DISSERTATION

Ultraviolet/Infrared Mixing & Non-Commutative Instanton Calculus

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WENN MAN ALLES UNWAHRSCHEINLICHE AUSSCHLIESST, MUSS DAS, WAS ÜBRIG BLEIBT, UND SEI ES AUCH NOCH SO UNWAHRSCHEINLICH, DIE WAHRHEIT SEIN.

WHEN YOU HAVE ELIMINATED THE IMPOSSIBLE, THAT WHICH REMAINS, HOWEVER IMPROBABLE, MUST BE THE TRUTH.

Sherlock Holmes to Dr. Watson in 'The Sign of Four'
by Sir Arthur Conan Doyle

Kurzfassung

Nichtkommutative Feldtheorien (NKFT) unterlagen in den letzten Jahren sehr großem Interesse, um ein besseres Verständnis diverser Probleme in Quantenfeldtheorie und Stringtheorie zu bekommen.

Die Idee Raum-Zeit eine immanente Längenskala zu geben, um punktartige Wechselwirkungen, die Divergenzen in der üblichen Quantenfeldtheorie verursachen, zu vermeiden, geht zurück auf die Fünfziger Jahre. Diese fundamentale Längenskala wurde über eine Unschärferelation, welche die Nichtkommutativität der Raum-Zeit impliziert, eingeführt.

Man mußte beinahe fünfzig Jahre auf die Wiederbelebung dieses Konzeptes warten, bis sich die NKFT als ein bestimmter niederenergetischer Grenzwert der Stringtheorie zeigte. Großer Aufwand wurde betrieben, um alle neuen Eigenschaften der NKFT zu verstehen: UV/IR-Mischung, Nichtlokalität, Brechung der Lorentz-Invarianz, Frage nach Unitarität und Renormierbarkeit, Seiberg–Witten-Abbildung, Morita-Dualität, usw.

Diese Doktorarbeit beschäftigt sich hauptsächlich mit einer Untersuchung der Mischung von ultravioletten und infraroten Freiheitsgraden in nichtkommutativen Yang–Mills-Theorien (NKYMT) und ihre Auswirkung auf die Vakuumenergie dieser Theorien.

Die UV/IR-Mischung zerstört normalerweise die Renormierbarkeit von NKFT. Ein-Schleifen-Rechnungen führen zu neuen quadratischen und linearen infraroten Divergenzen, die in höheren Schleifen-Ordnungen nicht aufintegriert werden können. Desweiteren zeigt die ein-schleifen-korrigierte Dispersionsrelation eine tachyonische Instabilität bei niedrigen Energien. Aber dies ist nicht das Ende der Geschichte — Supersymmetrie hilft aus.

In supersymmetrischen nichtkommutativen Feldtheorien (SUSY NKFT) findet man nur logarithmische UV/IR-Mischungseffekte, die in geeignetem Maße behandelt werden können. Es gibt keine gefährlich quadratisch oder linear divergenten Terme. Der Grund liegt in der üblichen Auslöschung zwischen fermionischen und bosonischen Freiheitsgraden. Aber nachdem SUSY im niederenergetischen Bereich nicht realisiert ist, muß man über mögliche Brechungsszenarien nachdenken. Eine Möglichkeit besteht in schwach gebro-

chener SUSY, wo der Superpartner des Photons, das sogenannte Photino, Masse erhält. Man weiß von gewöhnlichen kommutativen Theorien, dass trotz dieses Massenterms nachwievor alle führenden Divergenzen verschwinden.

Im ersten Teil dieser Doktorarbeit berechnen wir die Ein-Schleifen-Selbstenergie des Photons in schwach gebrochener Abelscher SUSY NKYMT und finden hier ebenfalls die oben erwähnten Auslöschungseffekte. Wir erhalten nur logarithmische UV/IR-Mischungen, die eine Modifikation der Beta-Funktion bewirken, aber die Renormierbarkeit nicht verletzen. Andererseits ergeben die Berechnungen einen tachyonischen Pol in der Dispersionsrelation des Photons, der proportional zur Photinomasse, aber unabhängig vom Nichtkommutativitätsparameter ist. Dies schließt die Möglichkeit einer Realisierung von schwach gebrochener Abelscher SUSY NKYMT in der Natur aus, da man nicht mehr mit einem sehr kleinen Nichtkommutativitätsparameter argumentieren kann, der die tachyonischen Moden unterdrücken würde.

Im zweiten Teil dieser Doktorarbeit untersuchen wir Effekte der UV/IR-Mischung im nichtstörungstheoretischen Sektor von NKYMT, das heißt, wir berechnen den nichtkommutativen Instanton-Beitrag zur Vakuum–Vakuum-Übergangsamplitude. Wir wissen von gewöhnlichen Nicht-Abelschen Yang–Mills Theorien, dass Instantonen einen Größen-Modulus besitzen, der infrarote Divergenzen verursacht. Diese Divergenzen zeigen den Zusammenbruch der „Dünnes-Instanton-Gas-Näherung" bei großen Abständen.

Um die Ein-Schleifen-Instanton-Determinante in Abelscher NKYMT zu berechnen, wiederholen wir zuerst die ADHM-Konstruktion von Instantonen, die sich auf den nichtkommutativen Fall verallgemeinern läßt. Wir lösen diese deformierten ADHM-Gleichungen explizit für ein antiselbstduales Instanton. Das Instanton fällt mit einer bestimmten Potenz bei großen Abständen und besitzt signifikante Werte nur innerhalb einer Skala gegeben durch den Nichtkommutativitätsparameter, und zeigt daher keinen Größen-Modulus.

Wir hofften deshalb, dass nichtkommutative Instantonen besseres Verhalten bei großen Abständen zeigen als ihre kommutativen Verwandten. Aber wir zeigen im weiteren, dass im Fall von NKYMT die UV/IR-Mischungseffekte die „Dünnes-Instanton-Gas-Näherung" zerstören. Diesmal erhalten wir infrarote Divergenzen aber von der UV/IR-Mischung und nicht von der Integration über den Größen-Modulus. Oder anders ausgedrückt: *„Nichtkommutative Quantenfluktuationen blasen die klassisch endlichen Instantonen zu unendlicher Größe auf."*

Diese Berechnungen stellen einen weiteren Hinweis für die mögliche Inkonsistenz von NKFT dar. Aber wiederum hilft SUSY auch in diesem Fall, da sie den Ein-Schleifen-Instanton-Beitrag endlich macht. SUSY NKFT scheinen konsistente Quantenfeldtheorien zu sein, da sie nicht unter gefährlicher UV/IR-Mischung leiden.

Summary

Non-commutative field theories (NCFT) have shown up in recent years as a huge playground in order to get a much deeper understanding of various problems in quantum field theory and string theory.

The idea to give space-time an intrinsic length scale in order to avoid point-like interactions which cause divergences in standard quantum field theory, goes back to the fifties. This fundamental length scale has been introduced via an uncertainty relation which implements non-commutativity of space-time.

One had to wait almost fifty years for the revival of this concept where NCFT shows up as a certain low-energy limit of string theory. Much effort has been made in order to understand all the new features of NCFT: UV/IR mixing, non-locality, breaking of Lorentz invariance, question of unitarity and renormalizability, Seiberg–Witten map, Morita duality, etc.

The main concern of this thesis consists in an investigation of the mixing of ultraviolet and infrared degrees of freedom in non-commutative Yang–Mills theories (NCYMT) and its impact on the vacuum energy of these theories.

The UV/IR mixing normally spoils renormalizability of NCFT. One-loop calculations lead to new quadratic and linear infrared divergences which cannot be integrated over at higher loops. Furthermore, the one-loop corrected dispersion relation presents a tachyonic instability at low energies. But this is not the end of the story—supersymmetry helps out.

In supersymmetric non-commutative field theories (SUSY NCFT), one finds only logarithmic UV/IR mixing effects which could be handled in an appropriate way. There are no dangerous quadratic or linear divergent terms. The reason for that is the usual cancelation between fermionic and bosonic degrees of freedom. But, since supersymmetry is not realized at low-energy regimes, one has to think about possible breaking scenarious. One possibility consists in softly broken SUSY where the superpartner of the photon, the so-called photino, gets a mass. One knows from usual commutative theories that all leading divergences are still absent despite such a mass term.

In this thesis, we calculate the one-loop self-energy of the photon in softly

broken Abelian SUSY NCYMT and find that the above mentioned cancelation effects hold also in the non-commutative set-up. We only have to deal with logarithmic UV/IR mixing which goes into a modification of the beta-function but does not spoil renormalizability. On the other hand, we get a tachyonic pole in the dispersion relation of the photon which is proportional to the mass of the photino but this time independent of the non-commutativity parameter. Unfortunately, this rules out softly broken Abelian SUSY NCYMT in nature, because one cannot argue anymore with a very small non-commutativity parameter which would render the tachyonic modes negligible.

In the second part of this thesis, we investigate the effects of UV/IR mixing in the non-perturbative sector of NCYMT, *i.e.* we calculate the non-commutative instanton contribution to the vacuum-to-vacuum amplitude. We know from ordinary non-Abelian Yang–Mills theories that instantons exhibit a size modulus which produces infrared divergences. These divergences indicate the breakdown of the dilute instanton gas approximation at large scales.

To calculate the one-loop instanton determinant in the case of an Abelian NCYMT, we review first the ADHM construction of instantons which can be generalized to the non-commutative case. We solve these deformed ADHM equations for an anti-self-dual instanton. Having this explicit expression in hand, we recognize a power-like fall off of the instanton gauge field at large scales. Further, the instanton has significant values only at a scale given by the non-commutativity parameter. Therefore, non-commutative instantons do not have a size modulus.

This gave hope that non-commutative instantons are better behaved at large scales than their commutative counterparts. But, we show that UV/IR mixing effects ruin the dilute instanton gas calculations also for NCYMT. This time, infrared divergences arise due to the UV/IR mixing, and not from integration over a size modulus. Stated in another way: *"Non-commutative quantum fluctuations blow the classical finite size of the instanton up to infinity."*

These calculations are one hint more for the possible inconsistency of NCFT. But again, SUSY helps also in this case, rendering the one-loop instanton contribution finite. SUSY NCFT seem to be consistent quantum field theories, not suffering from dangerous UV/IR mixing.

Preface

This thesis presents work carried out during the period January 2001 – June 2004 at the Institute for Theoretical Physics, University of Technology, Vienna, Austria and at the Theory Division at CERN, Geneva, Switzerland. It was financed by the Austrian government 'Bundesministerium für Bildung, Wissenschaft und Kultur' and by the 'Fonds zur Förderung der wissenschaftlichen Forschung'.

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- Professor Manfred Schweda
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- Professor José L. F. Barbón
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Contents

Introduction

Non-commutative field theories have shown up in recent years as a huge playground in order to get a much deeper understanding of various problems in quantum field theory and string theory.

The idea to give space-time an intrinsic length scale in order to avoid point-like interactions which cause divergences in standard quantum field theory, goes back to the 1950's. This fundamental length scale has been introduced via an uncertainty relation which implements non-commutativity of space-time [1].

Non-commutativity of space-time

The basic relation implementing non-commutativity of space-time will be

$$[x^\mu\,,\,x^\nu] = \mathrm{i}\,\theta^{\mu\nu},$$

where $\theta^{\mu\nu}$ is a constant real-valued antisymmetric matrix. It has dimension of length squared and defines therefore a fundamental length scale of space-time. Since this *graininess* of space-time seems to be rather unnatural, we will give a simple heuristic argument [2, 3] that the concept of space-time has to be modified at very short distances.

In order to measure the distance d between two points with the help of an interference pattern, the wave length λ which will be used in the experiment has to be smaller than the distance d

$$\lambda < d.$$

Using the de Broglie relation we can assign a mass m to the incoming wave

$$\lambda = \frac{h}{p} = \frac{hc}{E} = \frac{h}{mc}.$$

The mass m causes a gravitational field and the smaller our distance d, the stronger the gravitational field will be. When this field becomes so strong as to prevent light or other signals from leaving the region in question, an operational meaning can no longer be attached to the measurement of the distance. Therefore, the distance d should be larger than the Schwarzschild radius $R_{\rm S}$

$$d > R_{\rm S}.$$

With the help of

$$R_{\rm S} = \frac{Gm}{c^2},$$

we get

$$d > \frac{Gm}{c^2} = \frac{Gh}{c^3}\frac{1}{\lambda} > \frac{Gh}{c^3}\frac{1}{d},$$

and therefore the condition

$$d > \sqrt{\frac{Gh}{c^3}} = \lambda_{\rm P} \simeq 10^{-33}\text{cm}.$$

With these heuristic arguments combining the principles of general relativity and quantum theory we have shown that at the Planck scale $\lambda_{\rm P}$ the concept of space-time as a differentiable manifold breaks down and we have to replace it by something else—perhaps non-commutative geometry [4, 5, 6, 7].

One had to wait almost 50 years for the revival of this concept where non-commutative field theories (NCFT) show up as a certain low-energy limit of string theory. They appeared as low-energy effective descriptions of open strings on a D-brane with a constant background B field [8]. Much effort has been made in order to understand all the new features of NCFT: UV/IR mixing, non-locality, breaking of Lorentz invariance, question of unitarity and renormalizability, Seiberg–Witten map, Morita duality, etc. For general reviews on non-commutative field theories we refer the reader to [9, 10, 11].

Planar and non-planar graphs

Due to the fundamental length scale of space-time, there has been hope that quantum field theories on non-commutative space-time do not show standard ultraviolet divergences. But, it was shown in [12] that there are still UV divergences in the so-called planar sector of the theory. All Feynman graphs in a NCFT can be divided into planar and non-planar graphs. The planar ones are equal to their commutative counterparts multiplied by a phase factor which depends only on external momenta. Therefore, from this sector we get the usual UV divergences which are handled with standard renormalization techniques.

UV/IR mixing

Non-planar graphs include phase factors like $\exp(\mathrm{i}\, k \cdot \theta \cdot p)$, with k the loop-momentum, p an external momentum and θ the NC parameter. For very high loop momenta, the phase factor oscillates very fast and renders the integral finite. There are no UV divergences coming from the non-planar sector. However, this is only valid for a non-vanishing external momentum $\theta \cdot p$. Taking the limit $\theta \cdot p \to 0$ brings back the infinity, but this time as an IR divergence. Therefore, UV modes do not decouple from IR modes in non-commutative field theories. This phenomenon is known under the name *UV/IR mixing*, and was recognized for the first time in [13, 14, 15].

Quantization & renormalizability

Quantization of non-commutative scalar field theories performed in [13, 15, 16] showed that these new UV/IR mixing effects can spoil renormalizability of a theory. One-loop calculations lead to new quadratic and linear IR divergences which cannot be integrated over at higher loops [17, 18]. Furthermore, the one-loop corrected dispersion relation can present a tachyonic instability at low energies.

At first, there was hope that gauge theories are better behaved with respect to UV/IR mixing phenomena. Due to gauge invariance, a gauge theory has only logarithmic UV divergences which would naively suggest that there are also only logarithmic IR divergences in the non-planar sector. But, it was shown in [14] that there are also dangerous quadratic and linear IR divergences spoiling therefore renormalizability of non-commutative Yang–Mills (NCYM) theory at higher loops. Interesting work in this context has been done in [19, 20, 21, 22, 23, 24, 25, 26, 27].

Supersymmetry

The next logical step was to consider supersymmetric models. And indeed, in supersymmetric non-commutative field theories (SUSY NCFT) one has only logarithmic UV/IR mixing effects which could be handled in an appropriate way—there are no dangerous quadratic and linear divergent terms [28, 29, 30, 31, 32, 33, 34, 35]. The reason for that is the usual cancelation between fermionic and bosonic degrees of freedom which hold also in a non-commutative set-up.

As mentioned above, non-commutative field theories can suffer from a tachyonic instability in the low-momentum regime. This is also true in the case of gauge theories [14], but again, supersymmetry can heal this problem in avoiding all quadratic UV/IR mixing effects in the one-loop corrected dispersion relation [36, 37, 38].

Softly broken SUSY

Since supersymmetry is not realized in real world at low-energy regimes, one has to think about possible breaking scenarious. One possibility consists in softly broken SUSY where the superpartner of the photon, the photino, gets a mass. It has been known from ordinary field theory that all leading divergences are still absent despite such a mass term.

The study of softly broken SUSY NCYM theory has shown that the above mentioned cancelation effects hold also in a non-commutative set-up. One only has to deal with logarithmic UV/IR mixing effects, but in contrast to the fully supersymmetric case, one gets a tachyonic pole in the dispersion relation of the photon which is proportional to the mass of the photino, but this time independent of the non-commutativity parameter [39, 40, 41, 42]. Unfortunately, this rules out softly broken SUSY NCYM theory in nature, because one cannot argue anymore with a very small non-commutativity parameter which would render the tachyonic modes negligible.

Seiberg–Witten map

There exists also a complete other philosophy of quantizing non-commutative gauge theories. This approach makes use of the so-called Seiberg–Witten map [8] which can be applied to expand a non-commutative gauge field in terms of its commutative counterpart [43, 44]. Quantization of the Seiberg–Witten map was first performed in [45] and put forward in [46, 47, 48, 49, 50, 51, 52, 53, 54, 55]. Roughly speaking, there are no UV/IR mixing effects in this approach, but one has to deal with an infinite amount of interaction vertices. Further interesting attempts to cure the UV/IR mixing can be found in [56, 57, 58, 59, 60, 61, 62, 63, 64, 65]. Here, we will not use the Seiberg–Witten map, but will refer at several stages to it.

Recently, it was found in [66, 67, 68, 69, 70] that non-commutative scalar field theory can be renormalized if one does not use the standard plane wave expansion for the fields. Despite of being non-supersymmetric and not Seiberg–Witten mapped, there are no dangerous quadratic or linear IR divergences in the game.

Non-commutative instantons

So far, we considered perturbative studies of non-commutative field theories. But, of course, it is worth to take a short look on the non-perturbative sector of the theory. Especially, we are interested here in non-commutative instantons which have been first constructed in [71] via a deformation of the standard ADHM construction [72]. A review can be found in [73]. Explicit solutions in four-dimensional non-commutative spaces have been derived and discussed in [74, 75, 76, 77, 78, 79].

In contrast to the usual commutative case, non-commutative instantons exist also in Abelian Yang–Mills theories. A further very important difference is the lack of a size modulus, since scale invariance is explicitly broken in non-commutative field theories. Hence, the instanton has significant values only at a scale given by the square root of the non-commutativity parameter. This gave hope that the dilute instanton gas approximation, which is rendered IR divergent in the commutative case through integration over the size modulus of the instanton, is better behaved in the non-commutative case.

Outline of the thesis

In the first chapter, we will review the basic concepts of non-commutative space-time which will be used for the study of non-commutative field theories. In chapter two, we will study non-commutative Yang–Mills theory, deriving Feynman rules and calculating explicitly the self-energy. Then, we discuss the effects coming from UV/IR mixing and consider also the supersymmetric and softly broken supersymmetric cases. Chapter three will be concerned with the construction of non-commutative instantons and the study of their properties. The last chapter is devoted to a study of the instanton-induced vacuum energy in non-commutative Abelian Yang–Mills theory. It is based on the last work of the author [80].

Chapter 1

Non-Commutative Space-Time

Non-commutative space-time $\mathbb{R}^D_\theta$ is defined by replacing space-time coordinates x^μ by Hermitian generators $\hat{x}^\mu$ of a non-commutative C^*-algebra of "functions on space-time" [4, 5, 6, 7] which obey the following commutation relations:

$$[\hat{x}^\mu\,,\,\hat{x}^\nu] = \mathrm{i}\,\theta^{\mu\nu}. \tag{1.1}$$

In the whole work, we will only consider the simplest special case of (1.1) where $\theta^{\mu\nu}$ is a constant, real-valued antisymmetric $D \times D$ matrix (D is the dimension of space-time) with dimensions of length squared. Since the coordinates no longer commute, they cannot be simultaneously diagonalized and the underlying space disappears, *i.e.* the space-time manifold gets replaced by a Hilbert space of states.

1.1 Weyl map

Within the framework of canonical quantization, Weyl introduced an elegant prescription for associating a quantum operator to a classical function of the phase space variables [81]. This technique can be used also here, since the $\hat{x}^\mu$ in (1.1) generate a non-commutative algebra of operators. Weyl quantization provides a one-to-one correspondence between the algebra of fields on $\mathbb{R}^D$ and this ring of operators, and it may be thought of as an analog of the operator-state correspondence of local quantum field theory.

Let us take a function $f(x)$ on $\mathbb{R}^D$ and its corresponding Fourier coefficients given by

$$\tilde{f}(k) = \int d^D x\; \mathrm{e}^{-\mathrm{i}k_\mu x^\mu}\, f(x), \tag{1.2}$$

with $\tilde{f}(-k) = \tilde{f}(k)^*$ whenever $f(x)$ is real-valued. Then, we introduce its *Weyl symbol* by

$$\mathfrak{W}[f] = \int \frac{d^D k}{(2\pi)^D} \, \mathrm{e}^{\mathrm{i}k_\mu \hat{x}^\mu} \, \tilde{f}(k), \tag{1.3}$$

where $\mathfrak{W}[f]$ is Hermitian if $f(x)$ is real-valued. Notice, that in the exponential of (1.3) appears the operator $\hat{x}^\mu$. Therefore, we have to take a certain operator ordering which in this case is just the symmetric Weyl ordering.

Next, we introduce the Hermitian operator $\Diamond(x)$ given by

$$\Diamond(x) = \int \frac{d^D k}{(2\pi)^D} \, \mathrm{e}^{\mathrm{i}k_\mu \hat{x}^\mu} \, \mathrm{e}^{-\mathrm{i}k_\mu x^\mu}, \tag{1.4}$$

which provides us with an explicit map between the fields $f(x)$ and the associated operators $\mathfrak{W}[f]$:

$$\mathfrak{W}[f] = \int d^D x \, f(x) \, \Diamond(x). \tag{1.5}$$

The expression (1.5) can be verified directly by inserting (1.4) and using (1.2) which gives us back the Weyl symbol (1.3).

Therefore, we can interpret the field $f(x)$ as the coordinate space representation of the Weyl operator $\mathfrak{W}[f]$. Notice here again, that (1.4) is a highly non-trivial field operator, which reduces only in the commutative limit $\theta^{\mu\nu} \to 0$ to a delta-function $\delta^D(\hat{x} - x)$.

Since we have established now the coordinate space representation of a Weyl operator, we would like to have a proper notion of derivation and integration on non-commutative space. We may introduce "derivatives" of operators through an anti-Hermitian linear derivation $\hat{\partial}_\mu$ which is defined by the commutation relations

$$\left[\hat{\partial}_\mu \, , \, \hat{x}^\nu\right] = \delta_\mu{}^\nu, \qquad \left[\hat{\partial}_\mu \, , \, \hat{\partial}_\nu\right] = 0. \tag{1.6}$$

With the definitions (1.6) and (1.4) it is easy to show that

$$\left[\hat{\partial}_\mu \, , \, \Diamond(x)\right] = -\partial_\mu \, \Diamond(x). \tag{1.7}$$

Using (1.7) and (1.5) we arrive upon integration by parts at

$$\left[\hat{\partial}_\mu \, , \, \mathfrak{W}[f]\right] = \int d^D x \, \partial_\mu f(x) \, \Diamond(x) = \mathfrak{W}[\partial_\mu f]. \tag{1.8}$$

Therefore, we see from equation (1.8) that the introduced "derivatives" on the operator side indeed correspond to the usual derivatives on the associated fields on non-commutative space-time.

Further, we can interpret the expression (1.7) as an infinitesimal translation generated by the operator $\hat{\partial}_\mu$ which we can exponentiate to a finite one given by

$$\mathrm{e}^{y^\mu \hat{\partial}_\mu} \lozenge(x)\, \mathrm{e}^{-y^\mu \hat{\partial}_\mu} = \lozenge(x+y), \tag{1.9}$$

where y is a finite position vector in $\mathbb{R}^D$. Defining now a cyclic trace $\mathrm{Tr}_{\mathfrak{W}}$ on the algebra of Weyl operators, and acting with it on (1.9) yields

$$\mathrm{Tr}_{\mathfrak{W}} \lozenge(x) = \mathrm{Tr}_{\mathfrak{W}} \lozenge(x+y). \tag{1.10}$$

Therefore, we have the very important property that $\mathrm{Tr}_{\mathfrak{W}} \lozenge(x)$ is independent of the position in space-time and we can normalize it by

$$\mathrm{Tr}_{\mathfrak{W}} \lozenge(x) = 1. \tag{1.11}$$

With the normalization (1.11) we get by taking the trace of (1.5) the following result:

$$\mathrm{Tr}_{\mathfrak{W}} \mathfrak{W}[f] = \int d^D x\, f(x). \tag{1.12}$$

This shows that the operator trace $\mathrm{Tr}_{\mathfrak{W}}$ is equivalent to integration over the non-commutative coordinates.

The next thing we would like to show is that the relation (1.5) is invertible, and the operator $\lozenge(x)$ provides therefore a one-to-one correspondence between fields and operators.

For this purpose we will calculate the product of operators $\lozenge(x)$ at distinct points. Using the Baker–Campbell–Hausdorff formula

$$\mathrm{e}^A\, \mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]}, \qquad \text{with } [A,[A,B]] = [B,[A,B]] = 0, \tag{1.13}$$

we first calculate the product

$$\mathrm{e}^{\mathrm{i}k_\mu \hat{x}^\mu}\, \mathrm{e}^{\mathrm{i}k'_\mu \hat{x}^\mu} = \mathrm{e}^{\mathrm{i}(k+k')_\mu \hat{x}^\mu}\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\theta^{\mu\nu} k_\mu k'_\nu}. \tag{1.14}$$

From the defintion (1.4) and the relation (1.14) we derive

$$\begin{aligned}
\lozenge(x)\,\lozenge(y) &= \iint \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\; e^{i(k+k')_\mu \hat{x}^\mu}\; e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\; e^{-ik_\mu x^\mu - ik'_\mu y^\mu} \\
&= \iiint \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\; d^Dz\; e^{i(k+k')_\mu z^\mu}\; \lozenge(z)\; e^{-\frac{i}{2}\theta^{\mu\nu}k_\mu k'_\nu}\; e^{-ik_\mu x^\mu - ik'_\mu y^\mu} \\
&= \iiint \frac{d^Dk}{(2\pi)^D}\frac{d^Dk'}{(2\pi)^D}\; d^Dz\; \lozenge(z)\; e^{-\frac{i}{2}k_\mu\theta^{\mu\nu}k'_\nu}\; e^{ik_\mu(z-x)^\mu}\; e^{ik'_\nu(z-y)^\nu} \\
&= \iint \frac{d^Dk}{(2\pi)^D}\; d^Dz\; \lozenge(z)\; \delta^D\big((z-y)^\nu - \frac{1}{2}k_\mu\theta^{\mu\nu}\big)\; e^{ik_\mu(z-x)^\mu} \\
&= \iint \frac{d^Dk}{(2\pi)^D}\; d^Dz\; \lozenge(z)\; \frac{2^D}{|\det\theta|}\,\delta^D\big(2(z-y)^\nu(\theta^{-1})_{\nu\mu} - k_\mu\big)\; e^{ik_\mu(z-x)^\mu} \\
&= \frac{1}{\pi^D\,|\det\theta|}\int d^Dz\; \lozenge(z)\; e^{2i(z-y)^\nu(\theta^{-1})_{\nu\mu}(z-x)^\mu}.
\end{aligned} \tag{1.15}$$

Let us give a few comments about certain steps in the calculation of (1.15). In the second line we used the fact that

$$\mathfrak{W}[e^{iq_\mu x^\mu}] = e^{iq_\mu \hat{x}^\mu} = \int d^Dx\; e^{iq_\mu x^\mu}\; \lozenge(x), \tag{1.16}$$

which can be seen from the equations (1.3) and (1.5). Further, we used the property $\delta(a\,x) = \delta(x)/|a|$, where we have assumed that $\theta^{\mu\nu}$ is an invertible matrix with $\theta^{\mu\nu}(\theta^{-1})_{\nu\rho} = \delta^\mu{}_\rho$. Of course, this can be true only in even space-time dimensions, since theta being antisymmetric. In case of uneven space-time dimensions or a degenerate theta, we perfom the calculation (1.15) only in a subspace where theta is non-degenerate.

Having established the explicit expression of the product $\lozenge(x)\,\lozenge(y)$, we can take the trace of (1.15) and get

$$\begin{aligned}
\mathrm{Tr}_{\mathfrak{W}}\Big(\lozenge(x)\,\lozenge(y)\Big) &= \frac{1}{\pi^D\,|\det\theta|}\int d^Dz\; \mathrm{Tr}_{\mathfrak{W}}\,\lozenge(z)\; e^{2i(z-y)^\nu(\theta^{-1})_{\nu\mu}(z-x)^\mu} \\
&= \frac{1}{\pi^D\,|\det\theta|}\int d^Dz\; e^{2i(z-y)^\nu(\theta^{-1})_{\nu\mu}(z-x)^\mu} \\
&= \frac{1}{\pi^D\,|\det\theta|}\int d^Dz\; e^{2i(\theta^{-1})_{\nu\mu}z^\mu(x-y)^\nu}\; e^{2i(\theta^{-1})_{\nu\mu}x^\mu y^\nu} \\
&= \frac{2^D}{|\det\theta|}\,\delta^D\big(2(\theta^{-1})_{\nu\mu}(x-y)^\nu\big)\; e^{2i(\theta^{-1})_{\nu\mu}x^\mu y^\nu} \\
&= \delta^D(x-y)\; e^{2i(\theta^{-1})_{\nu\mu}x^\mu y^\nu} \\
&= \delta^D(x-y),
\end{aligned} \tag{1.17}$$

where we used the normalization (1.11) and the antisymmetry of $(\theta^{-1})_{\nu\mu}$. We see from (1.17) that the operators $\lozenge(x)$ form an orthonormal set. This implies that the transformation (1.5) is invertible, where the inverse is given by

$$f(x) = \mathrm{Tr}_{\mathfrak{W}}\Big(\mathfrak{W}[f]\,\lozenge(x)\Big). \tag{1.18}$$

We can easily proof equation (1.18) by inserting (1.5) and using the orthonormal relation (1.17):

$$\begin{aligned} f(x) &= \mathrm{Tr}_{\mathfrak{W}}\Big(\mathfrak{W}[f]\,\lozenge(x)\Big) = \mathrm{Tr}_{\mathfrak{W}}\Big(\int d^D y\; f(y)\,\lozenge(y)\,\lozenge(x)\Big) \\ &= \int d^D y\; f(y)\,\mathrm{Tr}_{\mathfrak{W}}\Big(\lozenge(y)\,\lozenge(x)\Big) = \int d^D y\; f(y)\,\delta^D(y-x) = f(x). \end{aligned}$$

Therefore, we have established a one-to-one correspondence between operators and fields on non-commutative space—the so-called *Weyl map* given by

$$f(x) \quad \xleftrightarrow{\lozenge(x)} \quad \mathfrak{W}[f]. \tag{1.19}$$

1.2 Star-product

Since we are able now to represent the Weyl operator $\mathfrak{W}[f]$ via the function $f(x)$ in non-commutative coordinate space we would like to go one step further and try to find the appropriate representation for the product of two Weyl operators $\mathfrak{W}[f]\,\mathfrak{W}[g]$.

Therefore, we introduce the so-called *star-product*:

$$\mathfrak{W}[f \star g] = \mathfrak{W}[f]\,\mathfrak{W}[g]. \tag{1.20}$$

Inverting the definition (1.20) with the help of (1.18) yields

$$(f \star g)(x) = \mathrm{Tr}_{\mathfrak{W}}\Big(\mathfrak{W}[f]\,\mathfrak{W}[g]\,\lozenge(x)\Big), \tag{1.21}$$

which will be calculated in the following in order to get an explicit representation of the star-product in coordinate space.

Inserting (1.5) into (1.21) we get

$$(f \star g)(x) = \iint d^D y\, d^D z\; f(y)\,g(z)\,\mathrm{Tr}_{\mathfrak{W}}\Big(\lozenge(y)\,\lozenge(z)\,\lozenge(x)\Big). \tag{1.22}$$

Using (1.15) and (1.17) for the expression under the trace in equation (1.22) yields

$$
\begin{aligned}
&\mathrm{Tr}_{\mathfrak{W}}\Big(\Diamond(y)\,\Diamond(z)\,\Diamond(x)\Big) = \\
&\quad = \frac{1}{\pi^D\,|\det\theta|}\int d^Dw\;\mathrm{e}^{2\mathrm{i}(w-z)^\nu(\theta^{-1})_{\nu\mu}(w-y)^\mu}\;\mathrm{Tr}_{\mathfrak{W}}\Big(\Diamond(w)\,\Diamond(x)\Big) \\
&\quad = \frac{1}{\pi^D\,|\det\theta|}\;\mathrm{e}^{2\mathrm{i}(x-z)^\nu(\theta^{-1})_{\nu\mu}(x-y)^\mu}.
\end{aligned} \tag{1.23}
$$

Taking now the inverse Fourier transformation

$$
f(x) = \int \frac{d^Dk}{(2\pi)^D}\;\mathrm{e}^{\mathrm{i}k_\mu x^\mu}\,\tilde{f}(k), \tag{1.24}
$$

and the expression (1.23) we can write (1.22) as

$$
\begin{aligned}
(f\star g)(x) = \frac{1}{\pi^D\,|\det\theta|}\iiiint \frac{d^Dk}{(2\pi)^D}\,\frac{d^Dk'}{(2\pi)^D}\,d^Dy\,d^Dz\;\tilde{f}(k)\,\tilde{g}(k') \\
\times\,\mathrm{e}^{\mathrm{i}k_\mu y^\mu}\,\mathrm{e}^{\mathrm{i}k'_\mu z^\mu}\,\mathrm{e}^{2\mathrm{i}(x-z)^\nu(\theta^{-1})_{\nu\mu}(x-y)^\mu}.
\end{aligned} \tag{1.25}
$$

Rewriting the term in the last exponent of (1.25) as

$$
2\mathrm{i}(x-z)^\nu(\theta^{-1})_{\nu\mu}(x-y)^\mu = 2\mathrm{i}(\theta^{-1})_{\nu\mu}\big(z^\nu(y-x)^\mu - x^\nu y^\mu\big), \tag{1.26}
$$

we can perform the integration over z and y in (1.25) and get

$$
\begin{aligned}
(f\star g)(x) &= \frac{2^D}{|\det\theta|}\iiint \frac{d^Dk}{(2\pi)^D}\,\frac{d^Dk'}{(2\pi)^D}\,d^Dy\;\tilde{f}(k)\,\tilde{g}(k') \\
&\qquad\times\,\mathrm{e}^{\mathrm{i}k_\mu y^\mu}\,\mathrm{e}^{-2\mathrm{i}(\theta^{-1})_{\nu\mu}x^\nu y^\mu}\,\delta^D\big(k'_\nu + 2(\theta^{-1})_{\nu\mu}(y-x)^\mu\big) \\
&= \iiint \frac{d^Dk}{(2\pi)^D}\,\frac{d^Dk'}{(2\pi)^D}\,d^Dy\;\tilde{f}(k)\,\tilde{g}(k') \\
&\qquad\times\,\mathrm{e}^{\mathrm{i}k_\mu y^\mu}\,\mathrm{e}^{-2\mathrm{i}(\theta^{-1})_{\nu\mu}x^\nu y^\mu}\,\delta^D\big(\tfrac{1}{2}\,\theta^{\mu\nu}k'_\nu + (y-x)^\mu\big) \\
&= \iint \frac{d^Dk}{(2\pi)^D}\,\frac{d^Dk'}{(2\pi)^D}\;\tilde{f}(k)\,\tilde{g}(k') \\
&\qquad\times\,\mathrm{e}^{\mathrm{i}k_\mu(x^\mu-\frac{1}{2}\theta^{\mu\nu}k'_\nu)}\,\mathrm{e}^{-2\mathrm{i}(\theta^{-1})_{\nu\mu}x^\nu(x^\mu-\frac{1}{2}\theta^{\mu\rho}k'_\rho)} \\
&= \iint \frac{d^Dk}{(2\pi)^D}\,\frac{d^Dk'}{(2\pi)^D}\;\tilde{f}(k)\,\tilde{g}(k')\,\mathrm{e}^{\mathrm{i}(k+k')_\mu x^\mu}\,\mathrm{e}^{-\frac{\mathrm{i}}{2}k_\mu\theta^{\mu\nu}k'_\nu}.
\end{aligned} \tag{1.27}
$$

Looking at the result (1.27) we just recognize that it corresponds to the Fourier transform of the usual product $f(x)\,g(x)$ but with the additional phase factor $\exp(-\frac{\mathrm{i}}{2}\,k_\mu\theta^{\mu\nu}k'_\nu)$.

