DIPLOMARBEIT

Non-Commutative Two Plus One Dimensional Quantum Electrodynamics with Chern-Simons Term

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Contents

Introduction

The past decades have revealed several reasons for studying non-commutative field theories. Einstein's theory of general relativity together with quantum field theory, for instance, suggest the following problem [9, 10, 11]: According to Heisenberg's uncertainty relation, measuring the position of a point particle with high accuracy a will cause an uncertainty in momentum of the order $\frac{1}{a}$ (in natural units $\hbar = c = G = 1$). Therefore an energy of the order $\frac{1}{a}$ will be concentrated in the localized region, and the associated energy-momentum tensor $T_{\mu\nu}$ will generate a gravitational field which will be determined by solving Einstein's equations for the metric $g_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{i}$$

where $R_{\mu\nu}$ and R denote the Ricci tensor and the Ricci scalar, respectively. If the uncertainties Δx_μ in the measurement of coordinates become sufficiently small, which is the case near the Planck length

$$\Delta x_\mu \simeq \lambda_p = \sqrt{\frac{Gh}{c^3}} \simeq 10^{-33}\text{cm}, \tag{ii}$$

the gravitational field generated by the measurement will become so strong as to prevent light or other signals from leaving the region in question. To avoid black holes from being produced in the course of measurement one is tempted to introduce quantum, or non-commutative, space-time.

Apart from this motivation, there is also the fact that ultraviolet divergent terms appear in quantum field theories due to point particles interacting locally. This also suggests that at very small distances physical laws must be modified in such a way that interactions become *non-local*[1].

A possible way of achieving such behaviour is to replace space-time coordinates with operators $\hat{x}^\mu$ acting on a Hilbert space [13, 14, 15]. Classical fields can then be described by an isomorphism in which the operators $\hat{x}^\mu$ are

[1] An historical overview of the idea of non-commutative space-time can be found in [8].

replaced by classical coordinates x^μ on a θ-deformed space-time leading to the so-called star product or Weyl-Moyal product (see Chapter 1 for details). In the Feynman rules this deformation leads to modified vertices picking up phases, whereas propagators remain unchanged. These phase factors act as UV-regulators depending on external momenta p^μ. As these momenta tend to zero, the regularization scheme unfortunately breaks down and some UV-divergences are replaced by infrared divergences [22]. Furthermore, Feynman diagrams can be split up into parts with phases called *non-planar* graphs and parts independent of phases referred to as *planar* graphs[2]. Planar graphs must therefore be ultraviolet divergent, like their commutative counterparts. This behaviour leads to the so-called *UV/IR-mixing* problem.

Even though the singularities arising in quantum field theories cannot be eliminated through non-commutativity, there is still one further motivation to consider such models: It has recently been discovered that non-commutativity between coordinates appears in open string theories with a B-field background as well as in toroidal compactification of Matrix Theory [23, 24, 25, 26].

Here, our aim is to study non-commutative 2+1 dimensional quantum electrodynamics (QED) including fermions and see whether such a model remains finite, like its commutative counterpart [1]. This is mainly done (in Feynman gauge) in Chapter 2, after reviewing some basic properties of non-commutative space-time[3] in the first chapter. We then extend the model by adding a Chern-Simons term, which has the effect of making the gauge bosons A^μ *massive*, and discuss more general (Lorentz) gauges.

Finally, in Chapter 4 we add a further term to the action, namely the Slavnov term. This extension is motivated by the fact that infrared divergences in a non-commutative model can only be cancelled by introducing non-local counter terms in the renormalization scheme. This fact suggests that the model is incomplete. The Slavnov term was first introduced in [27] and mainly consists of a further Lagrange multiplier field λ multiplied with the field tensor contracted with $\theta^{\mu\nu}$, the matrix describing non-commutativity between (space) coordinates. It has the effect of constraining the so contracted field tensor to zero. Furthermore, it modifies the gauge boson propagator (in our case the photon propagator) in such a way that it becomes transversal with respect to momenta $\widetilde{p}^\mu \equiv \theta^{\mu\nu} p_\nu$. Then, if all infrared singu-

[2]Actually, in some cases even planar graphs may have phase factors but these must only depend on *external* momenta [19].

[3]In fact non-commutativity will be restricted to space coordinates in order to preserve causality and unitarity [21, 23].

lar terms in the vacuum polarization are proportional to $\widetilde{p}^{\mu}\widetilde{p}^{\nu}$ they will make no contribution to the correction of the photon propagator, as contraction with (transversal) internal bare propagators will amount to zero. Therefore, these singularities will no longer present a problem. Eventual ultraviolet singularities can (hopefully) be eliminated through standard renormalization as known from commutative models. In order to keep the main part of this study compact and to avoid getting lost in technical details, most calculations have been put into the appendices.

Chapter 1

The Star Product

In the Hilbert space of non-commutative flat Minkowski space $\mathbb{M}^n_{NC}$ the coordinates x^μ are replaced by selfadjoint operators $\hat{x}^\mu$ with $\mu = 0, 1, ..., (n-1)$. Following the work of Filk [13] by considering a canonical structure, these operators respect the commutation relations

$$\begin{aligned} [\hat{x}^\mu, \hat{x}^\nu] &= i\theta^{\mu\nu}, \\ [\theta^{\mu\nu}, \hat{x}^\rho] &= 0, \end{aligned} \tag{1.1}$$

where the matrix $\theta^{\mu\nu}$ is real, constant and antisymmetric[1]. In the special case of $2+1$ dimensions this matrix is given by[2]

$$\theta^{\mu\nu} = \theta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{1.2}$$

where θ is a real constant. In natural units where $\hbar = c = 1$ it has mass dimension $[\theta] = -2$. Note that this matrix means space-space non-commutativity, only. The reason for this choice is that space-time non-commutativity would lead to an acausal theory with non-unitary scattering matrices [21].

The product of functions on $\mathbb{M}^n_{NC}$ can then be viewed as a 'deformed' product called the 'star product' of ordinary functions on regular Minkowski space $\mathbb{M}^n$ [19]. First, one needs to define the operator

$$\hat{T}(k) = e^{ik_\mu \hat{x}^\mu}, \tag{1.3}$$

[1] Obviously, this construction leads to a violation of Lorentz symmetry. Discussions on this matter can be found in [16, 17, 18] and references therein.

[2] Any antisymmetric 3×3 matrix can be transformed into (1.2).

which has the properties

$$\hat{T}^\dagger(k) = \hat{T}(-k), \tag{1.4}$$

$$\hat{T}(k)\hat{T}(k') = \hat{T}(k+k')e^{-\frac{i}{2}k\times k'}, \tag{1.5}$$

$$\operatorname{tr}\hat{T}(k) = (2\pi)^n\,\delta^n(k_\mu), \tag{1.6}$$

where the abbreviation $k \times k' \equiv k_\mu \theta^{\mu\nu} k'_\nu$ has been used. From these relations one can furthermore derive

$$\operatorname{tr}\left(\hat{T}(k_1)\hat{T}(k_2)\cdots\hat{T}(k_m)\hat{T}^\dagger(k)\right) = (2\pi)^n\, e^{-\frac{i}{2}\sum\limits_{i<j}^{m} k_i\times k_j}\,\delta^n\left(\sum_{i=1}^{m} k_{i\mu} - k_\mu\right). \tag{1.7}$$

Next, one associates an operator $\hat{\Phi}(\hat{x})$ to the classical function $\Phi(x)$ in the following way[3]:

$$\begin{aligned}\hat{\Phi}(\hat{x}) &= \int d^n x \int \frac{d^n k}{(2\pi)^n}\hat{T}(k)e^{-ik_\mu x^\mu}\,\Phi(x)\\ &= \int \frac{d^n k}{(2\pi)^n}\hat{T}(k)\widetilde{\Phi}(k),\end{aligned} \tag{1.8}$$

where $\widetilde{\Phi}(k)$ denotes the Fourier transform of $\Phi(x)$. The classical function $\Phi(x)$ can then be recovered from $\hat{\Phi}(\hat{x})$ by using the trace:

$$\Phi(x) = \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu x^\mu}\operatorname{tr}\left(\hat{\Phi}(\hat{x})\hat{T}^\dagger(k)\right). \tag{1.9}$$

This relation suggests the following definition of the star product (also known as the Weyl-Moyal product):

$$\Phi_1(x)\star\Phi_2(x)\star\cdots\star\Phi_m(x) \equiv \int \frac{d^n k}{(2\pi)^n} e^{ik_\mu x^\mu}\operatorname{tr}\left(\hat{\Phi}_1(\hat{x})\hat{\Phi}_2(\hat{x})\cdots\hat{\Phi}_m(\hat{x})\hat{T}^\dagger(k)\right). \tag{1.10}$$

Inserting (1.8) and (1.7) into (1.10) leads to

$$\begin{aligned}\Phi_1(x)\star\Phi_2(x)\star\cdots\star\Phi_m(x) = \int \frac{d^n k_1}{(2\pi)^n}\int \frac{d^n k_2}{(2\pi)^n}\cdots\int \frac{d^n k_m}{(2\pi)^n} e^{i\sum\limits_{i=1}^{m} k_i^\mu x_\mu}\\ \times\,\widetilde{\Phi}_1(k_1)\widetilde{\Phi}_2(k_2)\cdots\widetilde{\Phi}_m(k_m)e^{-\frac{i}{2}\sum\limits_{i<j}^{m} k_i\times k_j},\end{aligned} \tag{1.11}$$

[3]Note that functions of operators must be understood as Taylor expansions in their arguments. The operator $\hat{T}(k)$ in (1.3) therefore actually reads $\hat{T}(k) = 1 + ik_\mu\hat{x}^\mu + \ldots$

where the $\widetilde{\Phi}_i$ again denote the Fourier transformed functions[4]. Using this formula one can easily verify the following properties of the star product:

$$\int d^n x \left(\Phi_1 \star \Phi_2\right)(x) = \int d^n x \Phi_1(x)\Phi_2(x), \tag{1.12a}$$

$$\int d^n x \left(\Phi_1 \star \Phi_2 \star \cdots \star \Phi_m\right)(x) = \int d^n x \left(\Phi_2 \star \cdots \star \Phi_m \star \Phi_1\right)(x), \tag{1.12b}$$

$$\frac{\delta}{\delta\Phi_1(y)} \int d^n x \left(\Phi_1 \star \Phi_2 \star \cdots \star \Phi_m\right)(x) = \left(\Phi_2 \star \cdots \star \Phi_m\right)(y). \tag{1.12c}$$

Furthermore, the star product is associative.

[4] Clearly, considering formula (1.11) the commutation relation $\left[x^\mu, x^\nu\right]_\star = i\theta^{\mu\nu}$ is fulfilled as well.

Chapter 2

Non-Commutative, Massless QED_3

Quantum Electrodynamics was first formulated in about 1950 and has been very successful in the past decades (i.e. it predicted the anomalous magnetic moment of the electron correctly to six decimal places) [32]. Ultraviolet divergences appearing in the model can be dealt with through *renormalization*, but the fact that they appear at all suggests that interactions should actually be *non-local*. Unfortunately, non-commutative QED in $3+1$ dimensions does not cure these divergences. Instead it leads to *UV/IR-mixing*, which is worse since the appearing infrared divergences would require non-local counter terms [2, 22]. On the other hand, if one reduces dimensions to $2+1$, QED becomes *finite* [1]. Therefore our aim in this chapter is to find out whether this is also true for *non-commutative* $2+1$ dimensional QED. We start with massless fermions and gauge bosons:

2.1 The model

The action of this model is given by

$$S = S_{inv} + S_{gf}, \tag{2.1}$$

with the gauge invariant part

$$S_{inv} = \int d^3x \left\{ -\frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} + i\bar{\psi} \star \not{D} \star \psi \right\}, \tag{2.2}$$

and the gauge fixing part[1]

$$S_{gf} = \int d^3x \left\{ B\partial^\mu A_\mu + \frac{\alpha}{2}B^2 - \bar{c}\star\partial^\mu\left(\partial_\mu c - ie[A_\mu, c]_\star\right)\right\}, \tag{2.3}$$

where B is the Lagrange multiplier field, c is the ghost and $\bar{c}$ is the antighost field. The electromagnetic field tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie[A_\mu, A_\nu]_\star, \tag{2.4}$$

and the covariant derivative is defined as

$$\not{D} = \gamma^\mu D_\mu \ , \qquad D_\mu = \partial_\mu - ieA_\mu. \tag{2.5}$$

Note that even in the Abelian theory the commutators $[A_\mu, A_\nu]_\star$ and $[c, A_\mu]_\star$ do not vanish due to the star product. The set of γ matrices has been chosen as

$$\gamma^\mu = (\sigma_1, i\sigma_2, -i\sigma_3)\,, \tag{2.6}$$

respecting the usual relations (clifford algebra)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2.7}$$

where $g^{\mu\nu} = \mathrm{diag}\,(1, -1 - 1)$ is the 3-dimensional Minkowski metric. The σ_i are the usual Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

Furthermore, the action (2.1) is invariant under the BRS-transformations

$$\begin{aligned} s\psi &= iec\star\psi, \\ s\bar{\psi} &= ie\bar{\psi}\star c, \\ sA_\mu &= \partial_\mu c - ie[A_\mu, c]_\star, \\ sc &= iec\star c, \\ s\bar{c} &= B, \\ sB &= 0, \end{aligned} \tag{2.9}$$

as shown in Appendix A. Note, however, that in contrast to ordinary QED, the field tensor $F_{\mu\nu}$ is not gauge invariant due to the non-vanishing commutator (see (2.4)). Instead it transforms (covariantly) as

$$sF_{\mu\nu} = -ie[F_{\mu\nu}, c]_\star, \tag{2.10}$$

[1]Because of relation (1.12a) the star product need not be written in the bilinear terms.

a result known from non-Abelian gauge theories. But, of course, the gauge field action $\int d^3xF^{\mu\nu}\star F_{\mu\nu}$ is invariant due to the property (1.12b) of the star product.
Finally, BRS-invariance of the action (2.1) restricts all coupling constants in the model to be identical (see Appendix A for details).

2.2 Propagators

In the path-integral formalism one derives propagators[2] from [32, 33]

$$\triangle^{ab} = \frac{1}{i^2 Z^0} \frac{\delta^2 Z}{\delta j^a \delta j^b}\Big|_{j=0}, \tag{2.11}$$

with

$$\begin{aligned}
&Z = e^{iZ^c[j^a]} = \int \mathcal{D}\Phi^a \exp\left(iS[\Phi^a] + i\int d^3x j^a \Phi^a \right), \\
&Z^c[j^a] = \Gamma[\Phi^a_c[j^a]] + \int d^3x j^a \Phi^a_c[j^a], \\
&Z^0 = Z\Big|_{j=0},
\end{aligned} \tag{2.12}$$

where Φ^a stands for any field of our model and j^a for its source. Z is the generating functional of the Green functions and $Z^c[j^a]$ is the one of the connected Green functions and can be written as the Legendre transform of the so-called effective action[3] $\Gamma[\Phi^a_c]$. The effective action depends on the classical fields Φ^a_c which are functionals of the sources j^a as defined by the Legendre transformation. In order to derive the propagators one only needs to consider the bilinear part of the action and in this lowest order one can show that $\Gamma_0[\Phi^a_c] = S_0[\Phi^a_c]$. Therefore, in our model Z^c_0 can be written as

$$Z^c_0[j^a] = S_0[\Phi^a[j^a]] + \int d^3x \left(j_{\bar\psi}\psi + j_\psi \bar\psi + j^\mu_A A_\mu + j_{\bar c} c + j_c \bar c \right), \tag{2.13}$$

with

$$\begin{aligned}
S_0[\Phi^a] = \int d^3x \Big\{ &-\frac{1}{2}\partial^\mu A^\nu \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) + i\bar\psi \partial\!\!\!/ \psi \\
&+ B\partial^\mu A_\mu + \frac{\alpha}{2}B^2 - \bar c \Box c \Big\}.
\end{aligned} \tag{2.14}$$

[2]Due to relation (1.12a) the same procedure is applicable for non-commutative field theories.

[3]$\Gamma[\Phi^a_c]$ is in fact the generating functional of another two-point function, namely the inverse of the propagator.

The source of the unphysical field B has been set to zero, which corresponds to integrating out the B-field.
From equation (2.11) follows

$$\triangle^{ab} = \frac{1}{i^2 Z^0}\left(i\frac{\delta^2 Z^c}{\delta j^a \delta j^b} Z\Big|_{j=0} - \frac{\delta Z^c}{\delta j^a}\frac{\delta Z^c}{\delta j^b} Z\Big|_{j=0}\right). \tag{2.15}$$

The second term does not contribute due to momentum conservation and therefore we get

$$\triangle^{ab}(x,y) = \frac{1}{i}\frac{\delta^2 Z^c}{\delta j^a \delta j^b} = -i\frac{\delta \Phi^b}{\delta j^a}. \tag{2.16}$$

From the inverse of the Legendre transformation (2.13) and the commutation properties of the fields follows

$$\frac{\delta S_0}{\delta \Phi^a} = -(-1)^{ab} j^a, \tag{2.17}$$

with a, b equal to zero for bosons and equal to 1 for fermions.
Performing the left-hand side variation yields the equations of motion for the free fields:

$$\frac{\delta S_0}{\delta B} = \partial^\mu A_\mu + \alpha B = 0, \tag{2.18a}$$

$$\frac{\delta S_0}{\delta \bar{c}} = -\Box c = j_c, \tag{2.18b}$$

$$\frac{\delta S_0}{\delta \bar{\psi}} = i\partial\!\!\!/ \psi = j_\psi, \tag{2.18c}$$

$$\frac{\delta S_0}{\delta A_\mu} = \Box A^\mu + \frac{1-\alpha}{\alpha}\partial^\mu \partial^\rho A_\rho = -j_A^\mu, \tag{2.18d}$$

where the first equation has been used to eliminate B in the last equation.
The fields can be expressed by formally inverting the differential operators acting upon them and for the propagators in Feynman gauge ($\alpha = 1$) follows

$$\triangle^{\bar{c}c}(x,y) = -i\frac{\delta c(x)}{\delta j_c(y)} = \frac{i}{\Box_x}\delta^3(x-y) = -\triangle^{c\bar{c}}(x,y), \tag{2.19a}$$

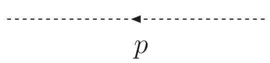
$$\triangle^{\bar{\psi}\psi}(x,y) = -i\frac{\delta \psi(x)}{\delta j_\psi(y)} = -\frac{1}{\partial\!\!\!/_x}\delta^3(x-y) = -\triangle^{\psi\bar{\psi}}(x,y), \tag{2.19b}$$

$$\triangle^{AA}_{\mu\nu}(x,y) = -i\frac{\delta A_\mu(x)}{\delta j_A^\nu(y)} = i\frac{g_{\mu\nu}}{\Box_x}\delta^3(x-y), \tag{2.19c}$$

from equation (2.16).
The ghost propagator can furthermore be written as

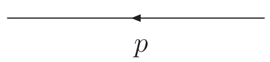
$$\triangle^{\bar{c}c}(x,y) = -\frac{i}{\Box_x}\int\frac{d^3p}{(2\pi)^3}e^{ip(x-y)} = \int\frac{d^3p}{(2\pi)^3}\frac{i}{p^2}e^{ip(x-y)}, \tag{2.20}$$

defining its Fourier transform

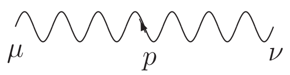


$$\widetilde{\triangle}^{c\bar{c}}(p) = \frac{i}{p^2}. \tag{2.21a}$$

The other propagators in momentum space can be calculated in a similar way and are thus given by



$$\widetilde{\triangle}^{\psi\bar{\psi}}(p) = -i\frac{\not{p}}{p^2}, \tag{2.21b}$$



$$\widetilde{\triangle}^{AA}_{\mu\nu}(p) = \frac{-ig_{\mu\nu}}{p^2}, \tag{2.21c}$$

where the property $\not{p}^2 = p^2$ has been used for the fermion propagator.

2.3 Vertices

The interaction part of the action, considering the antisymmetry of the electromagnetic field tensor, is given by

$$S_{int} = \int d^3x\Big\{ie\partial^\mu A^\nu \star [A_\mu, A_\nu]_\star + \frac{e^2}{2}A^\mu \star A^\nu \star [A_\mu, A_\nu]_\star$$
$$+e\bar{\psi}\star \not{A}\star\psi + ie\bar{c}\star\partial^\mu[A_\mu, c]_\star\Big\}. \tag{2.22}$$

Obviously, we are dealing with four different vertices: the 3-photon vertex, the 4-photon vertex, the fermion-photon vertex and the ghost-photon vertex.

First, we calculate the fermion-photon vertex (Figure 2.1a) in momentum space:

$$\widetilde{V}^{\bar{\psi}A\psi}_\mu(k_1,k_2,k_3) = i(2\pi)^9\frac{\delta}{\delta\widetilde{\psi}(-k_1)}\frac{\delta}{\delta\widetilde{A}^\mu(-k_2)}\frac{\delta}{\delta\widetilde{\bar{\psi}}(-k_3)}S_{int}. \tag{2.23}$$

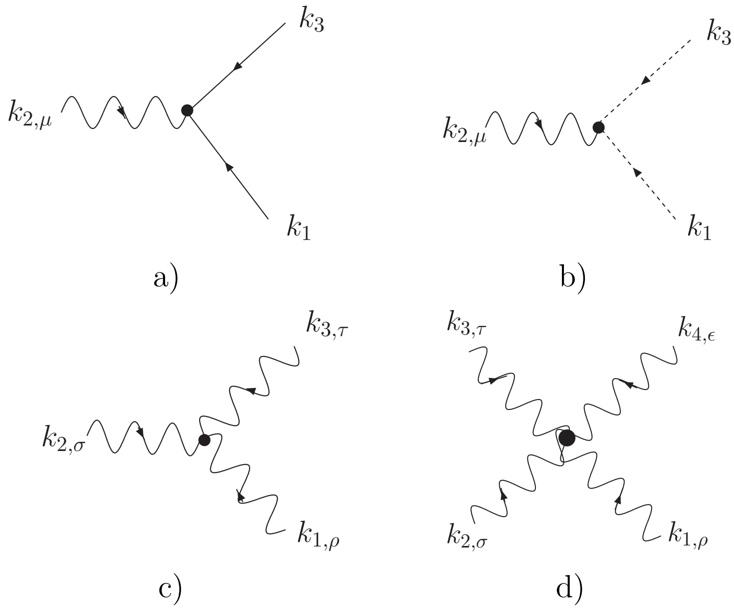


Figure 2.1: Vertices

Momenta pointing inwards (see Figure 2.1) are to be taken positive. Considering the definition of the star product (1.11), the fermion part of S_{int} becomes

$$\begin{aligned} S_{int}^{\bar{\psi}A\psi} &= e\int d^3x \int \frac{d^3q_{1-3}}{(2\pi)^9} e^{i\sum\limits_{i=1}^{3} q_i^\mu x_\mu} \widetilde{\bar{\psi}}(q_1)\gamma^\mu \widetilde{A}_\mu(q_2)\widetilde{\psi}(q_3) e^{-\frac{i}{2}\sum\limits_{j<i} q_j \times q_i} \\ &= e\int \frac{d^3q_{1-3}}{(2\pi)^6} \delta^3(q_1+q_2+q_3) \widetilde{\bar{\psi}}(q_1)\gamma^\mu \widetilde{A}_\mu(q_2)\widetilde{\psi}(q_3) e^{-\frac{i}{2}\sum\limits_{j<i} q_j \times q_i}, \end{aligned} \tag{2.24}$$

and for the vertex follows

$$\widetilde{V}_\mu^{\bar{\psi}A\psi}(k_1,k_2,k_3) = ie(2\pi)^3\delta^3(k_1+k_2+k_3)\gamma_\mu e^{-\frac{i}{2}(k_1\times k_3)}, \tag{2.25}$$

where k_2 has been eliminated in the exponent using momentum conservation expressed by the delta function and taking into account that $k_i \times k_i = 0$ due to the antisymmetry of $\theta^{\mu\nu}$.

The ghost-photon vertex (Figure 2.1b) follows from

$$\widetilde{V}_\mu^{\bar{c}\partial cA}(k_1,k_2,k_3) = i(2\pi)^9 \frac{\delta}{\delta\widetilde{c}(-k_1)} \frac{\delta}{\delta\widetilde{A}^\mu(-k_2)} \frac{\delta}{\delta\widetilde{\bar{c}}(-k_3)} S_{int}, \tag{2.26}$$

with

$$S_{int}^{\bar{c}\partial cA} = -ie\int d^3x \int \frac{d^3q_{1-3}}{(2\pi)^9} i q_1^\mu e^{i\sum\limits_{i=1}^{3} q_i^\mu x_\mu} \widetilde{\bar{c}}(q_1)\left(\widetilde{A}_\mu(q_2)\widetilde{c}(q_3)\right.$$
$$\left. -\widetilde{c}(q_2)\widetilde{A}_\mu(q_3)\right) e^{-\frac{i}{2}(q_1\times q_2+q_2\times q_3+q_1\times q_3)}$$
$$= e\int \frac{d^3q_{1-3}}{(2\pi)^6} q_1^\mu \delta^3(q_1+q_2+q_3)\left(\widetilde{\bar{c}}(q_1)\widetilde{A}_\mu(q_3)\widetilde{c}(q_2)e^{-\frac{i}{2}(q_1\times q_3)}\right.$$
$$\left. -\widetilde{\bar{c}}(q_1)\widetilde{c}(q_2)\widetilde{A}_\mu(q_3)e^{\frac{i}{2}(q_1\times q_3)}\right). \tag{2.27}$$

The result is

$$\widetilde{V}_\mu^{\bar{c}\partial cA}(k_1,k_2,k_3) = 2e(2\pi)^3\delta^3(k_1+k_2+k_3)k_{3\mu}\sin\left(\frac{k_1\times k_3}{2}\right), \tag{2.28}$$

where we used [34]

$$\frac{e^{ix}-e^{-ix}}{2i} = \sin x.$$

Next, we calculate the 3-photon vertex (Figure 2.1c):

$$\widetilde{V}_{\rho\sigma\tau}^{\partial AAA}(k_1,k_2,k_3) = i(2\pi)^9 \frac{\delta}{\delta\widetilde{A}^\rho(-k_1)}\frac{\delta}{\delta\widetilde{A}^\sigma(-k_2)}\frac{\delta}{\delta\widetilde{A}^\tau(-k_3)} S_{int}. \tag{2.29}$$

The relevant part of the action is given by

$$S_{int}^{\partial AAA} = 2ie\int \frac{d^3q_{1-3}}{(2\pi)^6}\delta^3\left(\sum_{i=1}^{3} q_i\right) q_1^\mu \widetilde{A}^\nu(q_1)\widetilde{A}_\mu(q_2)\widetilde{A}_\nu(q_3)\sin\left(\frac{q_1\times q_3}{2}\right), \tag{2.30}$$

and after a short calculation given in Appendix B.1 one gets for the vertex

$$\widetilde{V}_{\rho\sigma\tau}^{\partial AAA}(k_1,k_2,k_3) = -2e(2\pi)^3\delta^3(k_1+k_2+k_3)\left[(k_3-k_2)_\rho g_{\sigma\tau}\right.$$
$$\left. +(k_1-k_3)_\sigma g_{\rho\tau} + (k_2-k_1)_\tau g_{\rho\sigma}\right]\sin\left(\frac{k_1\times k_2}{2}\right). \tag{2.31}$$

Finally, the 4-photon vertex (Figure 2.1d) is calculated from

$$\widetilde{V}_{\rho\sigma\tau\epsilon}^{AAAA}(k_{1-4}) = i(2\pi)^{12}\frac{\delta}{\delta\widetilde{A}^\rho(-k_1)}\frac{\delta}{\delta\widetilde{A}^\sigma(-k_2)}\frac{\delta}{\delta\widetilde{A}^\tau(-k_3)}\frac{\delta}{\delta\widetilde{A}^\epsilon(-k_4)} S_{int}, \tag{2.32}$$

leading to

$$\begin{aligned}
&\widetilde{V}^{AAAA}_{\rho\sigma\tau\epsilon}(k_1,k_2,k_3,k_4) = -4ie^2(2\pi)^3\delta^3(k_1+k_2+k_3+k_4) \\
&\times \left[(g_{\rho\tau}g_{\sigma\epsilon} - g_{\rho\epsilon}g_{\sigma\tau})\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)\right. \\
&+(g_{\rho\sigma}g_{\tau\epsilon} - g_{\rho\epsilon}g_{\sigma\tau})\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right) \\
&\left.+(g_{\rho\sigma}g_{\tau\epsilon} - g_{\rho\tau}g_{\sigma\epsilon})\sin\left(\frac{k_2\times k_3}{2}\right)\sin\left(\frac{k_1\times k_4}{2}\right)\right]. \qquad (2.33)
\end{aligned}$$

The explicit calculation is given in Appendix B.2.
Notice that the fermion-photon vertex is the only one including a phase of the form $\exp(i\varphi)$, whereas all other vertices are proportional to $\sin\varphi$. This difference becomes clear when comparing the relevant terms in the action (2.22): The term including fermion fields ψ is the only one without a commutator, since our ψ-fields are in the *fundamental* representation.

