DISSERTATION

On two models for charged particle systems: The cometary flow equation and the Burgers-Poisson system.

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines Doktors der technischen Wissenschaften unter der Leitung von

Ao. Univ. Prof. Dr. techn. Christian Schmeiser
E 115/1
Institut für Angewandte und Numerische Mathematik

eingereicht an der Technischen Universität Wien
Fakultät für Technische Naturwissenschaften und Informatik

von

Dipl-Ing. Klemens Andreas Fellner
Matr. Nr. 9225574
Fernkorngasse 17/4/16
A-1100 Wien

Wien, am 10. Oktober 2002

Acknowledgment

I would like to thank my parents Friederike and Karl Fellner.

Further I wish to express my thanks and my deep respect to my advisor Christian Schmeiser. He taught me more than he knows.

I also want to thank Prof. Peter Markowich and Prof. Frederic Poupaud for the honor to work with them and to learn from them.

Thanx to God, Thomas, Stefan, Hannes, Ava, Sarah, Pauli, Franz, Prof. Kluwick, Prof. Mauser, Prof. Steindl, Prof. Szmolyan, Prof. Herfort, Rita, Christof, Yasmin, Astrid, Nikola, Martin, Ulla, P. Weikinger, Natasa, Margit, Erika Schöck, Fr. Jakob, Astrid, Thomas, David, Bernhard, Rudi, Günther, Stefan, Kati, Christa, Dorfner, KBE, Thomas, Erika, Franz, Otto Hederer, Robert, Andi, Othmar, Frederic, Wolfgang, Leo, Robert, Andres, Michael Werba, Paul Schürz, Johannes and Bojidara.

This work has been supported by the Austrian Science Foundation, grant nos. W008 "Wissenschaftskolleg Differential Equation Model in Science and Engineering" and P14876 "Fokker-Plank and Mean-Field Equations", by the *"Wittgenstein Award"* of Peter A. Markowich, by the Austrian government START project "Nonlinear Schrödinger and Quantum Boltzmann Equations" (FWF Y-137-TEC) of Norbert Mauser and by the TMR-project "Asymptotic Methods in Kinetic Theory".

Kurzfassung

Diese Arbeit beschäftigt sich mit zwei stark vereinfachten Modellgleichungen von Systemen geladener Teilchen.

Kapitel 1 behandelt ein Modell für Plasma in einem Kometenschweif:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q_{\mathbf{u}_f}(f) = P_{\mathbf{u}_f}(f) - f \tag{1}$$

Die kinetische Beschreibung der Teilchenverteilungsfunktion $f(t, \mathbf{x}, \mathbf{v})$ mit den unabhängigen Variablen Zeit, Ort und Geschwindigkeit erlaubt die Einbeziehung von mikroskopischen Streuprozessen. Der konkrete Term $Q_{\mathbf{u}_f}(f)$ ist eine Relaxationszeitapproximation komplizierter Streuintegrale [Stix], [EJM], die die Ablenkung von Teilchen aufgrund von zufälligen Unregelmäßigkeiten des umgebenden Magnetfeldes modellieren. Der nichtlineare Projektionsoperator (S^{d-1} bezeichnet die Einheitskugel in $\mathbb{R}^d$)

$$P_{\mathbf{u}_f}(f)(\mathbf{v}) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(\mathbf{u}_f + |\mathbf{v} - \mathbf{u}_f| \boldsymbol{\omega}) \, d\boldsymbol{\omega} \,,$$

bildet auf Verteilungsfunktionen ab, deren Geschwindigkeitsverteilung isotrop um die mittlere makroskopische Geschwindigkeit der Teilchen ist:

$$\mathbf{u}_f = \left(\int_{\mathbb{R}^d} \mathbf{v} f \, d\mathbf{v} \right) \Big/ \left(\int_{\mathbb{R}^d} f \, d\mathbf{v} \right) .$$

Ein verwandtes Modell stellt die BGK-Näherung der Boltzmann-Gleichung für die Dynamik von verdünnten Gasen dar [Per], [PP], [D].

Die erzielten Resultate basieren unter anderem auf [DP], [DLP], wo formale Eigenschaften und der formale hydrodynamische Limes von (1) zu den Euler-Gleichungen für ideale Gase gezeigt wurden. Die Existenz von schwachen Lösungen von (1) wurde in [DLPS] für solche Anfangsdaten gezeigt, die das Auftreten von Vakuum verhindern.

In Abschnitt 1.3 wird zunächst das Existenzresultat auf milde Lösungen mit möglichen Vakuumsanteilen und auf allgemeinere Anfangsverteilungen erweitert. Die Beweise sind für (1) auf einem beschränkten Gebiet mit glattem Rand ausgeführt, lassen sich jedoch direkt auf den Ganzraum $\mathbb{R}^d$ übertragen. Weiters werden lokale bzw. globale Erhaltungssätze für Masse, Impuls und Energie sowie Entropiedissipationsgleichungen für Anfangsverteilungen gezeigt, bei denen die auftretenden Flussterme wohldefiniert sind. Zentrale Beweismethoden stellen "Averaging Lemmata" [GLPS] für starke Konvergenz von Momenten schwach konvergierender Lösungsfolgen sowie die Lemmata von Egoroff und Gronwall dar. Abschnitt 1.4 zeigt die Konvergenz der Lösung auf kompakten, gegen unendlich verschobenen Zeitintervallen gegen eine Equilibriumsverteilung, d. h. eine Verteilungsfunktion die sowohl die linke als auch die rechte Seite von (1) annulliert. Dieses Resultat wird ergänzt durch

die Konstruktion aller glatten Equilibriumsverteilungen (Abschnitt 1.5) und deren Klassifizierung anhand ihrer geometrischen Struktur (Abschnitt 1.6). Abschnitt 1.7 diskutiert die Einschränkungen dieser Struktur, wenn zusätzlich reflektierende Randbedingungen erfüllt werden müssen. Im abschließenden Abschnitt 1.8 werden aufgrund der formalen Verbindung zu den Euler-Gleichungen für ideale Gase explizite Lösungen ebendieser konstruiert.

Das Thema des Kapitels 2 ist ein nichtlineares, dispersives Modellproblem

$$u_t + uu_x = \varphi_x, \tag{2}$$

$$\varphi_{xx} = \varphi + u, \tag{3}$$

(genannt Burgers-Poisson (BP)-System) für die Dichten u und φ abhängig von den skalaren Variablen Zeit t und Ort x. Man kann (2), (3) als Studienmodell von Zwei-Teilchen-Euler-Poisson-Systemen in der Plasmaphysik ansehen. Andererseits, wenn die Poissongleichung (3) mittels Green'scher Funktion gelöst und in die Burgers-Gleichung (2) eingesetzt wird, ergibt sich ein skalarer Erhaltungssatz, der auch als Modell von Flachwasserwellen auftritt [Whi, Kapitel 13.14].

Abschnitt 2.2 zählt zunächst formale Eigenschaften des BP-Systems auf: Die Erhaltung der Masse $\int_{\mathbb{R}} u\,dx$, die Entropie $\int_{\mathbb{R}} u^2\,dx$, die Hamilton-Struktur sowie die Galilei-Invarianz und die Verwandtschaft mit den Gleichungen von Camassa-Holm [CH] und Benjamin-Ono [Ben], [Ono]. In Abschnitt 2.3 werden die Wandernde-Wellen-Lösungen von (2), (3) konstruiert, die starke Ähnlichkeit zu jenen von Zwei-Teilchen-Euler-Poisson Systemen zeigen [CDMS1], [CDMS2]. In Abschnitt 2.4 wird zunächst die Existenz und Eindeutigkeit von lokalen, glatten Lösungen für Anfangsdaten in $H^k(\mathbb{R})$ mit $k > \frac{3}{2}$ gezeigt. Des weiteren wird die Existenz von globalen, schwachen Entropielösungen für Anfangsdaten mit beschränkter Variation bewiesen. Im letzten Abschnitt 2.5 wird der Limes $\varepsilon \to 0$ in BP-system mit umskalierter Poissongleichung

$$\varepsilon^2 \varphi^\varepsilon_{xx} = \varphi^\varepsilon + u^\varepsilon$$

betrachtet. Eine Chapman-Enskog-Entwicklung ergibt als Grenzgleichung eine Burgersgleichung mit dispersivem Störterm. Die Korteweg-deVries-Gleichung tritt als Limes eines räumlich und zeitlich umskalierten BP-Systems auf. Solange die Lösung der Grenzgleichung glatt bleibt, kann die Konvergenz der Lösungen im Grenzübergang $\varepsilon \to 0$ gezeigt werden.

Contents

Chapter 1

The Cometary Flow Equation

1.1 Introduction

We investigate the following kinetic equation (called *cometary flow equation*)

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = Q_{\mathbf{u}_f}(f) = P_{\mathbf{u}_f}(f) - f\,, \tag{1.1}$$

where $f(t,\mathbf{x},\mathbf{v})$ denotes a nonnegative particle distribution function, depending on time $t > 0$, on position $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ ($d \in \mathbb{N}$, Ω is either $\mathbb{R}^d$ or a bounded domain with piecewise C^1 boundary), and on velocity $\mathbf{v} \in \mathbb{R}^d$. The collision operator $Q_{\mathbf{u}_f}$ is used in quasi-linear plasma theory as a simplified model for wave-particle interaction in cometary flows. The map $P_{\mathbf{u}}$ is a projection onto the set of distribution functions isotropic around the mean velocity $\mathbf{u}$:

$$P_{\mathbf{u}}(f)(\mathbf{v}) = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(\mathbf{u} + |\mathbf{v}-\mathbf{u}|\boldsymbol{\omega})\,d\boldsymbol{\omega}\,,$$

with S^{d-1} and $|S^{d-1}|$ denoting the unit sphere in $\mathbb{R}^d$ and its $(d-1)$-dimensional Lebesgue measure, respectively. By $\mathbf{u}_f(t,\mathbf{x})$ we denote the mean velocity of the distribution function f, which is defined as the fraction of the momentum density $\mathbf{m}_f(t,\mathbf{x})$ through the mass density $\rho_f(t,\mathbf{x})$:

$$\rho_f = \int_{\mathbb{R}^3} f\,d\mathbf{v}, \qquad \mathbf{m}_f = \rho_f\mathbf{u}_f = \int_{\mathbb{R}^3} \mathbf{v}f\,d\mathbf{v}, \qquad \rho_f e_f = \int_{\mathbb{R}^3} \frac{|\mathbf{v}-\mathbf{u}_f|^2}{2} f\,d\mathbf{v}. \tag{1.2}$$

In the last therm of (1.2), $e_f(t,\mathbf{x})$ denotes the specific internal energy. The kinetic equation (1.1) is considered subject to initial conditions

$$f(0,\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v})\,, \tag{1.3}$$

for $(\mathbf{x},\mathbf{v}) \in \Omega \times \mathbb{R}^d$. For the *initial data* we shall use at least the following assumptions:

$$\exists\, p > 1 : \quad f_0 \in L^p(\Omega \times \mathbb{R}^d)\,, \quad f_0 \geq 0\,, \tag{1.4}$$

$$\exists\, r > 1 : \quad (1 + |\mathbf{v}|^r) f_0 \in L^1(\Omega \times \mathbb{R}^d)\,, \tag{1.5}$$

while for a well-defined internal energy $r \geq 2$ is required. We impose reflecting *boundary conditions*

$$f(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x}, \mathbf{v}'), \tag{1.6}$$

for $t > 0$, $\mathbf{x} \in \partial\Omega$, with specular or reverse reflection, i.e.,

$$\text{a) } \mathbf{v}' = \mathbf{v} - 2(\mathbf{n}(\mathbf{x})\cdot\mathbf{v})\mathbf{n}(\mathbf{x}), \qquad \text{or b) } \mathbf{v}' = -\mathbf{v}, \tag{1.7}$$

where $\mathbf{n}(\mathbf{x})$ denotes a unit normal along $\partial\Omega$.

For the physics modeled by (1.1), we refer, for example, to [EJM], [WJ1], [WJ2] and [WSJG]. Let us only remark that the collision operator describes the scattering of cosmic rays (energetic particles) against irregularities of the ambient magnetic field in an astrophysical plasma [EJM]. These irregularities are usually caused by plasma electromagnetic turbulence which generates a random spectrum of waves and the resulting collision operator is thus referred to as a waves-particle collision operator. The quasi-linear theory of plasma [Stix] provides complex expressions of such operators. Here, we shall consider a simplified relaxation time model, comparable to the BGK model of gas dynamics [Per], [PP]. We are treating a dimensionless version where the relaxation time has been chosen as reference time.

A mathematical treatment of (1.1) has been started in [DP] and [DLP], where (among other macroscopic limits) the hydrodynamic limit (i.e. the limit $\varepsilon \to 0$ after rescaling the variable $t \to t/\varepsilon$ and $\mathbf{x} \to \mathbf{x}/\varepsilon$) has been computed formally. They showed that the distribution functions f_ε converge in the limit towards a function $g(t, \mathbf{x}, |\mathbf{v} - \mathbf{u}|^2/2)$ with the remarkable property that the density ρ_g and the internal energy e_g together with $\mathbf{u}$ satisfy the Euler equations for ideal gases. In [DLPS], the whole space problem ($\Omega = \mathbb{R}^d$) was considered and existence of weak solutions of the initial value problem is proven. Also, macroscopic conservation laws, an entropy dissipation equality, and the propagation of higher order moments is shown. For the initial data, several strong assumptions are used in [DLPS], such as boundedness, existence of moments up to the second order and, most importantly, a positivity assumption guaranteeing that vacuum is avoided.

In this chapter, we state firstly in section 1.2 *Preliminaries*, several previous results on the cometary flow equation. In section 1.3 *Existence and Conservation Laws*, an existence theorem under milder assumptions than in [DLPS] is proven. In particular, the occurrence of vacuum is allowed. Also the result is stronger compared to that in [DLPS] in the sense that weak solutions are shown to be mild solutions. Although the proofs are carried out for bounded position domains with reflective boundaries, our results can be directly transfered to the whole space problem. Moreover in section 1.3, we prove results on the propagation of moments and on the validity of macroscopic conservation laws, formally derived in section 1.2.

In section 1.4 *Convergence to equilibrium*, we prove that, on a compact time interval shifted to infinity, the solution of (1.1)–(1.6) converges to an equilibrium distribution satisfying the boundary conditions (1.6). Equilibrium distributions of the cometary flow equation are functions, which annihilate separately both sides of (1.1), i.e. satisfy the free-streaming equation and lie in the null-space of the collision operator $Q_{\mathbf{u}_f}(f)$.

This result is complemented in section 1.5 *Smooth equilibrium solutions* by the computation of all smooth equilibrium solutions of (1.1). Assuming differentiability, we show, that the equilibria of the cometary flow equation are generalizations of the Maxwellian equilibria of the gas dynamics Boltzmann equation [D], [Cer] in the sense that they involve an arbitrary smooth function ψ instead of the exponential function of the Maxwellians. Contrary to the Boltzmann equation, where Desvillettes [D] showed that the arbitrarily large velocities components of the Maxwell distributions prohibit vacuum, the equilibria of the cometary flow equation with compactly supported ψ may include vacuum regions.

In the section 1.6 *Classification*, we classify all smooth equilibria in $\mathbb{R}^3$ by distinguishing four types according to different geometrical appearances. Symmetry transformations are exploited to state simplified normal forms of each type. For each type we discuss the behavior of the solutions especially with respect to possible vacuum parts.

In section 1.7 *Equilibria touching reflective boundaries*, we identify firstly all smooth equilibria on a bounded domain which satisfy the boundary conditions (1.6) in a non-vacuum way. Secondly, we investigate equilibria where non-vacuum parts are only locally in contact the boundaries.

It is a consequence of the hydrodynamic limit of the cometary flow equation that solutions of the Euler equations for ideal gases can be obtained as moments (density, mean velocity and internal energy) of any cometary flow equilibrium distribution. In section 1.8 *Explicit solutions of 3D Euler equations*, we construct explicit solutions of the 3D Euler equations for ideal gases and discuss them using the previous results.

1.2 Preliminaries

First, we collect some formal properties of the linear and nonlinear collision operator (see, e.g., [DLPS]).

Lemma 1 ***(Linear collision operator)*** *For arbitrary* $\mathbf{u} \in \mathbb{R}^d$, f, $g \in \mathcal{D}(\mathbb{R}^d)$, $\varphi \in C^\infty([0,\infty))$,

(i) $\varphi(|\mathbf{v}-\mathbf{u}|)$ *is a collision invariant of* $Q_{\mathbf{u}}$*:*

$$\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f)(\mathbf{v})\varphi(|\mathbf{v}-\mathbf{u}|)\,d\mathbf{v} = 0, \tag{1.8}$$

(ii) $Q_{\mathbf{u}}$ *is symmetric with respect to the* $L^2(\mathbb{R}^d)$*-inner product:*

$$\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f)g\,d\mathbf{v} = -\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f)Q_{\mathbf{u}}(g)\,d\mathbf{v} = \int_{\mathbb{R}^d} fQ_{\mathbf{u}}(g)\,d\mathbf{v} \tag{1.9}$$

(iii) $P_{\mathbf{u}}$ has the monotonicity property

$$a \le f(\mathbf{v}) \le b \quad \Longrightarrow \quad a \le P_{\mathbf{u}}(f)(\mathbf{v}) \le b. \tag{1.10}$$

Lemma 2 ***(Nonlinear collision operator)*** *For arbitrary $f \in \mathcal{D}(\mathbb{R}^d)$ with $\rho_f > 0$, $\varphi \in C^\infty([0,\infty))$,*

(i) $\varphi(|\mathbf{v} - \mathbf{u}_f|)$ and $\mathbf{v}$ are collision invariants of $Q_{\mathbf{u}_f}$:

$$\int_{\mathbb{R}^d} Q_{\mathbf{u}_f}(f)(\mathbf{v})\varphi(|\mathbf{v}-\mathbf{u}_f|)\, d\mathbf{v} = \int_{\mathbb{R}^d} Q_{\mathbf{u}_f}(f)(\mathbf{v})\mathbf{v}\, d\mathbf{v} = 0, \tag{1.11}$$

(ii) Equilibrium: $Q_{\mathbf{u}_f}(f) = 0$, if and only if there exist $\mathbf{u} \in \mathbb{R}^d$ and $F \in C_0^\infty([0,\infty))$, such that $f(\mathbf{v}) = F(|\mathbf{v}-\mathbf{u}|^2/2)$.

(iii) the following H-theorem holds: for monotonically increasing χ,

$$\int_{\mathbb{R}^d} Q_{\mathbf{u}}(f)\chi(f)\, d\mathbf{v} = -\int_{\mathbb{R}^d} [f - P_{\mathbf{u}}(f)]\,[\chi(f) - \chi(P_{\mathbf{u}}(f))]\, d\mathbf{v} \le 0\,.$$

The special choice $\chi = pf^{p-1}$, $p > 1$, leads to the entropy dissipation equality

$$\partial_t \int_\Omega \int_{\mathbb{R}^d} f^p\, d\mathbf{v}d\mathbf{x} = -p\int_\Omega \int_{\mathbb{R}^d} [f - P_{\mathbf{u}_f}(f)]\,[f^{p-1} - P_{\mathbf{u}}(f)^{p-1}]\, d\mathbf{v}d\mathbf{x}\,, \tag{1.12}$$

playing a central role in our study of the convergence to equilibrium below.

Remark 1 *The statements of the above lemmata can be extended for less regular functions by density arguments, whenever the involved integrals are well defined. This is the way those results and similar lemmata will be used in the following.*

Since the collision invariants of the form $\varphi(|\mathbf{v} - \mathbf{u}_f|)$ depend non-locally on the distribution function they do not lead to conservation laws. The only f-independent collision invariants of $Q_{\mathbf{u}_f}$ are linear combinations of 1, the components of $\mathbf{v}$, and $|\mathbf{v}|^2$ since $|\mathbf{v}|^2 = |\mathbf{v}-\mathbf{u}_f|^2 - |\mathbf{u}_f|^2 + 2\mathbf{u}_f\cdot\mathbf{v}$. Local conservation laws (for mass ρ_f, momentum $\mathbf{m}_f$, and energy $E_f = \int_{\mathbb{R}^d} \frac{|\mathbf{v}|^2}{2} f\, d\mathbf{v}$) are only produced by these:

$$\partial_t \begin{pmatrix} \rho_f \\ \mathbf{m}_f \\ E_f \end{pmatrix} + \nabla_{\mathbf{x}}\cdot \int_{R^d} \begin{pmatrix} \mathbf{v} \\ \mathbf{v}\otimes\mathbf{v} \\ \mathbf{v}|\mathbf{v}|^2/2 \end{pmatrix} f\, d\mathbf{v} = 0\,. \tag{1.13}$$

Note that, by the symmetry of the momentum flux tensor, we also have conservation of the $d(d-1)/2$ components of angular momentum:

$$\partial_t \int_{\mathbb{R}^d} (x_i v_j - x_j v_i) f \, d\mathbf{v} + \nabla_{\mathbf{x}} \cdot \int_{R^d} \mathbf{v}(x_i v_j - x_j v_i) f \, d\mathbf{v} = 0 \,, \tag{1.14}$$

$1 \leq i < j \leq d$. For the determination of globally conserved quantities in (1.1)–(1.6), we have to consider the effect of the reflexive boundary. The boundary conditions (1.6) conserve mass and energy such that these quantities are globally conserved:

$$\int_\Omega \begin{pmatrix} \rho_f \\ E_f \end{pmatrix} d\mathbf{x} = \int_\Omega \begin{pmatrix} \rho_{f_0} \\ E_{f_0} \end{pmatrix} d\mathbf{x} \,. \tag{1.15}$$

For reverse reflexive boundaries (1.7) b) no other conserved quantities are known. In the case of specular reflection (1.7) a) the component of angular momentum corresponding to the index pair (i,j) is globally conserved, if Ω has the corresponding rotational symmetry, i.e., for $(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_d) \in \Omega$, all points $(x_1,\ldots,\widetilde{x}_i,\ldots,\widetilde{x}_j,\ldots,x_d)$, $\widetilde{x}_i = \sqrt{x_i^2+x_j^2}\cos\varphi$, $\widetilde{x}_j = \sqrt{x_i^2+x_j^2}\sin\varphi$ and $\varphi \in \mathbb{R}$, also belong to Ω. Then, for the flux of angular momentum through the boundary $\partial\Omega$ we have

$$\int_{\partial\Omega}\int_{\mathbb{R}^d} (\mathbf{n}\cdot\mathbf{v})(x_i v_j - x_j v_i) f \, d\mathbf{v} d\sigma = 2\int_{\partial\Omega} (x_i n_j - x_j n_i) \int_{\mathbb{R}^d} (\mathbf{n}\cdot\mathbf{v})^2 f \, d\mathbf{v} d\sigma = 0 \,,$$

since $x_i n_j - x_j n_i = 0$ in the rotationally symmetric case. Consequently,

$$\int_\Omega \int_{\mathbb{R}^d} (x_i v_j - x_j v_i) f \, d\mathbf{v} d\mathbf{x} = \int_\Omega \int_{\mathbb{R}^d} (x_i v_j - x_j v_i) f_0 \, d\mathbf{v} d\mathbf{x} \,. \tag{1.16}$$

More generally, every $(d-2)$-dimensional affine space in $\mathbb{R}^d$ can serve as 'rotation axis' instead of the subspace $\{x_i = x_j = 0\}$. Summarizing, the number of globally conserved quantities in (1.1)–(1.6) is between 2 and $2 + d(d-1)/2$ (the latter, when Ω is a ball with specularly reflecting boundary).