Transforming the expression (1.27) back from momentum space into coordinate space we get the following formula for the star-product:

$$(f \star g)(x) = f(x) \exp\left(\frac{\mathrm{i}}{2} \overleftarrow{\partial_\mu} \theta^{\mu\nu} \overrightarrow{\partial_\nu}\right) g(x). \tag{1.28}$$

The star-product (1.28) is associative but non-commutative , and it is defined for a constant, possibly degenerate matrix $\theta^{\mu\nu}$. In the commutative limit $\theta^{\mu\nu} \to 0$, it reduces to the ordinary product of functions. Further, it is a highly *non-local* product, because it includes an infinite number of derivatives which can be seen by expanding the exponential in (1.28):

$$\begin{aligned}(f \star g)(x) &= f(x)\, g(x) \\ &+ \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \cdots \theta^{\mu_n\nu_n}\, \partial_{\mu_1} \cdots \partial_{\mu_n} f(x)\, \partial_{\nu_1} \cdots \partial_{\nu_n} g(x).\end{aligned} \tag{1.29}$$

More exactly, the product (1.28) is called Groenewold–Moyal star-product, and it is a particular example of a star-product which is normally defined in deformation quantization [82, 83].

A generalization of the expression (1.21) via

$$(f_1 \star \cdots \star f_n)(x) = \mathrm{Tr}_{\mathfrak{W}}\Big(\mathfrak{W}[f_1] \cdots \mathfrak{W}[f_n]\, \lozenge(x)\Big), \tag{1.30}$$

leads with an analogous calculation as above to the following formula for the star-product of an arbitrary number of functions:

$$(f_1 \star \cdots \star f_n)(x) = \prod_{i<j}^{n} \exp\left(\frac{\mathrm{i}}{2} \theta^{\mu\nu} \frac{\partial}{\partial x_i^\mu} \frac{\partial}{\partial x_j^\nu}\right) f_1(x_1) \cdots f_n(x_n)\Bigg|_{x_1 = \cdots = x_n = x}. \tag{1.31}$$

Therefore, we can draw the very important conclusion that the space-time non-commutativity may be encoded through ordinary products in the non-commutative C^*-algebra of Weyl operators, or equivalently through the deformation of the product of the commutative C^*-algebra of functions on space-time to the non-commutative star-product.

Before concluding this section we would like to state that cyclicity of the operator trace $\mathrm{Tr}_{\mathfrak{W}}$ implements cyclicity of the star-product under space-time integration. One can see this by the identity

$$\mathrm{Tr}_{\mathfrak{W}}\Big(\mathfrak{W}[f_1] \cdots \mathfrak{W}[f_n]\Big) = \mathrm{Tr}_{\mathfrak{W}}\Big(\mathfrak{W}[f_1 \star \cdots \star f_n]\Big) = \int d^D x\, (f_1 \star \cdots \star f_n)(x), \tag{1.32}$$

which follows from (1.20) and (1.12). But note the important point, that (1.32) is only invariant under cyclic permutations, and not under arbitrary ones.

Further, the star-product of two functions reduces under space-time integration to the usual commutative product:

$$\int d^D x\, f(x) \star g(x) = \int d^D x\, f(x)\, g(x). \tag{1.33}$$

This can be directly verified by inserting (1.28) and integrating by parts,[1] or by using the momentum space representation of the star-product (1.27) and momentum conservation. But note, that this works only for the product of two functions.

Next, we define the Moyal commutator and anti-commutator given by the following two formulas, respectively:

$$\begin{aligned} [f(x)\,,\, g(x)]_\star &= f(x) \star g(x) - g(x) \star f(x) \\ &= 2\mathrm{i}\, f(x)\, \sin\left(\frac{1}{2}\overleftarrow{\partial_\mu}\,\theta^{\mu\nu}\,\overrightarrow{\partial_\nu}\right)\, g(x), \end{aligned} \tag{1.34}$$

$$\begin{aligned} \{f(x)\,,\, g(x)\}_\star &= f(x) \star g(x) + g(x) \star f(x) \\ &= 2\, f(x)\, \cos\left(\frac{1}{2}\overleftarrow{\partial_\mu}\,\theta^{\mu\nu}\,\overrightarrow{\partial_\nu}\right)\, g(x). \end{aligned} \tag{1.35}$$

The form of the bi-differential operators in (1.34) and (1.35) can be verified again with the explicit form of the star-product (1.28).

Furthermore, we would like to mention that the Moyal commutator (1.34) can be used to generate derivatives:

$$[x^\mu\,,\, f(x)]_\star = \mathrm{i}\,\theta^{\mu\nu}\,\partial_\nu f(x). \tag{1.36}$$

As a last point, we introduce for later purpose the matrix norm θ of the non-commutativity parameter $\theta^{\mu\nu}$:

$$\theta = \max|\theta^{\mu\nu}|. \tag{1.37}$$

This concludes our introduction in non-commutative space-time $\mathbb{R}^D_\theta$.

[1] We assumed here from the beginning that all functions are of Schwartz type, *i.e.* they are faster decreasing than any polynomial in x.

Chapter 2

Non-Commutative Gauge Theory

2.1 Non-commutative field theory

Having established in the previous chapter all the main features of non-commutative space-time, we can start thinking about field theories on such spaces, called *non-commutative field theories*. Despite the fact, that we will focus here on non-commutative gauge theories, we will start in considering very briefly the simpler case of a non-commutative scalar field theory.

2.1.1 Non-commutative scalar theory

To illustrate the general idea of obtaining the action of a non-commutative field theory, we consider first the case of a massive real scalar field theory with quartic interaction in D dimensions. To transform an ordinary scalar field theory into a non-commutative one, we may use the Weyl quantization procedure we introduced in chapter 1. Written in terms of the Hermitian Weyl operator $\mathfrak{W}[\phi]$ corresponding to a real scalar field $\phi(x)$ on $\mathbb{R}^D$, the action is given by

$$S[\phi] = \mathrm{Tr}_{\mathfrak{W}} \left(\frac{1}{2} \left[\hat{\partial}_\mu \, , \, \mathfrak{W}[\phi] \right]^2 + \frac{m^2}{2} \mathfrak{W}[\phi]^2 + \frac{\lambda}{4!} \mathfrak{W}[\phi]^4 \right). \tag{2.1}$$

We can rewrite the action (2.1) with the help of the map (1.5) and the relations (1.8), (1.17), (1.20) and (1.33) in coordinate space:

$$S[\phi] = \int d^D x \left[\frac{1}{2} \Big(\partial_\mu \phi(x) \Big)^2 + \frac{m^2}{2} \phi(x)^2 + \frac{\lambda}{4!} \phi(x) \star \phi(x) \star \phi(x) \star \phi(x) \right]. \tag{2.2}$$

By looking at the action (2.2) we can draw the following important conclusions, which are valid for every non-commutative field theory:

- There are no star-products in the bilinear expression of a non-commutative action, which follows from property (1.33). Therefore, all propagators of a non-commutative field theory are the same as in the corresponding ordinary commutative theory, because they are just given by the inverse of the operators between two fields in the theory.

- The interaction vertices get drastically modified due to the appearance of the star-products. They become highly non-local interactions and the corresponding Feynman rules will be modified by momentum dependent phase factors. This fact can be seen from the momentum space representation of the star-product in equation (1.27), but will become more clear in explicit calculations in section 2.2.

- The non-commutative version of a field theory can simply be obtained by replacing all products in the action by star-products.

After this very basic look on non-commutative field theories by the simple example of a scalar field theory, we will go on and consider non-commutative gauge theories. For the case of non-commutative scalar field theories we refer the reader to the literature [12, 13, 15, 16], where they have been extensively discussed. Excellent reviews on NCFT can be found in [9, 10, 11].

2.1.2 Non-commutative Yang–Mills theory

The Weyl quantization procedure of chapter 1 generalizes straightforwardly to the algebra of $N \times N$ matrix-valued functions on $\mathbb{R}^D$. The star-product then becomes the tensor product of matrix multiplication with the Groenewold–Moyal product (1.28) of functions which is still associative. We can therefore use this method to systematically construct non-commutative gauge theories [43].

We denote the Hermitian matrix-valued $U(N)$ gauge field by $A_\mu(x)$ and introduce its Weyl operator by generalizing equation (1.5):

$$\mathfrak{W}[A_\mu] = \int d^D x \; \Diamond(x) \otimes A_\mu(x), \tag{2.3}$$

where $\Diamond(x)$ is the Weyl map (1.4) and the tensor product between the coordinate and matrix representations is written explicitly for emphasis. With this in hand, we can write down the non-commutative version of the Yang–Mills action given by

$$S[A] = -\frac{1}{4g^2}\,\mathrm{Tr}_{\mathfrak{W}\otimes N}\left(\Big[\hat{\partial}_\mu\,,\,\mathfrak{W}[A_\nu]\Big] - \Big[\hat{\partial}_\nu\,,\,\mathfrak{W}[A_\mu]\Big] - \mathrm{i}\Big[\mathfrak{W}[A_\mu]\,,\,\mathfrak{W}[A_\nu]\Big]\right)^2, \tag{2.4}$$

where $\mathrm{Tr}_{\mathfrak{W}\otimes N}$ denotes the tensor product of the operator trace $\mathrm{Tr}_{\mathfrak{W}}$ (1.12) and the trace Tr_N in some representation of the gauge group $U(N)$. With the definition of the star-product (1.20) and the relations (1.8), (1.17), (2.3) we can rewrite the action (2.4) as

$$S[A] = -\frac{1}{4g^2}\int d^Dx\;\mathrm{Tr}_N\Big(F_{\mu\nu}(x)\star F^{\mu\nu}(x)\Big), \tag{2.5}$$

where we introduced the non-commutative field strength

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - \mathrm{i}\,[A_\mu(x)\,,\,A_\nu(x)]_\star\,. \tag{2.6}$$

Note the appearance of the star-product in the equations (2.5) and (2.6).

In the following, we will work with the action (2.5) and denote it as the *non-commutative Yang–Mills action.* As explained in section 1.2, the star-product does not play a role for bilinear expressions under space-time integrations. Therefore, when considering perturbation theory, the gauge field propagator will be the same as in usual gauge theory, since it is simply given by the inverse of the operator between two gauge fields.[1] On the other hand, the star-product seriously changes the vertices of the theory, and even in the case of the Abelian gauge group $U(1)$ we have a non-trivial interacting theory because of a non-vanishing Moyal commutator in (2.6). This is quite astonishing, since the commutative limit of non-commutative $U(1)$ Yang–Mills theory corresponds to the free Maxwell theory.

The action (2.5) is invariant under the following local star-gauge transformation parameterized by a function $\lambda(x)$ and given by

$$\delta_\lambda A_\mu(x) = \partial_\mu\lambda(x) + \mathrm{i}\,[\lambda(x)\,,\,A_\mu(x)]_\star\,, \tag{2.7}$$

where the field strength transforms covariantly with

$$\delta_\lambda F_{\mu\nu}(x) = \mathrm{i}\,[\lambda(x)\,,\,F_{\mu\nu}(x)]_\star\,. \tag{2.8}$$

We can verify the star-gauge invariance just by inserting (2.8) into (2.5) and using the cyclicity of the star-product under space-time integrations together with the cyclicity of the matrix trace Tr_N.

[1] We will discuss gauge fixing later on.

We should mention here that it is not possible to realize every gauge group G on a non-commutative manifold in a consistent way [44, 84, 85]. To see this, let us write all fields $\Phi(x)$ appearing in the theory in some basis spanned by the generators T^a of the Lie algebra of G:

$$\Phi(x) = \sum_a T^a \, \Phi^a(x). \tag{2.9}$$

Inserting (2.9) for $A_\mu(x)$ and $\lambda(x)$ in (2.7) yields

$$\begin{aligned}\delta_\lambda A_\mu(x) = T^a \, \partial_\mu \lambda^a(x) \\ &+ \frac{\mathrm{i}}{2} \left[T^a \, , \, T^b\right] \left(\lambda^a(x) \star A^b_\mu(x) + A^b_\mu(x) \star \lambda^a(x)\right) \\ &\quad + \frac{\mathrm{i}}{2} \left\{T^a \, , \, T^b\right\} \left(\lambda^a(x) \star A^b_\mu(x) - A^b_\mu(x) \star \lambda^a(x)\right) .\end{aligned} \tag{2.10}$$

Therefore, the non-commutative character of the star-product implies that star-gauge transformations depend on the anti-commutator $\left\{T^a \, , \, T^b\right\}$, together with the usual commutator terms $\left[T^a \, , \, T^b\right]$. In general, the anti-commutator of two generators belongs to the Lie algebra only in the case of unitary groups $U(N)$. Thus, the discussions of non-commutative Yang–Mills (NCYM) theories are normally restricted to $U(N)$ groups.[2]

A further very important property of star-gauge symmetry is the non-existence of naive local gauge-invariant operators like $\mathrm{Tr}_N \, F_{\mu\nu}(x) \star F^{\mu\nu}(x)$. Since the star-product is cyclic only under space-time integrations, we have to take operators like $\int d^D x \;\, \mathrm{Tr}_N \, F_{\mu\nu}(x) \star F^{\mu\nu}(x)$ which are gauge invariant. Therefore, standard local operators must be integrated over in order to remain gauge-invariant on non-commutative space-time.

2.1.3 Perturbative Quantization

In order to quantize a non-commutative field theory we can start by writing down its formal path integral given by

$$Z[J] = \int [d\Phi] \, \mathrm{e}^{\mathrm{i}\, S_\star[\Phi]} \, \mathrm{e}^{\mathrm{i} \int d^D x \; J \star \Phi}, \tag{2.11}$$

where $Z[J]$ is the generating functional of Green's functions, and J denotes the external classical sources of the fields Φ in the theory. The integration measure $[d\Phi]$ has to be defined properly. The action $S_\star[\Phi]$ is normally obtained by taking the usual commutative counterpart of the theory and replace

[2] There exists a possibility to work with other groups than $U(N)$ which will be discussed in the context of the Seiberg–Witten map in section 2.6.

all products in it with star-products. See, *e.g.* the NCYM action of section 2.1.2. For the time being, we will restrict ourselves to the perturbative evaluation of $Z[J]$.

Mainly, there exists two philosophies how to quantize a non-commutative field theory correctly:

$$\begin{tikzcd} \star\text{-FT} \arrow[r, "\text{expansion}", "\text{in } \theta"'] \arrow[d, "\hbar"'] & \theta\text{-FT} \arrow[d, "\hbar"] \\ \star\text{-QFT} \arrow[r, "???"'] & \theta\text{-QFT} \end{tikzcd}$$

- $\star$-FT $\xrightarrow{\hbar}$ $\star$-QFT:
  Take the non-commutative action $S_\star[\Phi]$ and calculate from it Feynman rules in the usual way where one just obtaines modifications of the vertices as mentioned previously in section 2.1.2. Then, take these modified Feynman rules and perform loop calculations, *i.e.* make a perturbation series in $\hbar$.

- $\star$-FT $\xrightarrow{\theta}$ θ-FT $\xrightarrow{\hbar}$ θ-QFT:
  Expand the star-product in $S_\star[\Phi]$ with the help of formula (1.29) and terminate the resulting infinite series in θ at a certain order. Then, evaluate Feynman rules from this expanded action and perform loop calculations. Here, we have a perturbation series both in θ and $\hbar$.

As everything in life, both approaches have their advantages and disadvantages. For example, $\star$-QFT seems to have a new kind of severe infrared problems, whereas θ-QFT do not. On the other hand, one has to deal with an infinite number of vertices in θ-QFT, contrary to $\star$-QFT. Furthermore, it is an open question if both approaches lead to the same result at the very end, *i.e.* after summing up the θ-series in θ-QFT. We will discuss further points later on and give instead some relevant references for this topic.

$\star$-QFT has been considered for scalar fields in [12, 13, 15, 16] and extended to the case of gauge theories in [14, 19, 20, 21, 22, 23, 24, 25, 26, 27, 30, 31, 32, 33, 34, 35, 36].

One of the main motivations for considering θ-QFT was the observation that one is not restricted to unitary gauge groups anymore (as mentioned in section 2.1.2), if one performs also a Seiberg–Witten map of the gauge field [44]. This means an expansion of the star-product and of the gauge field itself which turns out to be a polynomial in its commutative counterpart and the non-commutativity parameter (see section 2.6). The first perturbative analysis of θ-expanded Yang–Mills theory has been done in [45]. Further work on θ-QFT can be found in [43, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55].

Here, we will follow the lines of $\star$-QFT, *i.e.* we will not expand our action in terms of the non-commutativity parameter.

2.2 NC Abelian Yang–Mills theory

From now on, we will only consider non-commutative gauge theories. Further, we restrict ourselves to the case of a Yang–Mills theory with gauge group $U(1)$ since this is enough to show all the new interesting physics which comes from the non-commutativity of the theory. In [24] it has been shown that only the $U(1)$ part of a pure NC $U(N)$ YM theory shows all the new features which will be studied in the next sections where we will follow the lines of [21, 22].

2.2.1 Gauge fixing

Taking the gauge group $G = U(1)$ in (2.5) and (2.6) we have the following action of the NC $U(1)$ YM theory in $D = 4$ space-time dimensions:

$$S_{\text{YM}} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}\star F^{\mu\nu}, \tag{2.12}$$

with the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\, g\, [A_\mu\,,\, A_\nu]_\star\,, \tag{2.13}$$

and g the coupling constant.[3]

In order to calculate the Feynman rules of NC $U(1)$ YM theory we have to gauge fix the action (2.12). Due to the non-Abelian structure of the non-commutative $U(1)$ field strength (2.13) we have to use the BRS formalism [86, 87, 88, 89, 90] which works also in non-commutative gauge theories [27]. The BRS transformations are given by

$$\begin{aligned} sA_\mu &= D^\star_\mu c, & s\bar{c} &= B, \\ sc &= \mathrm{i}\, g\, c\star c, & sB &= 0, \end{aligned} \tag{2.14}$$

where c denotes the ghost field, $\bar{c}$ the anti-ghost field and B a multiplier field. Here, we introduced the non-commutative covariant derivative

$$D^\star_\mu = \partial_\mu - \mathrm{i}\, g\, [A_\mu\,,\ ]_\star\,, \tag{2.15}$$

[3]We have also made a rescaling of the gauge field $A_\mu \to gA_\mu$.

and the BRS operator s is still nilpotent:

$$s^2 = 0. \tag{2.16}$$

In order to keep BRS invariance of the gauge fixed action, we write the gauge fixing term as a BRS exact term:

$$\begin{aligned} S_{\mathrm{gf}} &= s \int d^4x \, \bar{c} \star \left(\partial_\mu A^\mu + \frac{\alpha}{2} B \right) \\ &= \int d^4x \, \left(B \star \partial_\mu A^\mu + \frac{\alpha}{2} B \star B - \bar{c} \star \partial^\mu D^\star_\mu c \right), \end{aligned} \tag{2.17}$$

where α is some gauge parameter. After integrating out the B field,[4] we end up with the following expression for the gauge fixing term:

$$S_{\mathrm{gf}} = \int d^4x \, \left(-\frac{1}{2\alpha} \partial_\mu A^\mu \star \partial_\nu A^\nu - \bar{c} \star \partial^\mu \partial_\mu c + \mathrm{i}\, g \, \bar{c} \star \partial^\mu \left[A_\mu \, , \, c \right]_\star \right). \tag{2.18}$$

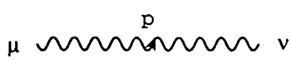
With (2.12) and (2.18), we have established the full action

$$S_{\mathrm{NCYM}} = S_{\mathrm{YM}} + S_{\mathrm{gf}}, \tag{2.19}$$

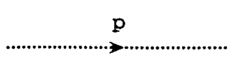
and can proceed now to calculate the Feynman rules.

2.2.2 Feynman rules

As explained in section 2.1.2, the bilinear part of the action (2.19) is the same as in the corresponding commutative Yang-Mills theory. Therefore, we can take the propagators of an ordinary $SU(N)$ YM theory from the textbooks, *e.g.* [92, 93, 94].[5] Taking the formal limit $SU(N) \to U(1)$ of these Feynman rules, we get for the photon propagator $D_{\mu\nu}(p)$ and the ghost propagator $G(p)$:



$$D_{\mu\nu}(p) = -\frac{\mathrm{i}}{p^2} \left[g_{\mu\nu} - (1-\alpha) \frac{p_\mu p_\nu}{p^2} \right], \tag{2.20}$$



$$G(p) = \frac{\mathrm{i}}{p^2}. \tag{2.21}$$

Now, we pass to the more interesting case of interactions. Looking at the interaction part of the action (2.19), one recognizes three kinds of interaction:

[4] Functional differentiation works as usual, despite the star-product [91].

[5] Of course, the reader can calculate the propagators directly from (2.19) just by using standard methods.

- three gauge interaction $\sim g\,\partial_\mu A_\nu \star [A^\mu\,,\,A^\nu]_\star + \text{perm.},$
- four gauge interaction $\sim g^2\,[A_\mu\,,\,A_\nu]_\star \star [A^\mu\,,\,A^\nu]_\star,$
- ghost gauge interaction $\sim g\,\partial^\mu \bar{c} \star [A_\mu\,,c]_\star.$

Under the integral $\int d^4x$ in the action (2.19) we can get rid of one star in each interaction term above. Therefore, we are left with Moyal commutators (1.34) which have the following form in momentum space:

$$\Phi_1(p)\,\sin\left(\frac{1}{2}p_\mu\theta^{\mu\nu}q_\nu\right)\,\Phi_2(q). \tag{2.22}$$

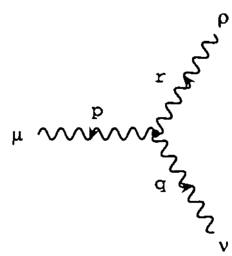
For obtaining the Feynman rules we use the standard procedure, *i.e.* we perform a Fourier transformation and differentiate with respect to the fields in the interaction part. In this way, we get the standard Feynman rules of $SU(N)$ YM, but with the structure constants replaced by some factors of the form[6]

$$\sin\left(\frac{1}{2}p_\mu\theta^{\mu\nu}q_\nu\right) = \sin\frac{p\,\tilde{q}}{2}, \tag{2.23}$$

where we defined

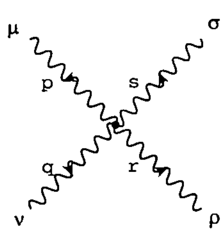
$$\tilde{k}^\mu = \theta^{\mu\nu}k_\nu. \tag{2.24}$$

The Feynman rules of the gauge three-vertex $Y_{\mu\nu\rho}$, the gauge four-vertex $X_{\mu\nu\rho\sigma}$, and the gauge ghost vertex V_μ are listed in the following:

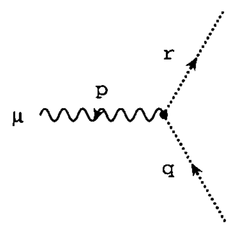


$$\begin{aligned} Y_{\mu\nu\rho}(p,q,r) &= 2g\,\sin\frac{p\,\tilde{q}}{2}\,\delta^4(p+q+r) \\ &\times\left[(p-q)_\rho g_{\mu\nu} + (q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu}\right], \end{aligned} \tag{2.25}$$

[6]Take the Feynman rules for non-Abelian gauge theories from *e.g.* [93], and make the replacement $f^{abc} \to 2\sin(p\tilde{q}/2)$.



$$X_{\mu\nu\rho\sigma}(p,q,r,s) = -4\mathrm{i}\, g^2\, \delta^4(p+q+r+s) \times \left[\sin\frac{p\,\tilde{q}}{2}\,\sin\frac{r\,\tilde{s}}{2}\,(g_{\mu\rho}\,g_{\nu\sigma} - g_{\mu\sigma}\,g_{\nu\rho}) + \sin\frac{r\,\tilde{p}}{2}\,\sin\frac{q\,\tilde{s}}{2}\,(g_{\mu\sigma}\,g_{\nu\rho} - g_{\mu\nu}\,g_{\rho\sigma}) + \sin\frac{p\,\tilde{s}}{2}\,\sin\frac{q\,\tilde{r}}{2}\,(g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma})\right], \tag{2.26}$$



$$V_\mu(p,q,r) = -2g\, r_\mu\, \sin\frac{p\,\tilde{q}}{2}\,\delta^4(p-q+r). \tag{2.27}$$

Here, one should recognize the non-locality of the vertices, because the expansion of the sinus function leads to an infinite series in the momenta. Further, all vertices disappear in the commutative limit $\theta^{\mu\nu} \to 0$ leading to a free Maxwell theory as expected.

2.2.3 Self-energy

Having all the Feynman rules in hand, we can start to calculate the one-loop corrections to the gauge field propagator, called *self-energy*. We have two contributions with a gauge loop: $\mathrm{i}\Pi^{(a)}_{\mu\nu}(p)$ (figure 2.1) and $\mathrm{i}\Pi^{(b)}_{\mu\nu}(p)$ (figure 2.2), and one contribution with a ghost loop: $\mathrm{i}\Pi^{(c)}_{\mu\nu}(p)$ (figure 2.3). Therefore, the self-energy $\mathrm{i}\Pi_{\mu\nu}(p)$ is given by:

$$\mathrm{i}\Pi_{\mu\nu}(p) = \mathrm{i}\Pi^{(a)}_{\mu\nu}(p) + \mathrm{i}\Pi^{(b)}_{\mu\nu}(p) + \mathrm{i}\Pi^{(c)}_{\mu\nu}(p). \tag{2.28}$$

To keep calculations as simple as possible, we introduce the definitions

$$k_+ = k + p/2, \qquad\qquad k_- = k - p/2, \tag{2.29}$$

and work in Feynman gauge $\alpha = 1$ (see (2.20)). Of course, physics is independent of the gauge choice also in the non-commutative case [32]. The explicit expressions for all the graphs read as follows:

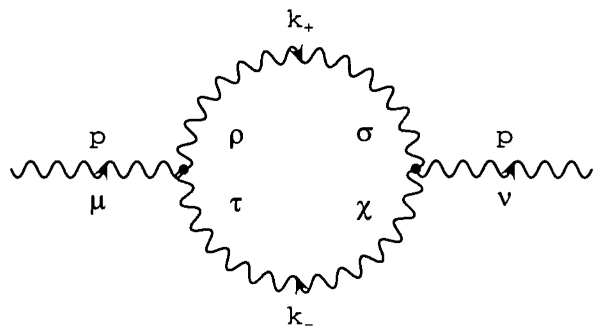


Figure 2.1: Self-energy contribution $i\Pi^{(a)}_{\mu\nu}(p)$

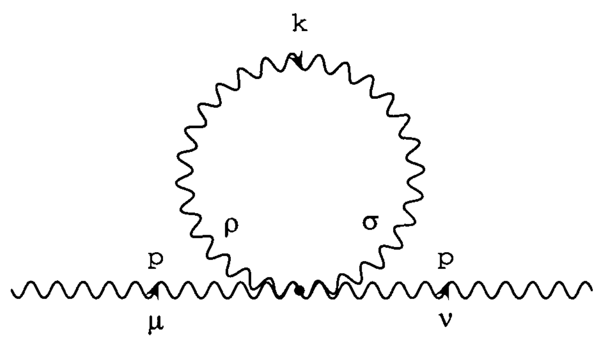


Figure 2.2: Self-energy contribution $i\Pi^{(b)}_{\mu\nu}(p)$

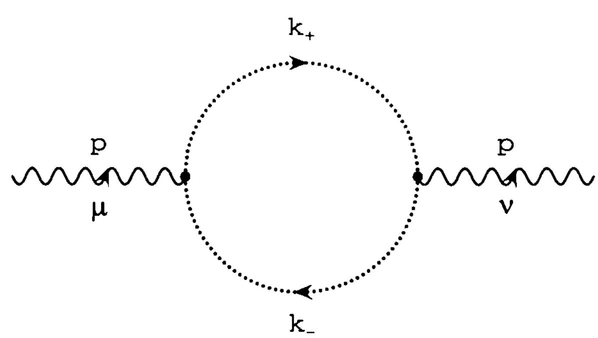


Figure 2.3: Self-energy contribution $i\Pi^{(c)}_{\mu\nu}(p)$

$$
\begin{aligned}
\mathrm{i}\Pi^{(a)}_{\mu\nu}(p) &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, D_{\rho\sigma}(k_+)\, D_{\chi\tau}(k_-)\, Y'_\mu{}^{\tau\rho}(-p,-k_-,k_+)\, Y'_\nu{}^{\sigma\chi}(p,-k_+,k_-)\\
&= 2g^2\int \frac{d^4k}{(2\pi)^4}\, \frac{\left(2k^2+\frac{9}{2}p^2\right) g_{\mu\nu} + 10k_\mu k_\nu - \frac{9}{2}p_\mu p_\nu}{k_+^2\, k_-^2}\, \sin^2\frac{\tilde{p}\,k}{2},
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
\mathrm{i}\Pi^{(b)}_{\mu\nu}(p) &= \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\, D_{\rho\sigma}(k)\, X'_{\mu\nu}{}^{\sigma\rho}(-p,p,-k,k)\\
&= -12g^2\int \frac{d^4k}{(2\pi)^4}\, \frac{g_{\mu\nu}}{k^2}\, \sin^2\frac{\tilde{p}\,k}{2},
\end{aligned}
\tag{2.31}
$$

$$
\begin{aligned}
\mathrm{i}\Pi^{(c)}_{\mu\nu}(p) &= -\int \frac{d^4k}{(2\pi)^4}\, G(k_+)\, G(k_-)\, V'_\mu(-p,k_-,k_+)\, V'_\nu(p,k_+,k_-)\\
&= -4g^2\int \frac{d^4k}{(2\pi)^4}\, \frac{k_{+\mu}\, k_{-\nu}}{k_+^2\, k_-^2}\, \sin^2\frac{\tilde{p}\,k}{2}.
\end{aligned}
\tag{2.32}
$$

To obtain the expressions (2.30), (2.31) and (2.32) we made use of the Feynman rules in section 2.2.2. The prime on Y, X and V denotes the fact that we can omit the delta-functions in the rules for the vertices, because we already considered momentum conservation in drawing the graphs in the figures 2.1, 2.2 and 2.3. Further, there is a symmetry factor 1/2 in (2.30) and (2.31), and a minus sign in (2.32) due to its fermionic ghost loop. Then, we have to integrate over the loop momentum k with integration measue $d^4k/(2\pi)^4$.

Further, we have the important property

$$
p_\mu\, \tilde{p}^\mu = p_\mu\, \theta^{\mu\nu}\, p_\nu = 0, \tag{2.33}
$$

due to the antisymmetry of the non-commutativity parameter $\theta^{\mu\nu}$. We used (2.33) to simplify the arguments of the sinus factors above.

Looking at the expressions (2.30), (2.31) and (2.32), we recognize that they all have a common factor $\sin^2(\tilde{p}k/2)$ which includes the non-commutativity parameter $\theta^{\mu\nu}$. Using the identity

$$
2\sin^2\frac{\tilde{p}\,k}{2} = 1-\cos\tilde{p}\,k, \tag{2.34}
$$

we can split every graph in a θ-dependent part and another part which is completely independent of θ. Therefore, we make the following definitions:

- *planar* graphs do not depend on the non-commutativity parameter θ,
- *non-planar* graphs depend on the non-commutativity parameter θ.

Now, we can write the self-energy (2.28) as a sum of a planar and a non-planar contribution, respectively:

$$\mathrm{i}\Pi_{\mu\nu}(p) = \mathrm{i}\Pi^{\mathrm{P}}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}}_{\mu\nu}(p). \tag{2.35}$$

The termini planar and non-planar come from the fact that it is possible to introduce some kind of 't Hooft's double line notation [95, 96] for non-commutative field theories [13].

The planar parts of the graphs in figure 2.1, 2.2 and 2.3 lead exactly to the same integrals as in the case of ordinary $SU(N)$ Yang–Mills theory in the formal limit $N \to 1$. They can be regularized and renormalized in the usual way [19, 24]. We will come back to these results later on, but now we will consider only the non-planar parts which contain possibly new physics.

Due to the oscillating factor $\cos \tilde{p}k$ in the non-planar parts of (2.30), (2.31) and (2.32), there is no need for an ultraviolet regularization. These integrals are completely finite in the UV. Let us calculate them explicitly.

In the following, we will use Schwinger parameterization [92]:

$$\frac{1}{k^2 - m^2 + \mathrm{i}\varepsilon} = -\int_0^\infty \mathrm{i}d\alpha\; \mathrm{e}^{\mathrm{i}\alpha(k^2 - m^2 + \mathrm{i}\varepsilon)}, \tag{2.36}$$

with an appropriate iε-prescription.[7] Further, we represent the non-commutative oscillating factor as

$$2\cos \tilde{p}k = \mathrm{e}^{\mathrm{i}\tilde{p}k} + \mathrm{e}^{-\mathrm{i}\tilde{p}k}. \tag{2.37}$$

Starting with the simplest contribution to the self-energy $\mathrm{i}\Pi^{\mathrm{NP}\,(b)}_{\mu\nu}(p)$ from (2.31) we get with (2.36) and (2.37) the expression

$$\mathrm{i}\Pi^{\mathrm{NP}\,(b)}_{\mu\nu}(p) = -3g^2\, g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \mathrm{i}d\alpha\; \mathrm{e}^{\mathrm{i}\alpha k^2} \left(\mathrm{e}^{\mathrm{i}\tilde{p}k} + \mathrm{e}^{-\mathrm{i}\tilde{p}k}\right). \tag{2.38}$$

Quadratic completion with the shift $k \to k \pm \tilde{p}/2\alpha$ in (2.38) leads to

$$\mathrm{i}\Pi^{\mathrm{NP}\,(b)}_{\mu\nu}(p) = -6\mathrm{i}g^2\, g_{\mu\nu} \int \frac{d^4k}{\mathrm{i}(2\pi)^4} \int_0^\infty \mathrm{i}d\alpha\; \mathrm{e}^{\mathrm{i}\alpha k^2 - \mathrm{i}\frac{\tilde{p}^2}{4\alpha}}. \tag{2.39}$$

Now, we can perform the Gaussian integration over the momentum k via the formula

$$\int \frac{d^dk}{\mathrm{i}(2\pi)^d}\, \mathrm{e}^{\mathrm{i}\alpha k^2} = \frac{1}{(4\pi\mathrm{i}\alpha)^{d/2}}, \tag{2.40}$$

[7]The iε-term simply acts as a damping factor as usual. We will not write it anymore in future calculations.

and get for (2.39) the following term:

$$\mathrm{i}\Pi^{\mathrm{NP}\,(b)}_{\mu\nu}(p) = -\frac{3\mathrm{i}g^2}{8\pi^2}\, g_{\mu\nu} \int_0^\infty \frac{\mathrm{i}d\alpha}{(\mathrm{i}\alpha)^2}\, \mathrm{e}^{\frac{\tilde{p}^2}{4\mathrm{i}\alpha}}. \tag{2.41}$$

The oscillating factor $\exp(-\mathrm{i}\tilde{p}^2/4\alpha)$ renders the integral in (2.41) finite and we get by applying the formula [97]:

$$\int_0^\infty \frac{dx}{x^2}\, \mathrm{e}^{a/x} = -1/a, \qquad\qquad [\, \Re(a) < 0\,], \tag{2.42}$$

the result

$$\mathrm{i}\Pi^{\mathrm{NP}\,(b)}_{\mu\nu}(p) = \frac{3\mathrm{i}g^2}{2\pi^2}\, g_{\mu\nu}\, \frac{1}{\tilde{p}^2}. \tag{2.43}$$

Because of the identity (2.33), the vector $\tilde{p}^\mu$ is space-like[8] and therefore $\tilde{p}^2 < 0$, which allowed us to use the formula (2.42) above.

With (2.43) we have calculated the non-planar part of figure 2.2, which is finite for non-vanishing $\tilde{p}$. But, we see that in the limit $\tilde{p} \to 0$ we get a quadratic singularity. Before discussing this point further, we will also derive the non-planar parts of figure 2.1 and figure 2.3.

Instead of performing the calculation of $\mathrm{i}\Pi^{\mathrm{NP}\,(a)}_{\mu\nu}(p)$ and $\mathrm{i}\Pi^{\mathrm{NP}\,(c)}_{\mu\nu}(p)$ in full detail we will give all necessary ingredients for this tedious but straightforward derivation in the following.