2.4 Power counting

Before we start to calculate one-loop corrections to the propagators, it would be useful to find the superficial degree of divergence of a particular graph. Several factors need to be considered: Every integral over 3 dimensional space-time gives a contribution of three powers of k in the numerator and therefore raises the degree of divergence. Furthermore, the ghost-photon vertex and the 3-photon vertex each raise the degree of divergence by one. On the other hand, the degree of divergence is reduced by the propagators: The fermion-propagator adds one power of k to the denominator and both ghost and photon propagators each add two powers of k to the denominator. Therefore, one can derive the following formula for the superficial degree of divergence $d(\gamma)$ for a certain Feynman graph

$$d(\gamma) = 3L - I_\psi - 2I_A - 2I_c + V_c + V_{3A}, \qquad (2.34)$$

where $I_{\psi,A,c}$ are the numbers of internal fermion, photon and ghost lines, respectively. V_c and V_{3A} are the number of ghost and 3-photon vertices and L denotes the number of loop integrations. Since there are I internal momenta as well as momentum conservation at each vertex and finally also overall momentum conservation, the number of independent momenta, which is L, is given by the relation

$$L = I_\psi + I_A + I_c - (V_\psi + V_c + V_{3A} + V_{4A} - 1)\,, \qquad (2.35)$$

where V_ψ and V_{4A} denote the number of fermion and 4-photon vertices, respectively. Elimination of L in (2.34) yields

$$d(\gamma) = 3 + 2I_\psi + I_A + I_c - 3V_\psi - 2V_c - 2V_{3A} - 3V_{4A}. \tag{2.36}$$

Finally, one needs relations between the number of vertices and the number of legs. External legs denoted by $E_{\psi,c,A}$ count once, whereas internal legs $I_{\psi,A,c}$ count twice as they are always connected to two vertices (or two legs at one vertex). Furthermore, one can treat the coupling constant e as an external field E_e because in 3-dimensional QED e has mass dimension $[e] = 1/2$ (compare the expression for the action (2.1) considering $[\partial_\mu] = [\psi] = [\bar{\psi}] = 1$, $[A_\mu] = 1/2$, $[c] = 0$ and $[\bar{c}] = 1$).
Thus, one finds the following relations:

$$\begin{aligned}
2V_\psi &= E_\psi + 2I_\psi, \\
2V_c &= E_c + 2I_c, \\
V_\psi + V_c + 3V_{3A} + 4V_{4A} &= E_A + 2I_A, \\
V_\psi + V_c + V_{3A} + 2V_{4A} &= E_e.
\end{aligned} \tag{2.37}$$

Elimination of the internal legs in equation (2.36) produces

$$d(\gamma) = 3 - E_\psi - \frac{1}{2}E_A - \frac{1}{2}E_c - \frac{1}{2}V_\psi - \frac{1}{2}V_c - \frac{1}{2}V_{3A} - V_{4A}, \tag{2.38}$$

and use of the last relation of (2.37) finally leads to

$$d(\gamma) = 3 - E_\psi - \frac{1}{2}E_A - \frac{1}{2}E_c - \frac{1}{2}E_e, \tag{2.39}$$

depending on the number of external legs, only. (In other words, instead of counting the different vertices as in (2.38) one merely counts the coupling constants, which are therefore treated as external legs.)
Obviously, E_e gets larger with growing loop order (which is equivalent to increasing number of vertices). Therefore, there is only a limited number of superficially divergent graphs in this model: Since every Feynman graph has $E_e \geq 2$ ($E_e = 2$ to one-loop order), the fermion self-energy ($E_\psi = 2$) appears to be, at most, logarithmically divergent, whereas superficially, the photon ($E_A = 2$) and ghost ($E_c = 2$) self-energies appear to be linearly divergent. Further superficially (logarithmically) divergent graphs are those including three external photon lines ($E_A = 3$, $E_e = 3$) and the (lowest order) correction to the ghost-photon vertex ($E_c = 2$, $E_A = 1$, $E_e = 3$). In commutative $2+1$ dimensional QED all vertex corrections converge (divergent contributions cancel each other) [32, 33]. Therefore it would also be interesting to

find out whether this remains true in the non-commutative model[4]. We will, however, concentrate on one-loop corrections to the propagators, since they include the worst possible divergences, and additional calculations of vertex corrections would exceed the scope of this study.

2.5 One-Loop calculations

Now that we have all the building blocks necessary for doing loop calculations, we will first take a look at the fermion self-energy at one-loop level:

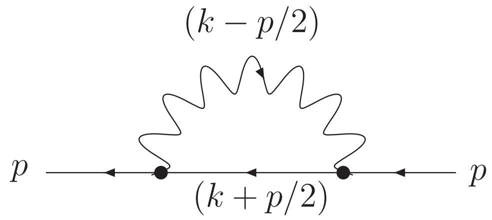


Figure 2.2: fermion self-energy

$$\Sigma(p) = e^2 \int \frac{d^3k}{(2\pi)^3} \gamma_\mu \frac{\not{k} + \not{p}/2}{(k+p/2)^2} \gamma^\mu \frac{1}{(k-p/2)^2}. \tag{2.40}$$

An explicit calculation of equation (2.40) is given in Appendix B.3. Obviously, this expression is completely independent of phases and there is *no modification* due to non-commutativity [2].
Equation (2.40) seems to be logarithmically divergent, which also follows from the previous chapter's considerations summarized in power counting formula (2.39), since $E_\psi = E_e = 2$.
From relation (2.7) follows

$$\gamma_\mu \gamma_\rho \gamma^\mu = \gamma_\rho (2 - \gamma_\mu \gamma^\mu) = -\gamma_\rho, \tag{2.41}$$

where $\gamma^\mu \gamma_\mu = 3$ in 3-dimensional space. The remaining integral is solved in (D.15) in Appendix D.2 using dimensional regularization and the result is

$$\Sigma(p) = -\frac{e^2}{16} \frac{\not{p}}{\sqrt{p^2}}, \tag{2.42}$$

[4]In fact, it was shown in [30] that in the case of 3 + 1 dimensional QED (without fermions) the ghost-photon vertex correction diverges only logarithmically (suggesting convergence in 2 + 1 dimensions) and that the sum of corrections to the 3-photon vertex includes a linear infrared divergence (suggesting a logarithmic infrared divergence in the 2 + 1 dimensional model).

which is in fact *finite* [1].

The photon self-energy at one-loop level has four contributions due to the star product: the fermion loop, the ghost loop and two different types of photon loops as shown in Figure 2.3. Remember that in ordinary QED one only has the fermion loop, the other contributions vanish.

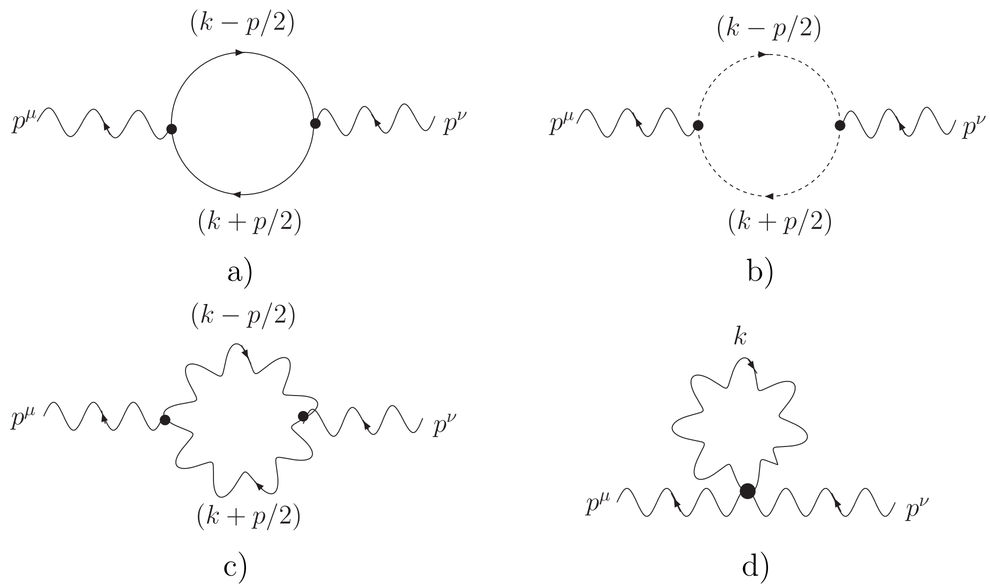


Figure 2.3: vacuum polarization

According to the Feynman rules one gets for the fermion-loop graph in Figure 2.3a (see Appendix B.4)

$$\Pi_a^{\mu\nu}(p) = -e^2 \mathrm{tr} \int \frac{d^3k}{(2\pi)^3} \gamma^\mu \frac{(\not{k}+\not{p}/2)}{(k+p/2)^2} \gamma^\nu \frac{(\not{k}-\not{p}/2)}{(k-p/2)^2}, \tag{2.43a}$$

and for the ghost-loop graph in Figure 2.3b (see Appendix B.5)

$$\Pi_b^{\mu\nu}(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(k+p/2)^\mu}{(k+p/2)^2} \frac{(k-p/2)^\nu}{(k-p/2)^2} \sin^2\left(\frac{k\times p}{2}\right). \tag{2.43b}$$

The photon loop including 3-photon vertices (Figure 2.3c) is given by

$$\Pi_c^{\mu\nu}(p) = 4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{3k^\mu k^\nu + k^2 g^{\mu\nu} + \frac{9}{4}\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)}{(k+p/2)^2 (k-p/2)^2} \sin^2\left(\frac{k\times p}{2}\right) \tag{2.43c}$$

as calculated in Appendix B.6 and the photon-tadpole loop (Figure 2.3d) becomes

$$\Pi_d^{\mu\nu}(p) = -8e^2 \int \frac{d^3k}{(2\pi)^3} \frac{g^{\mu\nu}}{k^2} \sin^2\left(\frac{k \times p}{2}\right), \tag{2.43d}$$

(see Appendix B.7). As expected from equation (2.39) all four graphs appear to be linearly divergent, superficially ($E_A = E_e = 2$).
Since [34]

$$\sin^2\left(\frac{k \times p}{2}\right) = \frac{1}{2}\left(1 - \cos\left(k \times p\right)\right), \tag{2.44}$$

all of these graphs have a planar (phase independent) and a non-planar (phase dependent) part (except for the fermion-loop graph which is completely planar). All planar contributions can be calculated with dimensional regularization (see Appendices D.1 and D.2).
In order to calculate the fermion-loop graph one needs a relation for the trace over four γ matrices. Such a relation can be found using equation (2.7):

$$\mathrm{tr}\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma = 2\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}\right). \tag{2.45}$$

This relation together with the integral formula (D.18) yields

$$\Pi_a^{\mu\nu}(p) = e^2 2\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\frac{p_\rho p_\sigma + p^2 g_{\rho\sigma}}{64\sqrt{p^2}}, \tag{2.46}$$

and finally

$$\Pi_a^{\mu\nu}(p) = \frac{e^2}{16}\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{\sqrt{p^2}}, \tag{2.47}$$

which is finite and transversal.

The planar part of the ghost-loop contribution can also be solved with formula (D.18):

$$\Pi_{b,pl}^{\mu\nu}(p) = \frac{e^2}{32}\frac{p^\mu p^\nu + p^2 g^{\mu\nu}}{\sqrt{p^2}}, \tag{2.48}$$

and is finite as well but not transversal.

For the calculation of the planar part of the photon-loop graph we need equations[5] (D.12) and (D.20) and arrive at

$$\begin{aligned}\Pi^{\mu\nu}_{c,pl}(p) &= e^2\left[6\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{64\sqrt{p^2}} + 2g^{\mu\nu}g_{\rho\sigma}\frac{p^\rho p^\sigma - p^2 g^{\rho\sigma}}{64\sqrt{p^2}} - 9\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{16\sqrt{p^2}}\right] \\ &= -\frac{15e^2}{32}\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{\sqrt{p^2}} - \frac{e^2}{16}\frac{p^2 g^{\mu\nu}}{\sqrt{p^2}}. \end{aligned} \tag{2.49}$$

So we find that all planar contributions of this model are finite, since the planar part of the photon-tadpole loop (2.43d) gives no contribution to the self-energy in dimensional regularization as shown in Appendix D.2 (equation (D.21) with $n = 3$). The planar part of the photon-loop graph (2.43c) is not transversal but the sum of ghost-loop (2.43b) and photon-loop (2.43c) contributions is.
The sum of all planar contributions to the vacuum polarization reads

$$\Pi^{\mu\nu}_{pl}(p) = -\frac{3e^2}{8}\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{\sqrt{p^2}}, \tag{2.50}$$

therefore being finite and transversal ($p_\mu \Pi^{\mu\nu}_{pl} = 0$).

The non-planar contributions of the graphs in Figure 2.3 can be evaluated using Schwinger parameterization as described in Appendix D.3. First, we write the phase factors as

$$-\frac{1}{2}\cos(k \times p) = -\frac{1}{4}\sum_{\eta=\pm 1} e^{i\eta k\widetilde{p}}, \tag{2.51}$$

where $\widetilde{p}^\mu$ stands for $\theta^{\mu\nu}p_\nu$. Employing formula (D.54) in Appendix D.3 we now find the following result for the non-planar ghost-loop contribution (Figure 2.3b):

$$\Pi^{\mu\nu}_{b,np}(p) = \frac{-e^2}{4\pi}\int\limits_0^1 d\xi\sqrt{\xi(1-\xi)}\left[\frac{p^\mu p^\nu}{\sqrt{p^2}} - \frac{p^2 g^{\mu\nu}}{\sqrt{p^2}}\frac{1}{z} + \frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}\frac{p^2}{\sqrt{p^2}}\left(\frac{1}{z}+1\right)\right]e^{-z(\xi)} \tag{2.52}$$

As defined in Appendix D.3, $z(\xi)$ is proportional[6] to $\sqrt{\widetilde{p}^2}$. Therefore, one can expand $e^{-z} \approx 1 - z + O(z^2)$ for small θ and arrives at

$$\Pi^{\mu\nu}_{b,np}(p) \approx -\frac{e^2}{4\pi\sqrt{\widetilde{p}^2}}\left(\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2} - g^{\mu\nu}\right) - \frac{e^2}{32}\frac{p^\mu p^\nu + p^2 g^{\mu\nu}}{\sqrt{p^2}} + O(\theta), \tag{2.53}$$

[5] For the second term we have to write $2k^2 g^{\mu\nu} = 2g^{\mu\nu}g_{\rho\sigma}k^\rho k^\sigma$

[6] $z(\xi) \equiv \sqrt{\xi(1-\xi)p^2\widetilde{p}^2}$

where equation (D.78b) was used to solve the remaining integral. We find that the second term exactly cancels the planar contribution of the ghost loop while the first term is proportional to $1/\sqrt{\widetilde{p}^2}$ and is therefore (linearly) infrared-divergent (for $p^2 \to 0$) and also tends to infinity in the commutative limit $\theta \to 0$.

The non-planar part of the photon-loop graph (Figure 2.3c) can be solved using equations (D.46) and (D.55) in Appendix D.3:

$$\begin{aligned}\Pi^{\mu\nu}_{c,np}(p) =& \frac{e^2}{4\pi}\int\limits_0^1 d\xi \sqrt{\xi(1-\xi)}e^{-z(\xi)}\left[\frac{3p^\mu p^\nu}{\sqrt{p^2}} + \frac{p^2 g^{\mu\nu}}{\sqrt{p^2}}\left(2-\frac{5}{z}\right)\right. \\ &\left. + \frac{3\widetilde{p}^\mu \widetilde{p}^\nu}{\widetilde{p}^2}\frac{p^2}{\sqrt{p^2}}\left(\frac{1}{z}+1\right)\right] + \frac{e^2}{8\pi}\frac{(3p^\mu p^\nu - 5p^2 g^{\mu\nu})}{\sqrt{p^2}}\int\limits_0^1 d\xi \frac{e^{-z(\xi)}}{\sqrt{\xi(1-\xi)}}.\end{aligned} \tag{2.54}$$

When expanding for small θ (considering equation (D.78a) and (D.78b)) this expression becomes

$$\Pi^{\mu\nu}_{c,np}(p) = \frac{e^2}{4\pi\sqrt{\widetilde{p}^2}}\left(\frac{3\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2} - 5g^{\mu\nu}\right) + \frac{e^2}{32}\frac{15p^\mu p^\nu - 13p^2 g^{\mu\nu}}{\sqrt{p^2}} + O(\theta), \tag{2.55}$$

where the second term cancels the planar contribution and the first term is again infrared-divergent.

Finally, we get the expression for the (non-planar) photon-tadpole loop (Figure 2.3d) from equation (D.58) in Appendix D.3 with $M = 0$:

$$\Pi^{\mu\nu}_{d,np}(p) = \frac{e^2 g^{\mu\nu}}{\pi\sqrt{\widetilde{p}^2}} \tag{2.56}$$

This graph is obviously also (linearly) infrared divergent.

If we sum up the IR divergent parts of the non-planar graphs we notice that the terms proportional to the metric cancel each other and what remains is the *transversal* expression

$$\Pi^{\mu\nu}_{IR-divergent}(p) = \frac{e^2}{2\pi\sqrt{\widetilde{p}^2}}\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}. \tag{2.57}$$

So in contrast to the commutative model non-commutative 3-dimensional QED is not finite but diverges for $p \to 0$ as well as in the limit $\theta \to 0$. From the superficial degree of divergence (2.39) we originally expected a linear ultraviolet divergence. The planar contributions, however, turned out to be finite in dimensional regularization (due to gauge symmetry). Now the non-planar contributions show an infrared divergence of exactly the same degree as the originally expected ultraviolet one, namely linear. This UV/IR-mixing is typical for non-commutative field theory since the (oscillating) phase factors (2.51) act as regulators depending on the external momenta p. However, as these external momenta get smaller, the phases become ineffective and the diagrams diverge at $p \to 0$. The same happens in the limit $\theta \to 0$. Due to this new divergence, taking the latter limit is non-trivial and does not recover the commutative theory [22].

One graph still remains: the ghost self-energy

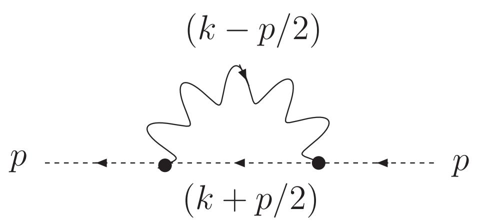


Figure 2.4: ghost self-energy

$$\Xi(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{p\,(k+p/2)}{(k+p/2)^2\,(k-p/2)^2} \sin^2\left(\frac{k \times p}{2}\right). \tag{2.58}$$

An explicit calculation of equation (2.58) is given in Appendix B.8. From our power counting formula (2.39) we expect linear divergence ($E_c = E_e = 2$) but in fact (2.58) seems to be only logarithmically divergent since one of the momenta in the numerator (coming from one of the vertices) is an *external* one.

Remembering relation (2.44) one sees that this graph also decomposes into a planar and a non-planar part. The planar part can be solved with equation (D.15) in Appendix D.2 and reads

$$\Xi_{pl}(p) = -\frac{e^2\sqrt{p^2}}{8}, \tag{2.59}$$

whereas the non-planar part is calculated using equation (D.50) in Appendix D.3:

$$\Xi_{np}(p) = \frac{e^2\sqrt{p^2}}{4\pi} \int_0^1 d\xi \sqrt{\frac{(1-\xi)}{\xi}}\, e^{-z(\xi)}, \tag{2.60}$$

with $z(\xi) = \sqrt{\xi(1-\xi)p^2\widetilde{p}^2}$. Both contributions are finite. Furthermore, in the limit $\theta \to 0$ one finds that planar and non-planar contributions cancel each other.

We see now that, apart from the infrared divergence (2.57), all contributions that do not appear in the commutative model are of the order $O(\theta)$ and we have

$$\begin{aligned}
\Sigma(p) &= -\frac{e^2}{16}\frac{\not{p}}{\sqrt{p^2}}, \\
\Pi^{\mu\nu}(p) &= \frac{e^2}{2\pi\sqrt{\widetilde{p}^2}}\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2} + \frac{e^2}{16}\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{\sqrt{p^2}} + O(\theta), \\
\Xi(p) &= O(\theta),
\end{aligned} \tag{2.61}$$

where the finite θ-independent term in the photon self-energy comes from the fermion-loop graph (Figure 2.3a).

Chapter 3

Non-Commutative QED$_3$ with Chern-Simons Mass Term

Now that we have seen that non-commutative *massless* quantum electrodynamics is infrared divergent, we would like to find an extension which renders the model finite. One could, for instance, try making the gauge bosons massive, since a photon propagator proportional to $1/\left(p^2 - M^2\right)$ would no longer be singular at $p \to 0$ and one could hope that the infrared singular term (2.57) in the self-energy becomes proportional to $1/\sqrt{\widetilde{p}^2 - M^2}$.
However, if one considers where the infrared divergence came from in the first place, one will realize that massive gauge bosons will probably not cure the divergence problem: As mentioned below equation (2.57), phases including $\widetilde{p}$ act as *UV-regulators* which become ineffective as $\widetilde{p}$ tends to zero. Therefore the infrared divergent term is actually related to an ultraviolet divergence which presumably cannot be eliminated with a mass parameter. In spite of this intuitive analysis it is still interesting to verify this claim through explicit calculations.
Unfortunately, a mass term of the type $MA^\mu A_\mu$ in the lagrangian is not gauge invariant, but it is possible to construct a term which is, namely the *Chern-Simons* term. Calculation of the new photon propagator will make clear why this term can be interpreted as a mass term even though it includes a derivative of the gauge field (see (3.1)). In order to be more general we will also make our fermions massive.

3.1 Propagators

If we add the (gauge invariant) mass terms

$$S_m = -\int d^3x \left\{ \frac{\mu_s}{2}\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + m_f \bar{\psi}\psi \right\} \tag{3.1}$$

to the action (2.1) we find the following equations of motion for the free fermion- and photon fields:

$$\frac{\delta S_0}{\delta \bar{\psi}} = \left(i\partial\!\!\!/ - m_f \right) \psi = j_\psi, \tag{3.2a}$$

$$\frac{\delta S_0}{\delta A_\mu} = \Box A^\mu + \frac{1-\alpha}{\alpha} \partial^\mu \partial^\rho A_\rho - \mu_s \epsilon^{\mu\nu\rho} \partial_\nu A_\rho = -j^\mu_A, \tag{3.2b}$$

where m_f and μ_s stand for the fermion mass and the Chern-Simons mass, respectively. The fermion propagator is then given by

$$\triangle^{\psi\bar{\psi}}(x,y) = i \frac{\delta \psi(x)}{\delta j_\psi(y)} = \frac{i}{i\partial\!\!\!/_x - m_f} \delta^3 \left(x - y \right), \tag{3.3}$$

or after Fourier transformation:

$$\widetilde{\triangle}^{\psi\bar{\psi}}(p) = -i \frac{p\!\!\!/ - m_f}{p^2 - m_f^2}. \tag{3.4}$$

In order to calculate the photon propagator one has to find the inverse of the operator (see (3.2b))

$$\mathcal{D}^{\mu\rho} = \Box g^{\mu\rho} + \frac{1-\alpha}{\alpha} \partial^\mu \partial^\rho - \mu_s \epsilon^{\mu\nu\rho} \partial_\nu. \tag{3.5}$$

One can make the ansatz

$$\mathcal{D}^{-1}_{\rho\alpha} = g_{\rho\alpha} A + \partial_\rho \partial_\alpha B + \epsilon_{\rho\alpha\tau} \partial^\tau C, \tag{3.6}$$

and calculate A, B and C from $\mathcal{D}^{\mu\rho}\mathcal{D}^{-1}_{\rho\alpha} = \delta^\mu_\alpha$:

$$\begin{aligned} \delta^\mu_\alpha = \mathcal{D}^{\mu\rho}\mathcal{D}^{-1}_{\rho\alpha} &= \Box \delta^\mu_\alpha A + \frac{1-\alpha}{\alpha} \partial^\mu \partial_\alpha A - \mu_s \epsilon^{\mu\nu}{}_\alpha \partial_\nu A + \Box \partial^\mu \partial_\alpha B \\ &+ \frac{1-\alpha}{\alpha} \partial^\mu \Box \partial_\alpha B - \mu_s \epsilon^{\mu\nu\rho} \partial_\nu \partial_\rho \partial_\alpha B + \Box \epsilon^\mu{}_{\alpha\tau} \partial^\tau C \\ &+ \frac{1-\alpha}{\alpha} \epsilon_{\rho\alpha\tau} \partial^\mu \partial^\rho \partial^\tau C - \mu_s \epsilon^{\mu\nu\rho} \epsilon_{\rho\alpha\tau} \partial_\nu \partial^\tau C. \end{aligned} \tag{3.7}$$

The second term depending on C and the third term depending on B vanish since $\epsilon^{\mu\nu\rho}\partial_\nu\partial_\rho = 0$ and with $\epsilon^{\mu\nu\rho}\epsilon_{\rho\alpha\tau} = \delta^\mu_\alpha\delta^\nu_\tau - \delta^\mu_\tau\delta^\nu_\alpha$ one finds the following three relations when comparing coefficients:

$$\delta^\mu_\alpha : \qquad \Box A - \mu_s \Box C = 1, \tag{3.8a}$$

$$\partial^\mu\partial_\alpha : \qquad \frac{1-\alpha}{\alpha} A + \frac{1}{\alpha}\Box B + \mu_s C = 0, \tag{3.8b}$$

$$\epsilon^\mu{}_{\alpha\tau}\partial^\tau : \qquad \mu_s A + \Box C = 0. \tag{3.8c}$$

Therefore one finds

$$A = \frac{1}{\Box + \mu_s^2}, \tag{3.9a}$$

$$B = \left(\frac{(\alpha-1)}{\Box} + \alpha\frac{\mu_s^2}{\Box^2}\right)\frac{1}{\Box + \mu_s^2}, \tag{3.9b}$$

$$C = -\frac{\mu_s}{\Box}\frac{1}{\Box + \mu_s^2}, \tag{3.9c}$$

and the photon propagator is given by[1]

$$\begin{aligned}\triangle^{AA}_{\mu\nu}(x,y) &= -i\frac{\delta A_\mu(x)}{\delta j^\nu_A(y)} = i\frac{\delta}{\delta j^\nu_A(y)}\left(\mathcal{D}^{-1}_{\mu\alpha} j^\alpha_A(x)\right) \\ &= i\left(g_{\mu\nu} + \left[(\alpha-1) + \alpha\frac{\mu_s^2}{\Box}\right]\frac{\partial_\mu\partial_\nu}{\Box} - \mu_s\frac{\epsilon_{\mu\nu\rho}\partial^\rho}{\Box}\right)\frac{\delta^3(x-y)}{\Box + \mu_s^2},\end{aligned} \tag{3.10}$$

or after Fourier transformation

$$\widetilde{\triangle}^{AA}_{\mu\nu}(p) = \frac{-i}{p^2 - \mu_s^2}\left(g_{\mu\nu} + \left[(\alpha-1) - \alpha\frac{\mu_s^2}{p^2}\right]\frac{p_\mu p_\nu}{p^2} + i\mu_s\frac{\epsilon_{\mu\nu\rho}p^\rho}{p^2}\right). \tag{3.11}$$

In Feynman gauge ($\alpha = 1$) this expression reduces to

$$\widetilde{\triangle}^{AA}_{\mu\nu}(p) = \frac{i}{p^2 - \mu_s^2}\left(-g_{\mu\nu} + \mu_s^2\frac{p_\mu p_\nu}{p^4} - i\mu_s\frac{\epsilon_{\mu\nu\rho}p^\rho}{p^2}\right). \tag{3.12}$$

We now continue with one-loop calculations to find out how (or if) the infrared divergent term (2.57) depends on the masses.