In the existence analysis, we shall use continuity properties of the collision operator (derived in [DLPS]):

Lemma 3 ***(Continuity and stability of the projection operator)*** *Let $\tau > 0$, $1 \leq p, q \leq \infty$, $f \in L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))$. Then,*

$$\textit{(Cont.)} \quad \|P_{\mathbf{u}}(f)\|_{L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))} \leq \|f\|_{L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d))} \,.$$

Let $1 \leq p, q < \infty$, $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{u}$ in $L^1((0,\tau)\times\Omega)^d$. Then,

$$\textit{(Stab.)} \qquad \lim_{n\to\infty} P_{\mathbf{u}_n}(f) = P_{\mathbf{u}}(f) \quad \textit{in } L^q((0,\tau);\ L^p(\Omega\times\mathbb{R}^d)) \,.$$

To remove the restriction of positive densities as in [DLPS], we extend the definition of the nonlinear projection operator.

Definition 1 ***(Extended projection operator)*** *Let f satisfy (1.4), (1.5). Then $Q(f)$ is defined as $P(f) - f$ with*

$$P(f) = \begin{cases} P_{\mathbf{u}_f}(f) & \text{for} \quad \rho_f > 0 \\ 0 & \text{for} \quad \rho_f = 0 \end{cases} \tag{1.17}$$

The pointwise definition of $P(f)$ is obviously also well-defined in the L^p sense. Although $\mathbf{u}_f$ is unbounded for $\rho_f \to 0$, the following lemma ensures the integrability of the extended projection operator.

Lemma 4 *Let $f \in L^p(\mathbb{R}^d)$ with $1 < p < \infty$. Then,*

$$\lim_{|\mathbf{u}|\to\infty} \|P_{\mathbf{u}}(f)\|_p = 0 \tag{1.18}$$

Proof: Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ be supported within a ball $B(\mathbf{v}_o, R)$. Then,

$$P_{\mathbf{u}}(\varphi)(\mathbf{v}) \leq \frac{\|\varphi\|_\infty}{|S|} \int_{S\cap B} d\boldsymbol{\omega} = O\left(\left(\frac{2R}{|\mathbf{v}_o - \mathbf{u}|}\right)^2\right) \quad \text{as } |\mathbf{v}_o - \mathbf{u}| \to \infty \tag{1.19}$$

implies $\lim_{|\mathbf{u}|\to\infty} \|P_{\mathbf{u}}(\varphi)\|_\infty = 0$. Since $\|P_{\mathbf{u}}(\varphi)\|_1 = \|\varphi\|_1$ for all $\mathbf{u} \in \mathbb{R}^d$, we know (1.18) for all testfunctions φ and the proof is completed by a density argument. $\square$

Lemma 5 *Let $f \in L^q((0,\tau);\ L^p(\Omega \times \mathbb{R}^d))$ for $1 < p, q < \infty$. Then, the continuity property Lemma 3 (Cont.) remains true for P (instead of $P_{\mathbf{u}}$). Also the statements of the lemmata 1 and 2 obviously remain true for Q (instead of $Q_{\mathbf{u}}$ and $Q_{\mathbf{u}_f}$).*

In the existence analysis below, we use the semigroup $T(t)$ generated by the free streaming equation

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = 0\,,$$

subject to the initial data (1.3) and the boundary conditions (1.6). As shown in [CIP], [BP], the unique solution $f(t, \mathbf{x}, \mathbf{v}) = (T(t)f_0)(\mathbf{x}, \mathbf{v})$ remains positive for positive initial data f_0 and satisfies $\|f(t, \mathbf{x}, \mathbf{v})\|_p \leq \|f_0(\mathbf{x}, \mathbf{v}\|_p$ for $1 \leq p \leq \infty$. For $1 \leq p < \infty$, the solution operator $T(t)$ is a strongly continuous positivity preserving contraction semigroup on $L^p(\Omega \times \mathbb{R}^d)$.

A mild formulation of (1.1)–(1.6) is then given by the Duhamel formula

$$f(t) = e^{-t}T(t)f_0 + \int_0^t e^{s-t}T(t-s)P_{\mathbf{u}_f}(f)(s)\,ds\,. \tag{1.20}$$

Note that solutions of (1.20) are also weak solutions on finite time intervals $(0, \tau)$ in the sense that [BP]

$$\begin{aligned} &\int_0^\tau \int_{\Omega\times\mathbb{R}^d} f(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\,d\mathbf{v}d\mathbf{x}dt + \int_\Omega \int_{\mathbb{R}^d} f_0\varphi(t=0)\,d\mathbf{v}d\mathbf{x} \\ &= \int_0^\tau \int_{\Omega\times\mathbb{R}^d} fQ_{\mathbf{u}_f}(\varphi)\,d\mathbf{v}d\mathbf{x}dt\,, \end{aligned} \tag{1.21}$$

for all testfunctions $\varphi \in \mathcal{D}_\tau$ which is defined as the set

$$\mathcal{D}_\tau = \left\{ \begin{array}{l} \varphi \in C_0^\infty([0,\tau) \times \overline{\Omega} \times \mathbb{R}^d), \\ \varphi \text{ satisfies the boundary conditions (1.6).} \end{array} \right\}$$

We remark that since the boundary conditions are included in the set of testfunctions, we avoid the question of traces of the weak solutions [CIP, section 9.2].

1.3 Existence and conservation laws

The existence proof follows the approach of [DLPS] extended by a final step where solutions with vacuum regions are allowed. As a first step we solve a linearized problem.

Lemma 6 ***(Existence of the linear problem)*** *Let (1.4) hold and* $\mathbf{u} \in L^\infty((0,\infty) \times \Omega)^3$ *be given. Then*

$$f(t) = e^{-t}T(t)f_0 + \int_0^t e^{s-t}T(t-s)P_{\mathbf{u}}(f)(s)\,ds \tag{1.22}$$

has a unique nonnegative solution $f \in C([0,\infty);\ L^p(\Omega \times \mathbb{R}^d))$ *satisfying*

$$\|f(t)\|_{L^p(\Omega\times\mathbb{R}^d)} \leq \|f_0\|_{L^p(\Omega\times\mathbb{R}^d)}\,. \tag{1.23}$$

Proof: Existence and uniqueness are the consequence of a standard contraction argument using the iteration

$$\begin{array}{c} g_1(t,\mathbf{x},\mathbf{v}) = e^{-t}(T(t)f_0)(\mathbf{x},\mathbf{v}) \\ g_k(t) = e^{-t}T(t)f_0 + \int_0^t e^{-t+s}T(t-s)P_{\mathbf{u}}(g_{k-1}(s))\,ds, \end{array}$$

where $T(t)$ denotes the solution operator of the free-streaming equation. The iteration process ensures the nonnegativity. The estimate (1.23) follows from the contractivity $\|T(t)\| \leq 1$, Lemma 3 (Cont.), and an application of the Gronwall lemma. Continuity in t is a straightforward consequence of (1.22) and of Lemma 3. $\square$

The next result is concerned with the propagation of moments in the linear problem.

Lemma 7 ***(Propagation of moments in the linear problem)*** *Let the assumptions of Lemma 6 and (1.5) hold. Then, for the solution of (1.22),* $(1+|\mathbf{v}|^r)f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega \times \mathbb{R}^d))$ *holds.*

Proof: In the weak formulation

$$\int_D f\,(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\,d\mathbf{v}d\mathbf{x}dt + \int_\Omega\int_{\mathbb{R}^d} f_0\varphi(t=0)\,d\mathbf{v}d\mathbf{x} \tag{1.24}$$
$$= \int_D Q_{\mathbf{u}}(f)\varphi\,d\mathbf{v}d\mathbf{x}dt\,,$$

of (1.22) we choose as a test function $\varphi(t,\mathbf{x},\mathbf{v}) = \theta(t)\Phi(|\mathbf{v}|^2/V)(1+|\mathbf{v}|^r) \in \mathcal{D}_\tau$ with $\theta \in C_0^\infty([0,\tau))$, $\Phi \in C_0^\infty([0,\infty))$, $\Phi(y) = 1$ for $y < 1$, $V > 0$. Using $(a+b)^r \le c_r(a^r + b^r)$, we estimate

$$\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)P_{\mathbf{u}}(f)\,d\mathbf{v} \le c_r\int_{\mathbb{R}^d}(1+|\mathbf{v}-\mathbf{u}|^r + |\mathbf{u}|^r)P_{\mathbf{u}}(f)\,d\mathbf{v} \tag{1.25}$$
$$= c_r\int_{\mathbb{R}^d}(1+|\mathbf{v}-\mathbf{u}|^r+|\mathbf{u}|^r)f\,d\mathbf{v} \le C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r+|\mathbf{u}|^r)f\,d\mathbf{v}\,,$$

where $C_r = c_r^2 + c_r$ is a constant depending only on r. Since u is bounded, we estimate further

$$C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r+|\mathbf{u}|^r)f\,d\mathbf{v} \le c\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f\,d\mathbf{v} \tag{1.26}$$

to show that the limit of (1.24) as $V \to \infty$ implies a differential inequality of the form

$$\frac{d}{dt}\int_\Omega\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f\,d\mathbf{v}d\mathbf{x} \le c\int_\Omega\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f\,d\mathbf{v}d\mathbf{x}\,.$$

The proof of the lemma is now completed by an application of the Gronwall lemma. □

Our first result for the nonlinear problem (1.1) assumes vacuum avoiding initial data (like in [DLPS]).

Theorem 1 ***(Existence on bounded domains without vacuum)***
Let (1.4), (1.5) hold. Moreover, assume $f_0(\mathbf{x},\mathbf{v}) \ge g(|\mathbf{v}|)$ with g having strictly positive density $\rho_g \ge \gamma > 0$. Then (1.1) has a mild, global, non-negative solution $f \in C([0,\infty);\ L^p(\Omega\times\mathbb{R}^d))$ satisfying (1.23), $(1+|\mathbf{v}|^r)f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega\times\mathbb{R}^d))$, and $\mathbf{u}_f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega))$.

Proof: As in [DLPS], we introduce a velocity truncation

$$\varphi_n(\mathbf{u}) = \begin{cases} \mathbf{u} & \text{for } |\mathbf{u}| \le n, \\ n\frac{\mathbf{u}}{|\mathbf{u}|} & \text{for } |\mathbf{u}| > n, \end{cases}$$

(see also the temperature truncation in [Per] for the BGK-model). Let $\mathcal{B}_n$ be the closed ball with center at the origin and with radius n in $L^\infty((0,n)\times\Omega)^d$. Then a fixed point map $G:\ \mathcal{B}_n \to \mathcal{B}_n$ is defined in the following way: For $\mathbf{u} \in \mathcal{B}_n$, let f denote the solution of the linear problem (1.22) on the time interval $(0,n)$. By Lemma 7, $\rho_f, \mathbf{m}_f \in L^\infty((0,n);\ L^1(\Omega))$ holds. By the positivity of the semigroup $T(t)$ and above positivity assumption we have

$$f(t) \geq e^{-t}T(t)f_0 \geq e^{-t}T(t)g = e^{-t}g$$

and, thus,

$$\rho_f(t) \geq e^{-t}\gamma\,, \tag{1.27}$$

implying $\mathbf{u}_f \in L^\infty((0,n);\ L^1(\Omega))$. We define then the image $G(\mathbf{u}) := \varphi_n(\mathbf{u}_f)$. As next step we prove compactness of the moments ρ_f and $\mathbf{m}_f$. By Lemma 3 (Cont.), we estimate for the solution of the linear problem (lemma 6) that the collision operator $Q_\mathbf{u}$ is uniformly bounded on $(0,n)$ in $L^p(\Omega \times \mathbb{R}^d)$ and therefore

$$\partial_t f + \mathbf{v}\cdot\nabla_\mathbf{x} f \in L^\infty((0,n);\ L^p(\Omega \times \mathbb{R}^d))\,.$$

Moreover, we have $|\mathbf{v}|^r f \in L^1((0,n) \times \Omega \times \mathbb{R}^d)$ (lemma 7) and a velocity averaging lemma [GLPS] implies that ρ_f and $\mathbf{m}_f$ belong to a compact set in $L^1((0,n) \times \Omega)$. Since ρ_f is bounded away from zero and the truncation φ_n is continuous, G is compact with respect to the $L^1((0,n) \times \Omega)$-topology. Continuity of G is a straightforward consequence of Lemma 3. The Schauder theorem now guarantees the existence of a fixed point of G, corresponding to a solution f_n of

$$f_n(t) = e^{-t}T(t)f_0 + \int_0^t e^{s-t}T(t-s)P_{\mathbf{u}_n}(f_n)(s)\,ds\,, \tag{1.28}$$

with $\mathbf{u}_n = \varphi_n(\mathbf{u}_{f_n})$. For passing to the limit $n \to \infty$, we need uniform bounds on moments of f_n. We proceed as in the proof of Lemma 7 with u_n and f_n instead of u and f and the testfunction $\varphi(t,\mathbf{x},\mathbf{v}) = \theta(t)\Phi(|\mathbf{v}|^2/V)(1+|\mathbf{v}|^r) \in \mathcal{D}_\tau$ with $\theta \in C_0^\infty([0,\tau))$, $\Phi \in C_0^\infty([0,\infty))$, $\Phi(y) = 1$ for $y < 1$, $V > 0$. The estimate (1.25) holds unchanged:

$$\begin{aligned}\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)P_{\mathbf{u}_n}(f_n)\,d\mathbf{v} &\leq c_r\int_{\mathbb{R}^d}(1+|\mathbf{v}-\mathbf{u}_n|^r+|\mathbf{u}_n|^r)P_{\mathbf{u}_n}(f_n)\,d\mathbf{v}\\ = c_r\int_{\mathbb{R}^d}(1+|\mathbf{v}-\mathbf{u}_n|^r+|\mathbf{u}_n|^r)f_n\,d\mathbf{v} &\leq C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r+|\mathbf{u}_n|^r)f_n\,d\mathbf{v}\,,\end{aligned}$$

where $C_r = c_r^2 + c_r$ is a constants only depending on r. Only the estimate (1.26) needs to be redone without using boundedness of the velocity $\mathbf{u}_n$. Since, obviously, $|\mathbf{u}_n| \leq |\mathbf{u}_{f_n}|$ holds, we estimate

$$C_r\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r+|\mathbf{u}_n|^r)f_n\,d\mathbf{v} \leq c\int_{\mathbb{R}^d}(1+|\mathbf{v}|^r)f_n\,d\mathbf{v}\,,$$

where we have used

$$\rho_f|\mathbf{u}_f|^r \leq \int_{\mathbb{R}^d}|\mathbf{v}|^r f\,d\mathbf{v}\,,$$

an application of Jensen's inequality (for the convex function $v \to |v|^r$ with the measure f/ρ_f) which applies here and avoids the use of more elaborated controls like Perthame and Pulvirenti in [PP] for the BGK-model. As in the proof of Lemma 7, the Gronwall lemma leads to uniform-in-n bounds for moments of f_n up to the order r in $L^\infty((0,\tau);\ L^1(\Omega))$. Since, by (1.27), $\rho_{f_n}(t) \geq e^{-\tau}\gamma$ for $0 < t < \tau$, $\mathbf{u}_{f_n}$ is also bounded in $L^\infty((0,\tau);\ L^1(\Omega))$ uniformly in n. Compactness of $\mathbf{u}_{f_n}$ is deduced as above using an averaging lemma and therefore a subsequence converges strongly in $L^1((0,\tau)\times\Omega)$. The convergence in $L^1((0,\tau)\times\Omega)$ of $\mathbf{u}_n = \varphi_n(\mathbf{u}_{f_n})$ to $\mathbf{u}_f$ (where f is the weak limit of f_n) is shown as in [DLPS]. The limit $n \to \infty$ in the weak version of (1.28) (compare (1.21)) can now be carried out (applying Lemma 3), and the proof is complete. ■

The next theorem removes the positivity assumption on the density.

Theorem 2 ***(General existence on bounded domains)*** *Let (1.4), (1.5) hold. Then (1.1) (with $Q_{\mathbf{u}_f}$ replaced by Q) has a mild, global, nonnegative solution $f \in C([0,\infty);\ L^p(\Omega\times\mathbb{R}^d))$ satisfying (1.23) and $(1+|\mathbf{v}|^r)f \in L^\infty_{loc}([0,\infty);\ L^1(\Omega\times\mathbb{R}^d))$.*

Proof: For $n \in \mathbb{N}$, the modified initial data

$$f_{0n}(\mathbf{x},\mathbf{v}) = f_0(\mathbf{x},\mathbf{v}) + \frac{1}{n}e^{-|\mathbf{v}|^2}$$

satisfy the assumptions of Theorem 1 guaranteeing the existence of a weak solution f_n of (1.20) (with f_0 replaced by f_{0n}). Note that f_{0n} satisfies (1.4) and (1.5) uniformly in n.

By (1.23), f_n is bounded in $L^\infty((0,\infty);\ L^p(\Omega\times\mathbb{R}^d))$ uniformly in n and, thus, a subsequence converges weakly to a limit f. Compactness of the moments up to order 1 is deduced as above using the bounds of the moments of the solution (theorem 1) and a velocity averaging. Hence, for a subsequence we have strong convergence of $\rho_{f_n} \to \rho_f$ and $\mathbf{m}_{f_n} \to \mathbf{m}_f$ in $L^1(G)$ with $G = (0,\tau)\times\Omega$. By the Egoroff theorem, we extract a further subsequence ρ_{f_n}, which converges almost uniformly on G, i.e., for every $\varepsilon > 0$ there exists $A_\varepsilon \subset G$ with $|A_\varepsilon| \leq \varepsilon$ such that ρ_{f_n} converges to ρ_f uniformly on $G\setminus A_\varepsilon$. The set $G\setminus A_\varepsilon$ is now further decomposed into subsets

$$V_\varepsilon := \{(t,\mathbf{x}) \in G : \rho_f(t,\mathbf{x}) \leq \varepsilon\}\ , \qquad N_\varepsilon := \{(t,\mathbf{x}) \in G : \rho_f(t,\mathbf{x}) > \varepsilon\}\,.$$

For a test function $\varphi \in \mathcal{D}_\tau$, the integral

$$\int_{G\times\mathbb{R}^d} (P(f_n) - P(f))\varphi\,dt\,d\mathbf{x}\,d\mathbf{v} \tag{1.29}$$

is split into three contributions according to the decomposition $G = V_\varepsilon \cup N_\varepsilon \cup A_\varepsilon$. For the first "almost vacuum" part, we have the estimate

$$\left|\int_{A_\varepsilon\times\mathbb{R}^d} (P(f_n) - P(f))\varphi\,dt\,d\mathbf{x}\,d\mathbf{v}\right| \leq (2\varepsilon + a_n)\tau|\Omega|\sup|\varphi|\,,$$

with $a_n \to 0$ as $n \to \infty$, by Lemma 3 (Cont.) and the uniform convergence of the density. In N_ε, $\mathbf{u}_n = \mathbf{m}_{f_n}/\rho_{f_n}$ is well defined for n large enough and converges to $\mathbf{u}_f$ in $L^1(N_\varepsilon)$. By the symmetry property Lemma 1 (ii), the second "non-vacuum" contribution to (1.29) can be written as

$$\int_{N_\varepsilon \times \mathbb{R}^d} (f_n P_{\mathbf{u}_n}(\varphi) - f P_{\mathbf{u}_f}(\varphi))\, dt\, d\mathbf{x}\, d\mathbf{v}$$

which converges to zero for $n \to \infty$ by the weak convergence of f_n, the strong convergence of $\mathbf{u}_n$, and by Lemma 3 (Stab.). Finally, the third contribution to (1.29) is estimated by

$$\left| \int_{A_\varepsilon \times \mathbb{R}^d} (P(f_n) - P(f))\varphi\, dt\, d\mathbf{x}\, d\mathbf{v} \right| \leq \sup |\varphi| \int_{A_\varepsilon} (\rho_{f_n} + \rho_f)\, dt\, d\mathbf{x}\, .$$

The right hand side and, thus, (1.29) tend to zero for $n \to \infty$, $\varepsilon \to 0$ by the convergence of ρ_{f_n} to ρ_f and by $|N_\varepsilon| \leq \varepsilon$. This proves that we can pass to the limit in the weak formulation of the problem for f_n. ■

In the last result of this section the formal computations of section 2 concerning entropy dissipation and conservation laws are justified.

Theorem 3 ***(Propagation of moments)*** *Let the assumptions of Theorem 2 hold. Then:*

(i) The entropy dissipation equation (1.12) holds.

(ii) Let $r \geq 2$ for the initial data in (1.5). Then, the global conservation of mass and energy (1.15) and the global conservation of angular momentum in the case of specular reflection and rotational symmetry (1.16) hold.

(iii) Let $r \geq 3$ for the initial data in (1.5). Then, the local conservation laws (1.13), (1.14) hold in the distributional sense.

Proof: To show the entropy dissipation equation (1.12) we note that the H-theorem (2) (iii) with $\chi = pf^{p-1}$ holds by lemma (3) (Cont.) since $f \in C([0,\infty); L^p(\Omega \times \mathbb{R}^d))$ and

$$\left| \int_{\mathbb{R}^d} Q(f) pf^{p-1}\, d\mathbf{v} \right| \leq p\|P(f)\|_p \|f^{p-1}\|_{\frac{p}{p-1}} + p\|f\|_p^p \leq 2p\|f\|_p^p.$$

and an analogous estimate for $\int (f - p(f))(f^{p-1} - P(f)^{p-1})\, d\mathbf{v}$ on the right hand side of (2) (iii). We multiply now (1.1) with pf^{p-1}. Since f and $\partial_t f + \mathbf{v}\cdot\nabla_\mathbf{x} f$ belong to $C([0,\infty); L^p(\Omega \times \mathbb{R}^d))$ we have as in [DLPS]

$$(\partial_t f + \mathbf{v}\cdot\nabla_\mathbf{x} f) pf^{p-1} = \partial_t f^p + \mathbf{v}\cdot\nabla_\mathbf{x} f^p.$$

Integration with respect to $\mathbf{x}$ and $\mathbf{v}$ gives then (1.12). As example for a global conservation laws we consider the conservation of energy in the weak formulation

$$\int_0^\infty \int_{\Omega\times\mathbb{R}^d} f(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\, dt\, d\mathbf{x}\, d\mathbf{v} + \int_0^\infty \int_{\Omega} \times\mathbb{R}^d Q(f)\, dt\, d\mathbf{x}\, d\mathbf{v} = \int_{\Omega\times\mathbb{R}^d} f_0\varphi(t=0)\, d\mathbf{x}\, d\mathbf{v}.$$

As in lemma 7 we choose as testfunction $\varphi(t,\mathbf{x},\mathbf{v}) = \theta(t)\Phi(|\mathbf{v}|^2/V)(1+|\mathbf{v}|^r) \in \mathcal{D}_\tau$ with $\theta \in C_0^\infty([0,\tau))$, $\Phi \in C_0^\infty([0,\infty))$, $\Phi(y) = 1$ for $y < 1$, $V > 0$. In the limit $V \to 0$ the second and the third term on the left hand side vanish which completes the proof. For the local conservation laws we remark that we need the assumption $r \geq 3$ to control the energy flux since a dispersion result as used in [DLPS] applies only for the whole space problem.

