DNS Points, DNS Thresholds and Regions of Multiple Optimal Solutions in the Light of Global and Local Optimality

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Nur wem es gelingt, seine Schwächen zu vergessen
und seine Ängste in tausend Stücke zu schlagen,
der überwindet die mächtigen Mauern
seiner eigenen Grenzen,
welche die Küsten
der Illusion und der Realität entzweien.
Wie glücklich sind die,
die auf den Bergen der Ewigkeit gewandert sind,
um die Geburt und den Tod
an einem einzigen Tag zu genießen
oder vielleicht auch weniger
als einen Tag lang

(Gertrude Reinisch, 1998)

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Chapter 1

Introduction

Using mathematical methods in modern physics is a great and successfull story. The most important turning-point was the development of the differential calculus (Fluxionsrechnung) and its application to the theory of celestial movement by Isaac Newton in his famous work (Newton, 1687). Within this calculus it was possible to formulate problems of dynamical systems and predict the systems movement for the future. Prediction implies the possibility of refutation by experimental data. One well known example is the refutation of the classical laws of Newton by Einsteins relativistic laws. This is the most used example for the strength of the natural sciences method (see e. g. Popper, 1935/1989). As the relation between (theoretical) physics and mathematics was that successfull, it is not surprising that the axiomatic method first introduced to geometry by Euklid and strongly formalized in the 19th century, found its counterpart in physics. Where the laws of mechanics are stated as axioms. The strenght of mathematical formalism and logic seems to be carried over to physics a science describing natural phenomena. This is a very rough outline of the intimate union of mathematics and physics, but it helps to understand, when León Walras formulates in his work (Walras, 1874/1954 S.71):

> ... pure mechanics surely ought to precede applied mechanics. Similarly, given the pure theory of economics, it must precede economics ...

where he is arguing for a strict mathematical formulation of (pure) economics. One should be able to deduce the economics systems behaviour if only the initila conditions are known. That Walras could argue in that way was foundec on his conviction of the existence of economic laws comparable to natural laws (Walras, 1874/1954 S.69):

> This does not mean that we have no control over prices. Because gravity is a natural phenomena and obeys natural laws, it does not follow that all we can do is to watch it operate. We can either resist it or give it free rein, whichever we please, but we *cannot change its essence or laws.*

In his view the laws describing economic processes cannot be deceived but understanding and formulating them as (mathematical) laws includes the possibility to use them for his own sake.

In a more recent work Oskar Morgenstern rates the axiomisation of economics as an aim although not the most important one (see Morgenstern, 1963 S. 15):

> Die Axiomatisierung wurde erst in jüngster Zeit und nur in beschränktem Ausmaß in die Wirtschaftswissenschaften eingeführt, und zwar deswegen, weil die meisten wirtschaftswissenschaftlichen Theorien *noch* nicht die Strenge aufweisen, die notwendig ist, bevor überhaupt eine Axiomatisierung vorgenommen werden kann.

But he also refers to the difference of mathematical axioms and axioms in empirical sciences (Morgenstern, 1963 S. 35):

> Wird ein Bereich der Mathematik axiomatisiert, dann werden die Axiome mathematische Sätze sein, ... Wenn es sich dagegen um ein empirisches Gebiet handelt, so werden die Axiome Behauptungen über einen Bereich der Wirklichkeit sein. ... In keinem Fall kommt den Axiomen irgendeine höhere Wahrheit zu.

This can be seen as the attempt to clarify the relation between a mathematical model and its application to real processes.

When Kenneth J. Arrow formulates his hope that social behaviour can be expressed in mathematical terms (see Arrow, 1951):

> "Mathematics, ... is a language." If this be true, any meaningful proposition can be expressed in a suitable mathematical form, and any generalizations about social behavior can be formulated mathematically.

than he regards the clarity of mathematical language as a great advantage:

> In the first place, clarity of thought is still a pearl of great price.

These examples show the economists hope finding answers for economical and sociological problems within a mathematical framework. When the authors of an economic textbook (see e. g. Felderer & Homburg, 2003) formulate the basic assumptions as axioms, generations of economists learn to think within these formal concepts.

Whatever the advantage of mathematics in sociological and economic sciences may be, one has to note that the mathematical language is per se not historical. Mathematical truth does not depend on time and place. The consequent elimination of empirical argumentations in mathematical proofs gave mathematics the strenght it is known for (see e. g. Mehrtens, 1990). But trying to copy this strategy must fail. In economics it is a weakness to formulate assumptions as axioms and trying to remove them in this way from a historical reflection. Concepts formulated as formal axioms loose their association to their historical origin. Using rather a formal language than an infomal is not only a question of taste. The used metaphers and concepts have direct influence to the areas of research and problems to be considered as Evelyn Fox Keller studied e. g. for biology (see Keller, 1998).

The importance of wrestling with meanings of terms and concepts should not be underestimated for a science which gives terms like value and utility the impression of neutrality. But that the usage of these terms has to be clarified for every different case becomes clear when we read Joan Robinson (Robinson, 1962 S. 46):

> Among all the various meanings of *value*, there has been one all the time under the surface, the old concept as Just Price ... Prices ought to be such (subject to political expediencey) that a day's work in the country and in the town brings in about the same income. But even when this is granted as an ideal there still remains the problem of calculating what is to be considered an equivalent income for individuals leading quite different kinds of live in different environments. *Value* will not help. It has no operational content. It is just a word.

