^0; z' + a', z'^0) &\equiv \theta(z^0 - z'^0) [\partial_\mu \phi^+(z + a), \phi^-(z' + a')] \\ &+ \theta(z'^0 - z^0) [\phi^+(z' + a'), \partial_\mu \phi^-(z + a)] . \end{aligned} \quad (2.75)$$

The result can now be obtained by a similar calculation as carried out above. But comparing with eq. (2.55), one sees that

$$\Delta_\mu(z + a, z^0; z' + a', z'^0) = \frac{\partial}{\partial a^\mu} \Delta(z + a, z^0; z' + a', z'^0) .$$

Thus, one can easily calculate  $\Delta_\mu$  using eq. (2.69):

$$\Delta_\mu(z + a, z^0; z' + a', z'^0) = \frac{i}{(2\pi)^4} \int d^4 q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} \sum_{\sigma \in \{1, -1\}} (q^\sigma)_\mu \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} e^{iq^\sigma(a-a')} . \quad (2.76)$$

It is remarkable that it is  $q^\sigma$  and not the off-shell momentum  $q$  occurring additionally. To see what is the limit for a local theory we set  $a = a' = 0$ :

$$\Delta_\mu(z, z^0; z', z'^0) = \frac{i}{(2\pi)^4} \int d^4 q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} \sum_{\sigma \in \{1, -1\}} (q^\sigma)_\mu \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} .$$

For  $\mu$  representing a space index  $j$ , the sum over sigma simply becomes one and for  $\mu = 0$ , one has

$$\sum_{\sigma \in \{1, -1\}} \sigma \omega_{\mathbf{q}} \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} = q_0 .$$

The local limit thus gives

$$\Delta_\mu(z, z^0; z', z'^0) = \frac{i}{(2\pi)^4} \int d^4 q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} q_\mu , \quad (2.77)$$

not violating Lorentz covariance.

The next step is to calculate the contractor of two field derivatives  $\partial_\mu \phi(z+a)$ ,  $\partial_\nu \phi(z'+a')$ , having time stamps  $z^0, z'^0$ , respectively, and one defines

$$-i\Delta_{\mu\nu}(z+a, z^0; z'+a', z'^0) \equiv \theta(z^0 - z'^0) [\partial_\mu \phi^+(z+a), \partial_\nu \phi^-(z'+a')] \quad (2.78)$$

$$+ \theta(z'^0 - z^0) [\partial_\nu \phi^+(z'+a'), \partial_\mu \phi^-(z+a)]$$

$$= -i \frac{\partial^2}{\partial a^\mu \partial a'^\nu} \Delta(z+a, z^0; z'+a', z'^0) . \quad (2.79)$$

The last identity again follows directly from the definition of the contractor given in eq. (2.55). One immediately obtains

$$\Delta_{\mu\nu}(z+a, z^0; z'+a', z'^0) = \frac{1}{(2\pi)^4} \int d^4 q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} \times \quad (2.80)$$

$$\sum_{\sigma \in \{1, -1\}} (q^\sigma)_\mu (q^\sigma)_\nu \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} e^{iq^\sigma(a-a')} .$$

from eq. (2.69). The limit of a local theory is again obtained by setting  $a = a' = 0$  leaving the sum

$$\sum_{\sigma \in \{1, -1\}} (q^\sigma)_\mu (q^\sigma)_\nu \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} = q_\mu q_\nu - \delta^0_\mu \delta^0_\nu (q^2 + m^2) .$$

This is obtained by going through the calculation for the several combinations of  $\sigma$ 's in  $(q^\sigma)_\mu (q^\sigma)_\nu$ . The term where  $\sigma$  occurs twice,  $\sigma^2 \omega_{\mathbf{q}}^2$ , causes the last summand which violates Lorentz covariance. The contributions with  $\sigma$  absent or occurring only once again turn out to be Lorentz covariant as we already saw when calculating  $\Delta_\mu$ . One then obtains for the local case

$$\Delta_{\mu\nu}(z, z^0; z', z'^0) = \frac{1}{(2\pi)^4} \int d^4 q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} q_\mu q_\nu + \delta^0_\mu \delta^0_\nu \delta^4(z-z') . \quad (2.81)$$

The specific case of a local theory is identical with the results already discussed in [29, 44].

#### 2.4.2 Contractors of Dirac spinors

Now, we turn our attention to free Dirac spinors. We write its components as

$$\psi_l(x) = \psi_l^+(x) + \psi_l^-(x) \quad (2.82)$$

with

$$\psi_l^+(x) = (2\pi)^{-3/2} \sum_{\lambda=\pm\frac{1}{2}} \int d^3 p u_l(p^+, \lambda) e^{ip^+ x} a(\mathbf{p}, \lambda) , \quad (2.83)$$

$$\psi_l^-(x) = (2\pi)^{-3/2} \sum_{\lambda=\pm\frac{1}{2}} \int d^3 p v_l(p^+, \lambda) e^{-ip^+ x} a^\dagger(\mathbf{p}, \lambda) . \quad (2.84)$$

The index  $l$  can have the 4 values  $0, \dots, 3$ . Unfortunately,  $\sigma$  is already occupied and we have chosen  $\lambda$  to represent the spin degree of freedom.  $a(\mathbf{p}, \lambda)$  and  $a^\dagger(\mathbf{p}, \lambda)$  represent the annihilation operators of an electron and a positron with momentum  $\mathbf{p}$  and spin  $\lambda$ , and

their adjoints stand for the corresponding creation operators. The only non-vanishing anti-commutation relations among these creation and annihilation operators are given by

$$\{a(\mathbf{p}, \lambda), a^\dagger(\mathbf{p}', \lambda')\} = \delta_{\lambda\lambda'} \delta^3(\mathbf{p} - \mathbf{p}') , \quad (2.85)$$

$$\{a^c(\mathbf{p}, \lambda), a^{c\dagger}(\mathbf{p}', \lambda')\} = \delta_{\lambda\lambda'} \delta^3(\mathbf{p} - \mathbf{p}') , \quad (2.86)$$

with the anti-commutator defined as usual. Hermitian interactions are constructed by including pairs of spinors  $\psi$  and  $\psi^\dagger$  (containing the four components), which also ensures commuting (bosonic) instead of anti-commuting (fermionic) interactions. Non-vanishing contractions now only arise from “contracting” a spinor  $\psi(x)$  with its adjoint  $\psi^\dagger(x')$  leaving over the distributions

$$\{\psi_l^+(x), \psi_m^{+\dagger}(x')\} = (2\pi)^{-3} \int d^3p e^{ip^+(x-x')} N_{lm}(\mathbf{p}) , \quad (2.87)$$

$$\{\psi_m^{-\dagger}(x'), \psi_l^-(x)\} = (2\pi)^{-3} \int d^3p e^{-ip^+(x-x')} M_{lm}(\mathbf{p}) , \quad (2.88)$$

with

$$N_{lm}(\mathbf{p}) \equiv \sum_{\lambda} u_l(\mathbf{p}, \lambda) u_m^*(\mathbf{p}, \lambda) , \quad (2.89)$$

$$M_{lm}(\mathbf{p}) \equiv \sum_{\lambda} v_l(\mathbf{p}, \lambda) v_m^*(\mathbf{p}, \lambda) . \quad (2.90)$$

The contractor is then defined in analogy to the propagator of Dirac fields by

$$-i\Delta_{lm}(x, t; x', t') \equiv \theta(t - t') \{\psi_l^+(x), \psi_m^{+\dagger}(x')\} - \theta(t' - t) \{\psi_m^{-\dagger}(x'), \psi_l^-(x)\} . \quad (2.91)$$

The minus sign relative between the two anti-commutators is due to the fact that we are dealing with fermions. Without loss of generality, one sets  $x = z + a$ ,  $t = z^0$  and does the same for the primed variables. The first term for the contractor can then be rewritten as

$$\frac{i}{(2\pi)^4} \int d^4p \frac{e^{ip(z-z')}}{p^0 - \omega_{\mathbf{p}} + i\epsilon} e^{ip^+(a-a')} N_{lm}(\mathbf{p}) ,$$

where the Fourier representation of the step function eq. (2.59) has been used. The second term is treated in analogy, but an additional integral transformation  $p \rightarrow -p$  is used:

$$\frac{i}{(2\pi)^4} \int d^4p \frac{e^{ip(z-z')}}{-p^0 - \omega_{\mathbf{p}} + i\epsilon} e^{ip^-(a-a')} M_{lm}(-\mathbf{p}) .$$

To proceed, it is helpful to use explicit expressions for the spin sums in eqs. (2.89, 2.90) taken from [29]:

$$N(\mathbf{p}) = \frac{1}{2\omega_{\mathbf{p}}} [-ip^{+\mu} \gamma_{\mu} + m] \beta , \quad (2.92)$$

$$M(\mathbf{p}) = \frac{1}{2\omega_{\mathbf{p}}} [-ip^{+\mu} \gamma_{\mu} - m] \beta . \quad (2.93)$$

The four four-matrices  $\gamma^\mu$  have been chosen as follows:

$$\gamma^0 = -i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^j = -i \begin{bmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{bmatrix}. \quad (2.94)$$

Here, 1 in the matrices stands for the unit  $2 \times 2$  matrix and the  $\sigma_j$ 's are the well known  $2 \times 2$  Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (2.95)$$

Furthermore,

$$\beta \equiv i\gamma^0. \quad (2.96)$$

The contractor for Dirac spinors is then obtained as

$$\begin{aligned} \Delta_{lm}(z+a, z^0; z'+a', z'^0) &= \frac{1}{(2\pi)^4} \int d^4q \frac{e^{iq(z-z')}}{q^2 + m^2 - i\epsilon} \times \\ &\quad \sum_{\sigma \in \{1, -1\}} ([-iq^\sigma \gamma + m]\beta)_{lm} \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} e^{iq^\sigma(a-a')}. \end{aligned} \quad (2.97)$$

The local limit is again easily obtained by setting  $a = a' = 0$ . Since  $p^\sigma$  occurs linear in the  $\sigma$ -sum one gets

$$\Delta_{lm}(z, z^0; z', z'^0) = \frac{1}{(2\pi)^4} \int d^4q e^{iq(z-z')} \frac{([-iq\gamma + m]\beta)_{lm}}{q^2 + m^2 - i\epsilon}. \quad (2.98)$$

Note that  $\beta$  still occurs due to the fact that we have considered the pairing of  $\psi(x)$  and  $\psi^\dagger(x')$ , but due to the interaction one has to contract  $\psi(x)$  and  $\bar{\psi}(x') \equiv \psi^\dagger(x')\beta$ .  $\beta$  then drops out due to  $\beta^2 = 1$ .

### 2.4.3 The photon contractor in Coulomb gauge

Photons are much more difficult to handle than the cases discussed above. The reason is the gauge invariance. In [29], it is introduced to construct a Lorentz covariant theory out of massless particles with helicity  $\pm 1$ , say photons. The gauge invariance causes constraints (see [45] for a detailed discussion) when studying the canonical quantisation of the theory. Within the framework of this chapter, the formalism of time-ordered perturbation theory, the first class constraints are assumed to be resolved by choosing a gauge. It is comfortable to use the Coulomb gauge (see [29] for a detailed discussion). The modern method of BRST-quantisation is not discussed here.

The free photon field operator is now written as

$$A^\mu(x) = A^{\mu+}(x) + A^{\mu-}(x), \quad (2.99)$$

with

$$A^{\mu+}(x) = (2\pi)^{-3/2} \sum_{\lambda=\pm 1} \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}} e^{ip^+x} e^\mu(\mathbf{p}, \lambda) a(\mathbf{p}, \lambda), \quad (2.100)$$

$$A^{\mu-}(x) = (2\pi)^{-3/2} \sum_{\lambda=\pm 1} \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}} e^{-ip^+x} e^{\mu*}(\mathbf{p}, \lambda) a^\dagger(\mathbf{p}, \lambda). \quad (2.101)$$

$a(\mathbf{p}, \lambda)$  and  $a^\dagger(\mathbf{p}, \lambda)$  are the annihilation and creation operators of a photon with momentum  $\mathbf{p}$  and helicity  $\lambda$ .  $\omega_{\mathbf{p}} = |\mathbf{p}|$  and  $e^\mu(\mathbf{p}, \lambda)$  are any two polarisation vectors satisfying

$$\mathbf{p} \cdot \mathbf{e}(\mathbf{p}, \lambda) = 0, \quad (2.102)$$

$$e^0(\mathbf{p}, \lambda) = 0. \quad (2.103)$$

Their normalisation is chosen to satisfy

$$\sum_{\lambda} e^i(\mathbf{p}, \lambda) e^{j*}(\mathbf{p}, \lambda) = \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}. \quad (2.104)$$

This implies the commutation relations

$$[a(\mathbf{p}, \lambda), a^\dagger(\mathbf{p}', \lambda')] = \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\lambda\lambda'}, \quad (2.105)$$

$$[a(\mathbf{p}, \lambda), a(\mathbf{p}', \lambda')] = [a^\dagger(\mathbf{p}, \lambda), a^\dagger(\mathbf{p}', \lambda')] = 0. \quad (2.106)$$

It is clear how to define the photon contractor (not to be confused with the contractor of two derivatives of scalar fields, which is written the same way):

$$-i\Delta^{\mu\nu}(x, t; x', t') \equiv \theta(t - t') [A^{\mu+}(x), A^{\nu-}(x')] + \theta(t' - t) [A^{\nu+}(x'), A^{\mu-}(x)]. \quad (2.107)$$

The two terms on the right hand-side can easily be calculated using the decomposition of eqs. (2.100, 2.101) for the field operators, the commutation relations given in eqs. (2.105, 2.106), and the Fourier representation of the step function eq. (2.59). Substituting  $x = z + a$ ,  $t = z^0$ ,  $x' = z' + a'$ ,  $t' = z'^0$ , one obtains

$$\frac{i}{(2\pi)^4} \int \frac{d^4 p}{2\omega_{\mathbf{p}}} \frac{e^{ip(z-z')}}{p^0 - \omega_{\mathbf{p}} + i\epsilon} e^{ip^+(a-a')} P_{\mu\nu}(\mathbf{p}) \quad (2.108)$$

and

$$\frac{i}{(2\pi)^4} \int \frac{d^4 p}{2\omega_{\mathbf{p}}} \frac{e^{ip(z-z')}}{-p^0 - \omega_{\mathbf{p}} + i\epsilon} e^{ip^-(a-a')} P_{\nu\mu}(-\mathbf{p}) \quad (2.109)$$

for the two summands in eq. (2.107), where

$$P_{\mu\nu}(\mathbf{p}) \equiv \sum_{\lambda} e^\mu(\mathbf{p}, \lambda) e^{\nu*}(\mathbf{p}, \lambda) \quad (2.110)$$

is already known from eqs. (2.104) and (2.103):

$$P_{ij}(\mathbf{p}) = \delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}, \quad (2.111)$$

and  $P_{0\mu}(\mathbf{p}) = 0$ . From this, one sees that  $P_{\mu\nu}(\mathbf{p}) = P_{\nu\mu}(\mathbf{p})$  and  $P_{\mu\nu}(-\mathbf{p}) = P_{\mu\nu}(\mathbf{p})$ . The whole photon contractor is now given by

$$\begin{aligned} \Delta_{\mu\nu}(z + a, z^0; z' + a', z'^0) &= \frac{1}{(2\pi)^4} \int d^4 p \frac{e^{ip(z-z')}}{p^2 + m^2 - i\epsilon} \times \\ &P_{\mu\nu}(\mathbf{p}) \sum_{\sigma \in \{1, -1\}} \frac{\omega_{\mathbf{p}} + \sigma p^0}{2\omega_{\mathbf{p}}} e^{ip^\sigma(a-a')}. \end{aligned} \quad (2.112)$$

$P_{\mu\nu}(\mathbf{p})$  does neither involve  $p^0$  nor  $\omega_{\mathbf{p}}$ . The local limit  $a = a' = 0$  yields

$$\Delta_{\mu\nu}(z, z^0; z', z'^0) = \frac{1}{(2\pi)^4} \int d^4 p \frac{e^{ip(z-z')}}{p^2 + m^2 - i\epsilon} P_{\mu\nu}(\mathbf{p}) , \quad (2.113)$$

which agrees with the calculation carried out in [29]. There, the covariant Feynman rules are derived from this non-covariant propagator by discussing the cancellation of the non-covariant terms with the non-covariant Coulomb potential present in the interaction Hamiltonian when calculated in the Coulomb gauge.

## 2.5 Applications

In this section, the vertex factors for three types of non-local interactions will be calculated making the Feynman rules of the last sections ready to use. First, the case of NCQFT will be treated, where a tadpole contribution will also be evaluated explicitly. Furthermore, we will examine the connected part of a two point Green function to first order in a quadratic interaction with vertex factor  $\chi_2$ . Then, the vertex factor will be calculated for the case of non-local interactions of Gaussian type introduced in [32]. Finally, we will present a simplified version of these interactions.

### 2.5.1 Non-commutative interactions

In this section, we study interactions of the following type:

$$V_k(z^0) = \frac{\kappa}{k!} \int d^3 z (\phi(z))^*{}^k , \quad (2.114)$$

where the  $*$  in the exponent indicates that the star product is to be used between the  $k$  fields. For  $k = 4$ , this is just the interaction density given in eq. (2.42) integrated over  $d^3 z$ . In general, one can write

$$V_k(z^0) = \frac{\kappa}{k!} \int d^3 z \prod_{i=1}^{k-1} \left( \int \frac{d^4 l_i d^4 s_i}{(2\pi)^4} e^{il_i s_i} \phi(z - \frac{1}{2} \tilde{l}_i + \sum_{j=1}^{i-1} s_j) \right) \phi(z + \sum_{j=1}^{k-1} s_j) . \quad (2.115)$$

These interactions are of the type as specified in eqs. (2.61-2.63) and thus obey momentum conservation. The factor  $\chi_k$  of eq. (2.71) can then be evaluated for the interactions  $V_k$  (abbreviating  $p_i \equiv q_i^{\sigma_i}$ ):

$$\chi_k(p_1, \dots, p_k) = \frac{\kappa}{k!} \sum_{Q \in S^k} \exp \left( -i \sum_{i < j} p_{Q_i} \wedge p_{Q_j} \right) \quad (2.116)$$

with

$$a \wedge b \equiv \frac{1}{2} \theta_{\mu\nu} a^\mu b^\nu = -\frac{1}{2} a \tilde{b} .$$

$\chi_2$  is simply

$$1/\kappa \chi_2(p_1, p_2) = \cos(p_1 \wedge p_2) \quad (2.117)$$

and

$$\begin{aligned}
3/\kappa \chi_3(p_1, p_2, p_3) &= \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_2 \wedge p_3) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 - p_2 \wedge p_3) \\
&+ \cos(p_1 \wedge p_2 - p_1 \wedge p_3 - p_2 \wedge p_3) .
\end{aligned} \tag{2.118}$$

Unfortunately,  $\chi_4$  becomes really lengthy:

$$\begin{aligned}
12/\kappa \chi_4(p_1, p_2, p_3, p_4) &= \\
&\cos(p_1 \wedge p_2 - p_1 \wedge p_3 - p_1 \wedge p_4 - p_2 \wedge p_3 - p_2 \wedge p_4 - p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 - p_1 \wedge p_4 - p_2 \wedge p_3 - p_2 \wedge p_4 - p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 - p_2 \wedge p_3 - p_2 \wedge p_4 - p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 - p_1 \wedge p_4 + p_2 \wedge p_3 - p_2 \wedge p_4 - p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 + p_2 \wedge p_3 - p_2 \wedge p_4 - p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 + p_2 \wedge p_3 + p_2 \wedge p_4 - p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 - p_1 \wedge p_3 - p_1 \wedge p_4 - p_2 \wedge p_3 - p_2 \wedge p_4 + p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 - p_1 \wedge p_3 + p_1 \wedge p_4 - p_2 \wedge p_3 - p_2 \wedge p_4 + p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 - p_2 \wedge p_3 - p_2 \wedge p_4 + p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 - p_1 \wedge p_3 + p_1 \wedge p_4 - p_2 \wedge p_3 + p_2 \wedge p_4 + p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 - p_2 \wedge p_3 + p_2 \wedge p_4 + p_3 \wedge p_4) \\
&+ \cos(p_1 \wedge p_2 + p_1 \wedge p_3 + p_1 \wedge p_4 + p_2 \wedge p_3 + p_2 \wedge p_4 + p_3 \wedge p_4) .
\end{aligned} \tag{2.119}$$

These results can be compared to [31], and they agree.

The result for  $k = 2$  is remarkable. The connected part of a two point Green function to first order in a quadratic interaction with vertex factor  $\chi_2$  can be written as

$$G_1^2(p, q)^{con} = \kappa \delta(p + q) \frac{(-i)^3 (2\pi)^4}{(p^2 + m^2 - i\epsilon)^2} \frac{\omega_{\mathbf{p}}^2 + p^{02} + (\omega_{\mathbf{p}}^2 - p^{02}) \cos(p^+ \wedge p^-)}{2\omega_{\mathbf{p}}^2} . \tag{2.120}$$

It is surprising that this term contains a phase factor which only vanishes for  $\theta_{0i} = 0$ . On-shell, it does not have any unusual effect since  $\omega_{\mathbf{p}}^2 = (p^0)^2$ . But off-shell it might be used as an extra counterterm within loops or other internal lines. If one starts with the naive Lagrangian approach, quadratic terms do not contain phases since

$$\int d^4x (f * g)(x) = \int d^4x f(x) g(x) .$$

This means that the free action is the same as ordinarily and the free Hamiltonian of non-commutative theory is the same as in the commutative case. Apparently, this might seem problematic for our work since we used  $H_0$  of the commutative case as free Hamiltonian. But this problem can be resolved the following way: Suppose, there was some non-commutative full Hamiltonian  $H_*$ . Then one can simply extract the interaction  $V$  by definition:

$$V \equiv H_* - H_0 .$$

The question of in- and out-states and their asymptotic behaviour, which is important for the physical meaning of the presented calculations, must be discussed elsewhere. The result



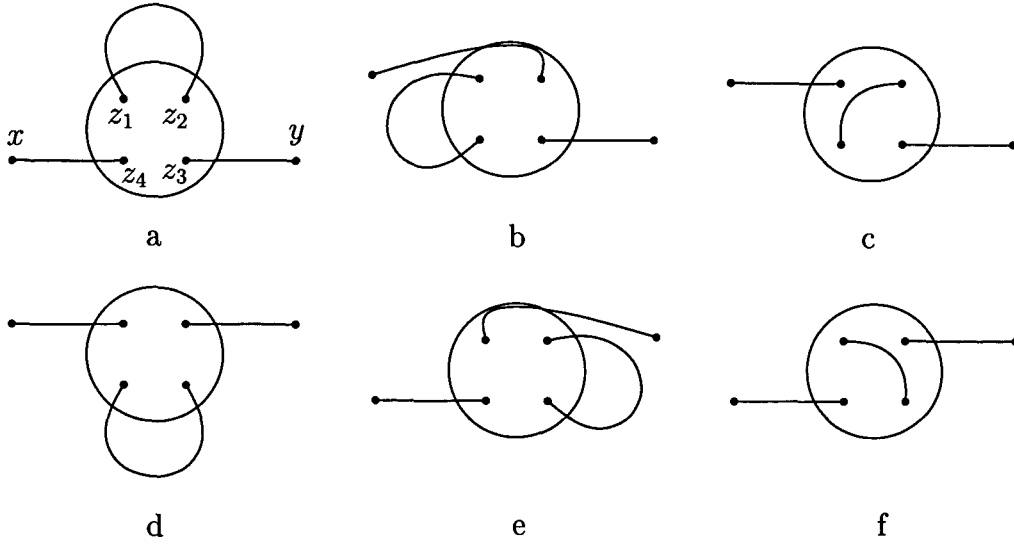


Figure 2.2: This figure shows half of the diagrams contributing to  $G_{tp}(x, y)$ . The crossed diagrams which are obtained by exchanging  $x$  and  $y$  have been omitted. Diagrams b - f are considered to be labelled like a. We have used the abbreviations  $z_1 = z - \frac{1}{2}\tilde{l}_1$ ,  $z_2 = z - \frac{1}{2}\tilde{l}_2 + s_1$ ,  $z_3 = z - \frac{1}{2}\tilde{l}_3 + s_1 + s_2$ ,  $z_4 = z + s_1 + s_2 + s_3$ .

presented in eq. (2.120) shows that the naive Lagrangian approach and time ordered perturbation theory for non-local interactions worked out here are not equivalent when non-locality also involves time ( $\theta_{0i} \neq 0$ ). For the case  $\theta_{0i} = 0$ , it is easy to verify the agreement of the rules given here and the usual treatment (see [19] for example) by simply realizing that  $a \wedge b$  is independent of  $a^0$  and  $b^0$ .

Finally, the calculation of a tadpole according to the coordinate space rules should be given, namely

$$G_{tp}(x, y) \equiv -i \int dz^0 \langle 0 | T \{ \phi(x) \phi(y) V_4(z^0) \} | 0 \rangle_H^{con}.$$

In this case, all connected contributions contain exactly one line starting and ending in the circle representing the interaction, which means that one only has diagrams containing tadpoles. Thus, we cannot use the momentum space rules which have not been worked out for tadpole contributions, but we have to take the somewhat lengthier coordinate space rules. We simply consider this as an exercise and check of our approach: using normal ordered interactions, there are no tadpole contributions. Therefore, we do not see any directly physical relevance of such contributions. The result is obtained by calculating the diagrams shown in fig. 2.2 and the corresponding crossed diagrams which are obtained by simply exchanging  $x$  and  $y$ . Diagram c and its crossed one give

$$-i \int d^4 z \prod_{i=1}^3 \left( \frac{d^4 l_i d^4 s_i}{(2\pi)^4} e^{il_i s_i} \right) [\Delta^+(z_2, z_4) (-i\Delta(x, x^0; z_1, z^0)) (-i\Delta(y, y^0; z_3, z^0)) + x \leftrightarrow y],$$

with  $z_1, \dots, z_4$  as described in fig. 2.2. Summing up all 12 diagrams and carrying out the

trivial integrations, one arrives at

$$\begin{aligned}
 G_{tp}(x, y) = & \frac{i\kappa}{4!} \int \frac{d^4 q}{(2\pi)^4 \omega_q^2} \frac{d^3 k}{(2\pi)^3 2\omega_k} e^{iq(x-y)} \\
 & \left[ \frac{2 + \cos(q^+ \tilde{k}^+)}{(q^0 - \omega_q + i\epsilon)^2} + \frac{2 + \cos(q^- \tilde{k}^+)}{(q^0 + \omega_q - i\epsilon)^2} \right. \\
 & \left. - \cos\left(\frac{1}{2} q^- \tilde{q}^+\right) \frac{3 + \cos(q^+ \tilde{k}^+) + \cos((q^+ - q^-) \tilde{k}^+) + \cos(q^- \tilde{k}^+)}{(q^{02} - \omega_q^2 + i\epsilon)} \right] .
 \end{aligned} \tag{2.121}$$

This is consistent with the result of [28]. Diagrams c and f (and their crossed counterparts) shown in fig. 2.2 are responsible for the contributions containing phase factors at the pole of order two at  $q^0 \rightarrow \omega_q$  of the integrand of eq. (2.121). Furthermore, it is interesting that the momentum  $q$  of external lines always occurs as  $q^\pm$  in the cos-functions. For  $\theta_{0i} = 0$ , it would not make any difference whether we write  $q^\pm \tilde{k}^\pm$  or  $q\tilde{k}$ , but for  $\theta_{0i} \neq 0$ , it definitely does. This new aspect also effects the UV/IR-mixing problem discussed in [46].