1.4 Convergence to equilibrium

The following theorem corresponds to the results of Desvillettes [D] for the gas dynamic Boltzmann equation.

Theorem 4 *Let the assumptions of Theorem 2 hold with $r \geq 2$ in (1.5). Then, for every sequence $t_n \to \infty$, there exists a subsequence (again denoted by t_n), such that, for every $T > 0$, $f_n(t,\mathbf{x},\mathbf{v}) := f(t_n+t,\mathbf{x},\mathbf{v})$ converges weakly in $L^p((0,T)\times\Omega\times\mathbb{R}^d)$ to $f_\infty(t,\mathbf{x},\mathbf{v})$. Moreover, $f_\infty(t,\mathbf{x},\mathbf{v})$ is an equilibrium solution of the free streaming equation satisfying the reflection boundary conditions:*

$$Q(f_\infty) = 0\,,$$
$$\int_{(0,T)\times\Omega\times\mathbb{R}^d} f_\infty(\partial_t\varphi + \mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)\, d\mathbf{v}d\mathbf{x}dt = 0\,,$$

for every $\varphi \in C_0^\infty((0,T)\times\overline{\Omega}\times\mathbb{R}^d)$ satisfying the boundary conditions (1.6).

Proof: We first prove the result with a subsequence possibly depending on T. Then the statement of the theorem follows from a diagonal procedure.

The weak convergence of f_n (up to a subsequence) to a limit f_∞ follows from the boundedness of $\|f(t,\cdot,\cdot)\|_{L^p(\Omega\times\mathbb{R}^d)}$ uniformly in time. By Theorem 3 (ii), ρ_{f_n} and E_{f_n} are bounded in $L^1((0,T)\times\Omega)$ uniformly in n. The same holds for $\mathbf{m}_{f_n}$ by the interpolation

$$|\mathbf{m}_f| \leq \sqrt{\rho_f E_f} \leq \frac{\rho_f + E_f}{2}\,.$$

For passing to the limit $n \to \infty$ we proceed exactly as in Theorem 2. Using a velocity averaging lemma for strong converges of moments and a vacuum-nonvacuum parts decomposition we prove that

$$\partial_t f_\infty + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_\infty = Q(f_\infty)$$

and the boundary conditions hold in the weak sense indicated in the formulation of the theorem. To prove f_∞ to be an equilibrium distribution, we apply the entropy dissipation equation (1.12) (valid by Theorem 3 (i)). As a first step we note that the assumptions (1.4), (1.5) imply the validity of assumption (1.4) also for every q between 1 and p by interpolation. Let us pick, in particular, $q = \min\{p, 2\}$. The entropy dissipation equation (1.12) then implies

$$\int_0^\infty \int_\Omega \int_{\mathbb{R}^d} [f - P(f)] \left[f^{q-1} - P(f)^{q-1}\right] d\mathbf{v}\, d\mathbf{x}\, dt < \infty\,,$$

and, hence,

$$\int_0^T \int_\Omega \int_{\mathbb{R}^d} [f_n - P(f_n)] \left[f_n^{q-1} - P(f_n)^{q-1}\right] d\mathbf{v}\, d\mathbf{x}\, dt$$

converges to zero. The convexity and definiteness of the function $C(x,y) = (x-y)(x^{q-1} - y^{q-1})$ for $1 < q \leq 2$ and the weak convergence of f_n and $P(f_n)$ allow to pass to the limit and conclude

$$f_\infty = P(f_\infty)\,,$$

completing the proof. ■

Let us remark that we could not show stronger convergence results including the rate of convergence by an entropy-entropy dissipation approach for non-homogeneous kinetic equations, recently developed and carried out for linear Fokker-Planck equations by Desvillettes and Villani [DV]. The reason relies on the nature of the collisions, for which in the homogeneous case all isotropic functions are solutions. As a consequence the set of equilibria, constructed in next section 1.5, is infinite dimensional. On the other hand, we have only a finite number of conserved quantities and the large time limit cannot be determined uniquely from the initial data which inhibits attempts like in [DV].

1.5 Smooth equilibrium solutions

In this section we compute all smooth solutions of the system

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f = 0\,, \qquad Q(f) = 0\,. \tag{1.30}$$

We remark that we do not see how to close the gap between the above weak convergence result and the consideration of smooth solutions.

We distinguish between subsets of $(t, \mathbf{x})$-space where the density ρ_f (and, thus, the distribution function f) vanishes and where ρ_f is positive. The following arguments holds for a connected component $D \subset \mathbb{R}^{d+1}$ of $\{\rho_f >$

$0\}$. Then, by $Q(f) = 0$, there exists a mean velocity $\mathbf{u}(t, \mathbf{x})$ and a function $F(t, \mathbf{x}, \xi)$ such that

$$f(t, \mathbf{x}, \mathbf{v}) = F\left(t, \mathbf{x}, \frac{|\mathbf{v} - \mathbf{u}(t, \mathbf{x})|^2}{2}\right).$$

Substituting this representation into the free streaming equation and introducing the change of variables $\mathbf{v} \mapsto (\xi, \boldsymbol{\omega}) \in [0, \infty) \times S^{d-1}$, defined by $\mathbf{v} = \mathbf{u} + \boldsymbol{\omega}\sqrt{2\xi}$, gives

$$-(2\xi\partial_\xi F)\boldsymbol{\omega}^{tr}(\nabla_{\mathbf{x}}\mathbf{u})\boldsymbol{\omega} + \sqrt{2\xi}\,\boldsymbol{\omega}\cdot(\nabla_{\mathbf{x}}F - \partial_\xi F D_t \mathbf{u}) + D_t F = 0\,, \tag{1.31}$$

where $\boldsymbol{\omega}^{tr}$ is the transpose of $\boldsymbol{\omega}$ and the material derivative is denoted by $D_t = \partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}$. We exploit this equation using the following simple linear algebra lemma.

Lemma 8 *Let* $\mathrm{A} \in \mathbb{R}^{d\times d}$ *and* $\mathbf{b} \in \mathbb{R}^d$. *Then*

$$\boldsymbol{\omega}^{tr}\mathrm{A}\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\mathbf{b} = 0 \qquad \text{for all } \boldsymbol{\omega} \in S^{d-1} \tag{1.32}$$

holds iff $\mathbf{b} = 0$ *and* A *is skew-symmetric.*

Proof: The matrix A can be replaced by its even part $\tilde{\mathrm{A}} = \frac{1}{2}(\mathrm{A} + \mathrm{A}^{tr})$ in (1.32). The odd part can be arbitrary. By a rotation of $\boldsymbol{\omega}$, $\tilde{\mathrm{A}}$ can be assumed as diagonal. The choices $\boldsymbol{\omega} = \pm\mathbf{e}_j$, $j = 1, \ldots, d$ prove $\tilde{\mathrm{A}} = \mathbf{b} = 0$. ■

Keeping $(t, \mathbf{x}, \xi)$ fixed and varying $\boldsymbol{\omega} \in S^{d-1}$ in (1.31), we deduce that

$$\nabla_{\mathbf{x}}F = \partial_\xi F D_t \mathbf{u} \tag{1.33}$$

holds, and that the matrix

$$\mathrm{A} = D_t F\,\mathrm{I} - 2\xi\partial_\xi F \nabla_{\mathbf{x}}\mathbf{u}$$

is skew-symmetric. Application of the curl to (1.33) leads to

$$0 = (\partial_\xi\partial_{x_i}F D_t u_j - \partial_\xi\partial_{x_j}F D_t u_i) + \partial_\xi F(\partial_{x_i}D_t u_j - \partial_{x_j}D_t u_i)\,,$$

for $1 \le i < j \le d$. Using (1.33) again for the computation of the components of $\nabla_{\mathbf{x}}F$ shows that the first term vanishes. Also the existence and positivity of the macroscopic density in D implies that for every $(t, \mathbf{x}) \in D$ there exists $\xi > 0$ such that $\partial_\xi F(t, \mathbf{x}, \xi) \neq 0$. Thus, the second term, curl of $D_t\mathbf{u}$ vanishes in D. This implies that locally in D a potential $\tilde{g}(t, \mathbf{x})$ exists such that

$$D_t\mathbf{u} = \nabla_{\mathbf{x}}\tilde{g}$$

holds. Inserting the above relation into (1.33) gives $\nabla_{\mathbf{x}}F - \partial_\xi F\nabla_{\mathbf{x}}\tilde{g} = 0$ for which the method of characteristics implies directly that F can be written as

$$F(t, \mathbf{x}, \xi) = F_0(t, z) \qquad \text{with} \qquad z = \xi + \tilde{g}(t, \mathbf{x})\,. \tag{1.34}$$

The fact that the diagonal elements of A vanish, imply that the diagonal elements of $\nabla_{\mathbf{x}}\mathbf{u}$ are identical: $\partial_{x_i}u_i(t,\mathbf{x}) = \sigma(t,\mathbf{x})$. Using $D_tF = \partial_tF_0 + \partial_zF_0D_t\tilde{g}$ and $\xi = z - \tilde{g}$ we calculate

$$\mathrm{A}_{ii} = \partial_tF_0 + \partial_zF_0(D_t\tilde{g} + 2\sigma\tilde{g} - 2\sigma z) = 0\,. \tag{1.35}$$

Similarly to above, we argue that for every t in the projection of D onto the t-axis there exists a z-interval of positive length, such that $\partial_zF_0(t,z) \neq 0$. This implies that the coefficients $D_t\tilde{g} + 2\sigma\tilde{g}$ and σ in (1.35) are independent of $\mathbf{x}$. We may apply now the following lemma

Lemma 9 *Let $\partial_tF_0 + \partial_zF_0\left((D_t\tilde{g} + 2\sigma\tilde{g})(t) - 2\sigma(t)z\right) = 0$. Then,*

$$F_0 = \psi(\alpha(t)z + \beta(t)), \tag{1.36}$$

where $|\alpha|(t) = \exp(-\int_0^t 2\sigma(\tau)\,d\tau)$ and thus cannot become zero.

Proof: Assume as Cauchy data $F_{0i}(\zeta)$ on a non characteristic curve $\Gamma = \{(t_0(\zeta), z_0(\zeta))\}$. With the notations $A(t) = \int_0^t 2\sigma(\tau)\,d\tau$ and $\beta(t) = \int_{t_0(\zeta)}^t (D_t\tilde{g} + 2\sigma\tilde{g})(\tau)e^{-A(\tau)}d\tau$, the method of characteristics defines the coordinate transformation

$$z(t,\zeta) = z_0(\zeta)e^{-A(t_0(\zeta)}e^{A(t)} + \beta(t)e^{A(t)}. \tag{1.37}$$

At least within a region where the method of characteristics succeeds, we may follow the characteristic lines to transform the general Cauchy data into equivalent data given on $t_0(\zeta) = const$ and the constant can be set to be zero. By (1.37) we have then $\zeta = z_0^{-1}(\alpha(t)z + \beta(t))$ and the method of characteristics implies $F_0 = F_{0i}(\zeta) = \psi(\alpha(t)z + \beta(t))$. $\square$

Combining the above lemma together with (1.34) shows that $F(t,\mathbf{x},\mathbf{v})$ can be written as function $\psi(\alpha(t)\xi + g(t,\mathbf{x}))$ with $g = \alpha\tilde{g} + \beta$. Returning to the equation (1.35) $\mathrm{A}_{ii} = 0$, we deduce by comparing the coefficients in ξ

$$\tfrac{d\alpha}{dt} = 2\sigma\alpha \tag{1.38}$$

$$D_tg = 0\,. \tag{1.39}$$

More information on the form of $\alpha(t)$ and $g(t,\mathbf{x})$ is derived similarly to Desvillettes [D]. We define the vector field $\mathbf{u}(t,\mathbf{x}) - \sigma(t)\mathbf{x}$ and observe by $\partial_{x_i}u_i = \sigma$ and the off-diagonal elements of A that the gradient of the vector field is skew-symmetric. Hence, lemma 1 in [D] ensures

$$\mathbf{u}(t,\mathbf{x}) = \sigma(t)\mathbf{x} + \Lambda(t)\mathbf{x} + \mathbf{C}(t)$$

with a skew-symmetric $\Lambda(t)$ and a vector field $\mathbf{C}(t)$. This representation is inserted in

$$\nabla_{\mathbf{x}}g = \alpha D_t\mathbf{u}\,, \tag{1.40}$$

which is a consequence of (1.33). It is then necessary for the right hand side of (1.40) to be a gradient that the skew-symmetric part must vanish. Using (1.38) and the abbreviation $\frac{d}{dt} =\,'$ this leads to:

$$\Lambda' + \frac{\alpha'}{\alpha}\Lambda = 0 \qquad \Rightarrow \qquad \Lambda(t) = \Lambda_0/\alpha(t),$$

where Λ_0 is an arbitrary constant skew-symmetric matrix. Next, (1.40) is integrated with respect to $\mathbf{x}$:

$$g(t,\mathbf{x}) = \left(\frac{\alpha''}{4} - \frac{(\alpha')^2}{8\alpha}\right)|\mathbf{x}|^2 - \frac{1}{2\alpha}(\Lambda_0\cdot\mathbf{x})^2 + \left(\alpha\mathbf{C}' + \frac{\alpha'}{2}\mathbf{C} + \Lambda_0\cdot\mathbf{C}\right)\cdot\mathbf{x} + g_0(t), \tag{1.41}$$

where $g_0(t)$ is the constant of integration. In a final step, (1.41) is substituted in (1.39) where the left hand side becomes a quadratic polynomial in $\mathbf{x}$. Equating coefficients to zero gives:

$$\begin{aligned} \alpha''' = 0 \qquad &\Rightarrow \qquad \alpha(t) = at^2 + 2bt + c \\ (\alpha\mathbf{C})'' = 0 \qquad &\Rightarrow \qquad \mathbf{C}(t) = \frac{1}{\alpha}(\mathbf{A} + \mathbf{B}(t+c)) \\ \frac{\partial g_0}{\partial t} + \frac{1}{2}\left(\alpha\mathbf{C}^2\right)' = 0 \qquad &\Rightarrow \qquad g_0(t) = -\frac{\alpha}{2}\mathbf{C}^2 + \text{const.} \end{aligned}$$

Altogether, a smooth equilibrium distribution function can be written in the form

$$f(t,\mathbf{x},\mathbf{v}) = \psi\left(\alpha(t)\frac{|\mathbf{v} - \mathbf{u}(t,\mathbf{x})|^2}{2} + g(t,\mathbf{x})\right). \tag{1.42}$$

There exist three constant scalars $a, b, c \in \mathbb{R}$, two constant vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^d$, and a constant skew-symmetric matrix $\Lambda_0 \in \mathbb{R}^{d\times d}$ such that

$$\begin{aligned} \alpha(t) &= at^2 + 2bt + c\,, \\ g(t,\mathbf{x}) &= \frac{ac - b^2}{\alpha(t)}\frac{|\mathbf{x}|^2}{2} - \frac{|\Lambda_0\mathbf{x}|^2}{2\alpha(t)} \\ &\quad + \left(\mathbf{A} + \frac{1}{\alpha(t)}(\Lambda_0 - at - b)(\mathbf{A}t + \mathbf{B})\right)\cdot\mathbf{x} - \frac{|\mathbf{A}t + \mathbf{B}|^2}{2\alpha(t)}\,, \\ \mathbf{u}(t,\mathbf{x}) &= \frac{1}{\alpha(t)}((at+b)\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{A}t + \mathbf{B})\,. \end{aligned} \tag{1.43}$$

So far this holds only locally in D. We intend to make the result global. Let us consider the intersection of the domains of two local representations of the form (1.42), (1.43). We shall prove that the function ψ and the constants have to be the same in both representations. However, an obvious source of non-uniqueness has to be eliminated first. We require a normalization of the coefficients of $\alpha(t)$: $a^2 + b^2 + c^2 = 1$. This can be achieved by a rescaling of the argument of ψ.

The mean velocity $\mathbf{u}(t,\mathbf{x})$ has to be the same in both representations and, thus, also the diagonal elements $\alpha'/(2\alpha)$ of its gradient. With the normalization condition this implies that $\alpha(t)$ and therefore also the coefficients a, b, and c are the same. Now it is an easy consequence of the formula for $\mathbf{u}$ in (1.43) that also the other coefficients Λ_0, $\mathbf{A}$, and $\mathbf{B}$ are the same. Finally, we conclude that the functions ψ in both representations have to be identical.

Summarizing, we have proven the following.

Theorem 5 *Let f be a smooth solution of (1.30), and let D be an open connected subset of $\mathbb{R}^{d+1}$ where ρ_f is positive. Then f can be written as*

$$F(t,\mathbf{x},\mathbf{v}) = \psi\left(\alpha(t)\xi(t,\mathbf{x},\mathbf{v}) + g(t,\mathbf{x})\right), \tag{1.44}$$

where $\psi \in C^1(\mathbb{R})$ is an arbitrary function. It depends on $\alpha(t)$

$$\alpha(t) = at^2 + 2bt + c\,, \qquad a,b,c \in \mathbb{R}, \tag{1.45}$$

the quadratic form $g(t,\mathbf{x})$

$$g(t,\mathbf{x}) = \frac{1}{2\alpha}\mathbf{x}^T\cdot\left(ac\mathrm{I} + \Lambda^2\right)\cdot\mathbf{x} + \left(\mathbf{A} + \frac{1}{\alpha}(\Lambda - at)(\mathbf{A}t + \mathbf{B})\right)\cdot\mathbf{x} - \frac{1}{2\alpha}\left(\mathbf{A}t + \mathbf{B}\right)^2, \tag{1.46}$$

where $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^3$ are two constant vectors and $\Lambda \in \mathbb{R}^{3\times 3}$ is a skew-symmetric constant matrix and on the velocity modulus

$$\xi(t,\mathbf{x},\mathbf{v}) = \frac{1}{2}|\mathbf{v} - \mathbf{u}(t,\mathbf{x})|^2. \tag{1.47}$$

The velocity dependence (1.47) reflects the isotropy of F around it's mean velocity

$$\mathbf{u}(t,\mathbf{x}) = \frac{1}{\alpha}\left(\Lambda\cdot\mathbf{x} + (at+b)\mathbf{x} + \mathbf{A}t + \mathbf{B}\right). \tag{1.48}$$

Remark 2 *The above formulas are only valid for times t such that $\alpha(t) \neq 0$. Without loss of generality, we restrict ourself to $\alpha(t) > 0$ which requires either $a > 0, c \in \mathbb{R}$ or $a = 0, c > 0$. If $c \leq 0$, we consider solutions only for times $t \in (\sqrt{|c|/a}, \infty)$.*

1.6 Classification of smooth equilibria in $\mathbb{R}^3$

The number of relevant parameters in (1.44)-(1.48) can be minimized by means of four *symmetry transformations*:

S1 Invariance under translations in time $t \to t + t_0$, $t_0 \in \mathbb{R}$,

S2 Invariance under rotations $\mathrm{R} \in O(3)$ in phase space $\mathbf{x} \to \mathrm{R}\cdot\mathbf{x}$,$\mathbf{v} \to \mathrm{R}\cdot\mathbf{v}$,

S3 Galilean invariance $\mathbf{x} \to \mathbf{x} + \mathbf{x}_0 + \mathbf{v}_0 t, \mathbf{v} \to \mathbf{v} + \mathbf{v}_0$, with $\mathbf{x}_0, \mathbf{v}_0, \in \mathbb{R}^3$ and

S4 Invariance under rescalings $(t, \mathbf{x}, \mathbf{v}) \to (k_2 t, k_1 k_2 \mathbf{x}, k_1 \mathbf{v})$ and $k_1, k_2 \in \mathbb{R}\backslash\{0\}$.

The invariance under time translations **S1** allows to set

$$b = 0$$

in (1.45).

A skew-symmetric matrix in 3D has an eigenvector $\mathbf{L}$ associated to the zero eigenvalue and defines the vector product $\mathbf{z} \times \mathbf{L} = \Lambda{\cdot}\mathbf{z}$ for $\mathbf{z} \in \mathbb{R}^3$ provided that we normalize $\mathbf{L}$ such that $|\mathbf{L}| = \|\Lambda\|$. The rotational symmetry **S2** permits to choose a coordinate system in which $\mathbf{L} = (\lambda, 0, 0)$ with $\lambda := |\mathbf{L}| = \|\Lambda\|$. It is useful to define a orthogonal decomposition for vectors $\mathbf{z}$ according to $\mathbf{L}$ and it's orthogonal complement:

$$\mathbf{z} = \mu_{\mathbf{z}} \mathbf{L} + P^{\perp}\mathbf{z}, \qquad \mu_{\mathbf{z}} = \mathbf{L}{\cdot}\mathbf{z}/\lambda^2,$$

where $P^{\perp}$ denotes the projection of $\mathbf{z}$ on the orthogonal complement of $\mathbf{L}$. For the matrix of the quadratic form (1.46), we use $\Lambda^2{\cdot}\mathbf{z} = -\lambda^2 P^{\perp}\mathbf{z}$ and obtain

$$\left(ac\mathrm{I} + \Lambda^2\right){\cdot}\mathbf{z} = ac\mu_{\mathbf{z}}\mathbf{L} + \left(ac - \lambda^2\right) P^{\perp}\mathbf{z}. \tag{1.49}$$

Hence, the matrix $ac\mathrm{I} + \Lambda^2$ is regular if and only if $ac \neq \lambda^2$ and $ac \neq 0$.

It is convenient to classify all equilibrium distributions into four types corresponding to the rank of $ac\mathrm{I} + \Lambda^2$ in $\{3, 2, 1, 0\}$. Each type will be further simplified to a normal form using the symmetry transformation **S3** and **S4**.