As I stated befor mathematics succeded in emptying the used terms from its meaning, the meaning of words has to be strengthend in empirical particularly sociological sciences. Within this context I want to give a short comment about the used diction of this thesis. The informal parts are considered from the authors personal point of view and therefor it is written in the first person. Whereas in the mathematical chapters the unpersonal third persons form is used. This diction reflects the aspect concerning the differences of the underlying concepts of truth. On the one side the ability to argue reasonable, convincing and tempting with the better, stronger or subtle arguments (see e. g. Feyerabend, 1976). On the other side the uncompromising power of logical deduction (see e. g. Mehrtens, 1990). But this power cannot be transformed from mathematics to its application.

What is the reason to state this remarks at the beginning of my thesis, although the considered problems are beyond the scope of this thesis? To answer this question I first have to explain what this thesis is all about.

One of the main attempts of this work is to clarify the concepts of so-called Dechert-Nishimura-Skiba (DNS) points. These points are not only of mathematical but also of economical interest. At a DNS point point a decision maker has multiple optimal choices. He/She is free in choosing one, all of them are equal from the point of optimality. It was of principal interest to find models as simple as possible, possessing the potential structure for the occurence of DNS points. Under these aspects three models were established, "A Model of Moderation" (MoM), "A Generalized Model of Moderation" (GMoM) and "A Model of Bridge Building" (MoBB), as promising candidates. Two of these models MoM and MoBB have also though very simple economic interpretations. Simplicity has the advantage that it was possible to do the analysis to a great extent analytically. Therefore analytic expressions of the systems steady states and bifuraction lines could be given. Hencefor the optimal policies are completely classified and related to regions seperated by the bifurcation lines. This gives complete insight to the models optimal behaviour, when for example changing a system's parameter. Phenomena like bifurcating DNS points were detected, regions with the same dynamical but different optimal behaviour and vice versa have been found. Moreover consise interpretations within the simplicity of its assumptions could be given.

On studying DNS points in more detail, points were detected, where it was not clear if the definiton of a DNS point does fit or not. Initially starting at such a point one has no different optimal choices but has to stay there forever. Nevertheless moving only an infinitesimally small distance to the left or to the right leads to different long run optimal states. The question about the nature of such points and if it is optimal to stay at such a point arises. Then Prof. Vladimir Veliov, whom I will thank at this place, proposes to study the local optimality of these points. This entailed myself to the concept of local optimality and subsequently to a new and important topic for my thesis.

Local optimality, at first a more mathematical concept as it is related to the solutions of differential equations of Riccati type and needs in general highly sophisticated mathematics, has a reasonable economic interpretation.Once again the simplicity of the models had the advantage to find explicit solutions of the Riccati differential equations, at least for the steady states. For the general case numerical solutions have been calculated and hencefor a complete classifiacation of the models local optimal behaviour had been given.

For local optimal solutions regions are found where up to three local optimal policies exist. Comparing these regions with the regions where DNS points exist, shed new light on the DNS points too.

Summarizing it can be said that the main idea of this thesis is the opposition of the two different optimality concepts, where one is searching for local optimal solutions while on the other hand one is searching for global optimal solutions. Under this point of view DNS points and regions with multiple local optimal solutions are studied. The mathematical results are stated at first formally and furthermore interpreted under the aspect of optimal policies.

But now back to the question about the relation of the introdutctory remarks and my thesis. The models are thus simple that terms like decision maker, optimality and utility do not become very problematic, they can be naivly interpreted. Therefore one can concentrate on the relation between the usage of mathematical tools to solve the problem and the economic interpretation. As the models behaviour is rich enough these models can serve as simple examples for the modeling of economic processes and its interpretation and the influence of different mathematical concepts to the interpretation. From this point of view these models serve as a first starting point for further investigations, where the problems stated at the beginning are of more relevance.

The second reason for these statements is a personal one, because this critical point of view in using mathematical methods in the context of social sciences is my main motivation to work on these issues.

Furthermore I want to mention another important aspect of this work and for research done in general on this field. It was necessary to do numerically calculations by the computer, where I used the software MATLAB©. Without help of a computer this work could not have been done, this relativizes the meaning of simplicity of these models. But what is realy worth to be noted is the usage of a "new" kind of mathematics, contrary to mathematics practised with "a pencil and a sheet of paper". When for example a differential equation is soved numerically and one plots the solution to a figure one is to take this representation as the solution itself. This impression will be destroyed if the figure is zoomed in up to the machines precision level. What looks continuous from the distant is discontinuous in "reality" and one has to accept that this is not the "real" exact solution of the given problem. This example shall only emphasis that we are arguing in a highly theoretical burdend system which has the power to seduction. Under this aspect interpretations often try to blur these dependences on underlying theories and concepts.

For the dayly work forgetting these dependencies is necessary but not to be aware of the presence of these conceptions seems negligent to me.

At the end I want to give a short outline about the structure of this work. In the first chapter I introduce the main concepts and the class of models I am going to analyse for three special cases. These three models are arranged in three chapters. Each chapter is of the same structure what facilitates comparison among the different models. Every of these chapters consists of sections, where at first a motivation, of the specified model is given. In this motivation the functional forms are defined, thereafter the models are analysed by the Pontryagins minimum principle. Furthermore second order conditions, that is the corresponding Riccati differential equation is stated. Subsequent to this section the existence of steady states is analysed and following the properties of these steady states are inspected. Following the second order conditions are analysed at the steady states and along the paths sufficing the necessary conditions. Then the occuring bifurcation lines and regions of different stability are described. After that the optimal strategies are classified and related to the regions of stability in the parameter space. Another section treats the different local optimal strategies and relates these strategies to the global optimal strategies. In a last section the change of optimal behaviour is described in a more descriptive way, when parameters vary. Each of these chapters contain figures with the different regions of the parameter space and phase portraits depicting the main global and local optimal solutions. In Tables the properties of stability and occurence of DNS thresholds are summarized.

