Spectral stability of small amplitude profiles of the Jin-Xin model

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Introduction

We consider $n \times n$ systems of hyperbolic conservation laws in one space variable $x \in \mathbb{R}$

$$u_t + f(u)_x = 0 \tag{1}$$

where the unknown function $u(x,t)$ takes its values in an open convex set $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is a given smooth vector field. The Cauchy problem is to find a solution $u : \mathbb{R} \times [0,T) \to U$, $T > 0$, which solves the system (1) and satisfies

$$u(x,0) = u_0(x) \tag{2}$$

for given initial data $u_0 \in \mathcal{C}^1(\mathbb{R})$. The problem is well posed if the system is (strictly) hyperbolic.
The calculus along characteristics shows that for some nonlinear functions f a classical solution exists only for a finite time. Therefore, one has to allow weak solutions, which are bounded measureable functions $u(x,t)$ satisfying (1), (2) in the distributional sense. However, weak solutions turn out to be not unique. An important class of solutions are *shock waves* $(u^-, u^+; s)$, which are defined as

$$u(x,t) := \begin{cases} u^-, & x < st, \\ u^+, & x > st, \end{cases} \tag{3}$$

where the constant vectors $u^-, u^+ \in \mathbb{R}^n$ together with the *wave speed* $s \in \mathbb{R}$ satisfy the *Rankine-Hugoniot* condition

$$s(u^+ - u^-) - \big(f(u^+) - f(u^-)\big) = 0. \tag{4}$$

Such a shock wave is a weak solution of system (1), which is piecewise constant with a single jump discontinuity moving with wave speed s. The quantity $|u^+ - u^-|$ is referred to as shock strength or amplitude.
In order to single out one of the solutions Gelfand proposed the *vanishing viscosity method*, where admissible solutions of system (1) correspond to limiting solutions of viscous conservation laws

$$u_t + f(u)_x = \epsilon\big(B(u)u_x\big)_x \tag{5}$$

as $\epsilon \to 0$. In the context of profiles this method is particularly simple. A *travelling wave* or *profile* is a smooth solution of system (5) depending on a single variable $\xi := \frac{x-st}{\epsilon}$, where $s \in \mathbb{R}$ is the speed of the wave. Majda and Pego [MP] proposed a stability criterion for the matrix $B(u)$ which guarentees for shock waves the existence of a travelling wave. The identity matrix I_n is admissible in the sense of [MP] and the viscous profiles are governed by the system of ordinary differential equations

$$-su_\xi + f(u)_\xi = u_{\xi\xi}. \tag{6}$$

The travelling waves are asymptotically constant and we get the boundary conditions $\lim_{\xi\to\pm\infty} u(\xi) = u^\pm$. We integrate the equation (6) once and obtain the viscous profile equation

$$u_\xi = f(u) - su - c, \tag{7}$$

where the boundary conditions determine the constant vector $c \in \mathbb{R}^n$ as $c = f(u^-) - su^- = f(u^+) - su^+$. The constant vectors $u^-, u^+ \in \mathbb{R}^n$ are the restpoints of the system (7) and a profile corresponds to a heteroclinic orbit connecting u^- with u^+.
The problem can also be considered as an evolutionary system and stability of solutions is an important admissibility criterion. A travelling wave is nonlinear stable if small perturbations vanish in time. Zumbrun [Z] was able to show that spectral stability of the linearized operator implies nonlinear stability. Hence the general stability problem for profiles is reduced to determine the spectrum of the linearized operator.
Freistühler and Szmolyan [FS] proved the spectral stability of small amplitude profiles in viscous conservation laws. In this paper they introduced the concept of Evans bundles and used geometric singular perturbation methods.

Jin and Xin introduced a relaxation model [JX] which is popular in the numerical studies of systems of hyperbolic conservation laws. For a given system (1) the corresponding Jin-Xin model is

$$\begin{array}{rcrcl} u_t & + & v_x & = & 0, \\ v_t & + & a^2 u_x & = & \frac{1}{\sigma}\left(f(u) - v\right), \end{array} \tag{8}$$

where σ is the (positive) relaxation parameter, a is a positive constant and $(u, v) \in U \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$. Taking the formal limit of (8) as σ tends to zero, one finds the equilibrium $v = f(u)$ and recovers the conservation law (1). We consider a shock wave $(u^-, u^+; s)$ satisfying the Rankine-Hugoniot condition (4). The corresponding profile $(u, v)^t(\theta)$, $\theta := \frac{x-st}{\sigma}$ should satisfy the nonlinear ordinary differential equation

$$\begin{array}{rcrcl} -su_\theta & + & v_\theta & = & 0, \\ -sv_\theta & + & a^2 u_\theta & = & f(u) - v, \end{array} \tag{9}$$

with boundary conditions

$$\lim_{\theta \to \pm\infty} u(\theta) = u^\pm, \qquad \lim_{\theta \to \pm\infty} v(\theta) = v^\pm. \tag{10}$$

We integrate the first equation of system (9), insert the solution $v = su + c$ into the second equation and obtain

$$(a^2 - s^2)u_\theta = f(u) - su - c. \tag{11}$$

For $a > |s|$ the travelling wave is governed by the same equation (7) as the viscous profile up to a positive constant. Hence for a small amplitude shock wave $(u^-, u^+; s)$ the existence of a profile is guarenteed [MP].

In this thesis we will consider the stability problem for the Jin-Xin model. Mascia and Zumbrun [MZ1], [MZ2] have further extended the work of Zumbrun and his collaborators [Z], [ZHo] to include relaxation models. Hence the stability problem is again reduced to determine the spectrum of the linearized operator.

The spectral stability of small amplitude profiles of the Jin-Xin model was established in the PhD thesis of Humpherys [Hu] by energy methods. We will instead follow the programme Freistühler und Szmolyan [FS] proposed in their paper on spectral stability of small amplitude profiles in viscous conservation laws. We will be able to construct the Evans bundles and to prove the spectral stability through a reduction of the problem. This can be viewed as a step towards a proof of spectral stability of small amplitude profiles in general relaxation systems.

In detail we will prove spectral stability for small amplitude profiles of a Jin-Xin model, which satisfies the following assumptions at a basepoint $u_* \in U$

(A1) The system (8) is strictly hyperbolic at u_*.

(A2) The eigenvalue λ_k is genuine nonlinear at u_*.

(A3) *subcharacteristic condition*: $a > \max\{\text{spec}(Df(u_*)), |s|\}$.

In Chapter 1 we review the existence of small amplitude travelling waves in the Jin-Xin model. In Chapter 2 we will give the definition of nonlinear and spectral stability and state the main result. It turns out that the proper notion of spectral stability is, that the linearized operator L has no spectrum in the domain $\mathbb{C}_\bullet^+ := \{\kappa \in \mathbb{C} | \Re\kappa \geq 0\} \backslash \{0\}$. The spectrum of L will consist of essential and point spectrum. The essential spectrum does not intersect $\mathbb{C}_\bullet^+$ and touches the imaginary axis only in 0. The point spectrum consists of isolated eigenvalues κ of finite multiplicity, i.e. we are searching for an eigenvalue κ and an eigenfunction w satisfying the eigenvalue equation associated with the differential operator L

$$Lw = \kappa w. \tag{12}$$

In Evans function theory the eigenvalue problem is considered as an first order differential equation with spectral parameter κ. We will construct the stable (unstable) Evans bundles, which are spaces of initial values of solutions decaying at plus (minus) infinity. Hence a nontrivial intersection of the bundles corresponds to an eigenfunction. The crucial region for the stability analysis is a small neighbourhood of $\kappa = 0$. We have to consider two regions, a inner problem corresponding to $|\kappa| \leq \varrho\epsilon^2$ and an outer problem $|\kappa| \geq \varrho\epsilon^2$ where ϵ corresponds to the shock strength and ϱ is a fixed large constant.

The inner problem is analyzed in Chapter 3, where it is shown, that the Evans bundles are essentially perturbations of the corresponding bundles for the viscous Burgers equation. From the well known stability of profiles of the Burgers equation we will be able to conclude that there is no point spectrum in the inner problem. In Chapter 4 we will prove that there is no point spectrum in the outer problem either. Thus the spectrum of the linearized operator does not intersect $\mathbb{C}_\bullet^+$ and the small amplitude profiles of the Jin-Xin model are spectrally stable. We present some results needed in the analysis in two appendices. In Appendix A we collect some facts about Grassmann manifolds. Appendix B gives a brief introduction to invariant manifold theory.

Chapter 1

Shock waves and profiles

We consider $n \times n$ systems of hyperbolic conservation laws in one space variable $x \in \mathbb{R}$

$$u_t + f(u)_x = 0, \tag{1.1}$$

where the unknown function $u(x,t)$ takes its values in an open convex set $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is a given smooth vector field. The Cauchy problem is to find a solution $u : \mathbb{R} \times [0,T) \to U$, $T > 0$, which solves the system (1.1) and satisfies

$$u(x,0) = u_0(x) \tag{1.2}$$

for given initial data $u_0 \in \mathcal{C}^1(\mathbb{R})$. The well-posedness of the problem in standard spaces, such as Hölder or Sobolev spaces, is only expected if the system is hyperbolic.

Definition. The system (1.1) is *hyperbolic* if at every point $u \in U$ the Jacobian $Df(u)$ is diagonalizable with real eigenvalues.

We will use an even stricter definition.

Definition. The system (1.1) is *strictly hyperbolic* if at every point $u \in U$ the Jacobian $Df(u)$ is diagonalizable with distinct, real eigenvalues.

In a strictly hyperbolic system (1.1) we denote with $\lambda_1(u) < \ldots < \lambda_n(u)$ the eigenvalues of $Df(u)$, $u \in U$ in ascending order, and with r_i, $i = 1, \ldots, n$ the associated eigenfields:

$$Df(u)r_i(u) = \lambda_i(u)r_i(u), \quad i = 1, \ldots, n.$$

The calculus along characteristics shows that for some non-linear functions f a classical solution exists only for a finite time. Therefore one has to allow weak solutions, which are bounded measureable functions $u : Q_T := \mathbb{R} \times [0,T) \to U$ satisfying (1.1), (1.2) in the distributional sense

$$\int_{Q_T} \big(u\partial_t\phi + f(u)\partial_x\phi\big)dxdt + \int_{\mathbb{R}} \phi(x,0)u_0(x)dx = 0, \quad \forall\phi \in \mathcal{C}^1\big(\mathbb{R}\times[0,T)\big).$$

Proposition 1.1. *Let Γ be a $\mathcal{C}^1$-curve within Q_T. Let u be $\mathcal{C}^1$ away from Γ, having continous left and right limits $u^\pm$ on Γ. Then u is a weak solution of the Cauchy problem if and only if*

1. *u is a classical solution away from Γ.*
2. *u satisfies along Γ the* Rankine-Hugoniot *condition*

$$(u^+ - u^-)\nu_t - \big(f(u^+) - f(u^-)\big)\,\nu_x = 0, \tag{1.3}$$

where ν denotes a normal vector field to Γ.

An important class of solutions are *shock waves* $(u^-, u^+; s)$, which are defined as

$$u(x,t) := \begin{cases} u^-, & x < st, \\ u^+, & x > st, \end{cases} \tag{1.4}$$

where the constant vectors $u^-, u^+ \in \mathbb{R}^n$ together with the *wave speed* $s \in \mathbb{R}$ satisfy the *Rankine-Hugoniot* condition

$$s(u^+ - u^-) - \big(f(u^+) - f(u^-)\big) = 0. \tag{1.5}$$

Such a shock wave is a weak solution of system (1.1), which is piecewise constant with a single jump discontinuity moving with wave speed s. The quantity $|u^+ - u^-|$ is referred to as shock strength or *amplitude*.

Definition. For $k = 1, \dots, n$ a k-Lax shock wave $(u^-, u^+; s)$ is a weak solution of (1.1) which satisfies

$$\lambda_{k+1}(u^+) > s > \lambda_k(u^+) \quad \text{and} \quad \lambda_k(u^-) > s > \lambda_{k-1}(u^-).$$

It turns out that weak solutions are not unique. In order to single out one of the solutions Gelfand proposed the *vanishing viscosity method*, which is based upon the idea that physically relevant (weak) solutions of system (1.1) should correspond to limiting solutions of viscous conservation laws

$$u_t + f(u)_x = \epsilon\big(B(u)u_x\big)_x \tag{1.6}$$

as ϵ tends to 0. In the context of travelling wave solutions the vanishing viscosity method is particularly simple.

Definition. A *travelling wave* or *profile* is a smooth solution of system (1.6) depending on a single variable $\xi := x - st$, where s is speed of the wave.

Majda and Pego [MP] proposed a stability criterion for the matrix $B(u)$ which guarentees for shock waves the existence of a travelling wave. The identity matrix I_n is admissible in the sense of [MP] and the viscous profiles are governed by the system of ordinary differential equations

$$-su_\xi + f(u)_\xi = u_{\xi\xi}. \tag{1.7}$$

The travelling waves under consideration are asymptotically constant and we get the boundary conditions $\lim_{\xi\to\pm\infty} u(\xi) = u^\pm$. We integrate the equation (1.7) once and obtain the viscous profile equation

$$u_\xi = f(u) - su - c, \tag{1.8}$$

where the boundary conditions determine the constant vector $c \in \mathbb{R}^n$ as $c = f(u^-) - su^- = f(u^+) - su^+$. The constant vectors $u^-, u^+ \in \mathbb{R}^n$ are the restpoints of the system (1.8) and a profile corresponds to a heteroclinic orbit connecting u^- with u^+.

1.1 Jin-Xin model

We will follow Pauline Godillon [G] in this introduction. Jin and Xin introduced a relaxation model [JX] which is popular in the numerical studies of systems of hyperbolic conservation laws. For a given system (1.1) the corresponding *Jin-Xin model* is

$$\begin{array}{ccccl} u_t & + & v_x & = & 0, \\ v_t & + & a^2 u_x & = & \frac{1}{\sigma}\left(f(u) - v\right), \end{array} \tag{1.9}$$

where σ is the (positive) relaxation parameter, a is a positive constant and the unknown function $(u, v) \in U \times \mathbb{R}^n \subseteq \mathbb{R}^{2n}$. The model was introduced by Jin and Xin [JX] in order to obtain nonoscillatory schemes for systems of conservation laws. Taking the formal limit of (1.9) as σ tends to 0, we recover the local equilibrium $v = f(u)$ and the *equilibrium system*

$$u_t + f(u)_x = 0, \tag{1.10}$$

which is in fact a system of conservation laws. Relaxation provides a dissipative mechanism similar to viscosity. This can be seen from a Chapman-Enskog type expansion (see Liu [L]), which shows that the solution $u(x,t)$ of (1.9) satisfies

$$u_t + f(u)_x = \sigma\Big(\underbrace{\left(a^2 I_n - Df(u)^2\right)}_{:=\beta(u)} u_x\Big)_x + O(\sigma^2).$$

In order that this equation is well posed the coefficient $\beta(u)$ has to be positive. This is guarenteed with the *subcharacteristic condition* [L]

$$a > \max\{\mathrm{spec}(Df(u)),\ u \in U\}. \tag{1.11}$$

The condition is equivalent to the characteristics of the equilibrium system being subcharacteristic to the characteristics with speeds $\pm a$ of the Jin-Xin model. The shock waves $(u^-, u^+; s)$ of the equilibrium system (1.10) are now related to travelling waves in the Jin-Xin model for $\sigma > 0$. For the existence of such profiles the wave speed has to be subcharacteristic and our condition (1.11) becomes

$$a > \max\{\mathrm{spec}(Df(u)), |s|\}. \tag{1.12}$$

A profile $u(\xi)$, $\xi := x - st$, which is a solution of system (1.9), is governed by the system of ordinary differential equations

$$\begin{array}{rcrcl} -su_\xi & + & v_\xi & = & 0, \\ -sv_\xi & + & a^2 u_\xi & = & \frac{1}{\sigma}\left(f(u) - v\right). \end{array}$$

The travelling waves under consideration are asymptotically constant and we get the boundary conditions

$$\lim_{\xi \to \pm\infty} u(\xi) = u^\pm, \quad \lim_{\xi \to \pm\infty} v(\xi) = v^\pm.$$

The points $(u, v)^\pm$ are rest points for this system of ordinary differential equations and a travelling wave corresponds to a heteroclinic orbit connecting (u^-, v^-) with (u^+, v^+). The system is autonomous in ξ and hence it is invariant under translations $\xi = \xi + z$, $z \in \mathbb{R}$. This means there is no unique solution $(\bar{u}, \bar{v})(\xi)$, since the shifted travelling waves $(\bar{u}, \bar{v})(\xi + z)$ solve the problem as well. As long as $\sigma > 0$ it is possible to introduce a new variable $\theta := \frac{\xi}{\sigma}$ and obtain an equivalent system

$$\begin{array}{rcrcl} -su_\theta & + & v_\theta & = & 0, \\ -sv_\theta & + & a^2 u_\theta & = & f(u) - v, \end{array} \tag{1.13}$$

which is independent of σ. The respective solutions $(u, v)(\xi)$, $(u, v)(\theta)$ are equivalent and we continue with system (1.13). The boundary conditions became

$$\lim_{\theta \to \pm\infty} u(\theta) = u^\pm, \quad \lim_{\theta \to \pm\infty} v(\theta) = v^\pm.$$

Remark 1.1. At the first glance system (1.13) is independent of σ, but we actually tied the relaxation parameter σ to the independent variable ξ. As we let σ tend to 0 we have

$$\lim_{\sigma \to 0} \theta = \lim_{\sigma \to 0} \frac{\xi}{\sigma} = \begin{cases} -\infty & \xi < 0, \\ 0 & \xi = 0, \\ +\infty & \xi > 0. \end{cases}$$

The variable ξ is defined as $\xi := x - st$ and we recover in the limit of the profile the shock wave $(u^-, u^+; s)$

$$\lim_{\sigma \to 0} u(\theta) = \lim_{\sigma \to 0} u\Big(\frac{\xi}{\sigma}\Big) = \begin{cases} u^- & x < st, \\ u^+ & x > st. \end{cases}$$

□

Integrating the first equation of system (1.13) once and substituting the result $v(\theta) = su(\theta) + c$ into the second equation of system (1.13) we derive

$$(a^2 - s^2)u_\theta = f(u) - su - c, \tag{1.14}$$

with boundary conditions $\lim_{\theta \to \pm\infty} u(\theta) = u^\pm$. The system is reduced to an ordinary differential equation in u with rest points $u^\pm$. The constant vector c is determined by $c = f(u^+) - su^+ = f(u^-) - su^-$, and we recover the Rankine-Hugoniot condition. The profiles of the Jin-Xin model are governed by the same equations as the viscous profiles in a viscous conservation law. In this case for small amplitude shock waves the existence of profiles is well known, see Majda and Pego [MP]. We will prove it by using a scaling which allows us to use methods from singular perturbation theory (Appendix B). The singular perturbation nature of the problem will be crucial to the proof of spectral stability in Chapter 3.
We choose a basepoint $u_* \in U$ and assume throughout this thesis

(A1) The system is strictly hyperbolic at u_*.

(A2) The eigenvalue λ_k is genuine nonlinear at u_*, i.e. $\nabla\lambda_k(u_*) \cdot r_k(u_*) \neq 0$.

(A3) *subcharacteristic condition*: $a > \max\big\{\mathrm{spec}\big(Df(u_*)\big), |s|\big\}$.

Theorem 1.1. *Under the assumptions (A2)+(A3) there exists for a small-amplitude shock wave* $(u^-, u^+; s)$ *a family of profiles satisfying system* (1.13).

Before we prove Theorem 1.1 we develop some tools and notation needed in the proof. We can assume without loss of generality

(S1) The basepoint u_* satisfies $u_* = 0$, otherwise substitute u with $u + u_*$.

(S2) The function f satisfies $f(0) = 0$, otherwise replace f with $f(u) - f(0)$.

(S3) The Jacobian Df satisfies $Df(0) = \text{diag}(\lambda_1(0), \dots, \lambda_n(0))$.
As $Df(0)$ is diagonalizable there exists a transformation matrix T with $T \circ Df(0) \circ T^{-1} = \text{diag}(\lambda_1(0), \dots, \lambda_n(0))$, and we replace u with $T^{-1}u$.

(S4) The genuine nonlinear eigenvalue λ_k satisfies $\lambda_k(0) = 0$, otherwise substitute f with $f(u) - \lambda_k(0)u$.