Type 1 denotes the case where $ac\mathrm{I} + \Lambda^2$ is regular. We firstly apply the Galilean transformation **S3** in order to determine $\mathbf{x}_0$ and $\mathbf{v}_0$ such that the linear $\mathbf{x}$-terms in (1.46) vanish. Using $(ac\mathrm{I} + \Lambda^2)^{-1} = \left(\mathrm{I} - \Lambda^2/(ac - \lambda^2)\right)/(ac)$ (a consequence of (1.49)), we calculate $\mathbf{x}_0 = (ac\mathrm{I} + \Lambda^2)^{-1}(-c\mathbf{A} - \Lambda\mathbf{B})$ and $\mathbf{v}_0 = (ac\mathrm{I} + \Lambda^2)^{-1}(a\mathbf{B} - \Lambda\mathbf{A})$ and obtain

$$g(t, \mathbf{x}) = \tfrac{1}{2\alpha}\mathbf{x}^T \cdot \left(ac\mathrm{I} + \Lambda^2\right){\cdot}\mathbf{x}, \tag{1.50}$$

$$\mathbf{u}(t, \mathbf{x}) = \tfrac{\alpha'}{2\alpha}\mathbf{x} + \tfrac{1}{\alpha}\Lambda{\cdot}\mathbf{x} + \mathbf{v}_0, \tag{1.51}$$

where we have dropped a constant term in (1.50), which can be drawn into ψ. Rescaling **S4** allows to normalize a, c and $\mathbf{v}_0$ in (1.50) and (1.51). In the generic case $\mathbf{v}_0 \in \mathbb{R}^3/\{\mathbf{0}\}$, we set $k_2 = \sqrt{|c|/a}$ and $k_1 = |\mathbf{v}_0|$. Then, after rescaling λ to $\sqrt{a|c|}\lambda$, the factor $|c|k_1^2$ appears as a common multiplier of $\alpha\xi$ and g in (1.44) and thus can be drawn into the arbitrary function ψ. In the special case $\mathbf{v}_0 = \mathbf{0}$, k_1 can be chosen arbitrarily. Altogether we have

Proposition 1 ***(Type 1: Normal Form)*** *Assume* $ac \neq 0$, $ac \neq \lambda^2$ *and* $s = \mathrm{sign}(c)$ *in (1.44)-(1.48). Then, by using the symmetry transformations* ***S1-S4*** *with* $\mathbf{L} = (\lambda, 0, 0)$, *the equilibrium distributions (1.44)-(1.48) can be*

Figure 1.1: **Type 1:** Hyperboloid shaped quadrics

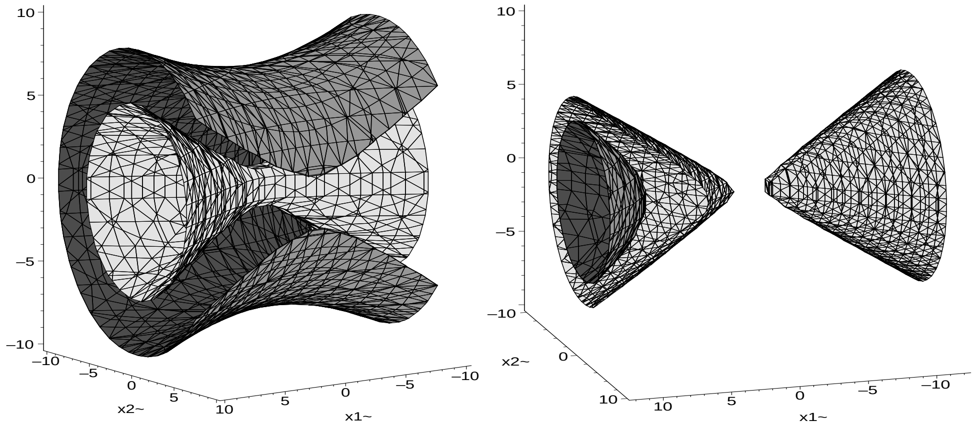

Fig. 1.1.1: Hyperboloid $Q(-6)$ for (1.52) with $\lambda = 2, s = 1, t_1 = 0, t_2 = 5$

Fig. 1.1.2: Hyperboloid $Q(1)$ for (1.52) with $\lambda = 2, s = 1, t_1 = 0, t_2 = 7.5$

transformed to

$$\begin{aligned} F &= \psi\left(\alpha\xi + g\right), \\ \alpha(t) &= (s + t^2), \qquad \xi = |\mathbf{v} - \mathbf{u}|/2, \\ \mathbf{u}(t, \mathbf{x}) &= \left(\tfrac{x_1 t}{\alpha}, \tfrac{x_2 t + \lambda x_3}{\alpha}, \tfrac{x_3 t - \lambda x_2}{\alpha}\right) + \mathbf{v}_0 \\ g(t, \mathbf{x}) &= \tfrac{1}{2\alpha}\left(s x_1^2 + (s - \lambda^2)(x_2^2 + x_3^2)\right), \end{aligned} \tag{1.52}$$

with the parameters $1 \neq \lambda \in \mathbb{R}_0^+$, $\mathbf{v}_0 \in S^2(\mathbb{R}^3) \cup \{\mathbf{0}\}$ *and* $s \in \{-1, 1\}$. *(1.52) is valid for times* t *such that* $\alpha(t) > 0$.

To visualize equilibrium distributions we use that $F(t, \xi, \mathbf{x})$ depends on $\mathbf{x}$ only through the quadratic form $g(t, \mathbf{x})$. Hence, the distribution function $F(t, \xi, \mathbf{x})$ is the same for all $x \in Q(g(t, \mathbf{x}))$, where

$$Q(g) := \left\{\mathbf{x} \in \mathbb{R}^3 : g(t, \mathbf{x}) = g\right\}$$

denotes the level sets of $g(t, \mathbf{x})$, which are closed *quadrics* for all types of $g(t, \mathbf{x})$ in (1.46). We remark that considering $F(t, \mathbf{v}, \mathbf{x})$ we know only $F(t, \mathbf{v}, \mathbf{x}) = F(t, \mathbf{w}, \mathbf{y})$ with $|\mathbf{v} - \mathbf{u}(t, \mathbf{x})| = |\mathbf{w} - \mathbf{u}(t, \mathbf{y})|$.

In this work, we are especially interested in equilibrium distributions $F = \psi(\alpha\xi + g)$ with $\text{supp}(\psi) \subset (\infty, g_0]$ and $g_0 < \infty$, since then $\psi(\alpha\xi + g(t, \mathbf{x})) = 0$ for all $\mathbf{x}$ where $g(t, \mathbf{x}) \geq g_0$: i.e. vacuum expands from the boundary $Q(g_0)$ to the side where $g(t, \mathbf{x}) > g_0$.

For equilibrium distributions of type 1, the quadrics Q are *hyperboloids* for $\lambda^2 > s = 1$ (Fig. 1.1.1, Fig. 1.1.2) and *ellipsoids* for $\lambda^2 < s = 1$ (Fig. 1.2.1) and for $s = -1$ (Fig. 1.2.2). All these four pictures show one quadric at two times $t_1 < t_2$. Q at time t_1 is drawn in the brightest grey-level, while Q at t_2 is plotted with differently colored in- and outside to indicate side of possible vacuum (darkest gray-level). Fig. 1.1.1 shows expanding vacuum due to strong rotation around the x_1 axis. In Fig. 1.1.2, the vacuum is pushed outside by rotating particles expanding from the center. In Fig. 1.2.1, the influence of the rotation is weaker and the particles spread out into all directions. The case $s = -1$ (Fig. 1.2.2) is different and shows an expanding vacuum ellipsoid, since all particles have outgoing velocities. One could imaging an explosion at time $t = 1$ which forces the particles outward and leaves nothing than vacuum at the origin.

Figure 1.2: **Type 1:** Ellipsoid shaped quadrics

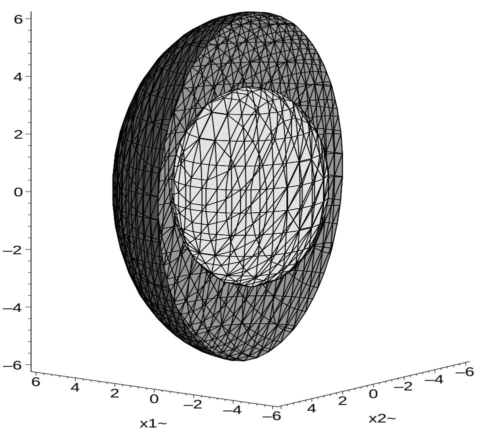

Fig. 1.2.1: Ellipsoid $Q(9)$ for (1.52) with $\lambda = 0.5, s = 1, t_1 = 0, t_2 = \sqrt{2}$

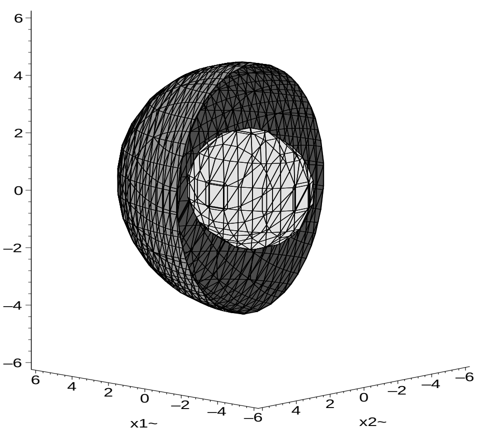

Fig. 1.2.2: Ellipsoid $Q(-20)$ for (1.52) with $\lambda = 1, s = -1, t_1 = 1.2, t_2 = 1.7$

Type 2 denotes the case $\mathrm{rk}(ac\mathrm{I} + \Lambda^2) = 2$, where $ac = 0$ and $\lambda^2 \neq 0$. By (1.49), the matrix $\left(ac\mathrm{I} + \Lambda^2\right)$ has a one-dimensional kernel in the direction of $\mathbf{L}$. As a consequence, the Galilean invariance only permits to find coordinates such that $P^{\perp}\mathbf{A}, P^{\perp}\mathbf{B}$ vanish in the linear part of (1.46).

Firstly, if $a = 0$ (implying $c > 0$) we set $\mathbf{x}_0 = \lambda^{-2}\left(cP^{\perp}\mathbf{A} + \Lambda\mathbf{B}\right)$ and $\mathbf{v}_0 = \lambda^{-2}\Lambda\mathbf{A}$ and obtain

$$\mathbf{u}(t,\mathbf{x}) = \tfrac{1}{c}\Lambda\cdot\mathbf{x} + \mathbf{v}_0 + \tfrac{1}{c}\left(\mu_{\mathbf{A}}\mathbf{L}t + \mu_{\mathbf{B}}\mathbf{L}\right), \tag{1.53}$$

$$g(t,\mathbf{x}) = \tfrac{1}{2c}\mathbf{x}^T\cdot\Lambda^2\cdot\mathbf{x} + \mu_{\mathbf{A}}\mathbf{L}\cdot\mathbf{x} - \tfrac{1}{2c}\left(+\mu_{\mathbf{A}}\mathbf{L}t + \mu_{\mathbf{B}}\mathbf{L}\right)^2, \tag{1.54}$$

where $\mu_{\mathbf{A}}, \mu_{\mathbf{B}} \in \mathbb{R}$ and $\mathbf{v}_0 \in P^{\perp}(\mathbb{R}^3)$. In the generic case $\mu_{\mathbf{A}} \neq 0$ and $\mu_{\mathbf{B}} \neq 0$, we normalize (1.53) and (1.54) by setting $k_1 = \mu_{\mathbf{A}}$, $k_2 = \mu_{\mathbf{B}}/\mu_{\mathbf{A}}$ and rescaling

λ to $|c/k_2|\lambda$. If either $\mu_{\mathbf{A}} = 0$ (where we set $k_1 = \mu_{\mathbf{B}}/c, k_2 = c$) or $\mu_{\mathbf{B}} = 0$ ($k_1 = \mu_{\mathbf{A}}, k_2 = c$) or $\mu_{\mathbf{A}} = 0 = \mu_{\mathbf{B}}$ ($k_1 = 1, k_2 = c$), we obtain normalized forms of (1.53) (1.54) without the corresponding vanishing terms.

Secondly, if $c = 0$ (and therefore $a > 0$), we choose $\mathbf{v}_0 = \lambda^{-2}\Lambda\mathbf{A} - \lambda^{-2}a\mathbf{B}$ and $\mathbf{x}_0 = \lambda^{-2}\Lambda\mathbf{B}$ and calculate

$$\begin{aligned} \mathbf{u}(t,\mathbf{x}) &= \tfrac{1}{at^2}\left(\Lambda + at\right)\cdot\mathbf{x} + \mathbf{v}_0, && (1.55)\\ g(t,\mathbf{x}) &= \tfrac{1}{2at^2}\mathbf{x}^T\cdot\Lambda^2\cdot\mathbf{x} - \tfrac{\mu_{\mathbf{B}}}{t}\mathbf{L}\cdot\mathbf{x}, && (1.56) \end{aligned}$$

where $\mu_{\mathbf{B}} \in \mathbb{R}$ and $\mathbf{v}_0 \in \mathbb{R}^3$. If $\mu_{\mathbf{B}} \neq 0$, we set $k_1 = \mu_{\mathbf{B}}\sqrt{a}$, $k_2 = 1/\sqrt{a}$ and rescale λ to $\lambda\sqrt{a}$ to normalize (1.55) and (1.56). If $\mu_{\mathbf{B}} = 0$, we set $k_1 = 1$. With the notation $m_A, m_B \in \{0,1\}$ as normalization of $\mu_{\mathbf{A}}, \mu_{\mathbf{B}}$, we collect the two above cases in

Proposition 2 ***(Type 2: Normal Forms)*** *Firstly, assume $a = 0$, $c > 0$, $\lambda^2 \neq 0$ and $s = \operatorname{sign}(\mu_{\mathbf{A}}\mu_{\mathbf{B}})$ in (1.44)-(1.48). Then, after applying the symmetries **S1-S4** with $\mathbf{L} = (\lambda, 0, 0)$, the equilibrium distributions (1.44)-(1.48) have the form*

$$\begin{aligned} F &= \psi\left(|\mathbf{v} - \mathbf{u}| + g\right), \\ \mathbf{u}(t,\mathbf{x}) &= (m_A\lambda t + m_B\lambda, s\lambda x_3, -s\lambda x_2) + \mathbf{v}_0 \qquad (1.57)\\ g(t,\mathbf{x}) &= 2m_A\lambda x_1 - \lambda^2(x_2^2 + x_3^2) - \lambda^2(m_A t + m_B)^2, \end{aligned}$$

with the parameters $\lambda \in \mathbb{R}^+$, $\mathbf{v}_0 = (0, v, w)$ with $v, w \in \mathbb{R}$, $m_A, m_B \in \{0,1\}$ and $s \in \{-1, 1\}$. Secondly, assume $a > 0$, $c = 0$ and $\lambda^2 \neq 0$. Then,

$$\begin{aligned} F &= \psi\left(t^2|\mathbf{v} - \mathbf{u}| + g\right), \\ \mathbf{u}(t,\mathbf{x}) &= \left(\tfrac{x_1}{t}, \tfrac{x_2 t + \lambda x_3}{t^2}, \tfrac{x_3 t - \lambda x_2}{t^2}\right) + \mathbf{v}_0 \qquad (1.58)\\ g(t,\mathbf{x}) &= -2m_B\lambda\tfrac{x_1}{t} - \tfrac{\lambda^2}{t^2}(x_2^2 + x_3^2), \end{aligned}$$

with the parameters $\lambda \in \mathbb{R}^+$, $\mathbf{v}_0 \in \mathbb{R}^3$, $m_B \in \{0,1\}$ and $t \neq 0$.

The quadrics Q of type 2 are either *paraboloids* ($m_A = 1$ in (1.57) and $m_B = 1$ in (1.58)) or *cylinders* ($m_A = 0$ in (1.57) and $m_B = 0$ in (1.58)). (1.57) covers situations with expanding particles while (1.58) deals with expanding vacuum solutions. As example, Fig. 1.3.1 shows a vacuum paraboloid being overwhelmed by particles from the right. The cylinders in Fig. 1.3.2 may include vacuum and expand due to the rotation of the particles outside.

Type 3 where $\operatorname{rk}(ac\mathrm{I} + \Lambda^2) = 1$ is complementary to the second type, since now $ac \neq 0$ and $ac = \lambda^2$. The Galilean transformation simplifies only in the direction of $\mathbf{L}$. We choose $\mathbf{v}_0 = \lambda^{-2}a\mu_{\mathbf{B}}\mathbf{L}$ and $\mathbf{x}_0 = -\lambda^{-2}c\mu_{\mathbf{A}}\mathbf{L}$ to obtain:

$$\begin{aligned} \mathbf{u}(t,\mathbf{x}) &= \tfrac{\alpha'}{2\alpha}\mathbf{x} + \tfrac{1}{\alpha}\Lambda\mathbf{x} + \mathbf{v}_0 + \tfrac{1}{\alpha}\left(P^\perp\left(\mathbf{A}t + \mathbf{B}\right)\right), \\ g(t,\mathbf{x}) &= \tfrac{1}{2\alpha}\left(\mathbf{x}\cdot\mathbf{L}\right)^2 + \tfrac{1}{\alpha}\left(-aP^\perp\mathbf{B}t + cP^\perp\mathbf{A} + \Lambda\left(\mathbf{A}t + \mathbf{B}\right)\right)\cdot\mathbf{x} - \\ &\quad \tfrac{1}{2\alpha}\left(P^\perp\left(\mathbf{A}t + \mathbf{B}\right)\right)^2, \end{aligned}$$

Figure 1.3: **Type 2:** Paraboloid and Cylinder shaped quadrics

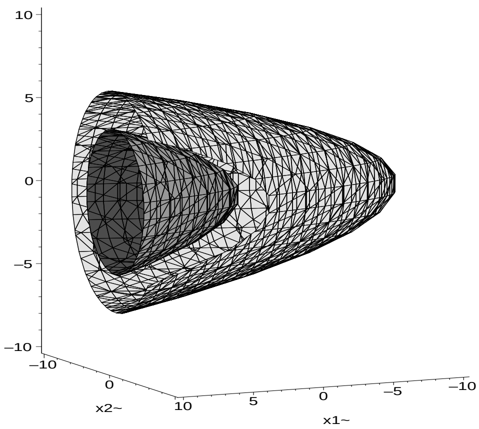

Fig. 1.3.1: Paraboloid $Q(-18)$ for (1.57) with $\lambda = 0.9, m_A = 1, m_B = 0, t_1 = 0, t_2 = 5$

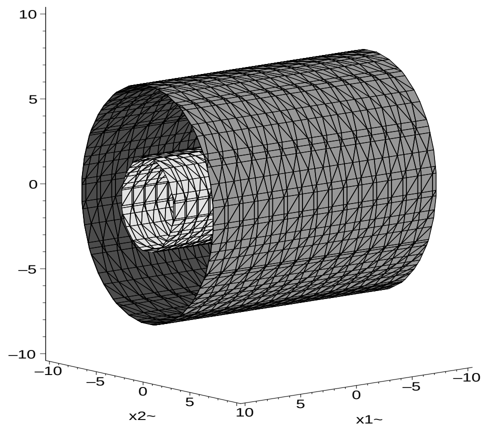

Fig. 1.3.2: Cylinder $Q(-2)$ for (1.58) with $\lambda = 1, m_B = 0, t_1 = 2, t_2 = 5$

with the parameters: $a > 0$, $c > 0$, $\mu_{\mathbf{B}} \in \mathbb{R}$ and $P^\perp \mathbf{A}$, $P^\perp \mathbf{B} \in P^\perp(\mathbb{R}^3)$. We normalize by setting $k_1 = 1/\sqrt{ac}$, $k_2 = \sqrt{c/a}$ in **S4**, where we rescale λ to $\sqrt{ac}\lambda$ and $\mathbf{B}$ to $k_2\mathbf{B}$.

Proposition 3 ***(Type 3: Normal Form)*** *Assume $ac \neq 0$ and $ac = \lambda^2$ in (1.44)-(1.48). Then, using the symmetries **S1-S4** and setting $\mathbf{L} = (\lambda, 0, 0)$, $P^\perp \mathbf{A} = (0, a_2, a_3)$, $P^\perp \mathbf{B} = (0, b_2, b_3)$ and $\mathbf{v}_0 = (v_0, 0, 0)$, the equilibrium distributions (1.44)-(1.48) have the normal form*

$$\begin{aligned} F &= \psi\left((1+t^2)|\mathbf{v}-\mathbf{u}| + g\right), \\ \mathbf{u}(t,\mathbf{x}) &= \left(\tfrac{x_1 t + v_0}{1+t^2}, \tfrac{x_2 t + x_3 + a_2 t + b_2}{1+t^2}, \tfrac{x_3 t - x_2 + a_3 t + b_3}{1+t^2}\right) \\ g(t,\mathbf{x}) &= \tfrac{1}{1+t^2}\Big(\lambda^2 x_1^2 + 2((a_3 - b_2)t + a_2 + b_3)x_2 \\ &\quad -2((a_2 + b_3)t - a_3 + b_2)x_3 - (a_2 t + b_2)^2 - (a_3 t + b_3)^2\Big), \end{aligned} \tag{1.59}$$

with the parameters $a_2, a_3 \in \mathbb{R}$, $b_2, b_3 \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$ and $v_0 \in \mathbb{R}$.

As shown in Fig. 1.4.1 for $(a_2, a_3) = (1, 0)$ and $(b_2, b_3) = (0, 1)$, the quadrics Q of type 3 are *parabolic shaped hyperplanes*, which rotate around an axis orthogonal to the symmetry plane of the parables. Vacuum is created and erased by the rotation.

In **type 4** with $\mathrm{rk}(ac\mathrm{I} + \Lambda^2) = 0$, the quadratic form $g(t, \mathbf{x})$ degenerates to a linear function. In the case $a = 0$, we use the rotational symmetry such

Figure 1.4: **Type 3 and 4:** Parabolic shaped hyperplanes and planes

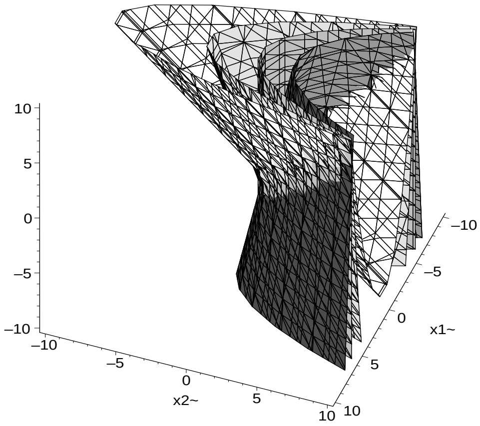

Fig. 1.4.1: $Q(-2)$ for (1.59) at times $t_1 = 0, t_2 = 0.5, t_3 = 1, t_4 = 1.5$

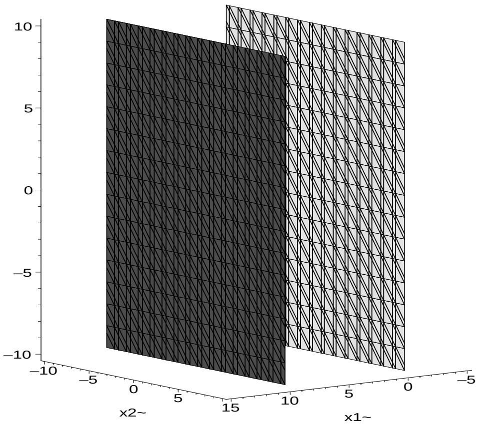

Fig. 1.4.2: Planes $Q(0)$ for (1.60) with $m_a = 1$ at times $t_1 = 0, t_2 = 4.5$

that $\mathbf{A} = (|\mathbf{A}|, 0, 0)$ and set $\mathbf{v}_0 = \frac{1}{c}\mathbf{B}$ for the Galilean invariance to obtain

$$\mathbf{u} = \mathbf{v}_0 + \frac{1}{c}\mathbf{A}t, \qquad g(t, \mathbf{x}) = |A|x_1 - \frac{1}{2c}|A|^2 t^2.$$

If $|\mathbf{A}| \neq 0$, we normalize applying rescaling **S4** with $k_1 = 1$ and $k_2 = c/|\mathbf{A}|$.