[1] $\mathcal{D}^{-1}_{\mu\alpha}$ denotes the inverse of the operator $\mathcal{D}^{\mu\rho}$ in (3.5). Furthermore, all derivatives are to be taken with respect to the variable x and act on the delta function.

3.2 One-Loop calculations

Following the same steps as in Appendix B.3 but with our modified propagators (3.4) and (3.12) one finds for the fermion self-energy at one loop level

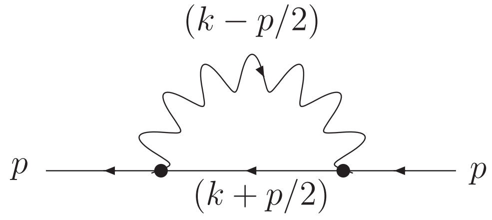


Figure 3.1: fermion self-energy

$$\Sigma(p) = e^2 \int \frac{d^3k}{(2\pi)^3} \gamma^\mu \frac{\not{k} + \not{p}/2 - m_f}{(k+p/2)^2 - m_f^2} \gamma^\nu \frac{1}{(k-p/2)^2 - \mu_s^2}$$
$$\times \left[g_{\mu\nu} - \mu_s^2 \frac{(k-p/2)_\mu (k-p/2)_\nu}{(k-p/2)^4} + i\mu_s \frac{\epsilon_{\mu\nu\sigma} (k-p/2)^\sigma}{(k-p/2)^2} \right], \quad (3.13)$$

which in the limit $m_f \to 0$ and $\mu_s \to 0$ reduces to (2.40). In the following, we only consider possible divergences: According to power counting, the only superficially divergent term in (3.13) is the one proportional to the metric reading

$$\Sigma^\infty(p) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\not{k} + \not{p}/2 + 3m_f}{(k+p/2)^2 - m_f^2} \frac{1}{(k-p/2)^2 - \mu_s^2}, \quad (3.14)$$

where the properties $\gamma^\mu\gamma_\mu = 3$ and (2.41) of the γ matrices have been used. This integral can be solved with the integral formulas (D.24) and (D.27) leading to

$$\Sigma^\infty(p) = \frac{-e^2}{8\pi\sqrt{p^2}} \left[\not{p}\sqrt{2A(\widetilde{m}_f, \widetilde{\mu}_s)}\sqrt{1-z^2}\Big|_{a_-}^{a_+} \right.$$
$$\left. + \left(\frac{\not{p}}{2}(1 + \widetilde{m}_f^2 - \widetilde{\mu}_s^2) + 3m_f \right) \arcsin z \Big|_{a_-}^{a_+} \right], \quad (3.15)$$

with

$$
\begin{aligned}
A(\widetilde{m}_f, \widetilde{\mu}_s) &= \frac{1}{8}\left[\left(1-\widetilde{m}_f^2+\widetilde{\mu}_s^2\right)^2 - 4\widetilde{\mu}_s^2\right], \\
a_\pm &= \frac{\pm 1 + \widetilde{m}_f^2 - \widetilde{\mu}_s^2}{\sqrt{\left(1-\widetilde{m}_f^2+\widetilde{\mu}_s^2\right)^2 - 4\widetilde{\mu}_s^2}} \quad \text{and} \quad \begin{aligned}\widetilde{m}_f^2 &= m_f^2/p^2 \\ \widetilde{\mu}_s^2 &= \mu_s^2/p^2\end{aligned}
\end{aligned} \tag{3.16}
$$

This expression is in fact finite (in accordance to the massless model discussed in the previous chapter). In the limit $m_f, \mu_s \to 0$ (3.15) reduces to (2.42).

Next we will take a look at the four contributions to the photon self-energy:

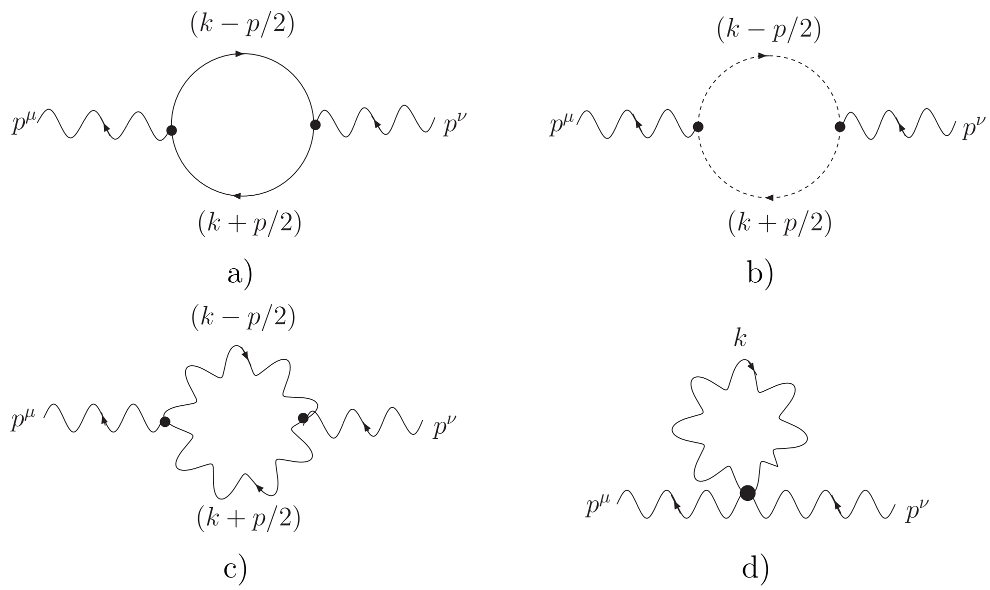


Figure 3.2: vacuum polarization

Following the steps given in Appendix B.4 but with the massive fermion propagator (3.4) the fermion-loop graph (Figure 3.2a) reads

$$
\Pi_a^{\mu\nu}(p) = -e^2 \mathrm{tr} \int \frac{d^3k}{(2\pi)^3} \gamma^\mu \frac{\not{k} + \not{p}/2 - m_f}{(k+p/2)^2 - m_f^2} \gamma^\nu \frac{\not{k} - \not{p}/2 - m_f}{(k-p/2)^2 - m_f^2}. \tag{3.17}
$$

Using the trace properties of the γ matrices

$$
\begin{aligned}
&\mathrm{tr}\gamma^\mu\gamma^\nu = 2g^{\mu\nu}, \\
&\mathrm{tr}\gamma^\mu\gamma^\rho\gamma^\nu = 2i\epsilon^{\mu\rho\nu}, \\
&\mathrm{tr}\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma = 2\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}\right),
\end{aligned} \tag{3.18}
$$

one finds

$$\operatorname{tr}\left[\gamma^{\mu}\left(\not{k}+\not{p}/2-m_f\right)\gamma^{\nu}\left(\not{k}-\not{p}/2-m_f\right)\right]=2g^{\mu\nu}m_f^2-2im_f\epsilon^{\mu\rho\nu}p_\rho$$
$$+2\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}\right)(k+p/2)^\rho(k-p/2)^\sigma\,. \quad (3.19)$$

Considering this relation one can now solve the integral (3.17) with formulas (D.24) and (D.30) when setting $m_1 = m_2 = m_f$:

$$\begin{aligned}\Pi_a^{\mu\nu}(p)=&\frac{e^2}{4\pi\sqrt{p^2}}\Bigg\{\Big[-im_f\epsilon^{\mu\nu\rho}p_\rho+2\widetilde{m}_f^2\left(p^\mu p^\nu-p^2g^{\mu\nu}\right)\Big]\arcsin z\Big|_{a_-}^{a_+}\\&+\left(p^\mu p^\nu-p^2g^{\mu\nu}\right)\left(\frac{1}{4}-\widetilde{m}_f^2\right)\left[z\sqrt{1-z^2}\Big|_{a_-}^{a_+}+\arcsin z\Big|_{a_-}^{a_+}\right]\Bigg\},\end{aligned} \quad (3.20)$$

with

$$a_\pm=\frac{\pm1}{\sqrt{1-4\widetilde{m}_f^2}} \quad \text{and} \quad \widetilde{m}_f^2=m_f^2/p^2\,. \quad (3.21)$$

This expression is finite as well and in the limit $m_f \to 0$ (3.20) reduces to (2.47).

Since the ghost propagator (2.21a) has not changed, the ghost-loop graph (Figure 3.2b) remains the same as in (2.43b).
The expression for the photon-loop graph (Figure 3.2c) can be derived similarly to Appendix B.6 when replacing the massless photon propagator (2.21c) with the massive one in (3.12):

$$\begin{aligned}\Pi_c^{\mu\nu}(p)=&2e^2\int\frac{d^3k}{(2\pi)^3}\frac{\sin^2\left(\frac{k\times p}{2}\right)}{\left[(k+p/2)^2-\mu_s^2\right]\left[(k-p/2)^2-\mu_s^2\right]}\\&\times\Big[-(k-3p/2)^\rho g^{\mu\tau}+2k^\mu g^{\rho\tau}-(k+3p/2)^\tau g^{\rho\mu}\Big]\\&\times\Big[-(k+3p/2)^\sigma g^{\nu\epsilon}+2k^\nu g^{\sigma\epsilon}-(k-3p/2)^\epsilon g^{\sigma\nu}\Big]\\&\times\left\{-g_{\tau\sigma}+\mu_s^2\frac{(k+p/2)_\tau(k+p/2)_\sigma}{(k+p/2)^4}-i\mu_s\frac{\epsilon_{\tau\sigma\eta}(k+p/2)^\eta}{(k+p/2)^2}\right\}\\&\times\left\{-g_{\epsilon\rho}+\mu_s^2\frac{(k-p/2)_\epsilon(k-p/2)_\rho}{(k-p/2)^4}-i\mu_s\frac{\epsilon_{\epsilon\rho\eta}(k-p/2)^\eta}{(k-p/2)^2}\right\}.\end{aligned} \quad (3.22)$$

The product of the two propagators in this expression leads to nine terms. However, only three of them are superficially divergent: the term proportional to $g_{\tau\sigma} g_{\epsilon\rho}$ appears to be linearly divergent and the other two being proportional to

$$\left(i g_{\tau\sigma} \mu_s \frac{\epsilon_{\epsilon\rho\eta} \left(k - p/2\right)^\eta}{\left(k - p/2\right)^2} + i g_{\epsilon\rho} \mu_s \frac{\epsilon_{\tau\sigma\eta} \left(k + p/2\right)^\eta}{\left(k + p/2\right)^2} \right)$$

seem to be only logarithmically divergent.
The term proportional to $g_{\tau\sigma} g_{\epsilon\rho}$ looks almost like expression (2.43c) except for the mass terms in the denominator and can therefore be evaluated using formulas (D.24), (D.33), (D.59) and (D.63). The results are

$$\begin{aligned} \Pi^{\mu\nu,\infty}_{c,pl}(p) = & \left[\frac{9e^2}{16\pi} \frac{p^2 g^{\mu\nu} - p^\mu p^\nu}{\sqrt{p^2}} + \frac{(1 - 4\widetilde{\mu}_s^2)\, e^2}{32\pi \sqrt{p^2}} \left(3p^\mu p^\nu - 5p^2 g^{\mu\nu}\right) \right] \arcsin z \Big|_{a_-}^{a_+} \\ & - \frac{e^2 \sqrt{-\widetilde{\mu}_s^2}}{8\pi \sqrt{p^2}} \left(3p^\mu p^\nu + 7p^2 g^{\mu\nu}\right), \end{aligned} \tag{3.23}$$

with

$$a_\pm = \frac{\pm 1}{\sqrt{1 - 4\widetilde{\mu}_s^2}} \quad , \quad \widetilde{\mu}_s^2 = \mu_s^2 / p^2 \tag{3.24}$$

and

$$\begin{aligned} \Pi^{\mu\nu,\infty}_{c,np}(p) = & \frac{e^2}{4\pi} \int_0^1 d\xi \sqrt{\xi(1-\xi) - \widetilde{\mu}_s^2}\, e^{-z(\xi)} \left[\frac{3p^\mu p^\nu}{\sqrt{p^2}} + \frac{p^2 g^{\mu\nu}}{\sqrt{p^2}} \left(2 - \frac{5}{z}\right) \right. \\ & \left. + \frac{3\widetilde{p}^\mu \widetilde{p}^\nu}{\widetilde{p}^2} \frac{p^2}{\sqrt{p^2}} \left(\frac{1}{z} + 1\right) \right] + \frac{e^2}{8\pi} \frac{\left(3p^\mu p^\nu - 5p^2 g^{\mu\nu}\right)}{\sqrt{p^2}} \int_0^1 d\xi \frac{e^{-z(\xi)}}{\sqrt{\xi(1-\xi) - \widetilde{\mu}_s^2}} \\ & + \frac{\widetilde{\mu}_s^2 e^2}{4\pi \sqrt{p^2}} \int_0^1 d\xi \frac{3p^\mu p^\nu + p^2 g^{\mu\nu}}{\sqrt{\xi(1-\xi) - \widetilde{\mu}_s^2}} e^{-z(\xi)}, \end{aligned} \tag{3.25}$$

with

$$z(\xi) = \sqrt{\left(\xi(1-\xi) - \widetilde{\mu}_s^2\right) p^2 \widetilde{p}^2}, \tag{3.26}$$

for the planar and the non-planar parts, respectively. (3.23) is finite whereas (3.25) shows a (linear) infrared divergence, as can be seen when expanding

(3.25) for small $\widetilde{p}^2$:

$$\Pi^{\mu\nu,\infty}_{c,np}(p) = \frac{e^2}{4\pi\sqrt{\widetilde{p}^2}}\left(\frac{3\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2} - 5g^{\mu\nu}\right) + \frac{e^2\sqrt{-\widetilde{\mu}_s^2}}{8\pi}\frac{3p^\mu p^\nu + 7p^2 g^{\mu\nu}}{\sqrt{p^2}}$$
$$+ \left[\frac{e^2}{32\pi}\frac{15p^\mu p^\nu - 13p^2 g^{\mu\nu}}{\sqrt{p^2}} + \frac{\widetilde{\mu}_s^2 e^2}{8\pi}\frac{3p^\mu p^\nu - 5p^2 g^{\mu\nu}}{\sqrt{p^2}}\right] \arcsin z\Big|_{a_-}^{a_+} + O(\theta). \tag{3.27}$$

As in the massless model the (lowest order) finite terms in this expression cancel the corresponding planar contributions, leaving exactly the same infrared divergent term as before. In the limit $\mu_s \to 0$ (3.23) and (3.25) tend to (2.49) and (2.54), respectively.

Due to symmetry properties

$$\begin{aligned}\epsilon_{\rho\mu\nu}\left(k \pm \frac{3p}{2}\right)^\mu \left(k \pm \frac{p}{2}\right)^\nu &= \pm\epsilon_{\rho\mu\nu}p^\mu k^\nu,\\ \epsilon_{\rho\mu\nu}\left(k \mp \frac{3p}{2}\right)^\mu \left(k \pm \frac{p}{2}\right)^\nu &= \pm 2\epsilon_{\rho\mu\nu}k^\mu p^\nu,\end{aligned} \tag{3.28}$$

and the superficially logarithmically divergent part of (3.22) becomes

$$2i\mu_s e^2 \int \frac{d^3k}{(2\pi)^3}\frac{\sin^2\left(\frac{k\times p}{2}\right)}{\left[(k+p/2)^2 - \mu_s^2\right]\left[(k-p/2)^2 - \mu_s^2\right]}$$
$$\times\left[\frac{-\left(k+\frac{3p}{2}\right)^\mu \epsilon^{\nu\rho\sigma}k_\rho p_\sigma - \left(3k - \frac{3p}{2}\right)^\nu \epsilon^{\mu\rho\sigma}k_\rho p_\sigma + \left(k - \frac{3p}{2}\right)^2 \epsilon^{\mu\nu\rho}\left(k+\frac{p}{2}\right)_\rho}{(k+p/2)^2}\right.$$
$$\left.+\frac{\left(3k+\frac{3p}{2}\right)^\mu \epsilon^{\nu\rho\sigma}k_\rho p_\sigma + \left(k-\frac{3p}{2}\right)^\nu \epsilon^{\mu\rho\sigma}k_\rho p_\sigma - \left(k+\frac{3p}{2}\right)^2 \epsilon^{\mu\nu\rho}\left(k-\frac{p}{2}\right)_\rho}{(k-p/2)^2}\right]. \tag{3.29}$$

In fact, we notice that with

$$\left(k \pm \frac{3p}{2}\right)^2 = \left(k \mp \frac{p}{2}\right)^2 + 4\left(p^2 \pm kp\right), \tag{3.30}$$

the only two terms with three powers of k in the numerators cancel each other and this contribution is finite after all.

Finally, the photon tadpole-loop graph (Figure 3.2d) is given by (see Appendix B.7)

$$\Pi_d^{\mu\nu}(p) = 2e^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2-\mu_s^2} \left(g_{\sigma\tau} - \mu_s^2 \frac{k_\sigma k_\tau}{k^4} + i\mu_s \frac{\epsilon_{\sigma\tau\rho}k^\rho}{k^2} \right)$$
$$\times \left[g^{\mu\tau} g^{\nu\sigma} + g^{\nu\tau} g^{\mu\sigma} - 2g^{\mu\nu} g^{\sigma\tau} \right] \sin^2\left(\frac{k\times p}{2}\right), \tag{3.31}$$

leading to

$$\Pi_d^{\mu\nu}(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2\left(\frac{k\times p}{2}\right)}{k^2-\mu_s^2} \left(2g^{\mu\nu} + \mu_s^2 \frac{k^\mu k^\nu - k^2 g^{\mu\nu}}{k^4} \right), \tag{3.32}$$

because $\left[g^{\mu\tau} g^{\nu\sigma} + g^{\nu\tau} g^{\mu\sigma} - 2g^{\mu\nu} g^{\sigma\tau} \right] \epsilon_{\sigma\tau\rho} = 0$. Only the term proportional to $g^{\mu\nu}$ seems to be divergent, superficially, and can be evaluated with formulas (D.10a) with $n=3$, $q=0$, $L^2=\mu_s^2$ and $\alpha=1$, and (D.58) with $M^2=\mu_s^2$:

$$\Pi_d^{\mu\nu,\infty}(p) = \frac{e^2 g^{\mu\nu}}{\pi} \left(\sqrt{-\mu_s^2} + \frac{\exp\left[-i\sqrt{\mu_s^2 \tilde{p}^2}\right]}{\sqrt{\tilde{p}^2}} \right). \tag{3.33}$$

In the limit $\mu_s \to 0$ this infrared divergent expression tends to (2.56). Furthermore, expanding (3.33) for small $\tilde{p}^2$ (with $\exp\left[-i\sqrt{\mu_s^2\tilde{p}^2}\right] \approx 1 - i\sqrt{\mu_s^2\tilde{p}^2}$) leads to

$$\Pi_d^{\mu\nu,\infty}(p) = \frac{e^2 g^{\mu\nu}}{\pi\sqrt{\tilde{p}^2}} + O(\theta). \tag{3.34}$$

All in all, if one sums up all divergent terms of this model, one finds exactly the same expression (2.57) as in the massless model: As expected, adding masses m_f and μ_s did not eliminate the (linear) infrared divergence.

Finally, replacing the massless photon propagator (2.21c) with (3.12) in Appendix B.8 leads to the expression for the ghost self-energy at one-loop level:

$$\Xi(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2\left(\frac{k\times p}{2}\right)}{(k+p/2)^2\left[(k-p/2)^2-\mu_s^2\right]} \Bigg[p\,(k+p/2) -$$
$$-\mu_s^2 \frac{\left[p\,(k-p/2)\right]\left[k^2-p^2/4\right]}{(k-p/2)^4} + i\mu_s \frac{\epsilon_{\mu\nu\rho} p^\mu (k+p/2)^\nu (k-p/2)^\rho}{(k-p/2)^2} \Bigg], \tag{3.35}$$

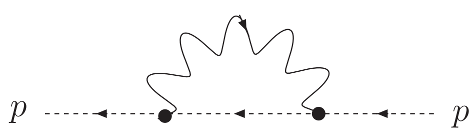


Figure 3.3: ghost self-energy

where only the first term is superficially (logarithmically) divergent. The planar part of this term can be evaluated with integral formula (D.27), identifying $m_1 = 0$ and $m_2 = \mu_s$ and leading to

$$\Xi_{pl}^{\infty}(p) = -\frac{e^2\sqrt{p^2}}{8\pi}\left[2\sqrt{-\widetilde{\mu}_s^2} + \left(1-\widetilde{\mu}_s^2\right)\arcsin z\Big|_{a_-}^{a_+}\right], \tag{3.36}$$

with

$$a_\pm = \frac{\pm 1 - \widetilde{\mu}_s^2}{\sqrt{\left(1+\widetilde{\mu}_s^2\right)^2 - 4\widetilde{\mu}_s^2}}. \tag{3.37}$$

So, (3.36) is also finite and in the limit $\mu_s \to 0$ tends to (2.59).
The non-planar part of the superficially divergent term can be calculated from equation (D.62) with $m_1 = 0$, $m_2 = \mu_s$ and reads

$$\Xi_{np}^{\infty}(p) = \frac{\sqrt{p^2}e^2}{4\pi}\int_0^1 d\xi\sqrt{\frac{(1-\xi)}{(\xi-\widetilde{\mu}_s^2)}}e^{-z(\xi)}, \tag{3.38}$$

with

$$z(\xi) = \sqrt{(\xi-\mu_s^2)(1-\xi)p^2\widetilde{p}^2}. \tag{3.39}$$

Here (3.38) is again finite and in the limit $\mu_s \to 0$ tends to (2.60). Furthermore, in the limit $\theta \to 0$ planar and non-planar contributions cancel each other.

3.3 Gauge-dependence of the model

So far we have done one-loop calculations in Feynman gauge ($\alpha = 1$). If one considers an arbitrary gauge parameter α one gets additional terms which might diverge.

In the fermion self-energy one has to deal with only one additional (superficially logarithmically divergent) term of the type

$$\begin{aligned}\Sigma^{\alpha,\infty}(p) &= -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(1-\alpha)\left(\not{k} - \not{p}/2\right)\left[\not{k} + \not{p}/2 - m_f\right]\left(\not{k} - \not{p}/2\right)}{\left[(k+p/2)^2 - m_f^2\right](k-p/2)^2\left[(k-p/2)^2 - \mu_s^2\right]} \\ &= -e^2(1-\alpha)\int \frac{d^3k}{(2\pi)^3}\left[\frac{\left(\not{k} + \not{p}/2\right) - m_f - 2\not{p}}{\left[(k+p/2)^2 - m_f^2\right]\left[(k-p/2)^2 - \mu_s^2\right]}\right. \\ &\quad \left. + \frac{\left(\not{k} - \not{p}/2\right)(2kp - p^2)}{\left[(k+p/2)^2 - m_f^2\right](k-p/2)^2\left[(k-p/2)^2 - \mu_s^2\right]}\right], \end{aligned} \tag{3.40}$$

where relation (2.7) was used for

$$\begin{aligned}\left(\not{k} - \not{p}/2\right)^2 &= (k-p/2)^2\,, \\ \left(\not{k} - \not{p}/2\right)\left(\not{k} + \not{p}/2\right)\left(\not{k} - \not{p}/2\right) &= 2\left(\not{k} - \not{p}/2\right)\left[(k+p/2)(k-p/2)\right] \\ &\quad - (k-p/2)^2\left(\not{k} + \not{p}/2\right). \end{aligned} \tag{3.41}$$

Furthermore, $(k+p/2)(k-p/2) = (k-p/2)^2 + kp - p^2/2$ was considered. The first term in (3.40) can be integrated using formulas (D.24) and (D.27) in Appendix D.2 and is therefore finite. The second term is finite as well, since there are only two powers of k left in the numerator.

Fermion and ghost-loop contributions to the vacuum polarization are obviously gauge-independent. The photon-loop graph gets the additional superficially divergent terms

$$\begin{aligned}\Pi_c^{\mu\nu,\alpha,\infty}(p) =& 2e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(\alpha-1)\sin^2\left(\frac{k\times p}{2}\right)}{\left[(k+p/2)^2 - \mu_s^2\right]\left[(k-p/2)^2 - \mu_s^2\right]} \\ &\times \Big[-(k-3p/2)^\rho g^{\mu\tau} + 2k^\mu g^{\rho\tau} - (k+3p/2)^\tau g^{\rho\mu}\Big] \\ &\times \Big[-(k+3p/2)^\sigma g^{\nu\epsilon} + 2k^\nu g^{\sigma\epsilon} - (k-3p/2)^\epsilon g^{\sigma\nu}\Big] \\ &\times \left\{ g_{\tau\sigma}\frac{(k-p/2)_\epsilon (k-p/2)_\rho}{(k-p/2)^2} + g_{\epsilon\rho}\frac{(k+p/2)_\tau (k+p/2)_\sigma}{(k+p/2)^2} \right. \\ &\quad \left. + (\alpha-1)\frac{(k-p/2)_\epsilon (k-p/2)_\rho (k+p/2)_\tau (k+p/2)_\sigma}{(k-p/2)^2 (k+p/2)^2} \right\}. \end{aligned} \tag{3.42}$$

Multiplying the three brackets and keeping only the highest powers of k in the numerator leads to

$$4e^2\int\frac{d^3k}{(2\pi)^3}\frac{(\alpha-1)\,(k^2g^{\mu\nu}-k^\mu k^\nu)}{\left[(k+p/2)^2-\mu_s^2\right]\left[(k-p/2)^2-\mu_s^2\right]}\sin^2\left(\frac{k\times p}{2}\right), \tag{3.43}$$

when considering $k^2=(k\pm p/2)^2\mp kp-p^2/2$. The planar part is finite, as can be seen from formula (D.33) in the Appendix, whereas the non-planar part, as calculated using formula (D.63), exhibits the infrared divergent term

$$\Pi_c^{\mu\nu,\alpha,\infty}(p)=-\frac{e^2(\alpha-1)}{4\pi\sqrt{\widetilde{p}^2}}\left(\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}+g^{\mu\nu}\right). \tag{3.44}$$

The photon-tadpole graph receives the additional superficially linearly divergent term

$$\begin{aligned}\Pi_d^{\mu\nu,\alpha,\infty}(p)&=2e^2(\alpha-1)\int\frac{d^3k}{(2\pi)^3}\frac{k_\sigma k_\tau\sin^2\left(\frac{k\times p}{2}\right)}{k^2\left(k^2-\mu_s^2\right)}\left(g^{\mu\tau}g^{\nu\sigma}+g^{\nu\tau}g^{\mu\sigma}-2g^{\mu\nu}g^{\sigma\tau}\right)\\&=4e^2(\alpha-1)\int\frac{d^3k}{(2\pi)^3}\frac{k^\mu k^\nu-k^2g^{\mu\nu}}{k^2\left(k^2-\mu_s^2\right)}\sin^2\left(\frac{k\times p}{2}\right).\end{aligned} \tag{3.45}$$

Comparing the term proportional to the metric with the one in (3.32) it can be seen that this contribution is the same as (3.33) multiplied with a factor $(\alpha-1)/2$. Finally, the first term can be evaluated using formulas (D.34) and (D.64) in the Appendix. The result for (3.45) then reads

$$\begin{aligned}\Pi_d^{\mu\nu,\alpha,\infty}(p)=\frac{e^2(\alpha-1)}{2\pi}\Bigg\{&g^{\mu\nu}\left(\frac{2}{3}\sqrt{-\mu_s^2}+\frac{\exp\left[-i\sqrt{\mu_s^2\widetilde{p}^2}\right]}{\sqrt{\widetilde{p}^2}}\right)\\&-\frac{1}{2}\int_0^1 d\xi\left[\frac{g^{\mu\nu}}{\sqrt{\widetilde{p}^2}}-\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{(\widetilde{p}^2)^{3/2}}\left(1+z(\xi)\right)\right]e^{-z(\xi)}\Bigg\},\end{aligned} \tag{3.46}$$

with $z(\xi)=\sqrt{-\mu_s^2(1-\xi)\widetilde{p}^2}$. Expanding this expression for small $\widetilde{p}^2$ finally yields

$$\Pi_d^{\mu\nu,\alpha,\infty}(p)=\frac{e^2(\alpha-1)}{4\pi\sqrt{\widetilde{p}^2}}\left(\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}+g^{\mu\nu}\right)+O(\theta), \tag{3.47}$$

where to order $O(0)$ planar and non-planar terms cancelled each other.
Obviously, the divergent terms (3.44) and (3.47) cancel each other and the infrared divergence of this model really is independent of the gauge parameter α. A similar result for $3+1$ dimensional QED was found in [30].