Remark 1.2. The subcharacteristic condition provides that $a^2 - s^2$ is a positive factor. Introducing the new variable

$$\tau := \frac{\theta}{a^2 - s^2}$$

leads to the profile equation

$$u_\tau = u_\theta \frac{d\theta}{d\tau} = f(u) - su - c, \tag{1.15}$$

with boundary conditions $\lim_{\tau \to \pm\infty} u(\tau) = u^\pm$. Note that since we scaled by a positive factor the direction of time remains unchanged. □

We denote with $' := \frac{d}{d\tau}$ differentiation with respect to the variable τ

$$u' = f(u) - su - c. \tag{1.16}$$

For $s = 0$, $c = 0$, the trivial solution $u = 0$ is an equilibrium solution of system (1.16), which is degenerated as $Df(0)$ has a simple eigenvalue $\lambda_k = 0$. We introduce the scaling

$$u = \epsilon\tilde{u}, \quad s = \epsilon\tilde{s}, \quad c = \epsilon^2\tilde{c}, \tag{1.17}$$

$0 < \epsilon \ll 1$, to analyze the system (1.16) for (u, s, c) close to $(0, 0, 0)$. In this way we get the *scaled profile equation*

$$\epsilon\tilde{u}' = f(\epsilon\tilde{u}) - \epsilon^2\tilde{s}\tilde{u} - \epsilon^2\tilde{c}. \tag{1.18}$$

We expand the function f around 0:

$$f(v) = f(0) + Df(0)v + \frac{1}{2}D^2f(0)(v, v) + O(|v|^3) \tag{1.19}$$

and rewrite the equation as

$$\begin{aligned}\epsilon\tilde{u}' &= f(0) + Df(0)\epsilon\tilde{u} + \frac{1}{2}D^2f(0)(\epsilon\tilde{u}, \epsilon\tilde{u}) + O(\epsilon^3) - \epsilon^2\tilde{s}\tilde{u} - \epsilon^2\tilde{c} \\ &= f(0) + \epsilon Df(0)\tilde{u} + \epsilon^2\frac{1}{2}D^2f(0)(\tilde{u}, \tilde{u}) - \epsilon^2\tilde{s}\tilde{u} - \epsilon^2\tilde{c} + O(\epsilon^3).\end{aligned}$$

The function f satisfies $f(0) = 0$ and we divide both sides with ϵ to obtain

$$u' = Df(0)u + O(\epsilon),$$

where we dropped the tilde for notational convenience. The Jacobian $Df(0)$ is diagonal and the system becomes

$$u_i' = \lambda_i(0)u_i + O(\epsilon), \tag{1.20}$$

$i = 1, \ldots, n$. The assumption $\lambda_k(0) = 0$ implies a slow-fast structure of the problem, see also Appendix B. Specifically system (1.20) is in the standard form of singularly perturbed ordinary differential equations on the fast time scale. Here u_k is the slow variable and u_i, $i = 1, \ldots, n$, $i \neq k$ are the fast variables. Taking the formal limit $\epsilon = 0$ we arrive at the *layer problem*

$$\begin{array}{rcl} u_i' & = & \lambda_i(0)u_i, \\ u_k' & = & 0, \end{array} \tag{1.21}$$

$i = 1, \ldots, n$, $i \neq k$, which is in equilibrium if $u_i = 0$, i.e. the u_k-axis is a manifold of equilibria. This critical manifold V_0 is normally hyperbolic since the eigenvalues $\lambda_i(0) \neq 0$, $i = 1, \ldots, n$, $i \neq k$. Therefore Fenichel theory ([F2],[J2]) is applicable. We infer that the critical manifold V_0 perturbs smoothly to a *slow manifold* V_ϵ, ϵ small, which has a parameterization

$$\begin{aligned} V_\epsilon := \{(u_i)_{i=1,\ldots,n} \big| u_k \in \mathbb{R}, \\ u_i = \epsilon h_i\big(u_k, s, c, \epsilon\big),\ h_i \text{ smooth},\ i \neq k\} \end{aligned} \tag{1.22}$$

and is invariant under the flow (1.18). We denote with " $\dot{}$ " differentiation with respect to the slow time scale $\epsilon\tau$. It remains to determine the evolution of u_k restricted to the slow manifold V_ϵ.

Lemma 1.1. *For* $0 < \epsilon \leq \epsilon_1$*, system* (1.18) *has a one-dimensional invariant slow manifold* V_ϵ *with a parameterization* (1.22)*. The flow on the slow manifold is described by*

$$\dot{u}_k = Au_k^2 - su_k - c_k + O(\epsilon), \tag{1.23}$$

where $A := \frac{1}{2}\frac{\partial^2 f_k}{\partial u_k^2}(0)$.

Proof. The slow flow on the manifold V_ϵ is given by

$$\dot{u}_k = \frac{1}{2}D^2 f(0)(u, u)_k - su_k - c_k + O(\epsilon).$$

The slow manifold V_ϵ has the parameterization (1.22) with $u_i = \epsilon h_i\big(u_k, s, c, \epsilon\big)$, h_i smooth, $i \neq k$ and we investigate the quantity

$$
\begin{aligned}
D^2 f(0)(u,u)_k &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f_k}{\partial u_i \partial u_j}(0) u_i u_j \\
&= \frac{\partial^2 f_k}{\partial u_k^2}(0) u_k^2 + \\
&\quad + \epsilon \sum_{i\neq k} \frac{\partial^2 f_k}{\partial u_i \partial u_k}(0) h_i u_k + \epsilon \sum_{j \neq k} \frac{\partial^2 f_k}{\partial u_k \partial u_j}(0) u_k h_j + \\
&\quad + \epsilon^2 \sum_{i\neq k}\sum_{j\neq k} \frac{\partial^2 f_k}{\partial u_i \partial u_j}(0) h_i h_j.
\end{aligned}
\tag{1.24}
$$

The term $\frac{\partial^2 f_k}{\partial u_k^2}(0)u_k^2$ is the only one which is $O(1)$. $\square$

Proof of Theorem 1.1. We assumed that the eigenvalue λ_k is genuine nonlinear at 0, i.e. $\nabla\lambda_k(0)\cdot r_k(0) \neq 0$ which is in our case $\lambda_k = \frac{\partial f_k}{\partial u_k}$ and $r_k = e_k$ equivalent to the condition $\frac{\partial^2 f_k}{\partial u_k^2}(0) \neq 0$. As $A := \frac{1}{2}\frac{\partial^2 f_k}{\partial u_k^2}(0)$ is non-zero we can find two restpoints connected through a heteroclinic orbit which corresponds to a travelling wave. Since such a profile for $\epsilon = 0$ exists by virtue of the intersection of stable and unstable manifolds, it perturbs smoothly to profiles lying in the one-dimensional slow manifold V_ϵ for $\epsilon > 0$. This finishes the proof. $\square$

Remark 1.3. The slow flow on V_0 is described by (1.23) for $\epsilon = 0$. Note that the wave speed s and the vector c depend on the small parameter ϵ, since they are determined through the Rankine-Hugoniot condition.

$$
\begin{aligned}
\dot{u}_k &= A u_k^2 - s u_k - c_k \\
&= A\left(\left(u_k - \frac{s}{2A}\right)^2 - \frac{c_k}{A} - \frac{s^2}{4A^2}\right)
\end{aligned}
$$

We define the variable v by $v := u_k - \frac{s}{2A}$ and obtain

$$
\dot{v} = A\Big(v^2 - B\Big)
$$

where $B := \frac{4Ac_k + s^2}{4A^2}$ is a positive number, since we have two restpoints. We introduce the variable $w := \frac{v}{\sqrt{B}}$ and get

$$
\dot{w} = A\sqrt{B}\big(w^2 - 1\big)
$$

We consider from now on only the case $A < 0$ as we can switch with the transformation $x := -w$ from one case to the other. Finally we rescale the time variable to $|A\sqrt{B}|\hat{\tau}$ and obtain

$$\dot{w} = 1 - w^2$$

In the same manner we can transform the flow on the manifold V_ϵ to obtain $u_k^\pm = \pm 1$ and get a simplified profile equation

$$\dot{u}_k = 1 - u_k^2 + O(\epsilon) = \big(1 - u_k^2\big) g_\epsilon(u_k) \tag{1.25}$$

with $g_\epsilon(u_k) = 1 + O(\epsilon)$. $\square$

Chapter 2

Spectral stability

In this introduction to the stability of profiles we will follow Humpherys [Hu]. We want to study the long term behavior of travelling waves. In this regard it is useful to describe our problem in the form of an evolutionary system $u_t = \mathcal{F}(u)$, defined in an appropriate Banach space $\mathcal{B}$. This is achieved by translating our problem (1.9) for $\sigma = 1$, see Remark 1.1, via the transformation $(x,t) \mapsto (\xi := x - s \cdot t, t)$ to

$$\left.\begin{aligned} u_t &= su_\xi - v_\xi \\ v_t &= -a^2 u_\xi + sv_\xi + f(u) - v \end{aligned}\right\} =: \mathcal{F}\left(\begin{pmatrix} u \\ v \end{pmatrix}\right). \tag{2.1}$$

In this setting a travelling wave $(\bar{u}, \bar{v})^t$ is a stationary solution. The stability is related to the long term behavior of solutions which are initially "close" to this equilibrium. More precisely, given an appropriate Banach space $\mathcal{B}$ with norm $\|.\|$ and an admissible subset $\mathcal{A} \subset \mathcal{B}$ of small perturbations, we consider the Cauchy problem for (2.1) with initial data

$$\begin{aligned} u(\xi, 0) &= \bar{u}(\xi) + p(\xi, 0), \\ v(\xi, 0) &= \bar{v}(\xi) + q(\xi, 0), \end{aligned}$$

with $(p,q)^t \in \mathcal{A}$ and $(\bar{u}, \bar{v})^t$ a stationary solution of (2.1). The evolution of $(p,q)^t(\xi,t)$ describes the difference between the stationary solution $(\bar{u}, \bar{v})^t$ and $(u,v)^t$. In general asymptotic stability is understood as

$$\left\| \begin{pmatrix} p \\ q \end{pmatrix}(\xi,t) \right\| = \left\| \begin{pmatrix} u \\ v \end{pmatrix}(\xi,t) - \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi) \right\| \to 0 \text{ as } t \to \infty.$$

However the stationary solution u is not unique, since the shifted profiles $(\bar{u}, \bar{v})^t(\xi + z)$, $z \in \mathbb{R}$ are solutions as well and we give the definition

Definition 2.1. A stationary solution $(\bar{u}, \bar{v})^t$ is *non-linearly stable* with respect to $\mathcal{A}$ if

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} (\xi, t) - \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} (\xi + z) \right\| \to 0$$

for a $z \in \mathbb{R}$ as $t \to \infty$, whenever $(u, v)^t(., 0) - (\bar{u}, \bar{v})^t(.) \in \mathcal{A}$.

In order to obtain an equation for $(p, q)^t(\xi, t)$ we first linearize the operator $\mathcal{F}$ around the solution $(\bar{u}, \bar{v})^t + (p, q)^t$ to find that small perturbations $(p, q)^t$ satisfy

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = \underbrace{\begin{pmatrix} sI_n & -I_n \\ -a^2 I_n & sI_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_\xi + \begin{pmatrix} 0_n & 0_n \\ Df(\bar{u}) & -I_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}}_{=: L\binom{p}{q}} + \begin{pmatrix} 0 \\ R(p) \end{pmatrix} \tag{2.2}$$

where $R(p) = O(|p|^2)$ is a nonlinear function. Results by Mascia and Zumbrum [MZ1], [MZ2] show that spectral stability (defined below) implies nonlinear stability, in the sense of Definition 2.1. Therefore we study the eigenvalue problem

$$L \begin{pmatrix} p \\ q \end{pmatrix} = \kappa \begin{pmatrix} p \\ q \end{pmatrix} \tag{2.3}$$

with boundary conditions

$$\lim_{\xi \to \pm\infty} p(\xi, .) = 0, \qquad \lim_{\xi \to \pm\infty} q(\xi, .) = 0.$$

Definition. The *spectrum* $\sigma(L)$ of L is the set of all $\kappa \in \mathbb{C}$ such that $L - \kappa I$ is not invertible, i.e. there does not exist a bounded inverse.

Definition. The *point spectrum* $\sigma_p(L)$ of L is the set of all isolated eigenvalues of L with finite multiplicity.

Definition. The *essential spectrum* $\sigma_e(L)$ of L is the entire spectrum less the point spectrum, i.e. $\sigma_e(L) = \sigma(L)/\sigma_p(L)$.

The next lemma shows that L has at least one eigenvalue:

Lemma 2.1 (Sattinger [St]). *The derivative of the profile $\binom{\bar{u}}{\bar{v}}'$ is an eigenfunction of L with eigenvalue 0.*

Proof. The translational invariance of the system is the reason that also the shifted shock waves $(\bar{u}, \bar{v})^t(\xi + z)$, $z \in \mathbb{R}$ are solutions, i.e. they satisfy $\mathcal{F}\left(\binom{\bar{u}}{\bar{v}}(\xi + z)\right) = 0$, $\forall z \in \mathbb{R}$. Hence differentiating with respect to z and evaluating at $z = 0$, yields $L\binom{\bar{u}'}{\bar{v}'} = \mathcal{F}\left(\binom{\bar{u}}{\bar{v}}\right)\binom{\bar{u}'}{\bar{v}'} = 0$. □

Spectral stability in a strong sense means that the whole spectrum lies to the left of the imaginary axis. Since $\kappa = 0$ is an eigenvalue we have to allow this one and give the following definition.

Definition. We say that the operator L, linearized about the profile $(\bar{u}, \bar{v})^t$, is *spectrally stable* if it has no spectrum in the punctured closed right half plane

$$\mathbb{C}^+_\bullet := \mathbb{C}^+ \backslash \{0\}$$

with $\mathbb{C}^+ := \{\kappa \in \mathbb{C} | Re(\kappa) \geq 0\}$ denoting the closed right half plane.

The variable ξ varies in $\mathbb{R}$, hence the operator L can have point and essential spectrum. We will start with the analysis of the essential spectrum and then turn our attention to the point spectrum.

2.1 Essential spectrum

The essential spectrum can be computed by using the following theorem.

Theorem 2.1 (Henry [He]). *The essential spectrum of L is sharply bounded to the right by*

$$\sigma_e(L^+) \cup \sigma_e(L^-),$$

where $L^\pm$ correspond to the operators obtained by linearizing about the constant solutions $\binom{\bar{u}}{\bar{v}} = \binom{u}{v}^\pm$ respectively.

Let $L^\pm$ be the limits of L at $\xi = \pm\infty$, i.e.

$$L^\pm \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} sI_n & -I_n \\ -a^2 I_n & sI_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_\xi + \begin{pmatrix} 0_n & 0_n \\ Df(u^\pm) & -I_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

A linear operator with constant coefficients has no point spectrum and $\sigma(L^\pm)$ satisfies $\sigma(L^\pm) = \sigma_e(L^\pm)$. We can determine $\sigma_e(L^\pm)$ by considering the Fourier transform

$$\left(\widehat{L^\pm - \kappa I_{2n}}\right) = \left(i\theta \begin{pmatrix} sI_n & -I_n \\ -a^2 I_n & sI_n \end{pmatrix} + \begin{pmatrix} -\kappa I_n & 0_n \\ Df(u^\pm) & -(\kappa+1)I_n \end{pmatrix} \right),$$

$\theta \in \mathbb{R}$. We loose invertibility of $L^\pm - \kappa I_{2n}$, when the right hand side is singular. Thus we conclude that $\kappa \in \sigma_e(L^\pm)$ if and only if

$$\det \left(i\theta \begin{pmatrix} sI_n & -I_n \\ -a^2 I_n & sI_n \end{pmatrix} + \begin{pmatrix} -\kappa I_n & 0_n \\ Df(u^\pm) & -(\kappa+1)I_n \end{pmatrix} \right) = 0, \qquad (2.4)$$

for some $\theta \in \mathbb{R}$. The condition (2.4) simplifies to

$$\det(\kappa(\kappa+1)I_n - i\theta s(2\kappa+1)I_n + (a^2-s^2)\theta^2 I_n + i\theta Df(u^\pm)) = 0.$$

We consider the matrices $Df(u^\pm)$ to be diagonalizable and obtain

$$\prod_{j=1}^{n}(\kappa(\kappa+1) - i\theta(2\kappa s + s - \lambda_j(u^\pm)) + (a^2-s^2)\theta^2) = 0. \tag{2.5}$$

This implies that there is a $j \in \{1,\dots,n\}$ such that

$$\kappa(\kappa+1) - i\theta(2\kappa s + s - \lambda_j(u^\pm)) + (a^2-s^2)\theta^2 = 0, \tag{2.6}$$

$\theta \in \mathbb{R}$. The equation (2.6) is a second order polynomial in κ

$$\kappa^2 + \kappa(1 - i2\theta s) - i\theta(s - \lambda_j(u^\pm)) + (a^2-s^2)\theta^2 = 0. \tag{2.7}$$

The spectral parameter $\kappa^\pm_{j,1/2}$ can be computed as

$$\begin{aligned}\kappa^\pm_{j,1/2} &= -\frac{1-i2\theta s}{2} \pm \sqrt{\frac{(1-i2\theta s)^2}{4} + i\theta(s-\lambda_j(u^\pm)) - (a^2-s^2)\theta^2} \\ &= \frac{-1 + i2\theta s \pm \sqrt{1 - 4a^2\theta^2 - i4\theta\lambda_j(u^\pm)}}{2},\end{aligned}$$

$j = 1,\dots,n$. This defines $4n$-curves $\kappa^\pm_{j,1/2}(\theta)$, $\theta \in \mathbb{R}$ corresponding to the eigenvalues $\lambda_j(u^\pm)$, $j = 1,\dots,n$. The essential spectrum $\sigma_e(L^\pm)$ is the union of these curves

$$\sigma_e(L^+) \cup \sigma_e(L^-) = \bigcup_{j=1}^{n} \kappa^+_{j,1/2}(\theta) \cup \bigcup_{j=1}^{n} \kappa^-_{j,1/2}(\theta).$$

We drop the subscripts for notational convenience. In the following lemma we will locate the spectrum $\sigma(L^\pm)$.

Lemma 2.2. *If* $a > \max\{\mathrm{spec}\left(Df(u^-)\right), \mathrm{spec}\left(Df(u^+)\right), |s|\}$ *it follows that*

$$\sigma(L^\pm) \subseteq \{\Re\kappa < 0\} \cup \{0\}.$$

Proof. The spectral parameter κ is at $\theta = 0$ computed as $\kappa_1(0) = -1$ and $\kappa_2(0) = 0$. Hence $\{0\}$ is part of $\sigma(L^\pm)$. We will prove that under the assumption for a and $\theta \neq 0$, the spectral parameter satisfies $\Re\kappa < 0$. We point out that the discriminant is a function in θ which is symmetric with

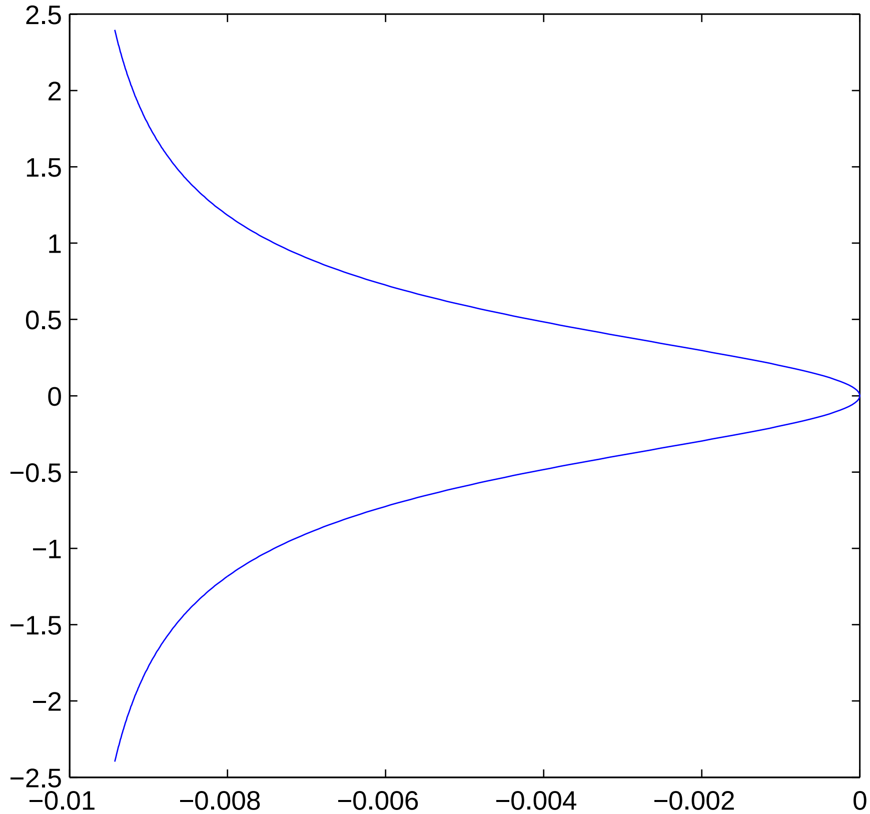

Figure 2.1: The eigenvalue $\kappa^+(\theta)$ as a function in θ.

respect to the real axis. Therefore we only consider the part of the function which is above the real axis. The discrimant z in polar coordinates is

$$r = \sqrt{(1 - 4a^2\theta^2)^2 + (4\theta\lambda)^2}$$
$$\phi = \arctan \frac{4\theta\lambda}{1 - 4a^2\theta^2}$$

We note that with $z = 1 - 4a^2\theta^2 - i4\theta\lambda$ we are only interested in the root of z with non-negative real part. The equation

$$\cos\phi = \frac{1 - 4a^2\theta^2}{r} = \frac{1 - 4a^2\theta^2}{\sqrt{(1 - 4a^2\theta^2)^2 + (4\theta\lambda)^2}}$$

implies

$$\Re\sqrt{z} = \sqrt{r}\cos\frac{\phi}{2} = \sqrt{\frac{1 - 4a^2\theta^2 + \sqrt{(1 - 4a^2\theta^2)^2 + (4\theta\lambda)^2}}{2}}.$$

The property $\Re\kappa < 0$ of the spectral parameter is equivalent to

$$\sqrt{\frac{1-4a^2\theta^2+\sqrt{(1-4a^2\theta^2)^2+(4\theta\lambda)^2}}{2}} < 1.$$

As the discriminant is non-negative we are allowed to square the inequality

$$\frac{1-4a^2\theta^2+\sqrt{(1-4a^2\theta^2)^2+(4\theta\lambda)^2}}{2} < 1$$

and obtain

$$\sqrt{(1-4a^2\theta^2)^2+(4\theta\lambda)^2} < 2-1+4a^2\theta^2.$$

Again we square the inequality

$$(1-4a^2\theta^2)^2+(4\theta\lambda)^2 < (1+4a^2\theta^2)^2$$

and obtain the condition

$$(4\theta\lambda)^2 < (4\theta a)^2.$$

The last inequality is satisfied with $\theta \neq 0$ and the assumption $a > \max\{\mathrm{spec}\big(Df(u^{\pm})\big), |s|\}$. □

Theorem 2.2. *The differential operator L associated with a small amplitude profile obtained in Theorem 1.1 has no essential spectrum in $\mathbb{C}^+_\bullet$ if the subcharacteristic condition*

$$a > \max\big(\mathrm{spec}\,\big(Df(u_*)\big), |s|\big)$$

is satisfied and ϵ is sufficiently small.