In the other case $c = 0$, we rotate the coordinates such that $\mathbf{B} = (|\mathbf{B}|, 0, 0)$ and set $\mathbf{x}_0 = -\frac{1}{a}\mathbf{A}$ which implies

$$\mathbf{u} = \frac{\mathbf{x}}{t} + \frac{1}{at^2}\mathbf{B}, \qquad g(t, \mathbf{x}) = -\frac{|B|}{t}x_1 - \frac{1}{2at^2}|B|^2.$$

If $|\mathbf{B}| \neq 0$, we normalize with $k_1 = |\mathbf{B}|$ and $k_2 = 1/\sqrt{a}$ in **S4**. With the notation $m_A, m_B \in \{0, 1\}$ as normalization of $|\mathbf{A}|$ and $|\mathbf{B}|$ we state

Proposition 4 ***(Type 4: Normal Forms)*** *Assume $a = 0$ and $\lambda = 0$ in (1.44)-(1.48). Then, the equilibrium distributions (1.44)-(1.48) reduces to*

$$F = \psi\left(|\mathbf{v} - \mathbf{u}| + g\right),$$
$$\mathbf{u}(t, \mathbf{x}) = (m_A t, 0, 0) + \mathbf{v}_0, \qquad g(t, \mathbf{x}) = 2m_A x_1 - m_A t^2, \tag{1.60}$$

with the parameters $m_A \in \{0, 1\}, \mathbf{v}_0 \in \mathbb{R}^3$. If $c = 0, \lambda = 0$ in (1.44)-(1.48), then

$$F = \psi\left(t^2|\mathbf{v} - \mathbf{u}| + g\right),$$
$$\mathbf{u}(t, \mathbf{x}) = \left(\tfrac{x_1 t + m_B}{t^2}, \tfrac{x_2}{t}, \tfrac{x_3}{t}\right), \qquad g(t, \mathbf{x}) = -2m_B\tfrac{x_1}{t} - \tfrac{m_B}{t^2}, \tag{1.61}$$

with the parameter $m_B \in \{0, 1\}$ and $t \neq 0$.

For $m_A = 1$ and $m_B = 1$, type 4 describes *planar fronts* of particle, as depicted in Fig. 1.4.2. In the trivial cases $m_A = 0$ and $m_B = 0$, the equilibrium is homogeneous.

For the sake of completeness we finally discuss briefly the coexistence of two (or more) equilibrium distributions where the non-vacuum regions are separated by vacuum. Suppose the equilibrium F is written as sum of two equilibria ψ_1 and ψ_2 which are disjunctly supported at some time t_0. Consider moreover the generic case where ψ_1 and ψ_2 are described by differently shaped quadrics $Q(g(t, \mathbf{x}))$ in the $(t, \mathbf{x}, \xi)$-variables. Then the particles cease to be in equilibrium as soon as the formerly separated particle regions merge. The argument is a contradiction which has to be expected since the nonlinear equilibrium condition $Q(f) = 0$ shall not permit superposition. Assume the merged particles are in equilibrium. Then, their equilibrium distribution is describted by quadrics defined by the intersection of the quadrics of ψ_1 and ψ_2. But since these quadrics are different, the intersection quadrics cannot coincide with the original quadrics outside.

A special case where the above contradiction does not apply is the example of a ball of expanding particle (as in Fig. 1.2.1 with $\lambda = 0$) centered inside a ball of expanding vacuum as in Fig. 1.2.2, since here the quadrics of ψ_1 and ψ_2 are identical. Let $r^p(t) = \sqrt{g_o^p}\sqrt{t^2+1}$ be the radius (see (1.52)) of the inner particle ball and $r^v(t) = \sqrt{-g_o^v}\sqrt{t^2-1}$ denote the radius of the expanding vacuum ball. At time $t_0 > 1$, we have that $r^p(t_0) < r^v(t_0)$, which implies

$$\sqrt{\frac{g_o^p}{-g_o^v}} < \sqrt{\frac{t_0^2-1}{t_0^2+1}} < 1.$$

But since two curves $k_1\sqrt{t^2+1}$ and $k_2\sqrt{t^2-1}$ with $k_1 < k_2$ intersect only once for $t > 1$ and $x^v(t_0)$ has already crossed $x^p(t_0)$, we conclude that the particle balls will stay separated by vacuum. The same is true for a cylinder of particles centered inside an expanding cylindric vacuum. By continuity, the above argument remains true for ellipsoids of particles inside expanding vacuum ellipsoids, hyperboloids or cylinders provided that the centers of the quadrics are not too far away from each other.

1.7 Equilibria touching reflective boundaries

We study the effects of reflective boundary conditions. In the work of Desvillettes [D] on the Boltzmann and the BGK equations it is shown that for Maxwellian equilibria (i.e., $\psi(y) = e^{-y}$) solving the free streaming problem within closed reflexive boundaries, vacuum cannot occur locally. The proof can easily be extended to any strictly positive ψ. It essentially relies on the fact that particles are spread with arbitrary velocities. In the more general situation discussed here, where ψ may have compact support, vacuum regions can be part of an equilibrium distribution. The following theorem shows that the presence of boundaries implies time independent equilibrium distributions by transferring the arguments of [D] to the present situation as far as possible.

Theorem 6 ***(Equilibria inside a bounded domain)*** *Let f be an equilibrium solution like in Theorem 5. Suppose the boundary of the $\mathbf{x}$-component of a non-vacuum region D (as above) $\tilde{\Omega}(t) = \{\mathbf{x} \in \Omega : \ (t, \mathbf{x}) \in D\}$ contains the boundary of Ω: that is $\partial\Omega \subset \partial\tilde{\Omega}(t)$, for $t_1 < t < t_2$. Then f satisfies the boundary conditions (1.6), if and only if it is of the form*

$$f(t, \mathbf{x}, \mathbf{v}) = \psi\left(|\mathbf{v}|^2\right), \qquad t_1 < t < t_2, \quad \mathbf{x} \in \Omega,$$

except in the case of specular reflecting boundaries on domains with rotational symmetries, whence

$$f(t, \mathbf{x}, \mathbf{v}) = \psi\left(|\mathbf{v}|^2 + \mathbf{v}^{tr}\Lambda_0(\mathbf{x} - \mathbf{x}_0)\right), \qquad t_1 < t < t_2, \quad (t, \mathbf{x}) \in D, \quad (1.62)$$

where the skew symmetric matrix Λ_0 and the point $\mathbf{x}_0$ can be chosen arbitrarily such that $\Lambda_0(\mathbf{x} - \mathbf{x}_0) = 0$ defines one of the symmetry axes of Ω. In particular, Λ_0 is an arbitrary skew symmetric matrix if Ω is a ball.

Proof: Firstly, in the case of the reverse reflexive boundary conditions (1.7) b), where $\mathbf{v}' = -\mathbf{v}$ relates the particle velocities before and after the contact with the boundary, it follows that $\mathbf{u}(t, \mathbf{x}) = \mathbf{0}$ the mean velocity vanishes along the boundary. In the general expression (1.48) this means

$$(at + b)\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{A}t + \mathbf{B} = 0, \qquad t_1 < t < t_2, \quad \mathbf{x} \in \partial\Omega,$$

and implies immediately $a\mathbf{x} + \mathbf{A} = 0$ for $\mathbf{x} \in \partial\Omega$, and, thus, $a = \mathbf{A} = 0$, as well as $b\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{B} = 0$ for $\mathbf{x} \in \partial\Omega$, and, thus, $b = \Lambda_0 = \mathbf{B} = 0$. As a consequence, the solution does not depend on $\mathbf{x}$ and, by our smoothness assumption, vacuum cannot occur.

The argument is a bit more involved for specular reflection (1.7) a), where $\mathbf{v}' = \mathbf{v} - 2(\mathbf{n}(\mathbf{x})\cdot\mathbf{v})\mathbf{n}(\mathbf{x})$ and $\mathbf{n}(x)$ denotes the unit inner normal along $\partial\Omega$. In this case the mean velocity satisfies the weaker condition $\mathbf{u}(t, \mathbf{x})\cdot\mathbf{n}(x) = 0$ along the boundary. Comparing the coefficients in time implies $(a\mathbf{x} + \mathbf{A})\cdot\mathbf{n}(x) = 0$ and $(b\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{B})\cdot\mathbf{n}(x) = 0$ for $\mathbf{x} \in \partial\Omega$. Therefore solutions of the ODEs

$$\frac{d\mathbf{x}}{ds} = a\mathbf{x} + \mathbf{A}, \qquad \frac{d\mathbf{x}}{ds} = b\mathbf{x} + \Lambda_0\mathbf{x} + \mathbf{B} \qquad (1.63)$$

with initial data on $\partial\Omega$ remain on $\partial\Omega$, which is a bounded set. However, solutions of (1.63) only remain bounded iff $a = b = \mathbf{A} = 0$ and $\mathbf{B} \in \mathrm{rg}\Lambda_0$. The solutions of the second ODE then describe rotations around an axis determined by $\Lambda_0(\mathbf{x} - \mathbf{x}_0) = 0$, where we have set $\mathbf{B} = -\Lambda_0\mathbf{x}_0$. ■

In the following we relax the condition that the complete boundary of a bounded domain is in contact with one non-vacuum part of an equilibrium distribution. We shall only assume that the particles in equilibrium touch locally a nonempty part ∂O of a not necessary closed boundary during a nontrivial time interval (t_1, t_2).

For reverse reflective boundaries this implies $\mathbf{u}(t,\mathbf{x}) = \mathbf{0}$, for $(t,\mathbf{x}) \in (t_1,t_2) \times \partial O$ and therefore as in the above theorem that all constants of $\mathbf{u}$ in (1.48) are zero. Hence, equilibria, which are locally in contact with reverse reflecting boundaries, must be stationary of the form $F = \psi(|\mathbf{v}|^2)$ and therefore global.

For specularly reflective boundaries, we have only $\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = 0$ locally for $(t,\mathbf{x}) \in (t_1,t_2) \times \partial O$. In particular, the blow up argument in the proof of theorem 6 cannot succeed. For a refined discussion of (1.63), it is convenient to treat seperatly the different types of equilibria as introduced in section 1.6. Comparing the coefficients in time of $\mathbf{u}(t,\mathbf{x})\cdot\mathbf{n}(\mathbf{x}) = 0$ leads to conditions restricting the admissible geometries of the boundaries and collected in table 1.7. As in the proof of theorem 6, it is useful to consider curves remaining on

	t^2	t	1
Type 1:	$\mathbf{v}_0\cdot\mathbf{n} = 0$	$\mathbf{x}\cdot\mathbf{n} = 0$	$\mathbf{x}\cdot\Lambda_0\cdot\mathbf{n} = 0$
Type 2a:		$\mu_{\mathbf{A}}\mathbf{L}\cdot\mathbf{n} = 0$	$-\mathbf{x}\cdot\Lambda_0\cdot\mathbf{n} + (\mathbf{v}_0 + \mu_{\mathbf{B}}\mathbf{L})\cdot\mathbf{n} = 0$
Type 2b:	$\mathbf{v}_0\cdot\mathbf{n} = 0$	$\mathbf{x}\cdot\mathbf{n} = 0$	$\mathbf{x}\cdot\Lambda_0\cdot\mathbf{n} = 0$
Type 3:	$\mathbf{v}_0\cdot\mathbf{n} = 0$	$\mathbf{x}\cdot\mathbf{n} + P^{\perp}\mathbf{A}\cdot\mathbf{n} = 0$	$-\mathbf{x}\cdot\Lambda_0\cdot\mathbf{n} + P^{\perp}\mathbf{B}\cdot\mathbf{n} = 0$
Type 4a:		$\mathbf{A}\cdot\mathbf{n} = 0$	$\mathbf{v}_0\cdot\mathbf{n} = 0$
Type 4b:		$\mathbf{x}\cdot\mathbf{n} = 0$	$\mathbf{B}\cdot\mathbf{n} = 0$

the boundary if the starting points $\mathbf{x}_0$ lies on the boundary. For the types 1, 2b, 3 and 4b, we set

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{x}(s) + \mathbf{X},$$

where $\mathbf{X}$ denotes different separately vectors for different types. The solutions

$$\mathbf{x}(s) = (\mathbf{x}_0 + \mathbf{X})e^s - \mathbf{X}$$

are lines spanning either a cone with center in $\mathbf{X}$ or a plane containing $\mathbf{X}$. Hence the boundary must be locally either a cone or a plane or equilibria of the types 1, 2b, 3, 4b cannot satisfy the specular reflective boundary condition. For the types 1, 2a, 2b and 3, we define

$$\frac{d\mathbf{x}(s)}{ds} = \Lambda\cdot(\mathbf{x}(s) + P^{\perp}\mathbf{X}) + \mu_{\mathbf{X}}\mathbf{L}$$

with solutions

$$\mathbf{x}(s) = e^{\Lambda s}\cdot(\mathbf{x}_0 + P^{\perp}\mathbf{X}) + \mu_{\mathbf{X}}\mathbf{L}s - P^{\perp}\mathbf{X}$$

describing either spiral lines (in the case $\mu_\mathbf{X} \neq 0$) or circles ($\mu_\mathbf{X} = 0$) both rotational symmetric around the **L**-axis. Hence the boundary has to be rotationally symmetric around the **L**-axis containing either spirals or circles, respectively.

We combine the above arguments in the following theorem:

Theorem 7 ***(Equilibria locally in touch with reflective boundaries)*** *Let F be an equilibrium distribution touching the boundary locally for $(t, \mathbf{x}) \in (t_1, t_2) \times \partial O$ where $t_1 < t_2$ and ∂O is a non empty part of a not necessary bounded, peace-wise C^1 boundary $\partial\Omega$.*

Then, for reverse reflective boundaries (1.7) b), F has to be of the form $F = \psi(|\mathbf{v}|^2)$.

For specularly reflective boundaries, we distinguish according to section 1.6:

Type 1, 2b: *Equilibria of type 1 (1.52) or type 2b (1.58) can satisfy specularly reflective boundary conditions if and only if $\mathbf{v}_0 = \mathbf{0}$ and the boundary is a rotationally symmetric cone with center $\mathbf{0}$.*

Type 2a: *In the case $\mu_\mathbf{A} = 0$ of type 2a (1.57) any rotationally symmetric boundary with axis through $P^\perp \mathbf{v}_0$ in the direction of $\mathbf{L}$ is admissible. In the other case $\mu_\mathbf{A} = 1$, the boundary has to be a plane containing $\mathbf{L}$.*

Type 3: *For equilibria of type 3 (1.59), the boundary has to be a cone with center $P^\perp \mathbf{A} = P^\perp \mathbf{B}$.*

Type 4a, 4b: *Admissible boundaries for equilibria of type 4a (1.60) are planes spanned by $\mathbf{A}$ and $\mathbf{v}_0$. In the special case $\mathbf{A} = \mathbf{v}_0 = \mathbf{0}$, where we have $F = \psi(|\mathbf{v}|^2)$, any boundary is admissible. For type 4b (1.61), the boundary has to be a plane containing $\mathbf{B}$.*

We remark that for specular reflective boundaries, equilibria of type 2a are the only space dependent equilibria, which are compatible with more complex geometries than planes or cones. This case covers examples like a ring of particles inside a spherical boundary, where the particles rotate around a cylinder of vacuum.

1.8 Explicit solutions of 3D Euler equations

We consider the 3D Euler equations for ideal gases with $p = \rho T$ and $e = \frac{3}{2}T$:

$$\begin{aligned} \rho_t + \nabla\cdot(\rho\mathbf{u}) &= 0, \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} &= -\tfrac{\nabla p}{\rho}, \\ e_t + \mathbf{u}\cdot\nabla e + \tfrac{p}{\rho}\nabla\cdot\mathbf{u} &= 0, \end{aligned} \tag{1.64}$$

As explained in the introduction, the moments ρ_F and e_F calculated from an equilibrium distributions F of the cometary flow equation

$$\rho_F = \int_{\mathbb{R}^3} F(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}, \qquad e_F = \int_{\mathbb{R}^3} \frac{|\mathbf{v}-\mathbf{u}_F|^2}{2} F(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}$$

together with the mean velocity $\mathbf{u}_F$ are solutions of the Euler equations (1.64). In theorem 5 (section 1.5), we showed that smooth equilibrium distributions have the form $F = \psi(\alpha\xi + g)$. For solutions with finite energy we shall assume $\int_{\mathbb{R}^3} |v|^2 F\,d\mathbf{v} < \infty$ which is equivalent to $\int_g^\infty \xi^{3/2}\psi(\xi)\,d\xi < \infty$ for all $g > -\infty$. The above moment integrals ρ_F and e_F suggests the introduction of a function $h(g)$ defined as

$$h(g) = 8\sqrt{2}\pi \int_0^\infty \xi^{\frac{3}{2}}\psi(\xi+g)\,d\xi\,, \tag{1.65}$$

since with

$$h'(g) = -4\sqrt{2}\pi \int_0^\infty \xi^{\frac{1}{2}}\psi(\xi+g)\,d\xi\,, \tag{1.66}$$

the solutions of (1.64) can be written as:

$$\rho = -\alpha(t)^{-\frac{3}{2}} h'\left(g(t,\mathbf{x})\right), \qquad \mathbf{u} = \mathbf{u}(t,\mathbf{x}), \qquad p = \rho T = \alpha(t)^{-\frac{5}{2}} h\left(g(t,\mathbf{x})\right). \tag{1.67}$$

The function h has the following properties:

Lemma 10 ***(Properties of h)*** *Let $\psi \in C^1(\mathbb{R})$ with $\int_c^\infty \xi^{3/2}\psi(\xi)d\xi < \infty$ for all $c > -\infty$ be positive with* $\operatorname{supp}\psi \in (a,b)$ *and $a < b \le \infty$. Define $h(g)$ as in (1.65). Then,*

(i) *$h(g) \in C^3(\mathbb{R})$ is strictly positive, strictly monotone decreasing and strictly convex on its support $(-\infty, b)$, such that $\lim_{g\to-\infty} h(g) = \infty$, $\lim_{g\to b} h(g) = 0$.*

(ii) *For $c > -\infty$ there is $\int_c^\infty (1+g)h''(g)\,dg < \infty$ as well as $\int_c^\infty (1+g+g^2)h'''(g)\,dg < \infty$, but $\int_{-\infty}^\infty h''(g)\,dg = \infty$.*

(iii) *We have the inversion formula*

$$\psi(t) = -\frac{1}{2\sqrt{2}\pi^2}\frac{d}{dt}\int_t^\infty \frac{h''(g)}{\sqrt{g-t}}dg\,. \tag{1.68}$$

Proof: (i) The signs $h > 0$ and $h' < 0$ on the support of h are clear from (1.65) and (1.66) since $\psi \ge 0$. Moreover, we have

$$h''(g) = 2\sqrt{2}\pi \int_0^\infty \xi^{-\frac{1}{2}}\psi(\xi+g)\,d\xi > 0, \qquad h'''(g) = 2\sqrt{2}\pi \int_0^\infty \xi^{-\frac{1}{2}}\psi'(\xi+g)\,d\xi,$$

where the last integral is bounded and thus continuous in g since $\psi \in C^1$ and $\int_g^\infty \xi^{\frac{3}{2}}\psi'(\xi)\,d\xi < \infty$ implies that also $\int_g^\infty \xi^{\frac{5}{2}}\psi'(\xi)\,d\xi < \infty$ and thus $h'''(g)$.
(ii) We substitute $(g,\xi) \to (r = \xi + g, s = \xi - g)$ and calculate for $c > -\infty$

$$\int_c^\infty (1+g)h''(g)\,dg = \int_c^\infty \int_0^\infty (1+g)\xi^{-\frac{1}{2}}\psi(\xi+g)\,d\xi\,dg =$$

$$\int_c^\infty \int_{-r}^r (1+\frac{r-s}{2})\left(\frac{r+s}{2}\right)^{-\frac{1}{2}}\psi(r)\,dr\,ds = \int_c^\infty O(r)^{\frac{3}{2}}\psi(r)\,dr < \infty.$$

On the other hand, $\int_{-\infty}^\infty h''(g)\,dg = \int_0^\infty \xi^{-1/2}\|\psi\|_{L^1(\mathbb{R})}\,d\xi = \infty$. An analogous argument shows $\int_c^\infty (1+g+g^2)h'''(g)\,dg < \infty$.
(iii) Since $h''(g) \in L^1((c,\infty)) \cap C^1(\mathbb{R})$ there is

$$2\sqrt{2}\pi \int_t^\infty \int_0^\infty \frac{1}{\sqrt{\xi}}\frac{1}{\sqrt{g-t}}\psi(\xi+g)\,d\xi\,dg = \int_t^\infty \frac{h''(g)}{\sqrt{g-t}}dg < \infty.$$

We substitute $\xi + g = z$ and change the parameterization of variable-range.

$$\int_t^\infty \int_g^\infty \frac{\psi(z)}{\sqrt{z-g}\sqrt{g-t}}dz\,dg \;= \int_t^\infty \psi(z)\left(\int_t^z \frac{dz}{\sqrt{z-g}\sqrt{g-t}}dz\right)\,dg$$

Using

$$\int_t^z \frac{dz}{\sqrt{y-t}\sqrt{z-t}} = \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi.$$

we conclude

$$\int_t^\infty \frac{h''(g)}{\sqrt{g-t}}dg = 2\sqrt{2}\pi^2 \int_t^\infty \psi(z)\,dz,$$

and the regularity of h'' allows to differentiate with respect to t. $\square$

In order to discuss solutions of the Euler equations it is convenient to consider directly functions $h(g)$ instead of calculating $h(g)$ for given distributions ψ. Therefore, we use the following lemma.

Lemma 11 ***(Inversion of h)*** *Let $h(g)$ satisfy the properties (i) and (ii) of lemma 10. Then,*

$$\psi(t) = -\frac{1}{2\sqrt{2}\pi^2}\frac{d}{dt}\int_t^\infty \frac{h''(g)}{\sqrt{g-t}}dg$$

is continuous and satisfies $\int_c^\infty t^{\frac{3}{2}}\psi(t)\,dt < \infty$.