First, we use two Schwinger parameters α_+ and α_- in (2.36) and formula (2.37) to rewrite the non-planar parts of the integrals (2.30) and (2.32):

$$\begin{aligned}&\mathrm{i}\Pi^{\mathrm{NP}\,(a)}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(c)}_{\mu\nu}(p) = \\ &= \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \mathrm{i}d\alpha_+ \int_0^\infty \mathrm{i}d\alpha_-\, \mathfrak{A}_{\mu\nu}(p,k)\, \mathrm{e}^{\mathrm{i}\alpha_+ k_+^2}\, \mathrm{e}^{\mathrm{i}\alpha_- k_-^2} \left(\mathrm{e}^{\mathrm{i}\tilde{p}k} + \mathrm{e}^{-\mathrm{i}\tilde{p}k}\right),\end{aligned} \tag{2.44}$$

with

$$\mathfrak{A}_{\mu\nu}(p,k) = -g^2\Big[\big(k^2 + \frac{9}{4}p^2\big)g_{\mu\nu} + 4k_\mu k_\nu - 2p_\mu p_\nu - \frac{1}{2}p_\mu k_\nu + \frac{1}{2}k_\mu p_\nu\Big]. \tag{2.45}$$

In order to perform the integration over k in (2.44) we write every k_μ in (2.45) as a derivative $\partial/\mathrm{i}\partial z^\mu$ acting on an additional factor $\exp(\mathrm{i}zk)$, which we introduce by hand. We can do this without changing the integral, since we will set this factor equal to 1 by evaluating the whole expression at $z = 0$ at the very end.

[8] The momentum p^μ is time-like for physical particles, of course.

The next step consists in a quadratic completion of the terms in the exponentials of (2.44) and the above introduced additional factor. Then, we can do the Gaussian k-integration (2.40), and perform afterwards the derivations with respect to z. After setting the auxiliary variable z equal to zero, we have the following expression:

$$\begin{aligned} \mathrm{i}\Pi^{\mathrm{NP}\,(a)}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(c)}_{\mu\nu}(p) = \frac{\mathrm{i}g^2}{16\pi^2}\Big[&\big(-5p^2 g_{\mu\nu} + 2p_\mu p_\nu\big)\,\mathfrak{I}_0(p) - 8\mathrm{i}g_{\mu\nu}\,\mathfrak{I}_1(p) \\ &- \big(\frac{1}{2}\tilde{p}^2 g_{\mu\nu} + 2\tilde{p}_\mu\tilde{p}_\nu\big)\,\mathfrak{I}_2(p) + \big(2p^2 g_{\mu\nu} + 8p_\mu p_\nu\big)\,\bar{\mathfrak{I}}(p)\Big], \end{aligned} \tag{2.46}$$

where we defined the integrals

$$\mathfrak{I}_\kappa(p) = \int_0^\infty \mathrm{i}d\alpha_+ \int_0^\infty \mathrm{i}d\alpha_-\, \frac{1}{(\mathrm{i}\beta)^2}\,\frac{1}{\beta^\kappa}\, \exp\Big(\mathrm{i}\frac{\alpha_+\alpha_-}{\beta}p^2 - \mathrm{i}\frac{1}{4\beta}\tilde{p}^2\Big), \tag{2.47}$$

$$\bar{\mathfrak{I}}(p) = \int_0^\infty \mathrm{i}d\alpha_+ \int_0^\infty \mathrm{i}d\alpha_-\, \frac{1}{(\mathrm{i}\beta)^2}\,\frac{\alpha_+\alpha_-}{\beta^2}\, \exp\Big(\mathrm{i}\frac{\alpha_+\alpha_-}{\beta}p^2 - \mathrm{i}\frac{1}{4\beta}\tilde{p}^2\Big), \tag{2.48}$$

with

$$\beta = \alpha_+ + \alpha_-. \tag{2.49}$$

In order to perform the integrals (2.47) and (2.48) we make the following reparameterization:

$$\mathrm{i}\alpha_+ = \chi\lambda, \qquad \mathrm{i}\alpha_- = (1-\chi)\lambda. \tag{2.50}$$

Then, we replace the integration variable λ through the variable η given by

$$\eta = -\chi(1-\chi)p^2\lambda, \tag{2.51}$$

and end up with

$$\mathfrak{I}_\kappa(p) = \int_0^1 d\chi\,\big(-\mathrm{i}\chi(1-\chi)p^2\big)^\kappa \int_0^\infty \frac{d\eta}{\eta^{1+\kappa}}\,\exp\big(-\eta - \frac{\upsilon^2}{\eta}\big), \tag{2.52}$$

$$\bar{\mathfrak{I}}(p) = \int_0^1 d\chi\,\chi(1-\chi) \int_0^\infty \frac{d\eta}{\eta}\,\exp\big(-\eta - \frac{\upsilon^2}{\eta}\big), \tag{2.53}$$

where

$$\upsilon^2 = \chi(1-\chi)\frac{p^2\tilde{p}^2}{4}. \tag{2.54}$$

The kind of integration over η in (2.52) and (2.53) can now be found in mathematical textbooks [97, 98, 99]. It is carried out with the formula

$$\int_0^\infty \frac{dx}{x^{1-\nu}} \exp\big(-bx-\frac{a}{x}\big) = 2\,\big(\frac{a}{b}\big)^{\frac{\nu}{2}}\,K_\nu\big(2\sqrt{ab}\big), \qquad [\,\Re(a)>0,\,\Re(b)>0\,], \tag{2.55}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind. Application of formula (2.55) to (2.52) and (2.53) yields

$$\mathfrak{I}_\kappa(p) = \int_0^1 d\chi\,\big(-\mathrm{i}\chi(1-\chi)p^2\big)^\kappa\,\frac{2}{\upsilon^\kappa}\,K_\kappa\big(2\upsilon\big), \tag{2.56}$$

$$\bar{\mathfrak{I}}(p) = \int_0^1 d\chi\,\chi(1-\chi)\,2\,K_0\big(2\upsilon\big), \tag{2.57}$$

where we used the property $K_\kappa(2\upsilon) = K_{-\kappa}(2\upsilon)$.

Since we are not able anymore to perform the integration over χ in (2.56) and (2.57) in closed form, we will make an expansion of the modified Bessel functions $K_\kappa(2\upsilon)$. Due to an exponential fall off for large υ

$$\lim_{\upsilon\to\infty} K_\kappa(2\upsilon) = \sqrt{\frac{\pi}{4\upsilon}}\,\mathrm{e}^{-2\upsilon}, \tag{2.58}$$

the only interesting regime will be the limit $\upsilon \to 0$. In the following, we list the necessary expansions ($\kappa = 0,1,2$) around $\upsilon \sim 0$ up to the order $\mathcal{O}(\upsilon^4)$:

$$\begin{aligned}
&2\,K_0\big(2\upsilon\big) \sim -\ln\upsilon^2\,\big(1+\upsilon^2\big) - 2\gamma_{\mathrm{E}} + 2\big(1-\gamma_{\mathrm{E}}\big)\upsilon^2,\\
&\frac{2}{\upsilon}\,K_1\big(2\upsilon\big) \sim \frac{1}{\upsilon^2} + \ln\upsilon^2\,\big(1+\frac{\upsilon^2}{2}\big) + \big(2\gamma_{\mathrm{E}}-1\big) + \big(\gamma_{\mathrm{E}}-\frac{5}{4}\big)\upsilon^2,\\
&\frac{2}{\upsilon^2}\,K_2\big(2\upsilon\big) \sim \frac{1}{\upsilon^4} - \frac{1}{\upsilon^2} - \ln\upsilon^2\,\big(\frac{1}{2}+\frac{\upsilon^2}{6}\big) + \big(\frac{3}{4}-\gamma_{\mathrm{E}}\big) + \big(\frac{17}{36}-\frac{\gamma_{\mathrm{E}}}{3}\big)\upsilon^2,
\end{aligned} \tag{2.59}$$

where $\gamma_{\mathrm{E}} \sim 0,577216$ denotes the Euler gamma. With the help of the expansions (2.59) we can do the integration over χ in (2.56) and (2.57) and get with (2.54) up to the order $\mathcal{O}(\tilde{p}^4)$:

$$
\begin{aligned}
\mathfrak{I}_0(p) &\sim 2\Big(1-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)+\frac{1}{12}\Big(\frac{11}{6}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2,\\
\mathfrak{I}_1(p) &\sim -\mathrm{i}p^2\Big[\frac{4}{p^2\tilde{p}^2}-\frac{1}{3}\Big(\frac{4}{3}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)\\
&\qquad -\frac{1}{120}\Big(\frac{61}{30}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2\Big],\\
\mathfrak{I}_2(p) &\sim p^4\Big[-\frac{16}{p^4\tilde{p}^4}+\frac{2}{3p^2\tilde{p}^2}-\frac{1}{30}\Big(\frac{23}{15}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)\\
&\qquad -\frac{1}{1680}\Big(\frac{457}{210}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2\Big],\\
\tilde{\mathfrak{I}}(p) &\sim \frac{1}{3}\Big(\frac{5}{6}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)+\frac{1}{60}\Big(\frac{107}{60}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2,
\end{aligned}
\tag{2.60}
$$

where we introduced the constant $\hat{\gamma}_{\mathrm{E}} = \gamma_{\mathrm{E}} - \ln 2 \sim -0,115932$. Further, the argument of the logarithm is always positive, *i.e.* we should have written more exactly $\ln(|x|)$ instead of $\ln(x)$. This remains true in all the following.

Finally, we can use the results (2.60) to get an explicit expression for the one-loop contributions of figure 2.1 and 2.3 to the non-planar part of the self-energy (2.46). We will summarize and discuss our results in the following.

The non-planar part of the self-energy is given by the sum of the terms (2.43) and (2.46):

$$
\mathrm{i}\Pi^{\mathrm{NP}}_{\mu\nu}(p) = \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{a})}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{b})}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{c})}_{\mu\nu}(p), \tag{2.61}
$$

and can be rewritten as

$$
\mathrm{i}\Pi^{\mathrm{NP}}_{\mu\nu}(p) = \Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\,\mathrm{i}\Pi^{\mathrm{NP}}(p) + \Big(\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\Big)\,\mathrm{i}\tilde{\Pi}^{\mathrm{NP}}(p), \tag{2.62}
$$

where up to the order $\mathcal{O}(\tilde{p}^4)$

$$
\begin{aligned}
\mathrm{i}\Pi^{\mathrm{NP}}(p) &\sim \frac{\mathrm{i}g^2}{16\pi^2}\Big[-\frac{20}{3}\Big(\frac{14}{15}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\\
&\qquad -\frac{3}{10}\Big(\frac{163}{90}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^4\tilde{p}^2\Big],
\end{aligned}
\tag{2.63}
$$

$$
\mathrm{i}\tilde{\Pi}^{\mathrm{NP}}(p) \sim \frac{\mathrm{i}g^2}{16\pi^2}\Big[\frac{32}{\tilde{p}^2}-\frac{4}{3}p^2+\frac{1}{15}\Big(\frac{23}{15}-\hat{\gamma}_{\mathrm{E}}-\frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^4\tilde{p}^2\Big]. \tag{2.64}
$$

Before discussing this result, let us write down the full one-loop self-energy, including also the contributions from the planar sector of the theory.

With (2.35) and (2.62) we get for the self-energy of non-commutative $U(1)$ Yang–Mills theory:

$$\mathrm{i}\Pi_{\mu\nu}(p) = \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left(\mathrm{i}\Pi^{\mathrm{P}}(p) + \mathrm{i}\Pi^{\mathrm{NP}}(p)\right) + \left(\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\right)\mathrm{i}\widetilde{\Pi}^{\mathrm{NP}}(p). \qquad (2.65)$$

The term $\mathrm{i}\Pi^{\mathrm{P}}(p)$ is just given by the logarithmic UV divergent part of the self-energy from ordinary $SU(N)$ Yang–Mills theory in the limit $N \to 1$:

$$\mathrm{i}\Pi^{\mathrm{P}}(p) \sim -\frac{\mathrm{i}g^2}{16\pi^2}\left(\frac{10}{3}\ln\left(p^2/\Lambda^2\right)\right)p^2, \qquad (2.66)$$

with Λ the UV cutoff scale. There is no corresponding piece $\mathrm{i}\widetilde{\Pi}^{\mathrm{P}}(p)$, of course.

Now, we know from gauge invariance of the action (2.12) that the self-energy has to be transversal. Since the two projection operators appearing in (2.65)

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad \mathcal{P}_{\mu\rho}\mathcal{P}^{\rho}{}_{\nu} = \mathcal{P}_{\mu\nu}, \qquad (2.67)$$

$$\widetilde{\mathcal{P}}_{\mu\nu} = \frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}, \qquad \widetilde{\mathcal{P}}_{\mu\rho}\widetilde{\mathcal{P}}^{\rho}{}_{\nu} = \widetilde{\mathcal{P}}_{\mu\nu}, \qquad (2.68)$$

have the properties[9]

$$p^\mu \mathcal{P}_{\mu\nu} = 0, \qquad p^\mu \widetilde{\mathcal{P}}_{\mu\nu} = 0, \qquad (2.69)$$

it is easy to see that the self-energy (2.65) of non-commutative $U(1)$ Yang–Mills theory is still transversal:

$$p^\mu\, \mathrm{i}\Pi_{\mu\nu}(p) = 0. \qquad (2.70)$$

This is a very important consistency check, because otherwise we would have an anomaly. The non-transversal part from the graph in figure 2.2 given by (2.43) was exactly canceled by the self-energy parts from the graphs 2.1 and 2.3 in (2.46).

We will discuss new physics coming from the non-planar parts $\mathrm{i}\Pi^{\mathrm{NP}}(p)$ and $\mathrm{i}\widetilde{\Pi}^{\mathrm{NP}}(p)$ of the self-energy (2.65) in section 2.3.

2.2.4 One-loop corrections

In order to be complete, we list here the one-loop corrections of all propagators and vertices of non-commutative $U(1)$ Yang–Mills theory (see section 2.2.2). We have calculated the vacuum polarization tensor $\mathrm{i}\Pi_{\mu\nu}(p)$ in

[9]See also equation (2.33).

section 2.2.3 in the Feynman gauge $\alpha = 1$, and take all the other results from [32]. There, dimensional regularization was used in the planar sector with $D = 4 + 2\epsilon$. We denote the gauge three-vertex by $\mathrm{i}\Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)$, the gauge four-vertex by $\mathrm{i}\Gamma_{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4)$, the ghost self-energy by $\mathrm{i}\Pi^{\mathrm{gh}}(p)$, and the gauge ghost vertex by $\mathrm{i}\Gamma_\mu^{\mathrm{gh}}(p_1, p_2, p_3)$. There are no more UV divergent one-loop corrections in the game.

In the following, all results are given modulo finite contributions:

$$\mathrm{i}\Pi_{\mu\nu}(p) \sim \frac{2\mathrm{i}g^2}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^4} - \frac{\mathrm{i}g^2}{16\pi^2}\left(\frac{13}{3} - \alpha\right)\left[\frac{1}{\epsilon} - \ln\left(p^2\tilde{p}^2\right)\right](p^2 g_{\mu\nu} - p_\mu p_\nu), \tag{2.71}$$

$$\begin{aligned}\mathrm{i}\Gamma_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) \sim{}& \frac{2g^3}{\pi^2}\cos\left(\frac{p_1\tilde{p}_2}{2}\right)\sum_{i=1}^{3}\frac{(\tilde{p}_i)_{\mu_1}(\tilde{p}_i)_{\mu_2}(\tilde{p}_i)_{\mu_3}}{(\tilde{p}_i)^4}\\ &+\frac{g^3}{16\pi^2}\sin\left(\frac{p_1\tilde{p}_2}{2}\right)\left(\frac{17}{3} - 3\alpha\right)\left[\frac{1}{\epsilon} - \frac{1}{3}\sum_{i=1}^{3}\ln\left((\tilde{p}_i)^2\right)\right]\\ &\times\Big[g_{\mu_1\mu_2}(p_1 - p_2)_{\mu_3} + g_{\mu_2\mu_3}(p_2 - p_3)_{\mu_1} + g_{\mu_3\mu_1}(p_3 - p_1)_{\mu_2}\Big],\end{aligned} \tag{2.72}$$

$$\begin{aligned}&\mathrm{i}\Gamma_{\mu_1\mu_2\mu_3\mu_4}(p_1, p_2, p_3, p_4) \sim\\ &\sim \frac{\mathrm{i}g^4}{16\pi^2}\Big[4\left(g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right)\sin\left(\frac{p_1\tilde{p}_2}{2}\right)\sin\left(\frac{p_3\tilde{p}_4}{2}\right)f_\epsilon(\tilde{p}_1, \tilde{p}_2)\\ &\qquad+4\left(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right)\sin\left(\frac{p_1\tilde{p}_3}{2}\right)\sin\left(\frac{p_2\tilde{p}_4}{2}\right)f_\epsilon(\tilde{p}_1, \tilde{p}_3)\\ &\qquad+4\left(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_3}g_{\mu_2\mu_4}\right)\sin\left(\frac{p_1\tilde{p}_4}{2}\right)\sin\left(\frac{p_2\tilde{p}_3}{2}\right)f_\epsilon(\tilde{p}_1, \tilde{p}_4)\Big],\end{aligned} \tag{2.73}$$

$$\mathrm{i}\Pi^{\mathrm{gh}}(p) \sim \frac{\mathrm{i}g^2}{16\pi^2}\frac{3-\alpha}{2}\left[\frac{1}{\epsilon} - \ln\left(\tilde{p}^2\right)\right]p^2, \tag{2.74}$$

$$\mathrm{i}\Gamma_\mu^{\mathrm{gh}}(p_1, p_2, p_3) \sim -\frac{\alpha g^3}{16\pi^2}\sin\left(\frac{p_2\tilde{p}_3}{2}\right)\left[\frac{1}{\epsilon} + \ln\left(\tilde{p}_1^2\right) - \ln\left(\tilde{p}_2^2\right) - \ln\left(\tilde{p}_3^2\right)\right](p_1)_\mu, \tag{2.75}$$

where the function $f_\epsilon(\tilde{p}_j, \tilde{p}_k)$ is given by

$$f_\epsilon(\tilde{p}_j, \tilde{p}_k) = \frac{1}{\epsilon}\left(\frac{4}{3} - 2\alpha\right) + \frac{1}{8}\left[(\alpha+3)(\alpha-1) + \frac{31}{3}\right] \sum_{i=1}^{4} \ln\left((\tilde{p}_i)^2\right)$$
$$- \frac{1}{2}\left[9 + (1-\alpha)^2\right] \ln\left((\tilde{p}_j + \tilde{p}_k)^2\right). \tag{2.76}$$

We will comment on these results in the next section and refer the interested reader for further details to [32].

2.3 UV/IR mixing — Part one

Let us take a closer look at the term $\widetilde{\mathcal{P}}_{\mu\nu}\,\widetilde{\Pi}^{\mathrm{NP}}(p)$ in the self-energy (2.65) of non-commutative $U(1)$ Yang–Mills theory which arises from the small-momentum regime of the non-planar sector.[10] Using the equations (2.64) and (2.68) we have

$$\widetilde{\mathcal{P}}_{\mu\nu}\,\widetilde{\Pi}^{\mathrm{NP}}(p) = \frac{2g^2}{\pi^2}\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^4} + \mathcal{O}(\tilde{p}^0). \tag{2.77}$$

We immediately recognize from expression (2.77) that we have a quadratic singularity, when either the non-planar projection of the external momentum goes to zero, *i.e.* $\tilde{p} \to 0$, or by taking the commutative limit $\theta \to 0$.

Therefore, the limit $\theta \to 0$ is not smooth anymore on the quantum level. On the tree level, we have got back free Maxwell theory from non-commutative $U(1)$ Yang–Mills theory in performing the commutative limit, but this is not true anymore after quantization. The non-commutative deformation and the quantum deformation do not commute anymore.

Moreover, at fixed non-commutativity parameter θ we have a new infrared singularity in the limit $\tilde{p} \to 0$. Where does this IR singularity come from?

Going back to the loop calculations of section 2.2.3, we see that all non-planar graphs are rendered finite due to oscillating factors like

$$\exp(\mathrm{i}\,\tilde{p}\,k), \tag{2.78}$$

where k is the loop-momentum and p the external momentum. Now, we see in taking the limit $\tilde{p} \to 0$ that the phase factor (2.78) goes to one, and there is no regulator anymore in the loop-integral over k. Therefore, we have to get back a singularity in the limit $\tilde{p} \to 0$. But this singularity coming from

[10] Remember, that there is no non-planar contribution coming from the large-momentum regime due to an exponential fall off of the modfied Bessel function in (2.58).

the ultraviolet regime of the loop-momentum, is now an infrared singularity of the external momentum. The UV and IR modes of the theory do not decouple anymore, hence the name *UV/IR mixing*.

One can see this phenomenon also by introducing a UV cutoff Λ for the non-planar graphs. Then, one gets together with the regulating phase factor (2.78) an effective cutoff scale Λ_{eff} given by

$$\Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda^2 + \tilde{p}^2}. \tag{2.79}$$

From the expression (2.79) we see that the UV limit $\Lambda \to \infty$ does not commute with the IR limit $\tilde{p} \to 0$, demonstrating again the interesting mixing of UV and IR modes.

2.3.1 Renormalizability

The UV/IR mixing was first discovered in [13] for the cases of non-commutative ϕ^4 theory in four dimensions and ϕ^3 in six dimensions. Naive power counting suggests in both theories a quadratic UV divergence of the one-loop planar two-point function. In the non-planar sector, one gets an IR divergence which has exactly the same degree as the corresponding UV divergence, therefore being also quadratic.

Quadratic UV/IR mixing effects in gauge theories were first stated in [14, 21, 22]. The degree of the new infrared divergences in the non-planar sector is again given by naive power counting of the corresponding UV divergence in the planar sector. Due to gauge invariance one has effectively only logarithmic planar UV divergences, but there are still quadratic non-planar IR divergences. Even more, there are also linear IR divergences of the three-point functions, and logarithmic ones of the four-point functions, corresponding again to naive power counting (see section 2.2.4).

These quadratic and linear UV/IR mixing effects are quite disastrous, because they spoil renormalizability of a quantum theory. Of course, as we have seen above, non-commutative $U(1)$ Yang–Mills theory is still one-loop renormalizable, because all UV divergences come from the planar sector and can be regularized and renormalized in the usual way. But, when we start to consider higher-loop corrections, the quadratic and linear IR divergences from the non-planar sector, formerly IR divergences of external momenta, become now divergences of loop-momenta. Therefore, they cannot be integrated over anymore yielding a non-renormalizable theory [17, 18].

Non-commutative Yang–Mills theory was also studied in [19, 20, 23, 24, 25, 26, 27]. It was shown in [24], that non-Abelian non-commutative gauge theories also suffer from quadratic UV/IR mixing, therefore being only one-loop renormalizable. Even more, the non-commutative $U(N)$ gauge group does not converge to the ordinary $SU(N) \times U(1)$ gauge group in the commutative limit. There is only one coupling constant, *i.e.* non-commutative gluons interact with non-commutative photons.

Now, the question rises, if there are non-commutative quantum field theories which are renormalizable. Mainly, there are two possibilities for a non-commutative theory to be renormalizable, either being topological or supersymmetric. Chern–Simons theory is free of radiative corrections at the one-loop level even in a non-commutative setting [100], and it was shown to be fully finite in [101]. The non-commutative Wess–Zumino model was studied in component formalism in [28], and in the superfield formalism in [29]. It was the first non-commutative field theory which was shown to be renormalizable to all orders, having only a logarithmic UV/IR mixing.

Later, we will see that it is a common feature of supersymmetric non-commutative field theories to suffer only from logarithmic UV/IR mixing which does not spoil renormalizability in general. We will discuss this point extensively in section 2.5, but will come back now again to non-supersymmetric gauge theories.

2.3.2 Running coupling

We still have not considered the new physics coming from the term $\mathrm{i}\Pi^{\mathrm{NP}}(p)$ in the self-energy (2.65) of non-commutative $U(1)$ Yang–Mills theory. From the expression (2.63) we have modulo finite terms

$$\mathrm{i}\Pi^{\mathrm{NP}}(p) \sim \frac{\mathrm{i}g^2}{16\pi^2}\left(\frac{10}{3}\ln\left(p^2\tilde{p}^2\right)\right)p^2. \tag{2.80}$$

Here, we get a logarithmic infrared singularity in the limit $\tilde{p} \to 0$, or alternatively $\theta \to 0$. Comparing (2.80) with its counterpart (2.66) coming from the planar sector, we see that both terms have the same coefficient 10/3, but with different signs. This is again an effect coming from UV/IR mixing. The coefficient of logarithmic IR divergences that arise from non-planar graphs is exactly opposite to that of the logarithmic UV divergences in the planar sector of the theory [21, 22, 24, 25].

We know from standard renormalization procedure that the logarithmic UV singularity (2.66) coming from the planar sector gets canceled by a counterterm corresponding to the wave function renormalization factor. This factor, together with another counterterm coming from the logarithmic divergent one-loop vertex correction, combines to the one-loop beta-function coefficient β_0 of the theory. Since the planar sector of NC $U(1)$ YM theory is identical to ordinary $SU(N)$ YM theory in the formal limit $N \to 1$, the beta-function of non-commutative $U(1)$ Yang–Mills theory is just given by[11]

$$\beta(g) = -\frac{g^3}{16\pi^2}\,\beta_0, \qquad\qquad \beta_0 = 22/3. \tag{2.81}$$

The commutative counterpart of non-commutative $U(1)$ Yang–Mills theory is the usual Maxwell theory, consisting only of free photons with no running of the gauge coupling constant, of course. Due to the new interactions coming from non-commutativity in NC $U(1)$ YM (see section 2.2.2), it is clear that we have now a gauge coupling constant which depends on the considered energy scale. Therefore, we have a non-vanishing beta-function in (2.81).

Inserting the expression (2.81) into the definition of the beta-function $\beta(g)$ with $g(p)$ denoting the coupling constant at momentum p:

$$\beta\big(g(p)\big) = \frac{\partial}{\partial(\ln p)}\, g(p), \tag{2.82}$$

we can integrate (2.82) with respect to the momentum p and get for the running of the coupling constant $g(p)$:

$$\frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} = \frac{1}{16\pi^2}\,\beta_0\,\ln\left(p^2/\Lambda^2\right), \tag{2.83}$$

with g_Λ the coupling constant at a certain scale Λ.

It is possible to get the result (2.83) directly by calculating the self-energy of NC $U(1)$ YM theory with the background field method [25, 34, 35]. Here, the Wilsonian effective coupling constant $g(p)$ is determined by the following equation [93]:

$$\frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} = \frac{1}{p^2}\left(\Pi^{\mathrm{P}}_{\mathcal{B}}(p) + \Pi^{\mathrm{NP}}_{\mathcal{B}}(p)\right). \tag{2.84}$$

The quantities $\Pi^{\mathrm{P}}_{\mathcal{B}}(p)$ and $\Pi^{\mathrm{NP}}_{\mathcal{B}}(p)$ denote the planar and non-planar parts of the self-energy in a background field $\mathcal{B}$, respectively. They are just given by the before calculated quantities (2.66) and (2.80), when we replace the coefficient $10/3$ by $-22/3$ and perform the rescaling $A_\mu \to g^{-1}A_\mu$.

Now, we have to consider two different energy regimes. For this purpose let us introduce the non-commutative scale given by $\Lambda_\theta^2 = 1/\theta$. In the regime with momenta much larger than the non-commutative scale, $|p| \gg \Lambda_\theta$, we

[11]The one-loop beta-function coefficient $\beta_0 = 22N/3$ for ordinary $SU(N)$ Yang–Mills theory [92, 93, 94], and also for non-commutative $U(N)$ Yang–Mills theory [24].

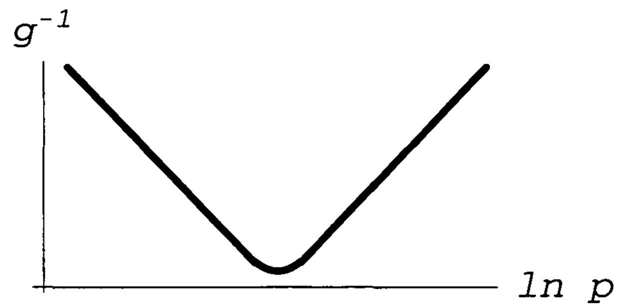


Figure 2.4: Running coupling of NC $U(1)$ YM theory

have only a planar contribution to (2.84). The non-planar part goes exponentially to zero (see section 2.2.3) and we get with the above mentioned modifications of (2.66) the relation

$$|p| \gg \Lambda_\theta : \qquad \frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} \sim \frac{1}{16\pi^2}\left(\frac{22}{3}\ln\left(p^2/\Lambda^2\right)\right). \tag{2.85}$$

Comparing (2.85) with (2.83) leads to $\beta_0 = 22/3$ exactly confirming the result in (2.81). Therefore, we see that due to its negative beta-function in the large-momentum regime, non-commutative $U(1)$ Yang–Mills theory is asymptotically free, or put another way, non-commutative photons are asymptotically free.

On the other hand, when momenta are much smaller than the non-commutative scale, *i.e.* in the regime $|p| \ll \Lambda_\theta$, we get contributions to (2.84) from the planar and the non-planar sector, leading with the appropriate forms of (2.66) and (2.80) to

$$|p| \ll \Lambda_\theta : \qquad \frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} \sim -\frac{1}{16\pi^2}\left(\frac{22}{3}\ln\left(\tilde{p}^2\Lambda^2\right)\right). \tag{2.86}$$

Looking at (2.86) and (2.85) we find that the running of the coupling constant in the infrared regime is completely similar to the one in the ultraviolet regime. But, there is a change in the sign of the beta-function which means that in the small-momentum regime the theory becomes again weakly coupled. We have drawn qualitatively the running of the coupling constant g with the energy scale p in figure 2.4 where the minimum sits at the non-commutative scale Λ_θ.

Therefore, we conclude that non-commutative $U(1)$ Yang–Mills theory is weakly coupled in the far infrared and ultraviolet. This kind of duality is a further characteristic feature of UV/IR mixing, *i.e.* mixing of ultraviolet and infrared degrees of freedom.

2.4 NC Abelian YM theory with matter

Having studied pure non-commutative Yang–Mills theory in the previous sections, we will go one step further now, and add some matter content to the theory. Here, we will study two interesting cases, namely adding fermions in the fundamental and the adjoint representation of the gauge group $U(1)$. Considering NC $U(1)$ YM theory with a fermion in the fundamental representation corresponds to non-commutative QED, *i.e.* we get non-commutative photons interacting with electrons. On the other hand, a massless fermion in the adjoint representation combines with the non-commutative photon into a supermultiplett yielding a supersymmetric NC $U(1)$ YM theory. We will also study the case with a massive fermion in the adjoint representation, giving a softly broken supersymmetric theory.

- NC $U(1)$ YM $\oplus$ fundamental fermion $\longrightarrow$ NC QED
- NC $U(1)$ YM $\oplus$ adjoint fermion
  - massless $\longrightarrow$ NC $U(1)$ SUSY YM
  - massive $\longrightarrow$ softly broken NC $U(1)$ SUSY YM

We will study both cases in the next two sections with special emphasis on the UV/IR mixing appearing in the self-energy of the gauge field.

2.4.1 Fundamental fermions

The interaction of non-commutative photons with fermions in the fundamental representation (electrons) is given by the non-commutative extension of the Dirac action [21, 22]:

$$S_\psi = \int d^4x\, \bar{\psi} \star (\mathrm{i}\not{D}^\star - M)\, \psi, \tag{2.87}$$

where ψ is the Dirac spinor, $\bar{\psi} = \psi^\dagger \gamma^0$ the Dirac adjoint spinor,[12] and M the electron mass. The covariant derivative $\not{D}^\star = D^\star_\mu \gamma^\mu$ acting on ψ and $\bar{\psi}$ is given by

$$D^\star_\mu \psi = \partial_\mu \psi - \mathrm{i}\, g\, A_\mu \star \psi, \qquad D^\star_\mu \bar{\psi} = \partial_\mu \bar{\psi} + \mathrm{i}\, g\, \bar{\psi} \star A_\mu. \tag{2.88}$$

Therefore, we have the total action

$$S_{\mathrm{NCQED}} = S_{\mathrm{NCYM}} + S_\psi, \tag{2.89}$$

[12]The Dirac matrices γ^μ, $(\mu = 0, 1, 2, 3)$ fulfill a Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\, \mathbb{1}_{4\times 4}$, and further properties can be found in standard textbooks [92, 93, 94].

where S_{NCYM} (2.19) denotes the pure gauge field and gauge fixing part (see section 2.2.1). The action (2.89) is again invariant under the BRS transformations (2.14) and

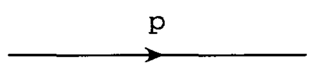
$$s\psi = \mathrm{i}\, g\, c \star \psi, \qquad\qquad s\bar{\psi} = -\mathrm{i}\, g\, \bar{\psi} \star c, \tag{2.90}$$

with c the ghost field. We should mention here that the charges of the fields ψ and $\bar{\psi}$ are restricted to 1 and -1, respectively. This is due to the non-Abelian structure of the star-gauge group and in contrast to ordinary QED where the charges can be any number [51].

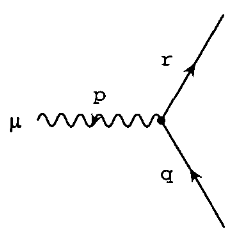
Using (2.88) to expand the matter term (2.87) we get

$$S_\psi = \int d^4x \left(\bar{\psi}\, (\mathrm{i}\not{\partial} - M)\, \psi + g\, \bar{\psi}\, \not{A} \star \psi \right), \tag{2.91}$$

where we used the property (1.33) to get rid of one star-product. Now, we can easily read off from (2.91) the Feynman rules of the matter part:



$$\Sigma(p) = \frac{\mathrm{i}}{\not{p} - M}, \tag{2.92}$$



$$\Upsilon_\mu(p,q,r) = \mathrm{i}\, g\, \gamma_\mu \exp\left(\frac{\mathrm{i}\, p\, \tilde{q}}{2}\right) \delta^4(p - q + r). \tag{2.93}$$

Note that in the vertex rule (2.93) appears an exponential factor rather than a sinus factor as in all other vertex rules (see section 2.2.2). This is due to the fact that the gauge field and the ghost transform in the adjoint representation producing therefore Moyal commutators (1.34) leading to sinus functions, and the fundamental fermion comes just with a star-product (1.28) giving the exponential factor above.

With these preparations in hand we can calculate the additional one-loop correction to the self-energy of the gauge field coming from the fermionic sector. We denote this contribution by $\mathrm{i}\Pi^{(d)}_{\mu\nu}(p)$ and the corresponding graph with a fermionic loop is given in figure 2.5. Application of the Feynman rules

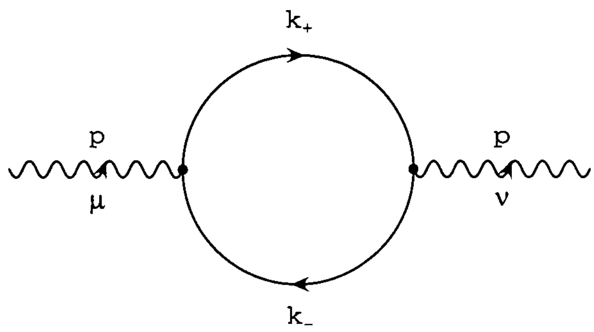


Figure 2.5: Self-energy contribution $\mathrm{i}\Pi^{(d)}_{\mu\nu}(p)$

(2.92) and (2.93) with $k_\pm = k \pm p/2$ leads to

$$\begin{aligned} \mathrm{i}\Pi^{(d)}_{\mu\nu}(p) &= -\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\Upsilon'_\mu(-p,k_-,k_+)\,\Sigma(k_+)\,\Upsilon'_\nu(p,k_+,k_-)\,\Sigma(k_-)\right] \\ &= -g^2\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_\mu\,\frac{1}{\not{k}_+ - M}\,\gamma_\nu\,\frac{1}{\not{k}_- - M}\right]\,\underbrace{\exp\left(\frac{\mathrm{i}\,\tilde{p}}{2}\,(k_- - k_+)\right)}_{1}, \end{aligned} \tag{2.94}$$

where the prime on Υ denotes again that we can omit the delta-function, because we already considered momentum conservation in drawing the graph in figure 2.5. Further, we have to take the trace over the spinor indices and get a minus sign from the fermionic loop.

Since $k_- - k_+ = -p$ and $\tilde{p}\,p = 0$ from (2.33), the exponential in (2.94) completely drops out, and we are left with a purely planar contribution, because there is no dependence on the non-commutativity parameter θ anymore. Therefore, we see that fermions in the fundamental representation of the gauge group contribute only to the planar part of the self-energy.