Proof. The subcharacteristic condition and ϵ sufficiently small guarentees that

$$a > \max\big(\mathrm{spec}\,\big(Df(\epsilon u^{\pm})\big), |s|\big)$$

is satisfied. Thus the requirements of Lemma 2.2 are fulfilled and we conclude that the curves $\kappa^{\pm}_{j,1/2}(\theta)$ are contained in the left half plane and touch the imaginary axis only in 0. Theorem 2.1 implies that the essential spectrum is bounded to the right by these curves. □

2.2 Point spectrum and the Evans function

In the last section we proved that under the subcharacteristic condition the essential spectrum does not intersect $\mathbb{C}^+_\bullet$. Hence the point spectrum will decide upon spectral stability.

Starting with the work of Evans (see [E] and references therein) it became popular to study the spectra of linearizations along travelling waves via a dynamical systems approach. The point spectrum consists of isolated eigenvalues of finite multiplicity, i.e. we are searching for functions $p, q : \mathbb{R} \to \mathbb{C}^n$ and $\kappa \in \mathbb{C}$ satisfying the eigenvalue equation associated with the differential operator L

$$L\begin{pmatrix} p \\ q \end{pmatrix} = \kappa \begin{pmatrix} p \\ q \end{pmatrix}$$

and boundary conditions $\lim_{\xi \to \pm\infty} p(\xi) = 0$, $\lim_{\xi \to \pm\infty} q(\xi) = 0$. The linear Operator L was defined in equation (2.2) and we obtain

$$\begin{pmatrix} sI_n & -I_n \\ -a^2 I_n & sI_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_\xi + \begin{pmatrix} 0_n & 0_n \\ Df(\bar{u}) & -I_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \kappa \begin{pmatrix} p \\ q \end{pmatrix}.$$

First we change the independent variable ξ to $\tau := \frac{\xi}{a^2 - s^2}$, ,see Remark 1.2,

$$\frac{1}{a^2 - s^2} \begin{pmatrix} sI_n & -I_n \\ -a^2 I_n & sI_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_\tau + \begin{pmatrix} 0_n & 0_n \\ Df(\bar{u}) & -I_n \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \kappa \begin{pmatrix} p \\ q \end{pmatrix}.$$

The eigenvalue problem can be rewritten as the following explicit first-order ordinary differential equation with variable coefficients obtained by solving for $(p, q)^t_\tau$

$$\begin{pmatrix} p \\ q \end{pmatrix}_\tau = \underbrace{\begin{pmatrix} Df(\bar{u}) - s\kappa I_n & -(\kappa + 1) I_n \\ sDf(\bar{u}) - a^2 \kappa I_n & -s(\kappa + 1) I_n \end{pmatrix}}_{=:\mathbb{A}_\kappa(\tau)} \begin{pmatrix} p \\ q \end{pmatrix} \tag{2.8}$$

The matrix $\mathbb{A}_\kappa$ is clearly analytic in κ and $\mathcal{C}^1$ in τ because f is smooth. The coefficients of $\mathbb{A}_\kappa(\tau)$ tend to constants as $\tau \to \pm\infty$, since $\lim_{\tau \to \pm\infty} Df(\bar{u}(\tau)) = Df(u^\pm)$ and we denote the limits of $\mathbb{A}_\kappa$ with $\mathbb{A}^\pm_\kappa$ at $\tau = \pm\infty$.

Starting with the work of Jones [J1] the connection between properties of the travelling wave problem and the eigenvalue problem became apparent. Alexander, Gardner and Jones [AGJ] developed a method which is used to analyze such problems, and is now known as Evans function theory. See also Gardner and Jones [GJ1] for a comprehensive introduction. We will follow the related concept of stable/unstable Evans bundles which was proposed by Freistühler and Szmolyan in [FS].

At first we will explain the Evans function and the Evans bundles for a k-Lax shock $\binom{u}{v}$ in a subset $\Omega \subset \mathbb{C}$ with consistent splitting.

Definition. $\Omega \subset \mathbb{C}$ has *consistent splitting* if there exists $l \in \mathbb{N}$ such that $\mathbb{A}_\kappa^\pm$ both have l (resp. $2n-l$) eigenvalues with positive (resp. negative) real part for all $\kappa \in \Omega$.

Remark. As usual the curves in $\mathbb{C}$ where $\mathbb{A}_\kappa^\pm$ have pure imaginary eigenvalues define the boundary of $\sigma_e(L)$. The essential spectrum is contained in the region to the left of the union of these curves and tangent to the imaginary axis at $\kappa = 0$. Thus Ω is contained in the set $\mathbb{C}/\sigma_e(L)$.
Additionally we get for such a domain Ω that $\mathbb{A}_\kappa^\pm$, $\kappa \in \Omega$ has n eigenvalues with positive respective negative real part. □

It is possible to write system (2.8) in the form

$$x' = \big(A + V(\tau)\big)x \tag{2.9}$$

with $A = \mathbb{A}_\kappa^\pm$ a constant matrix and $V(\tau) = \mathbb{A}_\kappa(\tau) - \mathbb{A}_\kappa^\pm$ a matrix which tends to zero in each coefficient as $\tau \to \pm\infty$. We state a theorem, whose proof can be found in [CoLe, Chapter 2, page 92], which characterizes the behavior of solutions of differential equations with asymptotic constant coefficients as in (2.9).

Theorem 2.3 (Coddington, Levinson [CoLe]). *Let A be a constant matrix with characteristic roots μ_j, $j = 1, \dots, n$, all of which are distinct. Let the matrix V be differentiable and satisfy*

$$\int_0^\infty \big|V'(t)\big| dt < \infty$$

and let $V(t) \to 0$ as $t \to \infty$. Let the roots of $\det(A + V(t) - \lambda I) = 0$ be denoted with $\lambda_j(t)$, $j = 1, \dots, n$. Clearly, by reordering the μ_j if necessary, $\lim_{t\to\infty} \lambda_j(t) = \mu_j$. For a given k, let

$$D_{kj}(t) = \Re\big(\lambda_k(t) - \lambda_j(t)\big).$$

Suppose all j, $j = 1, \dots, n$ fall into one of the two classes I_1 and I_2, where

$$j \in I_1, if \int_0^t D_{kj}(\tau)d\tau \to \infty \text{ as } t \to \infty \text{ and } \int_{t_1}^{t_2} D_{kj}(\tau)d\tau > -K \quad (t_2 \geq t_1 \geq 0)$$

$$j \in I_2, if \int_{t_1}^{t_2} D_{kj}(\tau)d\tau < K \quad (t_2 \geq t_1 \geq 0)$$

where k is fixed and K is a constant. Let p_k be a characteristic vector of A associated with μ_k, so that

$$Ap_k = \mu_k p_k$$

Then there is solution φ_k of (2.9) *and a t_0, $0 \leq t_0 \leq \infty$, such that*

$$\lim_{t\to\infty} \varphi_k(t) \exp\big[-\int_{t_0}^{t} \lambda_k(\tau) d\tau\big] = p_k.$$

As Ω has consistent splitting and f is smooth, we get the existence of n independent solutions of (2.8) $\eta^-_{i,\kappa}(\tau)$, $i = 1, \dots, n$ which decay to zero as $\tau \to -\infty$ and n solutions $\eta^+_{i,\kappa}(\tau)$, $i = n+1, \dots, 2n$ which decay to zero as $\tau \to +\infty$. In view of the hyperbolicity of $\mathbb{A}^{\pm}_{\kappa}$, κ is an eigenvalue of L if and only if (2.8) possess a nontrivial solution which decays to zero at both ends, $+\infty$ and $-\infty$. The existence of an eigenfunction is equivalent to the property that these two subspaces intersect nontrivially. This idea is explored in two ways: The *Evans function* is defined as

$$\begin{aligned} &E(\tau) : \Omega \to \mathbb{C} \\ &\qquad \kappa \to E(\tau, \kappa) = \det\big(\eta^-_{1,\kappa}(\tau), \dots, \eta^-_{n,\kappa}(\tau), \eta^+_{n+1,\kappa}(\tau), \dots, \eta^+_{2n,\kappa}(\tau)\big) \end{aligned}$$

which has the following properties (see [AGJ])

1. $E(\tau, \kappa)$ is analytic in κ for $\kappa \in \Omega$ and independent of τ.

2. $E(\tau, \kappa_0) = 0$ if and only if $\kappa_0 \in \sigma_p(L)$.

The *Evans bundles* are the result of our attempt to study the mentioned subspaces of solutions of system (2.8)

$$\begin{aligned} X^-_\kappa(\tau) &= \text{span}\{\eta^-_{1,\kappa}(\tau), \dots, \eta^+_{n,\kappa}(\tau)\}, \\ X^+_\kappa(\tau) &= \text{span}\{\eta^+_{n+1,\kappa}(\tau), \dots, \eta^+_{2n,\kappa}(\tau)\} \end{aligned}$$

and their possible intersection directly. We will now line out how we are going to construct these bundles and concentrate in this regard on the construction of the unstable one as the stable case differs only slightly. As the set $X^-_\kappa(\tau)$ is a family of n-dimensional subspaces of $\mathbb{C}^{2n}$ which depends continuously on parameters (τ, κ), it has the structure of a complex n-plane bundle over the base space $\mathbb{R} \times \Omega$. In order to be able to construct the subspaces $X^-_\kappa(\tau), X^+_\kappa(\tau)$ we augment the profile equation (1.15) with the system (2.8) to obtain an autonomous system of the form

$$\begin{aligned} u' &= f(u) - s \cdot u - c, \\ X'_\kappa &= \mathbb{A}_\kappa(u) X_\kappa, \end{aligned} \tag{2.10}$$

with $\mathbb{A}_\kappa$ analytic in κ and $\mathcal{C}^1$ in u. The idea is to apply the stable/unstable manifold theorem to the rest points, $(u^-, 0)$ and $(u^+, 0)$, of the augmented system (2.10). More precisely, as Ω has consistent splitting, there exist for $\kappa \in \Omega$ n-dimensional subspaces $U^{\pm}_{\kappa}$ (resp. $S^{\pm}_{\kappa}$) associated with the portion of spectra of $\mathbb{A}^{\pm}_{\kappa}$ with positive (resp. negative) real part. In the situation of a k-Lax shock the restpoint $(u^-, 0)$ of system (2.10) is hyperbolic with $n-k$ (real) unstable directions (from the u system) and n (complex) unstable directions (from the X-system). The unstable manifold theorem provides an unstable manifold $\mathcal{W}^u_-$ of solutions of (2.10) which tend to $(u^-, 0)$ in backward time. The projection of $\mathcal{W}^u_-$ into $\mathbb{C}^{2n}$ along the heteroclinic orbit $u(\tau)$ is denoted by X_κ.

The concept of the Grassmann manifold will be helpful to study the (global) behavior of these subspaces as an entity. The set $\mathcal{G}^d_m(\mathbb{C})$ of m-dimensional linear subspaces of $\mathbb{C}^d$ is a complex-analytic manifold of dimension $m(d-m)$. With respect to a given basis $\{e_1, \ldots, e_d\}$ of $\mathbb{C}^d$, the mapping

$$\begin{aligned} \phi : \mathbb{C}^{(d-m)\times m} &\to \mathcal{G}^d_m(\mathbb{C}), \\ T &\mapsto span\begin{pmatrix} I_m \\ T \end{pmatrix} \end{aligned}$$

is a local chart of $\mathcal{G}^d_m(\mathbb{C})$ with $\phi(0) = X_0 = span\{e_1, \ldots, e_m\}$. A linear autonomous system $Y' = AY$ with $Y \in \mathbb{C}^d$ induces a flow on $\mathcal{G}^d_m(\mathbb{C})$ which we denote with $\hat{Y}' = \Gamma^m A(\hat{Y})$. The solutions of the projectivized flow $\hat{Y}' = \Gamma^m A(\hat{Y})$ are obtained from the linear flow by forming the span of m independent solutions. An important observation is that an m-dimensional eigenspace E of A is A-invariant, and hence, E is a critical point of $\Gamma^m A$. Furthermore, if A is hyperbolic with m (resp. $d-m$) eigenvalues of positive (resp. negative) real part, and if E is the eigenspace associated with the unstable eigenvalues, then E is an attracting rest point for the projectivized flow on $\hat{Y}' = \Gamma^m A(\hat{Y})$. See also Appendix A.

For the asymptotic matrices $\mathbb{A}^{\pm}_{\kappa}$ of $\mathbb{A}_\kappa(u)$, the unstable subspaces $U^{\pm}_{\kappa}$ are attractors for the flow which $\mathbb{A}^{\pm}_{\kappa}$ induces on $\mathcal{G}^{2n}_n(\mathbb{C})$. We projectivize the eigenvalue equation in the augmented system (2.10) and the flow on $\mathbb{R}^n \times \mathcal{G}^{2n}_n(\mathbb{C})$ becomes

$$\begin{aligned} u' &= f(u) - su - c, \\ X'_\kappa &= \Gamma^n \mathbb{A}_\kappa(u)(X_\kappa). \end{aligned} \tag{2.11}$$

The solution $X^-_\kappa(u)$ tends to the unstable subspace U^-_κ of $\mathbb{A}^-_\kappa$ as $u \to u^-$. The *unstable Evans bundle* $\mathcal{H}^-$ will be defined as

$$\begin{aligned} \mathcal{H}^- : \Omega &\to \mathcal{G}^{2n}_n(\mathbb{C}), \\ \kappa &\mapsto X^-_\kappa(u(0)) \end{aligned}$$

and is analytic in κ.
In the same manner we construct the *stable Evans bundle*

$$\mathcal{H}^+ : \Omega \to \mathcal{G}_n^{2n}(\mathbb{C}),$$
$$\kappa \mapsto X_\kappa^+(u(0))$$

as we consider the restpoint $(u^+, 0)$ and refer to the stable objects (stable directions, stable manifold (theorem)). The existence of an eigenfunction for $\kappa \in \Omega$ is equivalent to the nontrivial intersection of the subspaces

$$X_\kappa^-(u(0)) \cap X_\kappa^+(u(0)) \neq \{0\}.$$

We note that $\tau = 0$ as intersection point with the spaces $X_\kappa^\pm(\tau)$ was chosen arbitrarily.

In order to prove that there is no point spectrum in the punctured closed right half plane $\mathbb{C}_\bullet^+$ we would need Ω to contain a neighbourhood of 0. In the last section we proved that the curves in $\mathbb{C}$ where $\mathbb{A}_\kappa^\pm$ have pure imaginary eigenvalues define the boundary of $\sigma_e(L)$ and intersect the closed right half plane $\mathbb{C}^+$ (only) in 0. In Lemma 2.1 we even showed that 0 is an eigenvalue. Thus consistent splitting breaks down in (a neighbourhood of) the origin.
We consider small amplitude shock waves which we discovered through a suitable scaling. We will show that the slow-fast structure of the profile equation carries over to the eigenvalue problem. The slow-fast structure depends on the size of the spectral parameter κ relative to the shock strength ϵ, introducing $\kappa = \epsilon^2\zeta$, we distinguish two regions in $\mathbb{C}^+$:

1. Inner problem: For $|\zeta| \leq \rho$, $\rho > 0$ the equations governing the eigenvalue problem have an intricate slow-fast structure and the Evans bundles can be constructed as Whitney sums of lower dimensional subbundles related to the different time scales

$$\mathcal{H}_\epsilon^+ = \mathcal{H}_\epsilon^{+,s} \oplus \mathcal{H}_\epsilon^{+,f}, \quad \mathcal{H}_\epsilon^- = \mathcal{H}_\epsilon^{-,s} \oplus \mathcal{H}_\epsilon^{-,f}.$$

   The slow subbundle $\mathcal{H}_\epsilon^{+,s}$ can be decomposed even further into $\mathcal{H}_\epsilon^{+,s} = \mathcal{H}_\epsilon^{+,ss} \oplus \mathcal{H}_\epsilon^{+,sf}$ and we will prove that the subbundle $\mathcal{H}_\epsilon^{+,sf}$ will carry the stability information.

2. Outer problem: $|\zeta| \geq \rho$, $\rho > 0$ This is similar to the usual situation in Evans function theory where $|\kappa| \geq c$, $c > 0$ large. For large modulus of the spectral parameter κ the eigenvalue problem is approximately constant coefficient [AGJ]. Due to the scaling the analysis is not trivial as the essential spectrum approaches the entire imaginary axis of the ζ-plane as ϵ tends to zero. However we will show that the Evans bundle don't intersect.

We allow the small parameter ϵ to tend to zero and compute the limits $\mathcal{H}_0^{\pm,s}$, $\mathcal{H}_0^{\pm,f}$ of the perturbed summands. They will be found to be suspensions of Evans bundles $\mathcal{H}_{0,red}^{\pm}$ of a smaller problem. The analytic convergence of the Evans bundles means that in fact we can study the zero limit objects to decide upon the stability of the small amplitude shock waves. We will state the main theorem which is the analogon of the result Freistühler and Szmolyan proved for viscous conservation laws in the paper [FS].

Theorem 2.4. *("Reduction Lemma", version for bundles)*
Consider a family

$$\phi_\epsilon : \mathbb{R} \to U, \phi_\epsilon(\pm\infty) = u_\epsilon^{\pm}, \quad 0 < \epsilon < \epsilon_0$$

of shock waves for (1.15) *whose end states satisfy*

$$u_\epsilon^{\pm} = u_* \pm \epsilon(r(u_*) + O(\epsilon))$$

and the associated unstable and stable Evans bundles $\mathcal{H}_\epsilon^-, \mathcal{H}_\epsilon^+ : \mathbb{C}^+ \to \mathcal{G}_n^{2n}(\mathbb{C})$. *Then the scaled versions* $H_\epsilon^{\pm} : \mathbb{C}^+ \to \mathcal{G}_n^{2n}(\mathbb{C})$ *defined by*

$$H_\epsilon^{\pm}(\zeta) = \mathcal{H}_\epsilon^{\pm}(\epsilon^2\zeta)$$

have the following properties.
(i) Let

$$\phi_0 : \mathbb{R} \to \mathbb{R}, \quad \phi_0(\pm\infty) = \pm 1,$$

be the shock wave for the scalar viscous conservation law (Burgers equation)

$$v_t - (v^2)_x = v_{xx}$$

and $\mathcal{H}_{0,red}^-, \mathcal{H}_{0,red}^+ : \mathbb{C}_\bullet^+ \to \mathcal{G}_1^2(\mathbb{C})$ *be unstable and stable Evans bundles for the scalar shock wave* ϕ_0. *The scaled Evans bundles* $H_\epsilon^{\pm}$ *converge as analytic functions*

$$\lim_{\epsilon\to 0} H_\epsilon^{\pm} = H_0^{\pm}$$

where $H_0^{\pm}$ *denote suspensions of* $\mathcal{H}_{0,red}^{\pm}$ *in* $\mathcal{G}_n^{2n}(\mathbb{C})$, *namely with respect to appropriate coordinates on* $\mathbb{C}^{2n}$

$$\begin{aligned} H_0^-(\zeta) &= (\mathcal{H}_{0,red}^-(\zeta) \times \{(0,0)\}^{n-1}) \oplus (\{(0,0)\} \times (\mathbb{C} \times \{0\})^{n-1}), \\ H_0^+(\zeta) &= (\mathcal{H}_{0,red}^+(\zeta) \times \{(0,0)\}^{n-1}) \oplus (\{(0,0)\} \times (\{0\} \times \mathbb{C})^{n-1}). \end{aligned}$$

(ii) There exist $\bar\rho > 0$ *and* $\bar\epsilon > 0$ *such that* $H_\epsilon^-(\zeta) \cap H_\epsilon^+(\zeta) = \{0\}$ *for all* ϵ *with* $0 \le \epsilon \le \bar\epsilon$ *and all* $\zeta \in \mathbb{C}^+$ *with* $|\zeta| \ge \bar\rho$.