Chapter 2

General Concepts and Models

One of the main interests of this work is focused on the interesting phenomena of multiple optimal solutions. This phenomena is known as so-called Dechert-Nishimura-Skiba (DNS) points or in another context shock. Where DNS thresholds are related to a global concept of optimality the term shock has its origin in the investigation of local optimal behaviour. The concept and motivation of local optimality in an economic context (multiple locally optimal solutions) was first introdueced in a talk given at the Institute of Econometrics, Operations Research and System Theory (Technical University of Vienna) by Vladimir Veliov in 2003. This chapter gives a short informal introduction to these different concepts of optimality and its interpretations in an economic framework. Furthermore the class of models I am going to analyse for three different specifications is defined. Another section contains the explication of necessary and sufficient conditions for this class of models.

2.1 Global Versus Local Optimality

The abstract idea behind using optimal control theory in an economic framework has the conception from a decision maker, who can influence an economic system by applying control(s) over some timespan. The term optimal assumes that there exists a measure to value the system at any instant. Finding the optimal control is the demand on the decision maker choosing at any time a control such that the summed measure of the system is optimal, where optimal means minimal or maximal depending on the model. As optimality is the crucial aspect in this context the underlying concepts behind choosing optimal controls have to be clarified.

Therefore I want to give a more informal than formal definition of the terms global and local optimality, DNS point versus DNS threshold and shock. As optimality can be defined as well in a static as in a dynamic framework, I want to emphasize, that in the following I only refer to dynamic models as is implicite assumed, when noting that the decision maker chooses a control at any instant. Hencefor this control can vary over time and is therefore essentially dynamic. Especially time varies continuously and the state(s) dynamic describing the systems behaviour are formulated as differential equations. For the beginning I have chosen a heuristic approach to grasp the main ideas. Therefore terms are used within their naive meaning, which is opposed to a rigour mathematical treatment, where one has to abstract from the naive context of terms and exact definitions have to be given. At the intersection line between strict mathematical formulation and real world applications this can lead to terminological problems, as will be explored in more detail considering the problem of finding a comprehensive definition of the term DNS point and DNS threshold (see Appendix D). But in the following these problems are neglected and terms are used in their colloquial context. Strict definitions are given in the Appendices A and C. There one can also find references treating these topics in more detail.

Concept 1 A global optimal policy is the policy minimizing (maximizing) the criterion of the system over all admissible policies.
A locally optimal policy is the policy minimizing (maximizing) the value measure of the system within a *certain range* of admissible policies.

The terms "admissible", "within a certain range" and "minimizing (maximizing) the value measure" have to be strictly defined within the mathematical theory of optimal control (see Appendix A).

The terms DNS point and DNS threshold have their origin in the works of Dechert and Nishimura (1983) and Skiba (1978). For a comprehensive literature survey concerning this topic see e. g. Deissenberg et al. (2004). These terms are restricted to the one dimensional case, where the modeled system is described by one state variable. In the following I give a definition of a DNS point and threshold, which are restricted to the one dimensional case. These definitions are not strictly mathematical but sufficies to understand the underlying idea. For the difficulties finding consistent mathematical definitions see Appendix D.

Concept 2 A DNS threshold separates regions of different global optimal policy.

The following definition of a DNS point is more restrictive than that of a DNS threshold although both have the same idea in mind.

Concept 3 At a DNS point exist at least two different globally optimal policy.

Both definitions are used in economic literature, but they are not equivalent. As Con. 2 accentuate the DNS character as a threshold, where one is interested what follows from starting left or right from this point. Concept 3 describes what happens if one starts exactly at this point. Examples of the difference between Con. 2 and Con. 3 can be found for all analysed models.

Anyhow for both definitions one has to find the global optimal solution, that is DNS pont and threshold are related to the concept of global optimality.

Contrary to the global optimality the term shock is based on the concept of local optimality (see Appendices A and C). A descriptive definition of shock is given by

Concept 4 A shock happens if there exist at least two different extremal policies.

Where an extremal policy sufficies only the necessary Pontryagins conditions. For a rigorous treatment of shocks see (Caroff & Frankowska, 1996). As will be seen in the analysis of the models of Chapter 3, 4, 5 the occurence of shocks give an indication for the existence of multiple local optimal policies.

In economy the question if using a local or global concept is often circumvented by assuming convex (concave) models. For in this case the uniqueness of an optimal solution, if any exists, can be shown. But if the assumption of convexity (concavity) is dropped the question of local or global optimality gets into account. Using convex (concave) models is more a question of easier mathematical handling than a theoretical compelling one and many "realistic" models lack this sort of mathematical property (see e. g. Skiba, 1978), at least in the field of Operations Research. But it should not be concealed that using convex (concave) models in mathematical economy is a principal issue in discussion. Therefore to strengthen my argument for general models in mathematical economy a more detailed investigation would be necessary.

Nevertheless as stated before in the field treated by this thesis non-concave (non-convex) models are of great importance.