Now, the expression $\mathrm{i}\Pi^{(d)}_{\mu\nu}(p)$ in (2.94) is exactly the same as the fermionic contribution in an ordinary $SU(N)$ Yang–Mills theory coupled to fermions in the formal limit $N \to 1$. There, the only effect is a modification of the coefficient β_0 of the one-loop beta-function (2.81) given by

$$\beta_0 = 22/3 - 4/3\,N_{\mathrm{F}}, \tag{2.95}$$

where N_{F} is the number of flavors in the game. Therefore, we still have asymptotic freedom with a sufficient small number of fermions. Comparing this result with the running of the coupling constant in pure NC $U(1)$ YM theory in section 2.3.2, we see that we have a modification in the UV due to an additional fermionic planar contribution, but in the IR we still have a running with the coefficient $-22/3$ lacking of any new non-planar contributions. Therefore, the rule that one can get the coefficient of the logarithmic divergence in the IR by changing the sign of the corresponding logarithmic UV divergent term fails in a theory with fundamental fermions.

Before closing this section and passing to the case of adjoint fermions, we have to mention here the very important point that there is no change of the quadratic IR divergence (2.77) in the self-energy of the gauge field. The same holds for the linear IR divergent vertex correction in section 2.2.4. Fundamental fermions do not change quadratic or linear UV/IR mixing effects, therefore they cannot improve renormalizability of non-commutative Yang–Mills theory.

2.4.2 Adjoint fermions

Now, we proceed the analogous way we have gone in the previous section by considering fermions transforming in the adjoint representation of the gauge group instead of taking fundamental fermions. Denoting the Weyl fermion by λ and its conjugate by $\bar{\lambda}$, we have the following matter Lagrangian:

$$S_\lambda = \int d^4x\, \bar{\lambda} \star (\mathrm{i}\not{D}^\star - m)\, \lambda, \tag{2.96}$$

with m the mass of the fermion. Here, the covariant derivative $\not{D}^\star = D^\star_\mu \gamma^\mu$ acts on λ and $\bar{\lambda}$ through a Moyal commutator:

$$D^\star_\mu \lambda = \partial_\mu \lambda - \mathrm{i}\, g\, [A_\mu\,,\, \lambda]_\star\,, \qquad D^\star_\mu \bar{\lambda} = \partial_\mu \bar{\lambda} + \mathrm{i}\, g\, [\bar{\lambda}\,,\, A_\mu]_\star\,, \tag{2.97}$$

contrary to (2.88) where we had just a star-product. The total action

$$S_{\mathrm{NCSYM}} = S_{\mathrm{NCYM}} + S_\lambda, \tag{2.98}$$

with S_{NCYM} given in (2.19) is invariant under the BRS transformations (2.14) and

$$s\lambda = \mathrm{i}\, g\, [c\,,\, \lambda]_\star\,, \qquad s\bar{\lambda} = -\mathrm{i}\, g\, [\bar{\lambda}\,,\, c]_\star\,, \tag{2.99}$$

with c the ghost. We have again a restriction of the charge of the fermion which in this case has to be zero [51].

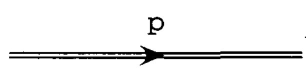
Furthermore, since the fermion transforms in the adjoint representation like the gauge field, λ and A_μ build a supermultiplett [102, 103, 104]. They are the component fields of an $\mathcal{N} = 1$ vector multiplett.[13] Of course, supersymmetry is only realized in the case $m = 0$ where we have a massless adjoint fermion, called photino, in the game. If the photino gets a mass, supersymmetry is broken, but in a so-called soft way.

One of the big advantages of supersymmetric field theories is the absence of dangerous quadratic and linear ultraviolet divergences due to cancelations between bosonic and fermionic degrees of freedom. Softly breaking of supersymmetry, *e.g.* by giving a mass to the superpartner of the photon, does not spoil these cancelation effects. The question is now, if this holds also in a supersymmetric theory on non-commutative space. We will investigate this point in the following by considering the case of supersymmetric NC $U(1)$ YM theory with the action already given in (2.98).

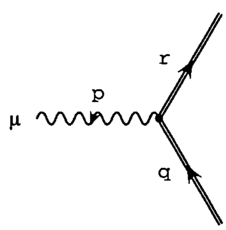
With (2.97) we can expand the matter term (2.96) and get

$$S_\lambda = \int d^4x \left(\bar\lambda \, (\mathrm{i}\not{\partial} - m) \, \lambda + g \, \bar\lambda \left[\not{A} \, , \, \lambda \right]_\star \right), \tag{2.100}$$

where we used again the property (1.33) to get rid of one star-product. Now, the Feynman rules of (2.100) have the following form:



$$\Xi(p) = \frac{\mathrm{i}}{\not{p} - m}, \tag{2.101}$$



$$\Psi_\mu(p, q, r) = -2g \, \gamma_\mu \, \sin \frac{p \tilde{q}}{2} \, \delta^4(p - q + r). \tag{2.102}$$

In (2.102) we recognize the familiar sinus factor coming from the Moyal commutator (1.34), which appears in the interaction part of (2.100).

Now, we can start to calculate the additional one-loop contribution to the photon self-energy coming from a photino running in the loop. The Feynman graph is drawn in figure 2.6 and the corresponding self-energy contribution $\mathrm{i}\Pi^{(e)}_{\mu\nu}(p)$ with the Feyman rules (2.101) and (2.102) is given by

$$\begin{aligned} \mathrm{i}\Pi^{(e)}_{\mu\nu}(p) &= - \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr} \left[\Psi'_\mu(-p, k_-, k_+) \, \Xi(k_+) \, \Psi'_\nu(p, k_+, k_-) \, \Xi(k_-) \right] \\ &= -4g^2 \int \frac{d^4k}{(2\pi)^4} \, \mathrm{tr} \left[\gamma_\mu \, \frac{1}{\not{k}_+ - m} \, \gamma_\nu \, \frac{1}{\not{k}_- - m} \right] \sin^2 \frac{\tilde{p} \, k}{2}, \end{aligned} \tag{2.103}$$

[13]There is also an auxiliary scalar field which can be integrated out.

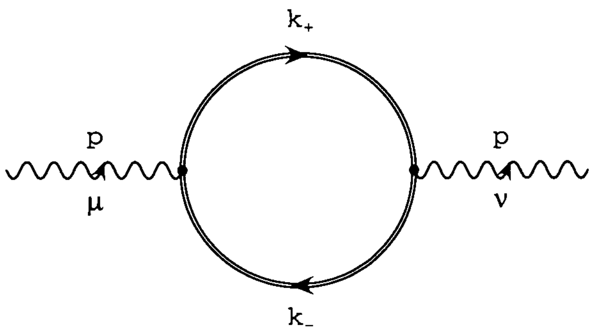


Figure 2.6: Self-energy contribution $\mathrm{i}\Pi^{(e)}_{\mu\nu}(p)$

where we used the same conventions as for the graph in section 2.4.1. Since the sinus factor in (2.103) does not vanish in general, there will be a non-planar contribution from the photino-loop in contract to the electron-loop of the previous section.

With (2.34) we can split (2.103) into a planar and a non-planar part

$$\mathrm{i}\Pi^{(e)}_{\mu\nu}(p) = \mathrm{i}\Pi^{\mathrm{P}\,(e)}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(e)}_{\mu\nu}(p). \tag{2.104}$$

Since we can achieve the contribution coming from the planar sector by considering ordinary $SU(N)$ Yang–Mills theory coupled to a massive adjoint fermion in the formal limit $N \to 1$, we will concentrate in the following on the non-planar part given by

$$\mathrm{i}\Pi^{\mathrm{NP}\,(e)}_{\mu\nu}(p) = 2g^2 \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_\mu \frac{1}{\not{k}_+ - m}\gamma_\nu \frac{1}{\not{k}_- - m}\right]\cos\tilde{p}\,k. \tag{2.105}$$

Multiplication by $(\not{k}_\pm + m)$ and using of $\not{k}\not{k} = k^2$ leads to

$$\mathrm{i}\Pi^{\mathrm{NP}\,(e)}_{\mu\nu}(p) = 2g^2 \int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{tr}\left[\gamma_\mu\,(\not{k}_+ + m)\,\gamma_\nu\,(\not{k}_- + m)\right]}{(k_+^2 - m^2)\,(k_-^2 - m^2)}\cos\tilde{p}\,k. \tag{2.106}$$

In doing some γ-matrix gymnastics via the relations[14]

$$\begin{aligned} \mathrm{tr}\left[\gamma_\mu\,\gamma_\nu\right] &= 2\,g_{\mu\nu}, \qquad\qquad \mathrm{tr}\left[\gamma_\mu\,\gamma_\nu\,\gamma_\rho\right] = 0,\\ \mathrm{tr}\left[\gamma_\mu\,\gamma_\nu\,\gamma_\rho\,\gamma_\sigma\right] &= 2\left(g_{\mu\nu}\,g_{\rho\sigma} - g_{\mu\rho}\,g_{\nu\sigma} + g_{\mu\sigma}\,g_{\nu\rho}\right), \end{aligned} \tag{2.107}$$

[14]Here, the γ-matrices are in a 2×2 representation and fulfill $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,\mathbb{1}_{2\times 2}$, because we are dealing with two-dimensional Weyl spinors instead of four-dimensional Dirac spinors.

we can rewrite (2.106) as

$$\mathrm{i}\Pi^{\mathrm{NP}\,(e)}_{\mu\nu}(p) = 4g^2 \int \frac{d^4k}{(2\pi)^4} \frac{k_{+\mu}k_{-\nu} + k_{+\nu}k_{-\mu} + g_{\mu\nu}\left(m^2 - k_{+\mu}k_-^{\mu}\right)}{\left(k_+^2 - m^2\right)\left(k_-^2 - m^2\right)} \cos\tilde{p}\,k. \tag{2.108}$$

Now, we will proceed close to the calculational steps we did already in section 2.2.3. The expression (2.108) reads with Schwinger parameterization (2.36) and formula (2.37) as follows:

$$\mathrm{i}\Pi^{\mathrm{NP}\,(e)}_{\mu\nu}(p) = \int \frac{d^4k}{(2\pi)^4} \int_0^\infty \mathrm{i}d\alpha_+ \int_0^\infty \mathrm{i}d\alpha_-\ \mathfrak{B}_{\mu\nu}(m,k) \\ \times\, \mathrm{e}^{\mathrm{i}\alpha_+(k_+^2-m^2)}\, \mathrm{e}^{\mathrm{i}\alpha_-(k_-^2-m^2)} \left(\mathrm{e}^{\mathrm{i}\tilde{p}k} + \mathrm{e}^{-\mathrm{i}\tilde{p}k}\right), \tag{2.109}$$

with

$$\mathfrak{B}_{\mu\nu}(m,k) = 2g^2\Big[k_{+\mu}k_{-\nu} + k_{+\nu}k_{-\mu} + g_{\mu\nu}\left(m^2 - k_{+\mu}k_-^{\mu}\right)\Big]. \tag{2.110}$$

Performing the trick $k_{\pm\mu} \to \partial/\mathrm{i}\partial z_\pm^\mu$ (see section 2.2.3) and doing the Gaussian integration (2.40) over the loop-momentum k leads to

$$\mathrm{i}\Pi^{\mathrm{NP}\,(e)}_{\mu\nu}(p) = \frac{\mathrm{i}g^2}{16\pi^2}\Big[4m^2 g_{\mu\nu}\,\mathfrak{L}_0(p,m) - 4\mathrm{i}g_{\mu\nu}\,\mathfrak{L}_1(p,m) \\ - \left(\tilde{p}^2 g_{\mu\nu} - 2\tilde{p}_\mu\tilde{p}_\nu\right)\mathfrak{L}_2(p,m) + \left(4p^2 g_{\mu\nu} - 8p_\mu p_\nu\right)\bar{\mathfrak{L}}(p,m)\Big], \tag{2.111}$$

with

$$\mathfrak{L}_\kappa(p,m) = \int_0^\infty \mathrm{i}d\alpha_+ \int_0^\infty \mathrm{i}d\alpha_-\ \frac{1}{(\mathrm{i}\beta)^2}\frac{1}{\beta^\kappa}\exp\left(\mathrm{i}\frac{\alpha_+\alpha_-}{\beta}p^2 - \mathrm{i}\beta m^2 - \mathrm{i}\frac{1}{4\beta}\tilde{p}^2\right), \tag{2.112}$$

$$\bar{\mathfrak{L}}(p,m) = \int_0^\infty \mathrm{i}d\alpha_+ \int_0^\infty \mathrm{i}d\alpha_-\ \frac{1}{(\mathrm{i}\beta)^2}\frac{\alpha_+\alpha_-}{\beta^2}\exp\left(\mathrm{i}\frac{\alpha_+\alpha_-}{\beta}p^2 - \mathrm{i}\beta m^2 - \mathrm{i}\frac{1}{4\beta}\tilde{p}^2\right), \tag{2.113}$$

and β given in (2.49). Comparing the integrals (2.112) and (2.113) with the expressions (2.47) and (2.48) we see that

$$\mathfrak{L}_\kappa(p,0) = \mathfrak{I}_\kappa(p), \qquad \bar{\mathfrak{L}}(p,0) = \bar{\mathfrak{I}}(p). \tag{2.114}$$

Therefore, the loop with a massless photino leads exactly to the same type of integrals as we had for the photon- and the ghost-loop in section 2.2.3. In the following, we consider the modifications coming from a non-zero photino-mass corresponding to softly broken supersymmetry. At the end, we can simply pass to the superymmetric case by setting the mass of the photino to zero.

Performing the reparameterization (2.50) and introducing the variable

$$\bar{\eta} = -\chi(1-\chi)p^2\lambda + m^2, \tag{2.115}$$

we get for (2.112) and (2.113) the expressions

$$\mathfrak{L}_\kappa(p,m) = \int_0^1 d\chi \left(-\mathrm{i}\big(\chi(1-\chi)p^2 - m^2\big)\right)^\kappa \int_0^\infty \frac{d\bar{\eta}}{\bar{\eta}^{1+\kappa}} \exp\Big(-\bar{\eta} - \frac{\bar{v}^2}{\bar{\eta}}\Big), \tag{2.116}$$

$$\bar{\mathfrak{L}}(p,m) = \int_0^1 d\chi\, \chi(1-\chi) \int_0^\infty \frac{d\bar{\eta}}{\bar{\eta}} \exp\Big(-\bar{\eta} - \frac{\bar{v}^2}{\bar{\eta}}\Big), \tag{2.117}$$

with

$$\bar{v}^2 = \big(\chi(1-\chi)p^2 - m^2\big)\frac{\tilde{p}^2}{4}, \tag{2.118}$$

in complete analogy to the equations (2.51)–(2.54) of section 2.2.3. Now, we can again use formula (2.55) to integrate over $\bar{\eta}$ in (2.116) and (2.117) and get

$$\mathfrak{L}_\kappa(p,m) = \int_0^1 d\chi \left(-\mathrm{i}\big(\chi(1-\chi)p^2 - m^2\big)\right)^\kappa \frac{2}{\bar{v}^\kappa} K_\kappa(2\bar{v}), \tag{2.119}$$

$$\bar{\mathfrak{L}}(p,m) = \int_0^1 d\chi\, \chi(1-\chi)\, 2\, K_0(2\bar{v}), \tag{2.120}$$

which correspond in the massless case exactly to (2.56) and (2.57).

Since the modified Bessel functions $K_\kappa(2\bar{v})$ fall off exponentially for large values of $\bar{v}$ (see (2.58)) the only non-vanishing contributions come from the region $\bar{v} \sim 0$. In this regime, the expansions of $K_\kappa(2\bar{v})$ are given by the equations (2.59) in replacing v by $\bar{v}$. With these expansions in hand, we can perform the remaining integration over χ and get for (2.116) with $\kappa = 0, 1, 2$

and (2.117) up to the order $\mathcal{O}(\tilde{p}^4, m^4)$:

$$
\begin{aligned}
\mathfrak{L}_0(p,m) &\sim \mathfrak{I}_0(p) - \frac{m^2}{p^2}\Big[2\ln\big(\frac{m^2}{p^2}\big) + \frac{1}{2}\Big(\frac{5}{3} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2\Big],\\
\mathfrak{L}_1(p,m) &\sim \mathfrak{I}_1(p) - im^2\Big[2\Big(\frac{7}{6} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)\\
&\qquad + \frac{1}{12}\Big(\frac{29}{15} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2\Big],\\
\mathfrak{L}_2(p,m) &\sim \mathfrak{I}_2(p) + m^2p^2\Big[-\frac{4}{p^2\tilde{p}^2} + \frac{1}{3}\Big(\frac{43}{30} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)\\
&\qquad + \frac{1}{120}\Big(\frac{221}{105} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2\Big],\\
\bar{\mathfrak{L}}(p,m) &\sim \bar{\mathfrak{I}}(p) + \frac{m^2}{p^2}\Big[\frac{2}{3} - \frac{1}{12}\Big(\frac{23}{15} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2\Big],
\end{aligned}
\tag{2.121}
$$

where the functions $\mathfrak{I}_\kappa(p)$, $\bar{\mathfrak{I}}(p)$ and the constant $\hat{\gamma}_{\mathrm{E}}$ are given in section 2.2.3. We see that the equations (2.121) justify the relations (2.114).

Now, we are able to write down the explicit expression for the non-planar part of the self-energy coming from the photino-loop in figure 2.6. We perform again a splitting (2.62) with the two projection operators (2.67) and (2.68) for the self-energy part $\mathrm{i}\Pi_{\mu\nu}^{\mathrm{NP}\,(e)}(p)$:

$$
\mathrm{i}\Pi_{\mu\nu}^{\mathrm{NP}\,(e)}(p) = \Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\,\mathrm{i}\Pi^{\mathrm{NP}\,(e)}(p) + \Big(\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^2}\Big)\,\mathrm{i}\tilde{\Pi}^{\mathrm{NP}\,(e)}(p), \tag{2.122}
$$

and get with the formulas (2.111) and (2.121) the following result:

$$
\begin{aligned}
\mathrm{i}\Pi^{\mathrm{NP}\,(e)}(p) \sim \frac{ig^2}{16\pi^2}\Big[&\frac{8}{3}\Big(\frac{5}{6} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2 + \frac{16}{3}m^2\\
&+ \frac{2}{15}\Big(\frac{107}{60} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^4\tilde{p}^2\\
&- \frac{2}{3}\Big(\frac{23}{15} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2m^2\Big],
\end{aligned}
\tag{2.123}
$$

$$
\begin{aligned}
\mathrm{i}\tilde{\Pi}^{\mathrm{NP}\,(e)}(p) \sim \frac{ig^2}{16\pi^2}\Big[&-\frac{32}{\tilde{p}^2} + \frac{4}{3}p^2 - 8m^2\\
&- \frac{1}{15}\Big(\frac{23}{15} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^4\tilde{p}^2\\
&+ \frac{2}{3}\Big(\frac{43}{30} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2m^2\Big],
\end{aligned}
\tag{2.124}
$$

again up to the order $\mathcal{O}(\tilde{p}^4, m^4)$. Since we are able to write the expression $\mathrm{i}\Pi_{\mu\nu}^{\mathrm{NP}\,(e)}(p)$ in the form given by (2.122), the non-planar part coming from the photino-loop is also transversal (see the relations (2.69)). This follows again from gauge symmetry.

Finally, we have to sum up all non-planar contributions coming from the Feynman graphs in the figures 2.1, 2.2, 2.3, and 2.6:

$$\mathrm{i}\Pi^{\mathrm{NP}}_{\mu\nu}(p)\big|^{m}_{\mathcal{N}=1} = \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{a})}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{b})}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{c})}_{\mu\nu}(p) + \mathrm{i}\Pi^{\mathrm{NP}\,(\mathrm{e})}_{\mu\nu}(p), \quad (2.125)$$

where the subscript shows the fact that we are dealing with an $\mathcal{N} = 1$ supersymmetric field content (one gauge field A_μ and one Weyl fermion λ), and the superscript denotes the non-vanishing mass m of the fermion.

Adding the photino-loop contribution (2.122) to the already obtained photon- and ghost-loop contributions (2.61) of section 2.2.3, we end up with the following non-planar self-energy part of softly broken $U(1)$ NCSYM theory up to the order $\mathcal{O}(\tilde{p}^4, m^4)$:

$$\mathrm{i}\Pi^{\mathrm{NP}}_{\mu\nu}(p)\big|^{m}_{\mathcal{N}=1} = \Big(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\Big)\,\mathrm{i}\Pi^{\mathrm{NP}}(p)\big|^{m}_{\mathcal{N}=1} + \Big(\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2}\Big)\,\mathrm{i}\widetilde{\Pi}^{\mathrm{NP}}(p)\big|^{m}_{\mathcal{N}=1}, \quad (2.126)$$

with

$$\begin{aligned}
\mathrm{i}\Pi^{\mathrm{NP}}(p)\big|^{m}_{\mathcal{N}=1} &\sim \frac{ig^2}{16\pi^2}\Big[-4\Big(1 - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2 + \frac{16}{3}m^2 \\
&\qquad - \frac{1}{6}\Big(\frac{11}{6} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^4\tilde{p}^2 \\
&\qquad - \frac{2}{3}\Big(\frac{23}{15} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2 m^2\Big], \quad (2.127) \\
\mathrm{i}\widetilde{\Pi}^{\mathrm{NP}}(p)\big|^{m}_{\mathcal{N}=1} &\sim \frac{ig^2}{16\pi^2}\Big[-8m^2 + \frac{2}{3}\Big(\frac{43}{30} - \hat{\gamma}_{\mathrm{E}} - \frac{1}{2}\ln\big(p^2\tilde{p}^2\big)\Big)p^2\tilde{p}^2 m^2\Big], \quad (2.128)
\end{aligned}$$

Let us discuss this result in the next section in the context of UV/IR mixing.

2.5 UV/IR mixing — Part two

Let us first investigate the leading part of $\widetilde{\mathcal{P}}_{\mu\nu}\,\widetilde{\Pi}^{\mathrm{NP}}(p)\big|^{m}_{\mathcal{N}=1}$ in the non-planar self-energy (2.126) of softly broken non-commutative $U(1)$ SUSY Yang–Mills theory by using the equation (2.128) and the projector (2.68):

$$\widetilde{\mathcal{P}}_{\mu\nu}\,\widetilde{\Pi}^{\mathrm{NP}}(p)\big|^{m}_{\mathcal{N}=1} = -\frac{g^2 m^2}{2\pi^2}\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^2} + \mathcal{O}(\tilde{p}^2). \quad (2.129)$$

The expression (2.129) is completely finite in the limit $\tilde{p} \to 0$ or $\theta \to 0$. There is no quadratic IR singularity anymore in the self-energy of the non-commutative photon in a softly broken supersymmetric set-up. The quadratic IR singular term in the self-energy of $U(1)$ NCYM in (2.77) is exactly canceled by the photino-loop contribution given by (2.124).

In a regime where supersymmetry is fully realized, *i.e.* when we can set the photino-mass $m = 0$, we get an even stronger result, namely

$$\widetilde{\mathcal{P}}_{\mu\nu}\,\widetilde{\Pi}^{\mathrm{NP}}(p)\big|^{m=0}_{\mathcal{N}=1} = 0. \tag{2.130}$$

Equation (2.130) holds to every order in $\tilde{p}$, it is an exact result. We can see this by looking at the sum of the unexpanded equations (2.46) and (2.111) where all terms proportional to $\widetilde{\mathcal{P}}_{\mu\nu}$ (2.68) are exactly canceled in the massless case,[15] where $\mathfrak{J}_2(p) = \mathfrak{L}_2(p, 0)$ (see (2.114)). Hence, the non-planar vacuum polarization tensor $\mathrm{i}\Pi^{\mathrm{NP}}_{\mu\nu}(p)\big|^{m=0}_{\mathcal{N}=1}$ in (2.126) is proportional to the ordinary transversal projection operator $\mathcal{P}_{\mu\nu}$ (2.67) for arbitrary values of the non-commutative momentum $\tilde{p}$.

Therefore, supersymmetry kills all quadratic UV/IR mixing effects of non-commutative Yang–Mills theory. Furthermore, it has been been shown in [32] that these cancelation effects remove also the linear IR divergence of the gauge three-point-function (2.72). We are only left with logarithmic IR divergences which are harmless with respect to higher loop integrations. Supersymmetry seems to be a necessary tool for rendering non-commutative gauge theories renormalizable.

2.5.1 Dispersion relation

We have argued in section 2.3.1 that the quadratic IR divergent term in the self-energy of non-commutative $U(1)$ Yang–Mills theory spoils renormalizability at higher loops. But, there is another crucial effect coming from this term. To see this, we generalize the expressions (2.77) and (2.129) to the following form [14, 32, 34, 35, 37, 38]:

$$\widetilde{\Pi}^{\mathrm{NP}}(p) = \frac{\mathfrak{C}}{\tilde{p}^2} + \mathcal{O}(\tilde{p}^0), \qquad \mathfrak{C} = \frac{2g^2}{\pi^2}\Big(1 - n_{\mathrm{f}} + \frac{n_{\mathrm{s}}}{2}\Big), \tag{2.131}$$

where n_{f} and n_{s} are the numbers of Weyl fermions and real scalars in the game.[16] We can understand expression (2.131) from the fact that fermions in the adjoint representation of the gauge group contribute to these IR divergent terms with an additional minus sign (see section 2.4.2), and scalars come with a positive sign, due to their bosonic nature. Therefore, we can write

[15] There are no terms proportional to $\widetilde{\mathcal{P}}_{\mu\nu}$ in $\mathrm{i}\Pi^{\mathrm{NP}\,(b)}_{\mu\nu}(p)$ given by (2.43).

[16] For $U(N)$ NCYM theory one has $\mathfrak{C} = \frac{2g^2}{\pi^2}(1 - n_{\mathrm{f}} + \frac{n_{\mathrm{s}}}{2})N$.

the constant $\mathfrak{C}$ in the sloppy but simple form:[17]

$$\mathfrak{C} \propto N_{\mathrm{B}} - N_{\mathrm{F}}. \tag{2.132}$$

The number of bosonic degrees of freedom N_{B} is always equal to the number of fermionic degrees of freedom N_{F} in a supersymmetric theory, hence the constant $\mathfrak{C}_{\mathrm{SUSY}} = 0$, and all quadratic IR divergences disappear. In non-supersymmetric theories there is a sign difference between the cases $N_{\mathrm{B}} > N_{\mathrm{F}}$ and $N_{\mathrm{B}} < N_{\mathrm{F}}$.

Now, we know from ordinary Yang-Mills theory that the polarization tensor modifies the photon propagator [92, 93, 94]. The one-loop corrected photon propagator has still a pole at $p^2 = 0$, yielding the standard dispersion relation for massless photons:

$$\text{commutative:} \qquad E^2 = \vec{p} \cdot \vec{p}. \tag{2.133}$$

In non-commutative Yang–Mills theory we have to take into account the non-planar part of the self-energy, where the quadratic IR divergent term yields an additional pole in the modified propagator. For explicit calculations of the resummation procedure we refer the reader to [34, 35]. From the non-planar polarization tensor (2.131) one gets the following dispersion relation:[18]

$$\text{non-commutative:} \qquad E^2 = \vec{p} \cdot \vec{p} - \frac{\mathfrak{C}}{|\theta \cdot \vec{p}|^2}. \tag{2.134}$$

The physical interpretation of (2.134) is very interesting and we draw this dispersion relation for all different cases qualitatively in the following graphs of the figures 2.7, 2.8, 2.9, 2.10.

The figure 2.7 corresponds to the case $N_{\mathrm{B}} > N_{\mathrm{F}}$ where we see that the energy is unbounded from below and we have a tachyonic instability in the low-momentum regime of the spectrum. This could be taken as a hint that we are expanding around the wrong vacuum, hence perturbation theory fails. But nobody comes up with a solution to this problem so far. Note the fact that this takes place also in pure non-commutative Yang–Mills theory which we considered in section 2.2.

The tachyonic instability disappears in the case $N_{\mathrm{B}} < N_{\mathrm{F}}$ of figure 2.8, but nevertheless the theory is oddly behaved at low-momentum where the energy increases again at low momentum and gets infinite at zero momentum.

[17] Remember, that each gauge boson and Weyl fermion has two degrees of freedom, whereas a real scalar field has just one degree of freedom.

[18] Here, we restrict ourselves to the case $\theta^{0i} = 0$, *i.e.* space-space non-commutativity. Therefore, $\tilde{p}^{\mu} = \theta^{\mu\nu} p_{\nu}$ has only spatial components and we can write it as $\theta \cdot \vec{p}$.

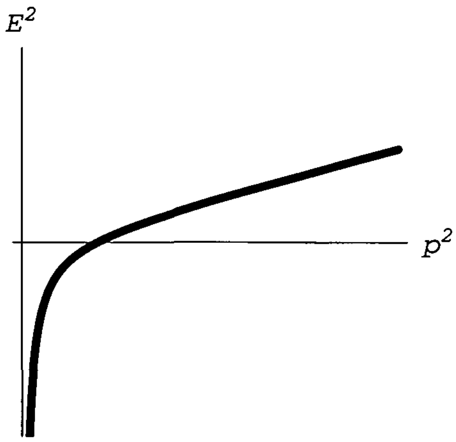


Figure 2.7: $N_{\rm B} > N_{\rm F}$

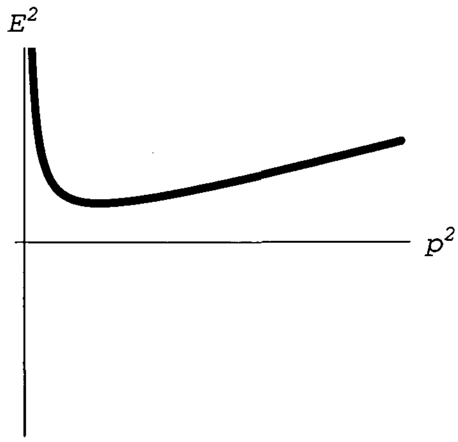


Figure 2.8: $N_{\rm B} < N_{\rm F}$

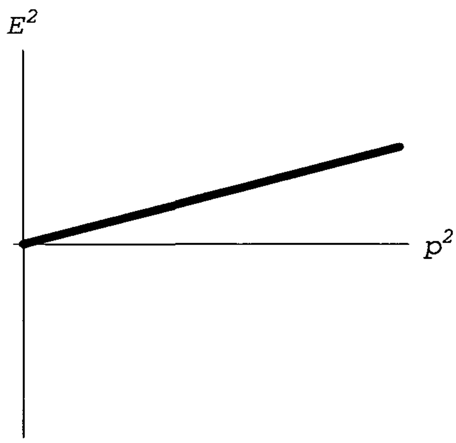


Figure 2.9: $N_{\rm B} = N_{\rm F}$ (SUSY)

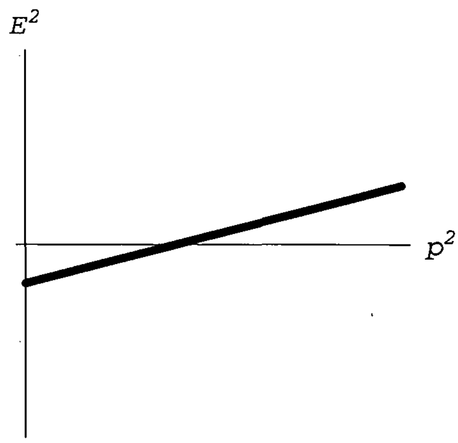


Figure 2.10: Softly broken SUSY

The only way out of this disaster seems to be supersymmetry where these IR divergences are exactly canceled and we get back the usual dispersion relation for the photon even in a non-commutative set-up (see figure 2.9).

The last figure 2.10 shows the dispersion relation of softly broken $\mathcal{N}=1$ NCSYM which is given through relation (2.129) by the following expression:[19]

$$\text{softly broken NC SUSY:} \qquad E^2 = \vec{p}\cdot\vec{p} - \frac{g^2}{2\pi^2}m^2. \tag{2.135}$$

Equation (2.135) implies that the photon has a negative mass squared at zero momentum. And, even worse, this tachyonic mass of the photon cannot be argued away by demanding a very small non-commutativity scale, because it is independent of θ. This fact rules out softly broken $\mathcal{N}=1$ NCSYM theory in nature [42].

On the other hand, it was argued in [34, 35] that soft breaking scenarious could make sense in higher supersymmetric theories. There, one has more fields in the game and the relation (2.135) is replaced by

$$\text{SB NC higher SUSY:} \qquad E^2 = \vec{p}\cdot\vec{p} - \frac{g^2}{2\pi^2}\Big(\sum_f m_f^2 - \frac{1}{2}\sum_s m_s^2\Big), \tag{2.136}$$

where m_f and m_s denote the masses of the fermions and scalars, respectively. Therefore, one can get a massless photon if $\sum_f m_f^2 = \frac{1}{2}\sum_s m_s^2$, but this would correspond to fine-tuning and seems to be unphysical.

2.5.2 Beta-function

Having discussed in the previous section the effects coming from the quadratic UV/IR mixing effects in the one-loop corrected two-point function, we would like to take a look now on the logarithmic IR divergent piece of (2.126). We know already from section 2.3.2 that non-planar logarithmic contributions give rise to a modification of the beta-function in the low-momentum regime $|p| \ll \Lambda_\theta$, with $\Lambda_\theta = 1/\sqrt{\theta}$ the non-commutativity scale.

In a supersymmetric set-up, *i.e.* with the photino mass $m=0$, we have the following relations for the high- and the low-momentum regime, respectively:

$$|p| \gg \Lambda_\theta : \qquad \frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} \sim \frac{1}{16\pi^2}\Big(\beta_0 \ln\left(p^2/\Lambda^2\right)\Big), \tag{2.137}$$

and

$$|p| \ll \Lambda_\theta : \qquad \frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} \sim -\frac{1}{16\pi^2}\Big(\beta_0 \ln\left(\tilde{p}^2\Lambda^2\right)\Big), \tag{2.138}$$

[19]Remember that $\tilde{p}^2 < 0$ (see section 2.2.3).

in complete analogy to (2.85) and (2.86), but with the one-loop beta-function coefficient β_0 modified to

$$\beta_0 = \frac{22}{3} - \frac{4}{3}n_{\mathrm{f}} - \frac{1}{3}n_{\mathrm{s}}, \tag{2.139}$$

again with n_{f} and n_{s} the numbers of Weyl fermions and real scalars in the game.[20] As in the non-supersymmetric case of section 2.3.2, we have a complete similar running of the gauge coupling constant $g(p)$ in the ultraviolet and infrared regime, yielding to a weakly coupled theory in the UV and IR, and confirming therefore again the UV/IR duality of non-commutative field theories.

Additionally, supersymmetric $\mathcal{N} = 4$ NCYM theory has a vanishing beta-function ($n_{\mathrm{f}} = 4$, $n_{\mathrm{s}} = 6$), suffering therefore not even from logarithmic UV/IR mixing in the two-point function and is believed to be ultraviolet finite [105, 106].

In the case of softly broken supersymmetric NC $U(1)$ YM theory, we have a non-vanishing photino mass giving us therefore an additional mass scale. The behaviour of the running coupling for momenta much smaller than the photino mass scale m is given by

$$|p| \ll m < \Lambda_\theta : \qquad \frac{1}{g^2(p)} - \frac{1}{g_\Lambda^2} \sim -\frac{1}{16\pi^2}\left(\frac{22}{3}\ln\left(\tilde{p}^2 m^2\right)\right). \tag{2.140}$$

In the large-momentum regime, the relation (2.137) is still valid (with $n_{\mathrm{f}} = 1$ and $n_{\mathrm{s}} = 0$), since m can be neglected in this case.

For an arbitrary number of Weyl fermions n_{f} and real scalars n_{s} with different masses m_f and m_s, we have to make the following replacement in equation (2.140):

$$m \longrightarrow m_{\mathrm{eff}} = \left(\prod_{f=1}^{n_{\mathrm{f}}} m_f^2 \prod_{s=1}^{n_{\mathrm{s}}} m_s^{\frac{1}{2}}\right)^{\frac{1}{2n_{\mathrm{f}}+n_{\mathrm{s}}/2}}, \tag{2.141}$$

in the regime where the momentum p is much smaller than all the masses in the game. This concludes our discussion of the beta-function of non-commutative gauge theories.

2.5.3 Heuristic explanation

To conclude the discussion of the UV/IR mixing, we would like to give a simple physical picture about its origin. We will closely follow the argumentation line given in [11].

[20] For $U(N)$ NCYM theory the expression for β_0 gets multiplied by N.