Remark 3 *The counterexamples $h(g) = e^{-g}(1 - c\sin(g))$ with $\sqrt{2}-1 < c < \frac{1}{2}$ show that the properties (i) and (ii) of lemma 10 do not imply the positivity of ψ. On the other side, the sufficient condition $h'''(g) < 0$ implying $\psi > 0$ is not satisfied for the example $\psi(\xi) = \xi^2\exp(-|\xi|)$.*

Proof: The continuity is clear. To show $\int_c^\infty t^{\frac{3}{2}}\psi(t) < \infty$, we substitute $r = z + t, s = z - t$ as in lemma 10 (ii)

$$\int_c^\infty t^{\frac{3}{2}} \int_0^\infty z^{-\frac{1}{2}} h'''(z+t)\, dz\, dt = \frac{1}{2}\int_c^\infty h'''(r) \int_{-r}^{r} (r-s)^{\frac{3}{2}}(r+s)^{-\frac{1}{2}}\, ds\, dr,$$

which is bounded due to lemma 10 (ii). □

The following examples of explicit solutions of the 3D Euler equations (1.64) are characterized by the geometry of the level sets $Q(g)$ (section 1.6) and by the profiles of the density $2h'(g)$ and the temperature $-h/h'$ describing the variation of ρ and T normal to the level sets $Q(g)$ for fixed time t. We start with the introductory example

$$h(g) = \begin{cases} (g-c)^4 & g < c \\ 0 & g > c \end{cases}$$

for a constant $c > 0$ and with $g(t, \mathbf{x})$ chosen from equilibria of type 1 (1.52) in the case $s = 1$ and $\lambda < 1$

$$g(t,\mathbf{x}) = \frac{1}{2(1+t^2)}\left(x_1^2 + (1-\lambda^2)(x_2^2 + x_3^2)\right) =: \frac{1}{\alpha(t)} G(\mathbf{x}) \geq 0, \tag{1.69}$$

such that the level sets $Q(g)$ are expanding particle ellipsoids as in Fig. 1.2.1. The profiles Fig. 1.5.1 and Fig. 1.5.2 determine the decay of ρ (cubical) and T (linear) from the center of the ellipsoid $Q(g = 0)$ to the boundary $Q(g = c)$. The solutions are

Figure 1.5: Example 1: algebraically decaying h

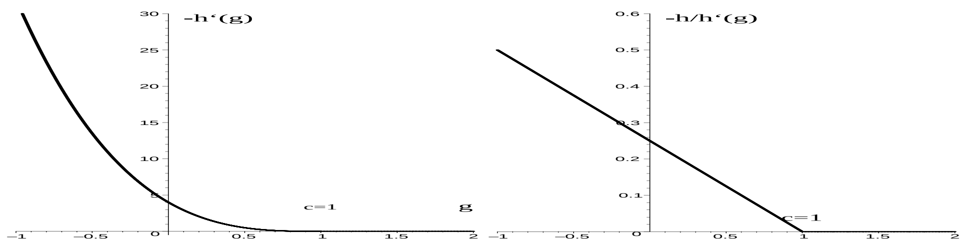

Fig. 1.5.1: Density profile

Fig. 1.5.2: Temperature profile

$$\rho = \begin{cases} -4\alpha(t)^{-\frac{3}{3}}(g(t,\mathbf{x}) - c)^3 & G(\mathbf{x}) < \alpha(t)c \\ 0 & G(\mathbf{x}) > \alpha(t)c \end{cases}$$

$$T = \begin{cases} -\frac{1}{4}(g(t,\mathbf{x}) - c)\alpha(t)^{-1} & G(\mathbf{x}) < \alpha(t)c \\ 0 & G(\mathbf{x}) > \alpha(t)c \end{cases}$$

$$\mathbf{u} = \frac{\mathbf{x}}{t} + \frac{1}{t^2}\Lambda\cdot\mathbf{x} + \mathbf{v}_0.$$

While the particles expand outwards, the density ρ and the temperature T at a fixed point $\mathbf{x}$ may increase intermediately as long as $g(t,\mathbf{x}) - c > -\frac{2}{\alpha(t)}$ and $g(t,\mathbf{x}) - c > g(t,\mathbf{x})$, respectively, while for $t \to \infty$ both ρ and t tend to zero. For instance if a vacuum point is reached by the expanding particles, ρ and T grow in the beginning continuously from zero until they start to decay due to the expansion process.

The associated velocity field $\mathbf{u}$ consists of three components: a linearly decelerating expansion $\frac{\mathbf{x}}{t}$, a quadratically decelerating rotation around the $\mathbf{L}$-axis and a constant offset velocity $\mathbf{v}_0 \in S^2(\mathbb{R}^3) \cup \{\mathbf{0}\}$.

For other choices of $h(g)$ with other density and temperature profiles, we observe different behavior of ρ and T during the expansion of the particles. The exponentially decaying $h(g) = \exp(-g)$ leads to an exponentially decaying density profile $-h' = \exp(-g)$ as in Fig. 1.6.1, while the temperature profile $-h/h' = 1$ is constant Fig. 1.6.2.

Figure 1.6: Example 2: exponentially decaying h

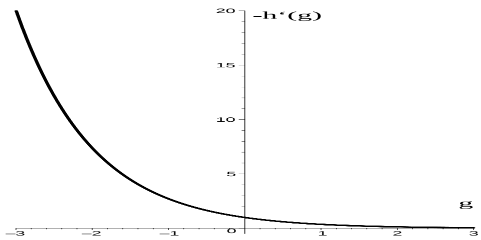

Fig. 1.6.1: Density profile

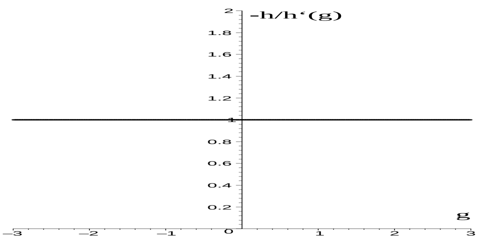

Fig. 1.6.2: Temperature profile

Since the support of $h(g)$ is $\mathbb{R}$, the solutions

$$\rho = \alpha(t)^{-\frac{3}{2}} e^{-g(t,\mathbf{x})}, \qquad \mathbf{u} = \frac{\mathbf{x}}{t} + \frac{1}{t^2}\Lambda\cdot\mathbf{x} + \mathbf{v}_0, \qquad T = \alpha(t)^{-1},$$

do not include a vacuum part. The density ρ may increase intermediately before tending to zero. The temperature T is homogeneous and decaying.

A third class of profiles is deduced from

$$h(g) = (g+c)^{-\gamma} \qquad g > -c + \delta$$

with $\gamma > 0$ and $c > \delta > 0$. For $g < -c + \delta < 0$, h is somehow extended to $h(-\infty) = \infty$ according to the properties of lemma 10.

Figure 1.7: Example 3: reciprocally decaying h

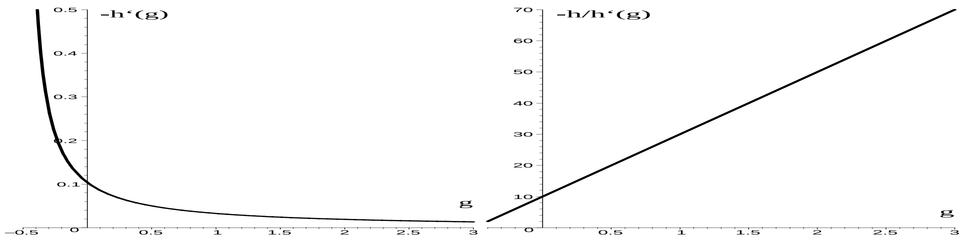

Fig. 1.7.1: Density profile   Fig. 1.7.2: Temperature profile

Since we have $g(t,\mathbf{x}) = \frac{G(\mathbf{x})}{\alpha(t)} > 0$ in (1.69), we explicitly calculate the solutions

$$\begin{aligned} \rho &= \gamma\alpha(t)^{-\frac{3}{2}} \left(\tfrac{G(\mathbf{x})}{\alpha(t)} + c\right)^{-\gamma-1}, \\ p &= \alpha(t)^{-\frac{5}{2}} \left(\tfrac{G(\mathbf{x})}{\alpha(t)} + c\right)^{-\gamma}, \\ T &= \tfrac{1}{\gamma}\alpha(t)^{-1} \left(\tfrac{G(\mathbf{x})}{\alpha(t)} + c\right). \end{aligned} \tag{1.70}$$

The temperature T is unbounded as $g(t,\mathbf{x}) \to \infty$ (Fig. 1.7.2). For fixed $\mathbf{x}$, it decays to zero independently of γ as $t \to \infty$ as well as the density ρ and the pressure p.

More general profiles of density and temperature can be composed from the above three examples 1.5-1.7 by peace-wise smooth connection according to the properties of lemma 10. In that way, we can produce temperature profiles which are locally decreasing, constant or increasing in g. As example, we can compose profiles, such that locally $h(g) = g^{-\gamma}$ for $g(t,\mathbf{x}) > \delta$. The associated solutions are found as limit $c \to 0$ in (1.70). For fixed $\mathbf{x}$,

$$\rho = O(\alpha(t))^{\gamma-\frac{1}{2}}, \qquad T = O(\alpha(t))^{-2}, \qquad p = O(\alpha(t))^{\gamma-\frac{5}{2}}.$$

Hence, the density ρ and the pressure p decay in time for $\gamma < \frac{1}{2}$. For $\frac{1}{2} < \gamma < \frac{5}{2}$, the density decreases and the pressure increases and for $\gamma > \frac{5}{2}$ both the density and the pressure increase in time.

All profiles, which are defined for $g \in \mathbb{R}$ such as 1.5 or 1.6 may be combined with other quadrics $Q(g)$ to compose solutions with different geometric structure. Taking

$$g(t,\mathbf{x}) = \frac{1}{2(1+t^2)} \left(x_1^2 - (\lambda^2 - 1)(x_2^2 + x_3^2)\right) \in \mathbb{R}, \tag{1.71}$$

with $\lambda > 1$, where $Q(g)$ define hyperboloids as plotted in Fig. 1.1.1 or Fig. 1.1.2, the properties of h in lemma 10 imply that the density ρ is unbounded for $g \to -\infty$. To illustrate these solutions, we remark that the level set $Q(0)$ is a cone separating the $\mathbf{x}$-space into two parts with $g > 0$ and $g < 0$, where ρ and T tend either to 0 or to ∞ according to their profiles. For fixed $\mathbf{x}$, since $g \to 0$ for $t \to \infty$, the density ρ and the temperature T decay asymptotically towards $\rho(0)$ and $T(0)$, the values at $Q(0)$, which tend to zero for $t \to \infty$.

Similar solutions are found in the case of the cylindrical quadrics of type 2b and equilibria of type 3 and type 4b, where also $g \to 0$ as $t \to \infty$ and $Q(0)$ may be consider as asymptotic quadric.

For solutions constructed from type 2a or type 4a, we have either $g(t, \mathbf{x}) \to -\infty$ for $t \to \infty$ (in the case $m_{\mathbf{A}} = 1$) or, rather uninteresting, $g(t, \mathbf{x})$ is constant in time ($m_{\mathbf{A}} = 0$). In the first case, the quadrics of type 2a are paraboliods as in Fig. 1.3.1 and the density increases increases as the particles approach from $\mathbf{x} = \infty$ where ρ is unbounded. In the other type 4a, the quadrics are planes Fig. 1.4.2 showing similar behavior.

1.8.1 More solutions

Straightforward insertion shows that functions of the form (1.67) are still strong solutions of the 3D Euler equations (1.64) provided only that $h(g)$ is in $C^2(\mathbb{R})$. This allows a much large variety of density and temperature profiles. As long as the inversion formula (1.68) is still well-defined, we can even calculate an "equilibrium" distribution associated to h, although ψ will no longer be positive or smooth in general.

In particular without the growing of h', we have bounded solutions for quadrics $Q(g)$ where $g \in \mathbb{R}$. Further, by choosing non-monotone functions h, h', we find oscillating solutions.

We can even construct weak solutions whenever h' is such that the involved integrals are defined. One example is $g = x_1 - \frac{1}{2}t^2$ from type 4 together with $h(g) = \max(0, -g)$, where we find

$$\rho = H\left(\frac{1}{2}t^2 - x_1\right), \qquad \mathbf{u} = (t, 0, 0), \qquad T = \max\left(0, \frac{1}{2}t^2 - x_1\right),$$

and H denotes the Heavyside function. In this solution, a planar front of particles spreads out into vacuum in the direction of the x_1-axis.

Although a large amount of solutions can be constructed like above, there exist more solutions, which seem related to the above structure, but not completely. As example

$$\rho = \frac{\alpha}{t^3}, \qquad T = \frac{\beta}{t^2}, \qquad \mathbf{u} = \frac{\mathbf{x}}{t},$$

with α, β arbitrary in $\mathbb{R}$, reminds of solutions constructed from type 2b with $\lambda = 0$ and $\mathbf{v}_0 = \mathbf{0}$ and $\alpha(t) = t^2$. Nevertheless, there is not function h such that $\alpha = -h'$ and $\beta = -h/h'$.

Chapter 2

The Burgers-Poisson system

2.1 Introduction and Motivation

In this chapter, a nonlinear dispersive model problem is proposed, the Burgers-Poisson (BP)-system:

$$u_t + uu_x = \varphi_x, \tag{2.1}$$

$$\varphi_{xx} = \varphi + u, \tag{2.2}$$

where u and φ depend on $(t, x) \in (0, \infty) \times \mathbb{R}$, and subscripts denote partial derivatives. The Burgers equation (2.1) is driven by the right hand side φ_x, which is determined by solving the Poisson equation (2.2).

Using the Green's function $G(x) = -\frac{1}{2}e^{-|x|}$ (of $\partial_{\mathbf{x}}^2 - 1$) to define the convolution operator

$$\varphi[u](x) = (G * u)(x) = \int_{\mathbb{R}} G(x-y)u(y)\,dy \tag{2.3}$$

the BP-system reduces to the single BP-equation

$$u_t + uu_x = \varphi_x[u], \tag{2.4}$$

with the obvious notation $\varphi_x[u] := (\varphi[u])_x$.

A rescaled version of (2.4) was considered by Whitham [Whi, Chapter 13.14] as a shallow water equation featuring weaker dispersivity than the Korteweg-de Vries (KdV)-equation.

The study of (2.1), (2.2) has been motivated by earlier work [CDMS1], [CDMS2] on two-species-Euler-Poisson (2SEP)-systems modeling the dynamics of 2 oppositely charged species of particles subject to Coulomb interaction. A simple version in dimensionless form is given by

$$\rho_t + (\rho u)_x = 0, \tag{2.5}$$

$$u_t + uu_x + \tfrac{\rho_x}{\rho} = \varphi_x, \tag{2.6}$$

$$\varepsilon\varphi_{xx} = \rho - e^{-\varphi}. \tag{2.7}$$

Here, the unknowns ρ, u and φ depend on position $x \in \mathbb{R}$ and time $t > 0$. The system (2.5), (2.6) are the isothermal Euler equations for the first species of particles with density ρ and velocity u. In the Poisson equation (2.7) for the electrostatic potential φ, the density of the second species is modeled by the equilibrium approximation $e^{-\varphi}$, resulting from an equation like (2.6) with the first two terms neglected and the opposite sign on the right hand side. The dimensionless parameter ε denotes the scaled Debye length.

In view of the formal similarities between the BP- and the 2SEP-system, we shall use the terms position, time, velocity and potential for the variables x, t, u and φ, respectively. Note that the velocity (instead of the density) appears on the right hand side of the Poisson equation. Nevertheless one can think the BP-system as a caricature of the 2SEP-system with Burgers equation replacing the Euler equations, and the potential terms in (2.2) coming from a linearization of the two-species Poisson equation (2.7).

The BP-system has a number of interesting formal properties, collected in section 2.2. In particular, we mention its relation to the Camassa-Holm equation [CH], [CHH] and the Benjamin-Ono equation [Ben], [Ono], its Galilean invariance, its Hamiltonian structure as well as the existence of an entropy.

In section 2.3, a general traveling wave analysis of the BP-system is performed recovering the result of Fornberg and Whitham [FW] in the particular case of solitary waves. It turns out that the traveling wave structure of the BP-system and several versions of the 2SEP-system (see [CDMS1], [CDMS2]) are qualitatively equivalent. The section is completed with some numerical experiments.

In section 2.4, existence and uniqueness of smooth solutions locally in time for smooth initial data are proven. Recently, for two related problems, the Euler-Poisson system [ELT] and the Camassa-Holm equation [CE], [RoB], global existence of smooth solutions has been shown under certain conditions on the initial data. The methods employed there do not apply to the BP-system. Also our numerical experiments with the BP-system indicate that comparable results are not true. A global existence result for weak entropy solutions with initial data in $BV(\mathbb{R})$ is also derived. A comparable result has recently been shown for a radiating gas model [LM], which is obtained from the BP-system replacing u by $-u_x$ in the right hand side of (2.2).

Finally, the rescaling $x \to x/\varepsilon, t \to t/\varepsilon, 0 < \varepsilon \ll 1$, is introduced in (2.1), (2.2), leading to

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varphi_x^\varepsilon, \tag{2.8}$$

$$\varepsilon^2 \varphi_{xx}^\varepsilon = \varphi^\varepsilon + u^\varepsilon. \tag{2.9}$$

A Chapman-Enskog expansion of (2.8), (2.9) recovers the Burgers equation with flux $(u^\varepsilon + 1)^2/2$ and the leading order perturbation $\varepsilon^2 u_{xxx}^\varepsilon$. For a rescaled system, there exists a direct asymptotic towards the KdV-equation for $\varepsilon \to 0$. The traveling wave analysis and numerical simulations suggest that the quasineutral limit $\varepsilon \to 0$ in general is a weak limit, both for the 2SEP- and the BP-systems. Here, a result is shown for the BP-system which has been proved for a 2SEP-system in [CG] and for the radiating gas model in [LM]: convergence to smooth solutions of the formal limit problem. In general this is only a local-in-time result since the limiting inviscid Burgers equation can develop singularities in finite time. The situation is the same for 2SEP-system, but not for the radiating gas model, where the limiting equation is the viscous Burgers equation with global smooth solutions.

2.2 Formal Properties

Firstly, we rewrite the BP-system as a single differential equation for u. By applying $1 - \partial_{\mathrm{x}}^2$ to (2.1) and using (2.2) on the resulting right hand side, we calculate:

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx}. \tag{2.10}$$

The terms in (2.10) correspond to those in the Camassa-Holm equation [CH]:

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{2.11}$$

where the constant $\kappa \geq 0$ is related to the critical shallow water wave speed. Vice versa, the Camassa-Holm equation (2.11) can be written in "BP form":

$$u_t + uu_x = \varphi_x \,, \tag{2.12}$$

$$\varphi_{xx} = \varphi + 2\kappa u + u^2 + \tfrac{1}{2}u_x^2 \,. \tag{2.13}$$

Note that for $\kappa = 1/2$ the BP-system is recovered by neglecting the two quadratic terms in (2.13).

The Camassa-Holm equation was introduced by Fokas and Fuchssteiner [FF] as formally integrable bi-Hamiltonian generalization of the Korteweg-DeVries equation. Camassa and Holm [CH] rediscovered it as shallow water equation by approximating the Hamiltonian for the vertically averaged incompressible Euler equations. By the bi-Hamiltonian property, they derived an infinite sequence of conservation laws and showed that the associated flows of this hierarchy are iso-spectral and, thus, completely integrable.

Most commonly (cf. [CHH], [CE], [Con], [RoB]), the Camassa-Holm equation (2.11) is considered with $\kappa = 0$. Then the Camassa-Holm equation possesses peaked soliton solutions (called peakons), attractive traveling waves of the form $u(x,t) = c\exp(-|x - ct|)$ and other breaking phenomena, which is desirable for a shallow water equation and in contrast to the KdV-behavior. For some initial data (e.g. with sufficiently large negative slope [Con], [RoB]) the solution develops verticality within finite time. On the other hand, global well-posedness was proved ([CE], [RoB]) for initial data $u_0 \in H^s(\mathbb{R})$ with $s > 3/2$ provided that $\int |u_0|\, dx < \infty$ and $(1 - \partial_{\mathrm{x}}^2)u_0$ does not change sign.

The Camassa-Holm equation is remarkable since it combines complete integrability with the formation of singularities.

In the existence analysis of section 2.4, (2.4) will be considered, subject to the initial conditions

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.14}$$

Note that (2.4) contains additional information compared to (2.1), (2.2), since for bounded u, (2.3) is the unique bounded solution of the Poisson equation (2.2). The properties of the solution operator of the Poisson equation in a L^2-setting will be used, in particular the smoothing

$$\|\varphi[u]\|_{H^{k+2}(\mathbb{R})} \leq \|u\|_{H^k(\mathbb{R})}, \qquad u \in H^k(\mathbb{R}), k \geq 0 \tag{2.15}$$

and the symmetry

$$\int_{\mathbb{R}} \varphi[u]v\, dx = \int_{\mathbb{R}} \varphi[v]u\, dx, \qquad u, v \in L^2(\mathbb{R}), \tag{2.16}$$

following from the evenness of the Green's function G.

The BP-equation (2.4) becomes the Benjamin-Ono equation when $\varphi[u]$ is replaced by $-2H[u_x]$, where H is the Hilbert transform:

$$H[u] = \text{p.v.}\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy$$

The Benjamin-Ono equation arises in the study of long internal gravity waves in stratified fluids of great depth [Ben], [Ono]. It is a completely integrable Hamiltonian system [KLM] which possesses multi-soliton solutions [BK], [Cas]. There exists also the analogue of the inverse scattering method [AF] and Bäcklund transformations [Nak]. Although the dispersive regularization by the Hilbert transform is weaker compared to KdV (cf. [KPV]), the dispersion is strong enough that the Benjamin-Ono equation has globally smooth solutions for initial data $u_0 \in H^k(\mathbb{R})$, $k \geq 3/2$ (see [Ior], [Pon]), and even for sublinearly growing initial data [FL].

The BP-system (2.1),(2.2) is *Galilean invariant*, i.e. it is invariant under changes of the reference frame of the form

$$x \to x + x_0 + u_0 t, \qquad u \to u + u_0, \qquad \varphi \to \varphi - u_0.$$

Note that the potential transforms like a velocity. The Galilean invariance will simplify the traveling wave analysis in section 2.3.

The *dispersion relation* of the BP-equation (2.4) linearized at $u = c$ is given by

$$\frac{\omega}{k} = c + \frac{1}{1+k^2}, \tag{2.17}$$

with the frequency ω and the wave number k. The group velocities lie between c and $c+1$, the limits for large and small wave numbers, respectively. The existence of a finite limit for large wave numbers indicates that (2.4) does not have smoothing properties.