The ghost self-energy graph gets the additional superficially logarithmically divergent contribution

$$
\begin{aligned}
\Xi^{\alpha,\infty}(p) &= \int \frac{d^3k}{(2\pi)^3} \frac{4e^2(1-\alpha)\left[p\,(k-p/2)\right]\left[k^2-p^2/4\right]}{(k+p/2)^2\,(k-p/2)^2\left[(k-p/2)^2-\mu_s^2\right]} \sin^2\left(\frac{k\times p}{2}\right) \\
&= 4e^2(1-\alpha) \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{p\,(k-p/2)}{(k+p/2)^2\left[(k-p/2)^2-\mu_s^2\right]} \right. \\
&\quad \left. + \frac{\left[p\,(k-p/2)\right]^2}{(k+p/2)^2\,(k-p/2)^2\left[(k-p/2)^2-\mu_s^2\right]} \right\} \sin^2\left(\frac{k\times p}{2}\right),
\end{aligned}
\tag{3.48}
$$

where $(k^2-p^2/4)=(k-p/2)^2+kp-p^2/2$ was considered. The second term is obviously finite since only two powers of k remain in the numerator. The first term looks almost like the first one in (3.35) and therefore gives $(\alpha-1)$ times the contributions (3.36) and (3.38) which in fact are finite as well.

Chapter 4

Adding the Slavnov Term

4.1 Feynman rules

In the previous chapters it has been shown that non-commutative QED_3 suffers from UV/IR-mixing and is therefore linearly infrared divergent, where the divergent term (2.57) of the one-loop vacuum polarization is independent of the gauge parameter and also of the Chern-Simons mass term. A possible way of eliminating the problems caused by the infrared divergence by adding the following (gauge invariant) term in the action was suggested by Slavnov [27]:

$$S_{Slavnov} = \int d^3x \beta \lambda(x) \theta^{\mu\nu} \star F_{\mu\nu}(x), \tag{4.1}$$

where β is an arbitrary parameter, λ is a further Lagrange multiplier field and the non-commutativity matrix $\theta^{\mu\nu}$ was defined in (1.2). We choose $\beta = 1/2$ to simplify the resulting Feynman rules. In the limit $\theta \to 0$ the Slavnov term (4.1) vanishes, leading to ordinary commutative[1] QED. Using the abbreviation

$$\theta^{\mu\nu} F_{\mu\nu} = 2\theta^{\mu\nu} \left(\partial_\mu A_\nu - ieA_\mu \star A_\nu\right) \equiv \widetilde{F}, \tag{4.2}$$

[1]Note that Slavnov [27] chooses $\beta = 1/\theta$, making the Slavnov term independent of the parameter θ. Therefore, in the limit $\theta \to 0$ such a model will not reduce to regular QED but instead describes a scalar field.

the complete action now reads

$$S = \int d^3x \Bigg\{ \bar\psi \star \left(i\not{D} - m_f\right) \star \psi - \frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} - \mu_s \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$
$$+ B\partial^\mu A_\mu + \frac{\alpha}{2} B^2 - \bar c \star \partial^\mu \left(\partial_\mu c - ie[A_\mu, c]_\star\right) + \frac{1}{2}\lambda \star \widetilde{F} \Bigg\}, \quad (4.3)$$

and remains BRS invariant if λ transforms as

$$s\lambda = -ie[\lambda, c]_\star \ , \quad (4.4)$$

(see Appendix A).
Obviously, variation of the action with respect to λ yields

$$\widetilde{F} = \operatorname{tr}\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -E_1 & -E_2 \\ E_1 & 0 & -B \\ E_2 & B & 0 \end{pmatrix} \right\} = B = 0, \quad (4.5)$$

and the magnetic field[2] is therefore constrained to zero.

When adding a source term for the Lagrange multiplier λ

$$\int d^3x j_\lambda \lambda \ ,$$

in the generating functional (2.13) the equations of motion for the free λ and photon fields are given by[3]

$$\frac{\delta S_0}{\delta \lambda} = -\widetilde{\partial}^\mu A_\mu = -j_\lambda, \quad (4.6a)$$
$$\frac{\delta S_0}{\delta A_\mu} = \Box A^\mu + \frac{1-\alpha}{\alpha} \partial^\mu \partial^\rho A_\rho - \mu_s \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \widetilde{\partial}^\mu \lambda = -j^\mu_A, \quad (4.6b)$$

where $\widetilde{\partial}^\mu$ denotes $\theta^{\mu\nu}\partial_\nu$ and the Lagrange multiplier B has been eliminated using its equation of motion (2.18a).
Applying $\widetilde{\partial}_\mu$ to (4.6b) and inserting (4.6a) one finds

$$\lambda = \frac{1}{\widetilde{\widetilde{\Box}}} \left(-\widetilde{\partial}_\mu j^\mu_A - \Box j_\lambda + \mu_s \epsilon^{\mu\nu\rho} \widetilde{\partial}_\mu \partial_\nu A_\rho \right), \quad (4.7)$$

[2] In equation (4.5), B denotes the magnetic field and not the Lagrange multiplier introduced earlier.

[3] S_0 is now the bilinear part of (4.3).

where $\widetilde{\Box} = \widetilde{\partial}^\mu \widetilde{\partial}_\mu$. Reinserting this equation into (4.6b) leads to

$$\begin{aligned}\mathcal{D}^{\mu\rho} A_\rho &\equiv \left(\Box g^{\mu\rho} + \frac{1-\alpha}{\alpha} \partial^\mu \partial^\rho - \mu_s \epsilon^{\mu\nu\rho} \partial_\nu + \frac{\mu_s \widetilde{\partial}^\mu}{\widetilde{\Box}} \epsilon^{\tau\nu\rho} \widetilde{\partial}_\tau \partial_\nu \right) A_\rho \\ &= -j_A^\mu + \frac{\widetilde{\partial}^\mu}{\widetilde{\Box}} \left(\widetilde{\partial}_\rho j_A^\rho + \Box j_\lambda \right). \end{aligned} \tag{4.8}$$

In order to express A_ρ in terms of the sources, we need to find the inverse of the operator $\mathcal{D}^{\mu\rho}$. Taking into account the new tensor structure in (4.8) including $\widetilde{\partial}_\mu$ (compare (3.6)), we make the ansatz

$$\begin{aligned}\mathcal{D}^{-1}_{\alpha\mu} = {} & g_{\alpha\mu} A + \partial_\alpha \partial_\mu B + \epsilon_{\alpha\mu\tau} \partial^\tau C + \widetilde{\partial}_\alpha \widetilde{\partial}_\mu D \\ & + \widetilde{\partial}_\alpha \epsilon_{\tau\epsilon\mu} \widetilde{\partial}^\tau \partial^\epsilon E + \widetilde{\partial}_\mu \epsilon_{\tau\epsilon\alpha} \widetilde{\partial}^\tau \partial^\epsilon F + \epsilon_{\alpha\epsilon\nu} \widetilde{\partial}^\epsilon \partial^\nu \epsilon_{\mu\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau G, \end{aligned} \tag{4.9}$$

and $\mathcal{D}^{-1}_{\alpha\mu} \mathcal{D}^{\mu\rho} = \delta^\rho_\alpha$ yields

$$\begin{aligned}\delta^\rho_\alpha = {} & \Box \delta^\rho_\alpha A + \frac{1-\alpha}{\alpha} \partial^\rho \partial_\alpha A - \mu_s \epsilon_\alpha{}^{\nu\rho} \partial_\nu A + \frac{\mu_s}{\widetilde{\Box}} \widetilde{\partial}_\alpha \epsilon^{\tau\nu\rho} \widetilde{\partial}_\tau \partial_\nu A + \frac{1}{\alpha} \Box \partial^\rho \partial_\alpha B \\ & + \Box \epsilon_\alpha{}^\rho{}_\tau \partial^\tau C - \mu_s \epsilon^{\mu\nu\rho} \epsilon_{\alpha\mu\tau} \partial_\nu \partial^\tau C + \frac{\mu_s}{\widetilde{\Box}} \epsilon^{\tau\nu\rho} \widetilde{\partial}_\tau \partial_\nu \epsilon_{\alpha\mu\sigma} \widetilde{\partial}^\mu \partial^\sigma C + \Box \widetilde{\partial}_\alpha \widetilde{\partial}^\rho D \\ & + \Box \widetilde{\partial}_\alpha \epsilon_{\tau\epsilon}{}^\rho \widetilde{\partial}^\tau \partial^\epsilon E - \mu_s \widetilde{\partial}_\alpha \epsilon^{\mu\nu\rho} \epsilon_{\tau\epsilon\mu} \partial_\nu \widetilde{\partial}^\tau \partial^\epsilon E + \Box \widetilde{\partial}^\rho \epsilon_{\tau\epsilon\alpha} \widetilde{\partial}^\tau \partial^\epsilon F \\ & + \Box \epsilon_{\alpha\epsilon\nu} \widetilde{\partial}^\epsilon \partial^\nu \epsilon^\rho{}_{\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau G - \mu_s \epsilon^{\mu\nu\rho} \epsilon_{\mu\sigma\tau} \partial_\nu \widetilde{\partial}^\sigma \partial^\tau \epsilon_{\alpha\epsilon\nu} \widetilde{\partial}^\epsilon \partial^\nu G. \end{aligned} \tag{4.10}$$

Remembering $\epsilon^{\rho\mu\nu} \epsilon_{\rho\sigma\tau} = \delta^\mu_\sigma \delta^\nu_\tau - \delta^\nu_\sigma \delta^\mu_\tau$ and comparing coefficients we find the following relations:

$$\delta^\mu_\alpha : \qquad \Box A - \mu_s \Box C = 1, \tag{4.11a}$$

$$\partial^\rho \partial_\alpha : \qquad \frac{1-\alpha}{\alpha} A + \frac{1}{\alpha} \Box B + \mu_s C = 0, \tag{4.11b}$$

$$\epsilon^\rho{}_{\alpha\tau} \partial^\tau : \qquad \mu_s A + \Box C = 0, \tag{4.11c}$$

$$\widetilde{\partial}^\rho \widetilde{\partial}_\alpha : \qquad \Box D + \mu_s \Box E = 0, \tag{4.11d}$$

$$\widetilde{\partial}_\alpha \epsilon^{\tau\nu\rho} \widetilde{\partial}_\tau \partial_\nu : \qquad \frac{\mu_s}{\widetilde{\Box}} A + \Box E = 0, \tag{4.11e}$$

$$\widetilde{\partial}^\rho \epsilon^{\tau\nu}{}_\alpha \widetilde{\partial}_\tau \partial_\nu : \qquad \Box F + \mu_s \Box G = 0, \tag{4.11f}$$

$$\epsilon_{\alpha\epsilon\nu} \widetilde{\partial}^\epsilon \partial^\nu \epsilon^\rho{}_{\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau : \qquad \frac{\mu_s}{\widetilde{\Box}} C + \Box G = 0, \tag{4.11g}$$

leading to

$$A = \frac{1}{\Box + \mu_s^2}, \tag{4.12a}$$

$$B = \left(\frac{(\alpha - 1)}{\Box} + \alpha \frac{\mu_s^2}{\Box^2} \right) \frac{1}{\Box + \mu_s^2}, \tag{4.12b}$$

$$C = -\frac{\mu_s}{\Box} \frac{1}{\Box + \mu_s^2}, \tag{4.12c}$$

$$D = \frac{\mu_s^2}{\Box \widetilde{\Box}} \frac{1}{\Box + \mu_s^2}, \tag{4.12d}$$

$$E = -\frac{\mu_s}{\Box \widetilde{\Box}} \frac{1}{\Box + \mu_s^2}, \tag{4.12e}$$

$$F = -\frac{\mu_s^3}{\Box^2 \widetilde{\Box}} \frac{1}{\Box + \mu_s^2}, \tag{4.12f}$$

$$G = \frac{\mu_s^2}{\Box^2 \widetilde{\Box}} \frac{1}{\Box + \mu_s^2}, \tag{4.12g}$$

and the operator we were after reads

$$\begin{aligned} \mathcal{D}^{-1}_{\alpha\mu} = \frac{1}{\Box + \mu_s^2} \Bigg(& g_{\alpha\mu} + \left[(\alpha - 1) + \alpha \frac{\mu_s^2}{\Box} \right] \frac{\partial_\alpha \partial_\mu}{\Box} - \mu_s \frac{\epsilon_{\alpha\mu\rho} \partial^\rho}{\Box} + \frac{\mu_s^2}{\Box \widetilde{\Box}} \widetilde{\partial}_\alpha \widetilde{\partial}_\mu \\ & - \frac{\mu_s}{\Box \widetilde{\Box}} \widetilde{\partial}_\alpha \epsilon_{\tau\epsilon\mu} \widetilde{\partial}^\tau \partial^\epsilon - \frac{\mu_s^3}{\Box^2 \widetilde{\Box}} \widetilde{\partial}_\mu \epsilon_{\tau\epsilon\alpha} \widetilde{\partial}^\tau \partial^\epsilon + \frac{\mu_s^2}{\Box^2 \widetilde{\Box}} \epsilon_{\alpha\epsilon\nu} \widetilde{\partial}^\epsilon \partial^\nu \epsilon_{\mu\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau \Bigg). \end{aligned} \tag{4.13}$$

We therefore have

$$\begin{aligned} A_\alpha = & - \left(\mathcal{D}^{-1}_{\alpha\mu} - \frac{\widetilde{\partial}_\alpha \widetilde{\partial}_\mu}{\Box \widetilde{\Box}} + \frac{\mu_s}{\Box^2 \widetilde{\Box}} \widetilde{\partial}_\mu \epsilon_{\alpha\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau \right) j^\mu_A \\ & + \frac{1}{\widetilde{\Box}} \left(\widetilde{\partial}_\alpha - \frac{\mu_s}{\Box} \epsilon_{\alpha\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau \right) j_\lambda, \end{aligned} \tag{4.14}$$

and

$$\lambda = -\frac{\Box + \mu_s^2}{\widetilde{\Box}} j_\lambda - \frac{1}{\widetilde{\Box}} \left(\widetilde{\partial}_\mu + \frac{\mu_s}{\Box} \epsilon_{\mu\sigma\tau} \widetilde{\partial}^\sigma \partial^\tau \right) j^\mu_A. \tag{4.15}$$

The propagators are then given by[4]

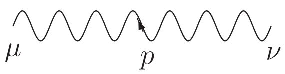
$$\triangle^{AA}_{\mu\nu}(x,y) = -i\frac{\delta A_\mu(x)}{\delta j^\nu_A(y)} = i\left(\mathcal{D}^{-1}_{\mu\nu} - \frac{\widetilde{\partial}_\mu\widetilde{\partial}_\nu}{\Box\widetilde{\Box}} + \frac{\mu_s}{\Box^2\widetilde{\Box}}\widetilde{\partial}_\nu\epsilon_{\mu\sigma\tau}\widetilde{\partial}^\sigma\partial^\tau\right)\delta^3(x-y), \tag{4.16a}$$

$$\triangle^{\lambda A}_{\mu}(x,y) = -i\frac{\delta A_\mu(x)}{\delta j_\lambda(y)} = \frac{i}{\widetilde{\Box}}\left(-\widetilde{\partial}_\mu + \frac{\mu_s}{\Box}\epsilon_{\mu\sigma\tau}\widetilde{\partial}^\sigma\partial^\tau\right)\delta^3(x-y), \tag{4.16b}$$

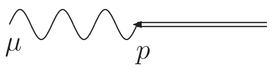
$$\triangle^{A\lambda}_{\mu}(x,y) = -i\frac{\delta \lambda(x)}{\delta j^\mu_A(y)} = \frac{i}{\widetilde{\Box}}\left(\widetilde{\partial}_\mu + \frac{\mu_s}{\Box}\epsilon_{\mu\sigma\tau}\widetilde{\partial}^\sigma\partial^\tau\right)\delta^3(x-y), \tag{4.16c}$$

$$\triangle^{\lambda\lambda}(x,y) = -i\frac{\delta \lambda(x)}{\delta j_\lambda(y)} = i\frac{\Box + \mu_s^2}{\widetilde{\Box}}\,\delta^3(x-y). \tag{4.16d}$$

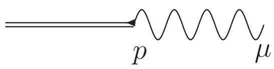
Since these propagators will lead to rather lengthy expressions in the corresponding one-loop graphs we will now set $\mu_s = 0$ for simplicity's sake. (Otherwise the photon propagator consists of nine terms!) Furthermore, we do not expect the Chern-Simons mass to have any effect on divergent terms as this was not the case in the previous model.
Fourier transformation then leads to



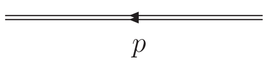
$$\widetilde{\triangle}^{AA}_{\mu\nu}(p) = \frac{-i}{p^2}\left(g_{\mu\nu} + (\alpha-1)\frac{p_\mu p_\nu}{p^2} - \frac{\widetilde{p}_\mu\widetilde{p}_\nu}{\widetilde{p}^2}\right), \tag{4.17a}$$



$$\widetilde{\triangle}^{\lambda A}_{\mu}(p) = -\frac{\widetilde{p}_\mu}{\widetilde{p}^2}, \tag{4.17b}$$



$$\widetilde{\triangle}^{A\lambda}_{\mu}(p) = \frac{\widetilde{p}_\mu}{\widetilde{p}^2}, \tag{4.17c}$$



$$\widetilde{\triangle}^{\lambda\lambda}(p) = \frac{ip^2}{\widetilde{p}^2}. \tag{4.17d}$$

Ghost and fermion propagators remain the same as in (2.21a) and (2.21b), respectively. The additional term in our new photon propagator (4.17a) leads to the nice property

$$\widetilde{p}^\mu\triangle^{AA}_{\mu\nu}(p) = 0. \tag{4.18}$$

In fact, the photon propagator is transversal with respect to $\widetilde{p}^\mu$ even when keeping $\mu_s \neq 0$. This property can render infrared singularities of the type

[4]Once more, all derivatives act on the delta functions and are to be taken with respect to the variable x.

(2.57) harmless, since when they are contracted with internal photon propagators in higher loop insertions one gets expressions including $p_\mu \widetilde{p}^\mu = 0$ [27].

Now that we know the propagators, we still need the new additional lambda-photon vertex (Figure 4.1):

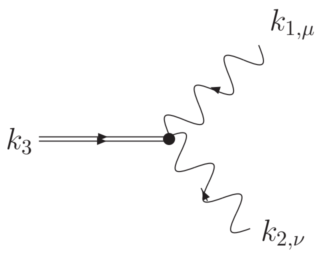


Figure 4.1: Lambda-photon vertex

$$\widetilde{V}^{\lambda AA}_{\mu\nu} = i(2\pi)^9 \frac{\delta}{\delta \widetilde{A}^\mu(-k_1)} \frac{\delta}{\delta \widetilde{A}^\nu(-k_2)} \frac{\delta}{\delta \lambda(-k_3)} S_{int}, \tag{4.19}$$

with

$$\begin{aligned} S^{\lambda AA}_{int} &= -ie\theta^{\rho\tau} \int d^3x \int \frac{d^3 q_{1-3}}{(2\pi)^9} e^{i \sum\limits_{i=1}^{3} q_i^\mu x_\mu} \widetilde{\lambda}(q_1) \widetilde{A}_\rho(q_2) \widetilde{A}_\tau(q_3) e^{-\frac{i}{2} \sum\limits_{j<i} q_j \times q_i} \\ &= -ie\theta^{\rho\tau} \int \frac{d^3 q_{1-3}}{(2\pi)^6} \delta^3(q_1+q_2+q_3) \widetilde{\lambda}(q_1) \widetilde{A}_\rho(q_2) \widetilde{A}_\tau(q_3) e^{-\frac{i}{2}(q_2 \times q_3)}, \end{aligned} \tag{4.20}$$

from the action (4.3). The result is therefore

$$\widetilde{V}^{\lambda AA}_{\mu\nu} = -2ie(2\pi)^3 \delta^3(k_1+k_2+k_3) \theta^{\mu\nu} \sin\left(\frac{k_1 \times k_2}{2}\right). \tag{4.21}$$

4.2 Power counting

Since we are now dealing with additional propagators and vertices we have to modify our power counting formula (2.39). The $A\lambda$ propagator as well as the λA propagator each reduce the degree of divergence by one, whereas the $\lambda\lambda$ propagator adds an equal number of powers of k to both numerator and denominator. Therefore equation (2.34) becomes

$$d(\gamma) = 3L - I_\psi - 2I_A - 2I_c - I_{\lambda A} + V_c + V_{3A}, \tag{4.22}$$

where $I_{\lambda A}$ denotes the number of both internal λA and $A\lambda$ lines. The number of loop integrations L is then

$$L = I_\psi + I_A + I_c + I_{\lambda\lambda} + I_{\lambda A} - (V_\psi + V_c + V_{3A} + V_{4A} + V_\lambda - 1)\,, \qquad (4.23)$$

which differs from (2.35) by the additional internal $\lambda\lambda$-lines $I_{\lambda\lambda}$, the internal λA-lines $I_{\lambda A}$ and the lambda-photon vertex V_λ.
Elimination of L in (4.22) yields

$$d(\gamma) = 3+2I_\psi+I_A+I_c+3I_{\lambda\lambda}+2I_{\lambda A}-3V_\psi-2V_c-2V_{3A}-3V_{4A}-3V_\lambda. \qquad (4.24)$$

Similar considerations as in Section 2.4 lead to the relations

$$\begin{aligned} 2V_\psi &= E_\psi + 2I_\psi, \\ 2V_c &= E_c + 2I_c, \\ V_\psi + V_c + 3V_{3A} + 4V_{4A} + 2V_\lambda &= E_A + 2I_A + I_{\lambda A}, \\ V_\lambda &= E_{\lambda\lambda} + 2I_{\lambda\lambda} + I_{\lambda A}, \\ V_\psi + V_c + V_{3A} + 2V_{4A} + V_\lambda &= E_e. \end{aligned} \qquad (4.25)$$

Now what about external λA and $A\lambda$ legs? The degree of divergence of a certain Feynman graph depends on which end of these legs couple to the vertices: If the lambda part couples to a vertex, such a leg will have the same effect as $E_{\lambda\lambda}$, whereas if the photon part is coupled to a vertex it will have the same effect as E_A. Therefore we can either count them as $E_{\lambda\lambda}$ or as E_A. In any case, it is not clear how such legs should be amputated.
Elimination of the internal legs in equation (4.24) produces

$$d(\gamma) = 3-E_\psi-\frac{1}{2}E_A-\frac{1}{2}E_c-\frac{3}{2}E_{\lambda\lambda}-\frac{1}{2}V_\psi-\frac{1}{2}V_c-\frac{1}{2}V_{3A}-V_{4A}-\frac{1}{2}V_\lambda, \qquad (4.26)$$

and use of the last relation of (4.25) finally leads to

$$d(\gamma) = 3 - E_\psi - \frac{1}{2}E_A - \frac{1}{2}E_c - \frac{3}{2}E_{\lambda\lambda} - \frac{1}{2}E_e, \qquad (4.27)$$

depending only on the number of external legs once more. In addition to the ones we had before, one further superficially divergent Feynman graph is the one-loop correction to the λA (or $A\lambda$) propagator, which appears to be logarithmically divergent ($E_A = E_\lambda = 1$, $E_e = 2$).

4.3 One-loop calculations

Due to the Slavnov term in the action (4.3) one gets six additional one-loop graphs: the lambda self-energy (Figure 4.2a), three further corrections to the vacuum polarization (Figures 4.2b-d) and two corrections to the lambda-photon propagator (Figures 4.2e,f).

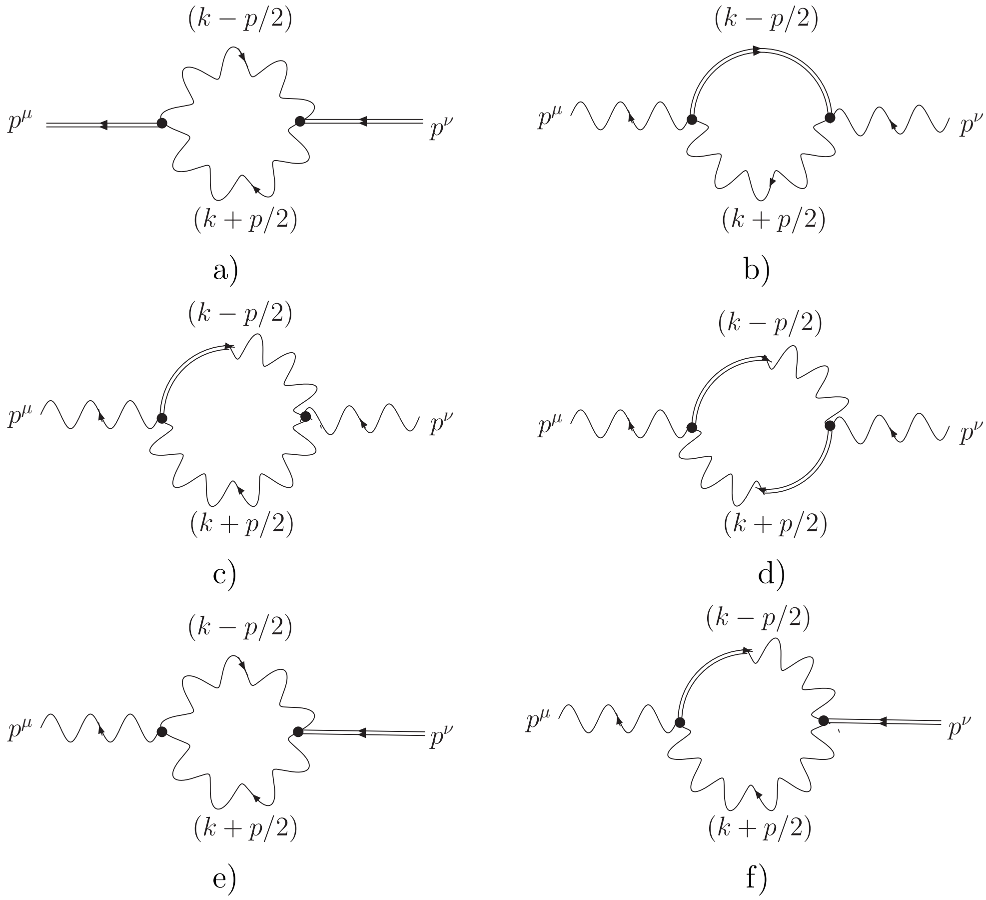


Figure 4.2: Additional graphs including the new lambda and lambda-photon propagators and the lambda-photon vertex

The lambda self-energy is obviously finite as follows from our power counting formula (4.27) ($E_{\lambda\lambda} = E_e = 2 \to d(\gamma) = -1$) and therefore need not be discussed here further.
The expression for the graph in Figure 4.2b as calculated in Appendix C.1

is given by[5]

$$\Pi_e^{\mu\nu}(p) = 4e^2 \int \frac{d^3k}{(2\pi)^3} \sin^2\left(\frac{k\times p}{2}\right) \frac{(k-p/2)^2}{(k+p/2)^2\left(\widetilde{k}-\widetilde{p}/2\right)^2}\Bigg\{ -\theta^2\bar{\mathbf{1}}^{\mu\nu}$$
$$-(\alpha-1)\frac{\left(\widetilde{k}+\widetilde{p}/2\right)^\mu\left(\widetilde{k}+\widetilde{p}/2\right)^\nu}{(k+p/2)^2} + \frac{\left(\bar{k}+\bar{p}/2\right)^\mu\left(\bar{k}+\bar{p}/2\right)^\nu}{\left(\widetilde{k}+\widetilde{p}/2\right)^2}\Bigg\}, \tag{4.28}$$

with the abbreviations (see (1.2))

$$\theta^2\bar{\mathbf{1}}^{\mu\nu} \equiv -\theta^\mu{}_\rho\theta^{\rho\nu} = \theta^2\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.29a}$$

$$\bar{k}^\mu \equiv \theta^{\mu\nu}\widetilde{k}_\nu = -\theta^2\begin{pmatrix} 0 \\ k^1 \\ k^2 \end{pmatrix}. \tag{4.29b}$$

Notice that $\bar{\mathbf{1}}^{ij} = g^{ij}$ where $i,j=1,2$. Therefore the matrix $\bar{\mathbf{1}}^{\mu\nu}$ can be used to pull up indices of all vectors with zero component of zero (e.g. $\widetilde{k}_\mu$, $\bar{k}_\mu$).