Consider a non-commutative field $\phi(x)$ which interacts through a Moyal commutator (1.34) with some other field $\psi(x)$. This resembles the cases we had for the interaction parts in non-commutative Yang–Mills theory in the previous sections. Therefore, we take the interaction Lagrangian

$$\mathcal{L}_{\text{int}} = [\psi(x)\,,\,\phi(x)]_\star, \tag{2.142}$$

which can be written for a plane wave configuration

$$\phi(x) \sim \mathrm{e}^{\mathrm{i}px}, \tag{2.143}$$

in the following form:

$$\mathcal{L}_{\text{int}} = \big(\psi(x - \tilde{p}/2) - \psi(x + \tilde{p}/2)\big)\,\mathrm{e}^{\mathrm{i}px}, \tag{2.144}$$

where we used the formula (1.28) for the star-product.

Hence, the non-commutative field $\phi(x)$ interacts with the field $\psi(x)$ effectively like an extended rigid object [107, 108, 109, 110] with length ℓ given by the quantity[21]

$$\ell = |\tilde{p}| = |\theta \cdot p|. \tag{2.145}$$

From relation (2.145) follows that the size ℓ of a non-commutative particle represented by the field $\phi(x)$ grows with its momentum p. Therefore, when a particle of momentum p circulates in a loop it can induce an effect at the distance $|\theta \cdot p|$. The high-momentum end (ultraviolet regime) of the loop-integrals give rise to long range forces (infrared regime) which are completely absent in the classical theory [14]. Hence, we get a mixing of ultraviolet and infrared degrees of freedom, called UV/IR mixing.

Furthermore, the non-locality of the interactions in non-commutative field theories leads to an interesting extension of the heuristic Heisenberg uncertainty principle. There, the extent Δx of a particle is given by[22]

$$\Delta x \sim 1/\Delta p. \tag{2.146}$$

Combining the two relations (2.145) and (2.146), we see that the effective size ℓ_{eff} of a non-commutative particle follows from

$$\ell_{\text{eff}} = \max\big(1/|p|, |\theta \cdot p|\big). \tag{2.147}$$

[21] Fields interacting through a star-product instead of a Moyal commutator have the effective length $\ell/2$ [111].

[22] Note, that $\hbar = c = 1$ in "God-given" units.

Therefore, the size of a particle grows in the infrared and the ultraviolet momentum-regime, taking a non-vanishing minimum in-between. This type of relation is known to appear also in string theory [112] where the non-commutativity scale θ is replaced by the Regge slope parameter α'. Hence, non-commutative field theory can also be seen as an interesting toy model for string theory.

2.6 Seiberg–Witten map

Despite the fact of being not in the scope of this work we would like to give a few comments about the *Seiberg–Witten map* which showed to be very important in the context of quantizing non-commutative gauge theories.

It was first derived in [8] via requirement of gauge equivalence between a commutative δ_λ and a non-commutative gauge transformation $\hat{\delta}_{\hat{\lambda}}$:

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}}\hat{A}(A) = \hat{A}(A + \delta_\lambda A), \tag{2.148}$$

where $\hat{A}(A)$ is a non-commutative gauge field which depends on a commutative gauge field A. Or, denoted graphically, equation (2.148) reads:

$$\begin{array}{ccccc} & \hat{A} & \longrightarrow & A & \\ \theta\neq 0 & \hat{\delta}_{\hat{\lambda}}\big\downarrow & & \big\downarrow\delta_\lambda & \theta=0 \\ & \hat{A}' & \longrightarrow & A' & \end{array}$$

A non-commutative gauge transformation on the left hand side has to be equivalent to a commutative gauge transformation on the right hand side. This demand for gauge equivalence follows from the fact that non-commutative and commutative gauge theories arise from the same two-dimensional field theory regularized in different ways [8], and leading therefore to the same physics given by gauge invariant operators.

We have shown in [49] that the Seiberg–Witten map results also from a covariant splitting of combined conformal transformations of the non-commutative Yang–Mills field $\hat{A}_\mu$ and of the non-commutativity parameter $\theta^{\mu\nu}$. The Seiberg–Witten differential equations describing the map can be computed as the missing piece to complete a covariant conformal transformation to an invariance of the Yang–Mills action. This approach does not require the usual ansatz of gauge equivalence.

The Seiberg–Witten differential equation for the non-commutative gauge field $\hat{A}_\mu$ is given by

$$\frac{\partial \hat{A}_\mu}{\partial\theta^{\alpha\beta}} = -\frac{1}{8}\left\{\hat{A}_\alpha, (\hat{F}_{\beta\mu} + \partial_\beta\hat{A}_\mu)\right\}_\star + \frac{1}{8}\left\{\hat{A}_\beta, (\hat{F}_{\alpha\mu} + \partial_\alpha\hat{A}_\mu)\right\}_\star, \tag{2.149}$$

with the non-commutative field strength

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - \mathrm{i}\left[\hat{A}_\mu , \hat{A}_\nu\right]_\star . \tag{2.150}$$

The differential equation (2.149) can be solved for the non-commutative gauge field $\hat{A}_\mu$ perturbatively in $\theta^{\mu\nu}$ yielding the following polynomial in the commutative gauge field A_μ:

$$\hat{A}_\mu(A) = A_\mu - \frac{1}{2}\theta^{\rho\sigma} A_\rho \left(\partial_\sigma A_\mu + F_{\sigma\mu}\right) + \mathcal{O}(\theta^2), \tag{2.151}$$

now with the usual commutative field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.152}$$

With the expansion of the non-commutative gauge field (2.151) and the expansion of the star-product (1.29), we can map the non-commutative Yang–Mills action (2.12) from section 2.2:

$$S = -\frac{1}{4g^2}\int d^4x \; F_{\mu\nu} \star F^{\mu\nu}, \tag{2.153}$$

into the following commutative action [45]:

$$S = -\frac{1}{4g^2}\int d^4x \left(F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\,\theta^{\alpha\beta}F_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + 2\,\theta^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}F^{\mu\nu}\right) + \mathcal{O}(\theta^2). \tag{2.154}$$

The action (2.154) is invariant under the usual Abelian gauge transformations

$$\delta_\lambda A_\mu = \partial_\mu \lambda, \tag{2.155}$$

and has in its full form, involving all orders of θ, infinitely many interactions at infinitely high order in the gauge field. Furthermore, since θ has dimension minus two, the theory is power-counting non-renormalizable in the traditional sense. Despite these facts, we started to do loop calculations in [45] and have been able to show in [47] that the non-commutative photon self-energy is renormalizable to all orders in θ and $\hbar$ via Seiberg–Witten map.

The Seiberg–Witten map of non-commutative QED was performed in [46] and later on, shown in [52] to be non-renormalizable due to a divergence in the electron four-point function which cannot be removed by field redefinitons in the context of the Seiberg–Witten map.

We refer the reader for further considerations on this interesting topic to the literature where a lot of work has been done [43, 44, 48, 50, 51, 53, 54, 55]. With this short trip into the world of quantization via Seiberg–Witten map we conclude the chapter about non-commutative gauge theories.

Chapter 3

Non-Commutative Instantons

3.1 Instantons in general

Instantons are finite-action solutions of the classical equation of motion in the Euclidean version of some theory. They play an important role in quantum field theory, describing phenomena which cannot be catched by perturbation theory. Excellent reviews on ordinary instantons can be found in Coleman's book [96] and in the review article by Shifman *et al.* [113].

3.1.1 Ordinary instantons in gauge theories

Before passing to non-commutative instantons we will give a very short survey of ordinary instantons in Euclidean four-dimensional Yang–Mills theory. The Euclidean action is given by

$$S = -\frac{1}{2g^2} \int d^4x \, \mathrm{Tr}\big(F_{\mu\nu} F_{\mu\nu}\big), \tag{3.1}$$

with g the coupling constant, Tr the trace in color space, and the Euclidean anti-Hermitian field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\,[A_\mu\,,\, A_\nu]\,, \tag{3.2}$$

taking values in some gauge group G. The action (3.1) implements the following equation of motion for the gauge field A_μ:

$$D_\mu F_{\mu\nu} = 0, \tag{3.3}$$

with the covariant derivative

$$D_\mu = \partial_\mu - \mathrm{i}\,[A_\mu\,,\ \ ]\,. \tag{3.4}$$

Now, we are after finite-action solutions of (3.3), but instead of trying to solve the non-linear differential equation (3.3), we will study the properties such solutions must have yielding us to a much simpler possibility to obtain them. In the following, we show that such solutions are of the form

$$A_\mu = \mathfrak{g}\partial_\mu \mathfrak{g}^{-1} + \mathcal{O}(1/r^2), \tag{3.5}$$

where r is the radial variable in Euclidean four-space, and $\mathfrak{g}$ is a gauge transformation from configuration space into the gauge group G of order one, depending on angular variables only.[1] Inserting (3.5) into (3.2), we see that

$$F_{\mu\nu} \sim \mathcal{O}(1/r^3), \tag{3.6}$$

because the first term in (3.5) is pure gauge. Therefore, $F_{\mu\nu}$ falls off faster than $1/r^2$ and we get a finite action in four-dimensional space.

Thus, with every finite-action field configuration there is associated a group-element-valued function of angular variables that is to say, a mapping of a three-dimensional hypersphere S^3 into the gauge group G. Due to a remarkable theorem (see [96] and references therein) we can cook down the whole problem to the case $G = SU(2)$, because any continuous mapping of S^3 into G can be continously deformed into a mapping of S^3 into an $SU(2)$ subgroup of G. Since $SU(2)$ is topologically a hypersphere S^3, we have to study mappings from S^3 into S^3. Now, we know from mathematicians that such mappings are divided into different homotopy classes. In our case, we have to consider the third homotopy group[2]

$$S^3 \to S^3 : \qquad \pi_3(S^3) \sim \mathbb{Z}. \tag{3.7}$$

Equation (3.7) means that every mapping $S^3 \to S^3$ has an integer $Q \in \mathbb{Z}$ associated with it, the so-called Pontryagin index or winding number[3] which is a homotopy invariant and uniquely defined. Each mapping belongs exactly to one and only one homotopy class, and one cannot pass from one homotopy class to another via continuous transformations.

[1] Written in equations: $g = \exp(i\lambda^a T^a)$ with λ^a some function in Euclidean four-space depending on angular variables only, and T^a the generators of the Lie algebra associated to the Lie group G.

[2] In the case of an Abelian gauge theory with $G = U(1)$, we would have the trivial mapping $S^3 \to S^1 : \pi_3(S^1) \sim \mathbb{1}$ without any winding number. There exist no instantons in ordinary Maxwell theory, contrary to its non-commutative counterpart (see section 3.3).

[3] Denoted picturally, it gives the number how often the first hypersphere is wrapped around the second one.

To summarize, we can characterize every finite-action solution (3.5) by an integer Q which can be calculated by the following formula:[4]

$$Q = -\frac{1}{16\pi^2} \int d^4x \,\mathrm{Tr}\big(F_{\mu\nu}\, \tilde{F}_{\mu\nu}\big), \tag{3.8}$$

with the dual field strength[5]

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, F_{\rho\sigma}. \tag{3.9}$$

Although we do not have an explicit form of the finite-action solution (3.5), we can nevertheless calculate the value of the action for it.[6] Rewriting the Euclidean action (3.1) by using (3.8) we obtain

$$\begin{aligned} S &= -\frac{1}{2g^2} \int d^4x \,\mathrm{Tr}\big(F_{\mu\nu}\, F_{\mu\nu}\big) \\ &= -\frac{1}{2g^2} \int d^4x \,\mathrm{Tr}\Big[\pm F_{\mu\nu}\, \tilde{F}_{\mu\nu} + \frac{1}{2}\,\big(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\big)^2\Big] \\ &= \pm\frac{8\pi^2}{g^2}\, Q - \frac{1}{4g^2} \int d^4x \,\mathrm{Tr}\Big[\big(F_{\mu\nu} \mp \tilde{F}_{\mu\nu}\big)^2\Big]. \end{aligned} \tag{3.10}$$

Since the second term in (3.10) is negative definite,[7] we obtain the minimum of the action if the solution fulfills the self-duality (anti-self-duality) condition

$$F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}, \tag{3.11}$$

where the positive (negative) sign holds for positive (negative) Q. Further, we show that gauge fields satisfying (3.11) are also solutions of the equation of motion. Taking the covariant derivative (3.4) of (3.11) and using (3.9) leads to

$$\begin{aligned} D_\mu\, F_{\mu\nu} = \pm D_\mu\, \tilde{F}_{\mu\nu} &= \pm\frac{1}{2}\, \epsilon_{\mu\nu\rho\sigma}\, D_\mu\, F_{\rho\sigma} \\ &= \pm\frac{1}{6}\, \epsilon_{\mu\nu\rho\sigma}\big(D_\mu\, F_{\rho\sigma} + D_\rho\, F_{\sigma\mu} + D_\sigma\, F_{\mu\rho}\big) = 0, \end{aligned} \tag{3.12}$$

where the last equality follows from the Bianchi identity. Hence, (anti-)self-dual gauge field configurations (3.11) fulfill the equation of motion (3.3).

---

[4]For brevity, it is given without any proof, but we refer the interested reader to [96, 113].

[5]The factor 1/2 is inserted in the definition so that $\tilde{\tilde{F}}_{\mu\nu} = F_{\mu\nu}$, and $\epsilon_{1234} = 1$ with antisymmetry in all indices.

[6]We have to disappoint the reader waiting for such an explicit solution, but we will not need it anyway.

[7]Remember that we have chosen A_μ anti-Hermitian.

Now, we can make the following statement: (Anti-)Self-dual gauge fields are finite-action solutions of the equation of motion, and establish therefore our searched instantons with the action

$$S_{\text{inst}} = \frac{8\pi^2}{g^2}\,|Q|, \tag{3.13}$$

where the index Q is also called *topological charge* of the instanton. Instantons with negative topological charge corresponding to anti-self-dual solutions are called anti-instantons in the following.

3.1.2 Review of the ADHM construction

Before going into details of the case of non-commutative instantons we will review very briefly the pioneering work [72] of constructing instantons in usual commutative space $\mathbb{R}^4$. This ADHM construction (named after Atiyah, Hitchin, Drinfeld, and Manin) has the advantage of solving quadratic matrix equations instead of non-linear differential equations,[8] in order to obtain (anti-)self-dual instantons [114]. Here, we will only state the main steps of this construction, *i.e.* giving just a recipe. For further details we refer the interested reader to the literature [115, 116, 117, 118, 119, 120].

In order to construct (anti-)self-dual $U(N)$ gauge fields corresponding to k instantons, one starts from the following ADHM data:

- A pair of complex hermitian vector spaces $V = \mathbb{C}^k$ and $W = \mathbb{C}^N$.
- The operators $B_1, B_2 \in Hom(V,V), I \in Hom(W,V), J \in Hom(V,W)$, which have to satisfy the equations:

$$\begin{aligned}\mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0,\\ \mu_c &= [B_1, B_2] + IJ = 0.\end{aligned} \tag{3.14}$$

For $z = (z_1, z_2) \in \mathbb{C}^2 \approx \mathbb{R}^4$ we define the Dirac operator $\mathcal{D}_z^\dagger : V \oplus V \oplus W \to V \oplus V$ by the formula:

$$\mathcal{D}_z^\dagger = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix}, \tag{3.15}$$

[8] However, it was due to Belavin, Polyakov, Shvarts, and Tyupkin [114] to construct the so-called BPST instantons out of the first-order differential equation (3.11) instead of using the second-order differential equation (3.3).

where

$$\tau_z = \begin{pmatrix} B_2 - \bar{z}_2 & B_1 + z_1 & I \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} -B_1 - z_1 \\ B_2 - \bar{z}_2 \\ J \end{pmatrix} \tag{3.16}$$

for self-dual (SD) instantons, and

$$\tau_z = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 & I \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} -B_1 + z_1 \\ B_2 - z_2 \\ J \end{pmatrix} \tag{3.17}$$

for anti-self-dual (ASD) instantons. The ADHM equations (3.14) are equivalent to the so-called factorization conditions:

$$\tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z, \qquad \tau_z \sigma_z = 0. \tag{3.18}$$

Given the matrices obeying all the conditions above the actual instanton solution is determined by the following formula:

$$A_\mu = \psi^\dagger \partial_\mu \psi, \tag{3.19}$$

where $\psi : W \to V \oplus V \oplus W$ is given by the N zero-modes of the Dirac operator

$$\mathcal{D}_z^\dagger \psi = 0. \tag{3.20}$$

The zero-modes ψ are normalized by

$$\psi^\dagger \psi = 1, \tag{3.21}$$

and have to fulfill the completeness relation

$$\mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} \mathcal{D}_z^\dagger + \psi \psi^\dagger = \mathbb{1}. \tag{3.22}$$

Note that the operators on the left hand side of (3.22)

$$\mathfrak{D} = \mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} \mathcal{D}_z^\dagger, \qquad \mathfrak{P} = \psi \psi^\dagger, \tag{3.23}$$

are Hermitian projection operators since they have the following behaviour in addition to (3.22):

$$\mathfrak{D}^2 = \mathfrak{D}, \qquad \mathfrak{P}^2 = \mathfrak{P}, \qquad \mathfrak{D} \cdot \mathfrak{P} = \mathfrak{P} \cdot \mathfrak{D} = 0. \tag{3.24}$$

So far we have been completely abstract, but in section 3.3.4 we will see that a vector ψ constructed by the formulas above will indeed correspond to an (anti-)self-dual gauge field which minimizes the Euclidean action, being therefore an instanton, as was explained in section 3.1.1.

3.2 Non-commutative ADHM construction

3.2.1 Parameterizing the non-commutative space

Before considering the generalization of the ADHM sheme to the non-commutative case, we will introduce the following notations.

We consider a four-dimensional non-commutative Euclidean space which is represented by coordinates x_μ obeying the following algebra:

$$[x_\mu, x_\nu] = \mathrm{i}\theta_{\mu\nu}, \tag{3.25}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric real matrix and $\mu, \nu = 1,2,3,4$ are the Euclidean Lorentz indices. Using Euclidean space-time rotations, $\theta_{\mu\nu}$ can be always brought to the form

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \theta_{12} & 0 & 0 \\ -\theta_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_{34} \\ 0 & 0 & -\theta_{34} & 0 \end{pmatrix}. \tag{3.26}$$

Passing to complex coordinates,

$$\begin{aligned} z_1 &= x_2 + \mathrm{i}x_1, & \bar{z}_1 &= x_2 - \mathrm{i}x_1, \\ z_2 &= x_4 + \mathrm{i}x_3, & \bar{z}_2 &= x_4 - \mathrm{i}x_3, \end{aligned} \tag{3.27}$$

we end up with the following commutator relations:

$$\begin{aligned} [\bar{z}_1, z_1] &= 2\theta_{12}, & [\bar{z}_2, z_2] &= 2\theta_{34}, \\ [z_i, z_j] &= 0, & [\bar{z}_i, z_{j\neq i}] &= 0, \end{aligned} \tag{3.28}$$

where $i, j = 1,2$ denote the indices for the complex coordinates.

Now, we have to distinguish between three important cases:

- the case of $\mathbb{R}^4$:
  Here, $\theta_{12} = \theta_{34} = 0$ and all the commutators vanish giving the ordinary commutative space. The corresponding gauge theory is the usual commutative gauge theory, and instanton solutions are given by the standard ADHM construction (see section 3.1.2).

- the case of $\mathbb{R}^2_\theta \times \mathbb{R}^2$:
  When either θ_{12} or θ_{34} vanishes, there is only one non-vanishing commutator in (3.28). Here, we have the direct product of two-dimensional

non-commutative space with two-dimensional ordinary space. For definiteness we set $\theta_{34} = 0$ and introduce the notation $\theta_{12} = \theta = \zeta/2$. Therefore, we get

$$[\bar{z}_1, z_1] = \zeta, \qquad [\bar{z}_2, z_2] = [z_i, z_j] = [\bar{z}_i, z_{j\neq i}] = 0. \tag{3.29}$$

This case corresponds to space–space non-commutativity[9] and therefore avoids unitarity problems of the associated Lorentzian theory [122, 123].

- the case of $\mathbb{R}^4_\theta$:
  The most general case is given by $\theta_{12} \neq 0$ and $\theta_{34} \neq 0$, and generates the non-commutative Euclidean space-time $\mathbb{R}^4_{NC} = \mathbb{R}^2_{NC} \times \mathbb{R}^2_{NC}$. Here, the corresponding gauge theory has non-commutative (Euclidean) time implementing all the various subtleties [122, 123]. Via appropriate rescalings[10] of the coordinates one can divide this case into two further subclasses:

  - the self-dual $\theta_{\mu\nu}$:
    Here, the condition $\theta_{12} = \theta_{34} = \zeta/4$ gives a self-dual theta, $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma} = \theta_{\mu\nu}$ implementing the following commutator relations:

$$[\bar{z}_i, z_j] = \delta_{ij}\zeta/2, \qquad [z_i, z_j] = 0. \tag{3.30}$$

  - the anti-self-dual $\theta_{\mu\nu}$:
    This case is given by $\theta_{12} = -\theta_{34} = \zeta/4$ corresponding to an anti-self-dual theta, $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma} = -\theta_{\mu\nu}$ and yielding to:

$$[\bar{z}_i, z_j] = (-)^{i-1}\delta_{ij}\zeta/2, \qquad [z_i, z_j] = 0. \tag{3.31}$$

Since we have settled now all the conventions for the different cases of non-commutativity in Euclidean four-dimensional space, we will proceed to the discussion of constructing instantons on this space.

[9] Using Euclidean space-time rotations, we can always transform a three-dimensional non-commutative space into the product space $\mathbb{R}^2_\theta \times \mathbb{R}^2$.

[10] Of course, physics of non-commutative space-time is determined by a constant $\theta_{\mu\nu}$ and therefore not invariant under dilatations and parity transformations. However, the general case can be always recovered from the simple case of a self-dual theta via opposite rescalings. The anti-self-dual theta can be obtained from this by a parity transformation of two coordinates.

3.2.2 Deformed ADHM equations

In order to generalize the ADHM equations $\mu_r = 0$ and $\mu_c = 0$ given in (3.14) to the case of a non-commutative space, Nekrasov & Schwarz [71] have deformed them in the following way:

$$\mu_r = \zeta \mathbb{1}, \qquad \mu_c = 0. \tag{3.32}$$

Here, ζ is just some deformation parameter, but we will see in a moment that it is exactly the non-commutativity parameter introduced in the previous section. One may think of deforming the second equation in (3.32) also, but this modification is equivalent to the one already considered by a linear transformation of the matrices in (3.14).

Now, suppose the data $B_{1,2}, B_{1,2}^\dagger, I, I^\dagger, J, J^\dagger$ obey the modified equations (3.32). Then, the factorization conditions (3.18) are no longer valid but they will be valid again if the coordinates $z_i, \bar{z}_i$ will not commute.

Let us study this in more detail. Inserting the data for an anti-self-dual instanton (3.17) into the factorization conditions (3.18) yields the following two equations

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = [\bar{z}_1, z_1]\mathbb{1} + [\bar{z}_2, z_2]\mathbb{1}, \tag{3.33}$$
$$[B_1, B_2] + IJ = [z_2, z_1]\mathbb{1}. \tag{3.34}$$

Since the matrices on the left hand side have to fulfill the deformed ADHM equations (see (3.32) and (3.14)) we get

$$\zeta = [\bar{z}_1, z_1] + [\bar{z}_2, z_2], \tag{3.35}$$
$$0 = [z_2, z_1]. \tag{3.36}$$

Therefore, we see that the factorization conditions for an anti-self-dual instanton hold in the case of $\mathbb{R}_\theta^2 \times \mathbb{R}^2$ (3.29) or $\mathbb{R}_\theta^4$ with self-dual theta (3.30). Doing the same calculation with the data for a self-dual instanton (3.16) leads to

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = [\bar{z}_1, z_1]\mathbb{1} - [\bar{z}_2, z_2]\mathbb{1}, \tag{3.37}$$
$$[B_1, B_2] + IJ = [z_1, \bar{z}_2]\mathbb{1}, \tag{3.38}$$

imposing

$$\zeta = [\bar{z}_1, z_1] - [\bar{z}_2, z_2], \tag{3.39}$$
$$0 = [z_1, \bar{z}_2]. \tag{3.40}$$

These conditions will be fulfilled on the spaces $\mathbb{R}_\theta^2 \times \mathbb{R}^2$ (3.29) and $\mathbb{R}_\theta^4$ with anti-self-dual theta (3.31).

To summarize, we have seen that we can construct non-commutative anti-self-dual and self-dual instantons on $\mathbb{R}^2_\theta \times \mathbb{R}^2$. On $\mathbb{R}^4_\theta$ with self-dual theta only the ADHM equations for anti-self-dual instantons are deformed, therefore we have non-commutative anti-self-dual instantons but usual commutative instantons in the self-dual sector. The analog holds in the case of $\mathbb{R}^4_\theta$ with anti-self-dual theta.

3.3 One ASD $U(1)$ instanton on $\mathbb{R}^2_\theta \times \mathbb{R}^2$

So far we have been completely general concerning the construction of instantons on non-commutative four-dimensional Euclidean space. Now, we will restrict ourselves to the most simple case of one instanton taking values in the gauge group $U(1)$ and living on a space with commutative time ($\mathbb{R}^2_\theta \times \mathbb{R}^2$). For definiteness we will consider anti-self-dual solutions but there is now obstruction in doing all the calculations in the self-dual case. Nevertheless, we will make some comments about instantons on $\mathbb{R}^4_\theta$ at appropriate stages of the calculations.

3.3.1 Operator formalism

To handle the non-commutativity of the coordinates $z_i, \bar{z}_i$ in an appropriate way we introduce the following operator formalism.

Regarding formula (3.29) we see that the only non-vanishing commutation relation on $\mathbb{R}^2_\theta \times \mathbb{R}^2$ is given by

$$[\bar{z}_1, z_1] = \zeta. \tag{3.41}$$

We easily realize the coordinates $\bar{z}_1$ and z_1 as an annihilation and a creation operator acting in a Fock space $\mathcal{H}$ for a simple harmonic oscillator[11] spanned by a basis $|n\rangle$ with $n \geq 0$:

$$\bar{z}_1 |n\rangle = \sqrt{\zeta n}\, |n-1\rangle\,, \qquad z_1 |n\rangle = \sqrt{\zeta(n+1)}\, |n+1\rangle\,. \tag{3.42}$$

On the other hand, the coordinates $\bar{z}_2$ and z_2 are still ordinary c-numbers (see (3.29)). Therefore, all fields on the space $\mathbb{R}^2_\theta \times \mathbb{R}^2$ will be described by operator-valued expressions on the non-commutative plane ($\bar{z}_1$,z_1) and ordinary functions on the commutative plane ($\bar{z}_2$,z_2).

Derivatives of a function $f(z_1, \bar{z}_1, z_2, \bar{z}_2)$ with respect to the non-commutative coordinates z_1 and $\bar{z}_1$ are defined by

$$\partial_1 f = \zeta^{-1} [\bar{z}_1, f]\,, \qquad \bar{\partial}_1 f = -\zeta^{-1} [z_1, f]\,. \tag{3.43}$$

[11]The space $\mathbb{R}^4_\theta = \mathbb{R}^2_\theta \times \mathbb{R}^2_\theta$ would require two oscillators.

Using the relations (3.29) it can be easily checked that (3.43) fulfill the standard requirements for $f = z_i$ or $f = \bar{z}_i$, as well as the chain rule. Further, we have the useful identities for the derivatives of the inverse function:

$$\partial_1 f^{-1} = -f^{-1}\,(\partial_1 f)\,f^{-1}, \qquad \bar{\partial}_1 f^{-1} = -f^{-1}\,(\bar{\partial}_1 f)\,f^{-1}, \tag{3.44}$$

where a simple proof of these equations can be found in appendix A.1.

Since we will make extensively use of the language of differential forms we introduce the differentials dz_i and $d\bar{z}_i$, $i \in (1,2)$ which anticommute with each other, but commute with z_j and $\bar{z}_j$. Next, we define the exterior derivative

$$d = dz_i\,\partial_i + d\bar{z}_i\,\bar{\partial}_i, \tag{3.45}$$

where ∂_1, $\bar{\partial}_1$ are given in (3.43), and $\partial_2 = \partial/\partial z_2$, $\bar{\partial}_2 = \partial/\partial \bar{z}_2$ as usual. Further, the operator d is nilpotent:

$$d^2 = 0. \tag{3.46}$$

For later convenience we define the following abbreviations:

$$\begin{aligned} \delta &= z_1\bar{z}_1 + z_2\bar{z}_2, \\ \Delta = \delta + \zeta, &\qquad \nabla = \delta - \zeta, \end{aligned} \tag{3.47}$$

and find the following very useful formulas in order to keep control over the operator ordering in the $(\bar{z}_1, z_1)$-plane:

$$\bar{z}_1 f(\delta) = f(\Delta)\,\bar{z}_1, \qquad z_1 f(\delta) = f(\nabla)\,z_1. \tag{3.48}$$

Nevertheless, the reader should notice that the Hermitian operators δ, Δ and ∇ commute under each other, a fact which will be used extensively throughout all calculations. The total exterior derivative of a function $f(\delta)$ is given by the formula:

$$\begin{aligned} d\,f(\delta) &= \zeta^{-1}\,(f(\Delta) - f(\delta))\,\bar{z}_1 dz_1 + \zeta^{-1}\,(f(\delta) - f(\nabla))\,z_1 d\bar{z}_1 \\ &+ \frac{\partial f(\delta)}{\partial \delta}\,(\bar{z}_2 dz_2 + z_2 d\bar{z}_2). \end{aligned} \tag{3.49}$$

A proof of (3.48) and (3.49) can be found in appendix A.1.

3.3.2 Solving the deformed ADHM equations

Using the preparation of the configuration space in the previous section we can solve the deformed ADHM equations (3.32) which will lead us to the wanted instanton solution. For further details, we refer the reader to the literature [74, 75, 76, 77, 78, 79], and references therein where this topic has been extensively discussed.

In order to build the instanton field (3.19) we have to find the zero-mode ψ of the Dirac operator $\mathcal{D}^\dagger_z$. Therefore, we write down equation (3.20) using (3.17) in its explicit matrix form:

$$\mathcal{D}^\dagger_z\psi = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 & I \\ -B_1^\dagger + \bar{z}_1 & B_2^\dagger - \bar{z}_2 & J^\dagger \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} = 0, \tag{3.50}$$

where $\psi = \psi_1 \oplus \psi_2 \oplus \xi$ as an element of $V \oplus V \oplus W$. In the case of one anti-self-dual $U(1)$ instanton, the ADHM data $B_{1,2}, B^\dagger_{1,2}, I, I^\dagger, J, J^\dagger$ above are just c-numbers.

We can make life easy by using translational invariance and set $B_1 = B_2 = 0$. Further, one can show that $J = 0$ in the case of a $U(1)$ gauge group [121]. From the deformed ADHM equations (3.32) we get $I = \sqrt{\zeta}$ and (3.50) reduces to

$$\mathcal{D}^\dagger_z\psi = \begin{pmatrix} -z_2 & -z_1 & \sqrt{\zeta} \\ \bar{z}_1 & -\bar{z}_2 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} = 0. \tag{3.51}$$

In order to see the above introduced operator formalism at work, we will perform the solution to (3.51) in full detail. Having the two equations

$$-z_2\psi_1 - z_1\psi_2 = -\zeta^{\frac{1}{2}}\xi, \tag{3.52}$$

$$\bar{z}_1\psi_1 - \bar{z}_2\psi_2 = 0, \tag{3.53}$$

we will multiply (3.52) with $\bar{z}_1$ from the left, and (3.53) with z_2. This yields

$$-\bar{z}_1 z_2\psi_1 - \bar{z}_1 z_1\psi_2 = -\bar{z}_1\zeta^{\frac{1}{2}}\xi, \tag{3.54}$$

$$z_2\bar{z}_1\psi_1 - z_2\bar{z}_2\psi_2 = 0. \tag{3.55}$$

Now, we make use of the commutation relations (3.29) and the fact that ζ is just a c-number:

$$-z_2\bar{z}_1\psi_1 - (z_1\bar{z}_1 + \zeta)\psi_2 = -\zeta^{\frac{1}{2}}\bar{z}_1\xi, \tag{3.56}$$

$$z_2\bar{z}_1\psi_1 - z_2\bar{z}_2\psi_2 = 0. \tag{3.57}$$

Adding (3.56) and (3.57) one gets

$$(z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta)\psi_2 = \Delta\psi_2 = \zeta^{\frac{1}{2}}\bar{z}_1\xi, \tag{3.58}$$

where we used (3.47). Multiplying by the operator Δ^{-1} from the left we get with (3.48) the equation:

$$\psi_2 = \zeta^{\frac{1}{2}}\Delta^{-1}\bar{z}_1\xi = \zeta^{\frac{1}{2}}\bar{z}_1\delta^{-1}\xi. \tag{3.59}$$

Putting (3.53) and (3.59) together we have

$$\bar{z}_1\psi_1 = \bar{z}_2\psi_2 = \zeta^{\frac{1}{2}}\bar{z}_1\bar{z}_2\delta^{-1}\xi, \tag{3.60}$$

and get also an equation for ψ_1:

$$\psi_1 = \zeta^{\frac{1}{2}}\bar{z}_2\delta^{-1}\xi. \tag{3.61}$$

The last thing we have to do is normalizing our solution with respect to equation (3.21):

$$\psi^\dagger\psi = \psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2 + \xi^\dagger\xi = 1. \tag{3.62}$$

Inserting (3.59) and (3.61) we get

$$\zeta\xi^\dagger\delta^{-1}z_2\bar{z}_2\delta^{-1}\xi + \zeta\xi^\dagger\delta^{-1}z_1\bar{z}_1\delta^{-1}\xi + \xi^\dagger\xi = 1. \tag{3.63}$$

Using (3.47), this reduces to

$$\zeta\xi^\dagger\delta^{-1}\xi + \xi^\dagger\xi = 1. \tag{3.64}$$

Multiplying (3.64) with $(\xi^\dagger)^{-1}$ from the left and with ξ^{-1} from the right leads to

$$(\zeta\delta^{-1} + 1) = (\xi\xi^\dagger)^{-1}. \tag{3.65}$$

Recognizing the identity

$$(\zeta\delta^{-1} + 1) = \delta^{-1}(\zeta + \delta) = \delta^{-1}\Delta, \tag{3.66}$$

we can rewrite (3.65) as

$$\delta\Delta^{-1} = \xi\xi^\dagger, \tag{3.67}$$

and end up with the following Hermitian expression for ξ:

$$\xi = \delta^{\frac{1}{2}}\Delta^{-\frac{1}{2}}. \tag{3.68}$$

Collecting all results (3.59), (3.61) and (3.68) we have the solution:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \quad \text{with} \quad \psi_1 = \bar{z}_2\sqrt{\frac{\zeta}{\delta\Delta}}, \quad \psi_2 = \bar{z}_1\sqrt{\frac{\zeta}{\delta\Delta}}, \quad \xi = \sqrt{\frac{\delta}{\Delta}}. \tag{3.69}$$

The reader should recognize at this stage that we are allowed to represent the solution (3.69) with the help of fractions, because we put only operators inside one fraction which commute under each other.

Before proceeding further in calculating the gauge field associated to ψ we should check if our solution (3.69) indeed fulfills the completeness relation (3.22). Taking the Dirac operator $\mathcal{D}_z^\dagger$ from equation (3.51) we see that

$$\mathcal{D}_z^\dagger \mathcal{D}_z = \Delta \mathbb{1}, \tag{3.70}$$

and therefore

$$\mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} \mathcal{D}_z^\dagger = \mathcal{D}_z \Delta^{-1} \mathcal{D}_z^\dagger. \tag{3.71}$$

With the expression (3.71) and the vector notation of (3.51) for ψ, we can write the completeness relation (3.22) as

$$\mathcal{D}_z \Delta^{-1} \mathcal{D}_z^\dagger + \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger & \xi^\dagger \end{pmatrix} = \mathbb{1}_{3\times 3}. \tag{3.72}$$

As an example we pick out the 11 matrix element of (3.72) which reads

$$\begin{aligned} \left(\mathcal{D}_z \Delta^{-1} \mathcal{D}_z^\dagger\right)_{11} + \psi_1 \psi_1^\dagger &= \bar{z}_2 \Delta^{-1} z_2 + z_1 \Delta^{-1} \bar{z}_1 + \zeta \bar{z}_2 \delta^{-1} \Delta^{-1} z_2 \\ &= (\zeta \delta^{-1} + 1) \Delta^{-1} z_2 \bar{z}_2 + \delta^{-1} z_1 \bar{z}_1 \\ &= (\zeta + \delta) \delta^{-1} \Delta^{-1} z_2 \bar{z}_2 + \delta^{-1} z_1 \bar{z}_1 \\ &= \delta^{-1} (z_2 \bar{z}_2 + z_1 \bar{z}_1) \\ &= 1, \end{aligned} \tag{3.73}$$

where we used the explicit expressions of $\mathcal{D}_z^\dagger$ (3.51) and ψ (3.69). The reader can convince himself that all the other matrix elements fulfill the relation (3.72) and ψ is indeed a solution of the ADHM equations.