The concept of global optimality demands of the decision maker to choose the optimal policy among all possible ones. The local concept of optimality can be interpreted in a twofold manner. First the decision maker knows the optimality measure for all possible policies, but decides to choose the one which takes lesser effort, where this effort is not considered explicitely in the model. This interpretation grants the decision-maker more freedom in choosing an optimal policy. Hencefor it is more flexible in reacting under different circumstances. Although these different circumstances could principally be modeled explicitly the basic approach is different. In a second interpretation the decision maker is not interested in a policy differing too much from his actual position. He is searching for an optimal solution only in a certain range of possible optimal policies. I think both interpretations can be found in real economic decision strategies and give justification for studying the models local optimal strategies of models.

2.2 Class of Models

As mathematics becomes quickly complex if one attempts to model realistic problems, it was tried to find as simple models as possible but with an interesting solution structure as described before. Therefore the models were restricted to one state variable x, one control u and only two external parameters r and c. Simplicity had the advantage that every possible optimal solution has been classified corresponding to deiiferent regions in parameter space. Most of the occuring bifurcation lines have been calculated as explicit analytical functions. Therefore the regions with different optimal and dynamical behaviour were explicitly specified.

Within the mentioned restrictions a class of simple models with complex mathematical behaviour was defined. This general class of models is given as

$$\begin{aligned}
&\min_u \int_0^\infty e^{-rt}\left(g(x)+cu^2\right)dt\\
&\text{s.t.}\quad \dot{x}=f(x)+u\\
&\text{and}\quad x(0)=x_0,
\end{aligned}\tag{GM}$$

where x denotes the state variable, u the control variable and $r > 0$ is a discounting rate, while $c > 0$ is interpreted as the cost of control u. As well state variable x as control variable u depend explicitly on time t and should therefore be written as $x(t)$ and $u(t)$. In order not to overload notation the functional argument t is omitted whenever there is no ambiguity to expect. The function $g(x) \geq 0$ denotes the cost of being at state x. Whereas $f(x)$ reflects the uncontroled state dynamic. Both functions $g(x)$ and $f(x)$ are analytic functions and therefore arbitrarily often differentiable. Under these general assumptions for $f(x)$ and $g(x)$ convexity for the Hamiltonian H cannot be assumed, which is a basic requirement for the occurence of multiple optimal solutions.

In the following chapters three special models within (GM) are specified. And at least two of these models have also an concistent economic interpretation within the limitations of simplicity.

2.3 Analysis of the General Model

This section consists of two subsections. In the first subsection the necessary optimality conditions are stated, whereas in the next subsection sufficient second order condtions for local optimality are stated. The analysis of the general model is done as far as possible to avoid that it has to be done for every specific model in the sequel.

2.3.1 Necessary Optimality Conditions

The optimal control problem persuted above is given by

$$\begin{aligned} &\min_u \int_0^\infty e^{-rt}\left(g(x) + cu^2\right) dt \\ &\text{s.t.} \quad \dot{x} = f(x) + u \\ &\text{and} \quad x(0) = x_0, \end{aligned} \tag{2.1}$$

with state variable x, control variable u, discount rate r and costs c. To solve this problem by applying Pontryagin's minimum principle (see, e. g. , Feichtinger & Hartl, 1986) we consider the current value Hamiltonian

$$H = \lambda_0 \left(g(x) + cu^2\right) + \lambda \left(f(x) + u\right) \tag{2.2}$$

where λ denotes the costate variable in current value terms and $\lambda_0 \geq 0$ is a real value. For infinite time horizon problems it cannot be excluded, that $\lambda_0 = 0$. Although $\lambda_0 = 0$ is considered as an anomalous case (see Leonard & Long, 1992) it has to be shown explicitly. But for the class (GM) of models it is easy to show that $\lambda_0 = 0$ can be excluded.

For assume $\lambda_0 = 0$, then $H = \lambda\,(f(x) + u)$. Minimizing H in respect to u we get $H_u = \lambda = 0$. But for $(\lambda_0, \lambda) = (0, 0)$ the necessary optimality conditions are violated (see, e. g. , Feichtinger & Hartl, 1986), which refutes the assumption $\lambda_0 = 0$.

From now on $\lambda_0 = 1$ can be assumed and Eq. 2.2 can be written as

$$H = g(x) + cu^2 + \lambda\,(f(x) + u) \tag{2.3}$$

Following the standard methods of optimal control theory we derive the necessary optimality condition

$$u^\star = \arg\min_u H,$$

Since H is differentiable in u and no control constraints have to be considered, minimizing H is equal to setting $H_u = 0$. As $H_u = 2cu + \lambda$ this implies

$$\lambda = -2cu \Leftrightarrow u = -\frac{\lambda}{2c}. \tag{2.4}$$

Since $H_{uu} = 2c > 0$ the Legendre-Clebsch condition is satisfied. Furthermore, considering that

$$H_x = g'(x) + \lambda f'(x)$$

the adjoint equation is given by

$$\begin{aligned} \dot{\lambda} &= r\lambda - H_x & (2.5)\\ &= \lambda r - g'(x) - \lambda f'(x) & \\ &= \lambda\,(r - f'(x)) - g'(x). & (2.6) \end{aligned}$$

Considering Eq. 2.4 and therefore $\dot{\lambda} = -2c\dot{u}$ we derive the differential equation for the control variable u

$$\begin{aligned} -2c\dot{u} &= -2cu\,(r - f'(x)) - g'(x) & \\ \dot{u} &= u\,(r - f'(x)) + \frac{g'(x)}{2c}. & (2.7) \end{aligned}$$

State equation Eq. 2.1 and the adjoint equation Eq. 2.6 or Eq. 2.7 for control u, yield the canonical system as necessary optimality conditions for (GM).