### 2.5.2 Non-local interactions of Gaussian type

It can easily be seen that the interaction of eq. (2.46) proposed in [32] is translation invariant in the sense of eqs. (2.61-2.63). So the momentum space rules are applicable. In order to specify perturbation theory further, one simply has to calculate  $\chi_k$  of eq. (2.71):

$$\begin{aligned}
 \chi_k(p_1, \dots, p_k) = & \kappa c_k \int d^4 a_1 \dots d^4 a_k \times \\
 & \exp \left\{ -\frac{1}{2} \sum_{j=1}^k (a_j^\mu)^2 \right\} \delta^4 \left( \frac{1}{k} \sum_{j=1}^k a_j \right) \sum_{Q \in S^k} \exp \left( -i \sum_{j=1}^k p_{Q_j} \zeta a_j \right) ,
 \end{aligned} \tag{2.122}$$

where we have abbreviated  $p_i \equiv q_i^{\sigma_i}$ . A lengthy but standard calculation involving a multiple Gaussian integral then yields (cf. [32])

$$\chi_k(p_1, \dots, p_k) = \kappa (2\pi)^{2k-2} (k-1)! c_k e^{-k \zeta^2 (\overline{(p^\mu)^2} - (\overline{p^\mu})^2)} , \tag{2.123}$$

with  $\overline{A}$  representing the mean value

$$\overline{A} \equiv \frac{1}{k} \sum_{j=1}^k A_j .$$

We have written  $\chi_k$  in such a statistical manner in order to see that the exponent in eq. (2.123) is always negative. Thus, it seems plausible that the exponential damping caused by  $\chi_k$  makes the contribution of all diagrams finite. Indeed,  $\int dx p_n(x) \exp(-x^2)$  is always finite for  $p_n$  a polynomial of order  $n$ . But one has to keep in mind that  $\chi_k$  never involves time components  $q_i^0$  of the four momenta  $q_i$ , but  $q_i^{\sigma_i}$  instead. So it is clear that one separately has to check whether the integrations over time components are finite or not. Concerning the integrations over the 3-momenta, it seems plausible that they are finite due to the exponential damping. Investigations concerning these aspects are still to be carried out within the presented formalism, especially also for theories involving other types of particles.

It was shown in [32] that if the interaction is adiabatically switched off, all expectation values are finite at any order of perturbative expansion due to the presence of the Gaussian kernels. In the proof, the Wick theorem was applied without the detour of giving explicit graphical rules.

### 2.5.3 Simplified non-local interactions of Gaussian type

It would be interesting to carry out perturbative calculations for diagrams involving at least one loop to explicitly see the UV-finiteness of the theory studied in the last section. Only the Feynman rules of the previous chapter and the vertex function given in eq. (2.123) are needed, and one can start calculating the remaining integrals for diagrams involving loops. However, these integrals immediately become very complicated. The main reason is the fact that the momenta fed into eq. (2.123) are not the off-shell four momenta  $p$ , but  $p^\sigma$ . This means that the exponent of  $\chi_k(p_1^{\sigma_1}, \dots, p_k^{\sigma_k})$  becomes complicated due to its dependence on on-shell energies:

$$\overline{((p^\sigma)^\mu)^2} - \left(\overline{(p^\sigma)^\mu}\right)^2 = m^2 + \frac{2}{k} \sum_{i=1}^k \mathbf{p}_i^2 - \left(\frac{1}{k} \sum_{i=1}^k \mathbf{p}_i\right)^2 - \left(\frac{1}{k} \sum_{i=1}^k \sigma_i \omega_{\mathbf{p}_i}\right)^2.$$

The undesired complications for calculations are represented by the last two terms which come from the square of the mean momenta. Carrying out the calculations of the last section in detail, one sees that this arises from the  $\delta$ -function in eq. (2.122). Therefore, we propose interactions of the type

$$\begin{aligned} V_k(z^0) = & \kappa c_k \int d^3 z \int d^4 a_1 \dots d^4 a_k \phi(z + \zeta a_1) \dots \phi(z + \zeta a_k) \times \\ & \exp \left\{ -\frac{1}{2} \sum_{j=1}^k (a_j^\mu)^2 \right\}. \end{aligned} \quad (2.124)$$

Clearly, this interaction factorises in terms of *smeared* field operators

$$\int d^4 a e^{-\frac{1}{2}(a^\mu)^2} \phi(z + \zeta a).$$

It turns out that simple calculations involving one loop are feasible for this approach due to the simpler vertex function with the exponent proportional to

$$-\overline{\mathbf{p}^2}.$$

It will be discussed slightly generalised and more detailed in chapter 3.

## 2.6 Summary

In this chapter, we have first repeated the framework of QFT as presented in [29], and especially the basis of time-ordered perturbation theory was discussed. It was seen that the approach via the Dyson series is also applicable to non-local interactions since the detailed structure of the interaction is not relevant for the derivation. Then, a general type of non-local interactions was introduced (see eq. (2.45)) for neutral, scalar particles, which is capable

of embedding NCQFT also for  $\theta_{0i} \neq 0$ , the non-local interactions of Gaussian type of [32], a simplified version thereof, and, as a matter of course, also local QFT. Beside these cases already studied here, there might emerge a lot of more examples the results of this chapter are applicable to. However, this is only valid as long as the in- and out-states evolve according to the usual free Hamiltonian and the full Hamiltonian must conserve energy. One can then use the Feynman rules in coordinate space given in section 2.3.1. The basis of these rules is a generalised version of the Wick theorem able to deal with vacuum expectation values of several field operators in arbitrary order. This is necessary to handle the more general time-ordering prescription. A full proof of the needed theorem is given in appendix A, where also the concept of the contractor is introduced for the general case in connection with the required time-ordering. This then directly leads to the Feynman rules in coordinate space, where the contractor (see eq. (2.55)) replaces the propagator. For the vertices, one has to carry out an additional integration over the non-localities, which the arguments of the contractors depend on. The advantage of these rules over other approaches [30, 28] is the fact that time-ordering is already included in the contractor, and a single diagram is needed to respect all different possibilities of time-ordering. However, an application to more complex diagrams is very complicated. This is because within a given vertex one has to consider all possibilities of contracting the fields with other ones, yielding a new diagram for each possibility.

For the case that the Hamiltonian is also invariant under space translations, the Feynman rules of momentum space given in section 2.3.4 can be applied. These are simpler to use since diagrams only differing by a rearrangement of fields within vertices are represented by a single graph, which is identical with the corresponding Feynman diagram of local QFT. This observation was also made one year later in [43], but with the only difference that those considerations are based on a symmetrisation of the interaction with respect to identical fields. But this is only possible when using a normal ordered product of field operators, and tadpole contributions are therefore excluded automatically. The connection of the Feynman rules in momentum space given here to the usual ones of local QFT is quite obvious. One uses pole factors (see expression (2.74)) instead of the propagators, and for each vertex, one has to include a vertex factor (see eq. (2.71)). Furthermore, an additional summation over  $\sigma$ -labels has to be carried out for each line. The rules are still more complicated to use than local Feynman rules. Loop calculations for example will be very complicated. This is mainly because the pole factors and the vertex factors explicitly depend on on-shell energies. However, we succeeded in reducing the tremendous, combinatorial amount of directly evaluating expectation values of time-ordered products of field operators considerably. In the limit of a local QFT, one easily recovers the usual rules, and also in the case of NCQFT with  $\theta_{0i} = 0$ , one obtains the simple rules given in [18, 19]. Beside NCQFT, the vertex factor has also been calculated for two other approaches, namely non-local interactions of Gaussian type [32] and simplified non-local interactions of Gaussian type to be discussed in the following chapter.

The concept of the contractor has also been extended intuitively to more complicated fields. Besides contractors of derivatives of scalar fields, this concept was applied to Dirac spinors and photons in Coulomb gauge. A more appropriate treatment of gauge fields has not been worked out so far, since the concept of BRST-quantisation is usually carried out within the Lagrangian formulation. But we are interested in the case where non-locality involves time and the simple Lagrangian approach cannot be used then.

## Chapter 3

# Towards UV-finite QFTs

A simplified version of the UV-finite QFT introduced in [32] has already been discussed in section 2.5.3. It led to the occurrence of smeared field operators in the interaction. Our approach will be presented in section 3.2, and the corresponding Feynman rules will also be given there. Another approach resulting in very similar Feynman rules is based on an oscillator representation of non-commutative space-time [47, 48, 49]. The following section will focus on the presentation given in [49]. In section 3.3, we will consider 1-loop corrections in order to extend the classical theory. We will see that these contributions are finite.

### 3.1 The oscillator representation

Scalar field theory in  $D = 2 + 1$  dimensions is considered. The time component seems artificial. In this sense, the results obtained in [47, 48] for four-dimensional Euclidean space agree, corresponding to  $D = 4 + 1$  dimensional Minkowski space according to [49].

In [49], time commutes with the spatial coordinates which satisfy the relation

$$[\hat{x}^i, \hat{x}^j] = i\theta\epsilon^{ij} , \quad (3.1)$$

$i, j = 1, 2$ . Further on, there are the usual commutation relations with the momenta,

$$[\hat{x}^i, \hat{p}_j] = i\delta^i_j , \quad [\hat{p}_i, \hat{p}_j] = 0 . \quad (3.2)$$

New coordinates  $\hat{z}$  and  $\hat{z}^\dagger$  are introduced [50],

$$\begin{aligned} \hat{z} &= \frac{1}{\sqrt{2}}(\hat{x}^1 + i\hat{x}^2) , \\ \hat{z}^\dagger &= \frac{1}{\sqrt{2}}(\hat{x}^1 - i\hat{x}^2) \end{aligned} \quad (3.3)$$

in order to obtain

$$[\hat{z}, \hat{z}^\dagger] = \theta . \quad (3.4)$$

$\hat{z}$  and  $\hat{z}^\dagger$  can be established as annihilation and creation operators of a harmonic oscillator, and coherent states can be used as a basis of the Fock space. Coherent states  $|z\rangle$  are eigenstates of the annihilation operator,

$$\hat{z} |z\rangle = z |z\rangle , \quad \langle z| \hat{z}^\dagger = \bar{z} \langle z| . \quad (3.5)$$

They are given by

$$|z\rangle = \exp\left(-\frac{z\bar{z}}{2\theta} - \frac{z}{\theta}\hat{z}^\dagger\right)|0\rangle, \quad (3.6)$$

satisfying the completeness relation

$$\frac{1}{\pi\theta} \int dz d\bar{z} |z\rangle \langle z| = 1. \quad (3.7)$$

Coherent states are not orthogonal, however,

$$\langle w|z\rangle = \exp\left(-\frac{|z|^2 + |w|^2}{2\theta} - \frac{\bar{w}z}{\theta}\right). \quad (3.8)$$

Via expectation values, one can assign ordinary functions to any operator  $F(\hat{x}^1, \hat{x}^2)$ ,

$$F(z) \equiv \langle z| F(\hat{x}^1, \hat{x}^2) |z\rangle. \quad (3.9)$$

The algebraic structure of the non-commutative algebra (3.1) is properly taken care of, which can be seen to first order in  $\theta$  from

$$\langle z| [\hat{x}^1, \hat{x}^2] |z\rangle = i\theta. \quad (3.10)$$

With the expansion of a real scalar free field operator <sup>1</sup>

$$\begin{aligned} \phi(t, z) &= \int \frac{d^2p}{2\pi} a(\mathbf{p}) \exp(-i\omega_{\mathbf{p}}t) \langle z| \exp(ip_j \hat{x}^j) |z\rangle + h.c., \\ (\square_x + m^2)\phi(t, x) &= 0, \end{aligned} \quad (3.11)$$

the propagator — defined as the expectation value of a time-ordered product of field operators — becomes

$$\begin{aligned} G(t_1 - t_2, z_1 - z_2) &= \langle 0| T\{\phi(t_1, z_1)\phi(t_2, z_2)\} |0\rangle \\ &= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{-1}{p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon} \exp\left(-\frac{\theta}{2}\mathbf{p}^2\right) \exp(-ip_0(t_1 - t_2)) \\ &\quad \times \exp\left(i\frac{p_1}{\sqrt{2}}(z_1 - z_2 + \bar{z}_1 - \bar{z}_2) + i\frac{p_2}{\sqrt{2}}(z_1 - z_2 - \bar{z}_1 + \bar{z}_2)\right), \end{aligned} \quad (3.12)$$

where we assume the usual commutation relation

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = \delta^2(\mathbf{p} - \mathbf{q}).$$

Note that an integration over  $p_0$  has been introduced to represent the step function of time-ordering in Fourier space. This propagator is the "Green function" of the ordinary Klein-Gordon equation, with the exception that the delta function is replaced by an approximate (smeared) delta function,

$$\begin{aligned} (\square_1 + m^2) G(t_1 - t_2, z_1 - z_2) &= (-\partial_{t_1}^2 + 2\partial_{z_1}\partial_{\bar{z}_1} + m^2) G(t_1 - t_2, z_1 - z_2) \\ &= \frac{2\pi\delta(t_1 - t_2)}{\theta} \exp\left(-\frac{1}{4\theta}(z_1 - z_2 + \bar{z}_1 - \bar{z}_2)^2 + \frac{1}{4\theta}(z_1 - z_2 - \bar{z}_1 + \bar{z}_2)^2\right). \end{aligned} \quad (3.13)$$

<sup>1</sup>In this section, we tolerate the different notation for 4-vectors, and the metric  $(x_0, x_1, x_2) = (x^0, -x^1, -x^2)$ .

In this case, the “free propagator” is modified. It experiences an exponential damping (3.13). It is important to note that the non-commutativity is related to exponentially damped propagators. This fact motivates our model.

In what follows, we will consider modifications in the interaction only. There, we replace the local field operators  $\phi$  by smeared, non-local fields  $\phi_M$ , as discussed in the next section. Therefore, the free propagators are not modified. Internal lines, however, might be considered to be modified by an exponential damping factor, similar to the propagator (defined as the vacuum expectation value of two time-ordered field operators) given above. Let us emphasise the difference again: in the above approach, the “free propagator” is damped, whereas our model possesses ordinary free propagators, but damped internal lines.

### 3.2 Smeared field operators

We want to study the effect of replacing the scalar field operators  $\phi(x)$  by blurred operators, smeared over space-time

$$\phi_M(x) \equiv N \int d^n b \, e^{-b^T b} \phi(x + Mb) , \quad (3.14)$$

where  $b$  is a real Euclidean  $n$ -dimensional vector,  $M$  is a real  $4 \times n$  matrix.  $N$  denotes a normalisation constant. The integration parameters  $b^i$  are assumed to be dimensionless. Therefore, the matrix elements of  $M$  have dimension of length. The non-vanishing matrix  $M$  generates the non-locality. We will denote Minkowski indices by Greek letters, Euclidean indices by Roman letters. Therefore, the index structure of  $M$  is  $M^\mu_i$ . However, the case  $n > 4$  can be reduced to the case  $n = 4$ . Due to the possibility of a QR-decomposition, the matrix  $M$  can be written as a product of a  $4 \times n$  matrix  $\tilde{R}$  and an orthogonal  $n \times n$  matrix  $Q$ . The first 4 columns of  $\tilde{R}$  contain a lower triangular  $4 \times 4$  matrix  $R$ , all other entries are zero,

$$M = \begin{bmatrix} R & 0 \end{bmatrix} Q \equiv \tilde{R} Q . \quad (3.15)$$

The orthogonal matrix  $Q$  can be absorbed in an integral transformation,  $\tilde{b} = Q b$ , and we get

$$\phi_M(x) = N \int d^n \tilde{b} \, e^{-\tilde{b}^T \tilde{b}} \phi(x + \tilde{R} \tilde{b}) . \quad (3.16)$$

Since  $\tilde{R}$  has the form shown in (3.15), the integration over the variables  $\tilde{b}_5, \dots, \tilde{b}_n$  are Gaussian integrals which merely redefine the normalisation constant. Hence, only 4 dimensions are left. From now on, we will stick to that case. Since the newly defined field operators  $\phi_M(x)$  are superpositions of the free field operators  $\phi(x)$ ,  $\phi_M(x)$  satisfies the free Klein-Gordon equation. Alternatively, one can demand that  $\phi_M(x)$  is a solution of the free Klein-Gordon equation,

$$(\Box_x - m^2) \phi_M(x) = 0 . \quad (3.17)$$

The Fourier transform is given by

$$\phi(x + Mb) = \int \frac{d^4 k}{(2\pi)^2} e^{ik(x+Mb)} \tilde{\phi}(k) . \quad (3.18)$$

Due to the Klein-Gordon equation, we can find a nice expression for the smeared field operators  $\phi_M(x)$ ,

$$\begin{aligned}\phi_M(x) &= (2\pi)^{-3/2} N \int \frac{d^3p}{\sqrt{2\omega_p}} \left[ a(\mathbf{p}) e^{ip^+x} + a^\dagger(\mathbf{p}) e^{-ip^+x} \right] \\ &\quad \times \int d^4b e^{-b^r b^r + i p_\mu^+ M^\mu{}_{\nu} b^\nu} \\ &= (2\pi)^{-3/2} \pi^2 N \int \frac{d^3p}{\sqrt{2\omega_p}} \left[ a(\mathbf{p}) e^{ip^+x} + a^\dagger(\mathbf{p}) e^{-ip^+x} \right] \\ &\quad \times \exp\left(-\frac{1}{4} p_\mu^+ p_\nu^+ \kappa^{\mu\nu}\right) .\end{aligned}\tag{3.19}$$

$a$  and  $a^\dagger$  obey the canonical commutation relations

$$[a(\mathbf{p}), a^\dagger(\mathbf{q})] = \delta^3(\mathbf{p} - \mathbf{q}) .$$

Summation over repeated indices is implied. Furthermore, we have used the definition

$$\kappa^{\mu\nu} \equiv M^\mu{}_\tau M^\nu{}_\tau = (M M^T)^{\mu\nu} .\tag{3.20}$$

The matrix  $\kappa$  is symmetric. For real  $M$ , its eigenvalues are always larger than or equal to zero, i.e.  $\kappa$  is positive semidefinite. The exponential factor in (3.19) is always damping,

$$\exp\left(-\frac{1}{4} p_\mu^+ p_\nu^+ \kappa^{\mu\nu}\right) \leq 1 .$$

As we will see below, it is  $\kappa^{\mu\nu}$  which characterises the perturbation theory, not  $M$  itself. Therefore, we only have to choose an appropriate matrix  $\kappa^{\mu\nu}$  in order to do perturbation theory, ensuring that the matrix can be reproduced by  $M M^T$ . A tempting choice is  $\kappa^{\mu\nu} \propto g^{\mu\nu}$ , but  $g$  is neither positive nor negative semidefinite. The choice  $\kappa = 0$  reproduces local field theory.

We want to study the perturbative quantisation of this kind of deformation, within time-ordered perturbation theory elaborated in chapter 2. The deformed Hamiltonian is defined as

$$H^* = H_0 + V^* ,\tag{3.21}$$

where  $H_0$  denotes the free undeformed Hamiltonian of the theory. We have replaced the scalar fields by the smeared fields (3.14),  $\phi \rightarrow \phi_M$  in the interaction part of the Hamiltonian only. The free Hamiltonian  $H_0$  is unaltered. Of course, it would be more natural to deform  $H_0 \rightarrow H_0^*$  also. Then the applicability of the perturbation theory elaborated in 2 is related to the question whether  $H_0^* = H_0$  is true or not. If  $H_0^* \neq H_0$ , we have to define the interaction Hamiltonian as  $\tilde{V} = V^* + (H_0^* - H_0)$ . In this case, we also have to make sure that the time dependence of  $\tilde{V}$  is given by

$$\tilde{V}(t) = e^{iH_0 t} \tilde{V}(0) e^{-iH_0 t} ,\tag{3.22}$$

and the asymptotic behaviour is still governed by  $H_0$  and not  $H_0^*$ . The question if  $H_0^* = H_0$  or not is discussed in more detail in chapter 4.



Let us examine perturbation theory arising from eq. (3.21), leaving the free Hamiltonian  $H_0$  undeformed by definition. The interaction corresponding to  $\phi^k$  is deformed as follows:

$$\begin{aligned} V^*(x^0) &\equiv \frac{\lambda}{k!} \int d^3x \phi_M^k(x) \\ &= \frac{\lambda}{k!} N^k \int d^3x \int d^4b_1 \dots d^4b_k \times \\ &\quad e^{-\sum_i b_i^T b_i} \phi(x + Mb_1) \dots \phi(x + Mb_k). \end{aligned} \quad (3.23)$$

This is obviously translation invariant. Therefore, we will first relate eq. (3.23) to the notation introduced in chapter 2 in order to apply the momentum space rules given there for a general non-local interaction. The interaction has the general form

$$V(z^0) = \int d^3z \int d\mu w(\mu) \phi(z + h_1(\mu)) \dots \phi(z + h_k(\mu)), \quad (3.24)$$

where

$$\begin{aligned} \underline{\mu} &= (b_1^1, b_1^2, b_1^3, b_1^4, b_2^1, \dots, b_k^3, b_k^4), \\ w(\underline{\mu}) &= e^{-\sum_{j=1}^k b_j^T b_j}, \\ h_s(\underline{\mu}) &= M \cdot b_s, s = 1, \dots, k. \end{aligned}$$

From this, the vertex function  $\chi_k$  can then immediately be calculated:

$$\begin{aligned} \chi_k(p_1^{\sigma_1}, \dots, p_k^{\sigma_k}) &= \frac{\lambda}{k!} N^k \int d^n b_1 \dots d^n b_k e^{-\sum_j b_j^T b_j} \sum_{Q \in S^k} \exp(-i \sum_j p_j^{\sigma_j} M a_{Q_j}) \\ &= \lambda \exp(-\frac{1}{4} \sum_i p_i^{\sigma_i T} \kappa p_i^{\sigma_i}), \end{aligned} \quad (3.25)$$

where we have summed over all permutations  $Q \in S^k$  of the external momenta. By definition, the above integral is independent of the order of the momenta  $p_i$ . Remarkably, there are only on-shell momenta involved because of eq. (3.19). We have chosen

$$N = \pi^{-2}.$$

Note that

$$p^T \kappa q = p_\mu q_\nu \kappa^{\mu\nu} \quad (3.26)$$

and

$$\kappa^{\mu\nu} = \kappa^{\nu\mu}.$$

Knowing the vertex function, the Feynman rules are now to be applied as already described in chapter 2:

- The first step is to draw all possible momentum space diagrams with  $E$  external legs. We have to label each line with its 4-momentum  $p_i$  including its direction and the variable  $\sigma_i$ .
- To each line - with labels  $p_i$  and  $\sigma_i$  - we have to assign a pole factor

$$\frac{-i}{p_i^2 + m_i^2 - i\epsilon} \frac{\omega_{\mathbf{p}_i} + \sigma_i p_i^0}{2\omega_{\mathbf{p}_i}}. \quad (3.27)$$

- For each vertex, write down the vertex factor

$$-i \chi_k(\dots)$$

with the rule to insert  $\pm(q_i^{\sigma_i}) = (\pm \mathbf{q}_i, \pm \sigma_i \omega_{\mathbf{q}_i})^T$  into  $\chi_k$  for each line (at the vertex) labelled  $q_i, \sigma_i$ ; the “+” sign for momenta flowing into the vertex and “−” otherwise.

- Additionally, we have to introduce the usual symmetry factor  $\frac{1}{S}$  and to assure overall momentum conservation,

$$(2\pi)^4 \delta^4(p_1 + \dots + p_E) . \quad (3.28)$$

- Finally, we have to integrate over all internal momenta  $q_r$  which are not fixed by momentum conservation

$$\prod_{r=1}^{\# \text{Loops}} \frac{d^4 q_r}{(2\pi)^4} \quad (3.29)$$

and sum over all  $\sigma_i$ 's.

As an example, let us consider the contribution of a line between two internal points belonging to different interaction regions (“internal propagator”), i.e. corresponding to different interaction potentials  $V(x^0), V(y^0)$  in the Gell-Mann-Low formula (2.43). Therefore, we have to account for two vertices characterised by  $\chi_k(q^\sigma, \dots)$  and  $\chi_k(-q^\sigma, \dots)$ , and the corresponding line labelled by  $q$  and  $\sigma$  connecting these two vertices. The vertex functions factorise with respect to their arguments associated with the several lines running into or out of them. Thus, it is possible to define a new “propagator”  $\Delta_M$  taking into account all terms depending on the variables  $q, \sigma$  associated with the corresponding line. Sticking everything together yields the following expression:

$$\Delta_M(x-y) = \frac{-i}{(2\pi)^4} \int d^4 q \frac{e^{-iq(x-y)}}{q^2 + m^2 - i\epsilon} \sum_{\sigma=\pm 1} \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} e^{-\frac{1}{2} q^\sigma T \kappa q^\sigma} . \quad (3.30)$$

Eq. (3.30) for the “internal propagator” can also be obtained by contracting two smeared field operators (3.19),

$$\langle 0 | T \{ \phi_M(x) \phi_M(y) \} | 0 \rangle = \Delta_M(x-y) . \quad (3.31)$$

The time-ordered product can easily be written as a sum of two terms

$$\begin{aligned} \langle 0 | T \{ \phi_M(x) \phi_M(y) \} | 0 \rangle &= \langle 0 | \phi_M(x) \phi_M(y) | 0 \rangle \theta(x^0 - y^0) \\ &+ \langle 0 | \phi_M(y) \phi_M(x) | 0 \rangle \theta(y^0 - x^0) . \end{aligned} \quad (3.32)$$

Inserting eq. (3.19) and the integral representation of the Heaviside step function in eq. (2.59) into (3.32) yields

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{-1}{2\pi i} \int \frac{d^3 k d\tau}{(2\pi)^3 2\omega_{\mathbf{k}}} e^{-k_\mu^+ k_\nu^+ \kappa^{\mu\nu}/2} &\left( e^{-i\omega_{\mathbf{k}}(x^0 - y^0) + i\mathbf{k}(\mathbf{x} - \mathbf{y})} \frac{e^{i\tau(x^0 - y^0)}}{\tau - i\epsilon} \right. \\ &\left. + e^{i\omega_{\mathbf{k}}(x^0 - y^0) - i\mathbf{k}(\mathbf{x} - \mathbf{y})} \frac{e^{i\tau(x^0 - y^0)}}{\tau + i\epsilon} \right) . \end{aligned} \quad (3.33)$$

The exponential damping is the only difference to the usual local calculation. After some substitutions and noting that for the substitution  $\mathbf{k} \rightarrow -\mathbf{k}$ , we get  $k^+ \rightarrow -k^-$  we obtain the desired result:

$$\langle 0|T\{\phi_M(x)\phi_M(y)\}|0\rangle = \frac{-i}{(2\pi)^4} \int d^4q \frac{e^{iq(x-y)}}{q^2 + m^2 - i\epsilon} \sum_{\sigma=\pm 1} \frac{\omega_{\mathbf{q}} + \sigma q^0}{2\omega_{\mathbf{q}}} e^{-\frac{1}{2}q^\sigma T \kappa q^\sigma}. \quad (3.34)$$

Eq. (3.31) allows also a different interpretation for the Feynman rules. Namely, we can attribute an exponential damping factor

$$e^{-\frac{1}{2}q^\sigma T \kappa q^\sigma} \quad (3.35)$$

to internal lines labelled by  $q, \sigma$ . The damping can be assigned either to the internal lines or to the vertices. Of course, the amplitudes are unaffected by this choice.