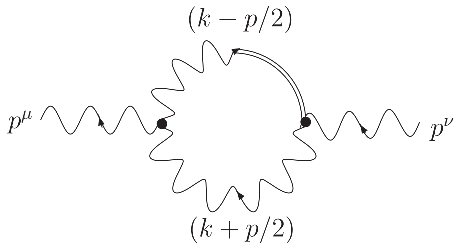


Figure 4.3: The graph including a photon-lambda propagator

In addition to the graph in Figure 4.2c there is actually also the possibility of replacing the lambda-photon propagator with a photon-lambda propagator

[5]Remember $\Pi^{\mu\nu}_{a-d}$ are the graphs represented in Figure 3.2.

as illustrated in Figure 4.3. The sum of both contributions is then given by

$$
\begin{aligned}
\Pi_f^{\mu\nu}(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3} & \frac{\sin^2\left(\frac{k\times p}{2}\right)}{\left(k+p/2\right)^2\left(\widetilde{k}-\widetilde{p}/2\right)^2} \\
& \times \Bigg\{ 2k\widetilde{p}\theta^{\mu\nu} + 2\left(\bar{k}-\bar{p}/2\right)^\mu k^\nu - \left(\widetilde{k}-3\widetilde{p}/2\right)^\mu \left(\widetilde{k}-\widetilde{p}/2\right)^\nu \\
& + (\alpha-1)\frac{\left(\widetilde{k}+\widetilde{p}/2\right)^\mu}{\left(k+p/2\right)^2}\left(k\widetilde{p}p^\nu - (k+p/2)(k-3p/2)\left(\widetilde{k}-\widetilde{p}/2\right)^\nu\right) \\
& - \frac{\left(\bar{k}+\bar{p}/2\right)^\mu}{\left(\widetilde{k}+\widetilde{p}/2\right)^2}\left(2k\widetilde{p}\widetilde{p}^\nu + 2\left(\widetilde{k}^2-\widetilde{p}^2/4\right)k^\nu\right)\Bigg\} + \mu\leftrightarrow\nu \quad , \qquad (4.30)
\end{aligned}
$$

(see Appendix C.2).
Furthermore, the contribution including two lambda-photon propagators (see Figure 4.2d) as calculated in Appendix C.3 is given by

$$
\Pi_g^{\mu\nu}(p) = 4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\left(\bar{k}-\bar{p}/2\right)^\mu \left(\bar{k}+\bar{p}/2\right)^\nu}{\left(\widetilde{k}+\widetilde{p}/2\right)^2\left(\widetilde{k}-\widetilde{p}/2\right)^2} \sin^2\left(\frac{k\times p}{2}\right) . \qquad (4.31)
$$

Finally, both photon-loop and photon-tadpole graphs (see Figures 3.2c and 3.2d in the previous chapter) receive additional terms due to our modified photon propagator (4.17a).

The photon-loop graph gets the additional terms[6]

$$
\begin{aligned}
\Pi^{\mu\nu}_{c,sl}(p) = -2e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\sin^2\left(\frac{k\times p}{2}\right)}{(k+p/2)^2\,(k-p/2)^2} \Bigg\{ & 8k^\mu k^\nu \\
& + \frac{\sum_{+/-}}{\left(\widetilde{k}\pm\widetilde{p}/2\right)^2} \Bigg[4(k\widetilde{p})^2 g^{\mu\nu} + \left(k+\frac{3p}{2}\right)^2 \left(\widetilde{k}\pm\frac{\widetilde{p}}{2}\right)^\mu \left(\widetilde{k}\pm\frac{\widetilde{p}}{2}\right)^\nu \\
& + 3k\widetilde{p} \left\{ \left(\widetilde{k}\pm\frac{\widetilde{p}}{2}\right)^\mu p^\nu + \mu\leftrightarrow\nu \right\} \Bigg] \\
& + \sum_{+/-} \frac{(\alpha-1)\left[\left(k^2 \pm kp - \frac{3p^2}{2}\right)\left(\widetilde{k}\pm\frac{\widetilde{p}}{2}\right)^\mu \mp k\widetilde{p}p^\mu\right][\mu\to\nu]}{\left(k\mp p/2\right)^2\left(\widetilde{k}\pm\widetilde{p}/2\right)^2} \\
& - 4\frac{\left[\left(\widetilde{k}^2-\widetilde{p}^2/4\right)k^\mu + k\widetilde{p}\widetilde{p}^\mu\right][\mu\to\nu]}{\left(\widetilde{k}+\widetilde{p}/2\right)^2\left(\widetilde{k}-\widetilde{p}/2\right)^2},
\end{aligned}
\tag{4.32}
$$

and the tadpole graph gets additionally

$$
\Pi^{\mu\nu}_{d,sl}(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3} \left(\frac{\widetilde{k}^\mu\widetilde{k}^\nu}{k^2\widetilde{k}^2} - \frac{g^{\mu\nu}}{k^2} \right) \sin^2\left(\frac{k\times p}{2}\right). \tag{4.33}
$$

We are now interested in possible infrared divergences coming from the non-planar sector. The relevant terms can be extracted by setting the external momentum $p=0$ but keeping $\widetilde{p}\neq 0$:

$$
\begin{aligned}
\Pi^{\mu\nu,\infty}_{e,np}(p) &= -e^2 \sum_{\eta=\pm1} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\eta k\widetilde{p}}}{\widetilde{k}^2} \left(-\theta^2\bar{\mathbf{1}}^{\mu\nu} - (\alpha-1)\frac{\widetilde{k}^\mu\widetilde{k}^\nu}{k^2} + \frac{\bar{k}^\mu\bar{k}^\nu}{\widetilde{k}^2} \right), \\
\Pi^{\mu\nu,\infty}_{f,np}(p) &= -2\alpha e^2 \sum_{\eta=\pm1} \int \frac{d^3k}{(2\pi)^3} e^{i\eta k\widetilde{p}} \frac{\widetilde{k}^\mu\widetilde{k}^\nu}{k^2\widetilde{k}^2}, \\
\Pi^{\mu\nu,\infty}_{g,np}(p) &= -e^2 \sum_{\eta=\pm1} \int \frac{d^3k}{(2\pi)^3} e^{i\eta k\widetilde{p}} \frac{\bar{k}^\mu\bar{k}^\nu}{\widetilde{k}^4}, \\
\Pi^{\mu\nu,\infty}_{c,sl,np}(p) &= e^2 \sum_{\eta=\pm1} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\eta k\widetilde{p}}}{k^2} \left(2\frac{k^\mu k^\nu}{k^2} + \alpha\frac{\widetilde{k}^\mu\widetilde{k}^\nu}{\widetilde{k}^2} \right),
\end{aligned}
$$

[6] $[\ldots][\mu\to\nu]$ means multiplication with the same term in brackets but with the index μ replaced by ν. Expression (4.32) is the result of algebraic manipulations of originally 36 terms.

$$\Pi^{\mu\nu,\infty}_{d,sl,np}(p) = e^2 \sum_{\eta=\pm1} \int \frac{d^3k}{(2\pi)^3} e^{i\eta k\tilde{p}} \left(\frac{\tilde{k}^\mu \tilde{k}^\nu}{k^2 \tilde{k}^2} - \frac{g^{\mu\nu}}{k^2} \right). \tag{4.34}$$

The sum of these terms is then

$$\Pi^{\mu\nu,\infty}_{sl,np}(p) = -e^2 \sum_{\eta=\pm1} \int \frac{d^3k}{(2\pi)^3} e^{i\eta k\tilde{p}} \left(-\frac{\theta^2 \bar{\mathbf{1}}^{\mu\nu}}{\tilde{k}^2} + 2\frac{\bar{k}^\mu \bar{k}^\nu}{\tilde{k}^4} + \frac{g^{\mu\nu}}{k^2} - 2\frac{k^\mu k^\nu}{k^4} \right), \tag{4.35}$$

which can be evaluated using formulas (D.55), (D.58), (D.65), (D.71) and considering $\tilde{k}^2 = -\theta^2 \vec{k}^2$. The divergent parameter integrals coming from the first two terms in (4.35) cancel each other and introducing the momentum cutoff $\int\limits_{-\infty}^{+\infty} dk^0 \to \int\limits_{-\Lambda}^{+\Lambda} dk^0$ we arrive at

$$\Pi^{\mu\nu,\infty}_{sl,np}(p) = \frac{e^2}{\pi} \left(\frac{\Lambda}{\pi} - \frac{1}{2\sqrt{\tilde{p}^2}} \right) \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2}. \tag{4.36}$$

Comparing this result with (2.57) we see that the infrared divergent (second) term of (4.36) cancels exactly the one we had in our previous model: There is no more infrared divergence left in the vacuum polarization of the photon. Instead we find the transversal linear ultraviolet divergence

$$\Pi^{\mu\nu}_{np,divergent}(p) = \lim_{\Lambda\to\infty} \Lambda \frac{e^2}{\pi^2} \frac{\tilde{p}^\mu \tilde{p}^\nu}{\tilde{p}^2}. \tag{4.37}$$

Note, however, that the cancellation of the infrared divergence only happens in 3 dimensions: The integrands producing these terms are $k^\mu k^\nu / k^4$ which appear only in the ghost-loop and the photon-loop graphs. As was shown in Appendix B.6 their fore factor depends on the trace of the metric, hence on the dimension in which the model is formulated, and happens to be zero in our case.

Next, let us take a look at the planar sector: Examining the integrands one notices several k^0-independent terms[7]. Therefore one expects further ultraviolet divergent terms coming from the planar part of the model. The

[7] The reason for the appearance of such integrals lies in the structure of the Slavnov term in the action (4.1).

most divergent terms[8] are:

$$
\begin{aligned}
\Pi^{\mu\nu,\infty}_{e,pl}(p) &= 2e^2 \int \frac{d^3k}{(2\pi)^3} \Bigg\{ \frac{-\theta^2 \bar{\mathbf{1}}^{\mu\nu}}{\left(\widetilde{k}-\widetilde{p}/2\right)^2} - (\alpha-1) \frac{\left(\widetilde{k}+\widetilde{p}/2\right)^\mu \left(\widetilde{k}+\widetilde{p}/2\right)^\nu}{(k+p/2)^2 \left(\widetilde{k}-\widetilde{p}/2\right)^2} \\
&\quad + \frac{\left(\bar{k}+\bar{p}/2\right)^\mu \left(\bar{k}+\bar{p}/2\right)^\nu}{\left(\widetilde{k}+\widetilde{p}/2\right)^2 \left(\widetilde{k}-\widetilde{p}/2\right)^2} \Bigg\}, \\
\Pi^{\mu\nu,\infty}_{f,pl}(p) &= 4e^2 \int \frac{d^3k}{(2\pi)^3} \Bigg\{ \frac{\left(\widetilde{k}-3\widetilde{p}/2\right)^\mu \left(\widetilde{k}-\widetilde{p}/2\right)^\nu}{(k+p/2)^2 \left(\widetilde{k}-\widetilde{p}/2\right)^2} \\
&\quad + (\alpha-1) \frac{\left(\widetilde{k}+\widetilde{p}/2\right)^\mu \left(\widetilde{k}-\widetilde{p}/2\right)^\nu}{(k+p/2)^2 \left(\widetilde{k}-\widetilde{p}/2\right)^2} \Bigg\}, \\
\Pi^{\mu\nu,\infty}_{g,pl}(p) &= 2e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\left(\bar{k}-\bar{p}/2\right)^\mu \left(\bar{k}+\bar{p}/2\right)^\nu}{\left(\widetilde{k}+\widetilde{p}/2\right)^2 \left(\widetilde{k}-\widetilde{p}/2\right)^2}, \\
\Pi^{\mu\nu,\infty}_{c,sl,pl}(p) &= -e^2 \int \frac{d^3k}{(2\pi)^3} \Bigg\{ \frac{4k^\mu k^\nu}{(k+p/2)^2 (k-p/2)^2} \\
&\quad + \alpha \sum_{+/-} \frac{\left(\widetilde{k}\pm\widetilde{p}/2\right)^\mu \left(\widetilde{k}\pm\widetilde{p}/2\right)^\nu}{(k\pm p/2)^2 \left(\widetilde{k}\pm\widetilde{p}/2\right)^2} \Bigg\}, \\
\Pi^{\mu\nu,\infty}_{d,sl,pl}(p) &= -2e^2 \int \frac{d^3k}{(2\pi)^3} \left(\frac{\widetilde{k}^\mu \widetilde{k}^\nu}{k^2 \widetilde{k}^2} - \frac{g^{\mu\nu}}{k^2} \right).
\end{aligned}
\tag{4.38}
$$

Summing up these terms, keeping only the superficially linearly divergent ones (logarithmic divergences will not be discussed in this chapter) and where convenient shifting $k \to k \pm p/2$ we arrive at

$$
\begin{aligned}
\Pi^{\mu\nu,\infty}_{sl,pl}(p) &= 2e^2 \int \frac{d^3k}{(2\pi)^3} \Bigg\{ -\frac{\theta^2 \bar{\mathbf{1}}^{\mu\nu}}{\widetilde{k}^2} + \frac{2\bar{k}^\mu \left(\bar{k}+\bar{p}/2\right)^\nu}{\left(\widetilde{k}+\widetilde{p}/2\right)^2 \left(\widetilde{k}-\widetilde{p}/2\right)^2} \\
&\quad + \frac{g^{\mu\nu}}{k^2} - \frac{2k^\mu k^\nu}{(k+p/2)^2 (k-p/2)^2} \Bigg\}.
\end{aligned}
\tag{4.39}
$$

[8]Remember $(k \pm p/2)^2 = (k \mp p/2)^2 \pm 2kp$.

The third and fourth terms are finite, as can be seen from formulas (D.20) and (D.21), whereas the first two terms are independent of k^0 and therefore one would naïvely expect them to diverge. But in fact the term proportional to $\bar{\mathbf{1}}^{\mu\nu}$ is zero in dimensional regularization:

$$-\int \frac{d^3k}{(2\pi)^3} \frac{\theta^2 \bar{\mathbf{1}}^{\mu\nu}}{\widetilde{k}^2} = \int \frac{dk^0}{2\pi} \int \frac{d^2k}{(2\pi)^2} \frac{\bar{\mathbf{1}}^{\mu\nu}}{\vec{k}^2}. \tag{4.40}$$

The integration over the space coordinates can be performed similarly to (D.21) in Appendix D.2 using dimensional regularization[9] and therefore this integral is zero.
Only the second term in (4.39) remains divergent and can be calculated using formula (D.77) in the Appendix with $n \to 3$ and taking into consideration that $\widetilde{k}^2 = -\theta^2 \vec{k}^2$. Introducing the momentum cutoff Λ as before, we get[10]

$$\Pi^{\mu\nu,\infty}_{sl,pl}(p) = \lim_{\Lambda\to\infty} \Lambda \lim_{\epsilon\to 0} \Gamma(\epsilon) \frac{e_\epsilon^2}{2\pi^2} \left(\frac{\bar{p}^\mu \bar{p}^\nu}{\bar{p}^2} - \bar{\mathbf{1}}^{\mu\nu} \right), \tag{4.41}$$

which is transversal since $p_\mu \bar{p}^\mu = -\bar{p}^2/\theta^2$ and $p_\mu \bar{\mathbf{1}}^{\mu\nu} = -\bar{p}^\nu/\theta^2$ (see definitions (4.29a) and (4.29b)). The coupling constant in this expression has mass dimension $(1+\epsilon)/2$ due to dimensional regularization and has therefore been denoted e_ϵ, which in the limit $\epsilon \to 0$ reduces to e. Introducing a new parameter μ with mass dimension 1 we can write $e_\epsilon^2 \equiv e^2 \mu^\epsilon$.

Together with (4.37) the (ultraviolet) divergent terms of this model now read

$$\Pi^{\mu\nu}_{divergent}(p) = \lim_{\Lambda\to\infty} \Lambda \frac{e^2}{\pi^2} \left[\frac{\widetilde{p}^\mu \widetilde{p}^\nu}{\widetilde{p}^2} + \lim_{\epsilon\to 0} \Gamma(\epsilon) \frac{\mu^\epsilon}{2} \left(\frac{\bar{p}^\mu \bar{p}^\nu}{\bar{p}^2} - \bar{\mathbf{1}}^{\mu\nu} \right) \right]. \tag{4.42}$$

(We have already seen that infrared divergent terms cancel each other.) Furthermore, one notices that the first term in (4.42) is transversal with respect to the photon propagator (4.17a). If the same is true for the second term, these divergences will not contribute to the one-loop correction of the photon propagator given by

$$\widetilde{\Delta}'^{AA}_{\mu\nu} = \widetilde{\Delta}^{AA}_{\mu\nu} + \Pi_{\mu\rho} \widetilde{\Delta}^{\rho\sigma,AA} \Pi_{\sigma\nu},$$

and therefore the Slavnov trick [27] should work in this model even though new ultraviolet divergent terms appear, provided that

$$\left(g_{\mu\nu} - \frac{\widetilde{p}_\mu \widetilde{p}_\nu}{\widetilde{p}^2} \right) \left(\frac{\bar{p}^\nu \bar{p}^\rho}{\bar{p}^2} - \bar{\mathbf{1}}^{\nu\rho} \right) \stackrel{?}{=} 0. \tag{4.43}$$

[9] Here we are dealing with a Euclidian integral, whereas the Appendix deals with Minkowski-type integrals. The difference in the results lies in some constant fore factor which in this case is, of course, irrelevant since this specific integral is zero anyway.

[10] $\epsilon \equiv 3 - n$ and therefore the limit $\epsilon \to 0$ corresponds to $n \to 3$.

Since $\bar{p}^\mu$ is proportional to p^μ with zero component $p^0 = 0$ (see equation (4.29b)), $\widetilde{p}_\nu \bar{p}^\nu = 0$ and one has

$$\left(g_{\mu\nu} - \frac{\widetilde{p}_\mu \widetilde{p}_\nu}{\widetilde{p}^2}\right)\left(\frac{\bar{p}^\nu \bar{p}^\rho}{\bar{p}^2} - \bar{\mathbf{1}}^{\nu\rho}\right) = \frac{\bar{p}_\mu \bar{p}^\rho}{\bar{p}^2} - \bar{\mathbf{1}}_\mu{}^\rho + \frac{\widetilde{p}_\mu \widetilde{p}^\rho}{\widetilde{p}^2}$$
$$= \frac{1}{\vec{p}^2}\left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & (p^1)^2 & p^1 p^2 \\ 0 & p^2 p^1 & (p^2)^2 \end{pmatrix} - \vec{p}^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & (p^2)^2 & -p^1 p^2 \\ 0 & -p^2 p^1 & (p^1)^2 \end{pmatrix}\right],$$

where the last line follows from $\bar{p}_\mu = -\bar{p}^\mu$, $\widetilde{p}_\mu = -\widetilde{p}^\mu = -\theta^{\mu\nu} p_\nu$, $\bar{\mathbf{1}}_\mu{}^\nu = -\bar{\mathbf{1}}^{\mu\nu}$ (numerically) and from inserting the equations (4.29a) and (4.29b). Since $(p^1)^2 + (p^2)^2 = \vec{p}^2$ all remaining terms cancel each other and (4.43) is really fulfilled.

Note, however, that there might still be logarithmic (UV-) divergences left in the model coming from three and more parameter integrals. These and also all finite terms must have the following structure to be transversal:

$$\Pi^{\mu\nu}(p) = e^2 \sqrt{p^2}\left[\left(\frac{p^\mu p^\nu}{p^2} - g^{\mu\nu}\right) A\left(p^2, \widetilde{p}^2\right) + \left(\frac{\bar{p}^\mu \bar{p}^\nu}{\bar{p}^2} - \bar{\mathbf{1}}^{\mu\nu}\right) B\left(p^2, \widetilde{p}^2\right) + \frac{\widetilde{p}^\mu \widetilde{p}^\nu}{\widetilde{p}^2} C\left(p^2, \widetilde{p}^2\right)\right]. \tag{4.44}$$

Because of (4.43), divergences in $B\left(p^2, \widetilde{p}^2\right)$ and $C\left(p^2, \widetilde{p}^2\right)$ will not contribute to the one-loop correction of the vacuum polarization, and those in $A\left(p^2, \widetilde{p}^2\right)$ can be dealt with through standard renormalization procedures.

Finally, corrections to the lambda-photon propagator appear to be only logarithmically divergent, as follows from our power counting formula (4.27). Fermion and ghost self-energy (Figures 3.1 and 3.3) also both get additional terms because of the modified photon propagator (4.17a), but one can easily show that both contributions are finite:

$$\Sigma^{sl}(p) = -e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\left(\tilde{\not{k}} - \tilde{\not{p}}/2\right)\left(\not{k} + \not{p}/2\right)\left(\tilde{\not{k}} - \tilde{\not{p}}/2\right)}{(k+p/2)^2 (k-p/2)^2 \left(\widetilde{k} - \widetilde{p}/2\right)^2}$$
$$= \int \frac{d^3k}{(2\pi)^3} \frac{e^2}{(k+p/2)^2 (k-p/2)^2}\left[\frac{2k\widetilde{p}\left(\tilde{\not{k}} - \tilde{\not{p}}/2\right)}{\left(\widetilde{k} - \widetilde{p}/2\right)^2} + \left(\not{k} + \not{p}/2\right)\right], \tag{4.45}$$

$$
\begin{aligned}
\Xi^{sl}(p) &= 4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(k+p/2)^\mu \left(\widetilde{k}-\widetilde{p}/2\right)_\mu p^\nu \left(\widetilde{k}-\widetilde{p}/2\right)_\nu}{(k+p/2)^2\,(k-p/2)^2\left(\widetilde{k}-\widetilde{p}/2\right)^2} \sin^2\left(\frac{k\times p}{2}\right) \\
&= 4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{(k\widetilde{p})^2}{(k+p/2)^2\,(k-p/2)^2\left(\widetilde{k}-\widetilde{p}/2\right)^2} \sin^2\left(\frac{k\times p}{2}\right) .
\end{aligned}
\tag{4.46}
$$

The only superficially divergent term is the second one in (4.45), which can be calculated with formula (D.15) in Appendix D.2 and is therefore finite as well.

It remains to be checked in future studies whether logarithmic divergences in the vacuum polarization (and in the corrections to the lambda-photon propagator) cancel each other and if the model really is renormalizable in case they do not.

Conclusion

By extending massless 2+1 dimensional QED to non-commutative Minkowski space $\mathbb{M}^3_{NC}$ we have shown that the resulting model is no longer finite but exhibits a linear infrared singularity

$$\Pi^{\mu\nu}_{IR-divergent}(p) = \frac{e^2}{2\pi\sqrt{\widetilde{p}^2}}\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}, \tag{iii}$$

in the vacuum polarization (whereas fermion and ghost self-energy do remain finite). Making fermions and photons massive (for photons this is achieved in a gauge invariant way via a Chern-Simons mass term) does not change this situation: The divergent term (iii) remains *independent* of all mass parameters. (A similar result is quoted in [22] for Φ^4-theory.) In Chapter 3.3 we then verified gauge independence of (iii).

As stated in the Introduction, it should be possible to render the infrared singular term harmless by extending the action by the so-called Slavnov term [27]. Such a model was discussed concerning divergences in Chapter 4. For simplicity's sake and since the Chern-Simons mass did not have any effect on the divergence in the previous model, the mass parameter μ_s was set to zero. As expected, the new photon propagator became transversal with respect to $\widetilde{p}^\mu$ (see Introduction and reference [27]). In addition, the infrared divergent term (iii) now was even *cancelled* out, a result unique in 3 dimensions (see Chapter 4.3). On the other hand, new UV-divergences appeared in the model:

$$\Pi^{\mu\nu}_{UV-divergent}(p) = \lim_{\Lambda\to\infty}\Lambda\frac{e^2}{\pi^2}\left[\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2} + \lim_{\epsilon\to 0}\Gamma\left(\epsilon\right)\frac{\mu^\epsilon}{2}\left(\frac{\bar{p}^\mu\bar{p}^\nu}{\bar{p}^2} - \bar{\mathbf{1}}^{\mu\nu}\right)\right], \tag{iv}$$

but these are again transversal (like the initial infrared divergence) with respect to the photon propagator and therefore do not seem to present a problem. The ultraviolet divergent terms (iv) appear due to some k^0-independent integrands which are the result of $\theta^{\mu\nu}$ not having full rank. ($\theta^{\mu\nu}$ was defined in Chapter 1 in equation (1.2).) Also note that the divergent terms (iii) and

(iv) are completely independent of the fermionic sector, since the Lagrange multiplier field λ introduced in the Slavnov term couples to the gauge bosons, only.

A matrix $\theta^{\mu\nu}$ with full rank would replace (some) ultraviolet divergent terms with infrared divergent ones (due to the famous UV/IR-mixing). This is, however, only possible in even dimensions and one would have to sacrifice causality [21] (which would not matter in a Euclidian model, for instance). Calculations [6, 7] have shown that in the case of four dimensional (Euclidian) QED coupled to scalar fields (including full-rank $\theta^{\mu\nu}$) the Slavnov term does *not* eliminate (or even change) the (quadratic) infrared divergent term

$$\Pi^{\mu\nu}_{IR-divergent}(p) \propto \frac{\widetilde{p}^{\mu}\widetilde{p}^{\nu}}{\widetilde{p}^{4}}. \tag{v}$$

Due to the fourth dimension, logarithmic ultraviolet divergences still appear in the model, however. But these can (hopefully) be dealt with through standard renormalization procedures.

Whether logarithmic divergences remain in 2+1 dimensional QED including the Slavnov term remains to be checked. However, divergences proportional to

$$\sqrt{p^2}\left(\frac{p^{\mu}p^{\nu}}{p^2} - g^{\mu\nu}\right),$$

can (hopefully) be dealt with through standard renormalization procedures since similar corrections appear in commutative 3+1 dimensional QED. All other divergences (in the vacuum polarization) must have the same structure as the linear ones due to transversality. What still remains to discuss is whether the Slavnov trick also works for higher-loop corrections and whether the theory really is renormalizable.

Appendix A

Showing BRS-Invariance

The complete action used throughout this study reads

$$S = \int d^3x \Bigg\{ \bar{\psi} \star \left(i\not{D} - m_f \right) \star \psi - \frac{1}{4} F^{\mu\nu} \star F_{\mu\nu} - \mu_s \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$+ B\partial^\mu A_\mu + \frac{\alpha}{2} B^2 - \bar{c} \star \partial^\mu \left(\partial_\mu c - ig[A_\mu, c]_\star \right) + \frac{1}{2} \lambda \star \widetilde{F} \Bigg\}, \quad \text{(A.1)}$$

with

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]_\star, \\ \widetilde{F} &= \theta^{\mu\nu} F_{\mu\nu}, \\ \not{D} &= \gamma^\mu \left(\partial_\mu - ieA_\mu \right). \end{aligned} \quad \text{(A.2)}$$

We will now prove its invariance under the BRS transformations

$$\begin{aligned} s\psi = iec \star \psi \,, \quad s\bar{\psi} &= ie\bar{\psi} \star c, \\ sA_\mu &= \partial_\mu c - ig[A_\mu, c]_\star, \\ sc &= igc \star c, \\ s\bar{c} = B \,, \quad sB &= 0, \\ s\lambda &= -ig[\lambda, c]_\star. \end{aligned} \quad \text{(A.3)}$$

Furthermore we will find the restriction $e = g$ for the coupling constants. Letting the BRS-operator s act on the fermionic part of the action one gets

$$\begin{aligned} s\left(\bar{\psi} \star \left(i\not{D} - m_f \right) \star \psi \right) &= ie\bar{\psi} \star c \star i\not{D} \star \psi - \bar{\psi} \star s\left(i\not{D} \star \psi \right) \\ &\quad - ie\bar{\psi} \star cm_f \star \psi + ie\bar{\psi} \star m_f c \star \psi \equiv 0. \end{aligned} \quad \text{(A.4)}$$

This equation is obviously fulfilled if $D_\mu \star \psi$ transforms the same way as ψ:

$$\begin{aligned} s\,(D_\mu \star \psi) &= -ie\,(sA_\mu) \star \psi + ie\,(\partial_\mu - ieA_\mu) \star c \star \psi \\ &= -ie\,(\partial_\mu c - igA_\mu \star c + igc \star A_\mu) \star \psi + ie\,(\partial_\mu - ieA_\mu) \star c \star \psi \\ &\equiv iec \star (\partial_\mu - ieA_\mu) \star \psi. \end{aligned} \tag{A.5}$$

This condition leads to the important constraint $g = e$ for the two coupling constants.