The profile equation for the travelling wave ϕ_0 with speed $s = 0$ of the Burgers equation is

$$u' = 1 - u^2. \tag{2.12}$$

The corresponding eigenvalue problem can be written as the first order system of ordinary differential equations

$$\begin{aligned} p' &= -2\phi_0 p - q, \\ q' &= -\kappa p, \end{aligned} \tag{2.13}$$

$\kappa \in \mathbb{C}$. The travelling waves for the viscous Burgers equation are known to be spectrally stable. Hence the Evans bundles for the profile ϕ_0 don't intersect. We conclude from Assertion (i) that also the Evans bundles of profiles for the Jin-Xin model don't intersect. We state the result

Corollary 2.1. *The small-amplitude profiles of the Jin-Xin model* (1.9) *are spectrally stable.*

The results of Mascia and Zumbrun [MZ1], [MZ2] prove that spectral stability of profiles of a relaxation model implies their nonlinear stability. Hence the small-amplitude profiles in the Jin-Xin model are even nonlinearly stable.

Chapter 3

Inner problem

In the first step we will show that the augmented system (2.10) is singularly perturbed and reveal a slow-fast structure with a suitable scaling. This allows us to decompose the stable space $S_\epsilon^+(\kappa)$ and the unstable space $U_\epsilon^-(\kappa)$ in fast and slow directions which carries over to the Evans bundles. The Evans bundles thus can be constructed as Whitney sums of subbundles related to the different time scales

$$H_\epsilon^+ = H_\epsilon^{+,s} \oplus H_\epsilon^{+,f}, \quad H_\epsilon^- = H_\epsilon^{-,s} \oplus H_\epsilon^{-,f}$$

In the first section we will make the slow-fast structure of the augmented system explicit, as we introduce a scaling to investigate the singular solution. We still assume A1-A3 and make use of the simplifications S1-S4.

3.1 Scaling and slow-fast system

The eigenvalue problem

$$\begin{aligned} p' &= Df(u)p - s\kappa p - (\kappa+1)q, \\ q' &= sDf(u)p - a^2\kappa p - s(\kappa+1)q, \end{aligned} \tag{3.1}$$

reduces at the point of interest $\kappa = 0$ and at the trivial solution $u \equiv 0$ of the profile equation for $s = 0$, $c = 0$ to

$$\begin{aligned} p' &= Df(0)p - q, \\ q' &= 0. \end{aligned} \tag{3.2}$$

The upper triangular matrix

$$\begin{pmatrix} Df(0) & -I_n \\ 0_n & 0_n \end{pmatrix}$$

is singular with 0 an eigenvalue of multiplicity $n+1$. In order to simplify matters we introduce a new variable $y := q - sp$. The equations for the new variables are

$$\begin{pmatrix} p \\ y \end{pmatrix}_\tau = \underbrace{\begin{pmatrix} Df(\bar{u}) - s(2\kappa+1)I_n & -(\kappa+1)I_n \\ -(a^2 - s^2)\kappa I_n & 0_n \end{pmatrix}}_{:=\mathbb{A}_\kappa(u,s)} \begin{pmatrix} p \\ y \end{pmatrix} \tag{3.3}$$

The boundary conditions are $\lim_{\tau\to\pm\infty} p(\tau) = 0$ and $\lim_{\tau\to\pm\infty} y(\tau) = 0$. The matrix $\mathbb{A}_\kappa$ is clearly analytic in κ and $\mathcal{C}^1$ in τ because f is smooth. The coefficients of $\mathbb{A}_\kappa(\tau)$ tend to constants as $\tau \to \pm\infty$, since $\lim_{\tau\to\pm\infty} Df(\bar{u}(\tau)) = Df(u^\pm)$ and we denote the limits of $\mathbb{A}_\kappa$ with $\mathbb{A}_\kappa^\pm$ at $\tau = \pm\infty$.
The profile equation (1.15) combined with the transformed eigenvalue problem (3.3) gives a coupled autonomous non-linear system of ordinary differential equations

$$\begin{aligned} u' &= f(u) - su - c, \\ p' &= Df(u)p - s(2\kappa+1)p - (\kappa+1)y, \\ y' &= -(a^2 - s^2)\kappa p. \end{aligned} \tag{3.4}$$

We introduce the scaling which extends (1.17) to the eigenvalue problem

$$u = \epsilon\tilde{u}, \quad s = \epsilon\tilde{s}, \quad c = \epsilon^2\tilde{c}, \quad y = \epsilon\tilde{y}, \quad \kappa = \epsilon^2\zeta \tag{3.5}$$

to investigate the situation in the neighbourhood of the origin. To be able to use perturbation arguments we need ζ to be bounded and with $\varrho_1 > 0$ chosen arbitrarly, we consider from now on only spectral parameters $\zeta \in D_{\varrho_1} := \{\zeta \in \mathbb{C}^+ \big| |\zeta| \le \varrho_1\}$. From

$$\begin{aligned} \epsilon\tilde{u}' &= f(\epsilon\tilde{u}) - \epsilon^2\tilde{s}\tilde{u} - \epsilon^2\tilde{c}, \\ p' &= Df(\epsilon\tilde{u})p - \epsilon\tilde{s}(\epsilon^2\zeta+1)p - \epsilon(\epsilon^2\zeta+1)\tilde{y}, \\ \epsilon\tilde{y}' &= -\big(a^2 - (\epsilon\tilde{s})^2\big)\epsilon^2\zeta p, \end{aligned} \tag{3.6}$$

and by using the Taylor expansion of f (1.19) we obtain

$$\begin{aligned} \tilde{u}' &= Df(0)\tilde{u} + O(\epsilon), \\ p' &= Df(0)p + O(\epsilon), \\ \tilde{y}' &= O(\epsilon). \end{aligned} \tag{3.7}$$

The system is in the standard form of singularly perturbed ordinary differential equations on the fast time scale. The Jacobian matrix $Df(0)$ is diagonal and for notational convenience we drop the tilde

$$\begin{aligned} u_i' &= \lambda_i(0)u_i + O(\epsilon), \\ p_i' &= \lambda_i(0)p_i + O(\epsilon), \\ y_i' &= O(\epsilon), \end{aligned} \tag{3.8}$$

$i = 1, \dots, n$. The assumption $\lambda_k(0) = 0$ implies a slow-fast structure of the problem. We get that u_k, p_k and y_i, $i = 1, \dots, n$ are the *slow variables* whereas u_i, p_i, $i = 1, \dots, n$, $i \neq k$ are the *fast variables*. The system (3.8) for $\epsilon = 0$, i.e. the *layer problem*, is in equilibrium if $u_i = 0$, $p_i = 0$, $i = 1, \dots, n$, $i \neq k$. This defines a manifold of equilibria

$$\mathcal{M}_{0,\zeta} = \{(u_i, p_i, y_i)_{i=1,\dots,n} \big| u_i = p_i = 0,\ i = 1, \dots, n;\ i \neq k\} \subset \mathbb{R}^n \times \mathbb{C}^{2n}.$$

The *critical manifold* $\mathcal{M}_{0,\zeta}$ is normally hyperbolic as the eigenvalues $\lambda_i(0)$, $i = 1, \dots, n$, $i \neq k$ are non-zero with the assumptions $\lambda_k(0) = 0$ and strict hyperbolicity. Thus Fenichel theory ([F2],[J2]) shows that $\mathcal{M}_{0,\zeta}$ smoothly perturbs to a *slow manifold* $\mathcal{M}_{\epsilon,\zeta}$ which has a parameterization

$$\begin{aligned} \mathcal{M}_{\epsilon,\zeta} = \{(u_i, p_i, y_i)_{i=1,\dots,n} \big| & u_k \in \mathbb{R},\ u_i = \epsilon h_i(u_k, s, c, \epsilon),\ h_i \text{ smooth}, \\ & p_k \in \mathbb{C},\ p_i = \epsilon P_i(u_k, p_k, y, \epsilon, \zeta),\ P_i \text{ smooth}, \\ & y \in \mathbb{C}^n,\ i = 1, \dots, n;\ i \neq k\} \subset \mathbb{R}^n \times \mathbb{C}^{2n} \end{aligned} \tag{3.9}$$

and is invariant under the flow of the scaled augmented system (3.6). We point out that the profile equation decouples from the rest of the system which is reflected in the independence of h_i from p, y. See also Chapter 1. In order to switch to the *slow time* scale $\hat{\tau}$ we rescale τ with ϵ and denote with "$\cdot$" the differentiation with respect to the slow time variable $\hat{\tau} = \epsilon\tau$. This yields the *slow system*

$$\begin{aligned} \dot{u} &= \frac{1}{\epsilon^2} f(\epsilon u) - su - c \\ \dot{p} &= \frac{1}{\epsilon} Df(\epsilon u)p - s(2\epsilon^2\zeta + 1)p - (\epsilon^2\zeta + 1)y \\ \dot{y} &= -\big(a^2 - (\epsilon s)^2\big)\zeta p \end{aligned}$$

The profile equation was analyzed in Lemma 1.1. In a similar way we simplify the eigenvalue problem and obtain

$$
\begin{aligned}
\epsilon \dot{u}_i &= \lambda_i(0) u_i + O(\epsilon),\\
\dot{u}_k &= \frac{1}{2}\frac{\partial^2 f_k}{\partial u_k^2}(0) u_k^2 - s u_k - c_k + O(\epsilon),\\
\epsilon \dot{p}_i &= \lambda_i(0) p_i + O(\epsilon),\\
\dot{p}_k &= \frac{\partial^2 f_k}{\partial u_k^2}(0) u_k p_k - s p_k - y_k + O(\epsilon),\\
\dot{y}_i &= O(\epsilon),\\
\dot{y}_k &= -a^2 \zeta p_k + O(\epsilon),
\end{aligned}
$$

$i = 1, \dots, n$, $i \neq k$. In Remark 1.3 we showed the problem can be transformed to the case $A = \frac{1}{2}\frac{\partial^2 f_k}{\partial u_k^2}(0) = -1$, $u_k^{\pm} = \pm 1$, $s = 0$ and $c_k = -1$. In the following proposition we will collect our results for this special case and restrict our attention to the part of the slow manifold V_ϵ from Lemma 1.1 with $u_k \in J = [-1, 1]$, that corresponds to the profiles and its endpoints.

Proposition 3.1. *(a) Let $\varrho > 0$ be arbitrary. There exists $\epsilon_1 > 0$ such that for every ζ with $|\zeta| \leq \varrho$ and every ϵ with $0 < \epsilon \leq \epsilon_1$. the system* (3.6) *possess a unique invariant manifold $\mathcal{M}_{\epsilon,\zeta}$ such that*

$$
\mathcal{M}_{\epsilon,\zeta} = \bigcup_{\nu \in J} \{u(\nu)\} \times \mathcal{M}_{\epsilon,\zeta}^{\nu},
$$

where $u(\nu)$ denotes the point on the profile at which $u_k = \nu$, and with linear spaces of the form

$$
\begin{aligned}
\mathcal{M}_{\epsilon,\zeta}^{\nu} = \{ \big((p_k, y_k), (p_i, y_i)_{i \neq k} \big) \,\big|\, \\
p_i = \epsilon P_i(\nu, p_k, y_1, \dots, y_n, \epsilon, \zeta), i \neq k \} \subset \mathbb{C}^{2n},
\end{aligned}
$$

where P_i are smooth functions, linear in $p_k, y_1, \dots, y_n$ and analytic in ζ.

(b) With respect to the coordinates $\big(u_k, p_k, y_k, (y_i)_{i \neq k}\big)$ running in the fixed domain $J \times \mathbb{C}^{n+1}$, the (slow) flow on $\mathcal{M}_{\epsilon,\zeta}$ is governed by the equations

$$
\begin{aligned}
\dot{u}_k &= -u_k^2 + 1 + O(\epsilon),\\
\dot{p}_k &= -2 u_k p_k - y_k + O(\epsilon),\\
\dot{y}_k &= -a^2 \zeta p_k + O(\epsilon),\\
\dot{y}_i &= O(\epsilon),
\end{aligned}
\tag{3.10}
$$

and depends smoothly on ϵ for $0 \leq \epsilon \leq \epsilon_1$. The limiting slow flow on $\mathcal{M}_{0,\zeta}$ is represented by

$$\begin{aligned} \dot{u}_k &= -u_k^2 + 1, \\ \dot{p}_k &= -2u_k p_k - y_k, \\ \dot{y}_k &= -a^2 \zeta p_k, \\ \dot{y}_i &= 0. \end{aligned} \tag{3.11}$$

(c) *The (fast) flow outside $\mathcal{M}_{\epsilon,\zeta}$ is governed by system* (3.6). *The limiting (fast) flow outside $\mathcal{M}_{\epsilon,\zeta}$ is represented by*

$$\begin{aligned} u_i' &= \lambda_i(0) u_i, \\ p_i' &= \lambda_i(0) p_i, \end{aligned}$$

$i = 1, \ldots, n$, $i \neq k$ *with* $u_k, p_k, y_k, y_i =$ *constant.*

Remark. The claimed uniqueness of $\mathcal{M}_{\epsilon,\zeta}$ follows from the requirement that the fibres $\mathcal{M}^{\nu}_{\epsilon,\zeta}$ are linear spaces. $\square$

Remark. We point out that the slow flow on $\mathcal{M}_{0,\zeta}$ (3.11) consists of the augmented system of a profile of the viscous Burgers equation combined with the trivial flow $y_i = 0$, $i = 1, \ldots, n$, $i \neq k$. $\square$

In the following we will take full advantage of the fact that the profile equation can be reduced to

$$\chi' = \epsilon \big(1 - \chi^2\big) g_\epsilon(\chi) \tag{3.12}$$

with $g_\epsilon(\chi) = 1 + O(\epsilon)$. See Remark 1.3. We denote with $\chi : \mathbb{R} \to (-\infty, +\infty)$ the solution of this equation with $\chi^{\pm} = \pm 1$ the respective endpoints at $\pm\infty$. We note that with a solution χ also the shifted versions are solutions of the problem and we choose the one satisfying $\chi(0) = 0$.
We introduce the notation

$$\mathbb{A}_{\epsilon,\zeta}[\chi] \equiv \mathbb{A}_{\epsilon,\zeta}\big(\epsilon u(\chi), s\big)$$

and rewrite the *scaled augmented system* as

$$\begin{aligned} \chi' &= \epsilon \big(1 - \chi^2\big) g_\epsilon(\chi) \\ \begin{pmatrix} p \\ y \end{pmatrix}' &= \mathbb{A}_{\epsilon,\zeta}[\chi] \begin{pmatrix} p \\ y \end{pmatrix}. \end{aligned}$$

In the next step we want to investigate the eigenvalues of $\mathbb{A}_{\epsilon,\zeta}$ and the related eigenvectors.

3.2 Eigenvalues of $\mathbb{A}_{\epsilon,\zeta}$

The matrix

$$\mathbb{A}_{\epsilon,\zeta}(\epsilon u, \epsilon s) := \begin{pmatrix} Df(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)I_n & -(\epsilon^2\zeta + 1)I_n \\ -(a^2 - (\epsilon s)^2)\epsilon^2\zeta I_n & 0_n \end{pmatrix}$$

has associated to it the eigenvalue equation

$$\det\big(\mathbb{A}_{\epsilon,\zeta}(\epsilon u, \epsilon s) - \mu I_{2n}\big) = 0,$$

which specializes with $Df(\epsilon u)$ $\mathbb{R}$-diagonalizable for ϵ small to

$$\prod_{j=1}^{n} \Big(\mu^2 - \big(\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)\big)\mu - \epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big)\Big) = 0.$$

Thus an eigenvalue $\mu_{j,\epsilon,\zeta},\ j = 1, \dots, n$ has to fulfill

$$\mu_{j,\epsilon,\zeta}^2 - \big(\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)\big)\mu_{j,\epsilon,\zeta} - \epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big) = 0. \quad (3.13)$$

We obtain for the eigenvalues

$$\begin{aligned} \mu_{j,\epsilon,\zeta}^{\pm}(\epsilon u, \epsilon s) &= \frac{\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \\ &\quad \pm \sqrt{\frac{\big(\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)\big)^2}{4} + \epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big)} \\ &= \frac{\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \\ &\quad \pm \sqrt{\frac{\big(\lambda_j(\epsilon u) - \epsilon s\big)^2 + 4\epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)a^2 - 4\lambda_j(\epsilon u)\epsilon^3 s\zeta}{4}}. \end{aligned}$$

These eigenvalues have the property

Lemma 3.1. *Under the subcharacteristic condition $a > \max\big(\operatorname{spec}\big(Df(u_*)\big), |s|\big)$ and ϵ sufficiently small the domain $\mathbb{C}^+_\bullet$ has consistent splitting, i.e.*

$$\Re\big(\mu_{j,\epsilon,\zeta}^{-}(\epsilon u, \epsilon s)\big) < 0 < \Re\big(\mu_{j,\epsilon,\zeta}^{+}(\epsilon u, \epsilon s)\big) \quad (3.14)$$

for ϵ sufficiently small, $\zeta \in \mathbb{C}^+_\bullet$.

Proof. The equation (3.13) is equivalent to the equation of the spectral parameter κ (2.6) except that the parameters are scaled. We conclude with Lemma 2.2 that there is no purely imaginary solution μ for $\zeta \in \mathbb{C}^+_\bullet$. The continuity of the roots in ζ proves that the number of eigenvalues of $\mathbb{A}^{\pm}_{\epsilon,\zeta}$ with positive (resp. negative) real part is constant and so are the dimensions of the stable and unstable eigenspaces. Whenever ζ is real and positive, we note that the coefficients of equation (3.13) are real and that the product of its roots

$$\mu^{+}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s)\mu^{-}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = -\epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big) < 0$$

is negative. Hence these roots are real and of opposite signs and this is true for all of $\mathbb{C}^{+}_{\bullet}$. $\square$

The eigenvalues $\mu^{\pm}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s)$, $j = 1, \dots, n$ have the associated eigenvectors

$$R^{\pm}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = \operatorname{span}\left\{ \begin{pmatrix} r_j(\epsilon u, \epsilon s) \\ \frac{\mu^{\mp}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s)}{\epsilon^2\zeta+1} r_j(\epsilon u, \epsilon s) \end{pmatrix} \right\} \subset \mathbb{C}^{2n} \tag{3.15}$$

where $r_j(\epsilon u, \epsilon s)$ are the smooth eigenvectors of $Df(\epsilon u)$ associated with the eigenvalues $\lambda_j(\epsilon u)$, $j = 1, \dots, n$. The eigenvalues $\mu^{\pm}_{j}(\epsilon u, \epsilon s)$ and the associated eigenvectors $R^{\pm}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s)$ are analytic functions in $\zeta \in \mathbb{C}^{+}$ as long as $\nu = \nu^{\pm}$.

Lemma 3.2. *As long as $\epsilon > 0$ and $\zeta \in \mathbb{C}^{+}_{\bullet}$*

$$S^{+}_{\epsilon} = \bigoplus_{j=1}^{n} R^{-}_{j,\epsilon,\zeta}[\nu^{+}] \text{ is the } n\text{-dimensional stable space of } \mathbb{A}_{\epsilon,\zeta}[\nu^{+}] \tag{3.16}$$

and

$$U^{-}_{\epsilon} = \bigoplus_{j=1}^{n} R^{+}_{j,\epsilon,\zeta}[\nu^{-}] \text{ is the } n\text{-dimensional unstable space of } \mathbb{A}_{\epsilon,\zeta}[\nu^{-}]. \tag{3.17}$$

Proof. We consider the stable space S^{+}_{ϵ}. The same arguments carry over to the case of the unstable space U^{-}_{ϵ}. With Lemma 3.1 we have that $\mu^{-}_{j,\epsilon,\zeta}[\nu^{+}]$ is the stable spectrum and the related eigenvectors $R^{-}_{j,\epsilon,\zeta}[\nu^{+}]$ span the stable space of $\mathbb{A}_{\epsilon,\zeta}[\nu^{+}]$. The eigenvectors $r_j[\nu]$ of $Df(\epsilon u)$ are linearly independent as Df is $\mathbb{R}$-diagonalizable. This property carries over to the eigenvectors $R_{i,\epsilon,\zeta}, R_{j,\epsilon,\zeta}, i \neq j$. Additionally $\mathbb{C}^{+}_{\bullet}$ has consistent splitting for $\epsilon > 0$ and the eigenvalues satisfy $\mu^{-}_{j,\epsilon,\zeta} \neq \mu^{+}_{j,\epsilon,\zeta}$, $j = 1, \dots, n$. Therefore $R^{-}_{j,\epsilon,\zeta}$ and $R^{+}_{j,\epsilon,\zeta}$ are linearly independent as well. $\square$

In the next step we want to shed some light on the order of magnitude of the eigenvalues.