Finally, we look for *conservation laws*. Obviously, (2.4) can be written in conservation from:

$$u_t + \left(\frac{u^2}{2} - \varphi[u]\right)_x = 0 \qquad (2.18)$$

As a consequence, $\int_{\mathbb{R}} u\,dx$ is conserved for weak solutions with sufficiently strong decay for $x \to \pm\infty$. Multiplication of (2.1) by $u = \varphi_{xx} - \varphi$ leads to the second conservation law

$$(u^2)_t + \left(\frac{2}{3}u^3 + \varphi^2 - \varphi_x^2\right)_x = 0. \qquad (2.19)$$

Since we shall consider weak solutions based on the conservation law (2.18), the quantity $\int_{\mathbb{R}} u^2\,dx$ will only be conserved for smooth solutions. By the boundedness of the operator $\varphi_x[\cdot]$, we expect (as for the Burgers equation) non-uniqueness of weak solutions, which can be eliminated by an entropy condition. In section 2.4, weak solutions will be constructed satisfying the *entropy condition* (2.19) with the equality sign replaced by $\le$. Thus, for weak solutions, the entropy $\int_{\mathbb{R}} u^2\,dx$ is non-increasing. Note that, in contrast to the Burgers equation, not every convex function of u is an entropy density.

The *jump conditions* for entropic shocks with velocity s are those of the Burgers equation:

$$s = \frac{1}{2}(u_l + u_r), \qquad u_l > u_r, \qquad (2.20)$$

where u_l and u_r denote the left-sided and, respectively, right-sided limit of u at the shock.

The BP-equation has an *Hamiltonian structure* similar to the Benjamin-Ono equation. The bi-Hamiltonian structure of the Camassa-Holm equation is completely destroyed by dropping the quadratic terms in (2.13). With the Hamiltonian

$$H(u) = \frac{1}{2}\int_{\mathbb{R}}\left(-\varphi[u]u + \frac{u^3}{3}\right)dx,$$

(2.4) can be written as

$$u_t + \left(\frac{\delta H}{\delta u}\right)_x = 0,$$

where $\frac{\delta H}{\delta u} = -\varphi[u] + \frac{u^2}{2}$ is the L^2-representation of the Frechet-derivative of H. Note that this relies on the symmetry property (2.16) of $\varphi[\cdot]$. Conservation of the quantity $\int_{\mathbb{R}} H(u)\,dx$ corresponds to the third local conservation law

$$\left(-\varphi u + \frac{u^3}{3}\right)_t + \left(\varphi_x\varphi_t - \varphi\varphi_{xt} - \left(-\varphi + \frac{u^2}{2}\right)^2\right)_x = 0$$

which (as (2.19)) can only be expected to hold for smooth solutions.

2.3 Traveling Wave Analysis

The results of this section should be compared to those of [CDMS1] and [CDMS2] for different versions of the Euler-Poisson system. The qualitative similarities of the results have been one of the main motivations for this work.

By the Galilean invariance it suffices to consider only traveling waves with velocity 0, i.e. steady states. Traveling waves with velocity c are then found by adding the constant c to the velocity u (and $-c$ to the potential φ). After integration of the steady state version of (2.1) and using the result in (2.2), the steady state equations can be written as

$$uu_x = E, \tag{2.21}$$

$$E_x = \frac{u^2}{2} + u - d, \tag{2.22}$$

where we have used the notation $\varphi_x = E$ and d is the constant of integration. The system (2.21), (2.22) will be studied in the (u, E)-phase-plane. We shall also allow shocks (of course with velocity $s = 0$) satisfying the jump conditions (2.20):

$$-u_r = u_l > 0.$$

By (2.22), E is continuous across shocks.

Also worth mentioning is the line of singularities $u = 0$. In general, trajectories end (or begin) there with square root behavior. By (2.21), smooth trajectories can only cross $u = 0$ through the origin of the (u, E)-plane. Our analysis will be restricted to $d > -1/2$, whence there are two stationary points

$$P_\pm = \begin{pmatrix} E_\pm \\ u_\pm \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \pm \sqrt{1+2d} \end{pmatrix}.$$

It is easily seen that u_- is always negative and a saddle. The second stationary point u_+ is negative and a center for $-1/2 < d < 0$. It becomes positive and a saddle for $d > 0$.

By separation of variable, a first integral of (2.21), (2.22) can be found:

$$A = \frac{E^2}{2} - \frac{u^4}{8} - \frac{u^3}{3} + \frac{du^2}{2} \tag{2.23}$$

Besides the stationary points, this family of curves (parameterized by A) has the origin as a critical point. Only away from the line $u = 0$, these curves can be seen as trajectories (with opposite orientation on opposite sides of $u = 0$).

Depending on the value of d, three generic cases of phase portraits occur.

Solitary Waves for $-1/2 < d \leq -4/9$:

The phase portraits for $-1/2 < d < -4/9$ are characterized by a homoclinic orbit (pulse, solitary wave) connecting P_- to itself (see Fig. 2.1.1, Fig. 2.1.2).

Figure 2.1: Solitary Wave

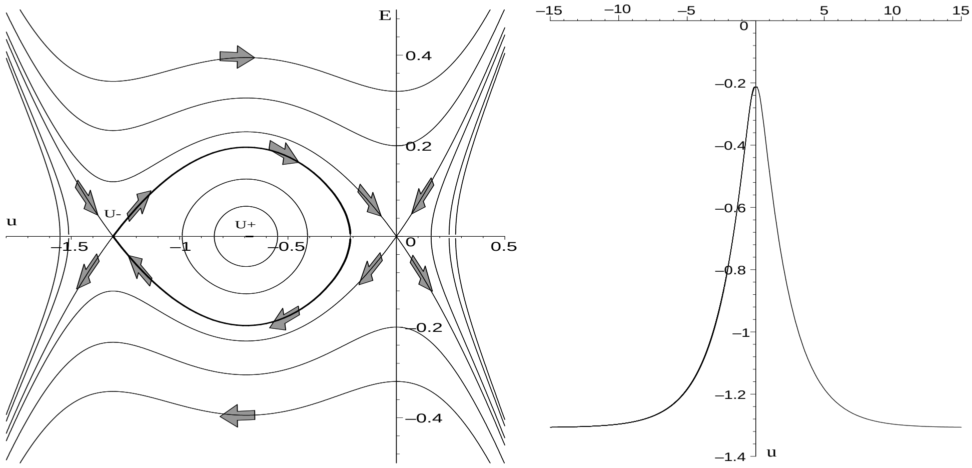

Fig. 2.1.1: Phase portrait for $-\frac{1}{2} < d \leq -\frac{4}{9}$

Fig. 2.1.2: Solitary wave

Its interior is filled with periodic solutions around P_+. These features are reminiscent of the KdV-equation. By the singularity, the origin is a point of non-uniqueness for the initial value problem. Taylor expansion shows a pair of smooth trajectories passing through the origin. An implicit formula for the solitary waves has already been calculated in [FW] together with a numerical simulation of the soliton like interaction of two solitary waves.

In the critical case $d = -4/9$, the trajectories through the origin coincide with the stable and unstable manifolds of P_-. As a consequence of the non-uniqueness, we can switch from the unstable to the stable manifold at the origin, producing a non-smooth solitary wave, reminiscent of the peakon solutions of the Camassa-Holm equation. It can be computed explicitly:

$$u(x) = \frac{4}{3}\left(e^{-|x|/2} - 1\right)$$

Peaked periodic solutions for $-4/9 < d < 0$:

In this case the solitary wave disappears and the trajectories passing through the origin connect to themselves (see Fig. 2.2.1). This closed curve in the left half plane corresponds to a peaked periodic solution (see Fig. 2.2.2). Again, these solutions can be computed explicitly. Taking the derivative of (2.21) and using (2.23) for the evaluation of u_x^2, we obtain (with A=0):

$$u_{xx} = \frac{u}{4} + \frac{1}{3}$$

Figure 2.2: Peaked periodic solution

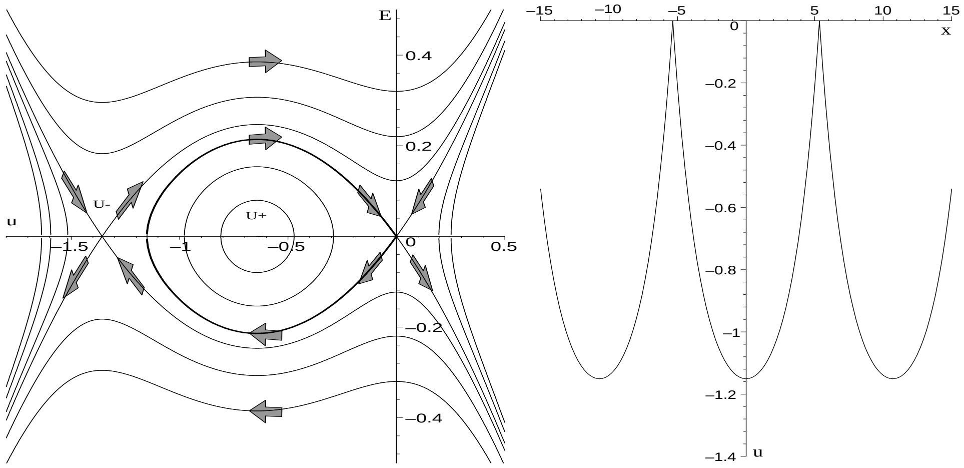

Fig. 2.2.1: Phase portrait for $-\frac{4}{9} < d < 0$ Fig. 2.2.2: Peaked periodic solution

The peaked periodic solution is given by

$$u(x) = \frac{4}{3}\left(\frac{\cosh(x/2)}{\cosh(p/2)} - 1\right)$$

for $-p < x < p$ and by periodic continuation with period $2p$. The length of the period is connected to the parameter d by

$$\cosh\left(\frac{p}{2}\right) = \left(1 + \frac{9}{4}d\right)^{-1/2}.$$

Heteroclinic connections for $d > 0$:

In this case, the stationary points are saddles and lie on opposite sides of the line $u = 0$ (see Fig. 2.3.1). A heteroclinic connection (front wave) between them can be constructed using an entropic shock. There is a unique pair of points $P_l = (u_l, E_l) = (\sqrt{1+2d}, \sqrt{11/12 + 2d})$, $P_r = (-u_l, E_l)$, satisfying the jump conditions, with P_l lying on the unstable manifold of P_+ and P_r on the stable manifold of P_-. The u-component of the heteroclinic solution is depicted in Fig. 2.3.2.

Remark 4 *The question arises, if two arbitrary constant states $u_{-\infty}$, u_∞ can be connected by a front wave. The answer is negative. The set of admissible pairs $(u_{-\infty}, u_\infty)$ is constructed by shifting pairs $(u_+(d), u_-(d))$, $d > 0$, (exploiting the Galilean invariance). This leads to the requirement $u_\infty - u_{-\infty} > 2$.*

Figure 2.3: Shock solution

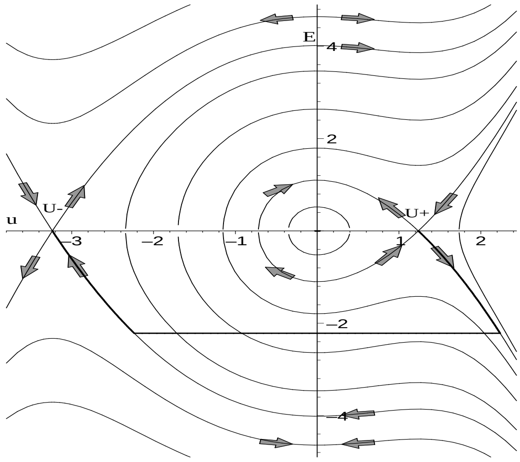

Fig. 2.3.1: Phase portrait for $d > 0$

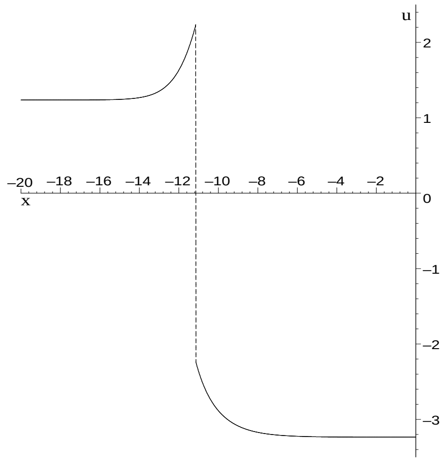

Fig. 2.3.2: Shock solution

Transient Behavior, Numerical Experiments

We have studied numerically the transient behavior of solutions of the BP-equation (2.4). A MATLAB program was written employing a straightforward explicit discretization: In a first step, the Poisson equation is solved for given u at the old time step. A centered difference scheme is used on a bounded interval with boundary conditions $\varphi + u = 0$ at both ends. The result is used for the evaluation of the right hand side of (2.4). Alternatively, we used an implicit spectral method (cf. [CS, Part II, Chapter 8]) and obtained very similar results. This spectral method is due to the use of FFT between 3-4 times faster but it applies only for specially periodic situations.

The Burgers-flux term is discretized by the Lax-Friedrichs method. Time steps are chosen according to the CFL-condition. As initial data, linear ramps connecting two constant states are prescribed. Recalling remark 4, we are interested in the behavior depending on the difference between the asymptotic states at $x = \pm\infty$.

Our results for downward ramps of height larger than 2 suggest the conjecture that the heteroclinic waves constructed above are attractive. For a typical example see Fig. 2.4.1 and Fig. 2.4.2. For a ramp with height 3 and the constant states lying symmetric with respect to $u = -1$, we observe numerical convergence to the stationary solution of type of Fig. 2.3.2. The development of shocks seems not to depend on the steepness of the ramp.

A completely different behavior is observed for initial ramps with a height less than 2: in this case, there exists no stationary solution connecting the

Figure 2.4: Time development of two initial ramps

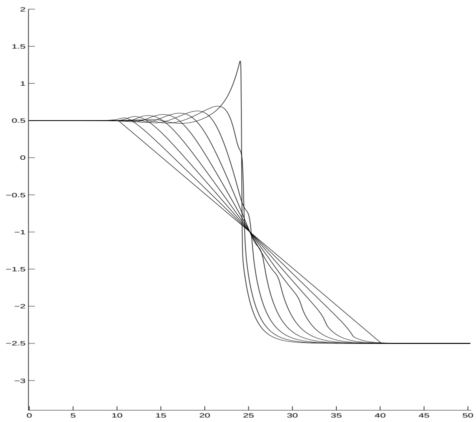

Fig. 2.4.1: Initial ramp: $0.5 \searrow -2.5$

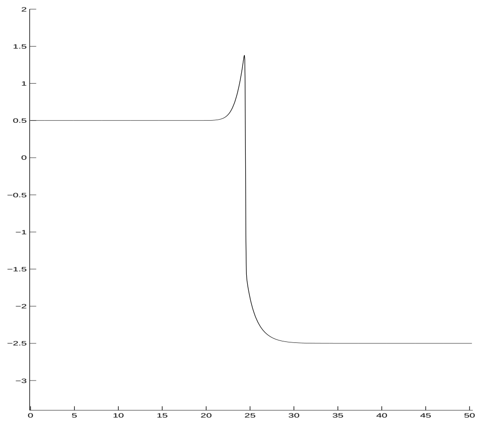

Fig. 2.4.2: Numerical stationary solution

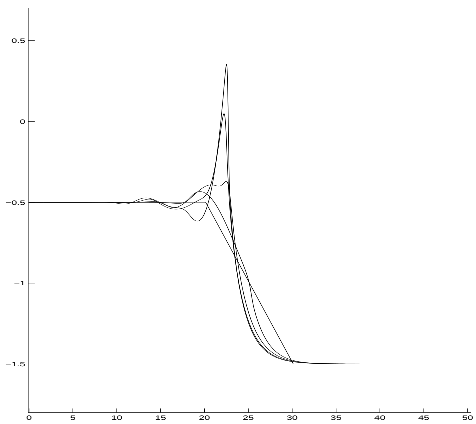

Fig. 2.4.3: Initial ramp: $-0.5 \searrow -1.5$

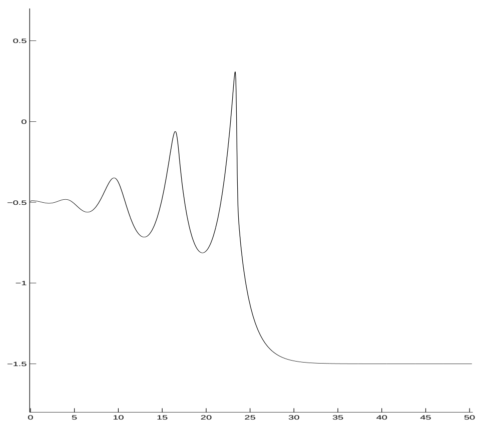

Fig. 2.4.4: Numerical quasistationary solution

asymptotic states. The observed behavior is a reminiscent of the KdV-equation and shows typical dispersive effects with oscillations at the left side of a smoothed ramp. This is in accordance with the dispersion relation (2.17) showing that high frequency components travel with lower velocities. The example depicted in Fig. 2.4.3 and Fig. 2.4.4 involves an initial ramp of height 1. Again, long time convergence has been observed. However, this is a numerical artefact, since no steady state solution of the continuous problem with the qualitative behavior shown in Fig. 2.4.4 exists.

2.4 Existence

In this section, the initial value problem

$$u_t + uu_x = \varphi_x[u], \qquad u(x,0) = u_0(x) \tag{2.24}$$

is considered, where the operator $\varphi[\cdot]$ is defined in (2.3).

Theorem 8 ***(Local strong solution)*** *Assume $u_0 \in H^k(\mathbb{R})$ with $k > \frac{3}{2}$. Then, there exists a time $T > 0$ and a unique solution*

$$u \in L^\infty\Big((0,T); H^k(\mathbb{R})\Big) \cap C\Big([0,T]; H^{k-1}(\mathbb{R})\Big)$$

of (2.24).

Proof: The proof is based on a contraction argument similar to [Tay, Section 16.1]. We define the iteration map S_T as follows: for any function $v \in B_T$ with

$$\begin{aligned} B_T \ &:= \ \Big\{ w \in L^\infty\Big((0,T), H^k(\mathbb{R})\Big) \cap C\Big([0,T]; H^{k-1}(\mathbb{R})\Big) \ : \\ &\qquad \sup_{t\in[0,T]} \|w(\cdot,t)\|_{H^k(\mathbb{R})} \le 2\|u_0\|_{H^k(\mathbb{R})} \Big\} \end{aligned}$$

the image $S_T(v)$ is the unique solution u of

$$u_t + uu_x = \varphi_x[v], \qquad u(t=0) = u_0. \tag{2.25}$$

First, we show that S_T maps B_T into itself for T small enough. We apply $\partial_{\mathbf{x}}^\alpha$ to (2.25) for $\alpha \le k$ and take the L^2-scalar product with $\partial_{\mathbf{x}}^\alpha u$:

$$\frac{1}{2}\frac{d}{dt}\big\|\partial_{\mathbf{x}}^\alpha u\big\|_{L^2(\mathbb{R})}^2 + \underbrace{\int_{\mathbb{R}} \partial_{\mathbf{x}}^\alpha (uu_x)\, \partial_{\mathbf{x}}^\alpha u \, dx}_{A} = \underbrace{\int_{\mathbb{R}} \varphi_x[\partial_{\mathbf{x}}^\alpha v]\, \partial_{\mathbf{x}}^\alpha u \, dx}_{B} \tag{2.26}$$

The first factor in the integrand of A is differentiated by the product rule:

$$\partial_{\mathbf{x}}^\alpha (uu_x) = u\, \partial_{\mathbf{x}}^{\alpha+1} u + \sum_{l=1}^{\alpha} \binom{\alpha}{l} \partial_{\mathbf{x}}^l u\, \partial_{\mathbf{x}}^{\alpha+1-l} u$$

Accordingly, A is split into two parts which are estimated separately:

$$\left|\int_{\mathbb{R}} u\,(\partial_{\mathbf{x}}^{\alpha+1}u)(\partial_{\mathbf{x}}^{\alpha}u)\,dx\right| = \frac{1}{2}\left|\int_{\mathbb{R}} u\,\partial_{\mathbf{x}}(\partial_{\mathbf{x}}^{\alpha}u)^2\,dx\right| \leq \frac{1}{2}\|u_x\|_{L^\infty(\mathbb{R})}\|u\|^2_{H^k(\mathbb{R})}. \tag{2.27}$$

For the second part of A, we use the Cauchy-Schwartz inequality and the interpolation estimate

$$\left\|\left(\partial_{\mathbf{x}}^{l-1}f_x\right)\left(\partial_{\mathbf{x}}^{\alpha-l}g_x\right)\right\|_{L^2(\mathbb{R})} \leq c\left(\|f_x\|_{L^\infty(\mathbb{R})}\|g\|_{H^\alpha(\mathbb{R})} + \|f\|_{H^\alpha(\mathbb{R})}\|g_x\|_{L^\infty(\mathbb{R})}\right)$$

(see [Tay, Chapter 13, Proposition 3.6]) to obtain:

$$\left|\int_{\mathbb{R}} \partial_{\mathbf{x}}^{\alpha}u\sum_{l=1}^{\alpha}\binom{\alpha}{l}\partial_{\mathbf{x}}^{l-1}u_x\,\partial_{\mathbf{x}}^{\alpha-l}u_x\,dx\right| \leq c\|u\|^2_{H^k(\mathbb{R})}\|u_x\|_{L^\infty(\mathbb{R})},$$

with some constant c which depends only on k. By the Sobolev embedding $W^{1,\infty}(\mathbb{R}) \hookrightarrow H^k(\mathbb{R})$ for $k > 3/2$, we calculate

$$|A| \leq c\|u\|^3_{H^k(\mathbb{R})}. \tag{2.28}$$

For B, we apply the Cauchy-Schwartz inequality and (2.15):

$$|B| \leq \|\varphi_x[\partial_{\mathbf{x}}^{\alpha}v]\|_{L^2(\mathbb{R})}\|\partial_{\mathbf{x}}^{\alpha}u\|_{L^2(\mathbb{R})} \leq c\|v\|_{H^k(\mathbb{R})}\|u\|_{H^k(\mathbb{R})}. \tag{2.29}$$

Using (2.28) and (2.29) in (2.26) gives

$$\frac{d}{dt}\|u\|_{H^k(\mathbb{R})} \leq c\left(\|u\|^2_{H^k(\mathbb{R})} + \|v\|_{H^k(\mathbb{R})}\right).$$

For T small enough, a comparison principle shows $\|u(\cdot,t)\|_{H^k(\mathbb{R})} \leq 2\|u_0\|_{H^k(\mathbb{R})}$ for $0 \leq t \leq T$. Since $u \in C\big([0,T];H^{k-1}(\mathbb{R})\big)$ is an obvious consequence of (2.25), we conclude that $S_T : B_T \to B_T$. In a second step, we prove S_T to be a strict contraction. Therefore, we consider two functions v_1 and v_2 in B_T and set $u_1 = S_T(v_1)$, $u_2 = S_T(v_2)$ and $u = u_1 - u_2$, $v = v_1 - v_2$. The difference of the equations for u_1, u_2 reads as

$$u_t + u(u_1)_x + u_2u_x = \varphi_x[v] \qquad u(t=0) = 0.$$

We proceed similarly to (2.25) using B from (2.26):

$$\tfrac{1}{2}\tfrac{d}{dt}\|\partial_{\mathbf{x}}^{\alpha}u\|^2_{L^2(\mathbb{R})} + \underbrace{\int_{\mathbb{R}}\partial_{\mathbf{x}}^{\alpha}(u\partial_{\mathbf{x}}u_1)\,\partial_{\mathbf{x}}^{\alpha}u\,dx}_{A_1} + \underbrace{\int_{\mathbb{R}}\partial_{\mathbf{x}}^{\alpha}(u_2u_x)\,\partial_{\mathbf{x}}^{\alpha}u\,dx}_{A_2} = B$$

In contrast to (2.27), the highest order term of A_1 is not bounded in terms of the $H^\alpha(\mathbb{R})$-norm. Therefore, we are obliged to reduce the order of differentiation to $\alpha \leq k-1$ and estimate as above for the second part of A:

$$\begin{aligned}|A_1| &\leq c\|u\|_{H^{k-1}(\mathbb{R})}\left(\|\partial_{\mathbf{x}}u_1\|_{L^\infty(\mathbb{R})}\|u\|_{H^{k-1}(\mathbb{R})} + \|u\|_{L^\infty(\mathbb{R})}\|u_1\|_{H^k(\mathbb{R})}\right)\\ &\leq c\|u\|^2_{H^{k-1}(\mathbb{R})}\|u_1\|_{H^k(\mathbb{R})},\end{aligned}$$

For A_2, we proceed as in (2.27)-(2.28).

$$
\begin{aligned}
|A_2| &= \left| \int_{\mathbb{R}} \left(u_2\, \partial_{\mathbf{x}} (\partial_{\mathbf{x}}^{\alpha} u)^2/2 + \partial_{\mathbf{x}}^{\alpha} u \sum_{l=1}^{\alpha} \binom{\alpha}{l} \partial_{\mathbf{x}}^{l} u_2\, \partial_{\mathbf{x}}^{\alpha-l} u_x \right) dx \right| \\
&\leq c \|u\|_{H^{k-1}(\mathbb{R})}^2 \|u_2\|_{H^k(\mathbb{R})}
\end{aligned}
$$

Since $\|u_{1,2}\|_{H^k(\mathbb{R})} \leq 2\|u_0\|_{H^k(\mathbb{R})}$, this leads to

$$
\frac{d}{dt} \|u\|_{H^{k-1}(\mathbb{R})} \leq c \left(\|u\|_{H^{k-1}(\mathbb{R})} + \|v\|_{H^{k-1}(\mathbb{R})} \right),
$$

and the Gronwall lemma implies that for T small enough, S_T is a strict contraction with respect to the topology in $C\big([0,T]; H^{k-1}(\mathbb{R})\big)$. ■

Theorem 9 ***(Global weak solution)*** *Assume $u_0 \in BV(\mathbb{R})$. Then, there exists a global weak solution*

$$
u \in L^{\infty}_{loc}\left([0,\infty); BV(\mathbb{R})\right) \tag{2.30}
$$

of (2.24), satisfying the entropy condition

$$
(u^2)_t + \left(\frac{2}{3} u^3 + \varphi^2 - \varphi_x^2 \right)_x \leq 0 \tag{2.31}
$$

in the distributional sense.