At the end of this section we should make a comment about the solution of (3.50) on $\mathbb{R}^4_\theta$ with self-dual theta. Having the commutation relations (3.30) in mind one can repeat the calculations (3.50-3.69) and get exactly the same result (3.69) as in the case of $\mathbb{R}^2_\theta \times \mathbb{R}^2$ before. But this naive approach will fail here, because we have to write the normalization condition (3.21) more accurately as

$$\langle 0 \left| \psi^\dagger \psi \right| 0 \rangle = 1, \tag{3.74}$$

since ψ is an operator on a Fock space $\mathcal{H}$ spanned by harmonic oszillator states (see section 3.3.1). Recognizing $\bar{z}_1$ and $\bar{z}_2$ as annihilation operators on

$\mathbb{R}^4_\theta$ we see that our naively obtained solution ψ (3.69) annihilates the vacuum and is therefore not properly normalized with respect to (3.74). To cure this problem one has to introduce a projection operator $p = \mathbb{1} - |0\rangle\langle 0|$ which projects out the vacuum state [74] and perform all further calculations with respect to the projected Fock space $p\mathcal{H}$.

In the previous case of $\mathbb{R}^2_\theta \times \mathbb{R}^2$ it is not necessary to project out the vacuum state, since $\bar{z}_1$ is the only annihilation operator in the game and ψ does not annihilate the vacuum and therefore being properly normalized. Further, it was shown in [77] that it is even not possible to project out the vacuum, because then the completeness relation (3.22) will not be fulfilled anymore. Therefore, in the case of $\mathbb{R}^2_\theta \times \mathbb{R}^2$ we are dealing with the full Fock space $\mathcal{H}$ which makes life more easy.

3.3.3 Instanton gauge field

Having the zero-mode ψ (3.69) in hand we can use equation (3.19) given as a differential form

$$A = \psi^\dagger d\psi, \tag{3.75}$$

to calculate an explicit expression for the anti-self-dual non-commutative $U(1)$ instanton. Expanding equation (3.75) into

$$A = \psi_1^\dagger d\psi_1 + \psi_2^\dagger d\psi_2 + \xi d\xi, \tag{3.76}$$

we can divide the following calculation into three parts.

Here, we will give the detailed calculation of $\psi_2^\dagger d\psi_2$ and leave the other two terms as an exercise to the reader.[12] Using the solution (3.69) and the differentiation rule (3.49) we get

$$\begin{aligned}
d\psi_2 &= \zeta^{\frac{1}{2}}\, d\Big(\bar{z}_1\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}\Big)\\
&= \zeta^{\frac{1}{2}}\left(d\bar{z}_1\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} + \bar{z}_1 d\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} + \bar{z}_1\delta^{-\frac{1}{2}}d\Delta^{-\frac{1}{2}}\right)\\
&= \zeta^{\frac{1}{2}}\Big[\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}d\bar{z}_1 + \bar{z}_1\Big(\zeta^{-1}(\Delta^{-\frac{1}{2}} - \delta^{-\frac{1}{2}})\bar{z}_1 dz_1 + \zeta^{-1}(\delta^{-\frac{1}{2}} - \nabla^{-\frac{1}{2}})z_1 d\bar{z}_1\\
&\qquad\qquad - \frac{1}{2}\delta^{-\frac{3}{2}}(\bar{z}_2 dz_2 + z_2 d\bar{z}_2)\Big)\Delta^{-\frac{1}{2}}\\
&\qquad + \bar{z}_1\delta^{-\frac{1}{2}}\Big(\zeta^{-1}((\Delta+\zeta)^{-\frac{1}{2}} - \Delta^{-\frac{1}{2}})\bar{z}_1 dz_1 + \zeta^{-1}(\Delta^{-\frac{1}{2}} - \delta^{-\frac{1}{2}})z_1 d\bar{z}_1\\
&\qquad\qquad - \frac{1}{2}\Delta^{-\frac{3}{2}}(\bar{z}_2 dz_2 + z_2 d\bar{z}_2)\Big)\Big].
\end{aligned}\tag{3.77}$$

[12] We choose $\psi_2^\dagger d\psi_2$ because ψ_2 depends explicitly on the non-commutative coordinate $\bar{z}_1$, building therefore the most difficult term to calculate in (3.76).

Moving the coordinate $\bar{z}_1$ with the rule (3.48) to the right side yields

$$\begin{aligned}
d\psi_2 &= \zeta^{\frac{1}{2}}\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}d\bar{z}_1 \\
&+ \zeta^{-\frac{1}{2}}((\Delta+\zeta)^{-\frac{1}{2}} - \Delta^{-\frac{1}{2}})(\Delta+2\zeta)^{-\frac{1}{2}}\bar{z}_1\bar{z}_1 dz_1 \\
&+ \zeta^{-\frac{1}{2}}(\Delta^{-\frac{1}{2}} - \delta^{-\frac{1}{2}})\Delta^{-\frac{1}{2}}\bar{z}_1 z_1 d\bar{z}_1 \\
&- \frac{1}{2}\zeta^{\frac{1}{2}}\Delta^{-\frac{3}{2}}(\Delta+\zeta)^{-\frac{1}{2}}\bar{z}_1(\bar{z}_2 dz_2 + z_2 d\bar{z}_2) \\
&+ \zeta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}((\Delta+2\zeta)^{-\frac{1}{2}} - (\Delta+\zeta)^{-\frac{1}{2}})\bar{z}_1\bar{z}_1 dz_1 \\
&+ \zeta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}((\Delta+\zeta)^{-\frac{1}{2}} - \Delta^{-\frac{1}{2}})\bar{z}_1 z_1 d\bar{z}_1 \\
&- \frac{1}{2}\zeta^{\frac{1}{2}}\Delta^{-\frac{1}{2}}(\Delta+\zeta)^{-\frac{3}{2}}\bar{z}_1(\bar{z}_2 dz_2 + z_2 d\bar{z}_2),
\end{aligned} \tag{3.78}$$

and reordering the resulting expression leads to

$$\begin{aligned}
d\psi_2 &= \zeta^{\frac{1}{2}}\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}d\bar{z}_1 \\
&+ \zeta^{-\frac{1}{2}}(\Delta+\zeta)^{-\frac{1}{2}}\left((\Delta+2\zeta)^{-\frac{1}{2}} - \Delta^{-\frac{1}{2}}\right)\bar{z}_1\bar{z}_1 dz_1 \\
&+ \zeta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}\left((\Delta+\zeta)^{-\frac{1}{2}} - \delta^{-\frac{1}{2}}\right)\bar{z}_1 z_1 d\bar{z}_1 \\
&- \frac{1}{2}\zeta^{\frac{1}{2}}\left(\Delta^{-\frac{3}{2}}(\Delta+\zeta)^{-\frac{1}{2}} + \Delta^{-\frac{1}{2}}(\Delta+\zeta)^{-\frac{3}{2}}\right)\bar{z}_1(\bar{z}_2 dz_2 + z_2 d\bar{z}_2).
\end{aligned} \tag{3.79}$$

Multiplying the Hermitian conjugate of ψ_2 given by

$$\psi_2^\dagger = \left(\zeta^{\frac{1}{2}}\bar{z}_1\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}}\right)^\dagger = \zeta^{\frac{1}{2}}\Delta^{-\frac{1}{2}}\delta^{-\frac{1}{2}}z_1, \tag{3.80}$$

with the expression for $d\psi_2$ in (3.79) yields

$$\begin{aligned}
\psi_2^\dagger d\psi_2 &= \zeta\Delta^{-\frac{1}{2}}\delta^{-1}\nabla^{-\frac{1}{2}}z_1 d\bar{z}_1 \\
&+ \left((\Delta+\zeta)^{-\frac{1}{2}}\Delta^{-1}\delta^{-\frac{1}{2}} - \Delta^{-1}\delta^{-1}\right)z_1\bar{z}_1\bar{z}_1 dz_1 \\
&+ \left(\Delta^{-1}\delta^{-1} - \Delta^{-\frac{1}{2}}\delta^{-1}\nabla^{-\frac{1}{2}}\right)z_1\bar{z}_1 z_1 d\bar{z}_1 \\
&- \frac{1}{2}\zeta\left(\Delta^{-1}\delta^{-2} + \Delta^{-2}\delta^{-1}\right)z_1\bar{z}_1(\bar{z}_2 dz_2 + z_2 d\bar{z}_2),
\end{aligned} \tag{3.81}$$

where we used again (3.48) for moving z_1 to the right side.

Analogous calculations lead to the following expressions for the other two terms in the sum for the instanton field (3.76):

$$\begin{aligned}\psi_1^\dagger d\psi_1 &= \zeta\Delta^{-1}\delta^{-1}z_2 d\bar{z}_2\\ &+\Big((\Delta+\zeta)^{-\frac{1}{2}}\Delta^{-1}\delta^{-\frac{1}{2}}-\Delta^{-1}\delta^{-1}\Big)z_2\bar{z}_2\bar{z}_1 dz_1\\ &+\Big(\Delta^{-1}\delta^{-1}-\Delta^{-\frac{1}{2}}\delta^{-1}\nabla^{-\frac{1}{2}}\Big)z_2\bar{z}_2 z_1 d\bar{z}_1\\ &-\frac{1}{2}\zeta\Big(\Delta^{-1}\delta^{-2}+\Delta^{-2}\delta^{-1}\Big)z_2\bar{z}_2(\bar{z}_2 dz_2+z_2 d\bar{z}_2),\end{aligned}\tag{3.82}$$

and

$$\begin{aligned}\xi^\dagger d\xi &= \zeta^{-1}\Big((\Delta+\zeta)^{-\frac{1}{2}}\delta^{\frac{1}{2}}-\Delta^{-1}\delta\Big)\bar{z}_1 dz_1\\ &+\zeta^{-1}\Big(\Delta^{-1}\delta-\Delta^{-\frac{1}{2}}\nabla^{\frac{1}{2}}\Big)z_1 d\bar{z}_1\\ &+\frac{1}{2}\Big(\Delta^{-1}-\Delta^{-2}\delta\Big)(\bar{z}_2 dz_2+z_2 d\bar{z}_2).\end{aligned}\tag{3.83}$$

Summing up the terms (3.81), (3.82) and (3.83) leads to

$$\begin{aligned}A &= \psi_1^\dagger d\psi_1+\psi_2^\dagger d\psi_2+\xi d\xi\\ &= \Big[(\Delta+\zeta)^{-\frac{1}{2}}\Delta^{-1}\delta^{\frac{1}{2}}-\Delta^{-1}+\zeta^{-1}(\Delta+\zeta)^{-\frac{1}{2}}\delta^{\frac{1}{2}}-\zeta^{-1}\Delta^{-1}\delta\Big]\bar{z}_1 dz_1\\ &+\Big[\Delta^{-1}-\Delta^{-\frac{1}{2}}\nabla^{-\frac{1}{2}}+\zeta^{-1}\Delta^{-1}\delta-\zeta^{-1}\Delta^{-\frac{1}{2}}\nabla^{\frac{1}{2}}\Big]z_1 d\bar{z}_1\\ &+\frac{1}{2}\Big[-\zeta\Delta^{-1}\delta^{-1}-\zeta\Delta^{-2}+\Delta^{-1}-\Delta^{-2}\delta\Big](\bar{z}_2 dz_2+z_2 d\bar{z}_2)\\ &+\Big[\zeta\Delta^{-\frac{1}{2}}\delta^{-1}\nabla^{-\frac{1}{2}}\Big]z_1 d\bar{z}_1+\Big[\zeta\Delta^{-1}\delta^{-1}\Big]z_2 d\bar{z}_2,\end{aligned}\tag{3.84}$$

where we used $\delta = z_1\bar{z}_1 + z_2\bar{z}_2$ several times. Since all operators in square brackets commute under each other, we can simplify them by writing each square bracket as a fraction with one common denominator. Here, we perform the explicit calculation for the expression proportional to $z_1 d\bar{z}_1$ given by the second and fourth square bracket, and leave the rest to the reader.[13]

$$\begin{aligned}
&\Delta^{-1} - \Delta^{-\frac{1}{2}}\nabla^{-\frac{1}{2}} + \zeta^{-1}\Delta^{-1}\delta - \zeta^{-1}\Delta^{-\frac{1}{2}}\nabla^{\frac{1}{2}} + \zeta\Delta^{-\frac{1}{2}}\delta^{-1}\nabla^{-\frac{1}{2}} \\
&\quad = \frac{1}{\Delta} - \frac{1}{\sqrt{\Delta\nabla}} + \frac{\delta}{\zeta\Delta} - \frac{\sqrt{\nabla}}{\zeta\sqrt{\Delta}} + \frac{\zeta}{\delta\sqrt{\Delta\nabla}} \\
&\quad = \frac{\zeta\delta\sqrt{\nabla} - \zeta\delta\sqrt{\Delta} + \delta^2\sqrt{\nabla} - \delta\nabla\sqrt{\Delta} + \zeta^2\sqrt{\Delta}}{\zeta\Delta\delta\sqrt{\nabla}} \\
&\quad = \frac{\Delta\delta\sqrt{\nabla} + (\zeta^2 - \delta^2)\sqrt{\Delta}}{\zeta\Delta\delta\sqrt{\nabla}}. \qquad (3.85)
\end{aligned}$$

Using the identity $\zeta^2 - \delta^2 = (\zeta + \delta)(\zeta - \delta) = -\Delta\nabla$ we can write (3.85) as

$$\frac{\Delta\delta\sqrt{\nabla} - \Delta\nabla\sqrt{\Delta}}{\zeta\Delta\delta\sqrt{\nabla}} = \frac{1}{\zeta}\left[1 - \frac{\sqrt{\Delta\nabla}}{\delta}\right]. \qquad (3.86)$$

Eventually, doing analogous transformations of the other terms we end up with the following explicit expression for the anti-self-dual non-commutative $U(1)$ instanton field:

$$\begin{aligned}
A = {} & \frac{1}{\zeta}\left[\frac{\sqrt{(\Delta + \zeta)\delta}}{\Delta} - 1\right]\bar{z}_1 dz_1 + \frac{1}{\zeta}\left[1 - \frac{\sqrt{\Delta\nabla}}{\delta}\right] z_1 d\bar{z}_1 \\
& - \left[\frac{\zeta}{2\Delta\delta}\right](\bar{z}_2 dz_2 - z_2 d\bar{z}_2). \qquad (3.87)
\end{aligned}$$

Before converting this expression back into real coordinates (x_1, x_2, x_3, x_4) and discussing its physical properties we will go on and calculate the corresponding field strength in the next section.

3.3.4 Field strength of the instanton

First of all, let us introduce the following notations for the gauge field

$$\begin{aligned}
A &= A_1 dx_1 + A_2 dx_2 + A_3 dx_3 + A_4 dx_4 \\
&= A_{z_1} dz_1 + A_{\bar{z}_1} d\bar{z}_1 + A_{z_2} dz_2 + A_{\bar{z}_2} d\bar{z}_2, \qquad (3.88)
\end{aligned}$$

[13]In order to be consistent, we again choose the most difficult term for presenting the detailed calculational steps.

and the field strength

$$\begin{aligned} F &= F_{12}dx_1dx_2 + F_{13}dx_1dx_3 + F_{14}dx_1dx_4 \\ &\quad + F_{23}dx_2dx_3 + F_{24}dx_2dx_4 + F_{34}dx_3dx_4 \\ &= F_{z_1z_2}dz_1dz_2 + F_{\bar{z}_1\bar{z}_2}d\bar{z}_1d\bar{z}_2 + F_{z_1\bar{z}_1}dz_1d\bar{z}_1 \\ &\quad + F_{z_2\bar{z}_2}dz_2d\bar{z}_2 + F_{z_1\bar{z}_2}dz_1d\bar{z}_2 + F_{\bar{z}_1z_2}d\bar{z}_1dz_2, \end{aligned} \tag{3.89}$$

where the transformation between the coordinates x and z is given in equation (3.27). With these notations in hand we can rewrite the (anti-)self-duality relation from (3.9) and (3.11) on x-space

$$F_{\mu\nu} = \pm\frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,F_{\rho\sigma}, \qquad (+\ldots\mathrm{SD}), \qquad (-\ldots\mathrm{ASD}), \tag{3.90}$$

to the equivalent conditions

$$\mathrm{SD:} \quad F_{12} - F_{34} = 0, \quad F_{13} + F_{24} = 0, \quad F_{14} - F_{23} = 0, \tag{3.91}$$

$$\mathrm{ASD:} \quad F_{12} + F_{34} = 0, \quad F_{13} - F_{24} = 0, \quad F_{14} + F_{23} = 0. \tag{3.92}$$

Transforming (3.91) and (3.92) with the help of (3.27) into the z-space yields respectively

$$\mathrm{SD:} \quad F_{z_1\bar{z}_1} - F_{z_2\bar{z}_2} = 0, \quad F_{z_1\bar{z}_2} = F_{\bar{z}_1z_2} = 0, \tag{3.93}$$

$$\mathrm{ASD:} \quad F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1z_2} = F_{\bar{z}_1\bar{z}_2} = 0. \tag{3.94}$$

Now, we will proof that a vector ψ fulfilling the relations (3.20), (3.21) and (3.22) will indeed lead to an (anti-)self-dual field strength. Using (3.75) the field strength is given by[14]

$$\begin{aligned} F &= dA + A \wedge A \\ &= d(\psi^\dagger d\psi) + (\psi^\dagger d\psi)(\psi^\dagger d\psi) \\ &= d\psi^\dagger(1 - \psi\psi^\dagger)d\psi, \end{aligned} \tag{3.95}$$

because $\psi^\dagger d\psi = -d\psi^\dagger\psi$ due to the normalization condition (3.21). With the help of the completeness relation (3.22) we can rewrite (3.95) and obtain

$$F = d\psi^\dagger \mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} \mathcal{D}_z^\dagger d\psi. \tag{3.96}$$

Taking the Dirac operator $\mathcal{D}_z^\dagger$ (3.15) and using the factorization conditions (3.18) we get

$$\mathcal{D}_z^\dagger \mathcal{D}_z = \Box_z \mathbb{1}, \tag{3.97}$$

[14]From time to time we omit the wedge symbol in order to improve readability.

where we introduced the notation $\tau_z\tau_z^\dagger = \Box_z$. Therefore, we have

$$\mathcal{D}_z \frac{1}{\mathcal{D}_z^\dagger \mathcal{D}_z} \mathcal{D}_z^\dagger = \mathcal{D}_z \Box_z^{-1} \mathcal{D}_z^\dagger = \tau_z^\dagger \Box_z^{-1} \tau_z + \sigma_z \Box_z^{-1} \sigma_z^\dagger, \tag{3.98}$$

and

$$F = d\psi^\dagger \left(\tau_z^\dagger \Box_z^{-1} \tau_z + \sigma_z \Box_z^{-1} \sigma_z^\dagger\right) d\psi. \tag{3.99}$$

Since ψ is a zero-mode of the Dirac operator $\mathcal{D}_z^\dagger$ fulfilling (3.20) which is equivalent to

$$\tau_z \psi = 0, \qquad \sigma_z^\dagger \psi = 0, \tag{3.100}$$

we have the relations

$$\tau_z d\psi = -d\tau_z \psi, \qquad \sigma_z^\dagger d\psi = -d\sigma_z^\dagger \psi. \tag{3.101}$$

Inserting (3.101) into (3.99) yields the following expression for the field strength

$$F = \psi^\dagger \left(d\tau_z^\dagger \Box_z^{-1} d\tau_z + d\sigma_z \Box_z^{-1} d\sigma_z^\dagger\right) \psi. \tag{3.102}$$

The components τ_z and $\sigma_z^\dagger$ of the Dirac operator $\mathcal{D}_z^\dagger$ are given by the formulas (3.16) and (3.17) in the SD and ASD case, respectively. Therefore, we have

$$\text{SD:} \quad d\tau_z = \begin{pmatrix} -d\bar{z}_2 & dz_1 & 0 \end{pmatrix}, \qquad d\sigma_z^\dagger = \begin{pmatrix} -d\bar{z}_1 & -dz_2 & 0 \end{pmatrix}, \tag{3.103}$$

$$\text{ASD:} \quad d\tau_z = \begin{pmatrix} -dz_2 & -dz_1 & 0 \end{pmatrix}, \qquad d\sigma_z^\dagger = \begin{pmatrix} d\bar{z}_1 & -d\bar{z}_2 & 0 \end{pmatrix}. \tag{3.104}$$

Insertion of (3.103) and (3.104) into (3.102) leads to

$$F_{\text{SD}} = \psi^\dagger \begin{pmatrix} dz_2 \Box_z^{-1} d\bar{z}_2 + dz_1 \Box_z^{-1} d\bar{z}_1 & -dz_2 \Box_z^{-1} dz_1 + dz_1 \Box_z^{-1} dz_2 & 0 \\ -d\bar{z}_1 \Box_z^{-1} d\bar{z}_2 + d\bar{z}_2 \Box_z^{-1} d\bar{z}_1 & d\bar{z}_1 \Box_z^{-1} dz_1 + d\bar{z}_2 \Box_z^{-1} dz_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi, \tag{3.105}$$

$$F_{\text{ASD}} = \psi^\dagger \begin{pmatrix} d\bar{z}_2 \Box_z^{-1} dz_2 + dz_1 \Box_z^{-1} d\bar{z}_1 & d\bar{z}_2 \Box_z^{-1} dz_1 - dz_1 \Box_z^{-1} d\bar{z}_2 & 0 \\ d\bar{z}_1 \Box_z^{-1} dz_2 - dz_2 \Box_z^{-1} d\bar{z}_1 & d\bar{z}_1 \Box_z^{-1} dz_1 + dz_2 \Box_z^{-1} d\bar{z}_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi. \tag{3.106}$$

Now, we can read off from the expressions (3.105) and (3.106) the relevant components in order to check their self-duality and anti-self-duality, respectively. Using the vector notation $\psi = \psi_1 \oplus \psi_2 \oplus \xi$ we get

$$\begin{aligned} (F_{\mathrm{SD}})_{z_1\bar{z}_1} &= (F_{\mathrm{SD}})_{z_2\bar{z}_2} = \psi_1^\dagger \Box_z^{-1}\psi_1 - \psi_2^\dagger \Box_z^{-1}\psi_2, \\ (F_{\mathrm{SD}})_{z_1\bar{z}_2} &= (F_{\mathrm{SD}})_{\bar{z}_1 z_2} = 0, \end{aligned} \tag{3.107}$$

$$\begin{aligned} (F_{\mathrm{ASD}})_{z_1\bar{z}_1} &= -(F_{\mathrm{ASD}})_{z_2\bar{z}_2} = \psi_1^\dagger \Box_z^{-1}\psi_1 - \psi_2^\dagger \Box_z^{-1}\psi_2, \\ (F_{\mathrm{ASD}})_{z_1 z_2} &= (F_{\mathrm{ASD}})_{\bar{z}_1\bar{z}_2} = 0. \end{aligned} \tag{3.108}$$

One can check immediately that (3.107) fulfills the self-duality conditions (3.93) and the same holds for (3.108) with (3.94). This closes the proof that a vector ψ constructed via the ADHM sheme (see section 3.1.2) indeed yields an (anti-)self-dual gauge field.

So far we have been completely generic about the form of the field strength of an instanton. Now, we proceed in calculating the explicit expression of the field strength of the anti-self-dual non-commutative $U(1)$ instanton constructed in section 3.3.3.

For this purpose we can go two different ways. We can either use the generic constructed formula (3.106) or calculate the field strength directly via its defining equation $F = dA + A \wedge A$ by using the explicit form of the instanton field A (3.87). Since this way is much longer we choose the first one, but it would be a nice exercise for the reader to do this brute force calculation by making extensively use of the derivative formula (3.49).

The components of the anti-self-dual field strength are given by (3.106) and read[15]

$$F_{z_1\bar{z}_1} = \psi_1^\dagger \Delta^{-1}\psi_1 - \psi_2^\dagger \Delta^{-1}\psi_2, \tag{3.109}$$

$$F_{z_2\bar{z}_2} = -\psi_1^\dagger \Delta^{-1}\psi_1 + \psi_2^\dagger \Delta^{-1}\psi_2, \tag{3.110}$$

$$F_{\bar{z}_2 z_1} = 2\psi_1^\dagger \Delta^{-1}\psi_2, \tag{3.111}$$

$$F_{\bar{z}_1 z_2} = 2\psi_2^\dagger \Delta^{-1}\psi_1, \tag{3.112}$$

where we used the fact that

$$\mathcal{D}_z^\dagger \mathcal{D}_z = \Box_z \mathbb{1} = \Delta \mathbb{1}, \tag{3.113}$$

which stems from comparison of (3.70) and (3.97). Using now the explicit expression for ψ (3.69) we can write (3.109) as

$$\begin{aligned} F_{z_1\bar{z}_1} &= \zeta\Delta^{-\frac{1}{2}}\delta^{-\frac{1}{2}} z_2 \Delta^{-1}\bar{z}_2 \delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} - \zeta\Delta^{-\frac{1}{2}}\delta^{-\frac{1}{2}} z_1 \Delta^{-1}\bar{z}_1 \delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} \\ &= \zeta\Delta^{-2}\delta^{-1} z_2\bar{z}_2 - \zeta\Delta^{-1}\delta^{-2} z_1\bar{z}_1. \end{aligned} \tag{3.114}$$

[15] We skip the substring ASD, because from now on we are only considering anti-self-dual quantities.

For (3.111) and (3.112) we get

$$\begin{aligned} F_{\bar{z}_2 z_1} &= 2\zeta\Delta^{-\frac{1}{2}}\delta^{-\frac{1}{2}}z_2\Delta^{-1}\bar{z}_1\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} \\ &= 2\zeta\Delta^{-2}(\Delta+\zeta)^{-\frac{1}{2}}\delta^{-\frac{1}{2}}z_2\bar{z}_1, \qquad (3.115) \\ F_{\bar{z}_1 z_2} &= 2\zeta\Delta^{-\frac{1}{2}}\delta^{-\frac{1}{2}}z_1\Delta^{-1}\bar{z}_2\delta^{-\frac{1}{2}}\Delta^{-\frac{1}{2}} \\ &= 2\zeta\delta^{-2}\Delta^{-\frac{1}{2}}\nabla^{-\frac{1}{2}}z_1\bar{z}_2. \qquad (3.116) \end{aligned}$$

Collecting all pieces together we find the following explicit expression for the anti-self-dual field strength corresponding to a non-commutative $U(1)$ instanton field:

$$\begin{aligned} F &= \frac{\zeta}{\Delta^2\delta^2}\left(\Delta z_1\bar{z}_1 - \delta z_2\bar{z}_2\right)\left(dz_2 d\bar{z}_2 - dz_1 d\bar{z}_1\right) \\ &+ \frac{2\zeta}{\Delta^2\sqrt{(\Delta+\zeta)\delta}} z_2\bar{z}_1 d\bar{z}_2 dz_1 + \frac{2\zeta}{\delta^2\sqrt{\Delta\nabla}} z_1\bar{z}_2 d\bar{z}_1 dz_2. \end{aligned} \qquad (3.117)$$

3.3.5 Topological charge

At the beginning of studying the properties of non-commutative instantons there has been some rumour in the literature that non-commutative instantons might have non-integer topological charge, whereas in the commutative sector the topological charge is always an integer at least on spaces with infinite volume.[16] Then, it has been shown in [77] due to a non-commutative generalization of Corrigan's identity [117] that every non-commutative instanton constructed in the ADHM regime has an integer topological charge. This result is independent of the rank of theta and applies equally to the cases of space-time ($\mathbb{R}^4_\theta$) and space-space non-commutativity ($\mathbb{R}^2_\theta \times \mathbb{R}^2$).

Here, we will evaluate the topological charge Q of the anti-self-dual non-commutative $U(1)$ instanton directly via formula (3.8) written in the language of differential forms[17]

$$Q = -\frac{1}{8\pi^2}\int F \wedge F, \qquad (3.118)$$

without making use of Corrigan's identity. Insertion of the field strength (3.117) into (3.118) yields after a tedious but straightforward calculation[18]

[16] On spaces with finite volume there exist so-called *fractional instantons* having non-integer topological charge.

[17] There is a relative factor of 2 coming from the trace and the wedge-product [124].

[18] After the explicit calculations of the previous sections the reader should be familiar now with the operator formalism introduced in section 3.3.1.

the following expression for the topological charge:

$$Q = \frac{\zeta^2}{4\pi^2} \int d^4z \left[\frac{(\Delta z_1\bar{z}_1 - \delta z_2\bar{z}_2)^2}{\Delta^4\delta^4} + \frac{2(z_1\bar{z}_1 + \zeta)z_2\bar{z}_2}{(\Delta+\zeta)\Delta^4\delta} + \frac{2z_1\bar{z}_1z_2\bar{z}_2}{\Delta\delta^4\nabla} \right], \tag{3.119}$$

with the integration measure

$$d^4z = dz_1d\bar{z}_1dz_2d\bar{z}_2 = -4\,dx_1dx_2dx_3dx_4 = -4\,d^4x. \tag{3.120}$$

Now, the question rises what does it mean to integrate over the operator-valued coordinates $z_1, \bar{z}_1$ (or equivalently x_1, x_2). The answer is a mapping of the integration over the non-commutative space to the trace in the Fock space $\mathcal{H}$ which we introduced in section 3.3.1:

$$\int d^{2d}x \quad \longrightarrow \quad (2\pi)^d\sqrt{\det\theta}\,\mathrm{Tr}_{\mathcal{H}}, \tag{3.121}$$

where $D = 2d$ is the dimension of the non-commutative space. In the Fock space representation we have d harmonic oszillators and can write the operator trace as

$$\mathrm{Tr}_{\mathcal{H}}\,\mathcal{O} = \sum_{n_1,n_2,\ldots,n_d=0}^{\infty} \langle n_1,n_2,\ldots,n_d|\,\mathcal{O}\,|n_1,n_2,\ldots,n_d\rangle\,. \tag{3.122}$$

Spezializing the formulas (3.121) and (3.122) to our case $\mathbb{R}^2_\theta \times \mathbb{R}^2$ we get

$$\int d^4z = -4\int d^4x \quad \longrightarrow \quad -4\pi\zeta\int d^2y\,\mathrm{Tr}_{\mathcal{H}}, \tag{3.123}$$

where we use $\sqrt{\det\theta} = \zeta/2$ (see section 3.2.1) and define $d^2y = dz_2d\bar{z}_2$ which can be integrated over in the usual sense.[19] With the simple harmonic oszillator representation (3.42) we end up with

$$\int d^4z\,\mathcal{O} \quad \longrightarrow \quad -4\pi\zeta\sum_{n=0}^{\infty}\int d^2y\ \langle n|\,\mathcal{O}\,|n\rangle\,. \tag{3.124}$$

Next, let us give an interpretation of the operators $z_1\bar{z}_1$ and $z_2\bar{z}_2$ which appear explicitly in the expression (3.119) for the topological charge. Using the operator relations (3.42) we get for the action of the operator $z_1\bar{z}_1$ on the state $|n\rangle$:

$$z_1\bar{z}_1\,|n\rangle = z_1\sqrt{\zeta n}\,|n-1\rangle = \zeta n\,|n\rangle\,. \tag{3.125}$$

[19]Remember that z_2 and $\bar{z}_2$ are commutative coordinates.

Therefore, we recognize $z_1\bar{z}_1$ being proportional to the number operator. Transforming $z_2\bar{z}_2$ via (3.27) back to the x-space:

$$z_2\bar{z}_2 = x_3^2 + x_4^2 \equiv \zeta r_y^2, \tag{3.126}$$

we see that it just gives the distance from the origin in the commutative plane $(\bar{z}_2, z_2)$ which we denote by the c-number r_y (given in units of the non-commutativity scale $\sqrt{\zeta}$). Since we have chosen the states $|n\rangle$ to span an orthonormal basis

$$\langle n|m\rangle = \delta_{nm}, \tag{3.127}$$

we have with (3.125) and (3.126) the following relations:

$$\langle n|\, z_1\bar{z}_1\, |n\rangle = \zeta n, \qquad \langle n|\, z_2\bar{z}_2\, |n\rangle = \zeta r_y^2. \tag{3.128}$$

Using the definitions (3.47) we get further

$$\begin{aligned} \langle n|\, \delta\, |n\rangle &= \zeta(n + r_y^2), \\ \langle n|\, \Delta\, |n\rangle &= \zeta(n + 1 + r_y^2), \\ \langle n|\, \nabla\, |n\rangle &= \zeta(n - 1 + r_y^2). \end{aligned} \tag{3.129}$$

With the mapping (3.124) and the identities (3.128) and (3.129) we can write the topological charge (3.119) as

$$\begin{aligned} Q = -\frac{1}{\pi\zeta}\sum_{n=0}^{\infty}\int d^2y \Bigg[&\frac{\left((n+1+r_y^2)n - (n+r_y^2)r_y^2\right)^2}{(n+1+r_y^2)^4(n+r_y^2)^4} \\ &+ \frac{2(n+1)r_y^2}{(n+2+r_y^2)(n+1+r_y^2)^4(n+r_y^2)} \\ &+ \frac{2nr_y^2}{(n+1+r_y^2)(n+r_y^2)^4(n-1+r_y^2)}\Bigg]. \end{aligned} \tag{3.130}$$

Since the integrand in (3.130) depends only on the radius r_y, we can integrate trivially over the angular variable in the plane $(\bar{z}_2, z_2)$ and replace the integration measure by

$$\int d^2y = 2\pi\zeta \int_0^\infty dr_y\, r_y. \tag{3.131}$$

Performing a shift $n \to n+1$ in the third term of (3.130) and using (3.131) we get

$$Q = -2\sum_{n=0}^{\infty}\int_0^{\infty} dr_y\, r_y \left[\frac{\left((n+1+r_y^2)n-(n+r_y^2)r_y^2\right)^2}{(n+1+r_y^2)^4(n+r_y^2)^4} \right.$$
$$\left. + \frac{4(n+1)r_y^2}{(n+2+r_y^2)(n+1+r_y^2)^4(n+r_y^2)}\right]. \quad (3.132)$$

From expression (3.132) we see that the topological charge Q does not depend on the non-commutativity scale ζ. This has to be the case, since the topological charge is just a number.

In order to improve our understanding of the various contributions to the instanton charge we define its density $Q_n(r_y)$:

$$Q = -\sum_{n=0}^{\infty}\int_0^{\infty} dr_y\, Q_n(r_y), \quad (3.133)$$

where the quantity $Q_n(r_y)$ can be read off from expression (3.132). Figure 3.1 shows the density of the instanton charge where the thick line denotes the contribution from $n=0$, the dashed line corresponds to $n=1$, and the dotted line to $n=2$. Since $Q_n(r_y)$ behaves like $1/n^4$ for large n, we omit the drawing of higher terms. Further, we see that the relevant contributions come from the region $0 \leq r_y \lesssim 1$. This is a typical feature of non-commutative instantons. They have a *finite size* given by the non-commutativity scale $\sqrt{\zeta}$ (the unit of r_y).

Now, we finish the calculation of the topological charge in performing the summation and integration in (3.132) with the help of the program *Mathematica*™. Doing first the summation we get

$$Q = -2\int_0^{\infty} dr_y\, r_y\Big(2\psi^{(1)}(r_y^2) + 4r_y^2\psi^{(2)}(r_y^2) + r_y^4\psi^{(3)}(r_y^2)\Big), \quad (3.134)$$

where $\psi^{(m)}(z)$ is the digamma function given by

$$\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z), \quad (3.135)$$

with $\Gamma(z)$ the usual gamma function. The integral in (3.134) can be done numerically and leads at the very end to the expected result:

$$Q = -1. \quad (3.136)$$

Therefore, we have shown that the anti-self-dual non-commutative $U(1)$ instanton has topological charge minus one.

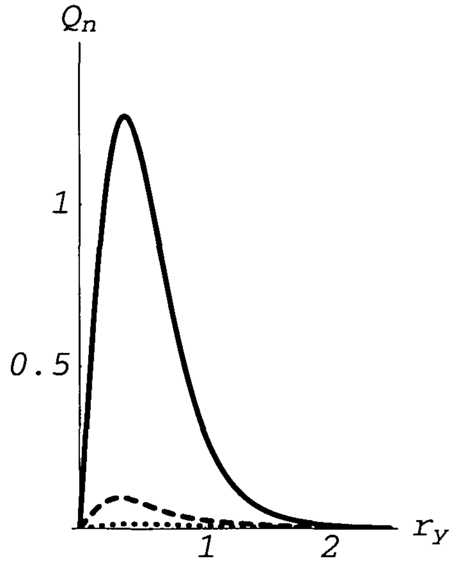


Figure 3.1: Density of instanton charge

3.3.6 Properties of the instanton

Established the expression (3.87) as the anti-selfdual one instanton solution of NC $U(1)$ YM theory in the space $\mathbb{R}^2_\theta \times \mathbb{R}^2$, we can proceed in taking a closer look at its properties.