Furthermore as these special class of models is given in autonomous form that is $g_t = 0$, $f_t = 0$ and $f \geq 0$ together with $0 \in \{f(\hat{x}, u) \mid u \in \mathbb{R}\}$ the transversality condition

$$\lim_{t \to \infty} e^{-rt} \lambda(t) = 0 \tag{2.8}$$

has to hold (see, e. g., Michel, 1982). Therefore possible candidates for optimal solutions are the stable manifolds of steady states of saddle type.

Since we made no assumptions on f and g nothing can be said in general about the usual (Mangasarian) sufficiency conditions.

For the sequel I give some notational remarks. Paths sufficing the necessary conditions Eqs. 2.1, 2.6 or 2.7 and 2.8 are referd to as extremal paths (see Definiton in Appendix A). Evaluating a function $F(x, \lambda)$ along an extremal path $(\hat{x}, \hat{\lambda})$ is abbreviated by $\hat{F} = F(\hat{x}, \hat{\lambda})$.

In the following chapters equilibria of the canonical systems are denoted by capital E. In order not to get confused by the term equilibria, which has different meaning in context of economic interpretations and dynamical systems, throughout this thesis the term steady state is used instead of equilibrium.

2.3.2 Sufficient Optimality Conditions

As was noted in the last section no sufficiency conditions for a global optimum can be given in this general case. Nevertheless as is shown in Appendix A, second order sufficient conditions for local optimality can be stated. A sufficient local optimality condition is the existence of a bounded solution of the following Riccati differential equation

$$\dot{p} = -e^{rt} \hat{H}^{\circ}_{\lambda x} p - p \hat{H}^{\circ}_{x \lambda} - p \hat{H}^{\circ}_{\lambda \lambda} p - e^{-rt} \hat{H}^{\circ}_{xx}, \tag{2.9}$$

where $\hat{H}^{\circ}$ is the minimized Hamiltonian H in respect to u and evaluated along an extremal path. The matrix function $R(x, \lambda)$ of Eq. A.21 in Appendix A reduces to a scalar function $p(x, \lambda)$.

For the class (GM) of models the minimized Hamiltonian H° is calculated considering the Hamiltonian H Eq. 2.3 and the minimum condition

Eq. 2.4 as

$$\begin{aligned} H^\circ &= g(x) + c\frac{\lambda^2}{4c^2} + \lambda\left(f(x) - \frac{\lambda}{2c}\right) \\ &= g(x) + \frac{\lambda^2}{4c} - \frac{\lambda^2}{2c} + \lambda f(x) \\ &= g(x) - \frac{\lambda^2}{4c} + \lambda f(x). \end{aligned} \tag{2.10}$$

Considering the partial derivatives of the minimized Hamiltonian H°

$$\begin{aligned} H^\circ_x &= g'(x) + \lambda f'(x) \quad \text{and} \\ H^\circ_\lambda &= -\frac{\lambda}{2c} + f(x), \end{aligned}$$

the second order partial derivatives are derived as

$$\begin{aligned} H^\circ_{xx} &= g''(x) + \lambda f''(x) \\ H^\circ_{x\lambda} &= f'(x) = H^\circ_{\lambda x} \\ H^\circ_{\lambda\lambda} &= -\frac{1}{2c}. \end{aligned}$$

Substituting these formulae in Eq. 2.9 the Riccati differential equation is written as

$$\dot{p} = e^{rt}\frac{1}{2c}p^2 - 2\hat{f}'(x)p - e^{-rt}\left(\hat{g}''(x) + \lambda\hat{f}''(x)\right),$$

or substituting λ by Eq. 2.4

$$\dot{p} = e^{rt}\frac{1}{2c}p^2 - 2\hat{f}'(x)p - e^{-rt}\left(\hat{g}''(x) - 2cu\hat{f}''(x)\right).$$

As it is shown in Appendix A these equations simplify to

$$\begin{aligned} \dot{q} &= \frac{1}{2c}q^2 - (2\hat{f}'(x) - r)q - \hat{g}''(x) - \lambda\hat{f}''(x), \\ \dot{q} &= \frac{1}{2c}q^2 - (2\hat{f}'(x) - r)q - \hat{g}''(x) + 2cu\hat{f}''(x), \end{aligned} \tag{2.11}$$

on setting $q = e^{-rt}p$.

In general it is not possible to find analytic solutions to these Riccati differential equations, because the extremal paths are only numerically given. Nevertheless it is possible to study the sufficient second order conditions analytically for steady states, as is shown in Appendix A. In this case the coefficients of the Riccati differential equation are reduced to simple scalars.

Chapter 3

A Model of Moderation

A simple model is considered that rewards "moderation" - finding the right balance between sliding down either of two "slippery slopes". Optimal solutions are computed as a function of two key parameters: (1) the cost of resisting the underlying uncontrolled dynamics and (2) the discount rate. Analytical expressions are derived for bifurcation lines separating regions where it is optimal to fight to stay balanced, to give in to the attraction of the "left" or the "right", or to decide based on one's initial state. The latter case includes situations both with and without DNS points or thresholds respectively defining optimal solution strategies. The model is unusual for having two DNS points in a one-state model, having a single DNS point that bifurcates into two DNS points, and for the ability to explicitly graph regions within which DNS points occur in the 2-D parameter space. The latter helps give intuition and insight concerning conditions under which these interesting points occur. Furthermore the concept of local optimality as is described by second order conditions is applied and interpreted for this model.