In the situation discussed here, free propagators are not changed, since

$$G(p) = \sum_{\sigma} \frac{-i}{p^2 + m^2 - i\epsilon} \frac{\omega_{\mathbf{p}} + \sigma p^0}{2\omega_{\mathbf{p}}} = \frac{-i}{p^2 + m^2 - i\epsilon}. \quad (3.36)$$

In the next section, we will examine 1-loop corrections and show that they are all finite. Let us first discuss specific choices of the matrix  $\kappa$ , respectively  $M$ . For simplicity, we concentrate on the case of a diagonal matrix  $\kappa$ .

The first choice we want to consider is the unit matrix,

$$(\kappa^{\mu\nu}) = 2\zeta \mathbf{1}. \quad (3.37)$$

This can be accomplished, for example by using the following matrix  $M$ :

$$(M^\mu{}_\nu) = \sqrt{2\zeta} \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix}. \quad (3.38)$$

The motivation to use an anti-symmetric Matrix  $M$  of full rank has already been stressed in [49]. We want to relate this approach to the non-commutativity of space-time. One of the  $2 \times 2$  matrices in the diagonal of (3.38) is related to the non-commutative structure in [49], cf. eq. (3.1) with  $\theta = 2\zeta$ . Explicitly, we have

$$p_\mu^+ \kappa^{\mu\nu} p_\nu^+ = 2\zeta(\mathbf{p}^2 + \omega_{\mathbf{p}}^2) = 2\zeta(2\mathbf{p}^2 + m^2), \quad (3.39)$$

where the mass term can be absorbed within the normalisation constant in (3.19). Therefore, this case is equivalent to choosing  $\kappa^{00} = 0$ . In general, the case  $\kappa^{0i} = 0$  is equivalent to the case  $\kappa^{0\mu} = 0$ .

The smearing of the field operators considered in the next section will only extend over the spatial dimensions, and the zero component of the 4-vector  $Ma$  in (3.14) vanishes. In this case, the Feynman rules become simpler. The factor  $\chi_k$  associated to vertices becomes

$$\chi_k(p_1^{\sigma_1}, \dots, p_k^{\sigma_k}) = \lambda \exp\left(-\frac{1}{4} \sum_i \mathbf{p}_i^T \tilde{\kappa} \mathbf{p}_i\right), \quad (3.40)$$

which only contains the spatial components of the incoming momenta. We will examine the cases

$$\begin{aligned}\tilde{\kappa} &= 2\zeta \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \\ \tilde{\kappa} &= 2\zeta \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \\ \tilde{\kappa} &= 2\zeta \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}.\end{aligned}\tag{3.41}$$

Therefore, the  $\sigma_i$ 's are only contained in the contributions assigned to lines, cf. eq. (3.27). They easily factorise, and

$$\sum_{\sigma_1} \frac{\omega_{\mathbf{q}_1} + \sigma_1 q_1^0}{2\omega_{\mathbf{q}_1}} \dots \sum_{\sigma_k} \frac{\omega_{\mathbf{q}_k} + \sigma_k q_k^0}{2\omega_{\mathbf{q}_k}} = 1.\tag{3.42}$$

Hence, we have to assign to every line the usual factor

$$\frac{-i}{p_i^2 + m_i^2 - i\epsilon}.\tag{3.43}$$

### 3.3 Perturbative corrections and power counting

In this section, some properties of perturbative calculations with damped scalar field propagators will be studied. First, we will elaborate a power counting criterion by examining tadpole loops as shown in figs. 3.1a, 3.2a and 3.2b. Finally, this criterion will be tested for various calculations in Euclidean as well as Minkowski space.

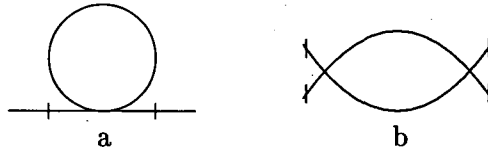


Figure 3.1: 1-loop contributions for  $\phi^4$ -theory.

As indicated in the previous section, the first  $j \leq 4 = d$  matrix elements in the diagonal of  $\kappa$  are assumed to be 1, whereas all the other elements are assumed to be zero.

For simplicity, the damping factor will be kept track of by putting it into the damped "internal propagator"

$$\Delta_j(k) \equiv \frac{e^{-\zeta \sum_{i=1}^j k_i^2}}{k^2 + m^2 - i\epsilon}\tag{3.44}$$

as already indicated by eq. (3.30).  $\zeta$  has dimension of  $[length]^2$ , possibly related to the deformation parameter of NCQFT [47, 48, 49].  $j$  denotes the number of damped dimensions.

The case  $j = d$  actually does not fit into our approach of smeared field operators, since the zero component of the occurring momenta are never involved in the damping factor, cf. eq. (3.19). Only on-shell momenta occur. However, we will also treat this case in the Euclidean theory since it is manifestly covariant, and it does not make much extra work. The definitions

$$\bar{k}^2 \equiv \sum_{i=1}^j k_i^2, \quad (3.45)$$

$$k'^2 \equiv \sum_{i=j+1}^d k_i^2 \quad (3.46)$$

will also be helpful.

In order to get a feeling for the power counting behaviour of perturbative calculations with damped internal scalar field propagators in a  $d$ -dimensional space-time, we present some general statements. Any vertex function is characterised by the number of external lines  $E$ , the number of internal lines  $I$  and the number of interaction vertices  $V$ .

A general vertex in coordinate space is of the form

$$V_i = \int d^d x \partial_x^{\delta_i} \phi^{b_i}(x), \quad (3.47)$$

where  $\delta_i$  counts the number of derivatives, and  $b_i$  stands for the number of scalar fields involved in the interaction.

### 3.3.1 Power counting for full damping

Let us first consider full damping, i.e.  $j = d$ . The "internal propagators" described in section 3.2 are assumed to have the following form in an Euclidean formulation

$$\int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \frac{1}{k^2 + m^2} e^{-\zeta k^2}, \quad (3.48)$$

neglecting some factors, which are not important for our considerations here, cf. eq. (3.34). In momentum space, this involves

$$\Delta_M(k) = \frac{1}{k^2 + m^2} e^{-\zeta k^2}, \quad (3.49)$$

where  $k^2 = (k^0)^2 + \mathbf{k}^2$ . For any given  $n$ , there is always a real number  $k_0$  that we have

$$e^{\zeta k^2} > (\zeta k^2)^n \text{ and } e^{-\zeta k^2} < (\zeta k^2)^{-n}$$

for all  $k > k_0$ . Since we are interested in the high momentum behaviour we can then make use of the following simplification

$$\Delta_M(k) \rightarrow \Delta_M^n(k) = \frac{1}{(\zeta k^2)^n k^2}$$

to investigate power counting. The superficial degree of divergence of any vertex graph  $\gamma$  is therefore given by

$$D_n(\gamma) = dL - (2n + 2)I + \sum_{i=1}^V \delta_i. \quad (3.50)$$

Note that for  $n$  we can choose any positive integer we want. We will see below how high we have to choose it at least to render certain diagrams finite. Using

$$L = I - (V - 1) \quad (3.51)$$

and the total number of all lines running to all vertices

$$\sum_i b_i = 2I + E, \quad (3.52)$$

we get for eq. (3.50)

$$D_n(\gamma) = d - \dim \phi E - \sum_i (d - d_i) - 2nI. \quad (3.53)$$

The dimension of the scalar field is given by

$$\dim \phi = \frac{d}{2} - 1, \quad (3.54)$$

and the corresponding dimension of the interaction vertex  $V_i$  is defined as

$$d_i \equiv \delta_i + \left(\frac{d}{2} - 1\right) b_i. \quad (3.55)$$

For  $n = 0$ , we have the usual power counting. Now, we are in the position to discuss specific models.

In  $d = 3$  space-time dimensions, we have two classical interactions

$$V_1^3 = \frac{\lambda_1}{4!} \int d^3x \phi^4(x) \quad \text{and} \quad V_2^3 = \frac{\lambda_2}{6!} \int d^3x \phi^6(x). \quad (3.56)$$

In this case,  $\dim \phi = 1/2$ . This implies that  $\lambda_1$  has dimension of a mass, and  $\lambda_2$  is dimensionless. The corresponding analogous interaction of a  $\phi^4$ -model in  $d = 4$  is ( $\dim \phi = 1$ )

$$V_3^4 = \frac{\lambda_3}{4!} \int d^4x \phi^4(x). \quad (3.57)$$

For  $d = 3$ , some perturbative corrections up to third order are shown in figs. 3.2-3.4.

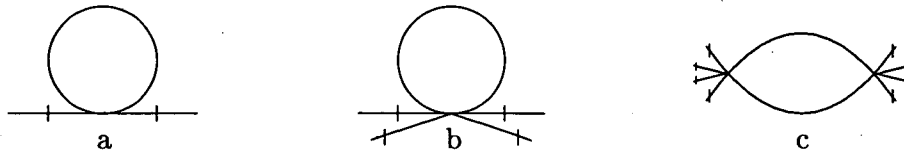
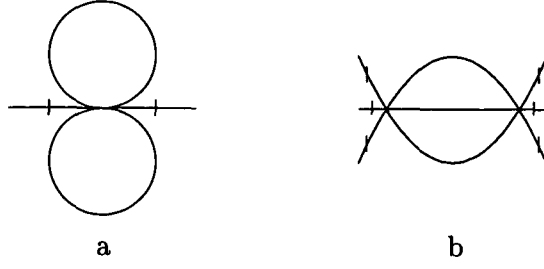


Figure 3.2: 1-loop graphs for a  $\phi^4$ - and a  $\phi^6$ -interactions.

Figure 3.3: 2-loop corrections for a  $\phi^6$ -interactions.Figure 3.4: 3-loop correction for a  $\phi^6$ -interactions.

According to eq. (3.53), we have the following degrees of divergence for these classes of radiative corrections <sup>2</sup>:

Fig. 3.2a	Fig. 3.2b	Fig. 3.2c
$D_n = 1 - 2n < 0, \forall n > 0$ $n = 0 : D_0 = 1$	$D_n = 1 - 2n < 0, \forall n > 0$ $n = 0 : D_0 = 1$	finite

Fig. 3.3a	Fig. 3.3b
$D_n = 2 - 4n < 0, \forall n > 0$ $n = 0 : D_0 = 2$	$D_n = -6n < 0, \forall n > 0$ $n = 0 : D_0 = 0$

Fig. 3.4

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$$D_n = 1 - 8n < 0, \forall n > 0$$

$$n = 0 : D_0 = 1$$

<sup>2</sup>The case  $n = 0$  is explicitly shown since this corresponds to the local case.

For  $d = 4$ , the corrections at the one-loop level are shown in fig. 3.1. The degrees of divergence are given by:

Fig. 3.1a	Fig. 3.1b
$D_n = 2 - 2n < 0, \forall n > 1$ $n = 0 : D_0 = 2$	$D_n = -2n < 0, \forall n > 0$ $n = 0 : D_0 = 0$

Thus, we have finiteness for all above mentioned graphs with fully damped propagators for  $n > 1$ .

### 3.3.2 Power counting for partial damping

In order to describe the power counting behaviour of the tadpole contribution with a partial damping in some directions in the Euclidean formulation, we have to consider the following integral

$$\Gamma_{tp}^j \equiv \frac{1}{(2\pi)^d} \int d^d k \Delta_j(k) . \quad (3.58)$$

This integral can be rewritten as

$$\Gamma_{tp}^j = \frac{1}{(2\pi)^d} \int d^j \bar{k} e^{-\zeta \bar{k}^2} \int d^{d-j} k' \frac{1}{\bar{k}^2 + k'^2 + m^2} . \quad (3.59)$$

The case  $j = d$  has already been discussed. Now, we approximate the exponential factor of the above integral in analogy to the above treatment

$$e^{-\zeta \bar{k}^2} < (\zeta \bar{k}^2)^{-l}$$

with  $l$  a positive integer. Then one gets

$$\Gamma_{tp}^{j,l} < \frac{1}{(2\pi)^d} \int d^j \bar{k} \frac{1}{(\zeta \bar{k}^2)^l} \int d^{d-j} k' \frac{1}{\bar{k}^2 + k'^2 + m^2} . \quad (3.60)$$

For  $0 < j \leq d$ , there exists always an  $l > 0$  such that the  $\bar{k}$ -integration converges. It remains to estimate the  $k'$  integration. Naive power counting can be applied. For the tadpole, we get

$$D_j = (d - j) - 2 , \quad (3.61)$$

For  $d = 3$  and  $j = 2$ , one has  $D_j = -1$ . This will be checked by explicit calculations in the following sections.

For  $d = 4$ , we conclude from eq. (3.61) that the degree of damping has to be  $j > 2$  in order to get convergence. We will see that these results are compatible with direct calculations.

Using the same philosophy, we can discuss an arbitrary  $L$ -loop contribution. We can estimate the naive power counting (assuming that the integration over the  $j$  damped directions is convergent) by

$$D_j = L(d - j) - 2I + \sum_i \delta_i , \quad (3.62)$$



implying

$$D_j = d - \dim \phi E - \sum_i (d - d_i) - jL, \quad (3.63)$$

with  $d_i$  given by eq. (3.55). Eq. (3.63) seems to imply that the superficial degree of divergence  $D_j$  linearly decreases with the number of loops  $L$ . But  $L$  and the number of vertices are related. We can rewrite eq. (3.63) in the following way:

$$D_j = d - j - E \dim \phi + \frac{Ej}{2} - \sum_i (d - d_i) - \sum_i \left( \frac{b_i}{2} - 1 \right) j. \quad (3.64)$$

We see that  $D_j$  decreases with the number of vertices and may increase with the number of external legs.

For  $j = 0$  (no damping), we get back the power counting behaviour of a local theory.

As a further consistency check, we discuss eq. (3.63) for the tadpole contribution with  $L = 1$ . For  $d = 3$ , we have the following:  $E = 2$  and  $d - d_i = 1$ , for the  $\phi^4$ -interaction;  $E = 4$  and  $d - d_i = 0$ , for the  $\phi^6$ -interaction. Therefore, both cases yield

$$D_j = 1 - j. \quad (3.65)$$

This implies convergence for  $j > 1$ .

For  $d = 4$ , we find

$$D_j = 2 - j, \quad (3.66)$$

meaning that convergence implies  $j > 2$ . The fact that the degree of divergence depends on the number of smeared dimensions has also been observed in [51] where the vacuum energy density has been discussed in the framework of the oscillator representation discussed above.

### 3.3.3 Explicit calculations in the Euclidean case

Let us consider the tadpole integral

$$\Gamma_{tp}^j = (2\pi)^{-d} \int d^d k \frac{e^{-\zeta \bar{k}^2}}{k^2 + m^2} \quad (3.67)$$

in  $d = 3$  and 4 dimensions. In 3 space-time dimensions, we have to solve the following integral

$$\Gamma_{tp}^j = (2\pi)^{-3} \int d^3 k \frac{e^{-\zeta \bar{k}^2}}{k^2 + m^2}. \quad (3.68)$$

The relevant loop graphs are fig. 3.2a and fig. 3.3a. We employ the Schwinger parametrisation

$$\frac{1}{k^2 + m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2 + m^2)}. \quad (3.69)$$

For  $j = 2$ , one obtains

$$\begin{aligned} \Gamma_{tp}^2 &= \frac{\pi^{3/2}}{(2\pi)^3} \int_0^\infty \frac{d\alpha}{\alpha^{1/2}(\alpha + \zeta)} e^{-\alpha m^2} \\ &= \frac{\pi^{3/2}}{(2\pi)^3} e^{\zeta m^2} \sqrt{\frac{\pi}{\zeta}} \Gamma(1/2, \zeta m^2). \end{aligned} \quad (3.70)$$

$\Gamma(z)$  is the ordinary Gamma-function, whereas  $\Gamma(1/2, z^2)$  denotes the "finite" incomplete Gamma-function,

$$\Gamma(1/2, \zeta m^2) = \sqrt{\pi} - \sum_{n=0}^{\infty} \frac{(-1)^n}{n! (n + 1/2)} (\zeta m^2)^{n+1/2}. \quad (3.71)$$

For  $j = 1$ , the tadpole contribution diverges.

In 4 dimensions, the calculations are a bit more involved. The tadpole integral is given by

$$\Gamma_{tp}^j = (2\pi)^{-4} \int d^4k \frac{e^{-\zeta k^2}}{k^2 + m^2}. \quad (3.72)$$

But already at this step, it is clear that UV/IR mixing (as it occurs in the usual NCQFT with complex phase factors) does not appear for the tadpole here. UV/IR mixing basically means that divergences due to UV-integrations arise for vanishing external momenta. But here, the integration does not even depend on the external momenta. Using the Schwinger parametrisation and carrying out the Gaussian integration, we obtain

$$\Gamma_{tp}^j = \frac{\pi^2}{(2\pi)^4} \int_0^\infty d\alpha \frac{e^{-\alpha m^2}}{\alpha^{2-j/2} (\alpha + \zeta)^{j/2}}. \quad (3.73)$$

The possible problems of the UV-integration are now hidden in the behaviour of this integral for  $\alpha \rightarrow 0$ . The power counting behaviour can be studied by regulating this expression which is done by restricting the integration to  $\alpha \in [1/\Lambda^2, \infty[$ . By dividing this area of integration into  $[1/\Lambda^2, a[$  and  $[a, \infty[$  with  $a \ll \zeta$ , we can read off the degree of divergence to be

$$D_j = 2 - j. \quad (3.74)$$

This agrees with the power counting formula given above and tells us that the tadpole is quadratically, linearly and logarithmically divergent for  $j = 0, 1, 2$ , respectively. The minimum damping rendering the tadpole contribution finite is given by  $j = 3$ . We obtain

$$\Gamma_{tp}^3 = \frac{2\pi^{5/2}}{(2\pi)^4 \zeta} U\left(\frac{1}{2}, 0, m^2 \zeta\right), \quad (3.75)$$

where  $U$  denotes the confluent hypergeometric function, with  $U(\frac{1}{2}, 0, 0) = \frac{2}{\sqrt{\pi}}$ . Of course, also the case  $j = 4$  gives a finite result:

$$\Gamma_{tp}^4 = \frac{\pi^2}{(2\pi)^4} \left[ \zeta^{-1} + m^2 e^{\zeta m^2} \text{Ei}(-\zeta m^2) \right], \quad (3.76)$$

where Ei is the exponential integral function with the following expansion for  $x < 0$ :

$$\text{Ei}(x) = \gamma + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k(k!)}, \quad (3.77)$$

where  $\gamma$  is the Euler-constant. Note that the parameter  $\zeta$  acts as a regulator. For any  $j$ , the integral (3.72) diverges quadratically for vanishing  $\zeta$ .

In a second step, we consider the one loop 4-point 1PI-vertex (see fig. 3.1b). The corresponding Feynman integral is

$$\Gamma_4^j(p) \equiv \frac{1}{(2\pi)^4} \int d^4k \Delta_j(k) \Delta_j(k+p) . \quad (3.78)$$

Using two Schwinger parameters  $\alpha, \beta$  and applying the integral transformation

$$\alpha = (1 - \xi)\lambda , \quad (3.79)$$

$$\beta = \xi\lambda , \quad (3.80)$$

we get

$$\Gamma_4^j(p) = \frac{\pi^2}{(2\pi)^4} \int_0^\infty d\lambda \int_0^1 d\xi \frac{e^{-\sum_{i=1}^j p_i^2 \frac{\xi(1-\xi)\lambda^2 + \zeta(\lambda+\zeta)}{\lambda+2\zeta} - (p^2 - p_i^2)\xi(1-\xi)\lambda - \lambda m^2}}{\lambda^{1-j/2}(\lambda+2\zeta)^{j/2}} . \quad (3.81)$$

A further evaluation of these integral is quite tricky. But the UV-behaviour can again be read off from the properties of the denominator

$$\lambda^{1-j/2}(\lambda+2\zeta)^{j/2}$$

for  $\lambda \rightarrow 0$ . The only problems might arise from the first factor and we do not expect UV-divergences for  $1 - j/2 < 1$  or  $j > 0$ . This means that at least one direction of space-time has to be damped in order to render the integral  $\Gamma_4^j$  finite, which again agrees with our power counting criterion  $D_j = -j < 0$ . A more detailed analysis of the integral was only possible for  $j = 4$ , where it could be rewritten after an appropriate transformation as

$$\Gamma_4^4(p) = -\frac{\pi^2}{(2\pi)^4} e^{2\zeta m^2} \int_0^1 d\xi \text{Ei}(-2[\xi(1-\xi)p^2 + m^2]\zeta) . \quad (3.82)$$

This expression is finite since Ei in the integrand is evaluated at negative values only, where it is well behaved, and the integral itself is over a finite interval.

### 3.3.4 Explicit calculations in the Minkowski case

We are ready to carry out a similar analysis for the Minkowski space. The tadpole diagram corresponds to the integral

$$\Gamma_{tp}^j \equiv \int d^4k \frac{e^{-\zeta \bar{k}^2}}{k^2 + m^2 - i\epsilon} . \quad (3.83)$$

The case of full damping (in all space-time directions) is omitted for the Minkowski space, where we would have to use  $\exp[-\zeta(k_1^2 + k_2^2 + k_3^2 + k_0^2)]$  as a damping factor to ensure finiteness. Wick rotation is not possible for the fully damped Minkowski situation since one would encounter exploding factors  $\exp(-\zeta k_0^2) \rightarrow \exp(\zeta k_4^2)$ . For the following discussion, the exponential is assumed not to depend on  $k_0$ . Hence, there are no obstacles opposing Wick rotation, and the results of the preceding discussion in the Euclidean space for  $j < 4$  are directly applicable.

We now turn to the more complicated kind of loops as shown in fig 3.1b. The interesting part of this diagram is given by the integral

$$\begin{aligned} \Gamma_4^j(p = p_1 + p_2) &\equiv \int d^4k \frac{e^{-\zeta \bar{k}^2}}{k^2 + m^2 - i\epsilon} \frac{e^{-\zeta \sum_i (k+p)_i^2}}{(k+p)^2 + m^2 - i\epsilon} \\ &\equiv \int d^4k f(\mathbf{k}, \mathbf{p}) g(k, p) , \end{aligned} \quad (3.84)$$

where

$$f(\mathbf{k}, \mathbf{p}) \equiv e^{-\zeta \bar{k}^2} e^{-\zeta \sum_i (k+p)_i^2} .$$

$f$  only depends on spatial momenta and not on their time component. The direct evaluation of  $\Gamma_4^j$  for arbitrary external momenta  $p$  seems to be rather tricky, and here we restrict ourselves to the UV-behaviour concerning the  $k$  integration. We want to give an upper bound for  $\Gamma_4^j$  and show that it is finite. But let us first get rid of the poles concerning the  $k^0$  integration. This is most easily accomplished by the residue theorem

$$I^0(\mathbf{k}, p) \equiv \int dk^0 g(k, p) = \pi i \frac{\left(\frac{1}{\omega_{\mathbf{k}}} + \frac{1}{\omega_{\mathbf{k}+\mathbf{p}}}\right)}{(\omega_{\mathbf{k}} + \omega_{\mathbf{k}+\mathbf{p}})^2 - p^0{}^2} . \quad (3.85)$$

The loop integral then reads

$$\Gamma_4^j(p) = \int d^3k f(\mathbf{k}, \mathbf{p}) I^0(\mathbf{k}, p) .$$

$I^0$  has the following bound

$$|I^0(\mathbf{k}, p)| \leq \frac{C}{|\mathbf{k}|^3} \quad \text{for } |\mathbf{k}| \geq r_p ,$$

where  $C > 0$  is some proportionality constant and

$$r_p \propto \max(|p_0|, |\mathbf{p}|) .$$

Defining the UV-part of the integration  $UV \equiv \{\mathbf{k} \in R^3 \mid |\mathbf{k}| \geq r_p\}$ , we thus conclude

$$\begin{aligned} |I^{UV}(p)| &\equiv \left| \int_{UV} d^3k f(\mathbf{k}, \mathbf{p}) I^0(\mathbf{k}, p) \right| \leq \int_{UV} d^3k |f(\mathbf{k}, \mathbf{p}) I^0(\mathbf{k}, p)| \\ &\leq \int_{UV} d^3k f(\mathbf{k}, \mathbf{p}) \frac{C}{|\mathbf{k}|^3} . \end{aligned} \quad (3.86)$$

This is finite as long as the sum over  $i$  within  $f$  involves at least one of the three spatial components, say  $j > 0$ . This is consistent with the results of the Euclidean discussion, where we concluded the same UV-behaviour by inspection of eq. (3.81) for  $\lambda \rightarrow 0$ . It again confirms our power counting criterion.

### 3.4 Summary

We have discussed a non-local, real, scalar field theory. The non-locality is located in the interaction where we have replaced the usual local fields by smeared field operators (3.14). The Feynman rules derived from the results of chapter 2 are shown in section 3.2. The free theory is not modified. Therefore, also the free propagators are unaltered. As a result of the smearing, the vertex contribution is exponentially damped by the incoming on-shell momenta (3.25). The fact that on-shell momenta enter the vertex contribution is of vital importance and a natural consequence of TOPT. In contrast to this result, the exponentially damped propagators obtained in [47, 48, 49] contain arbitrary momenta.

In section 3.3, we have carefully discussed UV properties of the model. We have derived a power counting formula (3.64) which provides the superficial degree of divergence for

theories with exponential damping in arbitrarily many dimensions. Explicit calculations of 1-loop diagrams in the Euclidean and Minkowski framework, done in sections 3.3.3 and 3.3.4, respectively, agree with the result from the generalised power counting formula. In  $d = 3$  space-time dimensions, the tadpole contributions shown in figs. 3.2a and 3.2b are finite if at least one dimension is damped, i.e.  $j > 1$ . The other loop contribution in fig. 3.2b is finite independently of  $j$ . In  $d = 4$  space-time dimensions, the tadpole contribution converges for  $j > 2$  and the 1-PI graph of fig. 3.1b for  $j \geq 1$ . The power counting formula shows that the presented model is UV finite to all orders in perturbation theory according to the proposed power counting formula. Notably, there is also no UV/IR mixing present at the 1-loop level.

## Chapter 4

# Special features of non-local perturbation theory

A convenient result of chapter 2 concerning time-ordered perturbation theory is that the same kind of Feynman diagrams as for local field theories can be used for non-local interactions which are translation invariant. We will concentrate on this case in this chapter. The relatively simple Feynman rules could be achieved by rewriting the  $\theta$ -functions of time-ordering in an appropriate way. Furthermore, all possibilities of connecting fields inside a vertex with the external ones have been considered for each vertex leading to the symmetrisation of the vertex factor. As a consequence, the ordinary Feynman diagrams of local theory also turn out to be sufficient to describe perturbation theory in the non-local case. Beside the higher generality, these are the main advantages compared to earlier prescriptions [31, 28] introduced in the context of NCQFT. There, all different configurations arising from time-ordering and the symmetrisation at each vertex have to be considered separately, whereas our approach includes all possibilities in a single term associated with an ordinary Feynman diagram. The only thing which is different from the diagrams of the local quantum field theory is the need for an extra label  $\sigma$  for each line.