Lemma 3.3. *For sufficiently small $\epsilon > 0$ and every $\varrho > 0$ we have with $\zeta \in D_\varrho$ and all $\nu \in J$: The n-1 fast eigenvalues*

$$\mu^-_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) \quad (j < k), \text{ and } \mu^+_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) \quad (j > k)$$

continue to satisfy the inequalities in Lemma 3.1 and are uniformly bounded away from 0, while the remaining n+1 slow eigenvalues

$$\mu^-_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) \quad (j \geq k), \text{ and } \mu^+_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) \quad (j \leq k)$$

are of order $O(\epsilon)$.

Proof. The eigenvalues $\mu^\pm_{j,\epsilon,\zeta}(\epsilon u, \epsilon s)$, $j = k+1, \dots, n$ are defined as

$$\mu^\pm_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = \frac{\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \pm \sqrt{\frac{\big(\lambda_j(\epsilon u) - \epsilon s\big)^2}{4} + \epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)a^2 - \lambda_j(\epsilon u)\epsilon^3 s\zeta}$$

The eigenvalues λ_j, $j = k+1, \dots, n$ satisfy $\lambda_j(0) > 0$ and we conclude $\lambda_j(\epsilon u) - \epsilon s = \lambda_j(0) + O(\epsilon) > 0$ for ϵ sufficiently small. We factor $\frac{\lambda_j(\epsilon u) - \epsilon s}{2}$ out of the squareroot and derive

$$= \frac{\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \pm \left|\frac{\lambda_j(\epsilon u) - \epsilon s}{2}\right| \sqrt{1 + \frac{4\epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)a^2 - 4\lambda_j(\epsilon u)\epsilon^3 s\zeta}{\big(\lambda_j(\epsilon u) - \epsilon s\big)^2}}.$$

We expand the squareroot

$$= \frac{\lambda_j(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \pm \frac{\lambda_j(\epsilon u) - \epsilon s}{2}\Big(1 + O(\epsilon^2|\zeta|)\Big)$$

and obtain

$$\mu^+_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = \lambda_j(0) + O(\epsilon),$$
$$\mu^-_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = O\big(\epsilon^2|\zeta|\big).$$

The eigenvalues $\lambda_j(0)$, $j = 1, \dots, k-1$ satisfy $\lambda_j(0) < 0$ and we conclude

$\lambda_j(\epsilon u) - \epsilon s = \lambda_j(0) + O(\epsilon) < 0$ for ϵ sufficiently small. In the same way as above we prove for the eigenvalues $\mu^{\pm}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s)$, $j = 1, \dots, k-1$ the results

$$\mu^{+}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = O\big(\epsilon^2|\zeta|\big),$$
$$\mu^{-}_{j,\epsilon,\zeta}(\epsilon u, \epsilon s) = \lambda_j(0) + O(\epsilon).$$

The eigenvalues $\mu^{\pm}_{k,\epsilon,\zeta}$ are defined as

$$\mu^{\pm}_{k,\epsilon,\zeta}(\epsilon u, \epsilon s) = \frac{\lambda_k(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \pm \sqrt{\frac{\big(\lambda_k(\epsilon u) - \epsilon s(2\epsilon^2\zeta + 1)\big)^2}{4} + \epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big)}$$

The eigenvalue λ_k satisfies $\lambda_k(0) = 0$ and we expand $\lambda_k(u) = \frac{\partial f_k}{\partial u_k}(u)$ around 0, $\lambda_k(\epsilon u) = \lambda_k(0) + \nabla\lambda_k(0)\epsilon u + O(\epsilon^2)$ with $\nabla\lambda_k(0) = \Big(\frac{\partial^2 f_k}{\partial u_1 \partial u_k}, \dots, \frac{\partial^2 f_k}{\partial u_n \partial u_k}\Big)(0)$. The parameterization of V_ϵ shows that $\frac{\partial^2 f_k}{\partial u_k^2}(0)u_k = 2Au_k$ is the leading term in magnitude and we derive $\lambda_k(\epsilon u) = 2A\epsilon u_k + O(\epsilon^2)$.

$$= \frac{\epsilon 2Au_k + O(\epsilon^2) - \epsilon s(2\epsilon^2\zeta + 1)}{2} \pm \sqrt{\frac{\big(\epsilon 2Au_k + O(\epsilon^2) - \epsilon s(2\epsilon^2\zeta + 1)\big)^2}{4} + \epsilon^2\zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big)}.$$

We factor out ϵ and obtain

$$= \epsilon\left(\frac{2Au_k - s + O(\epsilon)}{2} \pm \sqrt{\frac{\big(2Au_k - s + O(\epsilon)\big)^2}{4} + \zeta\big(\epsilon^2\zeta + 1\big)\big(a^2 - (\epsilon s)^2\big)}\right)$$

The expression in the parathensis is bounded since $|\zeta| < \varrho$. Thus the eigenvalues $\mu^{\pm}_{k,\epsilon,\zeta}$ satisfy

$$\mu^{\pm}_{k,\epsilon,\zeta}(\epsilon u, \epsilon s) = O(\epsilon).$$

$\square$

Remark 3.1. In Remark 1.3 we showed that our problem can be transformed to the case $A = -1$, $u_k^{\pm} = \pm 1$, $s = 0$ and $c_k = -1$. The above result for $\mu^{\pm}_{k,\epsilon,\zeta}[\nu^{\pm}]$ specializes to

$$\mu^{\pm}_{k,\epsilon,\zeta}[\nu^-] = \epsilon\Big(\big(1 + O(\epsilon)\big) \pm \sqrt{\big(1 + O(\epsilon)\big)^2 + \zeta\big(\epsilon^2\zeta + 1\big)a^2} \quad \Big)$$

The expression $1 + O(\epsilon)$ is positive for ϵ small and we factor it out of the squareroot

$$= \epsilon\big(1 + O(\epsilon)\big)\left(1 \pm \sqrt{1 + \frac{\zeta\big(\epsilon^2\zeta + 1\big)a^2}{(1 + O(\epsilon))^2}} \quad \right).$$

The expression $\frac{\zeta(\epsilon^2\zeta+1)a^2}{(1+O(\epsilon))^2}$ is $O(|\zeta|)$, as $(\epsilon^2\zeta + 1)$, a^2, $(1 + O(\epsilon))^2$ are non-vanishing for $\epsilon \to 0$ and/or $|\zeta| \to 0$

$$= \epsilon\big(1 + O(\epsilon)\big)\Big(1 \pm \sqrt{1 + O(|\zeta|)}\Big).$$

In the same manner we obtain

$$\mu^{\pm}_{k,\epsilon,\zeta}[\nu^+] = \epsilon\big(1 + O(\epsilon)\big)\Big(-1 \pm \sqrt{1 + O(|\zeta|)}\Big).$$

□

We will distinguish between fast and slow eigenvalues and let for such $\epsilon > 0$

$$S^{\nu,f}_{\epsilon}(\zeta), \quad N^{\nu}_{\epsilon}(\zeta), \quad U^{\nu,f}_{\epsilon}(\zeta)$$

denote those invariant spaces of $\mathbb{A}_{\epsilon,\zeta}[\nu]$ that are associated with the eigenvalue sets

$$\{\mu_j^-, j < k\},\ \{\mu_j^-, j \geq k\} \cup \{\mu_j^+, j \leq k\},\ \{\mu_j^+, j > k\}$$

respectively.

Despite the fact that some of the eigenvalues coincide in various ways, all eigenvalues $\mu^{\pm}_{j,\epsilon,\zeta}[\nu]$ and their associated eigenvectors $R^{\pm}_{j,\epsilon,\zeta}[\nu]$ are analytic functions in $\zeta \in \mathbb{C}^+$ as long as $\nu = \nu^{\pm}$. Consistent with the above definitions

$$S^{+,f}_{\epsilon}(\zeta) = \bigoplus_{j=1}^{k-1} R^-_{j,\epsilon,\zeta}[\nu^+], \qquad U^{-,f}_{\epsilon}(\zeta) = \bigoplus_{j=k+1}^{n} R^+_{j,\epsilon,\zeta}[\nu^-], \tag{3.18}$$

we define

$$S^{+,s}_{\epsilon}(\zeta) := \bigoplus_{j=k}^{n} R^-_{j,\epsilon,\zeta}[\nu^+], \qquad U^{-,s}_{\epsilon}(\zeta) := \bigoplus_{j=1}^{k} R^+_{j,\epsilon,\zeta}[\nu^-]. \tag{3.19}$$

With these definitions

$$S_\epsilon^+ = S_\epsilon^{+,f} \oplus S_\epsilon^{+,s} \quad \text{and} \qquad U_\epsilon^- = U_\epsilon^{-,f} \oplus U_\epsilon^{-,s} \tag{3.20}$$

are the stable and unstable spaces of $\mathbb{A}_{\epsilon,\zeta}[\nu^+]$ and $\mathbb{A}_{\epsilon,\zeta}[\nu^-]$ respectively, as long as $\epsilon > 0$ and $\zeta \in \mathbb{C}^+_\bullet$. We will construct the scaled Evans bundles $H_\epsilon^+, H_\epsilon^-$ as sums of subbundles that are associated with the slow and fast components of the n-dimensional invariant spaces S_ϵ^+ and U_ϵ^-.
In the Section 3.3 we consider the fast directions which correspond to the spaces $S_\epsilon^{\nu,f}, U_\epsilon^{\nu,f}$. As the related eigenvalues are uniformly bounded away from zero, consistent splitting persists for $\zeta \in D_\varrho$ as ϵ tends to 0. It will be easy to construct the corresponding fast subbundles of the Evans bundles. In the Section 3.4 we consider the slow directions corresponding to the space N_ϵ^ν. The eigenvalues are of order $O(\epsilon)$. However we will find an additional slow-fast structure in the slow problem which will enable us to get around this problem by geometric singular perturbation methods. As we are interested in both cases in the spaces as an entity, we will consider the Grassmann version of the problem, see Appendix A.

3.3 Fast subbundles $H_\epsilon^{+,f}, H_\epsilon^{-,f}$

Theorem 3.1. *For every ϱ there exists ϵ_1 such that, for all $\epsilon \in [0, \epsilon_1]$, there exist unique smooth mappings*

$$S_\epsilon^{+,f} : D_\varrho \to \mathcal{G}_{k-1}^{2n}(\mathbb{C}), \quad H_\epsilon^{+,f} : D_\varrho \to \mathcal{G}_{k-1}^{2n}(\mathbb{C})$$

all analytic in $\zeta \in D_\varrho$ such that

1. *For each $\epsilon \in [0, \epsilon_1]$ and each $\zeta \in D_\varrho$, the solution X^+ of the system*

$$\begin{aligned} \chi' &= \epsilon\big(1 - \chi^2\big) g_\epsilon(\chi), \\ X' &= \Gamma^{k-1} \mathbb{A}_{\epsilon,\zeta}[\chi]\big(X\big), \end{aligned} \tag{3.21}$$

assuming data $X^+(0) = H_\epsilon^{+,f}(\zeta)$, converges at the right end to

$$X^+(+\infty) = S_\epsilon^{+,f}(\zeta).$$

2. *The two bundles depend smoothly on $\epsilon \in [0, \epsilon_1]$ and $H_0^{+,f}(\zeta) = S_0^{+,f}(\zeta)$.*

Proof. The system (3.21) for $\epsilon = 0$ reads

$$\chi' = 0, \quad X' = \Gamma^{k-1} \mathbb{A}_0[\chi]\big(X\big) \tag{3.22}$$

where $\mathbb{A}_0 = \operatorname{diag}\big(df(0), 0\big)$. The X equation decouples from the χ equation as $\mathbb{A}_0$ is constant and independent of χ. An equilibrium of

$$\big(X\big)' = \Gamma^{k-1}\mathbb{A}_0[\chi]\big(X\big)$$

corresponds to an $\mathbb{A}_0$ invariant space with dimension $k-1$. This is the number of eigenvalues with negative real part of $\mathbb{A}_0$ and S_0^f denotes the constant stable subspace of $\mathbb{A}_0$. This is consistent with the above definition of $S_\epsilon^{\nu,f}(\zeta)$ (3.18) for $\epsilon = 0$, as we showed in Lemma 3.3 that the related eigenvalues are $O(1)$ with negative real part. Thus

$$\mathcal{C}_0^{+,f} \equiv J \times \hat{S}_0^f \subset J \times \mathcal{G}_{k-1}^{2n}(\mathbb{C})$$

is the critical manifold of system (3.22). The spectrally isolated point $\hat{S}_0^f$, $\operatorname{spec}(\mathbb{A}_0|S_0^f) < \operatorname{spec}(\mathbb{A}_0|N_0 \cup U_0^f)$, is by virtue of Lemma A.1, Appendix A, a repellor. Thus the critical manifold $\mathcal{C}_0^{+,f}$ is normally hyperbolic and perturbs by Fenichel theory ([F1],[F2]) smoothly to a unique repelling slow manifold $\mathcal{C}_{\epsilon,\zeta}^{+,f}$ for the system (3.22) with ϵ sufficiently small. The curve $\mathcal{C}_{\epsilon,\zeta}^{+,f}$ is a trace of a unique solution $\big(\chi, X_{\epsilon,\zeta}^+\big)$ with

$$X_{\epsilon,\zeta}^+(\pm\infty) = \hat{S}^{\pm,f}(\zeta).$$

The intersection of $\mathcal{C}_{\epsilon,\zeta}^{+,f}$ with $\chi = 0$ depends smoothly on ϵ and analytically on $\zeta \in D_\varrho$. By denoting this intersection point with $H_{\epsilon,\zeta}^{+,f}$, we have constructed the fast stable subbundle. The asserted properties are an immediate consequence of our construction. □

Theorem 3.2. *For every ϱ there exists ϵ_1 such that, for all $\epsilon \in [0, \epsilon_1]$, there exist unique smooth mappings*

$$U_\epsilon^{-,f} : D_\varrho \to \mathcal{G}_{n-k}^{2n}(\mathbb{C}), \quad H_\epsilon^{-,f} : D_\varrho \to \mathcal{G}_{n-k}^{2n}(\mathbb{C})$$

all analytic in $\zeta \in D_\varrho$ such that

1. *For each $\epsilon \in [0, \epsilon_1]$ and each $\zeta \in D_\varrho$, the solution X^- of the system*

$$\begin{aligned} \chi' &= \epsilon\big(1-\chi^2\big)g_\epsilon(\chi), \\ X' &= \Gamma^{n-k}\mathbb{A}_{\epsilon,\zeta}[\chi]\big(X\big), \end{aligned} \tag{3.23}$$

assuming data $X^-(0) = H_\epsilon^{-,f}(\zeta)$ converges at the left end to

$$X^-(-\infty) = U_\epsilon^{-,f}(\zeta).$$

2. The two bundles depend smoothly on $\epsilon \in [0, \epsilon_1]$ and $H_0^{-,f}(\zeta) = U_0^{-,f}(\zeta)$.

Proof. In the same way as for the fast stable subbundle we can show that

$$\mathcal{C}_0^{-,f} \equiv J \times \hat{U}_0^f \subset J \times \mathcal{G}_{n-k}^{2n}(\mathbb{C})$$

is a critical manifold for the system (3.23) for $\epsilon = 0$

$$\chi' = 0, \quad X' = \Gamma^{n-k}\mathbb{A}_0[\chi]\big(X\big). \tag{3.24}$$

The constant and spectrally isolated point $\hat{U}_0^f$,
$\operatorname{spec}(\mathbb{A}_0|U_0^f) > \operatorname{spec}(\mathbb{A}_0|N_0 \cup S_0^f)$, is now by Lemma A.1 an attractor. Thus $\mathcal{C}_0^{-,f}$ is normally hyperbolic and perturbs by Fenichel theory smoothly to an unique attracting slow manifold $\mathcal{C}_{\epsilon,\zeta}^{-,f}$ for ϵ sufficiently small. The curve $\mathcal{C}_{\epsilon,\zeta}^{-,f}$ is a trace of a unique solution $\big(\chi, X_{\epsilon,\zeta}^{-}\big)$ with

$$X_{\epsilon,\zeta}^{-}(\pm\infty) = \hat{U}^{\pm,f}(\zeta).$$

We get the fast unstable subbundle $H_{\epsilon,\zeta}^{-,f}$ as the intersection point at $\chi = 0$ and note that it is smooth in ϵ and analytic in $\zeta \in D_\varrho$. The stated properties are again a consequence of the construction. □

3.4 Slow subbundles $H_\epsilon^{+,s}, H_\epsilon^{-,s}$

The problem of the slow directions is that the eigenvalues tend to zero as ϵ tends to zero. This is the reason why the consistent splitting of D_ϱ breaks down. We combine these directions to the neutral space

$$N_\epsilon^\nu = S_\epsilon^{\nu,s} \oplus U_\epsilon^{\nu,s}$$

to obtain the result

Lemma 3.4. *Consider the system*

$$\begin{aligned} \chi' &= \epsilon\big(1 - \chi^2\big)g_\epsilon(\chi), \\ Y' &= \Gamma^{n+1}\mathbb{A}_{\epsilon,\zeta}[\chi](Y) \end{aligned} \tag{3.25}$$

For $\epsilon = 0$ the curve

$$\mathcal{M}_0 \equiv J \times N_0 \subset J \times \mathcal{G}_{n+1}^{2n}(\mathbb{C}) \tag{3.26}$$

is a normally hyperbolic critical manifold of the reduced system, where N_0 denotes the constant value of $N_\epsilon^\nu(\zeta)$ for $\epsilon = 0$.

Proof. The layer problem, $\epsilon = 0$, reads

$$\chi' = 0, \quad Y' = \Gamma^{k-1}\mathbb{A}_0[\chi](Y) \tag{3.27}$$

with $\mathbb{A}_0 = \operatorname{diag}\big(df(0), 0\big)$. The Y equation decouples from the χ equation as $\mathbb{A}_0$ is constant and independent of χ. An equilibrium of the layer system (3.27) corresponds to an $n+1$-dimensional, $\mathbb{A}_0$ invariant subspace. The eigenvalue 0 has multiplicity $n+1$ and we denote with N_0 the corresponding subspace. This is consistent with the above definition of $N_\epsilon^\nu(\zeta)$ for $\epsilon = 0$. Thus we get the critical manifold

$$\mathcal{M}_0 \equiv J \times \hat{N}_0 \subset J \times \mathcal{G}_{n+1}^{2n}(\mathbb{C})$$

of system (3.27). The point $\hat{N}_0$ satisfies $\operatorname{spec}(\mathbb{A}_0|S_0^f) < \operatorname{spec}(\mathbb{A}_0|N_0) < \operatorname{spec}(\mathbb{A}_0|U_0^f)$, and a slightly extended version of lemma A.1 implies that $\hat{N}_0$ is a hyperbolic saddle. Thus $\mathcal{M}_0$ is normally hyperbolic. □

The normally hyperbolic critical manifold $\mathcal{M}_0$ of the layer problem perturbs by Fenichel theory [F1, F2] to an unique invariant curve $\mathcal{M}_{\epsilon,\zeta}$ of the perturbed system (3.25). The slow manifold $\mathcal{M}_{\epsilon,\zeta}$ consists of an orbit $\big(\chi, Y\big)$ together with its α-limit $\{-1\} \times N_\epsilon^-$ and ω-limit $\{+1\} \times N_\epsilon^+$. We denote with $\mathcal{M}_{\epsilon,\zeta}^\nu \subset \mathbb{C}^{2n}$ the subspace which corresponds to the point $Y_{\epsilon,\zeta}\big(\chi^{-1}(\nu)\big)$. Hence the one-dimensional manifold $J \times \mathcal{G}_{n+1}^{2n}(\mathbb{C})$ is identical with the $n+2$-dimensional manifold $\mathcal{M}_{\epsilon,\zeta} \subset J \times \mathbb{C}^{2n}$, which we described in Proposition 3.1. Our goal is to decompose the slow manifold $\mathcal{M}_{\epsilon,\zeta}$ into the manifolds connected to the stable and unstable Evans bundles. We will restrict our attention to the slow flow on $\mathcal{M}_{\epsilon,\zeta}$

$$\begin{aligned} \dot{\chi} &= \big(1 - \chi^2\big) g_\epsilon(\chi), \\ \dot{Y} &= \frac{1}{\epsilon}\mathbb{A}_{\epsilon,\zeta}[\chi]\big(Y\big). \end{aligned} \tag{3.28}$$

The slow eigenvalues of $\frac{1}{\epsilon}\mathbb{A}_{\epsilon,\zeta}$ satisfy

$$\begin{aligned} \mu_{j,\epsilon,\zeta}^-[\chi^\pm] &= O(\epsilon|\zeta|), \qquad j > k \\ \mu_{j,\epsilon,\zeta}^+[\chi^\pm] &= O(\epsilon|\zeta|), \qquad j < k \\ \mu_{k,\epsilon,\zeta}^\pm[\nu^-] &= \big(1 + O(\epsilon)\big)\Big(1 \pm \sqrt{1 + O(|\zeta|)}\Big), \\ \mu_{k,\epsilon,\zeta}^\pm[\nu^+] &= \big(1 + O(\epsilon)\big)\Big(-1 \pm \sqrt{1 + O(|\zeta|)}\Big), \end{aligned}$$

with Lemma 3.3 and Remark 3.1. Note that we had to divide the approximations by $\epsilon \neq 0$ as we consider the slow flow. We conclude that the slow problem has again a slow-fast structure. The manifold $\mathcal{M}_{\epsilon,\zeta}$ is spanned by the eigenvectors $R^{-}_{j,\epsilon,\zeta}[\chi]$, $j = k, \dots, n$ and $R^{+}_{j,\epsilon,\zeta}[\chi]$, $j = 1, \dots, k$. The fast directions are $R^{-}_{k,\epsilon,\zeta}[\chi^{+}]$, $R^{+}_{k,\epsilon,\zeta}[\chi^{-}]$ and we define

$$S^{+,sf}_{\epsilon}(\zeta) := R^{-}_{k,\epsilon,\zeta}[\chi^{+}], \qquad U^{-,sf}_{\epsilon}(\zeta) := R^{+}_{k,\epsilon,\zeta}[\chi^{-}]. \tag{3.29}$$

The slow directions are $R^{-}_{j,\epsilon,\zeta}[\chi]$, $j = k+1, \dots, n$, $R^{+}_{j,\epsilon,\zeta}[\chi]$, $j = 1, \dots, k-1$ and the related eigenvalues $\mu^{-}_{j,\epsilon,\zeta}$, $j = k+1, \dots, n$, $\mu^{+}_{j,\epsilon,\zeta}$, $j = 1, \dots, k-1$ are sensitive as ϵ and/or ζ tend to zero. We will have to distinguish two cases

1. $\zeta \in D_{\varrho_1}$, $\varrho_1 > 0$ sufficiently small.
2. $\zeta \in D_{\varrho_2}/D_{\varrho_1}$, $0 < \varrho_1 < \varrho_2$.