Proof: The proof is based on the viscosity method similarly to [Tay]. Instead of (2.24), we consider the regularized equation

$$
u_t + u u_x = \varphi_x[u] + \nabla_{\mathbf{x}} \mathbf{u} u_{xx} \tag{2.32}
$$

with $\nabla_{\mathbf{x}} \mathbf{u} > 0$. Local existence of a unique smooth solution of the initial value problem for (2.32) with $u(t=0) = u_0 \in BV(\mathbb{R})$ can be shown by standard arguments. The next step is an L^1-stability result for (2.32). Let u_1, u_2 denote solutions of (2.32) with initial data $f_1, f_2 \in L^1(\mathbb{R})$. Then, the difference $v = u_1 - u_2$ satisfy

$$
v_t + (wv)_x = \varphi_x[v] + \nabla_{\mathbf{x}} \mathbf{u} v_{xx}, \qquad v(t=0) = f_1 - f_2, \tag{2.33}
$$

with $w = (u_1 + u_2)/2$. Let $\mathrm{abs}_\delta(\cdot)$ be a convex regularization of the modulus, uniformly converging to $|\cdot|$ as $\delta \to 0$, and satisfying $|\mathrm{abs}'_\delta(v)| \leq 1$. Multiplication of (2.33) with $\mathrm{abs}'_\delta(v)$ and integration with respect to x leads to

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} \mathrm{abs}_\delta(v)\, dx &= \int_{\mathbb{R}} wv\, \mathrm{abs}''_\delta(v) v_x\, dx - \int_{\mathbb{R}} \mathrm{abs}'_\delta(v) \varphi_x[v]\, dx \\
&\quad - \nabla_{\mathbf{x}} \mathbf{u} \int_{\mathbb{R}} \mathrm{abs}''_\delta(v)\, (v_x)^2\, dx.
\end{aligned} \tag{2.34}
$$

Since the function $\int_0^v s\,\mathrm{abs}_\delta''(s)\,ds$ converges to zero uniformly as $\delta \to 0$, we have for the first term on the right hand side of (2.34):

$$\int_{\mathbb{R}} wv\,\mathrm{abs}_\delta''(v) v_x\,dx = -\int_{\mathbb{R}} w_x \int_0^v s\,\mathrm{abs}_\delta''(s)\,ds\,dx \to 0 \qquad \text{as } \delta \to 0.$$

Boundedness of the operator $\varphi_x[\cdot] : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ can be shown easily. With the properties of abs_δ, we obtain from (2.34) in the limit $\delta \to 0$:

$$\frac{d}{dt}\|v\|_{L^1(\mathbb{R})} \le c\|v\|_{L^1(\mathbb{R})}. \tag{2.35}$$

Analogously to [Tay, Chapter 16, Lemma 6.1], it can be shown that

$$\frac{d}{dt}\|v\|_{BV(\mathbb{R})} \le c\|v\|_{BV(\mathbb{R})}.$$

holds for the solution of (2.32) as a consequence of (2.35). This is sufficient for proving that the solution of (2.32) is global and bounded in $L^\infty_{loc}([0,\infty); BV(\mathbb{R}))$ uniformly in $\nabla_{\mathbf{x}}\mathbf{u}$. This again is sufficient for passing to the limit $\nabla_{\mathbf{x}}\mathbf{u} \to 0$ in (2.32). For the details we refer to [Tay]. The entropy inequality (2.31) follows from a standard argument. ■

2.5 Asymptotics and the Quasi-neutral Limit

In this section, we investigate the rescaled ($x \to x/\varepsilon, t \to t/\varepsilon$) BP-system

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varphi_x^\varepsilon, \tag{2.36}$$

$$\varepsilon^2 \varphi_{xx}^\varepsilon = \varphi^\varepsilon + u^\varepsilon, \tag{2.37}$$

with $\varepsilon \ll 1$. In accordance with the terminology taken from the Euler-Poisson system, the limit $\varepsilon \to 0$ will be called the quasi-neutral limit. With the rescaled potential operator

$$\varphi^\varepsilon[u](x) = -\frac{1}{2\varepsilon}\int_{\mathbb{R}} \exp\left(-\frac{|x-y|}{2\varepsilon}\right) u(y)dy\,,$$

the initial value problem

$$u_t^\varepsilon + u^\varepsilon u_x^\varepsilon = \varphi_x^\varepsilon[u^\varepsilon]\,, \qquad u^\varepsilon(t=0) = u_0\,, \tag{2.38}$$

will be compared to its formal limit

$$u_t^0 + (u^0+1)u_x^0 = 0\,, \qquad u^0(t=0) = u_0\,. \tag{2.39}$$

The limit is the inviscid Burgers equation for the unknown $u^0 + 1$. Even for smooth initial data its solution can develop shocks in finite time. The traveling wave analysis of section 2.3 can be seen as an attempt to approximate solution profiles in the neighborhood of shocks of (2.39). The heteroclinic connections computed in section 2.3 are such profiles connecting two states u_l and u_r satisfying the jump conditions

$$s = \frac{1}{2}\left(u_l + u_r + 2\right), \qquad u_l > u_r .$$

However, these connections only exist for $u_l - u_r > 2$. For a better understanding of the situation, we expand the potential operator

$$\varphi^\varepsilon[u] = -u - \varepsilon^2 u_{xx} + O(\varepsilon^4) .$$

Thus, an $O(\varepsilon^4)$-approximation of (2.38) is given by the Korteweg-de-Vries equation (for the unknown $u+1$)

$$u_t + (u+1)u_x + \varepsilon^2 u_{xxx} = 0 . \tag{2.40}$$

Actually, the Korteweg-de-Vries equation can be obtained as a formal limit of (2.38). If (2.38) is rescaled by

$$t \to \frac{t}{\varepsilon^2}, \qquad u^\varepsilon \to -1 + \varepsilon^2 U ,$$

then the formal limit of the resulting equation for U is

$$U_t + UU_x + U_{xxx} = 0 .$$

In analogy to the well known results concerning the limit as $\varepsilon \to 0$ of (2.40), we expect that in general the limits of solutions of (2.38) are weak limits, which do not satisfy the formal limiting equation (2.39). In our last result, however, we prove that — as long as the solution of the limiting equation remains smooth — it is the strong limit of the solution of (2.38). For $u_0 \in C^{0,1}(\mathbb{R})$ there exists a $T > 0$, such that the Burgers equation (2.39) has a solution $u^0 \in C\left([0,T];\ C^{0,1}(\mathbb{R})\right)$. Also, if $u_0 \in H^s(\mathbb{R})$, then $u^0 \in L^\infty((0,T);\ H^s(\mathbb{R}))$.

Theorem 10 *Assume $u_0 \in C^{0,1}(\mathbb{R}) \cap H^s(\mathbb{R})$ with $s > 1$. Then, for a time interval $(0,T)$ as mentioned above, the solutions of (2.38) and (2.39) satisfy*

$$\|u^\varepsilon - u^0\|_{L^\infty((0,T);L^2(\mathbb{R}))} = O\left(\varepsilon^r\right), \qquad r = \min\{2, s-1\} .$$

Proof: Let $v = u^\varepsilon - u^0$ with initial data $v(t=0) = 0$. We subtract (2.39) from (2.38) to obtain an equation for v:

$$v_t + \left(\frac{v^2}{2} + u^0 v\right)_x = u^0_x + \varphi^\varepsilon[u^0_x] = \varepsilon^2 \varphi^\varepsilon_{xx}[u^0_x].$$

By taking the L^2-scalar product with v and by integration by parts, we calculate

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R})} + \int_{\mathbb{R}} \frac{v^2}{2}\, u^0_x \, dx = \varepsilon^2 \int_{\mathbb{R}} v\, \varphi^\varepsilon_{xx}[u^0_x]\, dx,$$

which implies

$$\frac{d}{dt}\|v\|_{L^2(\mathbb{R})} \leq \frac{1}{2}\|u_x^0\|_{L^\infty(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} + \|\varepsilon^2\varphi_{xx}^\varepsilon[u_x^0]\|_{L^2(\mathbb{R})}. \tag{2.41}$$

With the Fourier transform $\widehat{u^0}(k,t)$ of $u^0(x,t)$, the last term can be estimated:

$$\|\varepsilon^2\varphi_{xx}^\varepsilon[u_x^0]\|_{L^2(\mathbb{R})} = \left\|\frac{\varepsilon^2|k|^3|\widehat{u^0}|}{1+\varepsilon^2k^2}\right\|_{L^2(\mathbb{R})} \leq \sup_{k\in\mathbb{R}} \frac{\varepsilon^2|k|^3}{(1+\varepsilon^2k^2)(1+k^2)^{s/2}}\|u^0\|_{H^s(\mathbb{R})}$$

The factor on the right hand side is obviously $O(\varepsilon^2)$ for $s \geq 3$. For $s < 3$, it can be estimated by

$$\frac{\varepsilon^2|k|^3}{(1+\varepsilon^2k^2)|k|^s} = \varepsilon^{s-1}\frac{|\varepsilon k|^{3-s}}{1+|\varepsilon k|^2} = O(\varepsilon^{s-1}),$$

and, thus,

$$\|\varepsilon^2\varphi_{xx}^\varepsilon[u_x^0]\|_{L^2(\mathbb{R})} \leq c\varepsilon^r\|u^0\|_{H^s(\mathbb{R})}.$$

The statement of the theorem is now a direct consequence of the Gronwall lemma applied to (2.41). ■

Bibliography

[AF] M. J. Ablowitz, A. S. Fokas, *The inverse scattering transform for the Benjamin-Ono equation - a pivot to multidimensional problems*, Stud. Appl. Math. Vol. 68 **1** (1983), pp. 1-10

[AN] L. Arkeryd, A. Nouri, *Boltzmann asymptotics with diffuse reflection boundary conditions*, Monatsh. Math. **123** (1997), pp. 285-298

[BP] R. Beals, V. Protopopescu, *Abstract time-dependent transport equations*, J. Math. Anal. Appl. **121** (1987), pp. 370-405.

[Ben] T. B. Benjamin, *Internal waves of permanent form in fluids of great depth*, J. Fluid Mech., Vol. 29 **3** (1967), pp. 559-592

[BK] T. L. Bock, M. D. Kruskal *A two-parameter Miura Transfomation of the Benjamin-Ono equation*, Physics Letters, Vol. 74A **3,4** (1979), pp. 173-176

[CH] R. Camassa, D. D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. vol. 71 **11** (1993), pp. 1661-1664

[CHH] R. Camassa, D. D. Holm, J. M. Hyman, *A New Integrable Shallow Water Equation*, Adv. Appl. Mech. vol. 31 (1994), pp. 1-33

[Cas] K. M. Case, *Benjamin-Ono related equations and their solutions*, Proc. Natl. Acad. Sci. USA **76** (1979), pp. 1-3

[Cer] C. Cercigniani, *Equilibrium States and trend to Equilibrium in a Gas According to the Boltzmann Equation*, Rendi. Mat. Appl. **10** (1990), pp. 77-95

[CIP] C. Cercigniani, R. Illner and M. Pulvirenti, The Mathematical Theory of Dilute Gases, Springer, New-York, 1994.

[CS] C. Cercignani, D. H. Sattinger, Scaling Limits and Models in Physical Processes, DMV Seminar Band 28, Birkhäuser Verlag, Basel, (1998)

[Con] A. Constantin, *On the Cauchy problem for the periodic Camassa Holm equation*, J. Diff. Equ. **141** (2) (1997), pp. 218-235

[CE] A. Constantin, J. Escher *Well-Posedness, Global Existence, and Blowup Phenomena for a Periodic Quasi-Linear Hyperbolic Equation*, Comm. Pure Appl. Math **51** (5) (1998), pp. 475-504

[CDMS1] S. Cordier, P. Degond, P. Markowich, Ch. Schmeiser, *Travelling wave analysis and jump relations for the Euler-Possion model in the quasineutral limit*, Asymtotic Analysis **11** (1995), pp. 209-240

[CDMS2] S. Cordier, P. Degond, P. Markowich, Ch. Schmeiser, *Travelling wave analysis of an isothermal Euler-Poisson model*, Annales de la Faculté des Sciences de Toulouse, Vol. V, **4** (1996), pp. 599-643

[CG] S. Cordier, E. Grenier *Quasineutral limit of an Euler-Poisson system arasing form plasma physics*, Comm. Part. Diff. Equ. **25** (2000), no. 5-6, pp. 1099-1113

[DP] P. Degond and P. F. Peyrard, *Un modéle de collisions ondes-particules en physique des plasmas: application à la dynamique des gaz*, C. R. Acad. Sci. Paris **323** (1996), pp. 209-214.

[DLP] P. Degond, J. L. Lopez and P. F. Peyrard, *On the macroscopic dynamics induced by a model wave-particle collision operator*, J. Cont. Mech. Therm. **10** (1998), pp. 153-178.

[DLPS] P. Degond, J. L. Lopez, F. Poupaud and C. Schmeiser, *Existence of solutions of a kinetic equation modelling cometary flows*, J. Stat. Phys. **96** (1999), pp. 361-376.

[D] L. Desvillettes, *Convergence to equilibrium in large time for Boltzmann and B.G.K. equations*, Arch. Rat. Mech. Analysis **110** (1990), pp. 73-91.

[DV] L. Desvillettes, C. Villani, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math. **54** (2001), pp. 1–42.

[EJM] J.Earl, J. R. Jokipii and G. Morfill, *Cosmic ray viscosity*, The Astrophysical Journal **331** (1988), L91.

[ELT] S. Engelberg, H. Liu, E. Tadmor, *Critical thresholds in Euler-Poisson equations*, Indiana Univ. Math. J. **50** (2001), pp. 109–157.

[FS] K. Fellner, C. Schmeiser, *Equilibrium distributions of a kinetic equation modelling cometary flows and special solutions of 3D Euler equation*, in preparation

[1] K. Fellner, F. Poupaud and C. Schmeiser, *Existence and convergence to equilibrium of a kinetic equation for cometary flows*, preprint

[FF] A. Fokas, B. Fuchssteiner, *Sympletic structures, their Bäcklund transformations and hereditary symmetries.*, Physica 4 D (1981), pp. 47-66

[FL] G. Fonseca, F. Linares, *Benjamin-Ono equation with Unbounded Data*, J. Math. Anal. Appl. **247** (2000), pp. 426-447

[FW] B. Fornberg, G. B. Whitham, *A numerical and theoretical study of certain nonlinear wave phenomena*, Philos. Trans. R. Soc. London A **227** (1978), pp. 373-404

[GLPS] F. Golse, P. L. Lions, B. Perthame and R. Sentis, *Regularity of the moments of the solution of a transport equation*,

[Ior] R. J. Iório, *On the Cauchy Problem for the Benjamin-Ono Equation*, Comm. Part. Diff. Equa., Vol. 11 **10** (1986), pp. 1031-1081

[LeV] R. J. LeVeque, Numerical Methods for Conservation Laws, Birkhäuser Verlag, Basel (1990)

[KS] S. Kaniel and M. Shinbrot, *The Boltzmann equation. I, uniqueness and local existence*, Commun. Math. Phys. **58** (1978), pp. 65-84.

[KLM] D. J. Kaup, T. I. Lakoba, Y. Matsuno, *Complete integrability of the Benjamin-Ono equation by means of action-angle variables*, Physics Letters A, Vol. 238 (1998), pp. 123-133 J. Funct. Anal. **74** (1988), pp. 110-125.

[KPV] C. E. Kenig, G. Ponce, L. Vega, *On the generalized Benjamin-Ono equation*, Trans. AMS, Vol. 342 (1994), pp. 155-172

[LM] C. Lattanzio, P. Marcati, *Global well-posedness and relaxation limits of a model for radiating gas*, preprint, Univ. Aquila (2002).

[Nak] A. Nakamura *Bäcklund Transfomations and Conservation Laws of the Benjamin-Ono equation*, J. Phys. Soc. Japan, Vol. 47, **4** (1979), pp. 1335-1340

[Per] B. Perthame, *Global existence to the BGK model of Boltzmann equation*, J. Diff. Equs. **82** (1989), pp. 191-205.

[PP] B. Perthame, M. Pulvirenti, *Weighted L^∞ Bounds and Uniqueness for the Boltzmann BGK Model*, Arch. Rat. Mech. Anal. **125**(3) (1993), pp. 289-295.

[Pon] G. Ponce, *On the global well-posedness of the Bejamin-Ono equation*, Differential Equations **4** (1991), pp. 527-542

[Ono] H. Ono, *Algebraic solitary waves in stratified fluids*, J. Phys. Soc. Japan, Vol. 39 **4** (1975), pp. 1082-1091

[RoB] G. Rodríguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Analysis Vol. 46 (2001), pp. 309-327

[Stix] T. H. Stix, Waves in Plasmas, American Institute of Physics, New-York, 1992.

[Tay] M. E. Taylor, Partial Differential Equation III, Nonlinear Equations, Springer Series Applied Mathematical Sciences Vol. 117, New York, (1996)

[Whi] G. B. Whitham, Linear and Nonlinear Waves, John Wiley & Sons Inc., (1974)

[WJ1] L. L. Williams and J. R. Jokipii, *Viscosity and inertia in cosmic-ray transport: effects of an average magnetic field*, The Astrophysical Journal **371** (1991), pp. 639-647.

[WJ2] L. L. Williams and J. R. Jokipii, *A single-fluid, self-consistent formulation of fluid dynamics and particle transport*, The Astrophysical Journal **417** (1993), pp. 725-734.

[WSJG] L. L. Williams, N. Schwadron, J. R. Jokipii and T. I. Gombosi, *A unified transport equation for both cosmic rays and thermal particles*, The Astrophysical Journal **405** (1993), pp. L79-L81.

List of Figures

Index

CURRICULUM VITAE:

Name:	Klemens Andreas Fellner
Date of birth:	11. August 1973
Place of birth:	Bad Ischl, Austria
Citizenship:	Austria
1979 - 1983	Primary school in Ebensee
1983 - 1984	"Hauptschule" in Ebensee
1984 - 1992	Gymnasium in Gmunden:
June 1992	Matura with distinction
1992 - 1998	Studies of "Technischen Mathematik", TU Wien
1992 - 1998	Studies of bassoon at the conservatoire of the city of Vienna
1993 -	Studies of "Technische Physik", TU Wien
October 1997	First diploma "Technische Physik" with dist.
March 1998	Diploma in "Technische Mathematik" with dist. master thesis: "Transmission-Line Simulation meines Fagottes"
Apr. 1998 - Feb. 1999	Employeé at FWF project P11308-MAT: *Expansion Methods for the Semiconductor Boltzmann Equation*
April 1998 - Dec 2001	Member TMR Network ERB FMRX CT97 0157: *Asymptotic Methods in Kinetic Theory*
Oct. 1998 -	PhD studies "Technische Wissenschaften", TU Wien, Supervisor: Prof. Chr. Schmeiser
Mar. 1999 - Sep. 1999	Assistent of Prof. P. Markowich, Johannes Kepler university, Linz
Oct. 1999 - Sep. 2000	Civil service at the "Österreichischen Multiple Sklerose Gesellschaft"
Oct. 2000 - Feb. 2001	TMR-Predoc visit at the Laboratoire Dieudonné, Université de Nice Sophia-Antipolis, France
Mar. 2001 - Dec. 2001	Employeé at FWF project P14876-Mat: *Fokker-Planck and Mean-Field Equations*
July, August 2001	Visit á CMI, Univ. Aix-Marseille 1, France
Jan. 2002 -	PhD member of the "Wissenschaftskolleg" *Differential Equation Models in Science and Engineering*, no. W008