Time-ordered perturbation theory for non-local interactions elaborated in chapter 2 is based on the idea that the non-localities only effect the interaction, but the free Hamiltonian  $H_0$  is assumed to be unchanged. However, it seems to be natural that also the free Hamiltonian is treated the same way as the interaction. Thus, one would also have to deform it correspondingly yielding the deformed, free Hamiltonian  $H_0^*$ . Only for the case that this deformation does not alter the free Hamiltonian ( $H_0^* = H_0$ ), it is plausible to apply perturbation theory as presented in this work. Within the framework yielding the simplified rules of NCQFT [18], it was already shown that the free theory and consequently the free Hamiltonian do not change. The argument is based on the observation that one star can be omitted under the integral over 4-space. Therefore, the free action of NCQFT is equivalent to the free action of the commutative case. But the Hamiltonian is only an integral over space and not the full space-time. Furthermore, quadratic terms deformed with the star product give  $\theta$ -dependent contributions to Green functions when treated as interaction terms, as was already shown in section 2.5. An investigation of this behaviour with respect to the free Hamiltonian is carried out in the first section of this chapter. Besides the non-commutative case, general non-local deformations will also be considered.

Obviously, the results of time-ordered perturbation theory differ from the simple rules

first given in [18]. The question now is: How can these differences be explained? The answer will be given in the second section assuming that the simple rules are derived from a naive path integral approach to be defined in more detail below.

The Feynman rules for non-local theories which associate mathematical expressions with the common Feynman diagrams are slightly more complicated than in the local case. So far, the rules have only been given to evaluate expressions of the type

$$\int_{-\infty}^{\infty} dt_1 \dots \int_{-\infty}^{\infty} dt_m \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) V(t_1) \dots V(t_m) \} | 0 \rangle . \quad (4.1)$$

It remains to give a prescription for calculating  $S$ -matrix elements of physical particles. This will be accomplished in section 4.3 after defining the process of amputation of external legs in a meaningful way. Then, the concept of the self-energy will be studied for the non-local case.

Several symmetries, namely Hermiticity, unitarity, Lorentz invariance, parity, and time reversal will be considered in section 4.4 for the case of non-local Feynman rules.

In section 4.5, the difference between the simple rules and the time-ordered approach will be investigated in more detail. Diagrammatic quantities will be expanded in terms of the non-localities for this purpose yielding an interesting observation at first order in the non-locality of the time component.

## 4.1 Deformation of the free Hamiltonian

In this section, we consider deformed, free, scalar Hamiltonians. By this, it is meant that one starts with the free Hamiltonian of a scalar theory:

$$H_0(x^0) = \frac{1}{2} \int d^3x [\pi(x)^2 + (\nabla\phi(x))^2 + m^2\phi(x)^2] . \quad (4.2)$$

$\pi$  and  $\phi$  are considered to be free fields obeying the following equations of motion:

$$\dot{\phi}(x) = \pi(x) , \quad (4.3)$$

$$\dot{\pi}(x) = -\nabla^2\phi(x) - m^2\phi(x) . \quad (4.4)$$

From now on, one is not interested in the Hamiltonian as a functional generating the equations of motion. Thus, we can simply use the equations of motion to rewrite  $H_0$ . First,  $\pi$  is eliminated:

$$H_0(t) = \frac{1}{2} \int d^3x [\partial^\mu\phi(x)\partial_\mu\phi(x) + m^2\phi(x)^2] . \quad (4.5)$$

Here we have abbreviated  $t = x^0$  for better reading. It is clear that the Hamiltonian is not covariant. Thus, the partial derivatives do not occur in a covariant way. Furthermore, one gets for eq. (4.4)

$$[\partial^\mu\partial_\mu - m^2] \phi(x) = 0 . \quad (4.6)$$

For each (quadratic) product of field operators, one then introduces the deformation

$$f(x)g(x) \rightarrow \int d\underline{\mu} w(\underline{\mu}) f(x + h_1(\underline{\mu})) g(x + h_2(\underline{\mu})) . \quad (4.7)$$

It is important that this way all quadratic products are deformed in the same way (One could do it in another way and deform  $\phi^2$  different from  $(\partial^\mu \phi)^2$ , for example.). For  $H_0$  one thus gets

$$H_0(t) \rightarrow H_0^*(x^0) = \frac{1}{2} \int d^3x \int d\underline{\mu} w(\underline{\mu}) \times \quad (4.8)$$

$$[\partial^\nu \phi(x + h_1(\underline{\mu})) \partial^\nu \phi(x + h_2(\underline{\mu})) + m^2 \phi(x + h_1(\underline{\mu})) \phi(x + h_2(\underline{\mu}))] .$$

We still assume by definition that  $\phi$  denotes the free field operator obeying the free field eq. (4.6) with physical mass  $m$ .  $\phi$  can thus be Fourier transformed as usual with the coefficients being creation and annihilation operators  $a^\dagger(\mathbf{k})$  and  $a(\mathbf{k})$ , respectively. A simple calculation then yields

$$H_0^*(t) = \frac{1}{2} \int d^3k \int d\underline{\mu} w(\underline{\mu}) \omega_k \quad (4.9)$$

$$\left[ a(\mathbf{k}) a^\dagger(\mathbf{k}) e^{ik^+(h_1(\underline{\mu}) - h_2(\underline{\mu}))} + a^\dagger(\mathbf{k}) a(\mathbf{k}) e^{-ik^+(h_1(\underline{\mu}) - h_2(\underline{\mu}))} \right] .$$

It is interesting to remark that the terms proportional to  $a^\dagger(\mathbf{p}) a^\dagger(\mathbf{k})$  or  $a(\mathbf{p}) a(\mathbf{k})$  drop out because their coefficients vanish:

$$(-p^{+\mu} k^{+\mu} + m^2) \delta^3(\mathbf{p} + \mathbf{k}) = 0 ,$$

whereas the coefficient of  $a(\mathbf{p}) a^\dagger(\mathbf{k})$  and  $a^\dagger(\mathbf{p}) a(\mathbf{k})$  is proportional to

$$(p^{+\mu} k^{+\mu} + m^2) \delta^3(\mathbf{p} - \mathbf{k}) = 2\omega_p^2 \delta^3(\mathbf{p} - \mathbf{k}) .$$

Usually, the Hamiltonian is normal ordered. For  $H_0^*$ , this can also be done:

$$: H_0^*(t) := \int d^3k \omega_k a^\dagger(\mathbf{k}) a(\mathbf{k}) \xi(\mathbf{k}) \quad (4.10)$$

with

$$\xi(\mathbf{k}) \equiv \int d\underline{\mu} w(\underline{\mu}) \cos(k^+(h_1(\underline{\mu}) - h_2(\underline{\mu}))) . \quad (4.11)$$

Now, the question is, if  $\xi(\mathbf{k})$  is constant in  $\mathbf{k}$  or not. If it is,  $H_0^*$  can be made equal to  $H_0$  by introducing an appropriate normalisation into the deformation prescription. Another question is, which kind of deformations leave the free Hamiltonian unaltered. A trivial solution to this question would be  $h_1(\underline{\mu}) = h_2(\underline{\mu})$  and the requirement  $\int d\underline{\mu} w(\underline{\mu})$  to be finite. But this would mean that one actually does not have a non-local product of fields. There is also another case where  $H_0^* = H_0$ , namely NCQFT with constant  $\theta$ . One has  $\underline{\mu} = l, s$ ,  $w(\underline{\mu}) = \exp(i l s) / (2\pi)^4$ ,  $h_1(\underline{\mu}) = -\frac{1}{2} \tilde{l}$  and  $h_2(\underline{\mu}) = s$ , which yields

$$\xi(\mathbf{k}) = e^{-ik^+ \tilde{k}^+ / 2} = 1 .$$

For UV-finite theories, the situation seems to be different. According to the approach in [32], one would have  $\underline{\mu} = a_1, a_2$ ,  $h_1(\underline{\mu}) = \zeta a_1$ ,  $h_2(\underline{\mu}) = \zeta a_2$  and  $w(\underline{\mu}) = 2c_2 \exp(1/2(a_1^{\nu^2} + a_2^{\nu^2})) \delta^4(a_1 + a_2)$  which gives

$$\xi(\mathbf{k}) = 2c_2 \pi^2 e^{-\zeta^2 k_\nu^+{}^2} .$$



With appropriate choice of  $c_2$ , this would mean

$$: H_0^*(t) := \int d^3k \, \omega'_k \, a^\dagger(\mathbf{k}) a(\mathbf{k}) ,$$

and

$$\omega'_k \equiv \omega_k e^{-\zeta^2 k_\nu^2}$$

could in principle be interpreted as the new free particle energy. The simplified UV-finite QFT introduced in chapter 3 gives similar results. One has  $\underline{\mu} = b_1, b_2$  ( $b_i$  being an  $n$ -dimensional real vector),  $w(\underline{\mu}) = N^2 \exp(-b_1^2 - b_2^2)$ ,  $h_1(\underline{\mu}) = M b_1$  and  $h_1(\underline{\mu}) = M b_2$ . A simple calculation yields for this case

$$\xi(\mathbf{k}) = \pi^n N^2 e^{-\frac{1}{2} \mathbf{k}^+ T \kappa \mathbf{k}} = e^{-\frac{1}{2} \mathbf{k}^+ T \kappa \mathbf{k}} .$$

We have chosen the dimension of  $b_i$   $n = 4$  and used the normalisation convention  $N = \pi^{-2}$ . Both cases of UV-finite theories suffer from the following problems: First, it does not seem to be justified that one can apply the Gell-Mann-Low formula in the usual sense since it assumes that the asymptotic states are governed by  $H_0$  whereas they are apparently governed by  $H_0^* \neq H_0$  in the present cases. Second, the interpretation of  $\omega'_k$  as an energy of a physical state with momentum  $\mathbf{k}$  is troublesome. This can simply be seen considering that the energy would go to 0 for large  $\mathbf{k}$ . Now, one cannot say the way perturbation theory was applied to the UV-finite theories is wrong. However, this approach suffers from the blemish that by definition the deformation is to be restricted to the interaction and cannot be applied naturally to the whole Hamiltonian in the interaction picture. But the usual NCQFT has the nice feature that also the free Hamiltonian can be deformed without affecting perturbation theory since  $H_0^* = H_0$ . Furthermore, it was shown in [35] that NCQFT does not alter the free Hamiltonian of gauge field models.

## 4.2 Illuminating apparent discrepancies

In this section, we want to discuss discrepancies between TOPT and the naive path integral approach (NPIA). By TOPT we mean calculations based on the Dyson series given in eq. (2.28) with

$$V(x^0) \equiv \frac{\lambda}{k!} \int d^3x \, (\phi^*)^k(x) , \quad (4.12)$$

and  $\phi$  denoting the field operators in the Dirac picture. The resulting Feynman rules are given in [33]. By NPIA we mean that one calculates  $n$ -point functions according to the path integral

$$\int \mathcal{D}\phi \, \phi(x_1) \dots \phi(x_n) e^{iI[\phi]} , \quad (4.13)$$

with  $I$  denoting the corresponding action including  $i\epsilon$  terms:

$$I[\phi] = \int d^4x \, \left[ \mathcal{L}_0 + \frac{\lambda}{k!} (\phi^*)^k(x) + i\epsilon \right] . \quad (4.14)$$

The corresponding Feynman rules are the same as for the local theory but with phase factors to be included for vertices [18, 19]. Meanwhile, it is clear that these two approaches differ when non-commutativity involves time. The most striking problem is the unitarity violation [26] when applying these naive Feynman rules, which can be cured by a strict application of

TOPT [31]. Another mismatch was realised in section 2.5, where it turned out that the star product of NCQFT does not drop out of some quadratic terms which might be considered as counter terms for example. But in the action (NPIA), star products become redundant in any bilinear term.

For the local version of the model we are studying, both, the NPIA and TOPT give the same results. But for non-local theories as NCQFT, these approaches are not equivalent. To see the crucial points about these mismatches, we sketch how to pass from TOPT to the NPIA [29]. One starts with the Hamiltonian  $H \equiv H_D(0)$  written as a functional of canonical field operators  $\phi_D(\mathbf{x}, t), \pi_D(\mathbf{x}, t)$  in the Dirac picture <sup>1</sup>:

$$H_D(t) \equiv H[\phi_D, \pi_D; t]. \quad (4.15)$$

To be specific, our version of TOPT and NCQFT gives

$$H_D(t) = H_0 + V(t) \quad (4.16)$$

combined with eq. (4.12). The special notation of the functional (4.15) is due to the non-localities, especially the ones in time: it is not a functional depending just on field operators given at fixed time  $t$ , but possibly all times are involved (see also eq. (4.26) below). Furthermore, one assumes two complete basis sets  $|q; t\rangle$  and  $|p; t\rangle$  for each time  $t$  being eigenstates of the field operators in the Heisenberg picture  $\phi_H(\mathbf{x}, t)$  and  $\pi_H(\mathbf{x}, t)$ , respectively. The goal is to evaluate scalar products  $\langle q'; t' | q; t \rangle$  between basis vectors given at different times  $t < t'$ . Then, one sandwiches sums over complete basis sets belonging to intermediate times  $t_i$  with  $t < t_1 \dots < t_N < t'$ . So far, everything might also work for non-local field theories. Next, one has to evaluate matrix elements like

$$\langle q'; t + dt | q; t \rangle = \langle q'; t | e^{-iHdt} | q; t \rangle. \quad (4.17)$$

At this point,  $H$  is usually rewritten

$$H = e^{iHt} e^{-iHt} H[\phi_D, \pi_D; 0] = e^{iHt} H[\phi_D, \pi_D; 0] e^{-iHt}. \quad (4.18)$$

In a local theory, the functional just depends on field operators  $\phi_D(\mathbf{x}, 0)$  evaluated at time  $t = 0$  which can be simply replaced using  $\phi_D(\mathbf{x}, 0) = \phi_H(\mathbf{x}, 0)$ . As a matter of course, one thus rewrites this by sandwiching unit operators  $\exp(iHt) \exp(-iHt)$  as

$$e^{iHt} H[\phi_D, \pi_D; 0] e^{-iHt} = e^{iHt} H[\phi_H, \pi_H; 0] e^{-iHt} = H[\phi_H, \pi_H; t]. \quad (4.19)$$

However, these steps are problematic for theories which are non-local in time. To see this, we consider an operator  $O_D$  defined as a Moyal product of two operators  $A_D, B_D$  in the Dirac picture:

$$O_D(x) \equiv (A_D \star B_D)(x) = \int d^4s \int \frac{d^4l}{(2\pi)^4} e^{ils} A_D(x - \frac{1}{2}\tilde{l}) B_D(x + s). \quad (4.20)$$

The subscript  $D$  at  $O_D$  indicating that  $O_D$  has the time dependence of an operator in the Dirac picture is justified since

$$O_D(\mathbf{x}, t) = e^{iH_0t} O_D(\mathbf{x}, 0) e^{-iH_0t}$$

<sup>1</sup>As usual, this means that these operators satisfy the commutation relations  $[\pi_D(\mathbf{x}, t), \pi_D(\mathbf{x}', t)] = [\phi_D(\mathbf{x}, t), \phi_D(\mathbf{x}', t)] = 0$  and  $[\pi_D(\mathbf{x}, t), \phi_D(\mathbf{x}', t)] = -i\delta^3(\mathbf{x} - \mathbf{x}')$ . These relations also hold for the corresponding operators in the Heisenberg picture.

holds. The transition from the Dirac to the Heisenberg picture is now done as usual [29]:

$$O_H(\mathbf{x}, t) = e^{iHt} O_D(\mathbf{x}, 0) e^{-iHt}. \quad (4.21)$$

Substituting eq. (4.20), we get

$$\begin{aligned} O_H(\mathbf{x}, t) &= \int d^4s \int \frac{d^4l}{(2\pi)^4} e^{ils} e^{iHt} A_D(x_0 - \frac{1}{2}l) B_D(x_0 + s) e^{-iHt} \\ &= \int d^4s \int \frac{d^4l}{(2\pi)^4} e^{ils} W(t, t - \tilde{l}^0/2) A_H(x - \tilde{l}/2) \times \\ &\quad W(t - \tilde{l}^0/2, t + s^0) B_H(x + s) W(t + s^0, t). \end{aligned} \quad (4.22)$$

where  $x = (\mathbf{x}, t)$ ,  $x_0 = (\mathbf{x}, 0)$ , and

$$W(t, t_0) \equiv e^{iHt} e^{-iH_0(t-t_0)} e^{-iHt_0}. \quad (4.23)$$

$W(t, t_0)$  is unitary and  $W(t, t) = 1$ , but in general  $W(t, t_0)$  is not the unit operator. In order to stay consistent one thus has to redefine the non-commutative product with respect to Heisenberg fields correspondingly. For the Hamiltonian needed for path integrals, one could proceed with

$$e^{iHt} H[\phi_D, \pi_D; 0] e^{-iHt} \equiv H'[\phi_H, \pi_H; t]. \quad (4.24)$$

When non-locality involves time clearly  $H' \neq H$ , and it can be expected that dealing with  $H'$  is pretty hard. We assume that  $H' \neq H$  is the cause for the discrepancies between TOPT and the NPIA for systems described by Hamiltonians which are non-local in time. Further problems are expected when integrating out the conjugate momenta to pass from the Hamiltonian to the Lagrangian formulation, even if we accepted  $H$  instead of  $H'$ <sup>2</sup>. This might be due to the fact that the non-locality in time means that  $H$  depends on infinitely many time derivatives implying complicated equations of motion [52]. Furthermore, the equivalence of using the Lagrangian interaction or the Hamiltonian one is not justified anymore by path integrals. We believe that it is important to check this explicitly. Besides derivative couplings and counterterms, NCQED might also be affected due to the complicated quantisation procedure which involves derivatives through the constraints [45].

Now, we want to illustrate the differences of the NPIA and TOPT. The perturbation expansion of the NPIA is obtained by expanding the integrand in terms of the interaction leaving only the bilinear parts in the exponential. The resulting path integrals can be carried out and one gets Feynman rules which associate the usual propagators  $\Delta_F(x, y)$  (inverse of the bilinear parts) of the local theory with lines, and vertices contain 4-momentum dependent phase factors. A time-ordering interpretation of the resulting rules can be obtained by writing  $\Delta_F(x, y)$  in terms of time-ordered products of the free annihilation and creation fields,  $\phi^-$  and  $\phi^+$ , respectively:

$$\Delta_F(x, y) = \theta(x^0 - y^0) [\phi^-(x), \phi^+(y)] + \theta(y^0 - x^0) [\phi^-(y), \phi^+(x)]. \quad (4.25)$$

This indicates that the NPIA can be interpreted in the sense of a *total* time-ordering acting with respect to the time argument of each field. On the other hand, the time-ordering operator

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<sup>2</sup>Clearly, this would not be equivalent to our TOPT approach. It would mean that we started with  $H = H[\phi_H, \pi_H; t = 0]$ .

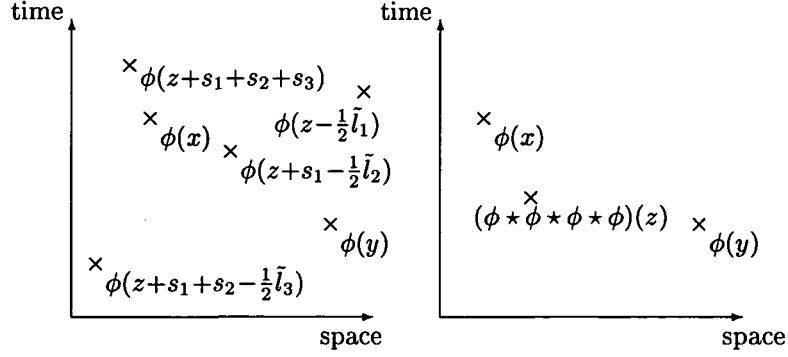


Figure 4.1: An example of a certain time-ordering configuration.

of TOPT just rearranges whole interactions. Now, let us consider  $\phi^4$ -theory. We have

$$(\phi \star \phi \star \phi \star \phi)(z) = \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{il_i s_i} \right) \times \quad (4.26)$$

$$\phi(z - \frac{1}{2}\tilde{l}_1) \phi(z + s_1 - \frac{1}{2}\tilde{l}_2) \phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3) \phi(z + s_1 + s_2 + s_3) ,$$

which clearly expresses the non-locality. The two-point function at first order in  $\lambda$  can then be written a bit sloppy

$$G(x, y) = \frac{g}{4!} \int d^4 z \left\langle 0 \left| T \{ \phi(x) \phi(y) (\phi \star \phi \star \phi \star \phi)(z) \} \right| 0 \right\rangle . \quad (4.27)$$

Let us discuss the total and the TOPT time-ordering for one particular geometrical situation with respect to (4.27): The arrangement of fields for the left diagram of fig. (4.1) corresponds to the following non-vanishing contribution to the total time-ordering of (4.27):

$$G(x, y) =$$

$$= \int d^4 z \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{il_i s_i} \right) \theta(s_1^0 + s_2^0 + s_3^0 + \frac{1}{2}\tilde{l}_1^0) \theta(z^0 - \frac{1}{2}\tilde{l}_1^0 - x^0)$$

$$\times \theta(x^0 - z^0 - s_1^0 + \frac{1}{2}\tilde{l}_2^0) \theta(z^0 + s_1^0 - \frac{1}{2}\tilde{l}_2^0 - y^0) \theta(y^0 - z^0 - s_1^0 - s_2^0 + \frac{1}{2}\tilde{l}_3^0)$$

$$\times \left\langle 0 \left| \phi(z + s_1 + s_2 + s_3) \phi(z - \frac{1}{2}\tilde{l}_1) \phi(x) \phi(z + s_1 - \frac{1}{2}\tilde{l}_2) \phi(y) \phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3) \right| 0 \right\rangle . \quad (4.28)$$

We find that there are  $6! = 720$  different contributions to (4.27) when interpreting the time-ordering in the Gell-Mann-Low formula as a total time-ordering of all field arguments. This kind of time-ordering guarantees that only causal processes contribute to the  $S$ -matrix.

In contrast to this total time-ordering, we now have *interaction point* time-ordering (see right diagram of fig. (4.1)), which is defined with respect to the interaction point:

$$G'(x, y) = \int d^4 z \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{il_i s_i} \right) \theta(x^0 - z^0) \theta(z^0 - y^0)$$

$$\times \left\langle 0 \left| \phi(x) \phi(z - \frac{1}{2}\tilde{l}_1) \phi(z + s_1 - \frac{1}{2}\tilde{l}_2) \phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3) \phi(z + s_1 + s_2 + s_3) \phi(y) \right| 0 \right\rangle . \quad (4.29)$$

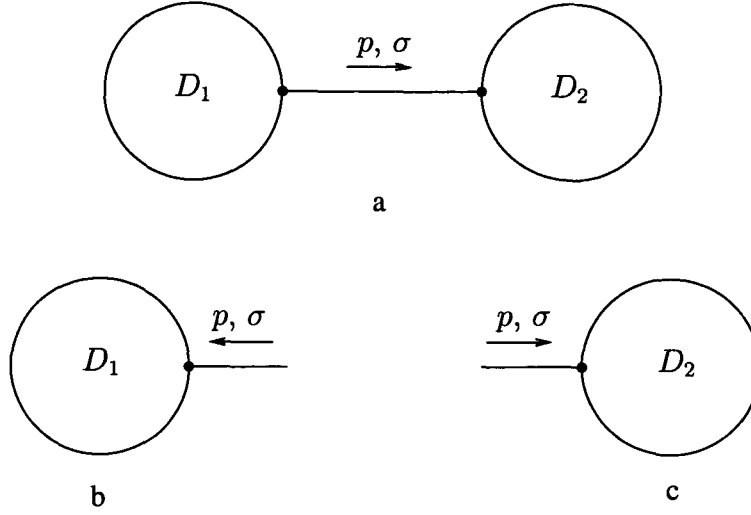


Figure 4.2: The upper diagram  $D_{1+2}$  is composed of the two lower ones,  $D_1$ , and  $D_2$ .

There are now only  $3! = 6$  different contributions of this type. For most contributions some of the fields are now at the “wrong” place with respect to the total time-order. Thus, the non-commutative version (4.29) of the Gell-Mann–Low formula violates causality but preserves unitarity (as we want to stress once more). After all, contributions to the Dyson series are precisely ordered only with respect to the time stamp of the interaction Hamiltonians.

### 4.3 Diagrammatic specialties of non-local interactions

A few fundamental, diagrammatic techniques are still to be discussed for the non-local Feynman rules in momentum space. First, we address the process of amputation. Then, we will elaborate the appropriate prescriptions for obtaining  $S$ -matrix elements from amputated diagrams. This will be needed in section 4.4, when discussing symmetries. Finally, the concept of the self-energy will be adopted for the non-local case.

#### 4.3.1 Amputation of external legs

The main advantage of amputated Feynman diagrams in local QFT is that they can be conveniently put together to obtain more complicated diagrams. As an example, consider a Feynman diagram with a structure as shown in fig. 4.2a. It is composed of two sub-diagrams, fig. 4.2b, and 4.2c, which are simply joined by one line. The corresponding expressions of these two diagrams  $D_i^c$  with  $i = 1, 2$  can be written as

$$D_i^c(p, \dots) \equiv \sum_{\sigma} D_i(p, \sigma, \dots) \frac{-i}{p^2 + m^2 - i\epsilon} \frac{\omega_p + \sigma p^0}{2\omega_p}. \quad (4.30)$$

$c$  indicates that external lines are included, and “...” stands for the dependence on possible other external legs we are not interested in here. The expression  $D_{1+2}$  for fig. 4.2a is then

given by

$$D_{1+2}(\dots) = \sum_{\sigma} \int \frac{d^4 p}{(2\pi)^4} D_1(-p, -\sigma) \frac{-i}{p^2 + m^2 - i\epsilon} \frac{\omega_p + \sigma p^0}{2\omega_p} D_2(p, \sigma) , \quad (4.31)$$

where the minus-signs in  $D_1$  occur due to the conventions of labelling chosen in fig. 4.2. The integration over the momentum  $p$  of the internal line is shown explicitly in order to turn the momentum conservation contained in the  $D_i$ 's to the correct overall momentum conservation. This example shows that for the purpose of building up more complicated diagrams it is useful to omit the elements associated with external legs when calculating the simpler building blocks. It is now clear how to perform amputation of external legs. The generalisation to diagrams with more than one external leg is also clear and the amputated version of a diagram is obtained as follows: *Write down the corresponding expression according to the Feynman rules, but leave out the pole factors and the  $\sigma$ -sums associated with external lines.*

The procedure of connecting diagrams at amputated legs also becomes clear by the considerations carried out above: *The expression of a diagram  $D_{1+2}$  consisting of two amputated diagrams  $D_1$ ,  $D_2$  is given by*

$$D_{1+2}(\dots) = \sum_{\sigma} \int \frac{d^4 p}{(2\pi)^4} D_1(\pm p, \pm \sigma, \dots) \frac{-i}{p^2 + m^2 - i\epsilon} \frac{\omega_p + \sigma p^0}{2\omega_p} D_2(\pm p, \pm \sigma, \dots) . \quad (4.32)$$

*The signs in the arguments of  $D_1$ ,  $D_2$  have to be chosen negative if the momentum of the amputated leg is chosen in the opposite direction as in the composite diagram, and positive otherwise.* Note that the two alternatives of choosing the sign in eq. (4.32) have to be considered separately for  $D_1$  and  $D_2$ . A '+' in  $D_1$  does not necessarily imply a '+' in  $D_2$ . As stated above, this depends on the convention of labelling the composite and the sub-diagrams. For diagrams, which are connected at more than one leg, this procedure then has to be carried out for each leg.

Our definition of amputation differs from the process carried out for local theories, where one just multiplies the complete expression with the inverse of the momentum space propagator of the corresponding external leg. This has also been applied for TOPT of NCQFT [28]. But here we want to stress that it seems to be more useful to define amputation as described above: leave out the  $\sigma$ -sums and the pole factors associated with external legs. Note that according to the above definition the momentum conserving  $\delta$ -functions are included in the amputated diagrams. Commonly, one defines *truncated* diagrams (see [44], for example) which means that also momentum conservation  $(2\pi)^4 \delta^4(\dots)$  is left out for the corresponding expressions.