First of all, we recognize the operator

$$\delta = z_1 \bar{z}_1 + z_2 \bar{z}_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \equiv r^2, \tag{3.137}$$

as the square of the distance r from the origin in $\mathbb{R}^2_\theta \times \mathbb{R}^2$ and rewrite the instanton field A (3.87) as

$$\begin{aligned} A = & \frac{1}{\zeta}\left[\frac{\sqrt{(r^2+2\zeta)r^2}}{r^2+\zeta} - 1\right]\bar{z}_1 dz_1 + \frac{1}{\zeta}\left[1 - \frac{\sqrt{r^4-\zeta^2}}{r^2}\right] z_1 d\bar{z}_1 \\ & - \left[\frac{\zeta}{2(r^2+\zeta)r^2}\right](\bar{z}_2 dz_2 - z_2 d\bar{z}_2)\,. \end{aligned} \tag{3.138}$$

From the expression (3.138) we see that we have a singularity at the origin, when we take the limit $r \to 0$:

$$\begin{aligned}\lim_{r\to 0} A = &-\frac{\mathrm{i}}{r^2}\, z_1 d\bar{z}_1 - \frac{1}{2r^2}\,(\bar{z}_2 dz_2 - z_2 d\bar{z}_2)\\ &-\frac{1}{\zeta}\,(\bar{z}_1 dz_1 - z_1 d\bar{z}_1) + \frac{1}{2\zeta}\,(\bar{z}_2 dz_2 - z_2 d\bar{z}_2).\end{aligned} \tag{3.139}$$

But this singularity is harmless since it can be removed by a singular gauge transformation like in the usual case of commutative instantons [77, 78]. It is a gauge artifact which disappears in gauge invariant quantities like $\mathrm{Tr}_{\mathcal{H}} F^n$. In the previous section we have seen this at work where no singularity in the expression for the topological charge appeared.

Further, we see from expression (3.139) that the commutative limit $\zeta \to 0$ gives also a singularity at the origin. This is consistent with the fact that there are no smooth instanton solutions in usual Maxwell theory which is the commutative limit of non-commutative $U(1)$ Yang–Mills theory (see [8, 79] for further discussions on this point).

Next, we will take a closer look at the instanton tail, *i.e.* the limit $r \to \infty$ which will be of great importance for the rest of our work. Performing this limit of the instanton solution (3.138) we get

$$\lim_{r\to\infty} A = \frac{\zeta}{2r^4}\,(-\bar{z}_1 dz_1 + z_1 d\bar{z}_1 - \bar{z}_2 dz_2 + z_2 d\bar{z}_2)\,. \tag{3.140}$$

Transforming (3.140) via (3.27) to the x-space yields

$$\lim_{r\to\infty} A = \frac{\mathrm{i}\zeta}{r^4}\,(-x_2 dx_1 + x_1 dx_2 - x_4 dx_3 + x_3 dx_4)\,. \tag{3.141}$$

Using the notation (3.88) and $\zeta = 2\,\theta$ (see section 3.2.1) we can rewrite (3.141) in the more convenient form

$$\lim_{r\to\infty} A_\mu(x) = -2\mathrm{i}\,\theta\,\epsilon_{\mu\nu}\,\frac{x_\nu}{r^4}, \tag{3.142}$$

where $\epsilon_{12} = \epsilon_{34} = 1$ and $\epsilon_{21} = \epsilon_{43} = -1$.

Therefore, the non-commutative instanton (3.142) decreases with $1/r^3$ very far from the origin and it can be shown that it has only significant values in a region $r \lesssim \sqrt{\theta}$ (see section 3.3.5). Furthermore, the instanton tail vanishes completely in the commutative limit $\theta \to 0$.

Chapter 4

Non-Commutative Vacuum Energy

In the previous chapter, we have established instanton solutions of non-commutative $U(1)$ Yang–Mills theory, and now we would like to know their contribution to the vacuum energy of the theory. This will be the aim of the following sections.

4.1 Review of ordinary Yang–Mills theory

As done already in previous sections, we would like to begin with a very short overview about the instanton-induced vacuum energy of ordinary non-Abelian Yang–Mills theory.[1] For a detailed investigation of the subject we refer the reader again to the excellent reviews [96] and [113].

Adding the gauge invariant topological term (3.8) to the Yang–Mills action (3.1) we get

$$S = -\frac{1}{2g^2}\int d^4x\, \mathrm{Tr}\big(F_{\mu\nu}F_{\mu\nu}\big) - \frac{\mathrm{i}\vartheta}{16\pi^2}\int d^4x\, \mathrm{Tr}\big(F_{\mu\nu}\tilde{F}_{\mu\nu}\big), \tag{4.1}$$

with ϑ the so-called *topological angle*.[2] It denotes the fact that the original gauge field theory splits up into a family of disconnected sectors, labeled by the angle ϑ, each with its own vacuum. Most of the time when Yang–Mills theory is considered, one ignors the second term in (4.1), because it is just a total derivative and thus has no effect on the equations of motion. But, it has topological consequences and contributes to the vacuum energy.

[1] As mentioned earlier there exist no instantons in Abelian Yang–Mills theory.

[2] The reader should not confuse the topological angle ϑ with the non-commutativity parameter θ.

The vacuum-to-vacuum amplitude is given by

$$\langle\vartheta|\, \mathrm{e}^{-HT}\, |\vartheta\rangle \propto \int [dA]\, \mathrm{e}^{-S}, \tag{4.2}$$

with H the corresponding Hamiltonian and T the Euclidean time extent of the four-dimensional box where we consider our theory for the moment. We will pass to infinite space at the end of our discussion.

Now, we know from section 3.1.1 that every instanton with topological charge Q has the action (3.13), and contributes therefore to (4.2) with a factor proportional to

$$\exp\big(-\frac{8\pi^2}{g^2}|Q| - \mathrm{i}\vartheta Q\big). \tag{4.3}$$

A summation over all possible instanton configurations leads in a semiclassical approximation to the following result for the vacuum energy:

$$\Delta E_{\text{inst}} = -2\, K/T\, \cos\vartheta. \tag{4.4}$$

A few explanations are necessary. Formula (4.4) was derived in the so-called *dilute instanton-gas approximation* where one assumes that all instantons contributing to the vacuum energy are widely separated, and thus can be regarded without any interactions. The factor K is given by the equation

$$K \propto V\, \Lambda_{\mathrm{UV}}^8 \left(\sqrt{S_{\mathrm{cl}}}\right)^8 \int_0^\infty d\rho\, \rho^3\, e^{-S_{\mathrm{cl}}}\, e^{-S_{\mathrm{eff}}(\mathcal{A})}, \tag{4.5}$$

with V the four-dimensional volume of the Euclidean box, and Λ_{UV} some ultraviolet momentum-cutoff. S_{cl} is the classical part of the action, coming from one instanton, therefore $S_{\mathrm{cl}} = 8\pi^2/g^2$, and S_{eff} is the perturbative effective action evaluated on the instanton field $\mathcal{A}$. The scale ρ denotes the size modulus of the instanton.

To leading order, the effective action S_{eff} is given by a ratio of determinants of the gauge field and ghost quadratic fluctuation operators.[3] Roughly, this can be seen by [93]

$$\mathrm{e}^{S_{\mathrm{eff}}(\Phi)} \propto \int [d\phi]\, \mathrm{e}^{-\frac{1}{2}\phi L(\Phi)\phi} \propto \big[\det L(\Phi)\big]^{\nu}, \tag{4.6}$$

with $\nu = -1/2$ for bosonic and $\nu = 1$ for fermionic quantum fields ϕ, and $L(\Phi)$ denotes the quadratic fluctuation operator evaluated on the background

[3] We have to gauge fix the action.

field Φ. As usual, symmetries of the action lead to zero-modes of the determinants in (4.6) and have to be treated separately. In the case of the instantons serving as background fields in the effective action of (4.5) the zero-modes are taken into account via so-called *collective coordinates.*

Here, we have eight collective coordinates of the instanton solution. First, there are four coordinates of the center of the instanton, then the scale ρ (the size of the instanton), and, finally, there are three Eulerian angles in four-dimensional space, specifying the orientation of the instanton in isospace.

Now, every zero-mode leads to a factor proportional to $\sqrt{S_{\mathrm{cl}}}$ and an integral with respect to a collective coordinate. Therefore, we can explain the origin of the pre-factors in (4.5). The volume V comes from translation invariance, the integration over ρ from scale invariance, and the factor ρ^3 from the Jacobian of the Eulerian angles. The momentum-cutoff Λ_{UV} arises due to a necessary regularization, but can also be found due to dimensional considerations.

The calculation [113] of the one-loop effective action in an instanton background of ordinary $SU(2)$ Yang–Mills theory leads to

$$S_{\mathrm{eff}}(\mathcal{A}) = -\frac{2}{3}\ln\left(\rho\,\Lambda_{\mathrm{UV}}\right). \tag{4.7}$$

Inserting (4.7) into (4.5) yields the following result:[4]

$$K \propto \left(\frac{8\pi^2}{g^2}\right)^4 V \int_0^\infty \frac{d\rho}{\rho^5}\exp\Big(-\frac{8\pi^2}{g^2}+\frac{22}{3}\ln\left(\rho\,\Lambda_{\mathrm{UV}}\right)\Big), \tag{4.8}$$

with the renormalization group invariant expression[5]

$$-\frac{8\pi^2}{g^2(\rho)} = -\frac{8\pi^2}{g^2} + \beta_0\ln\left(\rho\,\Lambda_{\mathrm{UV}}\right), \tag{4.9}$$

where $\beta_0 = 22/3$ is the one-loop beta-function coefficient of $SU(2)$ Yang–Mills theory.[6] Therefore, the zero-modes and the positive frequency modes of the quadratic fluctuation operators combine exactly to a renormalization group invariant expression (4.9) which determines the running of the gauge coupling.[7] One could have argued from the very beginning that (4.5) must have the form (4.8) as a direct consequence of renormalizability.

[4]The exact computation was done by 't Hooft [125], cited by Coleman [96] as a very hard worker.

[5]Here, g is the bare coupling constant at the cutoff-scale Λ_{UV}.

[6]The whole discussion works also for $SU(N)$ YM theory with $\beta_0 = 11N/3$, and $4N$ collective coordinates coming from additional isospace rotations.

[7]See also section 2.3.2.

The reader might have noticed that the integral over ρ in (4.8) is infrared divergent. The origin of the divergence is clear from the derivation of the integral, because the effective coupling constant $g(\rho)$ becomes large for large instantons, and this makes the integrand blow up. The subsequent integration over ρ yields an infrared divergence over moduli space. Therefore, the whole calculation holds only for very small instantons, and the dilute instanton gas approximation breaks down at large scales.

4.2 Vacuum energy of NC $U(1)$ YM theory

We will pass now to the study of the vacuum energy induced by instantons in non-commutative Maxwell theory, *i.e.* non-commutative $U(1)$ Yang–Mills theory, giving it a non-trivial vacuum-structure even in the case of an Abelian gauge group.

Proceeding in complete analogy to section 4.1, we write down the non-commutative version of (4.1):

$$S = -\frac{1}{2g^2}\int d^4x \; F_{\mu\nu} \star F_{\mu\nu} - \frac{i\vartheta}{16\pi^2}\int d^4x \; F_{\mu\nu} \star \tilde{F}_{\mu\nu}. \tag{4.10}$$

We refer the reader to chapter 2 where non-commutative $U(1)$ Yang–Mills theory with the action (4.10) has been extensively discussed.[8]

Now, the formulas (4.2), (4.3), (4.4) apply also in the non-commutative set-up, but there is a crucial change in the equation for the determinantal factor K which is given by

$$K \propto V\,\Lambda_{\mathrm{UV}}^4 \left(\sqrt{S_{\mathrm{cl}}}\right)^4 e^{-S_{\mathrm{cl}}}\, e^{-S_{\mathrm{eff}}(\mathcal{A})}. \tag{4.11}$$

The only collective coordinates we have in the game are the four position moduli of the instanton. There is no scale modulus anymore, because scale invariance is explicitly broken by the non-commutativity parameter θ and the size of the instanton is given by $\rho \sim \sqrt{\theta}$ (see section 3.3.5). Further, isospace is trivial in an Abelian gauge theory. Therefore, following the discussion in section 4.1, the four translational collective coordinates yield exactly the pre-factors in (4.11).

[8]The attentive reader would have noticed that there is relative factor of 2 in the expressions (2.5) and (4.10) for the action. Maybe this seems a bit confusing, but we will keep this difference, because this chapter is based on [80] where the numerical convention in (4.10) has been used. It is just an overall factor, but it goes for example into the definition of the β_0-coefficient. Nevertheless, all calculations are consistent within each chapter, of course.

4.2.1 One-loop instanton determinant

The procedure of obtaining the one-loop effective action of a theory with the background field method is standard in the literature [93], and we will be very briefly here in order to pass directly to the relevant calculations.

After a splitting of the gauge field and the ghost field into a classical background field and a quantum field, one obtains via integration over the quadratic part in the quantum fields the one-loop effective action

$$S_{\text{eff}}(\mathcal{A}) = -\frac{1}{2}\ln \det{}' L_{\text{gauge}} + \ln \det{}' L_{\text{ghost}}, \tag{4.12}$$

in complete analogy to equation (4.6). The determinants of the quadratic fluctuation operators in the gauge and ghost sectors are given by[9]

$$(L_{\text{gauge}})_{\mu\nu} = (D_\rho \star D_\rho)\,\delta_{\mu\nu} - 2\mathrm{i}\,F_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) D_\mu \star D_\nu, \tag{4.13}$$

$$L_{\text{ghost}} = D_\rho \star D_\rho, \tag{4.14}$$

where $D_\mu = \partial_\mu - \mathrm{i}\,[A_\mu,\ \ ]_\star$ and α denotes the gauge parameter.

Now, $\det'$ indicates that the zero modes have to be omitted when computing the determinants. The complete effective action, including the $\ln\det'$ contributions and the zero mode contributions can be written as a formal sum over all one-loop diagrams with external legs on the classical instanton profile $\mathcal{A}$. In this representation, the zero mode terms should arise as convenient resummations of infrared divergences to all orders.

At the level of planar diagrams we have a situation entirely similar to that of ordinary $SU(N)$ gauge theory, in the formal limit $N \to 1$. For example, the logarithmic dependence on the ultraviolet cutoff Λ_{UV} arises from the planar two-point function (see sections 2.2.3 and 4.1) given by the half[10] of (4.7) and combines with the explicit dependence from the zero modes in equation (4.11) to

$$K \propto \left(\frac{8\pi^2}{g^2}\right)^2 V\,\rho^{-4} \exp\Big(-\frac{8\pi^2}{g^2} + \frac{11}{3}\ln\left(\rho\,\Lambda_{\text{UV}}\right)\Big) + \dots. \tag{4.15}$$

Introducing the dynamical scale Λ via

$$\Lambda = \Lambda_{\text{UV}} \exp\Big(-\frac{8\pi^2}{\beta_0\, g^2}\Big), \tag{4.16}$$

[9]The commutative analog of (4.14) is given in [126], and can be derived directly from the action (4.10). But, we will not need the explicit expressions of the determinants in the following. We give them just for completeness.

[10]For $SU(N)$ the factor is $-N/3$.

we can write equation (4.15) as

$$K \propto \left(\frac{8\pi^2}{g^2}\right)^2 V \rho^{-4} (\rho\,\Lambda)^{\beta_0} + \ldots, \tag{4.17}$$

producing again a renormalization group invariant expression as expected. β_0 is the one-loop beta-function coefficient which is equal to 11/3 for the case of pure NC $U(1)$ YM theory (see section 2.3.2).[11]

Here, the size of the instanton $\rho \sim \sqrt{\theta}$ acts like an infrared cutoff, since the classical field $\mathcal{A}$ decays to zero on distances larger than the instanton size. Hence, even if we cannot calculate the numerical coefficients in a precise way, a combination of dimensional analysis and the general properties of the perturbative effective action allows us to determine the gross features of the planar contribution to the instanton measure.

The dots in Eq. (4.17) stand for other UV-finite perturbative contributions. Among those, the non-planar diagrams of low order have strong IR singularities as a result of the famous UV/IR mixing effects (see section 2.3). Despite the fact that the instanton profile vanishes at long distances, we must then check the infrared behaviour of the one-loop effective action. Here, in the non-planar sector, we do not have to implement the zero modes (corresponding to a summation over an infinite number of diagrams), because we focus on peculiar IR singularities that arise only from a finite number of diagrams which is enough to estimate their effect. We will split the analysis into two parts: we will consider the IR poles (I) and the IR logarithms (II) separately.

4.2.2 Infrared pole structure

The pole structure of non-planar n-point functions of pure NC $U(1)$ YM theory can be read off from the listed expressions in section 2.2.4. Or, to proceed another way, can be taken from the following gauge-invariant expression for the effective action [37]:

$$S^{\mathrm{I}}_{\mathrm{eff}}(\mathcal{A}) = \frac{1}{2\pi^2} \int \frac{d^4p}{(2\pi)^4}\, W'(-p)\, \frac{p^2}{\tilde{p}^2}\, K_2(\sqrt{p^2\tilde{p}^2})\, W'(p), \tag{4.18}$$

where $W'(p)$ denotes a truncated open Wilson line operator. It is a common feature of non-commutative gauge theories that one has to make use of such operators in order to write down gauge-invariant expressions [127, 128, 129].

[11] See also footnote 8 on page 86.

Because we are mainly interested in the IR regime of the theory we will expand the modified Bessel function $K_2(z)$ for small momenta[12]

$$\frac{p^2}{\tilde{p}^2}\, K_2(\sqrt{p^2\tilde{p}^2}) = \frac{2}{\tilde{p}^4} - \frac{p^2}{2\tilde{p}^2} + \mathcal{O}(\tilde{p}^0). \tag{4.19}$$

Insertion of the Wilson line operators (given in [37]) in the IR regime, with $\mathcal{A}$ denoting the classical background gauge field

$$W'(p) = \mathrm{i}\, \tilde{p}^\mu \mathcal{A}_\mu(p) - \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\, \tilde{p}^\mu \tilde{p}^\nu \mathcal{A}_\mu(p-q)\mathcal{A}_\nu(q) + \ldots, \tag{4.20}$$

and performing a Wick rotation leads to the following Euclidean expressions for the two- and three-point functions

$$S^{\mathrm{I},(2)}_{\mathrm{eff}}(\mathcal{A}) = \int \frac{d^4p}{(2\pi)^4}\, \mathcal{A}_\mu(p)\mathcal{A}_\nu(-p)\Pi^{\mathrm{I},(2)}_{\mu\nu}(p), \tag{4.21}$$

$$S^{\mathrm{I},(3)}_{\mathrm{eff}}(\mathcal{A}) = \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\, \mathcal{A}_\mu(p)\mathcal{A}_\nu(q)\mathcal{A}_\rho(-p-q)\Pi^{\mathrm{I},(3)}_{\mu\nu\rho}(p), \tag{4.22}$$

with[13]

$$\Pi^{\mathrm{I},(2)}_{\mu\nu}(p) = -\frac{1}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{\tilde{p}^4} + \mathcal{O}(\tilde{p}^0), \tag{4.23}$$

$$\Pi^{\mathrm{I},(3)}_{\mu\nu\rho}(p) = -\frac{1}{\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu\tilde{p}_\rho}{\tilde{p}^4} + \mathcal{O}(\tilde{p}^0). \tag{4.24}$$

It will be sufficient to consider these functions, since higher ones cannot lead to IR divergent terms. Looking at the expansions (4.19) and (4.20) we recognize that the n-point functions $\Pi^{\mathrm{I},(n)}(p)$ will lead to pole structures of the order of $\tilde{p}^{(n-4)}$. Furthermore, we have to check the IR structure of our background gauge field $\mathcal{A}$ which will be used to calculate the contributions to the effective action. The instanton field (3.142) obtained in section 3.3.6 will play this role. It reads

$$\lim_{r\to\infty} A_\mu(x) = -2\mathrm{i}\,\theta\,\epsilon_{\mu\nu}\,\frac{x_\nu}{r^4}, \tag{4.25}$$

with $\epsilon_{12} = \epsilon_{34} = 1$ and $\epsilon_{21} = \epsilon_{43} = -1$, and $r = \sqrt{x^2}$ the distance from the origin in four-dimensional Euclidean space. In order to proceed, we need the Fourier transform of (4.25) which is given in appendix A.2 by the equation[14]

$$\lim_{p\to 0} A_\mu(p) = 4\pi^2\,\theta\,\epsilon_{\mu\nu}\,\frac{p_\nu}{p^2}. \tag{4.26}$$

[12] See (2.59) and the textbooks [97, 98, 99] for the properties of $K_\kappa(z)$.

[13] Compare the expressions (4.23) and (4.24) with (2.71) and (2.72), respectively. Note also footnote 8 on page 86.

[14] We omit the usual tilde for denoting the Fourier transform for readability.

Now, every term of $S^{\mathrm{I}}_{\text{eff}}$ has the following form (we skip Lorentz indices):

$$S^{\mathrm{I},(n)}_{\text{eff}} \propto \int d^4p\, d^4q_1 \cdots d^4q_{n-2}\, \mathcal{A}(p)\mathcal{A}(q_1)\cdots\mathcal{A}(q_{n-2})\mathcal{A}(-p-\textstyle\sum_{i=1}^{n-2} q_i)\Pi^{\mathrm{I},(n)}(p). \tag{4.27}$$

Doing the power counting for the p-integration by taking the IR structure of the instanton (4.26) into account, we see that the first $\mathcal{A}$ field contributes with $\tilde{p}\, p^{-2}$, whereas the last one with p^{-2} for $n \geq 3$ (we have to pick out the most dangerous IR terms) and with $\tilde{p}\, p^{-2}$ for $n = 2$. Therefore, we have

$$n = 2: \qquad p^4\,(\tilde{p}\, p^{-2})\,(\tilde{p}\, p^{-2})\,\tilde{p}^{(2-4)} \quad \propto \quad \tilde{p}^0, \tag{4.28}$$

$$n \geq 3: \qquad p^4\,(\tilde{p}\, p^{-2})\, p^{-2}\,\tilde{p}^{(n-4)} \quad \propto \quad \tilde{p}^{(n-3)}. \tag{4.29}$$

The last thing we have to do is to check the remaining integrations over the q_i's. But they are harmless in the IR. With the same argumentation, it can be shown that they are linear in $\tilde{q}_i$:

$$n \geq 3: \qquad q_i^4\,(\tilde{q}_i\, q_i^{-2})\, q_i^{-2} \quad \propto \quad \tilde{q}_i, \qquad i = 1, \ldots, n-2. \tag{4.30}$$

This shows IR finiteness for four-point functions and higher terms. Furthermore, two- and three-point functions can at most lead to logarithmic IR divergences in the instanton background.

For the following calculations it will be useful to list all necessary tensor contractions (remember that we are considering a space with commuting time ($\mathbb{R}^2_\theta \times \mathbb{R}^2$) defined in section 3.3):

$$\begin{gathered}
\tilde{p}_\mu = \theta_{\mu\nu} p_\nu \quad \longmapsto \quad \tilde{p}_1 = \theta p_2, \quad \tilde{p}_2 = -\theta p_1, \quad \tilde{p}_3 = \tilde{p}_4 = 0, \\
p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2, \qquad \tilde{p}^2 = \theta^2(p_1^2 + p_2^2), \\
\tilde{p}_\mu \epsilon_{\mu\nu} p_\nu = \theta(p_1^2 + p_2^2), \qquad \tilde{p}_\mu \epsilon_{\mu\nu} q_\nu = \theta(p_1 q_1 + p_2 q_2), \\
p_\mu \epsilon_{\mu\nu} p_\nu = 0, \qquad \epsilon_{\rho\mu}\epsilon_{\rho\nu} = \delta_{\mu\nu}.
\end{gathered} \tag{4.31}$$

Let us start with the contribution of the two-point function (4.21). For the background gauge field $\mathcal{A}$, we insert the non-commutative instanton (4.26). As stated before, we are only interested in the IR regime, therefore taking only the small momentum approximation of the instanton solution. From (4.21), (4.23), (4.26), and (4.31) we get the fairly simple expression

$$S^{\mathrm{I},(2)}_{\text{eff}} = \frac{\theta^2}{\pi^2}\int d^4p\, \frac{\tilde{p}_\mu\, \epsilon_{\mu\rho}\, p_\rho\, \tilde{p}_\nu\, \epsilon_{\nu\sigma}\, p_\sigma}{p^4\, \tilde{p}^4} = \frac{\theta^2}{\pi^2}\int d^4p\, \frac{\theta^2(p_1^2+p_2^2)^2}{p^4\,\theta^4(p_1^2+p_2^2)^2} = \frac{1}{\pi^2}\int \frac{d^4p}{p^4}. \tag{4.32}$$

Since the integrand in (4.32) is invariant under $SO(4)$ rotations we can rewrite the integration measure with respect to the formula

$$\int d^D p = \Omega_D \int dp\, p^{D-1}, \qquad \Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}, \tag{4.33}$$

and get with $D = 4$ the expression:

$$S_{\text{eff}}^{\text{I},(2)} = 2 \int \frac{dp}{p}. \tag{4.34}$$

Being interested only in the infrared part of this integral, we evaluate it with an ultraviolet momentum cutoff at the instanton size $\Lambda_{\text{UV}} \sim 1/\sqrt{\theta}$ (see section 3.3.5) and an infrared momentum cutoff $\Lambda_{\text{IR}} \sim 1/L$ corresponding to a box of size L. The result is logarithmically divergent in the IR ($L \to \infty$):

$$S_{\text{eff}}^{\text{I},(2)} = 2 \int_{1/L}^{1/\sqrt{\theta}} \frac{dp}{p} = 2 \ln p \Big|_{1/L}^{1/\sqrt{\theta}} = \ln \frac{L^2}{\theta}. \tag{4.35}$$

We will postpone a further discussion of this point to section 4.2.4. Instead, we go straight to the appropriate calculations of the contribution from the three-point function to the one-loop effective action.

Applying again (4.31) and inserting (4.26) and (4.24) into (4.22), we get after lengthy but straightforward simplifications:

$$S_{\text{eff}}^{\text{I},(3)} = \frac{\theta^2}{2\pi^4} \int d^4p\, d^4q\, \frac{1}{p^2 q^2 (p+q)^2} \left[p_1 q_1 + p_2 q_2 + \frac{(p_1 q_1 + p_2 q_2)^2}{p_1^2 + p_2^2} \right]. \tag{4.36}$$

The structure of this integral suggests a splitting of the four-dimensional momenta into two two-dimensional subparts. In fact, it is clear, considering the underlying integration space $\mathbb{R}_\theta^2 \times \mathbb{R}^2$, that this procedure makes sense. Let us apply the substitutions

$$p = (s, t) \qquad \text{and} \qquad q = (u, v)\,, \tag{4.37}$$

to the integral (4.36)

$$S_{\text{eff}}^{\text{I},(3)} = \frac{\theta^2}{2\pi^4} \int d^2s\, d^2t\, d^2u\, d^2v\, \frac{(s \cdot u) + (s \cdot u)^2/s^2}{(s^2+t^2)(u^2+v^2)((s+u)^2+(t+v)^2)}. \tag{4.38}$$

Considering the two parts in the numerator separately, calling them I_1 and I_2, we introduce three Schwinger parameters α_i for the first part; for the second term, we use four of them via the relation

$$\frac{1}{k^2} = \int_0^\infty d\alpha\, e^{-\alpha k^2}. \tag{4.39}$$

Representing now the term $(s \cdot u)$ in both integrals with the help of a differential operator, we can perform all 2-dimensional Gaussian integrals over the momenta and end up with

$$I_1 = -\frac{\theta^2}{2}\int_0^\infty d\alpha\, d\beta\, d\gamma\, \frac{\gamma}{\tau^3}, \tag{4.40}$$

$$I_2 = \frac{\theta^2}{4}\int_0^\infty d\lambda\, d\alpha\, d\beta\, d\gamma\, \frac{\tau+\lambda(\beta+\gamma)+4\gamma^2}{\tau(\tau+\lambda(\beta+\gamma))^3}, \tag{4.41}$$

where we introduced

$$\tau = \alpha\beta + \alpha\gamma + \beta\gamma. \tag{4.42}$$

Using Schwinger cutoffs in the UV and IR by implementing factors such as

$$e^{-\frac{1}{\alpha\Lambda_{\mathrm{UV}}^2}-\alpha\Lambda_{\mathrm{IR}}^2}, \tag{4.43}$$

for every Schwinger parameter, we can perform the integrals I_1 and I_2 explicitly.[15] The results include only positive powers of Λ_{IR} and are therefore finite in the limit $\Lambda_{\mathrm{IR}} \to 0$. There is no IR singularity coming from the three-point function, as naive power counting in (4.29) would suggest.

4.2.3 Infrared logarithm structure

We have seen in section 2.3.2 that the UV/IR mixing of non-commutative gauge theories makes the coefficient of logarithmic IR divergences that arise from non-planar graphs exactly opposite to that of the logarithmic UV divergences in the planar sector of the theory. Therefore, we can write down the corresponding Euclidean contribution to the effective action in the region of small momenta $p \ll 1/\sqrt{\theta}$, given by

$$S_{\mathrm{eff}}^{\mathrm{II}}(\mathcal{A}) = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\frac{\beta_0}{(4\pi)^2}\ln(\Lambda_{\mathrm{UV}}^2\tilde{p}^2)\, F_{\mu\nu}(-p)\, F_{\mu\nu}(p) + \ldots, \tag{4.44}$$

where $\beta_0 = 11/3$ is the one-loop beta-function coefficient[16] for NC $U(1)$ YM theory and the ultraviolet cutoff is again given by the instanton size $\Lambda_{\mathrm{UV}} \sim 1/\sqrt{\theta}$ of section 3.3.5. The dots denote higher terms which have to be implemented in order to render the effective action gauge invariant. We must use again a generalization of open Wilson lines [37], but for our purpose the leading part is sufficient.

[15] We have done the parameter integrations with the program *Mathematica*™, but avoid to write down the results due to their lengthy unreadable structure.

[16] See also footnote 8 on page 86.

Let us first calculate the contribution of the two-point function to the effective action and then give arguments why higher-point functions are irrelevant to IR divergences. We rewrite (4.44) as

$$S_{\rm eff}^{\rm II}(\mathcal{A}) = \int \frac{d^4p}{(2\pi)^4}\, \mathcal{A}_\mu(p)\mathcal{A}_\nu(-p)\Pi_{\mu\nu}^{\rm II,(2)}(p) + \mathcal{O}(\mathcal{A}^3), \qquad (4.45)$$

with

$$\Pi_{\mu\nu}^{\rm II,(2)}(p) = -\frac{\beta_0}{(4\pi)^2}\ln(\Lambda_{\rm UV}^2\,\tilde{p}^2)\;(p^2\delta_{\mu\nu} - p_\mu p_\nu). \qquad (4.46)$$

Inserting in (4.45) the instanton field (4.26) and applying the relations (4.31) yields the following integral for the leading part of (4.45):

$$S_{\rm eff}^{\rm II,(2)} = \frac{\beta_0}{(4\pi)^2}\,\theta^2 \int d^4p\;\ln((p_1^2+p_2^2)\theta). \qquad (4.47)$$

Splitting again the four-dimensional momentum space into two two-dimensional parts via $p = (s,t)$ leads to

$$S_{\rm eff}^{\rm II,(2)} = \frac{\beta_0}{(4\pi)^2}\,\theta^2 \int d^2s\; d^2t\;\ln(s^2\theta). \qquad (4.48)$$

Making use of the formula [97, 98, 99]:

$$\ln\alpha = -\int_1^\infty \frac{e^{-\alpha\beta}}{\beta}d\beta - \gamma_E + \mathcal{O}(\alpha), \qquad (4.49)$$

and performing the Gaussian momentum integral over s shows that the expression (4.48) is completely finite in the IR. There are no IR divergences arising from the logarithmic piece (4.46) of the two-point function.

What happens in the case of higher-point functions? They are all safe in the IR regime ($\tilde{p}_i \to 0$) since, because of the $\star$-product, every ln-term gets multiplied by $\sin(\tilde{p}_i p_j/2)$ (see the explicit expressions in section 2.2.4) which renders the whole expression finite.

4.2.4 Discussion of the results

We have seen now from the sections 4.2.2 and 4.2.3 that the only IR divergent contribution to the one-loop effective action (4.12) in the instanton background (4.26) comes from the quadratic UV/IR mixing in the two-point function (4.23).

The resulting divergence is logarithmic (4.35) and signals the breakdown of the dilute instanton gas approximation also in non-commutative Yang–Mills theory. In the commutative case it was due to an IR divergent integration over the instanton size modulus, and now it is due to UV/IR mixing effects.

Since we know already that supersymmetric non-commutative gauge theories are much better behaved with respect to UV/IR mixing, we will comment on their one-loop instanton determinant in the next section.

4.3 Supersymmetry as an IR regulator

In section 2.5 we have discussed UV/IR mixing effects in supersymmetric non-commutative Yang–Mills theories and found that they do not show any quadratic or linear UV/IR mixing. Therefore, we expect that the instanton calculation outlined in the previous sections will be better behaved in the supersymmetric case.

To demonstrate this, we consider a theory with one gauge field, n_f Weyl fermions and n_s real scalars where we take again the simplest gauge group $U(1)$. The leading term of the non-planar two-point function of the gauge field in the IR regime is given by the Euclidean expression of (2.131) multiplied with (2.68) and reads[17]

$$\Pi^{\mathrm{I,SUSY}}_{\mu\nu}(p) = -\frac{1}{\pi^2}\left(1 - n_\mathrm{f} + \frac{n_\mathrm{s}}{2}\right)\frac{\tilde{p}_\mu \tilde{p}_\nu}{\tilde{p}^4} + \mathcal{O}(\tilde{p}^0), \tag{4.50}$$

which replaces the equation (4.23) in section 4.2.2. The expression in brackets is always zero for a theory with supersymmetric field content. Because for $\mathcal{N} = 1$ NCSYM theory we have $n_\mathrm{f} = 1$, $n_\mathrm{s} = 0$, whereas in the case of $\mathcal{N} = 2$ we have $n_\mathrm{f} = n_\mathrm{s} = 2$, and finally for $\mathcal{N} = 4$ we have to apply $n_\mathrm{f} = 4$ and $n_\mathrm{s} = 6$. This cancelation between bosonic and fermionic modes also takes place for higher-point functions. No IR divergences come from the pole-like structure of non-planar supersymmetric n-point functions, rendering therefore the one-loop instanton determinant IR finite.

Next, we have to generalize expression (4.44) to the supersymmetric case. This is again achieved very easily in replacing the coefficient of the beta-function by the corresponding expression (2.139) of section 2.5 which reads[18]

$$\beta_0 = \frac{1}{3}\left(11 - 2n_\mathrm{f} - \frac{n_\mathrm{s}}{2}\right). \tag{4.51}$$

[17]See also footnote 8 on page 86.

[18]There is again a relative factor of 2, explained in footnote 8 on page 86.

In the case of non-commutative $\mathcal{N} = 4$ theory, there is still a vanishing beta-function, whereas we get logarithmic non-planar corrections for $\mathcal{N} = 2$ and $\mathcal{N} = 1$ NCSYM theories. Nevertheless, as shown above, they are harmless with respect to the instanton determinant.

A last comment should be made on softly broken supersymmetric theories. These theories have different masses for the fermions and scalars which partly breaks supersymmetry. But, as shown in section 2.4.2, these soft breaking effects do not change the leading IR structure of non-planar n-point functions. The instanton determinant is again IR safe in these theories.

Hence, dilute instanton calculus in non-commutative Yang–Mills theory is ruined by the infrared catastrophe, unless we work in supersymmetric or softly broken supersymmetric theories.[19]

[19] Another interesting possibility consists in working in finite volume. See [130] for a discussion of such a case in a related context.

Conclusion

In this thesis we have reviewed the basic concepts of non-commutative field theories and have focused especially on the impact of UV/IR mixing effects. Non-commutative Yang–Mills theory is found to be non-renormalizable beyond one-loop due to quadratic and linear IR divergences which cannot be integrated over at higher loops. Further, we found a tachyonic mode in the dispersion relation and that non-commutative Yang–Mills theories are weakly coupled in the infrared and ultraviolet.