Next we calculate the transformation of the electromagnetic field tensor. As already mentioned, $F_{\mu\nu}$, having non-Abelian structure due to non-commutativity, will no longer be gauge invariant. It will instead transform the same way as λ (which is actually the reason why the BRS transformation $s\lambda$ was guessed in the first place).

$$\begin{aligned} sF_{\mu\nu} &= \partial_\mu\partial_\nu c - ig\partial_\mu[A_\mu, c]_\star - ig[\partial_\mu c, A_\nu]_\star + (ig)^2[[A_\mu, c]_\star, A_\nu]_\star - \mu \leftrightarrow \nu \\ &= -ig\,([\partial_\mu A_\nu - \partial_\nu A_\mu, c]_\star - ig[[A_\mu, c]_\star, A_\nu]_\star + ig[[A_\nu, c]_\star, A_\mu]_\star) \end{aligned} \tag{A.6}$$

Using the Jacobi identity

$$[[A_\mu, c]_\star, A_\nu]_\star + [[c, A_\nu]_\star, A_\mu]_\star = -[[A_\nu, A_\mu]_\star, c]_\star, \tag{A.7}$$

one finds

$$sF_{\mu\nu} = -ig[F_{\mu\nu}, c]_\star, \tag{A.8}$$

and therefore

$$\begin{aligned} \int d^3x s\,(F^{\mu\nu} \star F_{\mu\nu}) &= -ig \int d^3x\,([F^{\mu\nu}, c]_\star \star F_{\mu\nu} + F_{\mu\nu} \star [F^{\mu\nu}, c]_\star) \\ &= -ig \int d^3x\,(F^{\mu\nu} \star F_{\mu\nu} \star c - c \star F^{\mu\nu} \star F_{\mu\nu}) = 0. \end{aligned} \tag{A.9}$$

In the last step, property (1.12b) of the star product was used.

BRS transformation of the Chern-Simons term yields

$$\begin{aligned} \int d^3x s\,(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho) &= \int d^3x\,(\epsilon^{\mu\nu\rho}(sA_\mu)\partial_\nu A_\rho + \epsilon^{\mu\nu\rho} A_\mu \partial_\nu (sA_\rho)) \\ &= 2\int d^3x \epsilon^{\mu\nu\rho}(\partial_\mu c - igA_\mu \star c + igc \star A_\mu)\partial_\nu A_\rho = 0. \end{aligned} \tag{A.10}$$

which follows from partial integration, renaming of indices and equation (1.12b).

Next we prove BRS-invariance of the Slavnov term:

$$\int d^3x s\,(\lambda \star F_{\mu\nu}) = -ig[\lambda, c]_\star \star F_{\mu\nu} - ig\lambda \star [F_{\mu\nu}, c]_\star = 0, \tag{A.11}$$

which can be seen recalling (1.12b).
Transformation of the gauge fixing and ghost part of the action is given by

$$\int d^3x \left\{ B\partial^\mu sA_\mu - (s\bar{c}) \star \partial^\mu sA_\mu - \bar{c} \star \partial^\mu s^2 A_\mu \right\} = 0, \tag{A.12}$$

since the first and second term cancel each other, and the third term is zero because the BRS-operator s is nilpotent as we will now show:

$$\begin{aligned}
s^2 c &= ig(sc) \star c - igc \star sc = -g^2 c \star c \star c + g^2 c \star c \star c = 0, \\
s^2 \psi &= ie(sc) \star \psi - iec \star s\psi = -egc \star c \star \psi + e^2 c \star c \star \psi = 0, \\
s^2 \bar{\psi} &= ie(s\bar{\psi}) \star c - ie\bar{\psi} \star sc = -e^2 \bar{\psi} \star c \star c + eg\bar{\psi} \star c \star c = 0, \\
s^2 A_\mu &= s\left(\partial_\mu c - ig[A_\mu, c]_\star\right) \\
&= ig\partial_\mu (c \star c) + g^2 [A_\mu, c \star c]_\star - ig\left\{\partial_\mu c - ig[A_\mu, c]_\star, c\right\}_\star = 0, \\
s^2 \lambda &= -igs[\lambda, c]_\star = -g^2 \left\{[\lambda, c]_\star, c\right\}_\star + g^2 [\lambda, c \star c]_\star = 0,
\end{aligned} \tag{A.13}$$

and we see once more that $e = g$ is necessary.

Appendix B

Calculations for Chapter 2

B.1 The 3-photon vertex

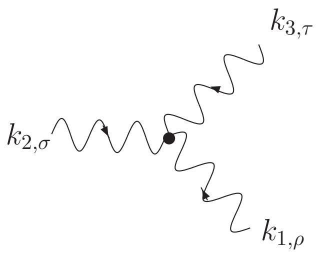


Figure B.1: 3-photon vertex

From equations (2.29) and (2.30) respecting the product rule for variations follows

$$
\begin{aligned}
\widetilde{V}^{\partial AAA}_{\rho\sigma\tau}(k_1,k_2,k_3) &= -2e(2\pi)^3\delta^3(k_1+k_2+k_3)\left[k_1^\mu\delta_\rho^\nu g_{\sigma\mu}g_{\tau\nu}\sin\left(\frac{k_1\times k_2}{2}\right)\right.\\
&\quad +k_1^\mu\delta_\rho^\nu g_{\sigma\nu}g_{\tau\mu}\sin\left(\frac{k_1\times k_3}{2}\right)+k_2^\mu\delta_\sigma^\nu g_{\rho\mu}g_{\tau\nu}\sin\left(\frac{k_2\times k_1}{2}\right)\\
&\quad +k_2^\mu\delta_\sigma^\nu g_{\tau\mu}g_{\rho\nu}\sin\left(\frac{k_2\times k_3}{2}\right)+k_3^\mu\delta_\tau^\nu g_{\rho\mu}g_{\sigma\nu}\sin\left(\frac{k_3\times k_1}{2}\right)\\
&\quad \left.+k_3^\mu\delta_\tau^\nu g_{\sigma\mu}g_{\rho\nu}\sin\left(\frac{k_3\times k_2}{2}\right)\right]\\
&= -2e(2\pi)^3\delta^3(k_1+k_2+k_3)\left[k_{1\sigma}g_{\rho\tau}-k_{1\tau}g_{\rho\sigma}-k_{2\rho}g_{\tau\sigma}\right.\\
&\quad \left.+k_{2\tau}g_{\rho\sigma}+k_{3\rho}g_{\tau\sigma}-k_{3\sigma}g_{\tau\rho}\right]\sin\left(\frac{k_1\times k_2}{2}\right).
\end{aligned}
\tag{B.1}
$$

B.2 The 4-photon vertex

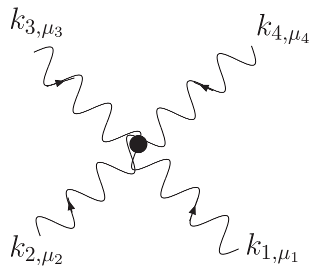


Figure B.2: 4-photon vertex

The relevant part of the action is

$$\begin{aligned}
S_{int}^{AAAA} =& \frac{e^2}{2} \int \frac{d^3 q_{1-4}}{(2\pi)^9} \delta^3 \left(\sum_{i=1}^{4} q_i \right) \widetilde{A}^\mu(q_1) \widetilde{A}_\nu(q_2) \left[\widetilde{A}_\mu(q_3) \widetilde{A}_\nu(q_4) \right. \\
& \left. - \widetilde{A}_\mu(q_4) \widetilde{A}_\nu(q_3) \right] e^{-\frac{i}{2}(q_1 \times q_2 + q_1 \times q_3 + q_1 \times q_4 + q_2 \times q_3 + q_2 \times q_4 + q_3 \times q_4)} \\
=& \frac{e^2}{2} \int \frac{d^3 q_{1-4}}{(2\pi)^9} \delta^3 \left(\sum_{i=1}^{4} q_i \right) \widetilde{A}^\mu(q_1) \widetilde{A}_\nu(q_2) \widetilde{A}_\mu(q_3) \widetilde{A}_\nu(q_4) e^{-\frac{i}{2}(q_1 \times q_2)} \\
& \times \left[e^{-\frac{i}{2}((q_1+q_2)\times q_3)} - e^{\frac{i}{2}((q_1+q_2)\times q_3)} \right] \\
=& - ie^2 \int \frac{d^3 q_{1-4}}{(2\pi)^9} \delta^3 (q_1 + q_2 + q_3 + q_4) \widetilde{A}^\mu(q_1) \widetilde{A}_\nu(q_2) \widetilde{A}_\mu(q_3) \widetilde{A}_\nu(q_4) \\
& \times e^{-\frac{i}{2}(q_1 \times q_2)} \sin \left(\frac{q_3 \times q_4}{2} \right) .
\end{aligned} \tag{B.2}$$

Inserting this expression into equation (2.32) leads to

$$\begin{aligned}
\widetilde{V}^{AAAA}_{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = e^2 (2\pi)^3 & \left[g_{\mu_1\mu_3} g_{\mu_2\mu_4} e^{-\frac{i}{2}(k_1 \times k_2)} \sin \left(\frac{k_3 \times k_4}{2} \right) \right. \\
& \left. + 23 \text{ index-permutations} \right] \delta^3 \left(\sum_{i=1}^{4} k_i \right) .
\end{aligned} \tag{B.3}$$

Since there are four photon-fields ($A_\mu(k_{1-4})$) in (B.2) there are 4! possible ways to do the variations according to (2.32). Because of the symmetry of the Minkowski metric one finds the relations $g_{\mu_1\mu_2} = g_{\mu_2\mu_1}$ and $g_{\mu_1\mu_3} g_{\mu_2\mu_4} = g_{\mu_2\mu_4} g_{\mu_1\mu_3}$. Therefore there are always 8 permutations leading to the same $g_{\mu\nu} g_{\rho\sigma}$-factor which one can group together. The following table shows the permutations for $g_{\mu_1\mu_3}g_{\mu_2\mu_4}$ as well as the resulting phase factors. Obviously, the sum of two successive lines can be written as a product of sine functions. In fact, all eight terms can be united into two such expressions.

$g_{\mu_1\mu_3}g_{\mu_2\mu_4}$:

$$\boxed{\begin{array}{ll}
\left.\begin{array}{l}
(1234) \to e^{-\frac{i}{2}(k_1\times k_2)}\sin\left(\frac{k_3\times k_4}{2}\right) \\
(2143) \to -e^{+\frac{i}{2}(k_1\times k_2)}\sin\left(\frac{k_3\times k_4}{2}\right)
\end{array}\right\} + & \longrightarrow \quad -2i\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right) \\
\left.\begin{array}{l}
(3412) \to e^{-\frac{i}{2}(k_3\times k_4)}\sin\left(\frac{k_1\times k_2}{2}\right) \\
(4321) \to -e^{+\frac{i}{2}(k_3\times k_4)}\sin\left(\frac{k_1\times k_2}{2}\right)
\end{array}\right\} + & \longrightarrow \quad -2i\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right) \\
\left.\begin{array}{l}
(3214) \to e^{-\frac{i}{2}(k_3\times k_2)}\sin\left(\frac{k_1\times k_4}{2}\right) \\
(2341) \to -e^{+\frac{i}{2}(k_3\times k_2)}\sin\left(\frac{k_1\times k_4}{2}\right)
\end{array}\right\} + & \longrightarrow \quad +2i\sin\left(\frac{k_2\times k_3}{2}\right)\sin\left(\frac{k_1\times k_4}{2}\right) \\
\left.\begin{array}{l}
(1432) \to -e^{-\frac{i}{2}(k_1\times k_4)}\sin\left(\frac{k_2\times k_3}{2}\right) \\
(4123) \to e^{+\frac{i}{2}(k_1\times k_4)}\sin\left(\frac{k_2\times k_3}{2}\right)
\end{array}\right\} + & \longrightarrow \quad +2i\sin\left(\frac{k_2\times k_3}{2}\right)\sin\left(\frac{k_1\times k_4}{2}\right)
\end{array}}$$

The remaining two groups of eight terms each can be calculated the same way and are given by

$g_{\mu_1\mu_2}g_{\mu_3\mu_4}$:

$$\boxed{\begin{array}{ll}
\left.\begin{array}{l}
(1324)+(2413)+ \\
+(3142)+(4231)
\end{array}\right\} & \longrightarrow \quad -4i\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right) \\
\left.\begin{array}{l}
(2314)+(1423)+ \\
+(3241)+(4132)
\end{array}\right\} & \longrightarrow \quad -4i\sin\left(\frac{k_2\times k_3}{2}\right)\sin\left(\frac{k_1\times k_4}{2}\right)
\end{array}}$$

and

$g_{\mu_1\mu_4}g_{\mu_2\mu_3}$:

$$\boxed{\begin{array}{ll}
\left.\begin{array}{l}
(1243)+(4312)+ \\
+(2134)+(3421)
\end{array}\right\} & \longrightarrow \quad +4i\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right) \\
\left.\begin{array}{l}
(4213)+(1342)+ \\
+(2431)+(3124)
\end{array}\right\} & \longrightarrow \quad +4i\sin\left(\frac{k_2\times k_4}{2}\right)\sin\left(\frac{k_1\times k_3}{2}\right)
\end{array}}$$

Gathering all 24 terms finally yields the result

$$\begin{aligned}
&\widetilde{V}^{AAAA}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4) = -4ie^2(2\pi)^3\delta^3(k_1+k_2+k_3+k_4) \\
&\quad\times\left[(g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3})\sin\left(\frac{k_1\times k_2}{2}\right)\sin\left(\frac{k_3\times k_4}{2}\right)\right. \\
&\quad+(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3})\sin\left(\frac{k_1\times k_3}{2}\right)\sin\left(\frac{k_2\times k_4}{2}\right) \\
&\quad\left.+(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_3}g_{\mu_2\mu_4})\sin\left(\frac{k_2\times k_3}{2}\right)\sin\left(\frac{k_1\times k_4}{2}\right)\right].
\end{aligned} \tag{B.4}$$

B.3 The one-loop fermion self-energy

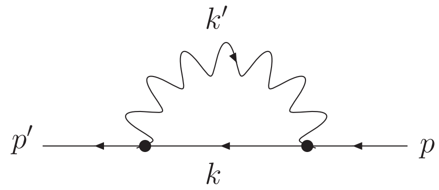


Figure B.3: fermion self-energy

With equations (2.21b),(2.21c) and (2.25), according to the Feynman rules, one arrives at

$$
\begin{aligned}
\Sigma(p) = & \int \frac{d^3k d^3k' d^3p'}{(2\pi)^9} \; ie(2\pi)^3\delta^3\left(-p' - k' + k\right)\gamma_\mu e^{\frac{i}{2}(k\times p')}\frac{-i\not{k}}{k^2} \\
& \times ie(2\pi)^3\delta^3\left(-k + k' + p\right)\gamma_\nu e^{\frac{i}{2}(p\times k)}\frac{-ig^{\mu\nu}}{k'^2} \\
= & \; e^2 \int \frac{d^3k d^3k'}{(2\pi)^3}\gamma_\mu e^{\frac{i}{2}(k\times(-k'))}\frac{\not{k}}{k^2}\delta^3\left(-k + k' + p\right)\gamma^\mu e^{\frac{i}{2}(p\times k)}\frac{1}{k'^2} \\
= & \; e^2 \int \frac{d^3k}{(2\pi)^3}\gamma_\mu e^{\frac{i}{2}(k\times p)}\frac{\not{k}}{k^2}\gamma^\mu e^{\frac{i}{2}(p\times k)}\frac{1}{(k-p)^2}. \qquad \text{(B.5)}
\end{aligned}
$$

Obviously, the phase factors cancel out, leading to the commutative result of regular 3-dimensional QED. Furthermore, one can shift $k \to k + p/2$ to make the integrand symmetric.

B.4 The photon line with fermion loop

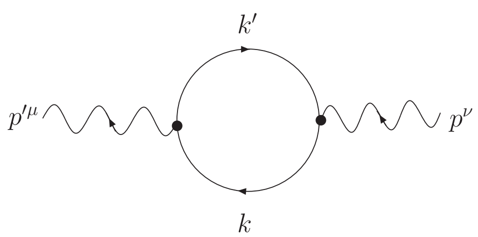


Figure B.4: fermion-loop graph

According to the Feynman rules, equations (2.21b) and (2.25) lead to

$$
\begin{aligned}
\Pi_a^{\mu\nu}(p) = & -\operatorname{tr}\int \frac{d^3k d^3k' d^3p'}{(2\pi)^9}\; ie(2\pi)^3\delta^3\left(k-p'-k'\right)\gamma^\mu e^{\frac{i}{2}(k'\times k)}\frac{-i\not{k}}{k^2} \\
& \times ie(2\pi)^3\delta^3\left(k'+p-k\right)\gamma^\nu e^{\frac{i}{2}(k\times k')}\frac{-i\not{k}'}{k'^2} \\
= & -\operatorname{tr}\int \frac{d^3k d^3k'}{(2\pi)^3} e^2\gamma^\mu \frac{\not{k}}{k^2}\delta^3\left(k'+p-k\right)\gamma^\nu\frac{\not{k}'}{k'^2} \\
= & -e^2\operatorname{tr}\int \frac{d^3k}{(2\pi)^3}\gamma^\mu\frac{\not{k}}{k^2}\gamma^\nu\frac{(\not{k}-\not{p})}{(k-p)^2}, \qquad \text{(B.6)}
\end{aligned}
$$

where again the phases cancelled. Shifting variables $k \to k + p/2$ finally yields the symmetric expression in (2.43a).

B.5 The photon line with ghost loop

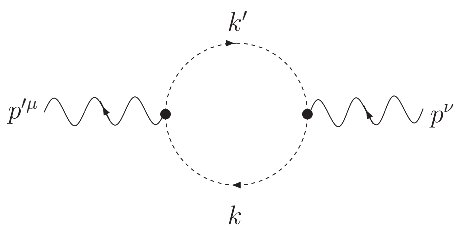


Figure B.5: ghost-loop graph

With equations (2.21a) and (2.28), according to the Feynman rules, one arrives at

$$
\begin{aligned}
\Pi_b^{\mu\nu}(p) = & -\int \frac{d^3k d^3k' d^3p'}{(2\pi)^9} 2e(2\pi)^3\delta^3\left(k-p'-k'\right)k^\mu \sin\left(\frac{k\times(-k')}{2}\right)\frac{i}{k'^2} \\
& \times 2e(2\pi)^3\delta^3\left(k'+p-k\right)k'^\nu \sin\left(\frac{k'\times(-k)}{2}\right)\frac{i}{k^2} \\
= & -4e^2\int \frac{d^3k d^3k'}{(2\pi)^3} k^\mu \frac{1}{k'^2}\delta^3\left(k'+p-k\right)k'^\nu \sin^2\left(\frac{k'\times k}{2}\right)\frac{1}{k^2} \\
= & -4e^2\int \frac{d^3k}{(2\pi)^3}\frac{k^\mu(k-p)^\nu}{k^2(k-p)^2}\sin^2\left(\frac{k\times p}{2}\right), \qquad \text{(B.7)}
\end{aligned}
$$

and a variable shift $k \to k+p/2$ leads to the symmetric version of this result. (Since $p \times p = 0$ the phase does not change.)

B.6 The photon line with photon loop

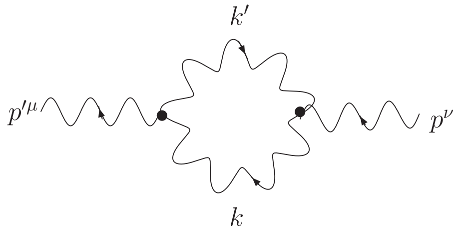


Figure B.6: photon-loop graph

According to the Feynman rules, equations (2.21c) and (2.31) lead to

$$\begin{aligned}
\Pi_c^{\mu\nu}(p) = & \frac{1}{2}\int \frac{d^3k d^3k' d^3p'}{(2\pi)^9}(-2e)(2\pi)^3\delta^3\left(k-p'-k'\right)\left[(-k'+p')^{\rho}g^{\mu\tau}\right.\\
& \left.+(k+k')^{\mu}g^{\rho\tau}+(-p'-k)^{\tau}g^{\rho\mu}\right]\sin\left(\frac{k\times(-p')}{2}\right)\frac{-ig_{\tau\sigma}}{k'^2}(-2e)(2\pi)^3\\
& \times\delta^3\left(k'+p-k\right)\left[(-k-p)^{\sigma}g^{\nu\epsilon}+(k'+k)^{\nu}g^{\sigma\epsilon}+(p-k')^{\epsilon}g^{\sigma\nu}\right]\\
& \times\sin\left(\frac{k'\times p}{2}\right)\frac{-ig_{\epsilon\rho}}{k^2}\\
= & -2e^2\int\frac{d^3kd^3k'}{(2\pi)^3}\left[(k-2k')^{\rho}\delta^{\mu}_{\sigma}+(k+k')^{\mu}\delta^{\rho}_{\sigma}+(k'-2k)_{\sigma}g^{\rho\mu}\right]\frac{1}{k'^2}\\
& \times\delta^3\left(k'+p-k\right)\left[-(k+p)^{\sigma}\delta^{\nu}_{\rho}+(k'+k)^{\nu}\delta^{\sigma}_{\rho}+(p-k')_{\rho}g^{\sigma\nu}\right]\frac{1}{k^2}\\
& \times\sin\left(\frac{k\times k'}{2}\right)\sin\left(\frac{k'\times p}{2}\right)\\
= & 2e^2\int\frac{d^3k}{(2\pi)^3}\left[(2p-k)^{\rho}\delta^{\mu}_{\sigma}+(2k-p)^{\mu}\delta^{\rho}_{\sigma}-(p+k)_{\sigma}g^{\rho\mu}\right]\frac{1}{(k-p)^2}\\
& \times\left[-(k+p)^{\sigma}\delta^{\nu}_{\rho}+(2k-p)^{\nu}\delta^{\sigma}_{\rho}+(2p-k)_{\rho}g^{\sigma\nu}\right]\frac{1}{k^2}\sin^2\left(\frac{k\times p}{2}\right).
\end{aligned} \tag{B.8}$$

After shifting variables $k \to k + p/2$, performing the multiplication of the two square brackets and considering $\delta^{\rho}_{\sigma}\delta^{\sigma}_{\rho} = 3$ when working in 3 dimensions, this expression becomes

$$\Pi_c^{\mu\nu}(p) = 4e^2\int\frac{d^3k}{(2\pi)^3}\frac{\left(k^2+\frac{9}{4}p^2\right)g^{\mu\nu}+3k^{\mu}k^{\nu}-\frac{9}{4}p^{\mu}p^{\nu}}{(k+p/2)^2(k-p/2)^2}\sin^2\left(\frac{k\times p}{2}\right). \tag{B.9}$$

B.7 The photon line with photon-tadpole loop

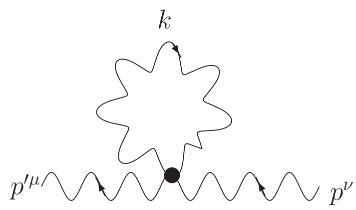


Figure B.7: tadpole graph

With equations (2.21c) and (2.33), according to the Feynman rules, one arrives at

$$\begin{aligned}\Pi_d^{\mu\nu}(p) = \frac{1}{2}\int \frac{d^3k d^3p'}{(2\pi)^6}(-4i)e^2(2\pi)^3\delta^3\left(-p'-k+k+p\right)\frac{-ig_{\sigma\tau}}{k^2} \\ \times\left[(g^{\mu\tau}g^{\sigma\nu}-g^{\mu\nu}g^{\sigma\tau})\sin\left(\frac{p'\times k}{2}\right)\sin\left(\frac{k\times p}{2}\right)\right. \\ \left.+(g^{\mu\sigma}g^{\tau\nu}-g^{\mu\nu}g^{\sigma\tau})\sin\left(\frac{(-p')\times k}{2}\right)\sin\left(\frac{(-k)\times p}{2}\right)\right].\end{aligned} \tag{B.10}$$

The square bracket contains only two terms, since the third term coming from the vertex is proportional to $\sin(k\times k/2)=0$. Using $g_{\sigma\tau}g^{\sigma\tau}=3$ (in 3 dimensions), and performing the integration over p' finally leads to

$$\Pi_d^{\mu\nu}(p) = -8e^2\int\frac{d^3k}{(2\pi)^3}\frac{g^{\mu\nu}}{k^2}\sin^2\left(\frac{k\times p}{2}\right). \tag{B.11}$$

B.8 The one-loop ghost self-energy

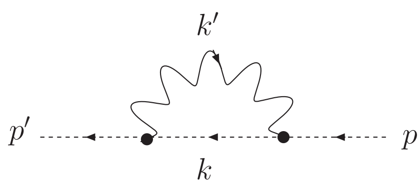


Figure B.8: ghost self-energy

With equations (2.21a), (2.21c) and (2.28), according to the Feynman rules, one arrives at

$$\begin{aligned}
\Xi(p) = & \int \frac{d^3k d^3k' d^3p'}{(2\pi)^9} 2e(2\pi)^3\delta^3\left(-p' - k' + k\right) k^\mu \sin\left(\frac{(-p')\times k}{2}\right) \\
& \times \frac{-ig_{\mu\nu}}{k'^2} 2e(2\pi)^3\delta^3\left(-k + k' + p\right) p^\nu \sin\left(\frac{(-k)\times p}{2}\right)\frac{i}{k^2} \\
= & \, 4e^2 \int \frac{d^3k d^3k'}{(2\pi)^3}\frac{p_\mu}{k'^2}\frac{k^\mu}{k^2}\delta^3\left(-k + k' + p\right)\sin\left(\frac{(k'\times k}{2}\right) \\
& \times \sin\left(\frac{(-k)\times p}{2}\right),
\end{aligned} \tag{B.12}$$

and finally

$$\Xi(p) = -4e^2 \int \frac{d^3k}{(2\pi)^3}\frac{p^\mu k_\mu}{k^2(k-p)^2}\sin^2\left(\frac{k\times p}{2}\right), \tag{B.13}$$

where again the shift $k \to k + p/2$ leads to the symmetric version given in (2.58).

Appendix C

Calculations for Chapter 4

C.1 The photon line with lambda propagator

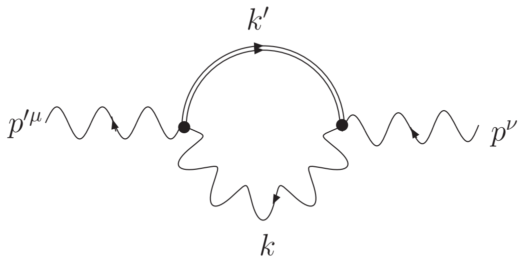


Figure C.1: photon line with lambda propagator

With equations (4.17a), (4.17d) and (4.21) one gets according to the Feynman rules

$$\begin{aligned}
\Pi_e^{\mu\nu}(p) = & \int \frac{d^3k d^3k' d^3p'}{(2\pi)^9} (-2ie)(2\pi)^3 \delta^3 \left(k - p' - k'\right) \theta^{\mu\rho} \sin\left(\frac{k \times (-p')}{2}\right) \\
& \times \frac{ik'^2}{\widetilde{k}'^2} (-2ie)(2\pi)^3 \delta^3 \left(-k + k' + p\right) \theta^{\sigma\nu} \sin\left(\frac{p \times (-k)}{2}\right) \\
& \times \left(\frac{-ig_{\rho\sigma}}{k^2} - (\alpha - 1)\frac{ik_\rho k_\sigma}{k^4} + \frac{i\widetilde{k}_\rho \widetilde{k}_\sigma}{k^2 \widetilde{k}^2}\right) \\
= & \, 4e^2 \int \frac{d^3k}{(2\pi)^3} \sin^2\left(\frac{k \times p}{2}\right) \frac{(k-p)^2}{k^2 \left(\widetilde{k} - \widetilde{p}\right)^2} \\
& \times \left(\theta^\mu{}_\sigma \theta^{\sigma\nu} - (\alpha - 1)\frac{\widetilde{k}^\mu \widetilde{k}^\nu}{k^2} + \frac{\theta^{\mu\rho}\widetilde{k}_\rho \theta^{\nu\sigma}\widetilde{k}_\sigma}{\widetilde{k}^2}\right). \qquad \text{(C.1)}
\end{aligned}$$

Shifting variables $k \to k + p/2$ finally yields the symmetric version (4.28).