### 4.3.2 Physical $S$ -matrix elements

Now it is time to investigate how  $S$ -matrix elements describing the scattering of physical particles are obtained. This can be done by evaluating simple vacuum expectation values where the external field operators are replaced by creation and annihilation operators  $a_{\mathbf{k}}^{\dagger}$ ,  $a_{\mathbf{k}}$ . After evaluating simple examples, a comparison with the expressions of amputated diagrams leads to the following prescription: *For each external leg, replace the corresponding momentum in the expression of the amputated diagram with  $\pm q^+$  and set the corresponding  $\sigma = \pm 1$ , where + is to be chosen when an incoming (outgoing) leg is replaced with an incoming (outgoing)*

particle with on-shell momentum  $q^+$ , and  $-$  otherwise. Multiply with a factor

$$\frac{1}{(2\pi)^{3/2} \sqrt{2\omega_q}} . \quad (4.33)$$

This prescription might sound confusing, but the following example should make things clear. Consider the scattering of two scalar particles to first order in a non-local  $\phi^4$  interaction with vertex factor  $\chi_4$ . The corresponding amputated diagram with all external legs incoming yields

$$-i(2\pi)^4 \delta^4(q_1 + q_2 + q_3 + q_4) \chi_4(q_1^{\sigma_1}, q_2^{\sigma_2}, q_3^{\sigma_3}, q_4^{\sigma_4}) .$$

The incoming particles are now assumed to have momenta  $p_1^+$ ,  $p_2^+$ , and  $p_3^+$ ,  $p_4^+$  stand for the outgoing particles. “ $-$ ”-signs are only to be chosen for the outgoing particles and the physical contribution to the  $S$ -matrix can be written as

$$-i \delta^4(q_1^+ + q_2^+ - q_3^+ - q_4^+) \frac{\chi_4(p_1^+, p_2^+, (-p_3)^-, (-p_4)^-)}{4(2\pi)^2 \sqrt{\omega_{q_1} \omega_{q_2} \omega_{q_3} \omega_{q_4}}} .$$

If we had adopted the convention that  $q_3$  and  $q_4$  are outgoing when calculating the amputated diagram, the vertex factor would be  $\chi_4(p_1^+, p_2^+, -(p_3^+), -(p_4^+))$ . But this is the same as given above since  $-(p^+) = (-p)^-$ . Generally, the physical results must not depend on the conventions we label our diagrams.

The prescription given above has been obtained by comparing physical  $S$ -matrix elements with Green function elements. There is a common formula to calculate  $S$ -matrix elements from Green functions within local QFTs. To be precise, it says that  $S$ -matrix elements can be calculated according to

$$\begin{aligned} S_{\beta\alpha} &= \langle \mathbf{k}'_1, \dots, \mathbf{k}'_N; \text{out} | \mathbf{k}_1, \dots, \mathbf{k}_M; \text{in} \rangle \\ &= \prod_{l=1}^M \left( i \int d^4 x_l f_{\mathbf{k}_l}(x_l) (-\square_{x_l} + m^2) \right) \prod_{j=1}^N \left( i \int d^4 y_j f_{\mathbf{k}'_j}^*(y_j) (-\square_{y_j} + m^2) \right) \\ &\quad \times \langle 0 | T \{ \phi_H(y_1) \dots \phi_H(y_N) \phi_H(x_1) \dots \phi_H(x_M) \} | 0 \rangle . \end{aligned} \quad (4.34)$$

$f_{\mathbf{k}}$  represents the wave function chosen as follows:

$$f_{\mathbf{k}}(x) = \frac{e^{ik^+x}}{(2\pi)^{3/2} \sqrt{2\omega_k}} . \quad (4.35)$$

We now want to check whether this formula is also applicable to non-local field theories. Therefore, we consider the perturbative expansion of the time-ordered vacuum expectation values of field operators given in eq. (2.43). The Fourier transforms of the perturbative contributions are evaluated according to the momentum space Feynman rules. Inverting these contributions in accordance with eq. (2.72), one obtains

$$\begin{aligned} G(y_1, \dots, x_1, \dots) &\equiv \langle 0 | T \{ \phi_H(y_1) \dots \phi_H(y_N) \phi_H(x_1) \dots \phi_H(x_M) \} | 0 \rangle \\ &= \sum_D \prod_{l=1}^M \left( \int \frac{d^4 p_l}{(2\pi)^4} e^{-ip_l x_l} \right) \prod_{j=1}^N \left( \int \frac{d^4 q_j}{(2\pi)^4} e^{-iq_j y_j} \right) \times \\ &\quad D^c(q_1, \dots, q_N, p_1, \dots, p_M) . \end{aligned} \quad (4.36)$$

The sum runs over all diagrams  $D$  contributing to  $G$ . The superscript  $c$  at  $D^c$  indicates that the external legs are included. All external legs are assumed to be incoming. As we have already seen above, this can be written as

$$D^c(q_1, \dots, p_1, \dots) = \frac{-i}{q_1^2 + m^2 - i\epsilon} \sum_{\sigma'_1} \frac{\omega_{\mathbf{q}_1} + \sigma'_1 q_1^0}{2\omega_{\mathbf{q}_1}} \dots \frac{-i}{p_1^2 + m^2 - i\epsilon} \sum_{\sigma_1} \frac{\omega_{\mathbf{p}_1} + \sigma_1 p_1^0}{2\omega_{\mathbf{p}_1}} \dots \times D(q_1, \sigma'_1, \dots, p_1, \sigma_1, \dots) \quad (4.37)$$

with  $D$  the amputated diagram. Now, we combine the above equations to see whether eq. (4.34) gives the correct prescription or not. Each operator  $i(-\square + m^2)$  produces a multiplication by  $i(p^2 + m^2)$  with  $p$  set to the momentum of the appropriate external leg cancelling the factors  $\frac{-i}{q_j^2 + m^2 - i\epsilon}$  or  $\frac{-i}{p_l^2 + m^2 - i\epsilon}$  in eq. (4.37). The integrations over  $x_l$  then yield  $(2\pi)^4 \delta^4(k_l^+ - p_l)$ , the integrations over  $y_j$  give  $(2\pi)^4 \delta^4(k_j^+ + q_j)$ . The  $\sigma'_j$  sums then become

$$\begin{aligned} \sum_{\sigma'_j} \frac{\omega_{\mathbf{q}_j} + \sigma'_j q_j^0}{2\omega_{\mathbf{q}_j}} D(\dots, q_j, \sigma'_j, \dots) &\rightarrow \\ \sum_{\sigma'_j} \frac{\omega_{\mathbf{k}'_j} - \sigma'_j \omega_{\mathbf{k}'_j}}{2\omega_{\mathbf{k}'_j}} D(\dots, -k_j^+, \sigma'_j, \dots) &= D(\dots, -k_j^+, -, \dots) \end{aligned} \quad (4.38)$$

and the sums over  $\sigma_l$  yield

$$\sum_{\sigma_l} \frac{\omega_{\mathbf{p}_l} + \sigma_l p_l^0}{2\omega_{\mathbf{p}_l}} D(\dots, p_l, \sigma_l, \dots) \rightarrow D(\dots, k_l^+, +, \dots). \quad (4.39)$$

Finally, eq. (4.34) gives the following expression for the  $S$ -matrix elements:

$$S_{\alpha\beta} = \frac{1}{(2\pi)^{3/2} \sqrt{2\omega_{\mathbf{k}'_1}}} \dots \frac{1}{(2\pi)^{3/2} \sqrt{2\omega_{\mathbf{k}_1}}} \dots D(-k_1^+, -, \dots, k_1^+, +, \dots). \quad (4.40)$$

Note that  $k_j^+$  stands for an outgoing and  $k_l^+$  for an incoming particle. Together with the fact that all external lines were chosen incoming into  $D$ , one sees that the prescription of eq. (4.34) indeed gives the correct  $S$ -matrix elements in agreement with the results worked out at the beginning of this section.

### 4.3.3 The self-energy matrix

We study the propagation of a single particle only interacting with itself. This is done by summing up all connected contributions with two external lines:

$$G_2(p_1, p_2) \equiv \sum_{D \in \text{Con}} D(p_1, p_2) \equiv -i(2\pi)^4 \delta^4(p_1 - p_2) \sum_{\sigma_1 \sigma_2} \Delta'(p_1)_{\sigma_1 \sigma_2}. \quad (4.41)$$

where Con denotes the set of all connected diagrams with 2 external lines (not amputated) and the matrix  $\Delta'$  was introduced for convenience and will be called *full propagator matrix*. For local QFT, it is well known that the sum of all connected diagrams can be written as (see [29])

$$\Delta'(q) = \frac{1}{q^2 + m^2 - \pi^*(q^2) - i\epsilon}, \quad (4.42)$$



where  $\Delta'$  and  $\pi^*$  are single numbers.  $\pi^*$  is called the self-energy with  $i\pi^*$  standing for the sum over all truncated 1-particle-irreducible (1PI) diagrams. Such a sum can also be defined in analogy for the kind of non-local perturbation theory developed here. The main difference lies in the fact that truncated diagrams in our context have an extra dependence on  $\sigma_1$  and  $\sigma_2$ , the additional labels of the incoming and outgoing lines, respectively. These will be interpreted as the indices of a matrix:

$$i\Pi^*(p)_{\sigma_1\sigma_2} \equiv \sum_{D \in \text{1PI}} D(p, \sigma_1, \sigma_2) . \quad (4.43)$$

with  $\Pi^*$  called the *self-energy matrix* from now on. 1PI denotes the set of all 1PI, truncated diagrams with one incoming and one outgoing leg. In order to obtain all diagrams to be summed up, one has to consider chains of 1PI diagrams. It is therefore useful to cast the pole factors into a matrix:

$$P(p)_{\sigma\sigma'} \equiv \delta_{\sigma\sigma'} \frac{-i}{p^2 + m^2 - i\epsilon} \frac{\omega_p + \sigma p^0}{2\omega_p} . \quad (4.44)$$

The full propagator matrix can now be rewritten comfortably:

$$\begin{aligned} \Delta'(p) &= iP + (iP)\Pi^*(iP) + (iP)\Pi^*(iP)\Pi^*(iP) + \dots \\ &= \left[ 2\omega_p \begin{pmatrix} \omega_p + p^0 & 0 \\ 0 & \omega_p - p^0 \end{pmatrix} - \Pi^* - i\epsilon \right]^{-1} . \end{aligned} \quad (4.45)$$

We have omitted the momentum argument  $p$  on the right hand-side for convenience. The matrix has been written assuming the convention of  $\sigma = -1$  corresponding to the first row or first column. The case of a theory which is local in time can easily be obtained by realising that the components of the matrix  $\Pi^*$  are all identical,  $\Pi^*(p)_{\sigma\sigma'} = \pi^*(p)$ :

$$\sum_{\sigma\sigma'} \Delta'(p)_{\sigma\sigma'} = \frac{1}{p^2 + m^2 - \pi^*(p) - i\epsilon} . \quad (4.46)$$

## 4.4 Symmetries

In this section, various properties of the non-local Feynman rules in momentum space will be studied. First, the Hermiticity of the interaction and its consequences for the vertex factor will be examined. Then, we will address unitarity, Lorentz invariance, and finally parity and time reversal.

### 4.4.1 Hermiticity

Now, we study the information one obtains from the requirement of Hermiticity of the Hamiltonian. For the interaction this means

$$V(0)^\dagger = V(0) . \quad (4.47)$$

Since  $H_0$  is also Hermitian, we also have  $V(t)^\dagger = V(t)$ . Now, one can directly try to apply this to eq. (2.64) to get

$$\int d^3z \int d\mu w(\mu) \phi(z + h_1(\mu)) \dots \phi(z + h_k(\mu)) = \quad (4.48)$$

$$\int d^3z \int d\mu w^*(\mu) \phi(z + h_k(\mu)) \dots \phi(z + h_1(\mu)) . \quad (4.49)$$

A class of Hermitian interactions is trivially given by the requirements  $w = w^*$  and  $h_k = h_1, h_{k-1} = h_2, \dots$ . But these conditions are quite strict and are not satisfied for the cases of interest so far. Thus, Hermiticity is examined through

$$\langle \phi | V(t) | \psi \rangle = \langle \psi | V(t) | \phi \rangle^* . \quad (4.50)$$

$|\phi\rangle$  and  $|\psi\rangle$  are now taken to be free particle states

$$|\phi\rangle = a_{\mathbf{q}_1}^\dagger \dots a_{\mathbf{q}_j}^\dagger |0\rangle , \quad (4.51)$$

$$|\psi\rangle = a_{\mathbf{q}_{j+1}}^\dagger \dots a_{\mathbf{q}_l}^\dagger |0\rangle . \quad (4.52)$$

Furthermore, we assume that the normal ordered part of  $V(t)$  may be considered separately. So we get the condition

$$\langle 0 | a_{\mathbf{q}_1} \dots a_{\mathbf{q}_j} : V(t) : a_{\mathbf{q}_{j+1}}^\dagger \dots a_{\mathbf{q}_l}^\dagger | 0 \rangle = \langle 0 | a_{\mathbf{q}_{j+1}} \dots a_{\mathbf{q}_l} : V(t) : a_{\mathbf{q}_1}^\dagger \dots a_{\mathbf{q}_j}^\dagger | 0 \rangle^* \quad (4.53)$$

The interesting case is now  $l = k$ , the number of field operators occurring in  $V(t)$ . The vacuum expectation value of the left hand-side

$$\langle 0 | a_{\mathbf{q}_1} \dots a_{\mathbf{q}_j} : \phi(x + h_1(\underline{\mu})) \dots \phi(x + h_k(\underline{\mu})) : a_{\mathbf{q}_{j+1}}^\dagger \dots a_{\mathbf{q}_k}^\dagger | 0 \rangle$$

can be evaluated systematically by applying the commutation relations

$$[a_{\mathbf{q}}, \phi^-(x)] = \frac{e^{-iq^+x}}{\sqrt{2\omega_k}(2\pi)^{\frac{3}{2}}} , \quad (4.54)$$

$$[\phi^+(x), a_{\mathbf{q}}^\dagger] = \frac{e^{+iq^+x}}{\sqrt{2\omega_k}(2\pi)^{\frac{3}{2}}} . \quad (4.55)$$

A comparison with eq. (2.71) then yields

$$\frac{e^{-it(\omega_{\mathbf{q}_1} + \dots + \omega_{\mathbf{q}_j} - \omega_{\mathbf{q}_{j+1}} - \dots - \omega_{\mathbf{q}_k})}}{\sqrt{2\omega_{\mathbf{q}_1} \dots 2\omega_{\mathbf{q}_k}}(2\pi)^{\frac{3k}{2}}} (2\pi)^3 \delta^3(\mathbf{q}_1 + \dots + \mathbf{q}_j - \mathbf{q}_{j+1} - \dots - \mathbf{q}_k) \times \\ \chi_k(q_1^+, \dots, q_j^+, -q_{j+1}^+, \dots, -q_k^+) .$$

The right hand-side of eq. (4.53) is dealt with in analogy. Provided the conservation of 3-momentum  $\mathbf{q}_1 + \dots + \mathbf{q}_j = \mathbf{q}_{j+1} + \dots + \mathbf{q}_k$  required by the  $\delta^3$ -function in the above expression, we can read off the relation

$$\chi_k(q_1^+, \dots, q_j^+, -q_{j+1}^+, \dots, -q_k^+) = \chi_k^*(-q_1^+, \dots, -q_j^+, q_{j+1}^+, \dots, q_k^+) \quad (4.56)$$

holding for  $j = 0, \dots, k$ . An expression of the type  $\chi_k(q_1^{\sigma_1}, \dots, q_k^{\sigma_k})$  can always be rewritten as  $\chi_k(p_1^+, \dots, p_j^+, p_{j+1}^-, \dots, p_k^-)$  with  $j$  denoting the number of “+”-signs occurring in  $\sigma_1, \dots, \sigma_k$ , and  $(\mathbf{p}_1, \dots, \mathbf{p}_k)$  is a permutation of the momenta  $(\mathbf{q}_1, \dots, \mathbf{q}_k)$ . Here, one has to make use of the fact that one can rearrange the arguments of  $\chi_k$  arbitrarily. Rewriting the 4-vectors with negative time component  $p^- = -(-p)^+$ , one can apply eq. (4.56) provided that  $\sum_i \mathbf{p}_i = 0$ . Rewriting  $(-p)^+ = -p^-$  and replacing the  $p$ ’s with the  $q$ ’s respectively, one finally gets

$$\chi_k(q_1^{\sigma_1}, \dots, q_k^{\sigma_k}) = \chi_k^*(-q_1^{\sigma_1}, \dots, -q_k^{\sigma_k}) . \quad (4.57)$$

Note that this only holds for  $\sum_i \mathbf{q}_i = 0$ . This condition does not restrict the applicability of eq. (4.57) in connection with the Feynman rules given in [33], since momentum conservation holds at each vertex anyway.

#### 4.4.2 Unitarity

The question of unitarity in connection with NCQFT was intensively investigated in the literature [31, 27, 42, 53] after a violation of this property was observed in [26] when applying the simple rules given in [18]. Here, two ways of examining unitarity will be studied. First, the proof of the unitarity of the  $S$ -matrix given in [29] using the *Lippmann-Schwinger equations* given below will be sketched. Second, the discussion on unitarity in [31] will be commented. It refers to the proof carried out in [54]<sup>3</sup>, where the problem is analysed perturbatively using the Feynman rules. But let us now turn to the Lippmann-Schwinger equations,

$$|\alpha; \pm\rangle = |\alpha\rangle + \int d\beta \frac{T_{\beta\alpha}^{\pm} |\beta\rangle}{E_{\alpha} - E_{\beta} \pm i\epsilon} \quad (4.58)$$

with

$$T_{\beta\alpha}^{\pm} \equiv \langle \beta | V | \alpha; \pm \rangle . \quad (4.59)$$

For better reading, we have used the notation  $|\alpha; +\rangle \equiv |\alpha; \text{in}\rangle$  and  $|\alpha; -\rangle \equiv |\alpha; \text{in}\rangle$ . *Old-fashioned perturbation theory* can then be derived from this by applying the above equations iteratively. According to [29], one examines the matrix elements  $\langle \beta; \pm | V | \alpha; \pm \rangle$  using eq. (4.58) on either the left- or right-hand side. After inserting sums over the complete free particle states, one obtains the relation

$$T_{\alpha\beta}^{\pm *} - T_{\beta\alpha}^{\pm} = - \int d\gamma T_{\gamma\beta}^{\pm *} T_{\gamma\alpha}^{\pm} \left( \frac{1}{E_{\alpha} - E_{\gamma} \pm i\epsilon} - \frac{1}{E_{\beta} - E_{\gamma} \mp i\epsilon} \right) . \quad (4.60)$$

Multiplying this by  $\delta(E_{\alpha} - E_{\beta})$ , the unitarity of the  $S$ -matrix, which can also be written as

$$S_{\beta\alpha} = \delta(\beta - \alpha) - 2i\pi\delta(E_{\alpha} - E_{\beta})T_{\beta\alpha}^{+} , \quad (4.61)$$

can be shown. Special properties of  $V$  like locality have not been used. Only the Hermiticity of  $V$  was needed. It is therefore obvious, that TOPT should give a unitary  $S$ -matrix.

In [54], one starts from the Feynman-rules in momentum space. Then, one integrates out all time components of the momenta associated with the lines for an arbitrary diagram (after expressing the energy conserving  $\delta$ -functions by Fourier transforms). Here, this will also be done for the non-local Feynman rules of chapter 2. We assume that the line we consider is labelled with 4-momentum  $p$ , and  $\sigma$ , and connects two vertices simply named  $a$  and  $b$ . The integral over  $p^0$  for this line is then given by

$$\int dp^0 \frac{-ie^{-ip^0(\tau_a - \tau_b)}}{p^2 - p^{02} + m^2 - i\epsilon} \sum_{\sigma} \frac{\omega_p + \sigma p^0}{2\omega_p} f(p^{\sigma}, \dots) . \quad (4.62)$$

$\tau_a$  and  $\tau_b$  come from the energy conservation at the vertices  $a$  and  $b$ , respectively, and  $f$  contains the corresponding vertex factors. Note, that it only depends on  $\sigma$  and the 3-momentum  $p$ , but not on  $p^0$ . Therefore, the residue theorem can be applied easily, and the above integral can be written as

$$\frac{\pi}{\omega_p} \sum_{\sigma} f(p^{\sigma}, \dots) \theta(\sigma(\tau_a - \tau_b)) e^{-i\sigma(\omega_p - i\epsilon)(\tau_a - \tau_b)} . \quad (4.63)$$

<sup>3</sup>In [54], time-ordered perturbation theory actually refers to the usual perturbation theory with Feynman rules, but with the time components of momenta already integrated out. Note that this is not the case here where TOPT means calculations according to the Dyson series.

In comparison to a local QFT, the only difference is the additional vertex factor. Even the  $\sigma$ -sum is already present for a local theory (see [54], eq. (9.66) therein). The proof of unitarity given in [54] will therefore work as for the local theory. It must be emphasised that the above steps do not work generally for the simple rules given in [18]. This is because the vertex factors then generally depend on  $p^0$  (usually as soon as non-locality involves time) and the residue theorem cannot be carried out as done above. The decision, which complex half-plane is to be chosen for closing the integration path over  $p^0$ , will not depend only on  $\tau_a - \tau_b$ , but also on the type of  $f$  (given by the vertex factors) and possibly the other arguments fed into  $f$  indicated with “...” above. The weird dependence of the  $p^0$ -integration on the vertex factors and other momenta is thus the reason, why the proof of unitarity given in [54] is not applicable for the simple Feynman rules. In contrast to this, the above considerations show that it will work for the framework of TOPT presented here.

In [53], one carried out perturbative calculations within TOPT for NCQED, but one simply used the Lagrangian interaction instead of the Hamiltonian one, which is not correct [29]. One has to keep in mind that according to the result of section 4.2, this cannot be justified with the usual argument of the path integral formalism anymore. The claimed violation of unitarity [53] within TOPT is therefore baseless.

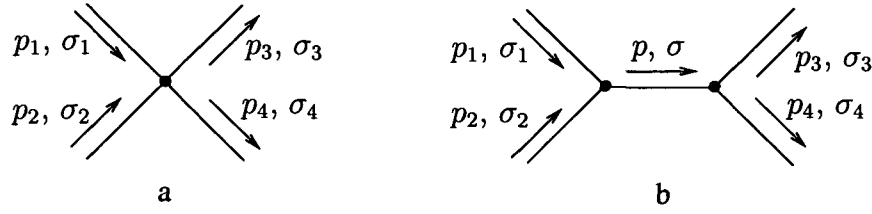
#### 4.4.3 Lorentz invariance

In this section, we want to discuss the properties of the Feynman rules given in chapter 2 with respect to Lorentz transformations. Since the connection of our approach to the Lagrangian formulation of field theory is not clear, we cannot carry out the usual discussion based on the Noether theorem. Instead, we will start from the Lorentz invariance of the  $S$ -matrix and examine the non-local Feynman rules in this context. We will follow the discussion presented in [29] here. Momentum conservation is satisfied within our approach since the momenta are conserved at each vertex as we have already seen in chapter 2. Therefore, we do not have to care about the properties concerning translations anymore. Furthermore, we restrict ourselves to a single kind of particles, namely real, scalar particles, which have spin zero. Thus, we can drop the labels specifying the kind of particle, and spin labels are also left out. The Lorentz group is represented by a unit matrix in this case. For the  $S$ -matrix, one then obtains the following relation:

$$S_{\mathbf{p}'_1, \mathbf{p}'_2, \dots; \mathbf{p}_1, \mathbf{p}_2, \dots} = \sqrt{\frac{\omega_{\Lambda \mathbf{p}_1} \omega_{\Lambda \mathbf{p}_2} \dots \omega_{\Lambda \mathbf{p}'_1} \omega_{\Lambda \mathbf{p}'_2}}{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \dots \omega_{\mathbf{p}'_1} \omega_{\mathbf{p}'_2}}} S_{\Lambda \mathbf{p}'_1, \Lambda \mathbf{p}'_2, \dots; \Lambda \mathbf{p}_1, \Lambda \mathbf{p}_2, \dots}, \quad (4.64)$$

which is obtained from the assumption that both, in- and out- states, transform like products of single particle states. Primes indicate out-states and  $\Lambda \mathbf{p}$  is actually a short notation for the 3-vector  $(\Lambda \mathbf{p})_i \equiv (\Lambda p^+)_i$ .

Now,  $S_{\alpha\beta}$  is calculated perturbatively according to the rules given in chapter 2 and the prescription for calculating  $S$ -matrix elements as described above. We will assume that eq. (4.64) has to hold independently of the actual value of the coupling constant, which suggests that it has to hold for each diagram separately. By inspection of the Feynman rules for non-local interactions, one is then confronted with the following two problems possibly violating Lorentz invariance: First, the  $S$ -matrix elements might depend, besides the momenta of particles, on additional parameters contained in the vertex functions. This is the case for the usual NCQFT with phase factors depending on the tensor  $\theta$ , for example. The second violation is

Figure 4.3: Scattering of two particles of a  $\phi^4$ -theory (a) and  $\phi^3$ -theory (b).

due to the pole factors. The problem concerning the additional parameters can be resolved by considering them as background fields which are to be changed in an appropriate way when changing the observer. To be complete, we want to mention the three ways of interpreting Lorentz transformations given in [55]. First, “Lorentz transformations in special relativity relate physical observations made in two inertial reference frames characterised by different velocities and orientations”, we will refer to as *observer Lorentz transformations*. Alternatively, transformations can relate two particles with different momenta or helicities within one specific inertial frame, known as *particle Lorentz transformations*. Without background fields, both approaches are equivalent. Thirdly, having background fields, one may consider transformations of all fields within a specific inertial frame, including the background field. These transformations are referred to as *(inverse) active Lorentz transformations*. These are then equivalent to observer Lorentz transformations. We adopt the first (or equivalently the third) approach implying the transformation of  $\theta$  like a background field. The fact that such fields and their influence on Lorentz transformations have not been observed yet implies that they must be very small.

As an example, let us consider the scattering of two scalar particles with some non-local  $\phi^4$ -interaction as shown in fig. 4.3a. The contribution to the  $S$ -matrix of this process to first order is then given by

$$S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(1)} = -i\delta^4(p_1^+ + p_2^+ - p_3^+ - p_4^+) \frac{\chi_4(p_1^+, p_2^+, -p_3^+, -p_4^+)}{\sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}}, \quad (4.65)$$

where  $p_1, p_2$  are the incoming and  $p_3, p_4$  are the outgoing momenta. This is now put into eq. (4.64) assuming that it has to hold for this order separately. The energies associated with external legs now cancel and for the case of momentum conservation, one is left with the relation

$$\chi_4(p_1^+, p_2^+, -p_3^+, -p_4^+) = \chi_4((\Lambda p_1)^+, (\Lambda p_2)^+, -(\Lambda p_3)^+, -(\Lambda p_4)^+). \quad (4.66)$$

Let us now consider the case of NCQFT with  $\chi_4$  given by eq. (2.116). Clearly, the above relation can only be satisfied if we admit that the right hand-side is calculated with a different  $\theta$ , say  $\theta'$  satisfying

$$\theta_{\rho\lambda} = \theta'_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda. \quad (4.67)$$

One would have to replace  $S$  in eq. (2.116) with  $S^\theta$  and  $S^{\theta'}$ , respectively in order to implement these background fields correctly.