3.1 Motivation

It is often both difficult and advantageous to maintain a position of moderation between two opposing factions, both of which seek to win the allegiance of those who remain uncommitted. Such situations present an interesting challenge: When and how should one remain unaligned if that is indeed the best option? More colloquially, how does one stay on top of a slippery slope? The solution to a simple model of this problem proves insightful both for the original context and as a window into quite interesting behaviour concerning multiple DNS points (Dechert & Nishimura, 1983; Skiba, 1978) within a one-dimensional model. We motivate the problem by sketching its application in three disparate contexts.

"Swing Voters"

Consider a legislature debating an issue for which there will be an up or down vote and for which the vote will be close. Both the "yes" and "no" lobbies court "uncommitted" representatives, and these uncommitted "swing voters" can trade (log roll) their cooperation for considerable benefit (cf. Ordeshook, 1986; Schickler & Rich, 1997). Politicians who are already firmly in the "yes" or "no" camp - e. g. , because they have made public commitments to their constituency about their vote - do not have similar leverage, and those who begin to lean one way or the other may quickly come to be known ("pigeon-holed") as belonging to that camp and not the other. In terms of conventional power indices (e. g. , the Shapley-Shubik or Banzhaf index) one can think of this as having most others "vote first" and cancel (balance out) each others' votes, leaving those few in the middle to determine the winning coalitions. Similar arguments apply to Supreme Court Justices (Blasecki, 1990) and blocks of voters in general elections. Of course, independents do not always reap rewards; in "machine politics" parties dole out favors to loyalists and independents are disadvantaged. The claim here is only that there are some situations in which unaligned moderates have leverage, not that it is always the case. Dixit and Londregan (1996) characterize when each situation pertains.

"Arbitration" - both Formal and Less Formal

Formal arbitrators wield considerable power, regularly resolving multi-million dollar commercial disputes, particularly in labor-management relations. Arbitrators jeopardize their ability to obtain that power if they become tainted by bias (Kaufman & Duncan, 1992). For example, the neutrality of the arbitrator is a non-negotiable precondition mentioned explicitly in the National Arbitration Forum's "Arbitration Bill of Rights". Less formally, a neutral third party can command respect in a partisan dispute. Statements from either side of the dispute are given little credence; they are perceived as mere posturing. In contrast, the opinions of an unaligned third party who has protected a reputation for impartiality are taken seriously. Consultants are sometimes hired to obtain such an independent perspective, and however poor the methodology (Clarke, 2002), US News and World Report rankings of universities are taken more seriously than individual school's own self-serving claims to offer the best education. At a macro level, India sought leadership among third world nations by remaining unaligned in the Cold War (Willetts, 1978). In highly charged contexts such that "he who is not my friend is my enemy", third parties may find it hard to resist becoming aligned with one side or the other, but may preserve a unique power if they can do so.

"Preserving a Balanced World View"

A more abstract example is an individual seeking to maximize intellectual honesty by remaining open-minded. Psychologists have shown that people often suffer from a "confirmation bias" (Gilovich, 1991). Once we begin to believe one thing, we are most attentive to evidence that reinforces that prior belief and are more skeptical toward contrary evidence. So with respect to any issue - e.g., "public schools are (or are not) doing a good job" - if one does not exert effort to avoid it, any bias in one direction or another may tend to be amplified over time.

What these situations have in common is that maintaining a "middle ground" is valuable but difficult. I.e., it can require effort, which presumably is costly. Visually, one can imagine this as standing on top of a narrow hilltop, with "slippery slopes" leading off to the left and to the right. Remaining atop the hill confers benefit, and at the peak the pulls from either side are balanced. However, if one moves a bit to the left or to the right, one has to fight (gravity) to keep from slipping further. If one falls, one does not fall forever. That is, the cross-section of the hill looks more like a bell-curve than an upside down U. Once one has moved from the middle all the way to one side or the other, there is little pressure to move beyond that point. E.g., there is not in general a force that would tend to push a Congressperson to be more conservative (or liberal) than the conservative (liberal) party whip wants that Congressperson to be.

A stylized model of this has a state variable x denoting the decision maker's position, with -1 and 1 representing the positions of the two opposing sides and $x = 0$ being the sought-for middle ground. If the state is currently between -1 and 1 there is tendency to be pulled out to the end of that range that is closest. If for some reason the decision maker moved beyond -1 or 1 (i.e.,$|x| > 1$), there would be a tendency to drift back to the nearest pole. Those tendencies could, however, be moderated by exerting effort (control variable u) to adjust the state, but at a (convex) cost. Various functional forms might be suitable. One that is analytically convenient is

$$\begin{aligned} &\min_u \int_0^\infty e^{-rt}\left(x^2 + cu^2\right) dt \\ &\text{s.t.} \quad \dot{x} = x - x^3 + u \\ &\text{and} \quad x(0) = x_0, \end{aligned} \tag{MoM}$$

where $r > 0$ is a discount rate and parameter c governs the cost of adjusting one's position.

3.2 Analysis of the Model

This section consists of five subsections. In the first subsection the optimality conditions are specified for model (MoM). After that the Riccati differential equation as sufficient second order condition is stated. Following an analysis of the existence and properties of steady states is given. Thereafter the regions of stability and the separating bifurcation lines are studied. In the last subsection the second order conditions for the occuring extremal paths are analysed.

3.2.1 Necessary Optimality Conditions

For model (MoM) the cost function g and the uncontrolled state dynamic f are specified as

$$g(x) = x^2 \quad \text{and} \quad f(x) = x - x^3,$$

with derivatives

$$g'(x) = 2x \quad \text{and} \quad f'(x) = 1 - 3x^2,$$

Following the analysis of Subsection 2.3.1 the current value Hamiltonian H is written as

$$H(x, u, \lambda) = x^2 + cu + \lambda\left(x - x^3 + u\right),$$

where λ denotes the costate variable in current value terms.