C.2 The photon line with lambda-photon propagator

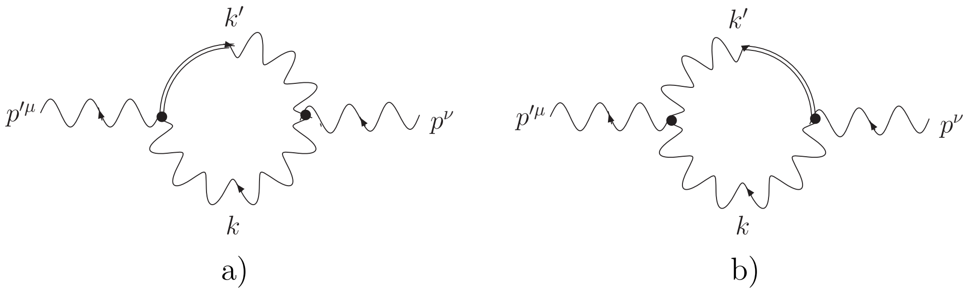


Figure C.2: photon line with lambda-photon propagator

With equations (4.17b) and (4.21) one gets according to the Feynman rules

$$
\begin{aligned}
\Pi^{\mu\nu}_{f1}(p) = &\int \frac{d^3k d^3k' d^3p'}{(2\pi)^9} (-2ie)(2\pi)^3 \delta^3\left(k - p' - k'\right) \theta^{\epsilon\mu} \sin\left(\frac{k \times (-p')}{2}\right) \\
&\times \frac{-\widetilde{k}'_\rho}{\widetilde{k}'^2} (-2e)(2\pi)^3 \left[(-k-p)^\rho g^{\nu\tau} + (k'+k)^\nu g^{\rho\tau} + (p-k')^\tau g^{\rho\nu}\right] \\
&\times \delta^3\left(-k + k' + p\right) \sin\left(\frac{k' \times p}{2}\right) \left(\frac{-ig_{\epsilon\tau}}{k^2} - (\alpha - 1)\frac{ik_\epsilon k_\tau}{k^4} + \frac{i\widetilde{k}_\epsilon \widetilde{k}_\tau}{k^2 \widetilde{k}^2}\right) \\
= 4e^2 &\int \frac{d^3k}{(2\pi)^3} \left[(k+p)^\rho g^{\nu\tau} - (2k-p)^\nu g^{\rho\tau} + (k-2p)^\tau g^{\rho\nu}\right] \\
&\times \frac{\theta^{\mu\epsilon}\left(\widetilde{k} - \widetilde{p}\right)_\rho}{k^2 \left(\widetilde{k} - \widetilde{p}\right)^2} \left(g_{\epsilon\tau} + (\alpha - 1)\frac{k_\epsilon k_\tau}{k^2} - \frac{\widetilde{k}_\epsilon \widetilde{k}_\tau}{\widetilde{k}^2}\right) \sin^2\left(\frac{k \times p}{2}\right), \quad \text{(C.2)}
\end{aligned}
$$

which is the expression for the graph in Figure C.2a. A similar calculation leads to the expression for the graph including a photon-lambda propagator (Figure C.2b). One finds almost the same result but with the indices μ, ν exchanged. Therefore the sum of both graphs, after shifting variables $k \to k + p/2$ and performing the multiplication of the two brackets in each expression, becomes

$$
\begin{aligned}
\Pi^{\mu\nu}_{f}(p) = -4e^2 &\int \frac{d^3k}{(2\pi)^3} \frac{\sin^2\left(\frac{k \times p}{2}\right)}{\left(k + p/2\right)^2 \left(\widetilde{k} - \widetilde{p}/2\right)^2} \\
&\times \Bigg\{ 2k\widetilde{p}\theta^{\mu\nu} + 2\theta^{\mu\epsilon}\left(\widetilde{k} - \widetilde{p}/2\right)_\epsilon k^\nu - \left(\widetilde{k} - 3\widetilde{p}/2\right)^\mu \left(\widetilde{k} - \widetilde{p}/2\right)^\nu \\
&+ (\alpha - 1)\frac{\left(\widetilde{k} + \widetilde{p}/2\right)^\mu}{(k + p/2)^2} \left(k\widetilde{p}p^\nu - (k + p/2)(k - 3p/2)\left(\widetilde{k} - \widetilde{p}/2\right)^\nu\right) \\
&- \frac{\theta^{\mu\epsilon}\left(\widetilde{k} + \widetilde{p}/2\right)_\epsilon}{\left(\widetilde{k} + \widetilde{p}/2\right)^2} \left(2k\widetilde{p}\widetilde{p}^\nu + 2\left(\widetilde{k}^2 - \widetilde{p}^2/4\right) k^\nu\right) \Bigg\} + \mu \leftrightarrow \nu \quad .
\end{aligned}
$$

(C.3)

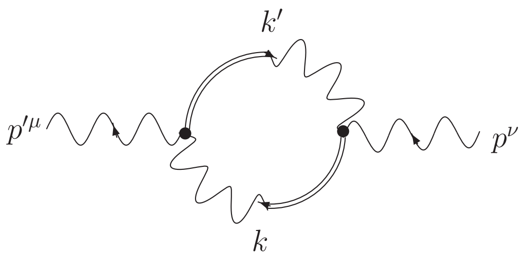


Figure C.3: photon line with two lambda-photon propagators

C.3 The photon line with two lambda-photon propagators

With equations (4.17b) and (4.21), according to the Feynman rules, one arrives at

$$\begin{aligned}
\Pi_g^{\mu\nu}(p) &= \int \frac{d^3k d^3k' d^3p'}{(2\pi)^9} (-2ie)(2\pi)^3 \delta^3\left(k - p' - k'\right) \theta^{\rho\mu} \sin\left(\frac{k \times (-p')}{2}\right) \\
&\quad \times \frac{-\widetilde{k}'_\rho}{\widetilde{k}'^2} (-2ie)(2\pi)^3 \delta^3\left(-k + k' + p\right) \theta^{\sigma\nu} \sin\left(\frac{k' \times p}{2}\right) \frac{-\widetilde{k}_\sigma}{\widetilde{k}^2} \\
&= 4e^2 \int \frac{d^3k}{(2\pi)^3} \frac{\theta^{\mu\rho}\left(\widetilde{k} - \widetilde{p}\right)_\rho \theta^{\nu\sigma}\widetilde{k}_\sigma}{\widetilde{k}^2\left(\widetilde{k} - \widetilde{p}\right)^2} \sin^2\left(\frac{k \times p}{2}\right).
\end{aligned} \tag{C.4}$$

Shifting variables $k \to k + p/2$ finally leads to the symmetric version (4.31).

Appendix D

Integrals

D.1 Dimensional regularization

We follow the derivation given in reference [32] on pages 382-385: Working in n-dimensional Minkowski space we start with integrals of the type

$$I_n(q) = \int \frac{d^n k}{(k^2 + 2kq - L^2)^\alpha}, \tag{D.1}$$

where $k = (k_0, r, \phi, \theta_1, \theta_2, \dots, \theta_{n-3})$ in polar coordinates and the volume element is therefore given by

$$d^n k = dk_0 r^{n-2} dr d\phi \prod_{i=1}^{n-3} \sin^i \theta_i d\theta_i. \tag{D.2}$$

Shifting variables $k'_\mu = k_\mu + q_\mu$ and using

$$\int_0^{2\pi} d\phi \prod_{i=1}^{n-3} \int_0^{\pi} \sin^i \theta_i d\theta_i = \frac{2\pi^{\frac{(n-1)}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}, \tag{D.3}$$

we arrive at

$$I_n(q) = \frac{2\pi^{\frac{(n-1)}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} \frac{r^{n-2} dr}{(k_0^2 - r^2 - (q^2 + L^2)^\alpha}, \tag{D.4}$$

where the primes have been dropped. Since the integrand depends only quadratically on k_0 we can replace $\int_{-\infty}^{\infty} dk_0 \to 2 \int_0^{\infty} dk_0$.

To evaluate the remaining integrals we use the Euler beta function [35]

$$B(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^\infty dt t^{2x-1}(1+t^2)^{-x-y}, \qquad \begin{matrix} \Re(x) > 0 \\ \Re(y) > 0 \end{matrix} \tag{D.5}$$

Putting

$$x = \tfrac{1+\beta}{2}, \qquad y = \alpha - \tfrac{1+\beta}{2}, \qquad t = \tfrac{s}{M}, \tag{D.6}$$

leads to

$$\int_0^\infty ds \frac{s^\beta}{(s^2+M^2)^\alpha} = \frac{\Gamma\left(\frac{1+\beta}{2}\right)\Gamma\left(\alpha - \frac{1+\beta}{2}\right)}{2\left(M^2\right)^{\alpha-(1+\beta)/2}\Gamma(\alpha)}, \tag{D.7}$$

and if we identify $\beta = 0$ and $M^2 = -r^2 - (q^2 + L^2)$ we can use this formula to perform the integration over k_0 in (D.4):

$$I_n(q) = \frac{2\pi^{\frac{(n-1)}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty dr \frac{r^{n-2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha - \frac{1}{2}\right)}{\left(-r^2 - (q^2+L^2)\right)^{\alpha-1/2}\Gamma(\alpha)}. \tag{D.8}$$

By identifying $\alpha' = \alpha - \frac{1}{2}$, $\beta = n-2$ and $M^2 = q^2 + L^2$ we can once again use formula (D.7) to perform the remaining integral:

$$I_n(q) = (-1)^{\frac{1}{2}-\alpha} \frac{\pi^{\frac{n}{2}}}{\Gamma(\alpha)} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\left(q^2+L^2\right)^{\alpha - \frac{n}{2}}} = (-1)^{\frac{n}{2}} i \pi^{\frac{n}{2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} (-q^2 - L^2)^{\frac{n}{2}-\alpha}. \tag{D.9}$$

Therefore the result is

$$I_n(q) = \int \frac{d^n k}{\left(k^2 + 2kq - L^2\right)^\alpha} = (-1)^{\frac{n}{2}} i \pi^{\frac{n}{2}} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} (-q^2 - L^2)^{\frac{n}{2}-\alpha}. \tag{D.10a}$$

Differentiating both sides with respect to q_μ and redefining α as well as using the property of the Gamma-function $x\Gamma(x) = \Gamma(1+x)$ leads to

$$I_n^\mu(q) = \int d^n k \frac{k^\mu}{\left(k^2+2kq-L^2\right)^\alpha} = (-q^\mu) I_n(q), \tag{D.10b}$$

and further differentiation with respect to q_ν yields

$$I_n^{\mu\nu}(q) = \int d^n k \frac{k^\mu k^\nu}{\left(k^2+2kq-L^2\right)^\alpha} = \left(q^\mu q^\nu + \frac{g^{\mu\nu}\left(-q^2-L^2\right)}{2\alpha - n - 2}\right) I_n(q). \tag{D.10c}$$

D.2 Integrals and Feynman trick

Feynman's formula[1] is given by

$$\frac{1}{ab} = \int_0^1 \frac{dz}{\left(az + b(1-z)\right)^2}, \tag{D.11}$$

and is a useful trick in order to bring the following integrals into the form $I_n(0)$ or $I_n^{\mu\nu}(0)$ as defined in Appendix D.1.

The first integral we want to calculate is

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(k+\frac{p}{2}\right)^2 \left(k-\frac{p}{2}\right)^2} = \frac{1}{8\sqrt{p^2}}. \tag{D.12}$$

Applying formula (D.11) leads to

$$\begin{aligned} &\int_0^1 dz \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left\{\left(k+\frac{p}{2}\right)^2 z + \left(k-\frac{p}{2}\right)^2 (1-z)\right\}^2} \\ &= \int_0^1 dz \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left\{\left(k - p\left(\frac{1}{2}-z\right)\right)^2 + p^2 z(1-z)\right\}^2}. \end{aligned} \tag{D.13}$$

Now we can shift variables to $k' = k - p\left(\frac{1}{2}-z\right)$ and use formula (D.10a) in order to perform the integration over k'. Identifying $(-L^2) = p^2 z(1-z)$, $q = 0$ and $n = 3$ leads to

$$\begin{aligned} \int_0^1 dz \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\left\{k'^2 + p^2 z(1-z)\right\}^2} &= \int_0^1 dz \frac{(-1)^{\frac{3}{2}} i\pi^{\frac{3}{2}}}{(2\pi)^3} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \left(p^2 z(1-z)\right)^{-\frac{1}{2}} \\ &= \frac{1}{8\sqrt{p^2}}, \end{aligned} \tag{D.14}$$

where formula (D.78a) was used for the remaining z-integral.

In a similar way we can derive

$$\int \frac{d^3k}{(2\pi)^3} \frac{\left(k+\frac{p}{2}\right)^\mu}{\left(k+\frac{p}{2}\right)^2 \left(k-\frac{p}{2}\right)^2} = \frac{p^\mu}{16\sqrt{p^2}}. \tag{D.15}$$

[1] The proof of this integral formula is straightforward: simply substitute $z' = (a-b)z-b$.

Applying formula (D.11) and shifting variables $k' = k - p\left(\frac{1}{2} - z\right)$ leads to

$$\int_0^1 dz \int \frac{d^3k'}{(2\pi)^3} \frac{k'^\mu + p^\mu(1-z)}{\left\{k'^2 + p^2 z(1-z)\right\}^2}, \tag{D.16}$$

which can be solved with equations (D.10a) and (D.10b). (Note that the first term including k'^μ does not contribute since $I_n^\mu(0) = 0$ in (D.10b).) With $(-L^2) = p^2 z(1-z)$, $q = 0$ and $n = 3$ one gets

$$\begin{aligned} \int_0^1 dz \int \frac{d^3k'}{(2\pi)^3} \frac{p^\mu(1-z)}{\left\{k'^2 + p^2 z(1-z)\right\}^2} &= \int_0^1 dz \frac{(-1)^{\frac{3}{2}} i\pi^{\frac{3}{2}}}{(2\pi)^3} \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} p^\mu \sqrt{\frac{1-z}{zp^2}} \\ &= \frac{p^\mu}{16\sqrt{p^2}}. \end{aligned} \tag{D.17}$$

Our next integral is

$$\int \frac{d^3k}{(2\pi)^3} \frac{\left(k + \frac{p}{2}\right)^\mu \left(k - \frac{p}{2}\right)^\nu}{\left(k + \frac{p}{2}\right)^2 \left(k - \frac{p}{2}\right)^2} = -\frac{p^\mu p^\nu + p^2 g^{\mu\nu}}{64\sqrt{p^2}}. \tag{D.18}$$

Following the same steps as before one gets

$$\begin{aligned} &\int_0^1 dz \int \frac{d^3k'}{(2\pi)^3} \frac{\left(k'^\mu + p^\mu(1-z)\right)\left(k'^\nu - p^\nu z\right)}{\left\{k'^2 + p^2 z(1-z)\right\}^2} \\ &= \int_0^1 \frac{dz}{(2\pi)^3} \frac{\pi^2\sqrt{z(1-z)}}{\sqrt{p^2}} \left(-g^{\mu\nu}p^2 - p^\mu p^\nu\right) = -\frac{p^\mu p^\nu + p^2 g^{\mu\nu}}{64\sqrt{p^2}}, \end{aligned} \tag{D.19}$$

where additionally formulas (D.10c) and (D.78b) were used.
In the same way we find

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu k^\nu}{\left(k + \frac{p}{2}\right)^2 \left(k - \frac{p}{2}\right)^2} = \frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{64\sqrt{p^2}}. \tag{D.20}$$

The last of these integrals is

$$\int \frac{d^nk}{(2\pi)^n} \frac{1}{k^2} = 0. \tag{D.21}$$

To prove this formula we use the following trick:

$$\begin{aligned} \int \frac{d^nk}{(2\pi)^n} \frac{1}{k^2} &= \int \frac{d^nk}{(2\pi)^n} \frac{1}{k^2} \frac{k^2 - M^2}{k^2 - M^2} \\ &= \int \frac{d^nk}{(2\pi)^n} \left[\frac{1}{k^2 - M^2} - \int_0^1 dz \frac{M^2}{\left(k^2 - M^2(1-z)\right)^2} \right], \end{aligned} \tag{D.22}$$

where M is an arbitrary parameter and formula (D.11) has been used for the second term. The integral over k can be solved with equation (D.10a) leading to

$$\frac{(-1)^{\frac{n}{2}} i\pi^{\frac{n}{2}}}{(2\pi)^n}\left[\frac{\Gamma\left(1-\frac{n}{2}\right)}{\Gamma(1)}(-M^2)^{\frac{n}{2}-1} - M^2\int_0^1 dz \frac{\Gamma\left(2-\frac{n}{2}\right)}{\Gamma(2)}(-M^2(1-z))^{\frac{n}{2}-2}\right]$$
$$= \frac{(-1)^{\frac{n}{2}} i(-M^2)^{\frac{n}{2}-1}}{(2\pi^2)^{\frac{n}{2}}}\Gamma\left(\frac{2-n}{2}\right)\left[1+\frac{2-n}{2}\frac{2}{n-2}\right] = 0. \tag{D.23}$$

The next set of integrals will contain masses as appear in Chapter 3.2. The first one[2] is

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]} = \frac{\arcsin a_+ - \arcsin a_-}{8\pi\sqrt{p^2}}, \tag{D.24}$$

with

$$a_\pm = \frac{\pm 1 + \widetilde{m}_1^2 - \widetilde{m}_2^2}{\sqrt{\left(1-\widetilde{m}_1^2+\widetilde{m}_2^2\right)^2 - 4\widetilde{m}_2^2}}. \tag{D.25}$$

To prove this formula we use (D.11) and (D.10a) and get

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]}$$
$$= \int_0^1 dz \int \frac{d^3k'}{(2\pi)^3}\frac{1}{\left\{k'^2 + p^2 z(1-z) - m_1^2 z - m_2^2(1-z)\right\}^2}$$
$$= \frac{1}{8\pi\sqrt{p^2}}\int_0^1 \frac{dz}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)}}, \tag{D.26}$$

where $k' = k - p\left(\frac{1}{2} - z\right)$ and $\widetilde{m}_{1,2}^2 = m_{1,2}^2/p^2$. Applying formula (D.84) finally yields the result.

[2]Note that if one takes the limit $m_{1,2} \to 0$ the parameters $a_\pm$ become ± 1 and the numerator reduces to π leading to the same expression as in formula (D.12).

Next we show

$$
\int \frac{d^3k}{(2\pi)^3} \frac{\left(k+\frac{p}{2}\right)^\mu}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]}
$$
$$
= \frac{p^\mu}{8\pi\sqrt{p^2}} \left[\sqrt{2A(\widetilde{m}_1, \widetilde{m}_2)}\sqrt{1-z^2}\Big|_{a_-}^{a_+} + \frac{1}{2}(1+\widetilde{m}_1^2 - \widetilde{m}_2^2)\arcsin z\Big|_{a_-}^{a_+}\right], \tag{D.27}
$$

with

$$
A(\widetilde{m}_1, \widetilde{m}_2) = \frac{1}{8}\left[\left(1-\widetilde{m}_1^2+\widetilde{m}_2^2\right)^2 - 4\widetilde{m}_2^2\right]. \tag{D.28}
$$

Applying formulas (D.11),(D.10a) and (D.10b) yields

$$
\int \frac{d^3k}{(2\pi)^3} \frac{\left(k+\frac{p}{2}\right)^\mu}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]}
$$
$$
= \int_0^1 dz \int \frac{d^3k'}{(2\pi)^3} \frac{k'^\mu + p^\mu(1-z)}{\left\{k'^2 + p^2 z(1-z) - m_1^2 z - m_2^2(1-z)\right\}^2}
$$
$$
= \frac{p^\mu}{8\pi\sqrt{p^2}} \int_0^1 dz \frac{(1-z)}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)}}, \tag{D.29}
$$

which can be solved with (D.88).

Our next integral is

$$
\int \frac{d^3k}{(2\pi)^3} \frac{\left(k+\frac{p}{2}\right)^\mu\left(k-\frac{p}{2}\right)^\nu}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]}
$$
$$
= \frac{-1}{8\pi\sqrt{p^2}} \Bigg\{ p^\mu p^\nu (\widetilde{m}_2^2 - \widetilde{m}_1^2)\sqrt{2A(\widetilde{m}_1, \widetilde{m}_2)}\sqrt{1-z^2}\Big|_{a_-}^{a_+}
$$
$$
+ p^\mu p^\nu \left(\frac{\widetilde{m}_1^2 + \widetilde{m}_2^2}{2} + \widetilde{m}_1^2\widetilde{m}_2^2 - \frac{\widetilde{m}_1^4 + \widetilde{m}_2^4}{2}\right)\arcsin z\Big|_{a_-}^{a_+}
$$
$$
+ A(\widetilde{m}_1, \widetilde{m}_2)\left(p^\mu p^\nu + p^2 g^{\mu\nu}\right)\left[z\sqrt{1-z^2}\Big|_{a_-}^{a_+} + \arcsin z\Big|_{a_-}^{a_+}\right]\Bigg\}. \tag{D.30}
$$

From formulas (D.11) and (D.10a) – (D.10c) follows

$$\int \frac{d^3k}{(2\pi)^3} \frac{\left(k+\frac{p}{2}\right)^\mu \left(k-\frac{p}{2}\right)^\nu}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]}$$

$$= \int_0^1 dz \int \frac{d^3k'}{(2\pi)^3} \frac{k'^\mu k'^\nu - p^\mu p^\nu z(1-z)}{\{k'^2 + p^2 z(1-z) - m_1^2 z - m_2^2(1-z)\}^2}$$

$$= \frac{-1}{8\pi\sqrt{p^2}} \int_0^1 dz \Bigg\{ \frac{p^\mu p^\nu z(1-z)}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)}}$$

$$+ p^2 g^{\mu\nu} \sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)} \Bigg\}. \tag{D.31}$$

This integral can be rewritten to

$$\frac{-1}{8\pi\sqrt{p^2}} \int_0^1 dz \Bigg\{ \frac{p^\mu p^\nu \left(\widetilde{m}_1^2 z + \widetilde{m}_2^2(1-z)\right)}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)}}$$

$$+ \left(p^\mu p^\nu + p^2 g^{\mu\nu}\right) \sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)} \Bigg\}, \tag{D.32}$$

which can be solved with equations (D.88), (D.89) and (D.91).
Similarly we evaluate

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu k^\nu}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]}$$

$$= \frac{-1}{8\pi\sqrt{p^2}} \Bigg\{ p^\mu p^\nu (\widetilde{m}_2^2 - \widetilde{m}_1^2) \sqrt{2A(\widetilde{m}_1, \widetilde{m}_2)} \sqrt{1-z^2}\Big|_{a_-}^{a_+}$$

$$+ p^\mu p^\nu \left(\frac{\widetilde{m}_1^2 + \widetilde{m}_2^2}{2} + \widetilde{m}_1^2 \widetilde{m}_2^2 - \frac{\widetilde{m}_1^4 + \widetilde{m}_2^4}{2} - \frac{1}{4} \right) \arcsin z'\Big|_{a_-}^{a_+}$$

$$+ A(\widetilde{m}_1, \widetilde{m}_2)\left(p^\mu p^\nu + p^2 g^{\mu\nu}\right) \left[z\sqrt{1-z^2}\Big|_{a_-}^{a_+} + \arcsin z'\Big|_{a_-}^{a_+} \right] \Bigg\}. \tag{D.33}$$

Finally we need

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu k^\nu}{k^2\left(k^2-M^2\right)} = \int_0^1 dz \int \frac{d^3k}{(2\pi)^3} \frac{k^\mu k^\nu}{\left(k^2-M^2(1-z)\right)^2}$$
$$= \frac{-g^{\mu\nu}}{8\pi} \int_0^1 dz \sqrt{-M^2(1-z)}$$
$$= \frac{-g^{\mu\nu}\sqrt{-M^2}}{12\pi}, \tag{D.34}$$

where formulas (D.11) and (D.10a) have been used.

D.3 Integrals and Schwinger parameterization

In order to solve the following integrals we use Schwinger parameterization[3]:

$$\frac{1}{\left(k \pm \frac{p}{2}\right)^2 + i\epsilon} = -i \int_0^\infty d\alpha e^{i\alpha\left(k\pm\frac{p}{2}\right)^2 - \alpha\epsilon}, \tag{D.35}$$

where $\epsilon > 0$ is an arbitrary small parameter needed to regularize the integral on the right hand side.
Furthermore, we have to solve some complex Gauss-integrals in n-dimensional Minkowski space:

$$\int d^nk e^{i\alpha k^2} = \lim_{\epsilon\to 0} \int d^nk e^{-(\epsilon - i\alpha)k_0^2 - (\epsilon + i\alpha)\vec{k}^2} = \sqrt{\frac{\pi}{-i\alpha}} \left(\frac{\pi}{i\alpha}\right)^{\frac{n-1}{2}}, \tag{D.36}$$

and therefore

$$\int \frac{d^nk}{(2\pi)^n} e^{i\alpha k^2} = \frac{-i}{(4i\pi\alpha)^{\frac{n}{2}}}. \tag{D.37}$$

Finally we need the following integral formula [35]:

$$\int_0^\infty d\rho \rho^{-(1-\nu)} e^{-\tau\rho - \frac{\sigma}{\rho}} = 2\left(\frac{\sigma}{\tau}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\sigma\tau}\right), \qquad \begin{matrix} \Re(\tau) > 0 \\ \Re(\sigma) > 0 \end{matrix} \tag{D.38}$$

[3] The same method for solving non-planar graphs was used in e.g. [2].

where the $K_\nu(z)$ are the modified Bessel functions. The only ones we need are [35]

$$\begin{aligned} K_{\pm 1/2}(z) &= \sqrt{\frac{\pi}{2z}}e^{-z} \qquad \text{and} \\ K_{-3/2}(z) &= \sqrt{\frac{\pi}{2}}\left(z^{-3/2}+z^{-1/2}\right)e^{-z}. \end{aligned} \tag{D.39}$$

The first integral we want to calculate is

$$\begin{aligned} \int \frac{d^3k}{(2\pi)^3} \frac{\sum\limits_{\eta=\pm 1} e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2} &= -\lim_{\epsilon\to 0}\sum_{\eta=\pm 1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\int\frac{d^3k}{(2\pi)^3}\exp\left[i\alpha\left(k+\frac{p}{2}\right)^2\right. \\ &\qquad \left. +i\beta\left(k-\frac{p}{2}\right)^2+i\eta k\widetilde{p}-(\alpha+\beta)\epsilon\right] \\ &= -\lim_{\epsilon\to 0}\sum_{\eta=\pm 1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\int\frac{d^3k'}{(2\pi)^3}\exp\left[i(\alpha+\beta)k'^2\right. \\ &\qquad \left. +i\frac{\alpha\beta p^2}{\alpha+\beta}-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}-(\alpha+\beta)\epsilon\right], \end{aligned} \tag{D.40}$$

where the integral over k has been shifted to $k' = k + \frac{(\alpha-\beta)p+\eta\widetilde{p}}{2(\alpha+\beta)}$. (Remember $\widetilde{p}p = \theta_{\mu\nu}p^\mu p^\nu = 0$ and $\eta^2 = 1$.) Making use of formula (D.37) and taking the limit $\epsilon \longrightarrow 0$ afterwards leads to

$$\frac{i}{4\,(i\pi)^{\frac{3}{2}}}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta(\alpha+\beta)^{-\frac{3}{2}}\exp\left[i\frac{\alpha\beta p^2}{\alpha+\beta}-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}\right]. \tag{D.41}$$

A factor 2 comes from the sum over η (since the integrand does not depend on η any longer). Next, we make the following substitutions:

$$\left.\begin{aligned} \alpha &= \tfrac{\rho}{(1-\xi)p^2} \\ \beta &= \tfrac{\rho}{\xi p^2} \end{aligned}\right\} \quad \longrightarrow \quad d\alpha d\beta = \frac{\rho}{\left(\xi(1-\xi)p^2\right)^2}d\xi d\rho \tag{D.42}$$

Therefore we have

$$\begin{aligned} \alpha\beta &= \frac{\rho^2}{\xi(1-\xi)p^4}, \\ \alpha+\beta &= \frac{\rho}{\xi(1-\xi)p^2}, \end{aligned} \tag{D.43}$$

and the integral becomes

$$\frac{i}{4\left(i\pi\right)^{\frac{3}{2}}}\int\limits_0^1 d\xi\int\limits_0^\infty d\rho\frac{\rho^{-1/2}}{\sqrt{\xi(1-\xi)p^2}}\exp\left[i\rho-\frac{i\xi(1-\xi)p^2\widetilde{p}^2}{4\rho}\right]. \qquad \text{(D.44)}$$