The second problem consists of the  $\sigma$ -dependent term in

$$\frac{-i}{q^2 + m^2 - i\epsilon} \frac{\omega_q + \sigma q^0}{2\omega_q}.$$

Obviously, the last fraction violates Lorentz invariance. In contrast to the above problem, where it turned out that  $\theta$  has to be treated as a background field, this is now a qualitatively new aspect. To examine it, a diagram containing at least one internal line has to be investigated. We again consider the scattering of two scalar particles as an example, but interacting through a  $\phi^3$ -potential. Actually, there are three connected diagrams to second order in the coupling constants. The one shown in fig. 4.3b combines  $p_1$ ,  $p_2$  and the inner momentum  $p$  at one vertex, where  $p$  is given by  $p = p_1 + p_2$ . This is generally referred to as the  $s$ -channel, since  $p$  corresponds to the Mandelstam variable  $s$ . The diagram where  $p_1$ ,  $p_3$  and  $p = p_1 - p_3$  are attached to one vertex is referred to as the  $t$ -channel. The third possible combination involves  $p_1$ ,  $p_4$  and  $p = p_1 - p_4$  at one vertex referred to as the  $u$ -channel. Applying the Feynman rules and the prescriptions to calculate  $S$ -matrix elements, one then obtains for the  $s$ -channel

$$S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,s)} = -\frac{\delta^4(p_1^+ + p_2^+ - p_3^+ - p_4^+)}{4(2\pi)^2 \sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}} \sum_{\sigma} \frac{-i}{(p_1^+ + p_2^+)^2 + m^2 - i\epsilon} \times \frac{\omega_{\mathbf{p}_1 + \mathbf{p}_2} + \sigma(\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2})}{2\omega_{\mathbf{p}_1 + \mathbf{p}_2}} \chi_3(p_1^+, p_2^+, -(p_1 + p_2)^\sigma) \chi_3(-p_3^+, -p_4^+, (p_1 + p_2)^\sigma). \quad (4.68)$$

Similarly, one gets for the  $t$ -channel

$$S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,t)} = -\frac{\delta^4(p_1^+ + p_2^+ - p_3^+ - p_4^+)}{4(2\pi)^2 \sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}} \sum_{\sigma} \frac{-i}{(p_1^+ - p_3^+)^2 + m^2 - i\epsilon} \times \frac{\omega_{\mathbf{p}_1 - \mathbf{p}_3} + \sigma(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_3})}{2\omega_{\mathbf{p}_1 - \mathbf{p}_3}} \chi_3(p_1^+, -p_3^+, -(p_1 - p_3)^\sigma) \chi_3(p_2^+, -p_4^+, (p_1 - p_3)^\sigma), \quad (4.69)$$

and for the  $u$ -channel

$$S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,u)} = -\frac{\delta^4(p_1^+ + p_2^+ - p_3^+ - p_4^+)}{4(2\pi)^2 \sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}} \sum_{\sigma} \frac{-i}{(p_1^+ - p_4^+)^2 + m^2 - i\epsilon} \times \frac{\omega_{\mathbf{p}_1 - \mathbf{p}_4} + \sigma(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_4})}{2\omega_{\mathbf{p}_1 - \mathbf{p}_4}} \chi_3(p_1^+, -p_4^+, -(p_1 - p_4)^\sigma) \chi_3(p_2^+, -p_3^+, (p_1 - p_4)^\sigma). \quad (4.70)$$

We then assume that eq. (4.64) has to hold for these three diagrams together:

$$S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,s)} + S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,t)} + S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,u)} = \sqrt{\frac{\omega_{\Lambda \mathbf{p}_1} \omega_{\Lambda \mathbf{p}_2} \omega_{\Lambda \mathbf{p}_3} \omega_{\Lambda \mathbf{p}_4}}{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}} \times \left( S_{\Lambda \mathbf{p}_3, \Lambda \mathbf{p}_4; \Lambda \mathbf{p}_1, \Lambda \mathbf{p}_2}^{\prime(2,s)} + S_{\Lambda \mathbf{p}_3, \Lambda \mathbf{p}_4; \Lambda \mathbf{p}_1, \Lambda \mathbf{p}_2}^{\prime(2,t)} + S_{\Lambda \mathbf{p}_3, \Lambda \mathbf{p}_4; \Lambda \mathbf{p}_1, \Lambda \mathbf{p}_2}^{\prime(2,u)} \right). \quad (4.71)$$

The prime at the  $S$  matrix indicates that it is to be calculated with the appropriately transformed background fields, which means that we have to use  $\chi'_3$  instead of  $\chi_3$  when calculating  $S'$ . The remaining relation then reads

$$\begin{aligned} & \sum_{\sigma} \left( \frac{\omega_{\mathbf{p}_1 + \mathbf{p}_2} + \sigma(\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2})}{2\omega_{\mathbf{p}_1 + \mathbf{p}_2}} - \frac{\omega_{\Lambda \mathbf{p}_1 + \Lambda \mathbf{p}_2} + \sigma(\omega_{\Lambda \mathbf{p}_1} + \omega_{\Lambda \mathbf{p}_2})}{2\omega_{\Lambda \mathbf{p}_1 + \Lambda \mathbf{p}_2}} \right) f_s(\sigma, \mathbf{p}_1, \dots, \mathbf{p}_4) \\ & + \sum_{\sigma} \left( \frac{\omega_{\mathbf{p}_1 - \mathbf{p}_3} + \sigma(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_3})}{2\omega_{\mathbf{p}_1 - \mathbf{p}_3}} - \frac{\omega_{\Lambda \mathbf{p}_1 - \Lambda \mathbf{p}_3} + \sigma(\omega_{\Lambda \mathbf{p}_1} - \omega_{\Lambda \mathbf{p}_3})}{2\omega_{\Lambda \mathbf{p}_1 - \Lambda \mathbf{p}_3}} \right) f_t(\sigma, \mathbf{p}_1, \dots, \mathbf{p}_4) \\ & + \sum_{\sigma} \left( \frac{\omega_{\mathbf{p}_1 - \mathbf{p}_4} + \sigma(\omega_{\mathbf{p}_1} - \omega_{\mathbf{p}_4})}{2\omega_{\mathbf{p}_1 - \mathbf{p}_4}} - \frac{\omega_{\Lambda \mathbf{p}_1 - \Lambda \mathbf{p}_4} + \sigma(\omega_{\Lambda \mathbf{p}_1} - \omega_{\Lambda \mathbf{p}_4})}{2\omega_{\Lambda \mathbf{p}_1 - \Lambda \mathbf{p}_4}} \right) f_u(\sigma, \mathbf{p}_1, \dots, \mathbf{p}_4) \\ & = 0, \end{aligned} \quad (4.72)$$

which has to hold for all configurations satisfying momentum conservation  $p_1^+ + p_2^+ = p_3^+ + p_4^+$ . The  $f$ 's are abbreviations for the internal propagator and the vertex factors of the corresponding channels. Note that this can be written in such a way since the propagator is Lorentz invariant and the vertex factors are to be calculated with appropriate background fields, which gives  $\chi'_3(\Lambda p_1, \Lambda p_2, \Lambda p_3) = \chi_3(p_1, p_2, p_3)$ . Clearly, the coefficients of the  $f$ 's do not vanish for general Lorentz transformations  $\Lambda$ , which simply expresses the fact that  $(\omega_q + \sigma q^0)/(2\omega_q)$  is not Lorentz invariant. The above relation is satisfied as soon as the vertex function  $\chi_3$  does not depend on  $\sigma$  since then the  $f$ 's are independent of  $\sigma$  and

$$\sum_{\sigma} \left( \frac{\omega_{\mathbf{p}_1 + \mathbf{p}_2} + \sigma(\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2})}{2\omega_{\mathbf{p}_1 + \mathbf{p}_2}} - \frac{\omega_{\Lambda \mathbf{p}_1 + \Lambda \mathbf{p}_2} + \sigma(\omega_{\Lambda \mathbf{p}_1} + \omega_{\Lambda \mathbf{p}_2})}{2\omega_{\Lambda \mathbf{p}_1 + \Lambda \mathbf{p}_2}} \right) = 0.$$

Even when having a closer look at eq. (4.72), one does not find other reasonable solutions to this problem. Furthermore, there are more complicated diagrams of higher order or different scattering processes, which all have to be Lorentz invariant yielding lots of similar conditions to be satisfied. We therefore conjecture: *Lorentz invariance might only be satisfied when the vertex factors are independent of the  $\sigma$ -variables.* It should be emphasised here that this statement only effects non-local scalar field theory treated within the framework of time-ordered perturbation theory worked out in an earlier chapter.

In the case of NCQFT with constant  $\theta$ , the requirement of  $\sigma$ -independence then means that  $\theta_{0i} = 0$ . But unfortunately, as we have seen above, different observers have to transform the background field  $\theta$  according to eq. (4.67), and the condition of vanishing  $\theta'_{0i}$  (the transformed background field) does not hold anymore. This is only one example of a non-local theory which is not Lorentz invariant, even when accepting  $\theta$  to be considered as a background field. The same problems affect the other suggested non-local field theories discussed in section 2.5. This leaves the conjecture that non-local QFTs cannot be constructed in such a way to satisfy Lorentz invariance. However, one should finally keep in mind that these results are obtained in the framework of time-ordered perturbation theory. Other schemes of setting up non-local QFT might be more successful. The path integral formulation for example gives the covariant Feynman rules much easier in the case of local QFTs. But this approach then has the drawback of not being automatically unitary. A corresponding violation of this property was for example found in [26] using the rules worked out in [18]. This would rather support the application of time-ordered perturbation theory or possibly other approaches being a priori unitary.

Finally, we want to mention an alternative, modifying the traditional approaches of Lorentz invariance [56, 57, 58]. These papers deal with the case that  $\theta$  represents universal constants, which are not interpreted as background fields, whereas the Poincaré algebra is deformed.

#### 4.4.4 $\mathcal{P}, \mathcal{T}$

Now, the symmetries of the parity transformation  $\mathcal{P}$  and time reversal  $\mathcal{T}$  will be studied. These act on 4-vectors  $x$  as follows:

$$\mathcal{P}x = (-\mathbf{x}, x^0)^T, \quad (4.73)$$

$$\mathcal{T}x = (\mathbf{x}, -x^0)^T. \quad (4.74)$$

The parity of scalar particles this discussion is restricted to is 1. Similarly, the time reversal does not leave any phase at scalar particle states. Assuming parity to be a symmetry of the

system, one therefore gets the following condition for the  $S$ -matrix of scalar particles:

$$S_{\mathbf{p}'_1, \mathbf{p}'_2, \dots; \mathbf{p}_1, \mathbf{p}_2, \dots} = S_{-\mathbf{p}'_1, -\mathbf{p}'_2, \dots; -\mathbf{p}_1, -\mathbf{p}_2, \dots} \quad (4.75)$$

Assuming the symmetry of time reversal then gives

$$S_{\mathbf{p}'_1, \mathbf{p}'_2, \dots; \mathbf{p}_1, \mathbf{p}_2, \dots} = S_{-\mathbf{p}_1, -\mathbf{p}_2, \dots; -\mathbf{p}'_1, -\mathbf{p}'_2, \dots} \quad (4.76)$$

Note, that the role of in- and out-states is interchanged here. Studying tree diagrams of first order as we did in eq. (4.65), it is obvious to demand

$$\chi_k(p_1, \dots, p_k) = \chi_k(\mathcal{P}p_1, \dots, \mathcal{P}p_k) \quad (4.77)$$

to satisfy parity, and

$$\chi_k(p_1, \dots, p_k) = \chi_k(\mathcal{T}p_1, \dots, \mathcal{T}p_k) \quad (4.78)$$

to satisfy time reversal. Both relations guarantee that eqs. (4.75) and (4.76) are satisfied when considering the scattering of  $k$  particles at a single vertex. We will only consider the case of NCQFT with constant  $\theta$ . Eqs. (4.77) and (4.78) are in general satisfied as soon as one has

$$(\mathcal{P}p)(\mathcal{P}\theta)(\mathcal{P}q) = \pm p\theta q, \quad (4.79)$$

$$(\mathcal{T}p)(\mathcal{T}\theta)(\mathcal{T}q) = \pm p\theta q. \quad (4.80)$$

Again,  $\theta$  is considered as a background field. The signs on the right hand-side can be chosen freely within the considerations carried out here. This is because the phase factor does not change when transforming  $\theta \rightarrow -\theta$ . However, one can also study gauge theories involving charged particles. Then it makes sense to consider charge conjugation  $\mathcal{C}$  also. In [59, 60], one constructed theories invariant under  $\mathcal{CPT}$ . The result was to choose a “+” for parity in eq. (4.79), and a “−” for time reversal in eq. (4.80). With this assumption, one can then read off the following transformation properties for  $\theta$ :

$$\begin{aligned} -(\mathcal{T}\theta)_{ij} &\equiv (\mathcal{P}\theta)_{ij} \equiv \theta_{ij}, \\ -(\mathcal{T}\theta)_{0j} &\equiv (\mathcal{P}\theta)_{0j} \equiv -\theta_{0j}. \end{aligned}$$

Taking the rule for charge conjugation  $\mathcal{C}\theta = -\theta$  from [59, 60], it can be seen that the combined transformation  $\mathcal{CPT}$  leaves  $\theta$  unchanged.

The UV-finite theories studied in section 2.5 and chapter 3 both satisfy eqs. (4.77) and (4.78), whereas for the simplified version also treated in chapter 3,  $\kappa$  is to be chosen diagonal. Otherwise, one would have to transform it correspondingly.

Finally, we want to consider the  $s$ -channel of 2-particle scattering in  $\phi^3$ -theory, already shown in fig. 4.3b. This should give an answer to the question whether the pole factors cause problems to these symmetries or not. The result of the corresponding contribution to the  $S$ -matrix is already given in eq. (4.68). Eq. (4.75) then becomes

$$\begin{aligned} S_{-\mathbf{p}_3, -\mathbf{p}_4; -\mathbf{p}_1, -\mathbf{p}_2}^{(2,s)} &= -\frac{\delta^4(p_1^+ + p_2^+ - p_3^+ - p_4^+)}{4(2\pi)^2 \sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}} \sum_{\sigma} \frac{-i}{(p_1^+ + p_2^+)^2 + m^2 - i\epsilon} \times \\ &\quad \frac{\omega_{\mathbf{p}_1 + \mathbf{p}_2} + \sigma(\omega_{\mathbf{p}_1} + \omega_{\mathbf{p}_2})}{2\omega_{\mathbf{p}_1 + \mathbf{p}_2}} \chi_3(\mathcal{P}p_1^+, \mathcal{P}p_2^+, -\mathcal{P}(p_1 + p_2)^{\sigma}) \times \\ &\quad \chi_3(-\mathcal{P}p_3^+, -\mathcal{P}p_4^+, \mathcal{P}(p_1 + p_2)^{\sigma}) \\ &= S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,s)}. \end{aligned} \quad (4.81)$$



Only the relation shown in eq. (4.77) is needed to satisfy this condition. Therefore, we shall assume that the pole factors do not cause any problems to the parity symmetry. The same question can be asked concerning the reversal of time:

$$\begin{aligned}
S_{-\mathbf{p}_1, -\mathbf{p}_2; -\mathbf{p}_3, -\mathbf{p}_4}^{(2,s)} &= -\frac{\delta^4(p_3^+ + p_4^+ - p_1^+ - p_2^+)}{4(2\pi)^2 \sqrt{\omega_{\mathbf{p}_1} \omega_{\mathbf{p}_2} \omega_{\mathbf{p}_3} \omega_{\mathbf{p}_4}}} \sum_{\sigma} \frac{-i}{(p_3^+ + p_4^+)^2 + m^2 - i\epsilon} \times \\
&\quad \frac{\omega_{\mathbf{p}_3+\mathbf{p}_4} + \sigma(\omega_{\mathbf{p}_3} + \omega_{\mathbf{p}_4})}{2\omega_{\mathbf{p}_3+\mathbf{p}_4}} \chi_3(\mathcal{P}p_3^+, \mathcal{P}p_4^+, -\mathcal{P}(p_3 + p_4)^{\sigma}) \times \\
&\quad \chi_3(-\mathcal{P}p_1^+, -\mathcal{P}p_2^+, \mathcal{P}(p_3 + p_4)^{\sigma}) \\
&= S_{\mathbf{p}_3, \mathbf{p}_4; \mathbf{p}_1, \mathbf{p}_2}^{(2,s)} .
\end{aligned} \tag{4.82}$$

Due to the interchange of in- and out-states, the last step is slightly more difficult to realise. First, one replaces  $\mathbf{p}_3 + \mathbf{p}_4$  with  $\mathbf{p}_1 + \mathbf{p}_2$  in the vertex factors, enabled by momentum conservation. Next, one makes use of  $-\mathcal{P}x = \mathcal{T}x$ . Finally, one has to realise that the two vertex factors have been interchanged. We therefore conclude that the pole factors also do not harm the symmetry of time reversal, and non-local theories can satisfy both, the symmetry of time reversal and of parity, within TOPT.

## 4.5 A perturbative comparison

In this section, we want to compare the perturbation theory given in [18], or more general the NPIA, with our approach. First, it should be noted that for the case of locality in time, say  $h_i(\mu)^0 = 0$  for  $i = 1, \dots, k$ , the vertex factor  $\chi_k$  given in eq. (2.71) is independent of the time components of its arguments, and therefore, it is independent of the  $\sigma$ 's. Furthermore, the vertex factors occurring in any diagram are identical of TOPT and the NPIA:

$$\chi_k(p_1^{\sigma_1}, \dots, p_k^{\sigma_k})|_{h_i^0=0, i=1, \dots, k} = \chi_k(p_1, \dots, p_k)|_{h_i^0=0, i=1, \dots, k} . \tag{4.83}$$

It should be emphasised that both expressions now only depend on the 3-momenta  $\mathbf{p}_i$ , and neither on the time components of the off-shell momenta  $p_i^0$ , nor on the variables  $\sigma$  any more. The only  $\sigma$ -dependence is now included in the pole factors, which directly drops out:

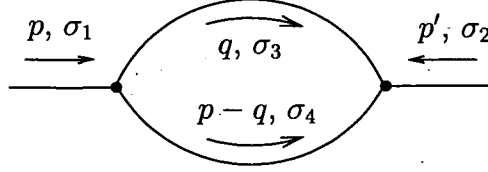
$$\sum_{\sigma_i} \frac{\omega_{\mathbf{p}_i} + \sigma_i p_i^0}{2\omega_{\mathbf{p}_i}} = 1 .$$

Consequently, TOPT is equivalent to the NPIA as soon as the non-localities in time vanish,  $h_i^0 = 0$ , for  $i = 1, \dots, k$ . It thus does not matter whether we calculate a certain diagram  $D$  according to the TOPT rules (subscript  $T$ ) or the NPIA (subscript  $N$ ). This is expressed as

$$D_T(p_1, \sigma_1; \dots; p_k, \sigma_k)|_{h^0=0} = D_N(p_1, \dots, p_k)|_{h^0=0} , \tag{4.84}$$

where  $h^0 = 0$  stands for locality in time,  $h_i^0 = 0$  with  $i = 1, \dots, k$ .

As a next step, it would be interesting to consider an expansion in the time component of the non-locality around  $h^0 = 0$ . Therefore, we will first consider a special example, namely the diagram shown in fig. (4.4). Using the rules given in chapter 2, one can directly write

Figure 4.4: 1-loop self-energy contribution of  $\phi^3$  theory.

down the result for TOPT referred to as  $D_T^L$ :

$$\begin{aligned}
 D_T^L(p, p') &= (-i)^2 \frac{(2\pi)^4}{2} \delta^4(p + p') \left( \frac{-i}{p^2 + m^2 - i\epsilon} \right)^2 \times \\
 &\sum_{\sigma_1, \sigma_2} \frac{\omega_p + \sigma_1 p^0}{2\omega_p} \frac{\omega_p + \sigma_2 p^0}{2\omega_p} \times \\
 &\int d^4q \frac{-i}{q^2 + m^2 - i\epsilon} \frac{-i}{(p - q)^2 + m^2 - i\epsilon} \times \\
 &\sum_{\sigma_3, \sigma_4} \frac{\omega_q + \sigma_3 q^0}{2\omega_q} \frac{\omega_{p-q} + \sigma_4 (p^0 - q^0)}{2\omega_{p-q}} \times \\
 &\chi_3(p^{\sigma_1}, -q^{\sigma_3}, -(p - q)^{\sigma_4}) \chi_3(-p^{\sigma_2}, q^{\sigma_3}, (p - q)^{\sigma_4}).
 \end{aligned} \tag{4.85}$$

Here, momentum conservation  $p' = -p$  was used and  $\sigma_2$  was transformed  $\sigma_2 \rightarrow -\sigma_2$ . This expression should now be expanded in terms of  $h^0$  around  $h^0 = 0$ . Therefore,

$$\left. \frac{\delta D_T^L(p, p')}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0}$$

is to be calculated. This operation just effects the last line of eq. (4.85), which becomes

$$\begin{aligned}
 &\left[ \frac{\delta \chi_3(p^{\sigma_1}, -q^{\sigma_3}, -(p - q)^{\sigma_4})}{\delta h_i(\underline{\mu}')^0} \chi_3(-p^{\sigma_2}, q^{\sigma_3}, (p - q)^{\sigma_4}) \right]_{h^0=0} \\
 &+ \left[ \chi_3(p^{\sigma_1}, -q^{\sigma_3}, -(p - q)^{\sigma_4}) \frac{\delta \chi_3(-p^{\sigma_2}, q^{\sigma_3}, (p - q)^{\sigma_4})}{\delta h_i(\underline{\mu}')^0} \right]_{h^0=0}.
 \end{aligned} \tag{4.86}$$

This derivative of the vertex factor defined in eq. (2.71) can in general be written as

$$\left. \frac{\delta \chi_k(q_1^{\sigma_1}, \dots, q_k^{\sigma_k})}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0} = +i w(\underline{\mu}') \sum_{Q \in S^k} \sigma_{Q_j} \omega_{\mathbf{q}_{Q_j}} \exp(-i \sum_i \mathbf{q}_{Q_j} \mathbf{h}_i(\underline{\mu}')). \tag{4.87}$$

The important point about this relation is the fact that  $\sigma$  occurs only linear in the above relation, accompanied by the energy of a single particle. Due to the expansion around  $h^0 = 0$ , it does not occur anywhere else. In expression (4.86), the  $\sigma$ 's then also occur only linear. Therefore, the  $\sigma$ -sums in eq. (4.85) can easily be performed. Out of the various  $\sigma$ 's, there is only one, say  $\sigma_{Q_j}$  with a factor  $\omega_{\mathbf{q}_{Q_j}}$ , for each of the summands resulting from the product rule of differentiation (see expression (4.86)) and the sum over permutations (see eq. (4.87)) at each vertex, giving the simple result

$$\sum_{\sigma_1 \dots \sigma_M} \frac{\omega_{\mathbf{q}_1} + \sigma_1 q_1^0}{2\omega_{\mathbf{q}_1}} \dots \frac{\omega_{\mathbf{q}_{Q_j}} + \sigma_{Q_j} q_{Q_j}^0}{2\omega_{\mathbf{q}_{Q_j}}} \sigma_{Q_j} \omega_{\mathbf{q}_{Q_j}} \dots \frac{\omega_{\mathbf{q}_M} + \sigma_M q_M^0}{2\omega_{\mathbf{q}_M}} = q_{Q_j}^0, \tag{4.88}$$

where  $M$  denotes the number of lines (internal and external ones,  $M = 4$  in this special case), and  $q_i$  is a systematic notation for their momenta. To compare to the NPIA, we write down the corresponding expression  $D_N^L$  of fig. (4.4) for this approach:

$$D_N^L(p, p') = (-i)^2 \frac{(2\pi)^4}{2} \delta^4(p + p') \left( \frac{-i}{p^2 + m^2 - i\epsilon} \right)^2 \times \int d^4 q \frac{-i}{q^2 + m^2 - i\epsilon} \frac{-i}{(p - q)^2 + m^2 - i\epsilon} \times \chi_3(p, -q, -(p - q)) \chi_3(-p, q, p - q). \quad (4.89)$$

The expansion we are interested in is

$$\left. \frac{\delta D_N^L(p, p')}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0}.$$

The differentiation again only effects the last line of eq. (4.89), giving expression (4.86), but the  $\sigma$ 's are dropped. Furthermore, one has

$$\left. \frac{\delta \chi_k(q_1, \dots, q_k)}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0} = +i w(\underline{\mu}') \sum_{Q \in S^k} q_{Q_j}^0 \exp(-i \sum_i \mathbf{q}_{Q_j} \mathbf{h}_i(\underline{\mu}')). \quad (4.90)$$

This is already the same as one obtains for eq. (4.87) after carrying out the  $\sigma$ -sums (see eq. (4.88)). Putting everything together, one can now write

$$\left. \frac{\delta D_T^L(p, p')}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0} = \left. \frac{\delta D_N^L(p, p')}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0}. \quad (4.91)$$

Inspecting the steps needed above (product rule, expansion of  $\chi_k$ ,  $\sigma$ -sums), it is clear that this procedure works for any diagram  $D$  with  $E$  external lines labelled with momenta  $p_1, \dots, p_E$ , and we can therefore generalise the above relation

$$\left. \frac{\delta D_T(p_1, \dots, p_E)}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0} = \left. \frac{\delta D_N(p_1, \dots, p_E)}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0}, \quad (4.92)$$

or summarising eqs. (4.84) and (4.92) in words: *TOPT and NPIA are identical up to first order in the time component of the non-locality  $h^0$ . Any differences of these two approaches are of order  $\mathcal{O}((h^0)^2)$ . In fact, one can show by studying simple examples that both approaches differ at order  $\mathcal{O}((h^0)^2)$ :*

$$\left. \frac{\delta D_T(p_1, \dots, p_E)}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0} - \left. \frac{\delta D_N(p_1, \dots, p_E)}{\delta h_i(\underline{\mu}')^0} \right|_{h^0=0} = \mathcal{O}((h^0)^2). \quad (4.93)$$

Actually, these results were motivated by first considering an expansion of the diagram in fig. (4.4) within NCQFT around  $\theta_{0i} = 0$ . It seemed to be miraculous as it turned out that TOPT and the NPIA are identical up to first order in  $\theta_{0i}$ . This can be seen by directly differentiating eq. (2.116) with respect to  $\theta_{0i}$ :

$$\left. \frac{\partial \chi_k(p_1, \dots, p_k)}{\partial \theta_{0i}} \right|_{\theta_{0i}=0} = \frac{-i\kappa}{k!} \sum_{Q \in S^k} \exp \left( -i \sum_{i < j} p_{Q_i} \wedge p_{Q_j} \right) \Big|_{\theta_{0i}=0} \times \sum_{i < j} \frac{1}{2} (p_{Q_i}^0 p_{Q_j}^i - p_{Q_i}^i p_{Q_j}^0). \quad (4.94)$$

The fact that the time components of momenta only occur linear is all we need and the argumentation carried out above works essentially the same and we obtain

$$\left. \frac{\partial D_T(p_1, \dots, p_E)}{\partial \theta_{0i}} \right|_{\theta_{0i}=0} - \left. \frac{\partial D_N(p_1, \dots, p_E)}{\partial \theta_{0i}} \right|_{\theta_{0i}=0} = \mathcal{O}((\theta_{0i})^2). \quad (4.95)$$

Since TOPT respects unitarity, a possible violation of this principle within the NPIA can only occur at order  $\mathcal{O}((\theta_{0i})^2)$  or higher, provided that Taylor expansions around  $\theta_{0i}$  are possible.