We consider the Grassmann versions of the slow flow on $\mathcal{M}_{\epsilon,\zeta}$ in order to construct the slow subbundles $H^{+,s}_{\epsilon}$ and $H^{-,s}_{\epsilon}$.

Slow subbundle $H^{+,s}_{\epsilon}$

Theorem 3.3. *For every ϱ there exists ϵ_1 such that for all $\epsilon \in [0, \epsilon_1]$, there exist unique smooth mappings*

$$S^{+,s}_{\epsilon} : D_{\varrho} \to \mathcal{G}^{2n}_{n-k+1}(\mathbb{C}), \quad H^{+,s}_{\epsilon} : D_{\varrho} \to \mathcal{G}^{2n}_{n-k+1}(\mathbb{C})$$

all analytic in $\zeta \in D_{\varrho}$. Additionally for each $\epsilon \in [0, \epsilon_1]$ and each $\zeta \in D_{\varrho}$, the solution Y^{+} of the system

$$\begin{aligned} \chi' &= \epsilon\big(1 - \chi^2\big) g_{\epsilon}(\chi), \\ Y' &= \Gamma^{n-k+1}\mathbb{A}_{\epsilon,\zeta}[\chi]\big(Y\big) \end{aligned} \tag{3.30}$$

assuming data $Y^{+}(0) = H^{+,s}_{\epsilon}(\zeta)$, converges at the right end to

$$Y^{+}(+\infty) = S^{+,s}_{\epsilon}(\zeta).$$

Proof. We first consider the case $\zeta \in D_{\varrho_1}$, $\varrho_1 > 0$ sufficiently small. In this regime we take $\varrho = |\zeta|$ as our small perturbation parameter, since with $\varrho = 0$ also the eigenvalue $\mu^{+}_{k,\epsilon,\zeta}[\nu^{+}] = \big(1 + O(\epsilon)\big)\big(-1 + \sqrt{1 + O(|\zeta|)}\big)$ is zero. Thus the layer problem is the one for $\varrho = |\zeta| = 0$. $S^{+,sf}_{\epsilon}(0)$ is spanned by the only vector whose related eigenvalue has nonzero realpart. Any $n-k+1$-dimensional subspace of eigenvectors of N_{ϵ} is invariant under the flow and thus a rest point for the Y equation. As we consider the slow flow on $\mathcal{M}_{\epsilon,0}$ we get that

$$\{\chi^{+}\} \times \{Y \in \mathcal{G}^{2n}_{n+k-1}(\mathbb{C}) \big| Y \subset N^{+}_{\epsilon}(0)\} \tag{3.31}$$

is a manifold of equilibria for the system, but is not normally hyperbolic. In order to apply the first theorem of Fenichel theory we need a subset of this manifold which is normally hyperbolic, i.e. the linearization at any point of the critical manifold of dimension m must have exactly m eigenvalues with real part 0. However as we want to construct the stable subbundle we want $S_\epsilon^{+,sf}(0)$ to be part of it anyway. If we consider

$$\mathcal{N}^+(\zeta) = \{\chi^+\} \times \{Y \in \mathcal{G}^{2n}_{n+k-1}(\mathbb{C}) \big| S_\epsilon^{+,sf}(\zeta) \subset Y \subset N_\epsilon^+(\zeta)\} \tag{3.32}$$

and

$$\mathcal{C}^{+,s}_{\epsilon,\zeta} = \{+1\} \times \mathcal{N}^+(\zeta), \tag{3.33}$$

we obtain that $\mathcal{C}^{+,s}_{\epsilon,0}$ is of dimension $(n-k)k$ and indeed normally hyperbolic. Specifically the critical manifold $\mathcal{C}^{+,s}_{\epsilon,0}$ is repelling inside $\{\chi = +1\}$ and attracting in the χ direction towards $\{\chi = +1\}$. Thus $\mathcal{C}^{+,s}_{\epsilon,0}$ perturbs smoothly to an invariant manifold $\mathcal{C}^{+,s}_{\epsilon,\zeta}$. Additionally the stable manifold $\mathcal{W}^s(\mathcal{C}^{+,s}_{\epsilon,\zeta})$ of $\mathcal{C}^{+,s}_{\epsilon,\zeta}$, restricted to the slow flow on $\mathcal{M}_{\epsilon,\zeta}$, possesses invariant foliations with one-dimensional leaves, with each leaf based at a point of the manifold $\mathcal{C}^{+,s}_{\epsilon,\zeta}$. Any leaf whose base point is a rest point is itself invariant under the flow (see Appendix B).
On the one hand the point $S_\epsilon^{+,s}(\zeta)$ satisfies for $\zeta \neq 0$, $\epsilon \neq 0$ with Lemma 3.1 $\mathrm{spec}(A|S_\epsilon^{+,s}(\zeta)) < \mathrm{spec}(A|U_\epsilon^{+,s}(\zeta))$. Hence it is by virtue of Lemma A.1 a repellor inside $\{\chi = +1\}$. On the other hand the point $\{+1\} \times S_\epsilon^{+,s}(\zeta)$ is attracting in the χ direction (in the slow flow). We conclude that $\{+1\} \times S_\epsilon^{+,s}(\zeta)$ is a restpoint and the leaf which is based in $\{+1\} \times S_\epsilon^{+,s}(\zeta)$ is indeed invariant. Thus we find a unique orbit $\left(\chi, Y^+_{\epsilon,\zeta}\right)$ with

$$Y^+_{\epsilon,\zeta}(+\infty) = S_\epsilon^{+,s}(\zeta).$$

In particular we denote its intersection point with $\chi = 0$ by $H_\epsilon^{+,s}$ and note that the bundle is smooth in ϵ and analytic in ζ. (Although from the theorem we only got that it perturbs smoothly in the radius $\varrho = |\zeta|$ we get still analyticity in ζ.)

Next we consider the second case $\zeta \in D_{\varrho_2}/D_{\varrho_1}$, $0 < \varrho_1 < \varrho_2$. In this case ϵ is the small perturbation parameter and we consider the layer problem for $\epsilon = 0$. For $\epsilon = 0$ we get that $\mu^\pm_{k,\epsilon,\zeta}[\nu^+]$ are the only eigenvalues which are non-zero. As we consider the slow flow on $\mathcal{M}_{0,\zeta}$ we get that

$$\{\chi^+\} \times \{Y \in \mathcal{G}^{2n}_{n+k-1}(\mathbb{C}) \big| Y \subset N_\epsilon^+(0)\} \tag{3.34}$$

is a manifold of equilibria for the system. In order to find a normally hyperbolic subset we still want $S_\epsilon^{+,sf}$ to be part of any point on the critical manifold. However the eigenvalue related to the vector spanning $U_\epsilon^{+,sf}$ has positive realpart and would not allow any point to be normally hyperbolic and thus should be excluded. Thus we define with $(\oplus_{j\geq k} R^-_{j,\epsilon,\zeta}[\chi^+]) \oplus (\oplus_{j<k} R^+_{j,\epsilon,\zeta}[\chi^+])$ instead of $N_\epsilon^+(\zeta)$

$$\begin{aligned} \mathcal{N}^+(\zeta) = \{\chi^+\} \times \{Y \in \mathcal{G}^{2n}_{n+k-1}(\mathbb{C}) \big| S_\epsilon^{+,sf}(\zeta) \subset Y, \\ Y \subset \big(\bigoplus_{j\geq k} R^-_{j,\epsilon,\zeta}[\chi^+]\big) \oplus \big(\bigoplus_{j<k} R^+_{j,\epsilon,\zeta}[\chi^+]\big)\} \end{aligned} \tag{3.35}$$

and

$$\mathcal{C}^{+,s}_{\epsilon,\zeta} = \{+1\} \times \mathcal{N}^+(\zeta), \tag{3.36}$$

The critical manifold $\mathcal{C}^{+,s}_{0,\zeta}$ is normally hyperbolic and has dimension $(n-k)(k-1)$. Indeed $\mathcal{C}^{+,s}_{0,\zeta}$ is repelling inside $\{\chi = +1\}$ and attracting in the χ direction towards $\{\chi = +1\}$. The critical manifold $\mathcal{C}^{+,s}_{0,\zeta}$ perturbs smoothly to an invariant manifold $\mathcal{C}^{+,s}_{\epsilon,\zeta}$ and in the same manner as above we construct the slow stable Evans subbundle $H_\epsilon^{+,s}$. □

Slow subbundle $H_\epsilon^{-,s}$

Theorem 3.4. *For every ϱ there exists ϵ_1 such that for all $\epsilon \in [0, \epsilon_1]$, there exist unique smooth mappings*

$$U_\epsilon^{-,s} : D_\varrho \to \mathcal{G}^{2n}_k(\mathbb{C}), \quad H_\epsilon^{-,s} : D_\varrho \to \mathcal{G}^{2n}_k(\mathbb{C})$$

all analytic in $\zeta \in D_\varrho$. Additionally for each $\epsilon \in [0, \epsilon_1]$ and each $\zeta \in D_\varrho$, the solution Y^- of the system

$$\begin{aligned} \chi' &= \epsilon\big(1-\chi^2\big) g_\epsilon(\chi), \\ Y' &= \Gamma^k \mathbb{A}_{\epsilon,\zeta}[\chi]\big(Y\big) \end{aligned} \tag{3.37}$$

assuming data $Y^-(0) = H_\epsilon^{-,s}(\zeta)$ converge at the left end to

$$Y^-(-\infty) = U_\epsilon^{-,s}(\zeta).$$

Proof. We will follow the program outlined in the construction of the slow stable Evans subbundle $H_\epsilon^{+,s}$ we will just point out the differences.
In case 1 we define

$$\mathcal{N}^-(\zeta) = \{\chi^-\} \times \{Y \in \mathcal{G}^{2n}_k(\mathbb{C}) \big| U_\epsilon^{-,sf}(\zeta) \subset Y \subset N_\epsilon^-(\zeta)\} \tag{3.38}$$

and consider the invariant manifold

$$\mathcal{C}^{-,s}_{\epsilon,\zeta} = \{-1\} \times \mathcal{N}^-(\zeta). \tag{3.39}$$

For $\zeta = 0$ the critical manifold $\mathcal{C}^{-,s}_{\epsilon,0}$ is normally hyperbolic. The critical manifold $\mathcal{C}^{-,s}_{\epsilon,0}$ is attracting inside $\{\chi = -1\}$ and repelling in the χ direction towards $\{\chi = -1\}$. $\mathcal{C}^{-,s}_{\epsilon,0}$ perturbs smoothly to an invariant manifold $\mathcal{C}^{-,s}_{\epsilon,\zeta}$ and the unstable manifold $\mathcal{W}^u(\mathcal{C}^{-,s}_{\epsilon,\zeta})$, restricted to the slow flow on $\mathcal{M}_{\epsilon,\zeta}$, possesses invariant foliations with one-dimensional leaves. The leaf based at the point $\{-1\} \times U^{-,s}_\epsilon(\zeta)$ is invariant under the slow flow. We find a unique orbit $\big(\chi, Y^-_{\epsilon,\zeta}\big)$ with the desired properties and denote the intersection point with $\chi = 0$ with $H^{-,s}_\epsilon$. This bundle is smooth in ϵ and analytic in ζ.

In case 2 both eigenvalues $\mu^\pm_{k,\epsilon,\zeta}[\chi^-]$ are O(1) and we will have to exclude $R^-_{k,\epsilon,\zeta}[\chi^-]$ for the same reasons from the definition of $\mathcal{N}^-(\zeta)$. We define

$$\begin{aligned} \mathcal{N}^-(\zeta) = \big\{\chi^-\big\} \times \{Y \in \mathcal{G}^{2n}_k(\mathbb{C}) \big| U^{-,sf}_\epsilon(\zeta) \subset Y, \\ Y \subset \big(\bigoplus_{j>k} R^-_{j,\epsilon,\zeta}[\chi^-]\big) \oplus \big(\bigoplus_{j\le k} R^+_{j,\epsilon,\zeta}[\chi^-]\big)\big\} \end{aligned} \tag{3.40}$$

and

$$\mathcal{C}^{-,s}_{\epsilon,\zeta} = \{-1\} \times \mathcal{N}^-(\zeta). \tag{3.41}$$

Again $\mathcal{C}^{-,s}_{0,\zeta}$ is a critical manifold for $\epsilon = 0$ which is normally hyperbolic. $\mathcal{C}^{-,s}_{0,\zeta}$ perturbs smoothly to an invariant manifold $\mathcal{C}^{-,s}_{\epsilon,\zeta}$ and like in case 1 we are able to construct the slow scaled unstable Evans subbundle $H^{-,s}_\epsilon$. □

Theorem 3.5. *For every ϱ there exists ϵ_1 such that the bundles $S^{+,s}_\epsilon$, $H^{+,s}_\epsilon$ of Theorem 3.3 and $U^{-,s}_\epsilon$, $H^{-,s}_\epsilon$ of Theorem 3.4 depend smoothly on $\epsilon \in [0, \epsilon_1]$ and*

$$\begin{aligned} H^{+,s}_0(\zeta) &= \tilde{\mathcal{H}}^{+,sf}_0(\zeta) \oplus S^{ss}_0(\zeta) \\ H^{-,s}_0(\zeta) &= \tilde{\mathcal{H}}^{-,sf}_0(\zeta) \oplus U^{ss}_0(\zeta). \end{aligned}$$

with $\tilde{\mathcal{H}}^{\pm,sf}_0(\zeta) \in N^{sf}_0$ and $\dim N^{sf}_0 = 2$. Where $\tilde{\mathcal{H}}^{\pm,sf}_0(\zeta) \cong \mathcal{H}^{\pm,sf}_0(\zeta) \in \mathbb{C}^{2n}$ via a coordinate transformation of $N^{sf}_0 \cong \mathbb{C}^2$.

Proof. The equations for the slow flow on $\mathcal{M}_{0,\zeta}$, see Proposition 3.1, are

$$\begin{aligned} \dot{u}_k &= 1 - u_k^2, \\ \dot{p}_k &= -2u_k p_k - y_k, \\ \dot{y}_k &= -a^2 \zeta p_k, \\ \dot{y}_i &= 0, \end{aligned} \tag{3.42}$$

$i = 1, \ldots, n$, $i \neq k$. That means

$$\dot{u}_k = 1 - u_k^2 \tag{3.43}$$

and

$$\dot{y}_i = 0 \qquad (i \neq k) \tag{3.44}$$

decouple from the rest of the system

$$\begin{aligned} \dot{p}_k &= -2u_k p_k - y_k \\ \dot{y}_k &= -a^2 \zeta p_k \end{aligned} \tag{3.45}$$

The slow manifold $\mathcal{M}_{0,\zeta}$ is spanned by the constant vectors

$$\big(\bigoplus_{j \geq k} R^-_{j,0,.}[.]\big) \oplus \big(\bigoplus_{j \leq k} R^+_{j,0,.}[.]\big),$$

which are independent of ζ and ν. See also the definition of μ_j and R_j. The solution space of system (3.44) is

$$\big(\bigoplus_{j > k} R^-_{j,0,.}[.]\big) \oplus \big(\bigoplus_{j < k} R^+_{j,0,.}[.]\big)$$

and the one for system (3.45)

$$R^-_{k,0,.}[.] \oplus R^+_{k,0,.}[.].$$

Y^+ converges at the right end to $S_0^{+,s}(\zeta) := \bigoplus_{j \geq k} R^-_{j,0,\zeta}[\nu^+]$. We note that the subspace

$$S_0^{ss} := \oplus_{j>k} R^-_{j,0,.}[.]$$

of constant vectors is invariant under the flow (3.44) and $H_0^{+,s}(\zeta)$ can be decomposed into

$$H_0^{+,s}(\zeta) = \tilde{\mathcal{H}}_0^{+,sf}(\zeta) \oplus S_0^{ss}$$

with $\tilde{\mathcal{H}}_0^{+,sf}(\zeta) \in N_0^{sf} := R^-_{k,0,.}[.] \oplus R^+_{k,0,.}[.]$. In the same way we show the decomposition of

$$H_0^{-,s}(\zeta) = \tilde{\mathcal{H}}_0^{-,sf}(\zeta) \oplus U_0^{ss}$$

with $U_0^{ss}(\zeta) := \oplus_{j<k} R^+_{j,0,.}[.]$ and $\tilde{\mathcal{H}}_0^{-,sf}(\zeta) \in N_0^{sf}$. □

Remark 3.2. The profile ϕ_0 with wave speed $s = 0$ is a solution of the viscous Burgers equation. The viscous profile equation is

$$u' = 1 - u^2. \tag{3.46}$$

The eigenvalue problem can be written as

$$\kappa v = v_t = v'' + 2uv', \tag{3.47}$$

which is equivalent to the first order system of ordinary differential equations

$$\begin{aligned} p' &= -2up - y, \\ y' &= -\kappa p. \end{aligned} \tag{3.48}$$

The augmented system for the profile ϕ_0 of the Burgers equation is

$$\begin{aligned} u' &= 1 - u^2, \\ p' &= -2up - y, \\ y' &= -\kappa p. \end{aligned} \tag{3.49}$$

We conclude that system (3.49) is exactly the nontrivial part of the slow flow on $\mathcal{M}_{0,\zeta}$. The travelling waves for the viscous Burgers equation are known to be spectrally stable. Hence the stable and unstable Evans bundles for the profile ϕ_0 don't intersect, which is also true for their suspensions in $\mathcal{G}_n^{2n}(\mathbb{C})$.

Proof of Assertion 1 of Theorem 2.4. We can find a small parameter $\epsilon_1 > 0$ such that we get from the Theorems 3.1, 3.2, 3.3, 3.4 the existence of the subbundles which span the stable Evans bundle

$$H_\epsilon^+ = H_\epsilon^{+,f} \oplus H_\epsilon^{+,s} \tag{3.50}$$

and the unstable Evans bundle

$$H_\epsilon^- = H_\epsilon^{-,f} \oplus H_\epsilon^{-,s}. \tag{3.51}$$

The Theorems 3.1 (Assertion 2), 3.2 (Assertion 2), 3.5 prove the analytic convergence of the Evans bundles. $\square$

Chapter 4

Outer problem

We will prove the second Assertion of Theorem 2.4 which states that there are no eigenvalue in the outer region. However an obstacle is that the essential spectrum is approaching the entire imaginary axis as ϵ tends to zero. Nonetheless we are going to prove that the two Evans bundles do not intersect for $|\zeta| \geq \varrho_1$ and $\epsilon \in [0, \epsilon_1]$, if ϱ_1 is chosen large enough and then ϵ_1 sufficiently small. The key ingredient in our argumentation is the following result for non-autonomous linear systems whose coefficient matrices possess a sufficiently slowly varying diagonalizer, which was formulated in the paper [FS] on "Spectral stability of small viscous profiles".