Now we can use formula (D.38) if we identify $\nu = 1/2$, $\tau = -i+\epsilon$, $\sigma = \frac{i}{4}\xi(1-\xi)p^2\widetilde{p}^2+\epsilon$ and take the limit $\epsilon\to 0$ after performing the integration over ρ. Setting $z(\xi)\equiv\sqrt{\xi(1-\xi)p^2\widetilde{p}^2}$ yields

$$\frac{i}{4\left(i\pi\right)^{\frac{3}{2}}}\int\limits_0^1 d\xi(\xi(1-\xi)p^2)^{-\frac{1}{2}}2\left(\frac{iz}{2}\right)^{1/2}K_{1/2}(z), \qquad \text{(D.45)}$$

and considering (D.39) the result is

$$\int\frac{d^3k}{\left(2\pi\right)^3}\frac{\sum\limits_{\eta=\pm1}e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2}=\frac{1}{4\pi\sqrt{p^2}}\int\limits_0^1 d\xi\frac{e^{-z(\xi)}}{\sqrt{\xi(1-\xi)}}. \qquad \text{(D.46)}$$

The next integral we need is

$$\begin{aligned}
&\sum_{\eta=\pm1}\int\frac{d^3k}{\left(2\pi\right)^3}\frac{\left(k+\frac{p}{2}\right)^\mu e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2}\\
&=i\lim_{\epsilon\to0}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\frac{\partial}{\partial y_\mu}\int\frac{d^3k}{\left(2\pi\right)^3}\exp\left[i\alpha\left(k+\frac{p}{2}\right)^2\right.\\
&\quad\left.\left.+i\beta\left(k-\frac{p}{2}\right)^2+i\eta k\widetilde{p}+iy\left(k+\frac{p}{2}\right)-(\alpha+\beta)\epsilon\right]\right|_{y=0}\\
&=i\lim_{\epsilon\to0}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\frac{\partial}{\partial y_\mu}\int\frac{d^3k'}{\left(2\pi\right)^3}\exp\left[i(\alpha+\beta)k'^2+i\frac{\alpha\beta p^2}{\alpha+\beta}\right.\\
&\quad\left.\left.-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}+i\frac{(2\beta p-\eta\widetilde{p})y}{2(\alpha+\beta)}-\frac{iy^2}{4(\alpha+\beta)}-(\alpha+\beta)\epsilon\right]\right|_{y=0}, \qquad \text{(D.47)}
\end{aligned}$$

where k has been shifted according to $k' = k+\frac{(\alpha-\beta)p+\eta\widetilde{p}+y}{2(\alpha+\beta)}$. Now we use equation (D.37), take the limit $\epsilon\to0$ and perform the differentiation with respect to y_μ and arrive at

$$\frac{i}{8\left(i\pi\right)^{\frac{3}{2}}}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\frac{(2\beta p-\eta\widetilde{p})^\mu}{2(\alpha+\beta)^{5/2}}\exp\left[i\frac{\alpha\beta p^2}{\alpha+\beta}-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}\right]. \qquad \text{(D.48)}$$

The term proportional to η cancels out when writing out the sum, and making the same substitutions (D.42) as before we get

$$\frac{i}{4\left(i\pi\right)^{\frac{3}{2}}}\int\limits_0^1 d\xi\int\limits_0^\infty d\rho\sqrt{\frac{1-\xi}{\xi}}\frac{p^\mu}{\sqrt{p^2}}\rho^{-1/2}\exp\left[i\rho-\frac{i\xi(1-\xi)p^2\widetilde{p}^2}{4\rho}\right]. \quad \text{(D.49)}$$

Finally, considering equations (D.38) and (D.39) the result is

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{\left(k+\frac{p}{2}\right)^\mu e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2}=\frac{p^\mu}{4\pi\sqrt{p^2}}\int\limits_0^1 d\xi\sqrt{\frac{(1-\xi)}{\xi}}e^{-z(\xi)}. \quad \text{(D.50)}$$

Next we calculate

$$\begin{aligned}
&\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{\left(k+\frac{p}{2}\right)^\mu\left(k-\frac{p}{2}\right)^\nu e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2}\\
&=\lim_{\epsilon\to0}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu}\int\frac{d^3k}{(2\pi)^3}\exp\left[i\alpha\left(k+\frac{p}{2}\right)^2\right.\\
&\quad\left.\left.+i\beta\left(k-\frac{p}{2}\right)^2+i\eta k\widetilde{p}+iy\left(k+\frac{p}{2}\right)+iz\left(k-\frac{p}{2}\right)-(\alpha+\beta)\epsilon\right]\right|_{y=z=0}\\
&=\lim_{\epsilon\to0}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu}\int\frac{d^3k'}{(2\pi)^3}\exp\left[i(\alpha+\beta)k'^2+i\frac{\alpha\beta p^2}{\alpha+\beta}\right.\\
&\quad\left.\left.-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}+i\frac{(2\beta p-\eta\widetilde{p})y-(2\alpha p+\eta\widetilde{p})z}{2(\alpha+\beta)}-\frac{i(y+z)^2}{4(\alpha+\beta)}-(\alpha+\beta)\epsilon\right]\right|_{y=z=0}
\end{aligned} \quad \text{(D.51)}$$

where k was shifted to $k'=k+\frac{(\alpha-\beta)p+\eta\widetilde{p}+y+z}{2(\alpha+\beta)}$. We use equation (D.37), take the limit $\epsilon\to0$ and perform the differentiations with respect to z_ν and y_μ

and get

$$\frac{-i}{8\,(i\pi)^{\frac{3}{2}}}\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\left[\frac{(2\beta p-\eta\widetilde{p})^\mu(2\alpha p+\eta\widetilde{p})^\nu}{4(\alpha+\beta)^{7/2}}-\frac{ig^{\mu\nu}}{2(\alpha+\beta)^{5/2}}\right]$$
$$\times\exp\left[i\frac{\alpha\beta p^2}{\alpha+\beta}-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}\right]$$
$$=\frac{-i}{4\,(i\pi)^{\frac{3}{2}}}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\left[\frac{\alpha\beta p^\mu p^\nu}{(\alpha+\beta)^{7/2}}-\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{4(\alpha+\beta)^{7/2}}-\frac{ig^{\mu\nu}}{2(\alpha+\beta)^{5/2}}\right]$$
$$\times\exp\left[i\frac{\alpha\beta p^2}{\alpha+\beta}-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}\right],\quad\text{(D.52)}$$

having taken into account in the last step that the terms proportional to η cancel out when writing out the sum. Making the substitutions (D.42) again yields

$$\frac{-i}{4\,(i\pi)^{\frac{3}{2}}}\int\limits_0^1 d\xi\int\limits_0^\infty d\rho\sqrt{\xi(1-\xi)}\left[\frac{p^\mu p^\nu}{\sqrt{p^2}}\rho^{-1/2}-\frac{ip^2g^{\mu\nu}}{2\sqrt{p^2}}\rho^{-3/2}\right.$$
$$\left.-\xi(1-\xi)\frac{(p^2)^{\frac{3}{2}}\,\widetilde{p}^\mu\widetilde{p}^\nu}{4\rho^{5/2}}\right]\exp\left[i\rho-\frac{i\xi(1-\xi)p^2\widetilde{p}^2}{4\rho}\right].\qquad\text{(D.53)}$$

Using equations (D.38) and (D.39) finally leads to

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{\left(k+\frac{p}{2}\right)^\mu\left(k-\frac{p}{2}\right)^\nu e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2}$$
$$=\frac{-1}{4\pi}\int\limits_0^1 d\xi\sqrt{\xi(1-\xi)}\left[\frac{p^\mu p^\nu}{\sqrt{p^2}}-\frac{p^2g^{\mu\nu}}{\sqrt{p^2}}\frac{1}{z}+\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}\frac{p^2}{\sqrt{p^2}}\left(\frac{1}{z}+1\right)\right]e^{-z(\xi)}.$$
(D.54)

Following the same steps we also find

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{k^\mu k^\nu e^{i\eta k\widetilde{p}}}{\left(k+\frac{p}{2}\right)^2\left(k-\frac{p}{2}\right)^2}$$

$$=\frac{-i}{4\,(i\pi)^{\frac{3}{2}}}\int\limits_0^\infty d\alpha\int\limits_0^\infty d\beta\left[\frac{-(\alpha-\beta)^2p^\mu p^\nu}{4(\alpha+\beta)^{7/2}}-\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{4(\alpha+\beta)^{7/2}}-\frac{ig^{\mu\nu}}{2(\alpha+\beta)^{5/2}}\right]$$

$$\times\exp\left[i\frac{\alpha\beta p^2}{\alpha+\beta}-\frac{i\widetilde{p}^2}{4(\alpha+\beta)}\right]$$

$$=\frac{-1}{4\pi}\int\limits_0^1 d\xi\left[\sqrt{\xi(1-\xi)}-\frac{1}{4\sqrt{\xi(1-\xi)}}\right]\frac{p^\mu p^\nu}{\sqrt{p^2}}e^{-z(\xi)}$$

$$+\sqrt{\xi(1-\xi)}\left[\frac{-p^2g^{\mu\nu}}{\sqrt{p^2}}\frac{1}{z}+\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}\frac{p^2}{\sqrt{p^2}}\left(\frac{1}{z}+1\right)\right]e^{-z(\xi)}. \tag{D.55}$$

Finally, we need to calculate the integral

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\eta k\widetilde{p}}}{k^2-M^2}=-i\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\frac{d^3k}{(2\pi)^3}\exp\left[i\alpha\left(k^2-M^2\right)+i\eta k\widetilde{p}\right]$$

$$=-i\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\frac{d^3k'}{(2\pi)^3}\exp\left[i\alpha k'^2-iM^2\alpha-\frac{i\widetilde{p}^2}{4\alpha}\right], \tag{D.56}$$

with $k'=k+\frac{\eta\widetilde{p}}{2\alpha}$. Using formula (D.37) and writing out the sum over η yields

$$\frac{-1}{4\,(i\pi)^{\frac{3}{2}}}\int\limits_0^\infty d\alpha\alpha^{-\frac{3}{2}}\exp\left[-iM^2\alpha-\frac{i\widetilde{p}^2}{4\alpha}\right]. \tag{D.57}$$

Identifying $\rho=\alpha$, $\nu=-\frac{1}{2}$, $\tau=iM^2+\epsilon$ and $\sigma=\frac{i\widetilde{p}^2}{4}+\epsilon$ one can use formulas (D.38) and (D.39) to solve the remaining integral. Taking the limit $\epsilon\to0$ finally leads to

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\eta k\widetilde{p}}}{k^2-M^2}=\frac{\exp\left[-i\sqrt{M^2\widetilde{p}^2}\right]}{2\pi\sqrt{\widetilde{p}^2}}. \tag{D.58}$$

Furthermore we need similar integrals including mass parameters for chapter 3.2. We start with

$$\int \frac{d^3k}{(2\pi)^3} \frac{\sum\limits_{\eta=\pm1} e^{i\eta k\widetilde{p}}}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]} = \frac{1}{4\pi}\int\limits_0^1 d\xi \frac{\sqrt{\widetilde{p}^2}}{z(\xi)} e^{-z(\xi)}, \quad \text{(D.59)}$$

where now

$$z(\xi) \equiv \sqrt{\left[\xi(1-\xi) - \xi\widetilde{m}_1^2 - (1-\xi)\widetilde{m}_2^2\right] p^2\widetilde{p}^2} \quad \text{and} \quad \widetilde{m}_{1,2}^2 = m_{1,2}^2/p^2. \quad \text{(D.60)}$$

This formula can be derived in the same way as (D.46) except that when using formula (D.38) one identifies

$$\tau = -i\left(1 - \frac{\widetilde{m}_1^2}{(1-\xi)} - \frac{\widetilde{m}_2^2}{\xi}\right) + \epsilon. \quad \text{(D.61)}$$

Following the same steps as in the derivation of (D.50) but with (D.60) and (D.61) we also find

$$\sum_{\eta=\pm1}\int \frac{d^3k}{(2\pi)^3} \frac{\left(k+\frac{p}{2}\right)^\mu e^{i\eta k\widetilde{p}}}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]} = \frac{p^\mu}{4\pi}\int\limits_0^1 d\xi \frac{(1-\xi)\sqrt{\widetilde{p}^2}}{z(\xi)} e^{-z(\xi)} \quad \text{(D.62)}$$

Similarly to (D.55) we evaluate

$$\begin{aligned}
&\sum_{\eta=\pm1}\int \frac{d^3k}{(2\pi)^3} \frac{k^\mu k^\nu e^{i\eta k\widetilde{p}}}{\left[\left(k+\frac{p}{2}\right)^2 - m_1^2\right]\left[\left(k-\frac{p}{2}\right)^2 - m_2^2\right]} \\
&= \frac{-i}{4\,(i\pi)^{\frac{3}{2}}}\int\limits_0^1 d\xi \int\limits_0^\infty d\rho \sqrt{\xi(1-\xi)}\Big[\left(1 - \frac{1}{4\xi(1-\xi)}\right)\frac{p^\mu p^\nu}{\sqrt{p^2}}\rho^{-1/2} - \frac{ip^2 g^{\mu\nu}}{2\sqrt{p^2}}\rho^{-3/2} \\
&\quad - \xi(1-\xi)\frac{(p^2)^{\frac{3}{2}}\,\widetilde{p}^\mu\widetilde{p}^\nu}{4\rho^{5/2}}\Big] \exp\left[i\left(1 - \frac{\widetilde{m}_1^2}{(1-\xi)} - \frac{\widetilde{m}_2^2}{\xi}\right)\rho - \frac{i\xi(1-\xi)p^2\widetilde{p}^2}{4\rho}\right] \\
&= \frac{-1}{4\pi}\int\limits_0^1 d\xi \left[\left(\xi(1-\xi) - \frac{1}{4}\right)\frac{\sqrt{\widetilde{p}^2}}{z(\xi)} p^\mu p^\nu - \frac{g^{\mu\nu}}{\sqrt{\widetilde{p}^2}} + \frac{\widetilde{p}^\mu\widetilde{p}^\nu}{(\widetilde{p}^2)^{3/2}}\left(1 + z(\xi)\right)\right] e^{-z(\xi)},
\end{aligned} \quad \text{(D.63)}$$

where $z(\xi)$ is once more given by (D.60).

The last of these integrals we need is

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{k^\mu k^\nu e^{i\eta k\widetilde{p}}}{k^2\left(k^2-M^2\right)}=\frac{1}{4\pi}\int\limits_0^1 d\xi\left[\frac{g^{\mu\nu}}{\sqrt{\widetilde{p}^2}}-\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\left(\widetilde{p}^2\right)^{3/2}}\left(1+z(\xi)\right)\right]e^{-z(\xi)},\tag{D.64}$$

with $z(\xi)=\sqrt{-M^2(1-\xi)\widetilde{p}^2}$. This result follows immediately from formula (D.63) when setting $m_2=M$, $m_1=p=0$ but keeping $\widetilde{p}\neq 0$.

Finally we need further integrals for Chapter 4.3, the first of which is

$$\begin{aligned}\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{e^{i\eta k\widetilde{p}}}{\vec{k}^2}&=\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha\int\frac{d^3k}{(2\pi)^3}e^{-\alpha\vec{k}^2+i\eta k\widetilde{p}}\\&=\frac{1}{4\pi^2}\int\limits_{-\infty}^{+\infty}dk^0\int\limits_0^\infty d\alpha\frac{\exp\left[-\widetilde{p}^2/4\alpha\right]}{\alpha}.\end{aligned}\tag{D.65}$$

Since $\vec{k}^2\equiv(k^1)^2+(k^2)^2$ includes only spacial components, we could this time use the parameterization

$$\frac{1}{\vec{k}^2}=\int\limits_0^\infty d\alpha e^{-\alpha\vec{k}^2},\tag{D.66}$$

which after completing the square in the exponent led to a 2-dimensional Gauss-integral (remember $\widetilde{p}^0=0$). The remaining integrals over k^0 and the parameter α do not converge.

The next integral is

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{\bar{k}^\mu\bar{k}^\nu e^{i\eta k\widetilde{p}}}{\theta^4\vec{k}^4}=\sum_{\eta=\pm1}\int\limits_0^\infty d\alpha d\beta\int\frac{d^3k}{(2\pi)^3}\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial y_\nu}e^{-(\alpha+\beta)\vec{k}^2+i\eta k\widetilde{p}-\vec{y}\vec{k}}\Bigg|_{y=0}\tag{D.67}$$

Completing the square, the exponent becomes

$$-(\alpha+\beta)\vec{k}'^2-\frac{(\widetilde{p}^i)^2}{4(\alpha+\beta)}-i\frac{\eta\widetilde{p}^iy^i}{2(\alpha+\beta)}+\frac{\vec{y}^2}{4(\alpha+\beta)},$$

where the sum over $i=1,2$ is to be taken. This leads to a 2-dimensional Gauss-integral. Performing the differentiations and taking the limit $y\to 0$

produces

$$-\int\limits_0^\infty d\alpha d\beta \int\limits_{-\infty}^{+\infty} \frac{dk^0}{8\pi^2}\left(\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{2(\alpha+\beta)^3}+\frac{\bar{\mathbf{1}}^{\mu\nu}}{(\alpha+\beta)^2}\right)\exp\left[-\frac{\widetilde{p}^2}{4(\alpha+\beta)}\right]. \tag{D.68}$$

The substitution

$$\left.\begin{array}{l}\alpha=\xi\lambda\\ \beta=(1-\xi)\lambda\end{array}\right\} \quad\rightarrow\quad d\alpha d\beta=\lambda d\xi d\lambda\ , \tag{D.69}$$

leads to

$$-\int\limits_0^\infty d\lambda \int\limits_{-\infty}^{+\infty} \frac{dk^0}{8\pi^2}\left(\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{2\lambda^2}+\frac{\bar{\mathbf{1}}^{\mu\nu}}{\lambda}\right)\exp\left[-\widetilde{p}^2/4\lambda\right], \tag{D.70}$$

and the result is

$$\sum_{\eta=\pm1}\int\frac{d^3k}{(2\pi)^3}\frac{\bar{k}^\mu\bar{k}^\nu e^{i\eta k\widetilde{p}}}{\theta^4\vec{k}^4} = \frac{-1}{4\pi^2}\int\limits_{-\infty}^{+\infty}dk^0\left(\frac{\widetilde{p}^\mu\widetilde{p}^\nu}{\widetilde{p}^2}+\int\limits_0^\infty d\lambda\frac{\bar{\mathbf{1}}^{\mu\nu}}{2\lambda}\exp\left[-\widetilde{p}^2/4\lambda\right]\right), \tag{D.71}$$

where the substitution $\lambda'=\frac{\widetilde{p}^2}{4\lambda}$ has been used for integration in the first term. The remaining integrals diverge once more.

Finally, we evaluate the similar (dimensionally regularized) integral[4]

$$\begin{aligned}
\int\frac{d^nk}{(2\pi)^n}&\frac{\bar{k}^\mu\left(\bar{k}+\bar{p}/2\right)^\nu}{\theta^4\left(\vec{k}+\vec{p}/2\right)^2\left(\vec{k}-\vec{p}/2\right)^2}\\
&=\int\limits_0^\infty d\alpha d\beta\int\frac{d^nk}{(2\pi)^n}\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu}\exp\Bigg[-\alpha\left(\vec{k}+\vec{p}/2\right)^2\\
&\qquad-\beta\left(\vec{k}-\vec{p}/2\right)^2-\vec{y}\vec{k}-\vec{z}\left(\vec{k}+\vec{p}/2\right)\Bigg]\Bigg|_{y=z=0}\\
&=\int\limits_0^\infty d\alpha d\beta\int\frac{d^nk}{(2\pi)^n}\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu}\exp\Bigg[-(\alpha+\beta)\vec{k}'^2\\
&\qquad-\frac{\alpha\beta\vec{p}^2}{(\alpha+\beta)}+\frac{\left((\alpha-\beta)\vec{y}-2\beta\vec{z}\right)\vec{p}}{2(\alpha+\beta)}+\frac{\left(\vec{y}+\vec{z}\right)^2}{4(\alpha+\beta)}\Bigg]\Bigg|_{y=z=0},
\end{aligned} \tag{D.72}$$

[4]Remember $\bar{k}^\mu=-\theta^2\left(0,\vec{k}\right)$ as defined in (4.29b).

with $\vec{k}' = \vec{k} + \frac{(\alpha-\beta)\vec{p}+(\vec{y}+\vec{z})}{2(\alpha+\beta)}$. Performing the Gauss-integration and the differentiations with respect to y and z yields

$$\int \frac{dk^0}{(2\pi)^n} \int_0^\infty d\alpha d\beta \frac{\pi^{\frac{n-1}{2}}}{(\alpha+\beta)^{\frac{n-1}{2}}} \left[\frac{\beta(\beta-\alpha)\bar{p}^\mu\bar{p}^\nu}{2(\alpha+\beta)^2\theta^4} - \frac{\bar{\mathbf{1}}^{\mu\nu}}{2(\alpha+\beta)} \right] e^{-\frac{\alpha\beta\vec{p}^2}{(\alpha+\beta)}}$$

$$= -\int \frac{dk^0}{2^n\pi^{\frac{n+1}{2}}} \int_0^\infty d\rho \int_0^1 d\xi \left(\xi(1-\xi)\vec{p}^2\right)^{\frac{n-3}{2}} \left[\frac{\xi-\frac{1}{2}}{(1-\xi)} \frac{\bar{p}^\mu\bar{p}^\nu}{\vec{p}^2}\rho + \frac{\bar{\mathbf{1}}^{\mu\nu}}{2} \right] \rho^{\frac{1-n}{2}} e^{-\rho}, \tag{D.73}$$

where substitutions (D.42) and the shift $\xi \to (1-\xi)$ have been used. Considering the definition of the Gamma-function[5]

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \tag{D.74}$$

this expression becomes

$$-\int dk^0 \int_0^1 d\xi \frac{(\xi(1-\xi)\vec{p}^2)^{\frac{n-3}{2}}}{2^{n+1}\pi^{\frac{n+1}{2}}} \left[\frac{2\xi-1}{(1-\xi)} \frac{\bar{p}^\mu\bar{p}^\nu}{\vec{p}^2} \Gamma\left(\frac{5-n}{2}\right) + \bar{\mathbf{1}}^{\mu\nu}\Gamma\left(\frac{3-n}{2}\right) \right], \tag{D.75}$$

and evaluating the remaining parameter integral with formula [35]

$$B(x,y) \equiv \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dt t^{x-1}(1-t)^{y-1}, \qquad \begin{matrix} \Re(x) > 0 \\ \Re(y) > 0 \end{matrix} \tag{D.76}$$

yields

$$\int \frac{d^n k}{(2\pi)^n} \frac{\bar{k}^\mu \left(\bar{k}+\bar{p}/2\right)^\nu}{\theta^4 \left(\vec{k}+\vec{p}/2\right)^2 \left(\vec{k}-\vec{p}/2\right)^2}$$

$$= \frac{(\vec{p}^2)^{\frac{n-3}{2}}}{2^{n+1}\pi^{\frac{n+1}{2}}} \int dk^0 \left[\frac{\bar{p}^\mu\bar{p}^\nu}{\vec{p}^2} - \bar{\mathbf{1}}^{\mu\nu} \right] \frac{\Gamma\left(\frac{3-n}{2}\right)\left(\Gamma\left(\frac{n-1}{2}\right)\right)^2}{\Gamma(n-1)}, \tag{D.77}$$

where the property $\Gamma(x+1) = x\Gamma(x)$ has been used in several places to simplify the resulting expression.

[5] see e.g. Bronstein [34]

D.4 Special integrals

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi, \tag{D.78a}$$

$$\int_0^1 \sqrt{x(1-x)}dx = \frac{\pi}{8}, \tag{D.78b}$$

$$\int_0^1 \sqrt{\frac{x}{1-x}}dx = \int_0^1 \sqrt{\frac{1-x'}{x'}}dx' = \frac{\pi}{2} \tag{D.78c}$$

proof:
The first of these three integrals can be easily verified using the substitution $y = (2x-1)$ which leads to

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_{-1}^1 \frac{dy}{\sqrt{1-y^2}} = \arcsin x\Big|_{-1}^{+1} = \pi. \tag{D.79}$$

The same substitution in equation (D.78b) yields

$$\int_0^1 \sqrt{x(1-x)}dx = \int_{-1}^1 \frac{1}{4}\sqrt{1-y^2}dy = \frac{1}{4}y\sqrt{1-y^2}\Big|_{-1}^{+1} - \int_{-1}^1 \frac{-y^2}{4\sqrt{1-y^2}}, \tag{D.80}$$

where partial integration has been used. Obviously, the boundary term is zero and one finds

$$\int_{-1}^1 \frac{1}{4}\sqrt{1-y^2}dy = -\int_{-1}^1 \frac{1}{4}\sqrt{1-y^2}dy + \frac{1}{4}\int_{-1}^1 \frac{1}{\sqrt{1-y^2}}. \tag{D.81}$$

The left hand side-integral appears again on the right hand side and therefore this equation can be rewritten as

$$\int_{-1}^1 \frac{1}{4}\sqrt{1-y^2}dy = +\frac{1}{8}\int_{-1}^1 \frac{1}{\sqrt{1-y^2}} = \frac{\pi}{8}, \tag{D.82}$$

where the remaining integral has been solved by recalling equation (D.79).

Finally, formula (D.78c) is again proved by substituting $y = (2x - 1)$:

$$\int_0^1 \sqrt{\frac{x}{1-x}} dx = \int_{-1}^1 \frac{y+1}{2\sqrt{1-y^2}} dy = \int_{-1}^1 \frac{1}{2\sqrt{1-y^2}} dy = \frac{\pi}{2}. \tag{D.83}$$

The term proportional to y in the numerator of the integrand does not contribute to the result since an uneven function is integrated over a symmetric interval.

Our next integral is

$$\int_0^1 \frac{dz}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2 (1-z)}} = \arcsin a_+ - \arcsin a_-, \tag{D.84}$$

with

$$a_\pm = \frac{\pm 1 + \widetilde{m}_1^2 - \widetilde{m}_2^2}{\sqrt{\left(1 - \widetilde{m}_1^2 + \widetilde{m}_2^2\right)^2 - 4\widetilde{m}_2^2}}, \tag{D.85}$$

which follows from making the substitution

$$z' = \frac{2z - (1 - \widetilde{m}_1^2 + \widetilde{m}_2^2)}{\sqrt{\left(1 - \widetilde{m}_1^2 + \widetilde{m}_2^2\right)^2 - 4\widetilde{m}_2^2}}, \tag{D.86}$$

leading to

$$\int_{a_-}^{a_+} \frac{dz'}{\sqrt{1 - z'^2}} = \arcsin a_+ - \arcsin a_-. \tag{D.87}$$

In the same way one can show that

$$\begin{aligned} &\int_0^1 dz \frac{1-z}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2 (1-z)}} \\ &= \sqrt{2A(\widetilde{m}_1, \widetilde{m}_2)} \sqrt{1-z^2} \Big|_{a_-}^{a_+} + \frac{1}{2}(1 + \widetilde{m}_1^2 - \widetilde{m}_2^2) \arcsin z \Big|_{a_-}^{a_+}, \end{aligned} \tag{D.88}$$

and

$$\begin{aligned} &\int_0^1 dz \frac{z}{\sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2 (1-z)}} \\ &= -\sqrt{2A(\widetilde{m}_1, \widetilde{m}_2)} \sqrt{1-z^2} \Big|_{a_-}^{a_+} + \frac{1}{2}(1 - \widetilde{m}_1^2 + \widetilde{m}_2^2) \arcsin z \Big|_{a_-}^{a_+}, \end{aligned} \tag{D.89}$$

with

$$A(\widetilde{m}_1, \widetilde{m}_2) = \frac{1}{8}\left[\left(1 - \widetilde{m}_1^2 + \widetilde{m}_2^2\right)^2 - 4\widetilde{m}_2^2\right]. \tag{D.90}$$

Finally we need

$$\begin{aligned} &\int_0^1 dz \sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)} \\ &= A(\widetilde{m}_1, \widetilde{m}_2)\Big[z\sqrt{1-z^2}\Big|_{a_-}^{a_+} + \arcsin z\Big|_{a_-}^{a_+}\Big]. \end{aligned} \tag{D.91}$$

Making the same substitution (D.86) as before we get

$$\int_0^1 dz \sqrt{z(1-z) - \widetilde{m}_1^2 z - \widetilde{m}_2^2(1-z)} = 2A(\widetilde{m}_1, \widetilde{m}_2)\int_{a_-}^{a_+} dz' \sqrt{1-z'^2}. \tag{D.92}$$

Using partial integration we find

$$\int_{a_-}^{a_+} dz' \sqrt{1-z'^2} = z'\sqrt{1-z'^2}\Big|_{a_-}^{a_+} + \int_{a_-}^{a_+} dz' \frac{1}{\sqrt{1-z'^2}} - \int_{a_-}^{a_+} dz' \sqrt{1-z'^2}, \tag{D.93}$$

and inserting this equation into (D.92) yields the result (D.91).

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