## 4.6 Summary

It was shown in the first section that NCQFT does not alter the free Hamiltonian. This applies to scalar QFT as well as for gauge field models [35]. However, the application of these techniques to Gaussian non-localities [32, 34] seems to be somewhat artificial since here we could show that the introduction of the non-localities into the free Hamiltonian would alter it significantly when assuming the usual free field equations. Thus, either one would have to leave the free Hamiltonian untouched [34] or one would have to develop an appropriate free theory. The effect of non-localities onto the free Hamiltonian has been studied on a general basis and can be quickly checked by applying eqs. (4.10) and (4.11).

Furthermore, the discrepancies between TOPT and the NPIA present for non-localities in time have been explained. The main reason is the problem of passing from non-local products given in the Dirac picture to the Heisenberg picture. One would have to alter these products to be consistent. But usually one compares situations where one deals with the same product in both pictures, which in general does not give the same as soon as non-localities in time are involved. We also want to point out that the use of the Lagrangian interaction instead of the Hamiltonian one in combination with TOPT is not justified by path integrals as soon as non-localities involve time. This explains the claim suggested in [53] that also TOPT would violate unitarity when applying it to non-commutative gauge field theory. One simply applied TOPT using the Lagrangian interaction instead of the Hamiltonian one, which is not justified.

The usual Feynman diagrams of local QFT can also be used in connection with non-local interactions. However, the rules are more complicated. Therefore, certain diagrammatic techniques have to be extended to the non-local case. In the third section, the process of amputation was investigated. It was defined such that complex diagrams consisting of sub-diagrams can be easily built up from these. One leaves out the  $\sigma$ -sums and the pole factors associated with the external legs to obtain an amputated version of a diagram. Next, we gave a prescription for obtaining physical  $S$ -matrix elements. It was seen that the common prescription of local QFT can be taken for non-local theories too. In the last part of section 4.3, the concept of self-energy was also extended to the non-local case. Even for scalar particles, the self-energy is a matrix whose indices correspond to the two  $\sigma$ -variables associated with the external legs.

In section 4.4, a few symmetries were studied. First, we started with the Hermiticity of the interaction, and worked out its meaning to the vertex factor. Then, the unitarity of TOPT was discussed. It can be proved by the Lippmann-Schwinger equation. Alternatively, one can directly study the diagrammatic expansion via cutting rules. Without going into detail, it was seen that the corresponding proof of unitarity [54, 31] will hold for TOPT, and fail for the simple rules. This is because the vertex factor of TOPT does only depend on the space components of the momenta, and the time components can then easily be integrated

out as usual. For the NPIA, this is problematic since the vertex factor then depends on the full 4-momenta. Next, Lorentz invariance of the  $S$ -matrix elements was examined according to [29]. It turned out that within TOPT, Lorentz invariance is violated for the non-local interactions studied here even when tolerating the presence of background fields. The results concerning this problem are not encouraging. But it has to be mentioned that this problem has not been treated from the most general point of view. A more general treatment would either show a way out of the violation of Lorentz invariance, or lead to a proof that it is impossible to satisfy this symmetry within TOPT except for local interactions. The situation is better for the symmetries of time reversal and parity. These are respected by all non-local interactions studied here.

An interesting observation was made in section 4.5. It says that the difference of the NPIA and TOPT is of second order in the time component of the non-locality. For certain calculations, it might therefore be sufficient to consider the simpler NPIA instead of TOPT, where especially loop calculations turn out to be much more complicated.

The main motivation for studying TOPT is the property of unitarity and the better behaving UV/IR mixing [46]. The disadvantages are the violation of causality and Lorentz invariance, and the higher complexity of the Feynman rules. The situation is vice versa for the NPIA, where the implied time-ordering respects causality. But the violation of unitarity in the NPIA is a severe problem, whereas it seems to be absent in TOPT.

## Chapter 5

# Adding a harmonic oscillator

In this chapter, we consider the effect of adding a harmonic oscillator potential to the  $\phi^4$  action. This model was already discussed in [36, 37, 38, 39, 40], where the oscillator term serves as a regulator for studying the renormalisation of NCQFT in the matrix base. One might expect that the oscillator term, which causes a cut-off at spatial infinity, provides a regularisation to the UV/IR-mixing problem. We also want to mention the duality property of the harmonic oscillator [61], which says that its action looks the same in momentum as well coordinate space (just the constants differ). However, translation invariance will be violated. The action we want to deal with, is now taken from [37]:

$$\begin{aligned} I[\phi] &= \int d^D x \left( \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) + \frac{m^2}{2} \phi^2 + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) (\tilde{x}^\mu \phi) + \frac{\lambda}{4!} (\phi^*)^4 \right) (x) \\ &\equiv I_0[\phi] + \int d^D x \frac{\lambda}{4!} (\phi^*)^4 (x) \end{aligned} \quad (5.1)$$

with  $\tilde{x}_\mu \equiv 2(\theta^{-1})_{\mu\nu} x^\nu$ .  $\theta$  is the constant, anti-symmetric tensor characterising non-commutativity of space-time, assumed to have full rank.  $\Omega$  is a dimensionless parameter for adjusting the strength of the harmonic oscillator. The propagator will be calculated by inverting the bilinear part of eq. (5.1) including also the harmonic oscillator term. This means that the propagator is not translation invariant, and its Fourier transform will thus depend on two instead of one 4-momentum. Energy-momentum conservation will however hold at each vertex, since the interaction does not violate translation invariance.

The space-time dimension is kept general (restricted to be even) in order to have a better insight when carrying out intermediate calculations. But finally,  $D$  is intended to be set to 4 (as it makes sense when considering a  $\phi^4$  theory). Furthermore, we will just consider the Euclidean case. For the Minkowski space, it is not known how to choose the harmonic oscillator term to be both, positive semi-definite and Lorentz invariant. We will also just derive the Feynman rules from a naive path integral approach, without properly carrying out TOPT. This will only give the same results as TOPT when  $\theta_{0i} = 0$ , but this case is excluded anyway since we have assumed  $\theta$  to have full rank. Within this approach, the omission of the star products in bilinear terms is also justified. The action given above will be referred to as the *dual action* due to the duality already mentioned at the beginning of this section.

## 5.1 The propagator

In order to simplify calculations, we restrict ourselves to frames where  $\theta_{2i-1,2i} = -\theta_{2i,2i-1} = \theta$  for  $i = 1, \dots, D/2$  and all other components vanish. Then,  $(\theta^{-1})_{\mu\nu}(\theta^{-1})^{\mu\rho} = \frac{1}{\theta^2} \delta_\nu^\rho$ .

The propagator  $\Delta$  is then obtained by inverting the bilinear part of eq. (5.1). Thus, the remaining problem is to solve

$$\frac{1}{2} [-\partial^2 + \Omega^2 \tilde{x}^2 + m^2] \Delta(x, y) = \delta^D(x - y) \quad (5.2)$$

for  $\Delta$ . Thanks to H. Grosse and R. Wulkenhaar, we know that this problem is related to some kernel of the harmonic oscillator, given by Mehler's formula [62]. What is meant by kernel and its connection to our problem will be enlightened in the following subsection.

### 5.1.1 Green functions and kernels

First of all, we rewrite the problem, we want to solve:

$$H_x \Delta(x, y) = \delta^D(x - y) . \quad (5.3)$$

$x, y$  are some vectors  $\in R^D$ , and  $H_x$  is assumed to be a positive definite operator depending on  $x$  and containing derivatives with respect to  $x$ . Now, consider a "time evolution" defined by

$$\psi(x, t) \equiv e^{-H_x t} \psi(x) . \quad (5.4)$$

Note that  $t$  does not represent a time with physical meaning. It is just a mathematical parameter. The physical time, as far as one can say time in the Euclidean space, is already contained in the  $D$ -dimensional vector  $x$ . Clearly,

$$\psi(x, 0) = \psi(x) , \quad (5.5)$$

and since  $H_x$  is assumed to be positive definite, we have

$$\lim_{t \rightarrow \infty} \psi(x, t) = 0 . \quad (5.6)$$

By differentiating eq. (5.4) with respect to  $t$ , one sees that  $\psi(x, t)$  satisfies the equation

$$\frac{d}{dt} \psi(x, t) = -H_x \psi(x, t) . \quad (5.7)$$

Besides the factor  $-i$  missing on the left hand-side, this corresponds to the Schrödinger equation of quantum mechanics with Hamiltonian  $H_x$ .

The effect of time evolution (5.4) can usually be described by a kernel  $Q_H(x, y, t)$  as

$$\psi(x, t) = \int d^D y Q_H(x, y, t) \psi(y) , \quad (5.8)$$

and if we are lucky,  $Q_H$  is even known. One can easily show that  $Q_H$  satisfies the relation

$$\frac{d}{dt} Q_H(x, y, t) = -H_x Q_H(x, y, t) . \quad (5.9)$$

We integrate out time in eq. (5.7) to get rid of the time dependence not apparent in the Green function we are looking for in eq. (5.3), and obtain

$$\psi(x, \infty) - \psi(x, 0) = - \int_0^\infty dt H_x \psi(x, t) = - \int d^D y \psi(y) H_x q(x, y) , \quad (5.10)$$

with

$$q(x, y) \equiv \int_0^\infty dt Q(x, y, t) . \quad (5.11)$$

Using eqs. (5.5) and (5.6), one gets

$$\psi(x) = \int d^D y \psi(y) H_x q(x, y) ,$$

which is valid for any test function  $\psi$ . Thus, we have

$$H_x q(x, y) = \delta^D(x - y) , \quad (5.12)$$

and with  $q = \Delta$  we have solved eq. (5.3).

### 5.1.2 The free Green function for the dual action

The kernel  $Q_h$  for the 1-dimensional harmonic oscillator with the time evolution operator

$$h_x = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 - \frac{1}{2} \quad (5.13)$$

with  $x \in \mathbb{R}$  is known from [62]:

$$Q_h(x, y, t) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp \left( -\frac{(x^2 + y^2)(1 + e^{-2t})/2 - 2e^{-t}xy}{1 - e^{-2t}} \right) . \quad (5.14)$$

This means that we can calculate the time evolution by

$$e^{-h_x t} \psi(x) = \int dy Q_h(x, y, t) \psi(y) . \quad (5.15)$$

The operator  $H_x$  of eq. (5.3) for the special frame sketched above can now be rewritten as

$$H_x \equiv \frac{1}{2} \left[ -\partial^2 + 4 \frac{\Omega^2}{\theta^2} x^2 + m^2 \right] = \frac{1}{c} \left( \sum_{i=1}^D h_{x_i/\lambda} + b \right) \quad (5.16)$$

with  $c = \frac{\theta}{2\Omega}$ ,  $\lambda^2 = c$ ,  $b = \frac{m^2}{2}c + \frac{D}{2}$ . Using eq. (5.16), the kernel  $Q_H$  of eq. (5.4) can be determined in terms of  $Q_h$  given by eq. (5.14):

$$Q_H(x, y, t) = \frac{e^{-\frac{tb}{c}}}{\lambda^D} \prod_{i=1}^D Q_h\left(\frac{x_i}{\lambda}, \frac{y_i}{\lambda}, \frac{t}{c}\right) . \quad (5.17)$$

From the last section, we know that we obtain the Green function for eq. (5.3) by integrating this kernel from 0 to  $\infty$ :

$$\begin{aligned} \Delta(x, y) &= \int_0^\infty dt Q_H(x, y, t) \\ &= \frac{c^{1-\frac{D}{2}}}{\pi^{\frac{D}{2}}} \int_0^\infty d\tau \frac{e^{-b\tau - \frac{(x^2+y^2)(1+e^{-2\tau})/2 - 2e^{-\tau}xy}{\lambda^2(1-e^{-2\tau})}}}{(1 - e^{-2\tau})^{\frac{D}{2}}} , \end{aligned} \quad (5.18)$$



where we have transformed  $t = \tau c$ . Note that in contrast to eq. (5.14),  $xy$  now means  $\sum_i x_i y_i$ . Next, we want to calculate the Fourier transform. This is a little bit different to the usual case since our propagator now is not translation invariant. Thus, we have to transform  $\Delta$  with respect to both arguments. To get

$$\Delta(x, y) = \int d^D p d^D q e^{ipx+iqy} \tilde{\Delta}(p, q) , \quad (5.19)$$

we define the Fourier transform of  $\Delta$  as

$$\tilde{\Delta}(p, q) \equiv (2\pi)^{-2D} \int d^D x d^D y e^{-ipx-iqy} \Delta(x, y) . \quad (5.20)$$

This task can easily be accomplished since only Gaussian integrals are involved:

$$\tilde{\Delta}(p, q) = \frac{c^{1+\frac{D}{2}}}{2^D \pi^{\frac{3}{2}D}} \int_0^\infty dt \frac{e^{-bt-c \frac{(p^2+q^2)(1+e^{-2t})}{(1-e^{-2t})} / 2 + 2e^{-t} pq}}{(1-e^{-2t})^{\frac{D}{2}}} . \quad (5.21)$$

## 5.2 Calculation of the tadpole graph

Now, one is interested in the effect of the newly added term on the UV/IR mixing behaviour. The simplest case in  $\phi^4$  theory where one can study this phenomenon is the tadpole graph already treated in section 2.5. This graph represents the term

$$\Gamma_{tp}(x, y) \equiv \frac{\lambda}{4!} \left\{ \int \mathcal{D}[\phi] \int d^4 z (\phi^*)^4(z) \phi(x) \phi(y) e^{-I_0[\phi]} \right\}^{con} . \quad (5.22)$$

Writing  $(\phi^*)^4(z)$  in a non-local way,

$$(\phi^*)^4(z) = \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{il_i s_i} \right) \phi(z - \frac{1}{2} \tilde{l}_1) \phi(z + s_1 - \frac{1}{2} \tilde{l}_2) \phi(z + s_1 + s_2 - \frac{1}{2} \tilde{l}_3) \phi(z + s_1 + s_2 + s_3) ,$$

one can refer to fig. 2.2 with  $z_1, \dots, z_4$  again labelling the several field arguments. Fig. 2.2 just shows 6 out of the twelve possible connected pairings. The six missing ones are obtained by “crossing” the external lines (exchanging  $x$  and  $y$ ). We first calculate the contribution due to fig. 5.1:

$$\begin{aligned} \Gamma_{tp,a}(x, y) &\equiv \frac{\lambda}{4!} \int d^4 z \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{il_i s_i} \right) \Delta(x, z_4) \Delta(y, z_3) \Delta(z_1, z_2) \\ &= \frac{\lambda}{4!} \int d^4 p d^4 q d^4 q_1 \dots d^4 q_4 e^{ipx+iqy} \tilde{\Delta}(p, q_4) \tilde{\Delta}(q_1, q_2) \tilde{\Delta}(q_3, q) \\ &\quad (2\pi)^4 \delta^4(q_1 + q_2 + q_3 + q_4) e^{i/2 \varphi(q_1, q_2, q_3, q_4)} , \end{aligned} \quad (5.23)$$

where

$$\begin{aligned} e^{i/2 \varphi(q_1, q_2, q_3, q_4)} &\times (2\pi)^4 \delta^4(q_1 + q_2 + q_3 + q_4) \\ &\equiv \int d^4 z \int \prod_{i=1}^3 \left( d^4 s_i \frac{d^4 l_i}{(2\pi)^4} e^{il_i s_i} \right) e^{i(q_1 z_1 + q_2 z_2 + q_3 z_3 + q_4 z_4)} \\ &= (2\pi)^4 \delta^4(q_1 + q_2 + q_3 + q_4) \exp \left( \frac{i}{2} \sum_{i < j} q_i \tilde{q}_j \right) . \end{aligned} \quad (5.24)$$

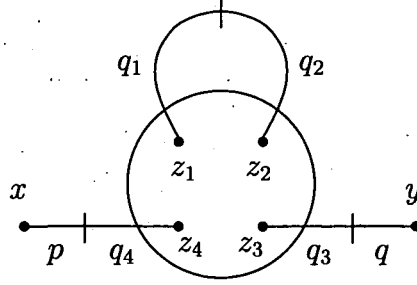


Figure 5.1: A special tadpole contribution: Again, we have used the abbreviations  $z_1 = z - \frac{1}{2}\tilde{l}_1$ ,  $z_2 = z - \frac{1}{2}\tilde{l}_2 + s_1$ ,  $z_3 = z - \frac{1}{2}\tilde{l}_3 + s_1 + s_2$ ,  $z_4 = z + s_1 + s_2 + s_3$ . The dashed of the propagators simply indicates their dependence on two 4-momenta instead of only one.

figure	uncrossed	crossed
a	$q_2\tilde{q}_3 + q_2\tilde{q}_4 + q_3\tilde{q}_4$	$q_2\tilde{q}_3 + q_2\tilde{q}_4 - q_3\tilde{q}_4$
b	$q_2\tilde{q}_3 + q_2\tilde{q}_4 - q_3\tilde{q}_4$	$q_2\tilde{q}_3 + q_2\tilde{q}_4 + q_3\tilde{q}_4$
c	$-q_2\tilde{q}_3 + q_2\tilde{q}_4 + q_3\tilde{q}_4$	$q_2\tilde{q}_3 - q_2\tilde{q}_4 - q_3\tilde{q}_4$
d	$q_2\tilde{q}_3 + q_2\tilde{q}_4 - q_3\tilde{q}_4$	$q_2\tilde{q}_3 + q_2\tilde{q}_4 + q_3\tilde{q}_4$
e	$q_2\tilde{q}_3 + q_2\tilde{q}_4 - q_3\tilde{q}_4$	$q_2\tilde{q}_3 + q_2\tilde{q}_4 + q_3\tilde{q}_4$
f	$-q_2\tilde{q}_3 + q_2\tilde{q}_4 + q_3\tilde{q}_4$	$q_2\tilde{q}_3 - q_2\tilde{q}_4 - q_3\tilde{q}_4$

Table 5.1: This table shows  $\varphi(\dots)$ . The figure labels refer to fig. (2.2).  $q_1$  was eliminated due to momentum conservation.

The labelling of momenta is shown in fig. 5.1. Applying the same labelling to the other diagrams shown in fig. 2.2 and to the uncrossed ones, the product of Mehler kernels in the integrand  $\tilde{\Delta}(p, q_4)\tilde{\Delta}(q_1, q_2)\tilde{\Delta}(q_3, q)$  remains unchanged and only the part containing the phase is modified. This part can always be rewritten in terms of  $\varphi(\dots)$ , fed with arguments  $q_i$  in corresponding order. Furthermore, one can make use of momentum conservation  $q_1 + q_2 + q_3 + q_4 = 0$  when calculating  $\varphi(\dots)$  due to the  $\delta$ -function always accompanying this factor. Table 5.1 summarises the resulting phase factors, where for convenience  $q_1$  has always been eliminated. One can now follow different strategies to evaluate  $\Gamma_{tp}$ . First, one can add up all phase factors (not eliminating  $q_1$  from the beginning) and solve the resulting integral. It is then helpful to make use of the fact that the result must not depend on interchanging the momenta  $q_1$  and  $q_2$ . In addition to summing up all 12 diagrams, one can thus also symmetrise with respect to  $q_1, q_2$ , which means full symmetrisation of the phase factor. One then just has to evaluate a single integral

$$\Gamma_{tp}(x, y) = \int d^D p d^D q d^D q_1 \dots d^D q_4 e^{ipx+iqy} \tilde{\Delta}(p, q_4)\tilde{\Delta}(q_1, q_2)\tilde{\Delta}(q_3, q) \quad (5.25)$$

$$(2\pi)^D \delta^D(q_1 + q_2 + q_3 + q_4) \chi_4(q_1, q_2, q_3, q_4) ,$$

with  $\chi_4$  defined as in eq. (2.116) ( $k = 4, \kappa = \lambda$ ). The dimension 4 is now written arbitrarily  $D$ . In contrast to the framework of time-ordered perturbation theory, the arguments of  $\chi_4$  are now the off-shell 4-momenta and one can make use of momentum conservation to eliminate

$q_1$  to obtain

$$\begin{aligned} \chi_4(-q_2 - q_3 - q_4, q_2, q_3, q_4) = & \frac{\lambda}{3} \left[ \cos \left( \frac{1}{2}(q_2 \tilde{q}_3 + q_2 \tilde{q}_4 + q_3 \tilde{q}_4) \right) + \cos \left( \frac{1}{2}(q_2 \tilde{q}_3 + q_2 \tilde{q}_4 - q_3 \tilde{q}_4) \right) \right. \\ & \left. + \cos \left( \frac{1}{2}(q_2 \tilde{q}_3 - q_2 \tilde{q}_4 - q_3 \tilde{q}_4) \right) \right] . \end{aligned} \quad (5.26)$$

The remaining integrals are the Gaussian accompanied by these cosine factors, which one would solve by decomposing the cosines into exponentials obtaining usual Gaussian integrals. But then, the connection of these contributions to the several diagrams gets lost without improving the calculational efforts.

So we follow another strategy evaluating the integrals for each phase factor in table (5.1) separately. There are only 4 different phase factors:

$$\varphi_1 = q_2(\tilde{q}_3 + \tilde{q}_4) + q_3 \tilde{q}_4 \quad (5.27)$$

occurs in au, bc, dc and ec <sup>1</sup>,

$$\varphi_2 = q_2(\tilde{q}_3 + \tilde{q}_4) - q_3 \tilde{q}_4 \quad (5.28)$$

in ac, bu, du and eu,

$$\varphi_3 = q_2(\tilde{q}_3 - \tilde{q}_4) - q_3 \tilde{q}_4 \quad (5.29)$$

in cc and fc, and

$$\varphi_4 = -q_2(\tilde{q}_3 - \tilde{q}_4) + q_3 \tilde{q}_4 \quad (5.30)$$

in cu and fu. The latter phase factor can be rewritten, since one has the freedom of interchanging  $q_1$  and  $q_2$  in fig. (5.1). After integrating out  $q_1$  through momentum conservation at the vertex, this corresponds to the transformation  $q_2 \rightarrow -q_2 - q_3 - q_4$ . Applying this transformation to  $\varphi_4$ , one arrives at  $\varphi_3$ :

$$\varphi_4 = -q_2(\tilde{q}_3 - \tilde{q}_4) + q_3 \tilde{q}_4 \rightarrow q_2(\tilde{q}_3 - \tilde{q}_4) - q_3 \tilde{q}_4 = \varphi_3 . \quad (5.31)$$

Thus, we drop  $\varphi_4$  and associate cc, fc, cu and fu with  $\varphi_3$ . The three remaining phases can be recognised from  $\chi_4$  in eq. (5.26). The remaining three integrals concerning  $q_2$  are

$$e^{\pm \frac{i}{2} q_3 \tilde{q}_4} \int d^D q_2 \tilde{\Delta}(-q_2 - q_3 - q_4, q_2) e^{\frac{i}{2} q_2 s} . \quad (5.32)$$

with

$$\begin{aligned} e^{\frac{i}{2} q_3 \tilde{q}_4}, s = \tilde{q}_3 + \tilde{q}_4 & \quad \text{for} \quad \varphi_1 , \\ e^{-\frac{i}{2} q_3 \tilde{q}_4}, s = \tilde{q}_3 + \tilde{q}_4 & \quad \text{for} \quad \varphi_2 , \\ e^{-\frac{i}{2} q_3 \tilde{q}_4}, s = \tilde{q}_3 - \tilde{q}_4 & \quad \text{for} \quad \varphi_3 . \end{aligned}$$

$q_2$  occurs quadratic in the exponent of the  $\tilde{\Delta}$  and it remains to carry out a Gaussian integral which gives

$$\frac{c}{(2\pi)^D} e^{\pm \frac{i}{2} q_3 \tilde{q}_4} e^{-\frac{i}{4} (q_3 + q_4) s} \int_0^\infty dt \frac{e^{-bt}}{(1 - e^{-t})^D} \exp \left( -\frac{\beta_t}{4} (c (q_3 + q_4)^2 + \frac{1}{4c} s^2) \right) , \quad (5.33)$$

<sup>1</sup>The second letter of these labels is either c or u referring to crossed and uncrossed, respectively.

where

$$\beta_t \equiv \frac{1 + e^{-t}}{1 - e^{-t}} = \coth\left(\frac{t}{2}\right).$$

Summing up all 12 integrals (each of the three phase combinations occurs four times), one gets

$$\begin{aligned} \Gamma_{tp}(x, y) = & \frac{c\lambda}{4!} \int dp^D dq^D dq_3^D dq_4^D e^{ipx+iqy} \tilde{\Delta}(p, q_4) \tilde{\Delta}(q_3, q) \\ & \int_0^\infty dt \frac{e^{-bt}}{(1 - e^{-t})^D} e^{-\frac{\beta_t}{4} c (q_3 + q_4)^2} \left[ 2 \cos\left(\frac{q_3 \tilde{q}_4}{2}\right) e^{-\frac{\beta_t}{16c} (\tilde{q}_3 + \tilde{q}_4)^2} + e^{-\frac{\beta_t}{16c} (\tilde{q}_3 - \tilde{q}_4)^2} \right]. \end{aligned} \quad (5.34)$$

One might expect that the cosine term corresponds to the non-planar contributions. But it originates from  $\phi_1, \phi_2$ , corresponding to the diagrams a, b, d, e, crossed and uncrossed respectively. The remaining term in square brackets is associated with diagrams c and f (crossed and uncrossed). And as was shown in section 2.5, the latter ones give the planar contributions! Thus, apparently counterintuitive, the first term in square brackets corresponds to the planar contribution, and the second one to the non-planar one. The fact that the cosine term has an additional factor 2 is also a typical hint that it is the planar part. And considering momentum conservation  $q_3 = -q_4$  as would be the case in a translation invariant theory, the first term simply becomes 2, again confirming our classification.

The second line stands for the amputated tadpole. Its integrand will be discussed now. The only problems are expected for  $t \rightarrow 0$ . Obviously,

$$\frac{e^{-bt}}{(1 - e^{-t})^D}$$

diverges like  $t^{-D}$ . These problems are possibly compensated by the exponential factors. To see this in detail, we consider the planar and the non-planar parts separately:

- The interesting term of the planar contribution is

$$e^{-\frac{\beta_t}{4} (q_3 + q_4)^2 \left(c + \frac{\theta^2}{4c}\right)}.$$

For small  $t$ , one can write  $\beta_t \approx 2/t$ . The integral for small  $t$  thus corresponds to

$$\int_0^\epsilon dt \frac{e^{-\frac{2f}{t}}}{t^D},$$

where  $f = (q_3 + q_4)^2 (c + \frac{\theta^2}{4c})/4$ . As long as  $f \neq 0$ , the limit for  $t \rightarrow 0$  of the integrand exists and is 0. However, for a configuration obeying momentum conservation,  $q_3 + q_4 = 0$ ,  $f$  vanishes and the integral over small  $t$  diverges.

- Similarly, the behaviour of the non-planar part is described by

$$\int_0^\epsilon dt \frac{e^{-\frac{2g}{t}}}{t^D},$$

where  $g = (c(q_3 + q_4)^2 + \theta^2/(4c)(q_3 - q_4)^2)/4$ . The limit is again well behaving except for  $g = 0$ . This will only be the case for  $q_3 = q_4 = 0$ . This reminds of the typical UV/IR-mixing problem, where one also encounters a divergence for vanishing external 4-momentum. The difference to the case treated here is basically the violation of the translation invariance so that two “external” 4-momenta have to vanish here.