Then, we studied the changes coming from implementing supersymmetry. In SUSY NCYM theories, all quadratic and linear IR divergences disappear and no tachyon shows up in the spectrum. Softly breaking of supersymmetry via a non-vanishing mass of the photino gives us back a tachyonic mode, whereas UV/IR mixing effects are still only logarithmic. The appearance of the tachyon is quite severe, because it is independent of the non-commutativity parameter and cannot be neglected anymore by arguing with a small NC parameter. Softly broken SUSY NCYM theory is ruled out in nature.

Afterwards, we passed to the non-perturbative sector of NCYM theories. More exactly, we studied non-commutative instantons which were constructed via deformed ADHM relations. We transformed the explicit solution of an anti-self-dual instanton in NC $U(1)$ YM theory given in an abstract operator formalism into an expression in non-commutative coordinate space. The instanton has a classical size of order $\sqrt{\theta}$ and decreases with $1/r^3$ away from the origin.

With this in hand, we studied the impact of UV/IR mixing on the one-loop instanton determinant of non-commutative $U(1)$ Yang–Mills theory. The instanton-induced vacuum energy is well behaved in the classical approximation of the theory. But, taking one-loop quantum effects into account, we found a logarithmic infrared divergence of the instanton determinant coming from the non-planar two-point function. *Non-commutative quantum fluctuations blow the classical finite size of the instanton up to infinity.*

It should be clarified here that the blow up of the instanton size to infinity is a metaphor, a way of conveying the message that at the end, the non-commutative Yang–Mills theory behaves similarly to the ordinary non-Abelian Yang–Mills theory, having an IR divergent dilute instanton measure, although for different reasons when it comes to the details. In the case of ordinary Yang–Mills theory, the IR problem comes from the integral over instanton sizes, whereas in the non-commutative case it is because of the UV/IR mixing effects.

Rescue is again provided by supersymmetry. All n-point functions are free of quadratic or linear IR divergences in supersymmetric or softly broken SUSY NCYM theories. Therefore, the instanton-induced vacuum energy is IR safe in these theories. This fact can be regarded as one hint more that the only consistent non-commutative field theories are theories with a supersymmetric field content.

Appendix A

Some Formulae

A.1 Operator formalism

- Proof of the formula $\partial_1 f^{-1} = -f^{-1}\,(\partial_1 f)\,f^{-1}$:

$$
\begin{aligned}
&0 = \partial_1 \mathbb{1} = \partial_1 (f^{-1} f) = \partial_1 f^{-1} f + f^{-1} \partial_1 f \\
\Rightarrow \quad &\partial_1 f^{-1} f = -f^{-1} \partial_1 f \qquad \Rightarrow \quad \partial_1 f^{-1} = -f^{-1} (\partial_1 f) f^{-1},
\end{aligned}
$$

  and analogous for $\bar{\partial}_1 f^{-1} = -f^{-1}\left(\bar{\partial}_1 f\right) f^{-1}$.

- Proof of the formula $\bar{z}_1 f(\delta) = f(\Delta)\,\bar{z}_1$:
  From the only non-vanishing commutator (3.29) we have

$$\bar{z}_1 z_1 = \zeta + z_1 \bar{z}_1.$$

  Now, we take $f(\delta) = \delta$, implying

$$\bar{z}_1 f(\delta) = \bar{z}_1 \delta = \bar{z}_1 (z_1 \bar{z}_1 + z_2 \bar{z}_2) = (\zeta + z_1 \bar{z}_1 + z_2 \bar{z}_2) \bar{z}_1 = \Delta \bar{z}_1 = f(\Delta) \bar{z}_1,$$

  which holds also for an arbitrary function of δ and works completely analogous for $z_1 f(\delta) = f(\nabla)\, z_1$.

- Proof of the following formula:

$$
\begin{aligned}
d\, f(\delta) = &\zeta^{-1} \left(f(\Delta) - f(\delta)\right) \bar{z}_1 dz_1 + \zeta^{-1} \left(f(\delta) - f(\nabla)\right) z_1 d\bar{z}_1 \\
&+ \frac{\partial f(\delta)}{\partial \delta} \left(\bar{z}_2 dz_2 + z_2 d\bar{z}_2\right).
\end{aligned}
$$

Using (3.41), (3.43), (3.45), (3.47) and (3.48) it is easy to show that

$$\begin{aligned}
d\,f(\delta) &= \left(dz_1\,\partial_1 + d\bar{z}_1\,\bar{\partial}_1 + dz_2\,\partial_2 + d\bar{z}_2\,\bar{\partial}_2\right) f(\delta) \\
&= \zeta^{-1}\left[\bar{z}_1, f(\delta)\right] dz_1 - \zeta^{-1}\left[z_1, f(\delta)\right] d\bar{z}_1 + \frac{\partial f(\delta)}{\partial z_2}\, dz_2 + \frac{\partial f(\delta)}{\partial \bar{z}_2}\, d\bar{z}_2 \\
&= \zeta^{-1}\left(\bar{z}_1 f(\delta) - f(\delta)\bar{z}_1\right) dz_1 + \zeta^{-1}\left(f(\delta) z_1 - z_1 f(\delta)\right) d\bar{z}_1 \\
&+ \frac{\partial f(\delta)}{\partial \delta}\frac{\partial \delta}{\partial z_2}\, dz_2 + \frac{\partial f(\delta)}{\partial \delta}\frac{\partial \delta}{\partial \bar{z}_2}\, d\bar{z}_2 \\
&= \zeta^{-1}\left(f(\Delta) - f(\delta)\right) \bar{z}_1 dz_1 + \zeta^{-1}\left(f(\delta) - f(\nabla)\right) z_1 d\bar{z}_1 \\
&+ \frac{\partial f(\delta)}{\partial \delta}\, \bar{z}_2 dz_2 + \frac{\partial f(\delta)}{\partial \delta}\, z_2 d\bar{z}_2 .
\end{aligned}$$

A.2 Fourier transform

Here, we perform the explicit Fourier transformation of the instanton field (3.142) given in section 3.3.6:

$$\lim_{r\to\infty} A_\mu(x) = -2\mathrm{i}\,\theta\,\epsilon_{\mu\nu}\,\frac{x_\nu}{r^4}, \tag{A.1}$$

with $\epsilon_{12} = \epsilon_{34} = 1$ and $\epsilon_{21} = \epsilon_{43} = -1$, and $r = \sqrt{x^2}$ the distance from the origin in four-dimensional Euclidean space. The Fourier transform of (A.1) is given by[1]

$$\lim_{p\to 0} A_\mu(p) = -2\mathrm{i}\,\theta\,\epsilon_{\mu\nu} \int d^4x\, \mathrm{e}^{\mathrm{i}px}\, \frac{x_\nu}{x^4} = -2\,\theta\,\epsilon_{\mu\nu}\,\partial^p_\nu \int d^4x\, \mathrm{e}^{\mathrm{i}px}\, \frac{1}{x^4}, \tag{A.2}$$

where ∂^p_ν denotes the derivative with respect to p. Therefore, we can reduce the problem to the Fourier transform of

$$\mathcal{F}_4\Big(\frac{1}{x^4}\Big) = \int d^4x\, \mathrm{e}^{\mathrm{i}px}\, \frac{1}{x^4}, \tag{A.3}$$

which can be obtained by the formula [131, 132]:

$$\mathcal{F}_d\big(x^\lambda\big) = 2^{\lambda+d}\pi^{\frac{d}{2}}\,\frac{\Gamma\big(\frac{\lambda+d}{2}\big)}{\Gamma\big(\frac{-\lambda}{2}\big)}\,p^{-\lambda-d}, \tag{A.4}$$

with d the space-time dimension and the momentum $p = \sqrt{p^2}$. Since the gamma function has poles at zero and the negative integers, we have to

[1] We omit the usual tilde for denoting the Fourier transform for readability.

regularize the expression (A.4) for $d = 4$ and $\lambda = -4$. Hence, we evaluate it in $d = 4 + \varepsilon$ dimensions with $\varepsilon > 0$ yielding

$$\mathcal{F}_{4+\varepsilon}\big(x^{-4}\big) = 2^{\varepsilon}\pi^{2+\frac{\varepsilon}{2}}\,\frac{\Gamma\big(\frac{\varepsilon}{2}\big)}{\Gamma(2)}\,p^{-\varepsilon}. \tag{A.5}$$

Performing a series expansion of (A.5) around $\varepsilon \sim 0$ leads to

$$\begin{aligned}\mathcal{F}_{4+\varepsilon}\big(x^{-4}\big) &= 2^{\varepsilon}\pi^{2+\frac{\varepsilon}{2}}\Big(\frac{2}{\varepsilon} - \gamma_{\mathrm{E}} + \frac{\varepsilon}{4}\big(\gamma_{\mathrm{E}}^2 + \frac{\pi^2}{6}\big) + \mathcal{O}(\varepsilon^2)\Big)\Big(1 - \varepsilon\,\ln(p) + \mathcal{O}(\varepsilon^2)\Big)\\ &= 2^{\varepsilon}\pi^{2+\frac{\varepsilon}{2}}\Big(\frac{2}{\varepsilon} - \gamma_{\mathrm{E}} - 2\,\ln(p) + \mathcal{O}(\varepsilon)\Big),\end{aligned} \tag{A.6}$$

with $\gamma_{\mathrm{E}} \sim 0,577216$ the Euler gamma. Combining now the expressions (A.2), (A.3), and (A.6) we get

$$\begin{aligned}\lim_{p\to 0} A_\mu(p) &= \lim_{\varepsilon\to 0}\Big(-2\,\theta\,\epsilon_{\mu\nu}\,\partial_\nu^p\,\mathcal{F}_{4+\varepsilon}\big(x^{-4}\big)\Big)\\ &= \lim_{\varepsilon\to 0}\Big(2^{2+\varepsilon}\pi^{2+\frac{\varepsilon}{2}}\,\theta\,\epsilon_{\mu\nu}\,\partial_\nu^p\,\ln(p) + \mathcal{O}(\varepsilon)\Big)\\ &= 4\pi^2\,\theta\,\epsilon_{\mu\nu}\,\partial_\nu^p\,\ln(p),\end{aligned} \tag{A.7}$$

where we can perform the limes $\varepsilon \to 0$ corresponding to four space-time dimensions at the end. Next, we calculate

$$\partial_\nu^p\,\ln(p) = \frac{1}{p}\,\partial_\nu^p\sqrt{p^2} = \frac{p_\nu}{p^2}, \tag{A.8}$$

and end up with the following expression for the Fourier transform of the instanton tail $\lim_{r\to\infty} A_\mu(x)$:

$$\lim_{p\to 0} A_\mu(p) = 4\pi^2\,\theta\,\epsilon_{\mu\nu}\,\frac{p_\nu}{p^2}. \tag{A.9}$$

Bibliography

[1] H. S. Snyder, *"Quantized space-time"*, Phys. Rev. **71** (1947) 38.

[2] S. Doplicher, K. Fredenhagen & J. E. Roberts, *"Space-time quantization induced by classical gravity"*, Phys. Lett. B **331** (1994) 39.

[3] S. Doplicher, K. Fredenhagen & J. E. Roberts, *"The quantum structure of space-time at the Planck scale and quantum fields"*, Commun. Math. Phys. **172** (1995) 187 [arXiv:hep-th/0303037].

[4] A. Connes, *"Non-commutative geometry"*, Academic Press (1994).

[5] G. Landi, *"An introduction to non-commutative spaces and their geometries"*, Springer Verlag (1997) [arXiv:hep-th/9701078].

[6] J. Madore, *"An introduction to non-commutative geometry and its physical applications"*, Cambridge University Press (1999).

[7] J. M. Gracia-Bondía, J. C. Várilly & H. Figueroa, *"Elements of non-commutative geometry"*, Birkhäuser (2000).

[8] N. Seiberg & E. Witten, *"String theory and non-commutative geometry"*, JHEP **9909** (1999) 032 [arXiv:hep-th/9908142].

[9] M. R. Douglas & N. A. Nekrasov, *"Non-commutative field theory"*, Rev. Mod. Phys. **73** (2001) 977 [arXiv:hep-th/0106048].

[10] R. J. Szabo, *"Quantum field theory on non-commutative spaces"*, Phys. Rep. **378** (2003) 207 [arXiv:hep-th/0109162].

[11] J. L. F. Barbón, *"Introduction to non-commutative field theory"*, Prepared for ICTP Summer School in Particle Physics, Trieste, Italy, 18 Jun - 6 Jul 2001

[12] T. Filk, *"Divergences in a field theory on quantum space"*, Phys. Lett. B **376** (1996) 53.

[13] S. Minwalla, M. Van Raamsdonk & N. Seiberg, *"Non-commutative perturbative dynamics"*, JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[14] A. Matusis, L. Susskind & N. Toumbas, *"The IR/UV connection in the non-commutative gauge theories"*, JHEP **0012** (2000) 002 [arXiv:hep-th/0002075].

[15] M. Van Raamsdonk & N. Seiberg, *"Comments on non-commutative perturbative dynamics"*, JHEP **0003** (2000) 035 [arXiv:hep-th/0002186].

[16] A. Micu & M. M. Sheikh-Jabbari, *"Non-commutative ϕ^4 theory at two loops"*, JHEP **0101** (2001) 025 [arXiv:hep-th/0008057].

[17] I. Chepelev & R. Roiban, *"Renormalization of quantum field theories on non-commutative $\mathbb{R}^d$. I: Scalars"*, JHEP **0005** (2000) 037 [arXiv:hep-th/9911098].

[18] I. Chepelev & R. Roiban, *"Convergence theorem for non-commutative Feynman graphs and renormalization"*, JHEP **0103** (2001) 001 [arXiv:hep-th/0008090].

[19] C. P. Martín & D. Sanchez-Ruiz, *"The one-loop UV divergent structure of $U(1)$ Yang–Mills theory on non-commutative $\mathbb{R}^4$"*, Phys. Rev. Lett. **83** (1999) 476 [arXiv:hep-th/9903077].

[20] T. Krajewski & R. Wulkenhaar, *"Perturbative quantum gauge fields on the non-commutative torus"*, Int. J. Mod. Phys. A **15** (2000) 1011 [arXiv:hep-th/9903187].

[21] M. Hayakawa, *"Perturbative analysis on infrared aspects of non-commutative QED on $\mathbb{R}^4$"*, Phys. Lett. B **478** (2000) 394 [arXiv:hep-th/9912094].

[22] M. Hayakawa, *"Perturbative analysis on infrared and ultraviolet aspects of non-commutative QED on $\mathbb{R}^4$"*, arXiv:hep-th/9912167.

[23] H. Grosse, T. Krajewski & R. Wulkenhaar, *"Renormalization of non-commutative Yang–Mills theories: A simple example"*, arXiv:hep-th/0001182.

[24] A. Armoni, *"Comments on perturbative dynamics of non-commutative Yang–Mills theory"*, Nucl. Phys. B **593** (2001) 229 [arXiv:hep-th/0005208].

[25] C. P. Martín & F. Ruiz Ruiz, *"Paramagnetic dominance, the sign of the beta-function and UV/IR mixing in non-commutative* $U(1)$*"*, Nucl. Phys. B **597** (2001) 197 [arXiv:hep-th/0007131].

[26] M. Hayakawa, *"Perturbative ultraviolet and infrared dynamics of non-commutative quantum field theory"*, arXiv:hep-th/0009098.

[27] C. P. Martín & D. Sanchez-Ruiz, *"The BRS invariance of non-commutative* $U(N)$ *Yang–Mills theory at the one-loop level"*, Nucl. Phys. B **598** (2001) 348 [arXiv:hep-th/0012024].

[28] H. O. Girotti, M. Gomes, V. O. Rivelles & A. J. da Silva, *"A consistent non-commutative field theory: The Wess–Zumino model"*, Nucl. Phys. B **587** (2000) 299 [arXiv:hep-th/0005272].

[29] A. A. Bichl, J. M. Grimstrup, H. Grosse, L. Popp, M. Schweda & R. Wulkenhaar, *"The superfield formalism applied to the non-commutative Wess–Zumino model"*, JHEP **0010** (2000) 046 [arXiv:hep-th/0007050].

[30] V. V. Khoze & G. Travaglini, *"Wilsonian effective actions and the IR/UV mixing in non-commutative gauge theories"*, JHEP **0101** (2001) 026 [arXiv:hep-th/0011218].

[31] D. Zanon, *"Non-commutative* $\mathcal{N} = 1, 2$ *super* $U(N)$ *Yang–Mills: UV/IR mixing and effective action results at one-loop"*, Phys. Lett. B **502** (2001) 265 [arXiv:hep-th/0012009].

[32] F. Ruiz Ruiz, *"Gauge fixing independence of IR divergences in non-commutative* $U(1)$*, perturbative tachyonic instabilities and supersymmetry"*, Phys. Lett. B **502** (2001) 274 [arXiv:hep-th/0012171].

[33] A. A. Bichl, M. Ertl, A. Gerhold, J. M. Grimstrup, H. Grosse, L. Popp, V. Putz, M. Schweda & R. Wulkenhaar, *"Non-commutative* $U(1)$ *super Yang–Mills theory: Perturbative self-energy corrections"*, arXiv:hep-th/0203141.

[34] L. Álvarez-Gaumé & M. A. Vázquez-Mozo, *"General properties of non-commutative field theories"*, Nucl. Phys. B **668** (2003) 293 [arXiv:hep-th/0305093].

[35] L. Álvarez-Gaumé & M. A. Vázquez-Mozo, *"Comments on non-commutative field theories"*, arXiv:hep-th/0311244.

[36] M. Van Raamsdonk, *"The meaning of infrared singularities in non-commutative gauge theories"*, JHEP **0111** (2001) 006 [arXiv:hep-th/0110093].

[37] A. Armoni & E. López, *"UV/IR mixing via closed strings and tachyonic instabilities"*, Nucl. Phys. B **632** (2002) 240 [arXiv:hep-th/0110113].

[38] A. Armoni, E. Lopez & A. M. Uranga, *"Closed strings tachyons and non-commutative instabilities"*, JHEP **0302** (2003) 020 [arXiv:hep-th/0301099].

[39] K. Landsteiner, E. López & M. H. G. Tytgat, *"Excitations in hot non-commutative theories"*, JHEP **0009** (2000) 027 [arXiv:hep-th/0006210].

[40] K. Landsteiner, E. López & M. H. G. Tytgat, *"Instability of non-commutative SYM theories at finite temperature"*, JHEP **0106** (2001) 055 [arXiv:hep-th/0104133].

[41] A. Anisimov, T. Banks, M. Dine & M. Graesser, *"Comments on non-commutative phenomenology"*, Phys. Rev. D **65** (2002) 085032 [arXiv:hep-ph/0106356].

[42] C. E. Carlson, C. D. Carone & R. F. Lebed, *"Supersymmetric non-commutative QED and Lorentz violation"*, Phys. Lett. B **549** (2002) 337 [arXiv:hep-ph/0209077].

[43] J. Madore, S. Schraml, P. Schupp & J. Wess, *"Gauge theory on non-commutative spaces"*, Eur. Phys. J. C **16** (2000) 161 [arXiv:hep-th/0001203].

[44] B. Jurco, S. Schraml, P. Schupp & J. Wess, *"Enveloping algebra valued gauge transformations for non-Abelian gauge groups on non-commutative spaces"*, Eur. Phys. J. C **17** (2000) 521 [arXiv:hep-th/0006246].

[45] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda & R. Wulkenhaar, *"Perturbative analysis of the Seiberg–Witten map"*, Int. J. Mod. Phys. A **17** (2002) 2219 [arXiv:hep-th/0102044].

[46] A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda & R. Wulkenhaar, *"Deformed QED via Seiberg–Witten map"*, arXiv:hep-th/0102103.

[47] A. A. Bichl, J. M. Grimstrup, H. Grosse, L. Popp, M. Schweda & R. Wulkenhaar, *"Renormalization of the non-commutative photon self-energy to all orders via Seiberg–Witten map"*, JHEP **0106** (2001) 013 [arXiv:hep-th/0104097].

[48] B. Jurco, L. Möller, S. Schraml, P. Schupp & J. Wess, *"Construction of non-Abelian gauge theories on non-commutative spaces"*, Eur. Phys. J. C **21** (2001) 383 [arXiv:hep-th/0104153].

[49] A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda & R. Wulkenhaar, *"Non-commutative Lorentz symmetry and the origin of the Seiberg–Witten map"*, Eur. Phys. J. C **24** (2002) 165 [arXiv:hep-th/0108045].

[50] J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda & R. Wulkenhaar, *"Non-commutative spin 1/2 representations"*, Eur. Phys. J. C **24** (2002) 485 [Eur. Phys. J. C **24** (2002) 491] [arXiv:hep-th/0110231].

[51] X. Calmet, B. Jurco, P. Schupp, J. Wess & M. Wohlgenannt, *"The standard model on non-commutative space-time"*, Eur. Phys. J. C **23** (2002) 363 [arXiv:hep-ph/0111115].

[52] R. Wulkenhaar, *"Non-renormalizability of θ-expanded non-commutative QED"*, JHEP **0203** (2002) 024 [arXiv:hep-th/0112248].

[53] V. Putz & R. Wulkenhaar, *"Seiberg–Witten map for non-commutative super Yang–Mills theory"*, Int. J. Mod. Phys. A **18** (2003) 3325 [arXiv:hep-th/0205094].

[54] J. M. Grimstrup & R. Wulkenhaar, *"Quantisation of θ-expanded non-commutative QED"*, Eur. Phys. J. C **26** (2002) 139 [arXiv:hep-th/0205153].

[55] R. Wulkenhaar, *"Quantum field theories on non-commutative $\mathbb{R}^4$ versus θ-expanded quantum field theories"*, arXiv:hep-th/0206018.

[56] A. Gerhold, J. M. Grimstrup, H. Grosse, L. Popp, M. Schweda & R. Wulkenhaar, *"The energy-momentum tensor on non-commutative spaces: Some pedagogical comments"*, arXiv:hep-th/0012112.

[57] I. Frühwirth, J. M. Grimstrup, Z. Morsli, L. Popp & M. Schweda, *"The role of the field redefinition in non-commutative Maxwell theory"*, arXiv:hep-th/0202092.

[58] J. M. Grimstrup, H. Grosse, L. Popp, V. Putz, M. Schweda, M. Wickenhauser & R. Wulkenhaar, *"IR singularities in non-commutative perturbative dynamics?"*, arXiv:hep-th/0202093.

[59] H. Bozkaya, P. Fischer, H. Grosse, M. Pitschmann, V. Putz, M. Schweda & R. Wulkenhaar, *"Space-time non-commutative field theories and causality"*, Eur. Phys. J. C **29** (2003) 133 [arXiv:hep-th/0209253].

[60] J. M. Grimstrup, B. Kloibock, L. Popp, V. Putz, M. Schweda & M. Wickenhauser, *"The energy-momentum tensor in non-commutative gauge field models"*, arXiv:hep-th/0210288.

[61] F. Aigner, M. Hillbrand, J. Knapp, G. Milovanovic, V. Putz, R. Schofbeck & M. Schweda, *"Technical remarks and comments on the UV/IR mixing problem of a non-commutative scalar quantum field theory"*, arXiv:hep-th/0302038.

[62] P. Fischer & V. Putz, *"No UV/IR mixing in unitary space-time non-commutative field theory"*, Eur. Phys. J. C **32** (2004) 269 [arXiv:hep-th/0306099].

[63] S. Denk & M. Schweda, *"Time ordered perturbation theory for non-local interactions: Applications to NCQFT"*, JHEP **0309** (2003) 032 [arXiv:hep-th/0306101].

[64] S. Denk, V. Putz, M. Schweda & M. Wohlgenannt, *"Towards UV finite quantum field theories from non-local field operators"*, arXiv:hep-th/0401237.

[65] S. Denk, V. Putz & M. Wohlgenannt, *"Consistent construction of perturbation theory on non-commutative spaces"*, arXiv:hep-th/0402229.

[66] H. Grosse & R. Wulkenhaar, *"Power-counting theorem for non-local matrix models and renormalisation"*, arXiv:hep-th/0305066.

[67] H. Grosse & R. Wulkenhaar, *"Renormalisation of ϕ^4 theory on non-commutative $\mathbb{R}^2$ in the matrix base"*, JHEP **0312** (2003) 019 [arXiv:hep-th/0307017].

[68] H. Grosse & R. Wulkenhaar, *"Renormalisation of ϕ^4 theory on non-commutative $\mathbb{R}^4$ in the matrix base"*, arXiv:hep-th/0401128.

[69] H. Grosse & R. Wulkenhaar, *"The beta-function in duality-covariant non-commutative ϕ^4 theory"*, arXiv:hep-th/0402093.

[70] H. Grosse & R. Wulkenhaar, *"Renormalisation of ϕ^4 theory on non-commutative $\mathbb{R}^4$ to all orders"*, arXiv:hep-th/0403232.

[71] N. Nekrasov & A. Schwarz, *"Instantons on non-commutative $\mathbb{R}^4$ and (2,0) superconformal six dimensional theory"*, Commun. Math. Phys. **198** (1998) 689 [arXiv:hep-th/9802068].

[72] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld & Y. I. Manin, *"Construction of instantons"*, Phys. Lett. A **65** (1978) 185.

[73] N. A. Nekrasov, *"Trieste lectures on solitons in non-commutative gauge theories"*, arXiv:hep-th/0011095.

[74] K. Furuuchi, *"Instantons on non-commutative $\mathbb{R}^4$ and projection operators"*, Prog. Theor. Phys. **103** (2000) 1043 [arXiv:hep-th/9912047].

[75] K. Y. Kim, B. H. Lee & H. S. Yang, *"Comments on instantons on non-commutative $\mathbb{R}^4$"*, J. Korean Phys. Soc. **41** (2002) 290 [arXiv:hep-th/0003093].

[76] K. Furuuchi, *"Topological charge of $U(1)$ instantons on non-commutative $\mathbb{R}^4$"*, Prog. Theor. Phys. Suppl. **144** (2001) 79 [arXiv:hep-th/0010006].

[77] C. S. Chu, V. V. Khoze & G. Travaglini, *"Notes on non-commutative instantons"*, Nucl. Phys. B **621** (2002) 101 [arXiv:hep-th/0108007].

[78] K. Y. Kim, B. H. Lee & H. S. Yang, *"Non-commutative instantons on $\mathbb{R}^2_{NC} \times \mathbb{R}^2$"*, Phys. Lett. B **523** (2001) 357 [arXiv:hep-th/0109121].

[79] P. Kraus & M. Shigemori, *"Non-commutative instantons and the Seiberg-Witten map"*, JHEP **0206** (2002) 034 [arXiv:hep-th/0110035].

[80] A. A. Bichl, *"A note on UV/IR mixing and non-commutative instanton calculus"*, arXiv:hep-th/0312230.

[81] H. Weyl, *"The theory of groups and quantum mechanics"*, Dover, New York (1931).

[82] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz & D. Sternheimer, *"Deformation theory and quantization. 1. Deformations of symplectic structures"*, Annals Phys. **111** (1978) 61.

[83] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz & D. Sternheimer, *"Deformation theory and quantization. 2. Physical applications"*, Annals Phys. **111** (1978) 111.

[84] S. Terashima, *"A note on superfields and non-commutative geometry"*, Phys. Lett. B **482** (2000) 276 [arXiv:hep-th/0002119].

[85] K. Matsubara, *"Restrictions on gauge groups in non-commutative gauge theory"*, Phys. Lett. B **482** (2000) 417 [arXiv:hep-th/0003294].

[86] C. Becchi, A. Rouet & R. Stora, *"The Abelian Higgs–Kibble model. Unitarity of the s-operator"*, Phys. Lett. **B52** (1974) 344.

[87] C. Becchi, A. Rouet & R. Stora, *"Renormalization of the Abelian Higgs–Kibble model"*, Commun. Math. Phys. **42** (1975) 127.

[88] C. Becchi, A. Rouet & R. Stora, *"Renormalization of gauge theories"*, Annals Phys. **98** (1976) 287.

[89] O. Piguet & S. P. Sorella, *"Algebraic renormalization: Perturbative renormalization, symmetries and anomalies"*, Berlin, Germany: Springer (1995).

[90] A. Boresch, S. Emery, O. Moritsch, M. Schweda, T. Sommer & H. Zerrouki, *"Applications of non-covariant gauges in the algebraic renormalization procedure"*, London, England: World Scientific (1998).

[91] A. A. Bichl, *"Non-commutative Chern–Simons theory"*, Diploma Thesis, Institute for Theoretical Physics, University of Technology, Vienna (2000).

[92] C. Itzykson & J. B. Zuber, *"Quantum field theory"*, New York, USA: McGraw–Hill (1980).

[93] M. E. Peskin & D. V. Schroeder, *"An introduction to quantum field theory"*, Westview Press (1995).

[94] L. H. Ryder, *Quantum field theory,* Cambridge, UK: University Press (1996).

[95] G. 't Hooft, *"A planar diagram theory for strong interactions"*, Nucl. Phys. B **72** (1974) 461.

[96] S. Coleman, *"Aspects of symmetry (Selected Erice lectures)"*, Cambridge University Press (1985).

[97] I. S. Gradshteyn & I. M. Ryzhik, *"Table of integrals, series and products"*, New York, USA: Academic Press (1969).

[98] N. N. Lebedev, *"Special functions and their applications"*, New York, USA: Dover (1972).

[99] M. Abramowitz & I. A. Stegun, *"Handbook of mathematical functions"*, New York, USA: Dover (1965).

[100] A. A. Bichl, J. M. Grimstrup, V. Putz & M. Schweda, *"Perturbative Chern–Simons theory on non-commutative $\mathbb{R}^3$"*, JHEP **0007** (2000) 046 [arXiv:hep-th/0004071].

[101] A. K. Das & M. M. Sheikh-Jabbari, *"Absence of higher order corrections to non-commutative Chern–Simons coupling"*, JHEP **0106** (2001) 028 [arXiv:hep-th/0103139].

[102] M. F. Sohnius, *"Introducing supersymmetry,"*, Phys. Rept. **128** (1985) 39.

[103] J. D. Lykken, *"Introduction to supersymmetry"*, arXiv:hep-th/9612114.

[104] J. Wess & J. Bagger, *"Supersymmetry and supergravity"*, Princeton, USA: Princeton University Press (1992).

[105] I. Jack & D. R. T. Jones, *"Ultraviolet finite non-commutative theories"*, Phys. Lett. B **514** (2001) 401 [arXiv:hep-th/0105221].

[106] I. Jack & D. R. T. Jones, *"Ultraviolet finiteness in non-commutative supersymmetric theories"*, New J. Phys. **3** (2001) 19 [arXiv:hep-th/0109195].

[107] C. S. Chu & P. M. Ho, *"Non-commutative open string and D-brane"*, Nucl. Phys. B **550** (1999) 151 [arXiv:hep-th/9812219].

[108] M. M. Sheikh-Jabbari, *"Open strings in a B-field background as electric dipoles"*, Phys. Lett. B **455** (1999) 129 [arXiv:hep-th/9901080].

[109] D. Bigatti & L. Susskind, *"Magnetic fields, branes and non-commutative geometry"*, Phys. Rev. D **62** (2000) 066004 [arXiv:hep-th/9908056].

[110] Z. Yin, *"A note on space non-commutativity"*, Phys. Lett. B **466** (1999) 234 [arXiv:hep-th/9908152].

[111] L. Álvarez-Gaumé & J. L. F. Barbón, *"Non-linear vacuum phenomena in non-commutative QED"*, Int. J. Mod. Phys. A **16** (2001) 1123 [arXiv:hep-th/0006209].

[112] T. Yoneya, *"String theory and space-time uncertainty principle"*, Prog. Theor. Phys. **103** (2000) 1081 [arXiv:hep-th/0004074].

[113] A. I. Vainshtein, V. I. Zakharov, V. A. Novikov & M. A. Shifman, *"ABC of instantons"*, Sov. Phys. Usp. **24** (1982) 195 [Usp. Fiz. Nauk **136** (1982) 553].

[114] A. A. Belavin, A. M. Polyakov, A. S. Shvarts & Y. S. Tyupkin, *"Pseudoparticle solutions of the Yang–Mills equations"*, Phys. Lett. B **59** (1975) 85.

[115] E. Corrigan, D. B. Fairlie, S. Templeton & P. Goddard, *"A Green's function for the general self-dual gauge field"*, Nucl. Phys. B **140** (1978) 31.

[116] N. H. Christ, E. J. Weinberg & N. K. Stanton, *"General self-dual Yang–Mills solutions"*, Phys. Rev. D **18** (1978) 2013.

[117] E. Corrigan, P. Goddard, H. Osborn & S. Templeton, *"Zeta function regularization and multi-instanton determinants"*, Nucl. Phys. B **159** (1979) 469.

[118] H. Osborn, *"Semiclassical functional integrals for self-dual gauge fields"*, Annals Phys. **135** (1981) 373.

[119] V. V. Khoze, M. P. Mattis & M. J. Slater, *"The instanton hunter's guide to supersymmetric $SU(N)$ gauge theory"*, Nucl. Phys. B **536** (1998) 69 [arXiv:hep-th/9804009].

[120] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis & S. Vandoren, *"Multi-instanton calculus and the AdS/CFT correspondence in $\mathcal{N} = 4$ superconformal field theory"*, Nucl. Phys. B **552** (1999) 88 [arXiv:hep-th/9901128].

[121] H. Nakajima, *"Instantons and affine Lie algebras"*, Nucl. Phys. Proc. Suppl. **46** (1996) 154 [arXiv:alg-geom/9510003].

[122] J. Gomis & T. Mehen, *"Space-time non-commutative field theories and unitarity"*, Nucl. Phys. B **591** (2000) 265 [arXiv:hep-th/0005129].

[123] L. Álvarez-Gaumé, J. L. F. Barbón & R. Zwicky, *"Remarks on time-space non-commutative field theories"*, JHEP **0105** (2001) 057 [arXiv:hep-th/0103069].

[124] M. Nakahara, *"Geometry, topology and physics"*, Bristol, UK: Hilger (1990).

[125] G. 't Hooft, *"Computation of the quantum effects due to a four-dimensional pseudoparticle"*, Phys. Rev. D **14** (1976) 3432 [Erratum, ibid. D **18** (1978) 2199].

[126] C. K. Lee, H. W. Lee & P. Y. Pac, *"Calculation of one-loop instanton determinants using propagators with space-time dependent mass"*, Nucl. Phys. B **201** (1982) 429.

[127] N. Ishibashi, S. Iso, H. Kawai & Y. Kitazawa, *"Wilson loops in non-commutative Yang–Mills"*, Nucl. Phys. B **573** (2000) 573 [arXiv:hep-th/9910004].

[128] S. R. Das & S. J. Rey, *"Open Wilson lines in non-commutative gauge theory and tomography of holographic dual supergravity"*, Nucl. Phys. B **590** (2000) 453 [arXiv:hep-th/0008042].

[129] D. J. Gross, A. Hashimoto & N. Itzhaki, *"Observables of non-commutative gauge theories"*, Adv. Theor. Math. Phys. **4** (2000) 893 [arXiv:hep-th/0008075].

[130] L. Álvarez-Gaumé & J. L. F. Barbón, *"Morita duality and large-N limits"*, Nucl. Phys. B **623** (2002) 165 [arXiv:hep-th/0109176].

[131] I. M. Gelfand & G. E. Schilow, *"Verallgemeinerte Funktionen (Distributionen)"*, Deutscher Verlag der Wissenschaft (1953).

[132] F. Constantinescu, *"Distributionen und ihre Anwendung in der Physik"*, Teubner Verlag (1974).

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"Perturbative analysis of the Seiberg–Witten map",
Int. J. Mod. Phys. A **17** (2002) 2219 [arXiv:hep-th/0102044]

4. A. A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda & R. Wulkenhaar,
"Deformed QED via Seiberg–Witten map",
arXiv:hep-th/0102103

5. A. A. Bichl, J. M. Grimstrup, H. Grosse, L. Popp, M. Schweda & R. Wulkenhaar,
"Renormalization of the non-commutative photon self-energy to all orders via Seiberg–Witten map", JHEP **0106** (2001) 013 [arXiv:hep-th/0104097]

6. A. A. Bichl, J. M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda & R. Wulkenhaar, *"Non-commutative Lorentz symmetry and the origin of the Seiberg–Witten map"*, Eur. Phys. J. C **24** (2002) 165 [arXiv:hep-th/0108045]

7. A. A. Bichl, M. Ertl, A. Gerhold, J. M. Grimstrup, H. Grosse, L. Popp, V. Putz, M. Schweda & R. Wulkenhaar, *"Non-commutative* $U(1)$ *super-Yang–Mills theory: Perturbative self-energy corrections"*, arXiv:hep-th/0203141

8. A. A. Bichl,
"A note on UV/IR mixing and non-commutative instanton calculus",
arXiv:hep-th/0312230

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