Substituting the concrete functional forms of f, f', g and g' in Eqs. 2.6-2.7 the adjoint differential equation is

$$\dot{\lambda} = \lambda\left(r + (3x^2 - 1)\right) - 2x. \tag{3.1}$$

or for control state u

$$\dot{u} = u\left(r + (3x^2 - 1)\right) + \frac{x}{c}. \tag{3.2}$$

Together with the state dynamics

$$\dot{x} = x - x^3 + u, \tag{3.3}$$

the transversality condition Eq. 2.8 and initial starting position $x(0) = x_0$ these equations yield the necessary optimality conditions for (MoM).

Since the Hamiltonian H is not convex with respect to the state variable the usual (Mangasarian) sufficiency conditions are not satisfied.

3.2.2 Sufficient Optimality Condition

Considering the analysis of Subsection 2.3.2 and the special forms of f and g the Riccati differential equation Eq. 2.9 is given by

$$\dot{p} = \frac{e^{rt}}{2c}p^2 - 2(1 - 3x^2)p - e^{-rt}(2 - 6\lambda x),$$

or by setting $q = e^{-rt}p$ we get

$$\dot{q} = \frac{q^2}{2c} - (2(1 - 3x^2) + r)q - (2 - 6\lambda x).$$

For the state control space Eq. 2.11 can be written as

$$\dot{p} = \frac{e^{rt}}{2c}p^2 - 2(1 - 3x^2)p - e^{-rt}(2 + 12cux), \tag{3.4}$$

or by the usual transformation

$$\dot{q} = \frac{q^2}{2c} - (2(1 - 3x^2) + r)q - (2 + 12cux). \tag{3.5}$$

3.2.3 Existence of Steady States

Considering the dynamical system

$$\begin{aligned} \dot{x} &= x - x^3 + u \\ \dot{u} &= u(r + 3x^2 - 1) + \tfrac{x}{c} \end{aligned} \tag{3.6}$$

the steady states must satisfy

$$\begin{aligned} u &= x^3 - x \\ u &= -\frac{x}{c(r + 3x^2 - 1)}. \end{aligned}$$

Setting these expressions for u equal to each other and assuming $x \neq 0$, we get

$$\begin{aligned} -1 &= c\left(r + \left(3x^2 - 1\right)\right)\left(x^2 - 1\right) \\ 0 &= c\left(3x^4 + (r-4)\,x^2 + 1 - r\right) + 1 \qquad \text{setting } y = x^2 \\ 0 &= c\left(3y^2 + (r-4)\,y + 1 - r\right) + 1, \end{aligned}$$

a quadratic in $y = x^2$, whose solutions are

$$y = \frac{4 - r \pm \sqrt{(r+2)^2 - 12/c}}{6}. \tag{3.7}$$

Abbreviating $w = \sqrt{(r+2)^2 - 12/c}$ the formal solutions of Eq. 3.7 together with $x = 0$ are

	$\hat{x}$	$\hat{u}$
E_1	0	0
E_2	$\sqrt{\frac{4-r+w}{6}}$	$-\sqrt{\frac{4-r+w}{6}}\left(\frac{r+2-w}{6}\right)$
E_3	$-\sqrt{\frac{4-r+w}{6}}$	$\sqrt{\frac{4-r+w}{6}}\left(\frac{r+2-w}{6}\right)$
E_4	$\sqrt{\frac{4-r-w}{6}}$	$-\sqrt{\frac{4-r-w}{6}}\left(\frac{r+2+w}{6}\right)$
E_5	$-\sqrt{\frac{4-r-w}{6}}$	$\sqrt{\frac{4-r-w}{6}}\left(\frac{r+2+w}{6}\right).$

Table 3.1: Steady states of the canonical system 3.6.

Comparing the x values of E_2-E_5 it is obvious that in absolute values the x coordinates of E_2 and E_3 are always larger then E_4 and E_5. This justifies referring to E_2 respectivly E_3 as outer steady states, wheras we refer to E_4 and E_5 as inner steady states. Furthermore it can be shown that for the outer steady states $\hat{x} < 1$ holds. Considering $w = \sqrt{(r+2)^2 - 12/c} < (r+2) \quad \forall c > 0$ we get

$$\hat{x} = \sqrt{\frac{4-r+w}{6}} < \sqrt{\frac{4-r+r+2}{6}} = 1,$$

which proofs the assertion.

Now we determine the regions of existence for the steady states. Considering Eq. 3.7 and noticing that $y = x^2$ the existence of steady states is determined by the two conditions $y \geq 0$ and $w \in \mathbb{R}$. These conditions can be written as

$$(r+2)^2 - 12/c \geq 0 \qquad (3.8)$$

$$4 - r \pm w \geq 0. \qquad (3.9)$$

Setting these conditions to zero, we get two boundary curves delimiting the regions to be considered. After a short calculation we get $c = 12/(c+2)^2$ and $c = 1/(r-1)$ from Eqs. 3.8-3.9. These two curves are tangent to each other at $r = 4$. Now every different case given by these solutions has to be analysed to prove the existence of the steady states. To cover every region with a different number of steady states we have to distinguish eight cases.

The solution is summarized in Tab. 3.2. In the following derivations we only consider the inner and outer steady states.

Case 1: $c < 12/(c+2)^2 \Rightarrow w \leq 0$ and therefore w is not real, and no steady states exist apart from the origin.