Lemma 4.1. *For every $n \in \mathbb{N}$ there exists a constant $c > 0$ with which the following holds. Let $A, R : J \to GL_{2n}(\mathbb{C})$ be smooth matrix functions such that*

$$R^{-1}AR = diag(\mu_1^+, \dots, \mu_n^+, \mu_1^-, \dots, \mu_n^-) \tag{4.1}$$

with

$$\Re\mu_j^+ > 0, \Re\mu_j^- < 0, \quad j = 1, \dots, n. \tag{4.2}$$

With $\chi : \mathbb{R} \to (-1, 1)$ the solution of

$$\chi' = (1 - \chi^2)g(\chi), \chi(0) = 0 \tag{4.3}$$

for some smooth $g : J \to (0, \infty)$, consider the equation

$$X' = \Gamma^n A(\chi)(X) \quad on\ \mathcal{G}_n^{2n}(\mathbb{C}) \tag{4.4}$$

which is associated with the non-autonomous linear system $\xi' = A(\chi)\xi$ on $\mathbb{C}^{2n}$. With $U(\tau), S(\tau)$ denoting the unstable resp. stable subspaces of $A(\tau), \tau \in J$, define $X^\pm : \mathbb{R} \to \mathcal{G}_n^{2n}(\mathbb{C})$ as the two solutions of (4.4) with

$$X^-(-\infty) = U(-1), \quad X^+(+\infty) = S(+1). \tag{4.5}$$

If furthermore

$$|(R(0))^{-1}\frac{d}{d\tau}R(\tau)| \leq c, \quad \tau \in J, \tag{4.6}$$

then these solutions satisfy also

$$X^{-}(+\infty) = U(+1), \quad X^{+}(-\infty) = S(-1). \tag{4.7}$$

Remark. The idea of the proof is that in the first step they construct a positively invariant set $\mathcal{N}$ for the augmented system of the profile equation and the projectivized eigenvalue problem with ω-limit $\{+1\} \times U(1)$. This is possible due to the slowly varying diagonalizer. As the orbit $O = \bigcup_{-1<\nu\leq 1}\left(\{\nu\} \times X^{-}(\chi^{-1}(\nu))\right.$ enters this positive invariant manifold it stays in there and we conclude $U(+\infty) = U(1)$, too. The entire proof can be found in [FS]. □

Remark. In a finite dimensional space the maximum norm is equivalent to any other norm and we will consider the maximum norm in our proofs. □

Proof of Assertion 2 of Theorem 2.4. We will show that Lemma 4.1 is applicable to

$$\begin{aligned} \chi' &= \epsilon\big(1-\chi^2\big)g_\epsilon(\chi), \\ X' &= \Gamma^n \mathbb{A}_{\epsilon,\zeta}(\chi)(X) \end{aligned} \tag{4.8}$$

uniformly for small $\epsilon > 0$ and sufficiently large $|\zeta|$. This would allow us to conclude that the Evans bundle don't intersect.

The problem is in the correct form and we will prove condition (4.6) for the matrix

$$R_{\epsilon,\zeta}[\nu] = \left(\begin{pmatrix} r_j[\nu] & r_j[\nu] \\ \frac{\mu^{+}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1}r_j[\nu] & \frac{\mu^{-}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1}r_j[\nu] \end{pmatrix}_{j=1,\dots,n}\right). \tag{4.9}$$

The inequality (4.6)

$$\left|(R(0))^{-1}\frac{d}{d\nu}R(\nu)\right| \leq \left|\big(R_{\epsilon,\zeta}[\nu]\big)^{-1}\right|\left|\frac{\partial}{\partial\nu}R_{\epsilon,\zeta}[\nu]\right| \tag{4.10}$$

holds especially if we can show

$$\left|\frac{\partial}{\partial\nu}R_{\epsilon,\zeta}[\nu]\right| \leq c\epsilon \tag{4.11}$$

and

$$\left|\left(R_{\epsilon,\zeta}[\nu]\right)^{-1}\right| \leq c \tag{4.12}$$

with a uniform constant $c > 0$ if $|\zeta| \geq \varrho_1$.
The coefficients of $\frac{\partial}{\partial\nu} R_{\epsilon,\zeta}[\nu]$ are

$$\frac{\partial}{\partial\nu} r_j(\epsilon u(\nu))$$

and

$$\frac{\partial}{\partial\nu}\left(\frac{\mu^{\pm}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1} r_j[\nu]\right) = \left(\frac{\partial}{\partial\nu}\mu^{\pm}_{j,\epsilon,\zeta}[\nu]\right)\frac{r_j[\nu]}{\epsilon^2\zeta+1} + \frac{\mu^{\pm}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1}\left(\frac{\partial}{\partial\nu} r_j(\epsilon u(\nu))\right).$$

Since $r_j[\nu]$ is an eigenvector associated to the Jacobian matrix of the smooth function f we observe that

$$\frac{\partial}{\partial\nu} r_j(\epsilon u(\nu)) = \left(\nabla r_j(\epsilon u)\right).\left(\epsilon\frac{du}{d\nu}\right) = O(\epsilon). \tag{4.13}$$

In order to prove a similar result for the other coefficient we will use $\epsilon^2\zeta \equiv \varrho e^{i\varphi}$ with $\varphi \leq |\frac{\pi}{2}|$.

$$\begin{aligned}
&\frac{\partial}{\partial\nu}\mu^{\pm}_{j,\epsilon,\zeta}[\nu] = \\
&= \frac{\partial}{\partial\nu}\Bigg(\frac{\lambda_j(\epsilon u) - \epsilon s(2\varrho e^{i\varphi}+1)}{2} \\
&\qquad \pm\sqrt{\frac{\left(\lambda_j(\epsilon u) - \epsilon s(2\varrho e^{i\varphi}+1)\right)^2}{4} + \varrho e^{i\varphi}\left(\varrho e^{i\varphi}+1\right)\left(a^2-(\epsilon s)^2\right)}\Bigg) \\
&= \frac{\partial}{\partial\nu}\left(\frac{\lambda_j(\epsilon u) - \epsilon s(2\varrho e^{i\varphi}+1)}{2}\right) \\
&\qquad \left(1 \pm \frac{\lambda_j(\epsilon u) - \epsilon s(2\varrho e^{i\varphi}+1)}{2\sqrt{\frac{\left(\lambda_j(\epsilon u)-\epsilon s\right)^2}{4} + \left(\varrho e^{i\varphi}\right)^2 + \varrho e^{i\varphi}\left(a^2 - 4\lambda_j(\epsilon u)\epsilon s\right)}}\right).
\end{aligned}$$

The second factor is bounded as the discriminant has positive real-part which is bounded away from zero. Whereas

$$\frac{\partial}{\partial\nu}\left(\frac{\lambda_j(\epsilon u) - \epsilon s(2\varrho e^{i\varphi}+1)}{2}\right) = \frac{1}{2}\frac{\partial}{\partial\nu}\lambda_j(\epsilon u) = \frac{1}{2}\nabla\lambda_j(\epsilon u).\frac{\partial\epsilon u}{\partial\nu} = O(\epsilon)$$

and we conclude

$$\frac{\partial}{\partial \nu}\mu^{\pm}_{j,\epsilon,\zeta}[\nu] = O(\epsilon). \tag{4.14}$$

Together with the result $\frac{\partial}{\partial\nu} r_j(\epsilon u(\nu)) = O(\epsilon)$ and that

$$\frac{r_j(\epsilon u(\nu))}{\varrho e^{i\varphi}+1} \text{ and } \frac{\mu^{\pm}_{j,\epsilon,\zeta}[\nu]}{\varrho e^{i\varphi}+1}$$

are smooth functions, we obtain that the coefficients of $\frac{\partial}{\partial\nu} R_{\epsilon,\zeta}[\nu]$ are $O(\epsilon)$. This proves (4.11).

To prove (4.12) we note that via a basis transformation modulo an order one, i.e. a transformation which is bounded and has a bounded inverse, R is similar to $\hat{R} := \operatorname{diag}(\hat{R}_1, \ldots, \hat{R}_n)$, with

$$\hat{R}_{j,\epsilon,\zeta}[\nu] = \begin{pmatrix} 1 & 1 \\ \frac{\mu^{+}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1} & \frac{\mu^{-}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1} \end{pmatrix} \tag{4.15}$$

and

$$\hat{R}^{-1}[\nu] = \operatorname{diag}(\hat{R}_1^{-1}, \ldots, \hat{R}_n^{-1}) \tag{4.16}$$

We derive

$$\hat{R}^{-1}_{j,\epsilon,\zeta}[\nu] = \frac{\epsilon^2\zeta+1}{\mu^{-}_{j,\epsilon,\zeta}[\nu] - \mu^{+}_{j,\epsilon,\zeta}[\nu]} \begin{pmatrix} \frac{\mu^{-}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1} & -1 \\ -\frac{\mu^{+}_{j,\epsilon,\zeta}[\nu]}{\epsilon^2\zeta+1} & 1 \end{pmatrix} \tag{4.17}$$

Thus we have to show

$$\left| \frac{\mu^{\pm}_{j,\epsilon,\zeta}[\nu]}{\mu^{-}_{j,\epsilon,\zeta}[\nu] - \mu^{+}_{j,\epsilon,\zeta}[\nu]} \right| \leq c \qquad j = 1, \ldots, n \tag{4.18}$$

and

$$\left| \frac{\epsilon^2\zeta+1}{\mu^{-}_{j,\epsilon,\zeta}[\nu] - \mu^{+}_{j,\epsilon,\zeta}[\nu]} \right| \leq c \qquad j = 1, \ldots, n \tag{4.19}$$

We will write the spectral parameter again as $\epsilon^2\zeta \equiv \varrho e^{i\varphi}$ with $\varphi \leq |\frac{\pi}{2}|$ and consider for any $\varrho_2 > 0$ two cases:

1. $0 < \varrho_1 \leq \varrho < \varrho_2$, $\varrho_1 > 0$.

2. $\varrho \geq \varrho_2$.

Case 1: $0 < \varrho_1 \le \varrho < \varrho_2,\ \varrho_1 > 0$.
For the eigenvalues $\mu^{\pm}_{j,\epsilon,\zeta},\ j = 1,\dots,n,\ j \neq k$ we obtain upper bounds

$$\begin{aligned}
\left|\mu^{\pm}_{j,\epsilon,\zeta}[\nu]\right| &= \Bigg|\frac{\lambda_j(\epsilon u) - \epsilon s(2\varrho e^{i\varphi}+1)}{2} \\
&\quad \pm \sqrt{\frac{\big(\lambda_j(\epsilon u)-\epsilon s\big)^2}{4} + a^2\big(\varrho e^{i\varphi}\big)^2 + \varrho e^{i\varphi}\big(a^2 - \lambda_j(\epsilon u)\epsilon s\big)}\Bigg| \\
&\le \underbrace{\left|\frac{\lambda_j(\epsilon u)-\epsilon s}{2}\right|}_{\le \gamma_1} + \underbrace{\left|\epsilon s \varrho e^{i\varphi}\right|}_{\le \gamma_2(\varrho_2)} \\
&\quad + \underbrace{\left|\pm\sqrt{\frac{\big(\lambda_j(\epsilon u)-\epsilon s\big)^2}{4} + a^2\big(\varrho e^{i\varphi}\big)^2 + \varrho e^{i\varphi}\big(a^2 - \lambda_j(\epsilon u)\epsilon s\big)}\right|}_{\le \gamma_3(\varrho_2)} \\
&= \gamma_1 + \gamma_2(\varrho_2) + \gamma_3(\varrho_2) = \gamma(\varrho_2)
\end{aligned}$$

Whereas the difference of two eigenvalues is bounded away from 0 since the discriminant has positive real part and $\lambda_j(0) \neq 0$

$$\begin{aligned}
\left|\mu^{+}_{j,\epsilon,\zeta}[\nu] - \mu^{-}_{j,\epsilon,\zeta}[\nu]\right| &= \left|\pm 2\sqrt{\frac{\big(\lambda_j(\epsilon u)-\epsilon s\big)^2}{4} + a^2\big(\varrho e^{i\varphi}\big)^2 + \varrho e^{i\varphi}\big(a^2 - \lambda_j(\epsilon u)\epsilon s\big)}\right| \\
&\ge \left|\lambda_j(\epsilon u) - \epsilon s\right| = \left|\lambda_j(0) + O(\epsilon)\right| > \beta > 0.
\end{aligned}$$

For $\mu^{\pm}_{k,\epsilon,\zeta}[\nu]$ we conclude from the approximations in Remark 3.1 that

$$\left|\mu^{\pm}_{k,\epsilon,\zeta}[\nu]\right| \le \epsilon\gamma(\varrho_2)$$

and for the difference

$$\begin{aligned}
\left|\mu^{+}_{k,\epsilon,\zeta}[\nu] - \mu^{-}_{k,\epsilon,\zeta}[\nu]\right| &= \left|\pm 2\sqrt{\epsilon^2\frac{\big(2Au_k - s + O(\epsilon)\big)^2}{4} + \varrho e^{i\varphi}\big(\varrho e^{i\varphi}+1\big)\big(a^2 - (\epsilon s)^2\big)}\right| \\
&\ge \epsilon\beta > 0.
\end{aligned}$$

From these bounds we conclude that the (4.12) holds with an uniform constant c.

Case 2: $\varrho \geq \varrho_2$.
In this case we have for the eigenvalues $\mu^{\pm}_{j,\epsilon,\zeta}$, $j = 1, \ldots, n$

$$\begin{aligned}
\big|\mu^{\pm}_{j,\epsilon,\zeta}[\nu]\big| &= \frac{\lambda_j(\epsilon u) - \epsilon s}{2} - \epsilon s \varrho e^{i\varphi} \\
&\quad \pm \sqrt{\frac{\big(\lambda_j(\epsilon u) - \epsilon s\big)^2}{4} + a^2\big(\varrho e^{i\varphi}\big)^2 + \varrho e^{i\varphi}\big(a^2 - \lambda_j(\epsilon u)\epsilon s\big)} \\
&= \frac{\lambda_j(\epsilon u) - \epsilon s}{2} - \epsilon s \varrho e^{i\varphi} \\
&\quad \pm a\varrho e^{i\varphi}\sqrt{1 + \frac{\big(\lambda_j(\epsilon u) - \epsilon s\big)^2 + 4\varrho e^{i\varphi}\big(a^2 - \lambda_j(\epsilon u)\epsilon s\big)}{4a^2\big(\varrho e^{i\varphi}\big)^2}} \\
&\approx (-\epsilon s \pm a)\varrho e^{i\varphi} \approx \pm \varrho e^{i\varphi}.
\end{aligned}$$

Thus we conclude that (4.12) holds again with an uniform constant c.

The inequality (4.6) holds and with Lemma 4.1 this proves Assertion 2 of Theorem 2.4. $\square$

Appendix A

Flows on Grassmann Manifolds

Certain properties of systems of linear autonomous or nonautonomous ordinary differential equations can be concisely expressed as properties of flows which these systems induce on certain Grassmann manifolds. Background material on Grassmann manifolds can be found in many geometry books, see e.g. [GrHa], however, we need only a few elementary notions.

In this section, we show in particular a simple normal form statement for the local behaviour of the Grassmann version of a linear constant-coefficients problem at a rest point that corresponds to a spectrally isolated invariant subspace of the original system. The proof of this proposition may serve those readers who feel unfamiliar with flows on Grassmann manifolds as a warmup example for the analysis in Chapter 3.

To begin with, we recall that for $m, d \in \mathbb{N}, m \leq d$, the set $\mathcal{G}_m^d(\mathbb{C})$ of m-dimensional linear subspaces of $\mathbb{C}^d$ is a complex-analytic manifold of dimension $m(d-m)$. With respect to a given basis $\{e_1, \ldots, e_d\}$ of $\mathbb{C}^d$, the mapping

$$\begin{aligned} \varphi : \mathbb{C}^{(d-m)\times m} &\to \mathcal{G}_m^d \\ T &\mapsto \operatorname{span}\begin{pmatrix} I_m \\ T \end{pmatrix} \end{aligned}$$

is a local chart of $\mathcal{G}_m^d$ with $\varphi(0) = X_0 = \operatorname{span}\{e_1, \ldots, e_m\}$. We call a chart related in this way to a basis of the underlying space $\mathbb{C}^d$ a canonical chart with respect to that basis. For any basis, different canonical charts resulting from permuting the basis vectors cover the whole manifold $\mathcal{G}_m^d$ and changes between these charts are analytic mappings between appropriate domains in $\mathbb{C}^{(d-m)\times m}$.

Lemma A.1. *(i) Fix a basis $\{e_1, \ldots, e_d\}$ of $\mathbb{C}^d$ and let $X_0 = span\{e_1, \ldots, e_m\}$.*

Consider a constant coefficient system

$$\xi' = A\xi \quad \text{on } \mathbb{C}^d. \tag{A.1}$$

and the associated system

$$X' = \Gamma^m A(X) \quad \text{on } \mathcal{G}_m^d, \tag{A.2}$$

and let φ be a canonical chart for $\mathcal{G}_m^d$ with respect to the basis $\{e_1, \ldots, e_d\}$. If

$$A = diag(\mu_1, \ldots, \mu_d).$$

then modulo φ^{-1}, the flow of (A.2) near X_0 obeys the linear system

$$t'_{ab} = (\mu_{m+a} - \mu_b)t_{ab} \quad \text{on } \mathbb{C}^{(d-m)\times m}. \tag{A.3}$$

(ii) If additionally

$$\Re(spec(A|X_0)) > \Re(spec(A|Y_0))$$

with $Y_0 = span\{e_{m+1}, \ldots, e_d\}$, then X_0 is a hyperbolic attractor for (A.2) and via φ, any sphere in $\mathbb{C}^{(d-m)\times m} \equiv \mathbb{C}^{(d-m)m}$ defines a positively invariant neighborhood of X in $\mathcal{G}_m^d$.

Proof of Lemma A.1. Represent any orbit $X : \mathbb{R} \to \mathcal{G}_m^d$ of (A.2) near X_0 also by some matrix-valued function $\Xi : \mathbb{R} \to \mathbb{C}^{d\times m}$ with Ξ of the form

$$\Xi = \begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_m \\ y_{11} & \cdots & y_{1m} \\ \vdots & & \vdots \\ y_{(d-m)1} & \cdots & y_{(d-m)m} \end{pmatrix} \text{ with } x_j \neq 0, j = 1, \ldots, m,$$

the columns spanning X as a subspace of $\mathbb{C}^d$ and thus satisfying (A.1). I. e.,

$$\begin{pmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_m \\ y_{11} & \cdots & y_{1m} \\ \vdots & & \vdots \\ y_{(d-m)1} & \cdots & y_{(d-m)m} \end{pmatrix}' = \begin{pmatrix} \mu_1 x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_m x_m \\ \mu_{m+1} y_{11} & \cdots & \mu_{m+1} y_{1m} \\ \vdots & & \vdots \\ \mu_d y_{(d-m)1} & \cdots & \mu_d y_{(d-m)m} \end{pmatrix}.$$

This yields

$$(t_{ab}) = \varphi^{-1}(X) = (y_{ab}/x_b)$$

with

$$\begin{aligned} (y_{ab}/x_b)' &= (x_b \mu_{m+a} y_{ab} - \mu_b x_b y_{ab})/x_b^2 \\ &= (\mu_{m+a} - \mu_b)(y_{ab}/x_b). \end{aligned}$$

This proves assertion (i). Under the assumptions of (ii) all eigenvalues $(\mu_{m+b} - \mu_a)$ have negative real part and (ii) follows. $\square$

Appendix B

Invariant manifold theory

This appendix contains some results from the geometric theory of dynamical systems needed in this work. For a general introduction, we refer to [Ar], [Ce], [Gu], [W], [F2], and [J2].

B.1 Center manifolds and invariant foliations

Center manifold theory is a tool for analyzing the local dynamics of non-hyperbolic equilibria of dynamical systems, providing a means for systematically reducing the dimension of problems near such equilibria. We will in the following consider systems in the standard (block diagonal) form

$$\begin{aligned} \mathbf{x}_s' &= A_s\mathbf{x}_s + \mathbf{f}_s(\mathbf{x}_s, \mathbf{x}_c, \mathbf{x}_u) \\ \mathbf{x}_c' &= A_c\mathbf{x}_c + \mathbf{f}_c(\mathbf{x}_s, \mathbf{x}_c, \mathbf{x}_u) \\ \mathbf{x}_u' &= A_u\mathbf{x}_u + \mathbf{f}_u(\mathbf{x}_s, \mathbf{x}_c, \mathbf{x}_u), \end{aligned} \tag{B.1}$$

where $\mathbf{x}_s \in \mathbb{R}^{n_s}, \mathbf{x}_c \in \mathbb{R}^{n_c}$, and $\mathbf{x}_u \in \mathbb{R}^{n_u}$ for $n_s, n_c, n_u \in \mathbb{N}$, $\mathbf{f}_s, \mathbf{f}_c$, and $\mathbf{f}_u$ are $\mathcal{O}\left(\|(\mathbf{x}_s, \mathbf{x}_c, \mathbf{x}_u)\|^2\right)$ and $\mathcal{C}^k$ $(k \in \mathbb{N})$ in all three arguments, and the prime denotes differentiation with respect to some $t \in \mathbb{R}$.[1] Moreover, we require the matrices A_s, A_c, and A_u to have only eigenvalues λ_s, λ_c, and λ_u with negative, zero and positive real parts, respectively.