Thus, the amputated tadpole is convergent for almost all configurations of the momenta  $q_3, q_4$  directly flowing into the vertex, but the planar part diverges for momentum conservation  $q_3 = q_4$ , and the non-planar one for vanishing momenta  $q_3 = q_4 = 0$ . This basically shows the same UV/IR mixing behaviour as the usual NCQFT without the harmonic oscillator term. A qualitative explanation for this similarity can also be given: A harmonic oscillator potential is very flat for small distances and therefore, it is not expected to influence the UV behaviour directly. The large distance behaviour is strongly modified, and the IR regime is commonly effected. However, an indirect influence onto the integrals for large momenta is possible due to the UV/IR mixing of NCQFT. The harmonic oscillator therefore does not directly avoid the UV/IR divergences, but it possibly circumvents them. This could be seen when carrying out the integrals over  $q_3$  and  $q_4$  in eq. (5.34). If the configuration  $q_3 = q_4 = 0$  is damped out sufficiently in the integrals over  $q_3$  and  $q_4$  in eq. (5.34), the UV/IR divergence will possibly be absent. Therefore, we will carry out the integral concerning the non-planar part. The evaluation is very lengthy. So we will focus our attention onto the non-planar part. Furthermore, the UV/IR mixing problem is expected for  $p = q = 0$ . The integral we are interested in is then

$$\begin{aligned}
 I^{np} &\equiv \int d^D q_3 d^D q_4 \tilde{\Delta}(0, q_4) \tilde{\Delta}(q_3, 0) \int_0^\infty dt \frac{e^{-bt}}{(1-e^{-t})^D} e^{-\frac{\beta_t}{4} c (q_3+q_4)^2} e^{-\frac{\beta_t \theta^2}{16c} (q_3-q_4)^2} \\
 &= c' \int_0^\infty du dv dt \frac{e^{-b(u+v+t)}}{(1-e^{-2u})^{D/2} (1-e^{-2v})^{D/2} (1-e^{-t})^D} \times \\
 &\quad \int d^D q_4 e^{-\alpha_{u,t} q_4^2} \int d^D q_3 e^{-\alpha_{v,t} q_3^2 - \gamma_t q_3 q_4}
 \end{aligned} \tag{5.35}$$

with

$$\begin{aligned}
 c' &\equiv \left[ \frac{c^{1+D/2}}{2^D \pi^{3D/2}} \right], \\
 \alpha_{u,t} &\equiv \beta_{2u} \frac{c}{2} + \frac{\beta_t}{4} \left( c + \frac{\theta^2}{4c} \right), \\
 \gamma_t &\equiv \frac{1}{2} \beta_t \left( c - \frac{\theta^2}{4c} \right).
 \end{aligned}$$

The last line of eq. (5.35) gives

$$\left( \frac{\pi^2}{\alpha_{v,t} \alpha_{u,t} - \gamma_t^2/4} \right)^{D/2}. \tag{5.36}$$

For small  $t$ , the denominator in the fraction goes like  $\theta/16\beta_t^2 \approx \theta^4/(4t^2)$ . Collecting the  $t$ -dependent terms of eq. (5.35) together, one then obtains for  $t \rightarrow 0$

$$\left[ \frac{1}{(\alpha_{v,t} \alpha_{u,t} - \gamma_t^2/4) (1-e^{-t})^2} \right]^{D/2} \rightarrow \left( \frac{4}{\theta^2} \right)^{D/2}. \tag{5.37}$$

This means that the  $t$ -integration converges. To see this, we had to integrate out the momenta  $q_3$  and  $q_4$ , which are already associated with the external lines, and the tadpole loop momenta  $q_1$  and  $q_2$  treated at the beginning.

Similar considerations can be carried out for the planar parts. Due to their length these calculations are skipped. They yield that the planar contributions diverge. This is not surprising. The harmonic oscillator does not regularise the UV integrations themselves. This is only done by the parameter  $\theta$  of non-commutativity explaining that the planar parts diverge. The regularisation due to non-commutativity however does not work for vanishing external momenta. But this is, where the harmonic oscillator comes into play. It apparently damps the problematic IR integration so that the UV/IR mixing does not end up with a divergence anymore. In this sense, these results are in agreement with the statements in [36, 37, 38, 39] that NCQFT can be renormalised by introducing the harmonic oscillator term.

# Conclusion

A very general type of non-local interactions for real, scalar particles was introduced at the beginning of this thesis. Feynman rules were derived within the framework of time-ordered perturbation theory. These rules hold also for the case where non-locality in the time direction is involved. It is surprising that it is possible to derive them for such a general type of non-locality. For translation invariant interactions, they are even relatively simple, and the limit of a local theory is easily recovered. The basic concept behind these rules is the contractor, which replaces the propagator as soon as the time component of the non-locality is present. The vertex factor, which is already present for the simpler Feynman rules derived within a naive version of the path integral approach, now only depends on the three space momenta, and not their time components anymore. This is a central property of time-ordered perturbation theory, which is important to unitarity as was shown in chapter 4. It even has an influence on the UV/IR mixing problem, and according to [46], the divergence for low momenta is also absent as long as the parameter of non-commutativity  $\theta$  has full rank. The UV/IR mixing problem was also a subject of the last chapter, but from a very different point of view. A harmonic oscillator term (violating translation invariance) was added to the action. The investigations were not carried out within time-ordered perturbation theory, but a naive path integral approach was chosen. It was shown for the special example of a tadpole loop that the divergence at vanishing external momenta is absent for the non-planar part. This is a hint that the UV/IR mixing problem is cured by the harmonic oscillator term. However, one has to accept the violation of translation invariance. Furthermore, the model was introduced in the Euclidean space, and it is not clear how to proceed in Minkowski space. The harmonic oscillator term either breaks Lorentz invariance or it is not positive definite anymore.

In chapter 3, a simplified version of non-local interactions of Gaussian type is motivated by replacing the field operators with smeared ones. As a result, one obtains Gaussian functions as vertex factors. It should be kept in mind that these only depend on the three space components of the momenta flowing in or out of the vertex. A simple power counting criterion could be established, and explicit calculations have confirmed it.

The diagrammatic techniques were elaborated in more detail in chapter 4. A remarkable result lies in the extension of the self-energy, which turned out to be a matrix even for scalar particles. Beside translation invariance, it could be shown that parity and time reversal can also be established as symmetries of non-local quantum field theories. However, a Lorentz invariant, non-local quantum field theory was not found within the framework of TOPT. We did not try the most general ansatz for the construction of a Lorentz invariant theory, but our investigations showed that it is at least difficult, if not impossible at all to unify this symmetry with a non-local interaction within the approach of time-ordered perturbation theory. The naive version of the path integral approach might do a better job in this respect, but there are evidently problems with unitarity as soon as non-locality involves time. The difference

between these two approaches was investigated perturbatively. Surprisingly, it turned out that the deviations are of second order in the time component of the non-locality.

Finally, it should be pointed out that it is hard to construct a working, non-local theory. The benefits like UV-finiteness and a possible link to gravitation are obvious, but the unification with fundamental, physical principles seems to be quite difficult. At least, non-local theories might be meaningful in connection with effective quantum field theories hiding unknown types of particles or newer concepts whatsoever.



## Appendix A

# Generalised Wick theorem

In this appendix, the time-ordered product of eq. (2.50)

$$\langle 0 | T \{ \phi(x_1) \dots \phi(x_n) V(t_{n+1}) \dots V(t_N) \} | 0 \rangle \quad (\text{A.1})$$

will be further processed for non-local interactions as given in eq. (2.45). To deal with the expression given above, it is quite useful to define the times  $t_1, \dots, t_n$  as  $x_1^0, \dots, x_n^0$  and

$$\theta_P(\underline{t}) \equiv \theta(t_{P_1} - t_{P_2}) \theta(t_{P_2} - t_{P_3}) \dots \theta(t_{P_{N-1}} - t_{P_N}) \quad (\text{A.2})$$

where  $\underline{t} \equiv (t_1, \dots, t_N)$  and  $P \in S^N$ , the group of permutations of  $(1, \dots, N)$ .  $P_i$  denotes the integer  $P(i)$ , the integer  $i$  is mapped to by the permutation  $P$  (see also [41] for more details on permutations). Inserting eq. (2.45), expression (A.1) can then be rewritten as

$$\int d\lambda_1 \dots d\lambda_m v(\lambda_1) \dots \sum_{P \in S^N} \theta_P(\underline{t}) \langle 0 | \phi(z_{\xi^{P_1}}) \dots \phi(z_{\xi^{P_M}}) | 0 \rangle, \quad (\text{A.3})$$

where  $M$  is the number of fields occurring in the vacuum expectation value and  $z_1, \dots, z_M$  are defined as the arguments put into the fields from left to right for the time-ordering  $\theta_P$  corresponding to the identity permutation  $P = I$  (which corresponds to the time-ordering  $t_1 > t_2 > \dots > t_N$ ).  $z_{\xi^{P_i}}$  denotes the four vector which is put into the  $i$ th field according to the time ordering  $\theta_P$ . Thus,  $\xi^P$  is implicitly defined as a permutation  $\in S^M$  for each  $P \in S^N$ , so that  $z_{\xi^{P_i}}$  is the argument put into the  $i$ th field (counting from left to right).

The vacuum expectation value in the above equation,

$$\langle 0 | \phi(z_{\xi^{P_1}}) \dots \phi(z_{\xi^{P_M}}) | 0 \rangle, \quad (\text{A.4})$$

can be rewritten in terms of commutators by applying the following algorithm:

- Substitute every field operator  $\phi(y)$  by the sum of its annihilation and creation part  $\phi^+(y) + \phi^-(y)$ .
- Expand the product into sums of products of  $\phi^+$  and  $\phi^-$ .
- Expand each product by replacing adjacent pairs of operators  $\phi^+(z_{\xi^{P_i}}) \phi^-(z_{\xi^{P_j}})$  by  $\phi^-(z_{\xi^{P_j}}) \phi^+(z_{\xi^{P_i}}) + D_P(\xi^P_i, \xi^P_j)$ , where<sup>1</sup>

$$D_P(\xi^P_i, \xi^P_j) \equiv \theta(j - i) \left[ \phi^+(z_{\xi^{P_i}}), \phi^-(z_{\xi^{P_j}}) \right]. \quad (\text{A.5})$$

<sup>1</sup>The omission of  $\theta(j - i)$  would not make any difference at this point since for all adjacent operators occurring during application of the given algorithm,  $i < j$ . Below, the purpose of this definition will become clear.

This step has to be repeated till no  $\phi^-$  can be found right of any  $\phi^+$ .

- Note that  $\dots \phi^+(x) |0\rangle = \langle 0| \phi^-(x) \dots = 0$ .

Now, the question is, which kind of terms remains after this procedure. First of all, it is clear that for  $M$  odd, the vacuum expectation value vanishes since for each summand only a single field operator would remain. So  $M$  even is the more interesting case. Some interesting properties for this case are:

- Each summand is a product of  $M/2$  functions  $D_P(\xi^P_i, \xi^P_j)$  with  $i < j$  which each integer  $\in \{1, \dots, M\}$  occurs exactly once in.
- Further on, all summands are different from each other, when considering  $D_P(i, j)$  and  $D_P(k, l)$  only identical for  $i = k$  and  $j = l$ .
- Each product of  $M/2$  functions  $D_P(\xi^P_i, \xi^P_j)$ , which each integer  $\in \{1, \dots, M\}$  occurs exactly once in, also represents a summand. The property  $i < j$  of the first item would also imply a restriction on the integers  $i, j$  occurring in the arguments of  $D_P$ . But due to the apparently useless  $\theta$ -function in eq. (A.5), terms not obeying this restriction are 0 which makes the restriction obsolete.

On the one hand, the first property says that one can write each summand  $S_i$  as

$$S_i = D_P(\xi^P_{Q_1}, \xi^P_{Q_2}) D_P(\xi^P_{Q_3}, \xi^P_{Q_4}) \dots D_P(\xi^P_{Q_{M-1}}, \xi^P_{Q_M}), \quad (\text{A.6})$$

where  $Q \in S^M$ . But since  $\xi^P_{Q_i} = \xi^P(Q(i))$  and  $\xi^P \in S^M$ , one can simply write  $Q'_i = Q'(i) = \xi^P(Q(i))$  instead of  $\xi^P_{Q_i}$ . On the other hand, according to the third property, it can be said that each term  $D_P(Q_1, Q_2) \dots D_P(Q_{M-1}, Q_M)$  with arbitrary  $Q$  occurs in the sum. The vacuum expectation value of eq. (A.4) can be written as

$$\langle 0| \phi(z_{\xi^P_1}) \dots \phi(z_{\xi^P_M}) |0\rangle_0 = \frac{1}{(M/2)!} \sum_{Q \in S^M} D_P(Q_1, Q_2) \dots D_P(Q_{M-1}, Q_M). \quad (\text{A.7})$$

One has to divide by  $(M/2)!$  since each product of  $M/2$  functions with 2 arguments is generated by  $(M/2)!$  permutations. Note that definition (A.5) is equivalent to

$$D_P(i, j) = \theta((\xi^P)^{-1}_j - (\xi^P)^{-1}_i) [\phi^+(z_i), \phi^-(z_j)]. \quad (\text{A.8})$$

To be complete, eq. (A.7) is now proven. First, we show the relation

$$\langle 0| \phi(y_1) \dots \phi(y_M) |0\rangle = \frac{1}{(M/2)!} \sum_{Q \in S^M} f_{\underline{y}}(Q_1, Q_2) \dots f_{\underline{y}}(Q_{M-1}, Q_M) \quad (\text{A.9})$$

with

$$f_{\underline{y}}(i, j) \equiv \theta(j - i) [\phi^+(y_i), \phi^-(y_j)] \quad (\text{A.10})$$

by induction. The dependence of  $f_{\underline{y}}$  on the  $M$  four vectors  $y_i$  has been indicated here explicitly in contrast to the usual notation in this appendix, since this will be useful below. Clearly, it only holds for  $M$  even, and only makes sense for  $M \leq 2$ . Thus, the induction start corresponds to  $M = 2$ . Both sides of eq. (A.9) give  $[\phi^+(y_1), \phi^-(y_2)]$ , verifying it for  $M = 2$ . The induction

step is trickier. We show that if it holds for  $M$ , it will also hold for  $M + 2$ . Therefore, we start with

$$\langle 0 | \phi(y_1) \dots \phi(y_{M+2}) | 0 \rangle . \quad (\text{A.11})$$

The number of field operators in this expression has to be reduced about 2. This can be done by splitting up any of the field operators, say  $\phi(y_i)$ , into its creation and annihilation part, which are then moved to the left or right, respectively. Here, it is important to play through this game not only with one field operator, but with all in a symmetric way:

$$\begin{aligned} \langle 0 | \phi(y_1) \dots \phi(y_{M+2}) | 0 \rangle = \\ \frac{1}{M+2} \sum_{i=1}^{M+2} [\langle 0 | \phi(y_1) \dots \phi^+(y_i) \dots \phi(y_{M+2}) | 0 \rangle + \langle 0 | \phi(y_1) \dots \phi^-(y_i) \dots \phi(y_{M+2}) | 0 \rangle] \end{aligned} \quad (\text{A.12})$$

Any of the summands is identical to the left hand-side. The symmetric treatment is important in this proof. Next, it is helpful to define the mapping  $\iota^{ij}$  for given  $i, j$ :

$$\begin{aligned} \iota^{ij} : \quad \{1, \dots, M\} &\rightarrow \{1, \dots, M\}, \\ n \rightarrow \iota^{ij}(n) \equiv \iota_n^{ij} &\equiv \begin{cases} n & ; 1 \leq n < \min(i, j) \\ n+1 & ; \min(i, j) \leq n < \max(i, j) \\ n+2 & ; \max(i, j) \leq n \end{cases} . \end{aligned} \quad (\text{A.13})$$

In eq. (A.12),  $\phi^+(y_i)$  and  $\phi^-(y_i)$  are then moved to the right and left, respectively, by using

$$[\phi^+(y_i), \phi(y_j)] = [\phi^+(y_i), \phi^-(y_j)] \equiv c_{ij} \quad (\text{A.14})$$

$$[\phi(y_j), \phi^-(y_i)] = c_{ji} . \quad (\text{A.15})$$

Eq. (A.12) thus becomes

$$\begin{aligned} &\frac{1}{M+2} \left[ \sum_{\substack{i,j=1 \\ j>i}}^{M+2} c_{ij} \langle 0 | \phi(y_{\iota^{ij}_1}) \dots \phi(y_{\iota^{ij}_M}) | 0 \rangle + \sum_{\substack{i,j=1 \\ j<i}}^{M+2} c_{ji} \langle 0 | \phi(y_{\iota^{ij}_1}) \dots \phi(y_{\iota^{ij}_M}) | 0 \rangle \right] \\ &= \frac{1}{M+2} \sum_{\substack{i,j=1 \\ j \neq i}}^{M+2} [f_{\underline{j}}(i, j) \langle 0 | \phi(y_{\iota^{ij}_1}) \dots \phi(y_{\iota^{ij}_M}) | 0 \rangle + f_{\underline{j}}(j, i) \langle 0 | \phi(y_{\iota^{ij}_1}) \dots \phi(y_{\iota^{ij}_M}) | 0 \rangle] \\ &= \frac{2}{M+2} \sum_{\substack{i,j=1 \\ j \neq i}}^{M+2} f_{\underline{j}}(i, j) \langle 0 | \phi(y_{\iota^{ij}_1}) \dots \phi(y_{\iota^{ij}_M}) | 0 \rangle \end{aligned}$$

First, the  $\theta$ -functions included in  $f_{\underline{j}}(i, j)$  were introduced instead of the restrictions  $j < i$  and  $j > i$ . In the last step, the symmetry  $\iota^{ji}(n) = \iota^{ij}(n)$  was used. Now, we can use eq. (A.9) to prove the induction step. Therefore, we define

$$y'_n \equiv y_{\iota^{ij}_n} \quad (\text{A.16})$$

with  $n = 1, \dots, M$ . Thus, one gets

$$\begin{aligned} & \frac{1}{\frac{M+2}{2}!} \sum_{\substack{i,j=1 \\ j \neq i}}^{M+2} \sum_{Q \in S^M} f_{\underline{y}'}(Q_1, Q_2) \dots f_{\underline{y}'}(Q_{M-1}, Q_M) f_{\underline{y}}(i, j) \\ &= \frac{1}{\frac{M+2}{2}!} \sum_{\substack{i,j=1 \\ j \neq i}}^{M+2} \sum_{Q \in S^M} f_{\underline{y}}(\iota^{ij}(Q_1), \iota^{ij}(Q_2)) \dots f_{\underline{y}}(\iota^{ij}(Q_{M-1}), \iota^{ij}(Q_M)) f_{\underline{y}}(i, j) . \end{aligned}$$

In the last step,  $f_{\underline{y}'}(m, n) = f_{\underline{y}}(\iota^{ij}(m), \iota^{ij}(n))$  was used which can be easily shown using  $\theta(n - m) = \theta(\iota^{ij}(n) - \iota^{ij}(m))$  ( $\iota^{ij}(n)$  is strictly monotonic increasing in  $n$ ). Finally, the induction step is completed by defining the permutation  $P \in S^{M+2}$

$$P(n) \equiv \begin{cases} \iota^{ij}(n) & ; 1 \leq n \leq M \\ i & ; n = M+1 \\ j & ; n = M+2 \end{cases} .$$

Thus,  $\sum_{\substack{i,j=1 \\ j \neq i}}^{M+2} \sum_{Q \in S^M} \dots$  is equivalent to  $\sum_{P \in S^{M+2}} \dots$ , and the induction step

$$\langle 0 | \phi(y_1) \dots \phi(y_{M+2}) | 0 \rangle = \frac{1}{\frac{M+2}{2}!} \sum_{P \in S^{M+2}} f_{\underline{y}}(P_1, P_2) \dots f_{\underline{y}}(P_{M+1}, P_{M+2}) \quad (\text{A.17})$$

is done and eq. (A.9) is proven. Eq. (A.7) then directly follows from eq. (A.9). One only has to set  $y_i = z_{\xi^P i}$  and shift the sum over permutations in eq. (A.9) like  $Q(i) \rightarrow (\xi^P)^{-1}(Q(i))$ , and substitute

$$f_{\underline{y}}((\xi^P)^{-1}_{Q_i}, (\xi^P)^{-1}_{Q_j}) = D_P(Q_i, Q_j) .$$

Combining eq. (A.7) with the sum over permutations  $P$  due to time-ordering, one gets

$$\begin{aligned} & \sum_{P \in S^N} \theta_P(\underline{t}) \sum_{Q \in S^M} D_P(Q_1, Q_2) \dots D_P(Q_{M-1}, Q_M) = \\ & \sum_{Q \in S^M} C(Q_1, Q_2) \dots C(Q_{M-1}, Q_M) \times \\ & \sum_{P \in S^N} \theta_P(\underline{t}) \theta \left( (\xi^P)^{-1}_{Q_2} - (\xi^P)^{-1}_{Q_1} \right) \dots \theta \left( (\xi^P)^{-1}_{Q_M} - (\xi^P)^{-1}_{Q_{M-1}} \right) , \end{aligned} \quad (\text{A.18})$$

where

$$C(i, j) \equiv \Delta^+(z_i, z_j) . \quad (\text{A.19})$$

The (only) nice thing about expression (A.18) is that the sum over  $P$  only depends on  $\theta$ -functions. For given times  $t_1, \dots, t_N$ ,  $\theta_P$  now is only survived by one permutation. The other  $\theta$ -functions oppose  $M/2$  conditions

$$(\xi^P)^{-1}_{Q_{2i-1}} < (\xi^P)^{-1}_{Q_{2i}} ,$$

or in other words: In expression (A.4), the field evaluated at  $z_{Q_{2i-1}}$  had to stand left of the field evaluated at  $z_{Q_{2i}}$ . This condition can easily be rewritten in terms of the time stamps

associated with the corresponding field arguments, provided the two stamps are not the same. Thus, it is comfortable to define  $\tau_i$  as the mapping

$$\tau : \{1, \dots, M\} \rightarrow \{t_1, \dots, t_N\} ,$$

where  $\tau_i$  is defined to be the time stamp associated with the field argument  $z_i$ . For  $\tau_{Q_{2i-1}} \neq \tau_{Q_{2i}}$ <sup>2</sup>

$$\theta \left( (\xi^P)^{-1}_{Q_{2i}} - (\xi^P)^{-1}_{Q_{2i-1}} \right) = \theta(\tau_{Q_{2i-1}} - \tau_{Q_{2i}}) .$$

$\tau_{Q_{2i-1}} = \tau_{Q_{2i}}$  means that the two time stamps are the same and come from one interaction, say  $D_P(Q_{2i-1}, Q_{2i})$  represents a tadpole loop. In this case, the  $\theta$ -function simply assures that  $\Delta^+(z_{Q_{2i-1}}, z_{Q_{2i}})$  is only taken into account when  $\phi(z_{Q_{2i-1}})$  really stands left of  $\phi(z_{Q_{2i}})$  in the interaction. It is useful to distinguish between sets of summands  $S_{nt}^M$  (no tadpole) where the above simplification is applicable and where it is not,  $S_t^M$  (tadpole):

$$\begin{aligned} S_{nt}^M &\equiv \{Q \in S^M \mid \forall i \in \{1, \dots, M/2\} : \tau_{Q_{2i-1}} \neq \tau_{Q_{2i}}\} , \\ S_t^M &\equiv S^M \setminus S_{nt}^M . \end{aligned} \quad (\text{A.20})$$

Summarising the book keeping carried out above, one gets<sup>3</sup>

$$\begin{aligned} \langle 0 | T \{ \phi(x_1) \dots \phi(x_n) V(t_{n+1}) \dots V(t_N) \} | 0 \rangle_0 &= \int d\lambda_1 \dots d\lambda_m v(\lambda_1) \dots \\ &\left[ \frac{1}{(M/2)! 2^{M/2}} \sum_{Q \in S_{nt}^M} (-i\Delta(Q_1, Q_2)) \dots (-i\Delta(Q_{M-1}, Q_M)) \right. \\ &\left. + \frac{1}{(M/2)!} \sum_{P \in S^N} \theta_P(\underline{t}) \sum_{Q \in S_t^M} D_P(Q_1, Q_2) \dots D_P(Q_{M-1}, Q_M) \right] \end{aligned} \quad (\text{A.21})$$

where  $\Delta(i, j)$  is the book keeping version of the contractor for integer arguments

$$-i\Delta(i, j) \equiv D_P(i, j) + D_P(j, i) = -i\Delta(z_i, \tau_i; z_j, \tau_j) .$$

The second sum still looks complicated, but one can use the contractor  $\Delta$  for all lines but the tadpole lines. For the latter one, one simply has to insert  $\Delta^+$  with arguments corresponding to the order in the interaction. Eq. (A.21) now represents the generalised Wick theorem. It is based on eq. (A.7), which is in principle applicable for an arbitrary ordering of fields specified by the left hand side. Combining this relation with the time-ordering supposed by the Gell-Mann-Low formula (2.43), one arrives at eq. (A.21), which we refer to the *generalised Wick theorem*. It is valid for non-local interactions, in particular, non-locality may also involve time. This means that  $\phi(z_i)$  is not necessarily ordered with respect to  $z_i^0$  but  $\tau_i$ . The ordinary Wick theorem is represented by the special case  $\tau_i = z_i^0$ . The contractor then becomes the usual propagator. For this case, our result agrees with the explicit Wick theorem (see [44]). As in this special case, the evaluation of eq. (A.21) for the general case can be simplified a lot by introducing diagrammatic rules. But in principle, it should be clear how this works and we confine ourselves to presenting the rules in section 2.3.

<sup>2</sup>This is meant to hold only for  $\forall i, j \in \{1, \dots, N\} : t_i \neq t_j$ . The other cases should be irrelevant: Those, which involve time stamps associated with interactions have 0 Lebesgue measure, and those, where the time stamps of two external lines are the same, can be omitted anyway.

<sup>3</sup>In the first sum, the factor  $1/(2^{M/2})$  arises from using the symmetric  $\Delta$  which corresponds to blowing up the sum over  $Q$  by replacing  $D_P(i, j)$  with  $D_P(i, j) + D_P(j, i)$ .

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# CURRICULUM VITAE

## STEFAN DENK

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### PERSONAL DATA

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nationality	Austrian
date of birth	25 <sup>th</sup> May, 1977
birthplace	Hallein
children	one son (Lorenz)

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### EDUCATION

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1987 - 1995	Realgymnasium in Ried/ Upper Austria
June 1995	maturity exam
October 1995	beginning studies of physics at the Johannes Kepler Universität Linz
November 1999 - March 2000	studying at the Rijksuniversiteit Groningen (Netherlands)
September 2001	second Diploma exam, passed with distinction; diploma thesis in quantum many body theories
April 2002	Wilhelm-Macke-Award for Diploma thesis
since October 2002	working on PhD thesis on non-commutative quantum field theory supervised by Prof. Schweda (TU Wien)

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### PROFESSIONAL ACTIVITIES

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summer 1996	VAI (Linz)
summer 1998	Siemens Matsushita (Munich)
summer 1999	Infineon Technologies (Munich)
2000 - 2001	tutor for the course series Theoretical physics
October 2001 - September 2002	civil service at Pro Mente in Linz
October 2002 - January 2003	fellow of the FWF-project nr. P15463
since February 2003	fellow of DOC (predoc program of the Österreichische Akademie der Wissenschaften)

Stefan Denk  
Leystr. 104/5/28  
A-1200 Wien  
*email:* denk@hep.itp.tuwien.ac.at  
*tel:* +43-1/3343131