For the linear part of (B.1) the following invariant subspaces can easily be identified: an n_s-dimensional *stable manifold* $\mathcal{E}^s$ given by $\{\mathbf{x}_c = 0 = \mathbf{x}_u\}$, which is the (generalized) eigenspace corresponding to the eigenvalues λ_s; an n_u-dimensional *unstable manifold* $\mathcal{E}^u$ given by $\{\mathbf{x}_s = 0 = \mathbf{x}_c\}$, which is the (generalized) eigenspace corresponding to the eigenvalues λ_u; and finally an n_c-dimensional *center manifold* $\mathcal{E}^c$ given by $\{\mathbf{x}_s = 0 = \mathbf{x}_u\}$, which is the

[1] Here $\|\cdot\|$ denotes a suitably chosen norm.

(generalized) eigenspace corresponding to the eigenvalues λ_c. Similarly, one can define the invariant subspaces $\mathcal{E}^{cs}$ (the *center-stable manifold* given by $\{\mathbf{x}_u = 0\}$) and $\mathcal{E}^{cu}$ (the *center-unstable manifold* given by $\{\mathbf{x}_s = 0\}$). To obtain the dynamics of the linear system in (B.1), one just has to combine the dynamics of the subsystems corresponding to these invariant subspaces.

For the full nonlinear system a similar rationale holds; indeed, one can show that there exist invariant submanifolds tangent to the invariant manifolds of the linear system, see [Ce] or [Gu]:

Theorem B.1. *Given system (B.1), there exists an $n_s(n_c, n_u)$-dimensional invariant $\mathcal{C}^k$-manifold $\mathcal{W}^s$ $(\mathcal{W}^c, \mathcal{W}^u)$ tangent to $\mathcal{E}^s$ $(\mathcal{E}^c, \mathcal{E}^u)$. Likewise, there are $(n_s + n_c)$- and $(n_c + n_u)$-dimensional invariant $\mathcal{C}^k$-manifolds $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to $\mathcal{E}^s \oplus \mathcal{E}^c$ and $\mathcal{E}^c \oplus \mathcal{E}^u$, respectively.*

For simplicity, let us now assume that $n_u = 0$; then system (B.1) takes the form

$$
\begin{aligned}
\mathbf{x}_s' &= A_s\mathbf{x}_s + \mathbf{f}_s(\mathbf{x}_s, \mathbf{x}_c)\\
\mathbf{x}_c' &= A_c\mathbf{x}_c + \mathbf{f}_c(\mathbf{x}_s, \mathbf{x}_c).
\end{aligned}
\tag{B.2}
$$

Since the center manifold $\mathcal{W}^c$ can be represented as a (local) graph

$$
\mathcal{W}^c = \{(\mathbf{x}_s, \mathbf{x}_c) |\ \mathbf{x}_s = \varphi(\mathbf{x}_c)\}, \qquad \varphi(\mathbf{0}) = \mathbf{0} = \frac{\partial \varphi}{\partial \mathbf{x}_c}(\mathbf{0}), \tag{B.3}
$$

the dynamics restricted to $\mathcal{W}^c$ is given by

$$
\mathbf{x}_c' = A_c\mathbf{x}_c + \mathbf{f}_c(\varphi(\mathbf{x}_c), \mathbf{x}_c), \tag{B.4}
$$

which implies the following result, see [Gu]:

Theorem B.2. *If the origin of (B.4) is locally asymptotically stable (unstable), then the origin of (B.2) also is locally asymptotically stable (unstable).*

Although in most cases φ cannot be computed exactly, it can often be approximated arbitrarily closely: substituting $\mathbf{x}_s = \varphi(\mathbf{x}_c)$ in the second equation of (B.2), one obtains

$$
\begin{aligned}
\mathbf{x}_s' = \frac{\partial \varphi(\mathbf{x}_c)}{\partial \mathbf{x}_c}\mathbf{x}_c' &= \frac{\partial \varphi(\mathbf{x}_c)}{\partial \mathbf{x}_c}\left[A_c\mathbf{x}_c + \mathbf{f}_c(\varphi(\mathbf{x}_c), \mathbf{x}_c)\right] =\\
&= A_s\varphi(\mathbf{x}_c) + \mathbf{f}_s(\varphi(\mathbf{x}_c), \mathbf{x}_c),
\end{aligned}
\tag{B.5}
$$

which motivates

Theorem B.3. *If a function $\varphi^N(\mathbf{x}_c)$ with $\varphi^N(\mathbf{0}) = \mathbf{0} = \frac{\partial \varphi^N}{\partial \mathbf{x}_c}(\mathbf{0})$ can be found such that (B.5) holds up to terms of $\mathcal{O}\left(\|\mathbf{x}_c\|^N\right)$ for some $N > 1$ as $\|\mathbf{x}_c\| \to 0$, then*

$$\varphi(\mathbf{x}_c) = \varphi^N(\mathbf{x}_c) + \mathcal{O}\left(\|\mathbf{x}_c\|^N\right). \tag{B.6}$$

A similar result can be derived for $n_u > 0$, see again [Gu].

In a neighbourhood of the center-stable and center-unstable manifolds $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$, one can describe the dynamics of (B.1) by the dynamics on so-called *invariant foliations*. In general, a foliation $\mathcal{F}$ of an n-dimensional manifold $\mathcal{W}$ consists of a family of m-dimensional connected submanifolds (*leaves* or *fibers*) $F(\mathbf{x})$, where $m, n \in \mathbb{N},\ m < n$, and $\mathbf{x} \in \mathcal{W}$. Any two leaves of $\mathcal{F}$ are either identical or disjoint (i.e., $F(\mathbf{x}) = F(\mathbf{y})$ or $F(\mathbf{x}) \cap F(\mathbf{y}) = \emptyset$ for $\mathbf{x}, \mathbf{y} \in \mathcal{W}$). Moreover, the foliation has to cover all of $\mathcal{W}$, i.e., $\mathcal{W} = \bigcup_{\mathbf{x} \in \mathcal{W}} F(\mathbf{x})$. An invariant foliation is a foliation which is invariant under the flow of (B.1), that is, any point $\mathbf{y}_0 \in F(\mathbf{x}_0)$ is mapped to the same leaf by the flow: $\mathbf{y}(t) \in F(\mathbf{x}(t))$, where $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{y}(0) = \mathbf{y}_0$.

Let $\beta > 0$ be such that $\Re(\lambda_s) < -\beta,\ \Re(\lambda_u) > \beta$; then there exist real constants $\alpha, \gamma > 0$ with $\alpha < \beta$ and $\alpha < \gamma < k\gamma < \beta$, where k is as above. The following result on the existence of invariant foliations for (B.1) can be found in [Ce]:

Theorem B.4. *There exists a stable (unstable) foliation $\mathcal{F}^s$ ($\mathcal{F}^u$) of $\mathbb{R}^n$ (with $n = n_s + n_c + n_u$) near $\mathcal{W}^{cu}$ ($\mathcal{W}^{cs}$) which is invariant under the flow of (B.1) and has the following properties:*

1. *Every leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$) has a unique transversal intersection with $\mathcal{W}^{cu}$ ($\mathcal{W}^{cs}$).*

2. *Every leaf $F^s(\mathbf{x})$ ($F^u(\mathbf{x})$) is a $\mathcal{C}^k$-manifold, which is, however, only Hölder continuous in its base point $\mathbf{x} \in \mathbb{R}^n$.*

3. *The distance of any two orbits starting in the same leaf of $\mathcal{F}^s$ ($\mathcal{F}^u$) is decaying (growing) exponentially fast with rate $e^{-\gamma t}$ ($e^{\gamma t}$).*

Remark B.1. For $n_u = 0$ ($n_s = 0$), the foliation $\mathcal{F}^s$ ($\mathcal{F}^u$) corresponding to $\mathcal{W}^{cs}$ ($\mathcal{W}^{cu}$) is in fact $\mathcal{C}^k$-smooth in its base points, see [T]. In that case the decay (growth) rate for orbits which start in the same leaf is $e^{-\beta t}$ ($e^{\beta t}$). $\square$

B.2 Fenichel theory

For the sake of completeness, we cite a few basic results from the pioneering work by [F2] on geometric singular perturbation theory. The equations from which we set out here are of the form

$$\begin{aligned} \mathbf{x}' &= \varepsilon \mathbf{f}(\mathbf{x}, \mathbf{y}, \varepsilon) \\ \mathbf{y}' &= \mathbf{g}(\mathbf{x}, \mathbf{y}, \varepsilon), \end{aligned} \tag{B.7}$$

where $0 < \varepsilon \ll 1$, $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$ for $m, n \in \mathbb{N}$, $\mathbf{f}$ and $\mathbf{g}$ are $\mathcal{C}^k$ $(k \in \mathbb{N})$ in all three arguments, and the prime denotes differentiation with respect to t. System (B.7) can be reformulated with a change of time-scale as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}, \varepsilon) \\ \varepsilon \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{x}, \mathbf{y}, \varepsilon); \end{aligned} \tag{B.8}$$

here the overdot denotes differentiation with respect to $\tau = \varepsilon t$. The time-scale given by t is said to be fast, whereas that for τ is slow. We thus call (B.7) the *fast system* and (B.8) the *slow system*. Similarly, $\mathbf{x}$ is usually referred to as the *slow variable*, whereas $\mathbf{y}$ is called the *fast variable*. Taking the limit $\varepsilon \to 0$ in both (B.7) and (B.8), one obtains two limiting systems, the *layer problem*

$$\begin{aligned} \mathbf{x}' &= \mathbf{0} \\ \mathbf{y}' &= \mathbf{g}(\mathbf{x}, \mathbf{y}, 0) \end{aligned} \tag{B.9}$$

and the *reduced problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{y}, 0) \\ \mathbf{0} &= \mathbf{g}(\mathbf{x}, \mathbf{y}, 0). \end{aligned} \tag{B.10}$$

One can think of the condition $\mathbf{g}(\mathbf{x}, \mathbf{y}, 0) = \mathbf{0}$ as determining the manifold $\mathcal{S}$ of equilibria of (B.9). A *normally hyperbolic* submanifold $\mathcal{S}_0$ of $\mathcal{S}$ consists of a connected compact subset of $\mathcal{S}$ where $\frac{\partial \mathbf{g}}{\partial \mathbf{y}}\big|_{\mathcal{S}_0}$ has no eigenvalues on the imaginary axis. There one can solve for $\mathbf{y} = \varphi_0(\mathbf{x})$ to obtain for the dynamics on this so-called *critical manifold* $\mathcal{S}_0$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \varphi_0(\mathbf{x}), 0). \tag{B.11}$$

The primary goal of Fenichel theory is to realize both the fast and the slow aspects of (B.7) simultaneously. Generally speaking, the fast dynamics is captured by (B.9), whereas the slow dynamics is characterized by (B.10). Given the above, it is often possible to reduce the analysis of the original problem to an analysis of these two lower-dimensional limiting problems.

The following two theorems give a precise description of the relation between the dynamics of (B.7) and the combined dynamics of (B.9) and (B.10). First, for $\varepsilon > 0$, $\mathcal{S}_0$ perturbs to a locally invariant manifold in the full problem, which we call the *slow manifold* $\mathcal{S}_\varepsilon$:

Theorem B.5. *For* $\mathbf{f}, \mathbf{g}$ $\mathcal{C}^k$ *in* $(\mathbf{x}, \mathbf{y}, \varepsilon)$ *and* $\mathcal{S}_0$ *a compact normally hyperbolic subset of* $\mathcal{S}$ *given by* $\mathcal{S}_0 = \{(\mathbf{x}, \varphi_0(\mathbf{x}))|\ \mathbf{x} \in U\}$ *with* U *compact, there is an* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *there exists a locally invariant* n*-dimensional* $\mathcal{C}^k$*-manifold* $\mathcal{S}_\varepsilon$ *given as a graph* $\mathcal{S}_\varepsilon = \{(\mathbf{x}, \varphi(\mathbf{x}, \varepsilon))|\ \mathbf{x} \in U\}$*, where* φ *is* $\mathcal{C}^k$ *in* $\mathbf{x}$ *and* ε *and* $\varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x})$.

The dynamics on $\mathcal{S}_\varepsilon$ thus can be described by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \varphi(\mathbf{x}, \varepsilon), \varepsilon), \tag{B.12}$$

which is a smooth perturbation of the reduced problem (B.10). Hence structurally stable properties of (B.10) persist for sufficiently small values of ε for the restriction of the full problem (B.7) to the slow manifold $\mathcal{S}_\varepsilon$.

For $\mathcal{S}_0$ given, let n_s (n_u) denote the number of the corresponding negative (positive) eigenvalues λ_s (λ_u). Close to $\mathcal{S}_0$ there exist two invariant manifolds for the layer problem (B.9), an $(m+n_s)$-dimensional stable manifold $\mathcal{W}^s(\mathcal{S}_0)$ and an $(m+n_u)$-dimensional unstable manifold $\mathcal{W}^u(\mathcal{S}_0)$, which intersect in $\mathcal{S}_0$. Provided that $\Re(\lambda_s) < -\alpha < 0$ and $\Re(\lambda_u) > \beta > 0$, one can characterize the flow off $\mathcal{S}_\varepsilon$ in terms of its stable and unstable manifolds $\mathcal{W}^s(\mathcal{S}_\varepsilon)$ and $\mathcal{W}^u(\mathcal{S}_\varepsilon)$, respectively:

Theorem B.6. *For* $\varepsilon \in (0, \varepsilon_0]$ *there exist a stable* $(m+n_s)$*-dimensional* $\mathcal{C}^k$*-manifold* $\mathcal{W}^s(\mathcal{S}_\varepsilon)$ *and an unstable* $(m+n_u)$*-dimensional* $\mathcal{C}^k$*-manifold* $\mathcal{W}^u(\mathcal{S}_\varepsilon)$*, which are both locally invariant and* $\mathcal{C}^k$*-close to* $\mathcal{W}^s(\mathcal{S}_0)$ *and* $\mathcal{W}^u(\mathcal{S}_0)$*, respectively. The dynamics in* $\mathcal{W}^s(\mathcal{S}_\varepsilon)$ *(*$\mathcal{W}^u(\mathcal{S}_\varepsilon)$*) is described by an invariant stable (unstable)* $\mathcal{C}^k$*-foliation* $\mathcal{F}^s$ *(*$\mathcal{F}^u$*) of* $\mathcal{W}^s(\mathcal{S}_\varepsilon)$ *(*$\mathcal{W}^s(\mathcal{S}_\varepsilon)$*) such that the distance between orbits which start in the same leaf of* $\mathcal{F}^s$ *(*$\mathcal{F}^u$*) is decaying (growing) exponentially fast with rate* $e^{-\alpha t}$ *(*$e^{\beta t}$*). The leaves of* $\mathcal{F}^s$ *(*$\mathcal{F}^u$*) are invariant under the flow, i.e., each leaf* $F^s(\mathbf{x}, \mathbf{y})$ *(*$F^u(\mathbf{x}, \mathbf{y})$*) is mapped to another leaf* $F^s(\mathbf{x}(t), \mathbf{y}(t))$ *(*$F^u(\mathbf{x}(t), \mathbf{y}(t))$*) by the flow in forward (backward) time* t.

We refer to the literature for a thorough discussion including proofs.

Bibliography

[AGJ] J. Alexander, R. Gardner and C. K. R. T. Jones, *A topological invariant arising in the analysis of traveling waves*, J. Reine Angew. Math. 410 (1990) 167–212.

[Ar] V.I. Arnol'd, *Geometrical Methods in the Theory of Ordinary Differential Equations,* Grundlehren der mathematischen Wissenschaften, Bd. 250, Springer-Verlag, New York, (1983).

[Ce] Chow, S.N., Chengzhi, L., Duo, W., *Normal Forms and Bifurcations of Planar Vector Fields*, Cambridge University Press, Cambridge, (1994).

[CoLe] E.A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill (1955).

[E] J.W. Evans, *Nerve axon equations: IV. The stable and the unstable impulse,* Ind. Univ. Math. J. 24 (1975) 1169–1190.

[F1] N. Fenichel, *Persistence and Smoothness of Invariant Manifolds and Flows*; Indiana University Math. J. 21, (1971), 193-226.

[F2] N. Fenichel, *Geometric singular perturbation theory*; Journal of Diff. Eq. 31, (1979), 53-98.

[FS] H. Freistühler and P. Szmolyan, Spectral stability of small shock waves, Arch. Rational Mech.Anal 164 (2002), 287-309.

[GJ1] R. Gardner, C. K. R. T. Jones, *Stability of one-dimensional waves in weak and singular limits. Viscous profiles and numerical methods for shock waves*, (Raleigh, NC, 1990), 32–48, SIAM, Philadelphia, PA, 1991.

[GJ2] R. Gardner, C. K. R. T. Jones, *Stability of travelling wave solutions of diffusive predator-prey systems*, Trans. Amer. Math. Soc. 327 (1991), no. 2, 465–524.

[G] P. Godillon, Linear stability of shock profiles for systems of conservation laws with semi-linear relaxation, Phys. D, 148(3-4):289-316, 2001.

[Go] J. Goodman, *Nonlinear asymptotic stability of viscous shock profiles for conservation laws,* Arch. Rational Mech. Anal. 95 (1986), no. 4, 325–344.

[GrHa] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley, New York (1978).

[Gu] J. Guckenheimer,P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences, Vol. 42, Springer-Verlag, New York, (1983).

[He] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Springer-Verlag, Berlin (1981).

[HPS] M.W. Hirsch, C.C. Pugh, M. Shub, *Invariant Manifolds*; Lecture Notes in Mathematics 583, Springer-Verlag New York, (1977).

[Hu] J. Humpherys, Spectral Energy methods and the stability of shock waves, PhD Thesis, Indiana University, (2002).

[J1] C. K. R. T. Jones, *Stability of the travelling wave solution of the FitzHugh–Nagumo system*, Trans. Amer. Math. Soc. 286 (1984), no. 2, 431–469.

[J2] C. K. R. T. Jones, *Geometric singular perturbation theory*; in Dynamical Systems, Springer Lecture Notes Math. 1609, (1995), 44-120.

[JGK] C. K. R. T. Jones, R. A. Gardner, and T. Kapitula, *Stability of travelling waves for non-convex scalar viscous conservation laws*, Comm. Pure Appl. Math. 46 (1993) 505–526.

[JX] S.Jin, Z.P.Xin, The relaxation schemes for systems of conservation laws in arbitrary space dimensions, Commun. Pure Appl. Math.48(3), (1995), 235-276.

[Ka] T. Kato, *Perturbation theory for linear operators*, Springer–Verlag, Berlin Heidelberg (1985).

[L] T.-P. Liu, *Nonlinear stability of shock waves for viscous conservation laws*, Mem. Amer. Math. Soc. 56 (1985), no. 328.

[MP] A. Majda, R. L. Pego, *Stable viscosity matrices for systems of conservation laws*, J. Differential Equations 56 (1985), no. 2, 229–262.

[MZ1] C.Mascia, K.Zumbrun, *Pointwise Green's bounds and stability of relaxation shocks*, Indiana Univ. Math. J. 51 (2002), no. 4, 773-904.

[MZ2] C.Mascia, K.Zumbrun, *Stability of small amplitude shock profiles of symmetric hyperbolic-parabolic systems*, Comm. Pure Appl. Math. 57 (2004), no. 7, 841-876.

[PW] R. L. Pego, M. I. Weinstein, *Eigenvalues, and instabilities of solitary waves*, Philos. Trans. Roy. Soc. London Ser. A 340 (1992), no. 1656, 47–94.

[Sd] B. Sandstede, *Stability of travelling waves*, in *Handbook of Dynamical Systems II*, (B. Fiedler, ed.). Elsevier, (2002) 983-1055.

[St] D. H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Advances in Math. 22 (1976), no. 3, 312–355.

[Sz] P. Szmolyan, *Transversal heteroclinic and homoclinic orbits in singular perturbation problems*, J. Diff. Eq. **92**, (1991), 252–281.

[SyX] A. Szepessy, Z.-P. Xin, *Nonlinear stability of viscous shock waves*, Arch. Rational Mech. Anal. 122 (1993), no. 1, 53–103.

[T] F. Takens, *Parially hyperbolic fixed points*, Topology, Vol. 10, (1971), 133-147.

[W] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Texts in Applied Mathematics, Vol. 2, Springer-Verlag, New York, (1990).

[Z] K. Zumbrun, *Multidimensional stability of planar viscous shock waves.* Advances in the theory of shock waves, 307–516, Progr. Nonlinear Differential Equations Appl., 47, Birkhäuser Boston, Boston, MA, (2001).

[ZHo] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves,* Indiana University Mathematics Journal 47, (1998), no. 4, 741–871.