Case 2: From $c = 12/(c+2)^2$ and $r < 4$ we get $w = 0$. This implies that inner and outer steady states coincide.

Case 3: For $r = 4$ and $c = 1/3$ all steady states coincide with the origin.

Case 4: Considering $r > 4$ and $c < 1/(r-1)$ we see that $4 - r - w < 0$ and therefore we only have to find the sign of $4 - r + w$. But for that case the following equivalence relation holds

$$\begin{aligned} c < \frac{1}{r-1} \quad &\Leftrightarrow \quad 12r - 12 - \frac{12}{c} < 0 \\ &\Leftrightarrow \quad (r+2)^2 - \frac{12}{c} < r^2 - 8r + 16 \\ &\Leftrightarrow \quad w^2 < (r-4)^2 \qquad r > 4!! \\ &\Leftrightarrow \quad w < r - 4 \\ &\Leftrightarrow \quad 4 - r + w = y < 0, \end{aligned}$$

and so no real solutions for $y = x^2$ exist and therefore we have no steady states apart from the origin in this case.

Case 5: For $c > 12/(r+2)^2$ and $r \leq 1$ we get $w > 0$ and $w \leq \sqrt{3(3-4/c)} < 3 \leq 4 - r$ implying $4 - r \pm w > 0$, therefore all five steady states exist and are distinct.

Case 6: Assuming $c > 12/(r+2)^2$, $c < 1/(r-1)$ for $1 < r < 4$, yields $3 - 1/c > w > 0$ and $4 - r > 3 - 1/c$. Therefore $4 - r \pm w > 0$, and as in the latter case all five steady states exist and are distinct.

Case 7: For $r < 4$ and $c = 1/(r-1)$ the following equation holds

$$\begin{aligned} w &= \sqrt{(r+2)^2 - \frac{12}{c}} \\ &= \sqrt{r^2 - 8r + 16} \\ &= |r - 4| \\ &= 4 - r, \end{aligned} \tag{3.10}$$

this yields $y = 4 - r - w = 0$ and $y = 4 - r + w > 0$. Therefore the inner steady states coincide with the origin, while the outer steady states exist and differ from the origin.

Case 8: For $r > 4$ and $c = 1/(r-1)$ considering Eq. 3.10, $y = 4 - r + w = 0$ and $y = 4 - r - w < 0$ holds. This means the outer steady states coincide with the origin but no inner steady states exist.

Case 9: $c > 1/(r-1)$ implies $w > |r-4|$ which in turn implies $4 - r + w > 0$ and $4 - r - w < 0$. Therefore only the outer steady states exist.

	Character of Steady States at:			
Region/Curve	Origin	Inner Steady States	Outer Steady States	# of DNS
I	saddle	–	–	–
II a	saddle	unstable focus	saddle	–
II b	saddle	unstable focus	saddle	2p
IIIa	saddle	unstable node	saddle	–
IIIb	saddle	unstable node	saddle	2p
IIIc	saddle	unstable node	saddle	2t
IV	unstable node	–	saddle	1t

Table 3.2: Number and properties of steady states. See Fig. 3.1 for definitions of regions and bifurcation curves.
t denotes a DNS threshold.
p denotes a DNS point.

3.2.4 Stability Properties

Knowing the number of steady states for the different regions, we analyse now their stability properties. The characterization of the steady state behaviour ensues from calculating the determinante, trace and discriminant of the Jacobi matrix J.

We get the common form of J, by linearizing the system of differential equations Eq. 3.6

$$J(x,u) = \begin{pmatrix} 1 - 3x^2 & 1 \\ 6xu + 1/c & r + 3x^2 - 1 \end{pmatrix} \tag{3.11}$$

calculating Δ, τ and D gives

$$\begin{aligned}
\tau &= r \\
\Delta &= -(3x^2-1)(r+3x^2-1)-(6xu+1/c) \\
D &= r^2-4\left((3x^2-1)(r+3x^2-1)-(6xu+1/c)\right)
\end{aligned}$$

with

$$\begin{aligned}
\tau &\ \ldots\ \mathrm{tr}(J) \\
\Delta &\ \ldots\ \det(J) \\
D &\ \ldots\ \tau^2-4\Delta.
\end{aligned}$$

In the following subsections these formal results will be analysed for the different steady states.

Origin

At the origin the Jacobi matrix Eq. 3.11 simplifies to

$$J(0,0)=\begin{pmatrix} 1 & 1 \\ 1/c & r-1 \end{pmatrix}, \tag{3.12}$$

and so we get

$$\begin{aligned}
\tau &= r & (3.13)\\
\Delta &= r-1-1/c & (3.14)\\
D &= r^2-4r+4+4/c & (3.15)
\end{aligned}$$

The stability properties are completely determined by the signs of the three parameters Δ, τ and D. As $\tau = r > 0$ always holds we only have to consider the occurrence of $\Delta = 0$ and $D = 0$. Furthermore it can be seen immediatly that $D > 0 \quad \forall r, c > 0$. Therefore only equation $\Delta = 0$ has to be analysed.

We distinguish three cases for $\mathrm{sgn}(\Delta)$ and therefore three regions in parameter space.

Case 1: $\Delta < 0 \Leftrightarrow r < 1 + 1/c$ together with $\tau > 0$ and $D > 0$ characterizes a saddle point.

Case 2: $\Delta = 0 \Leftrightarrow r = 1 + 1/c$ gives the critical case of a non-isolated steady state at the origin.

Case 3: $\Delta > 0 \Leftrightarrow r > 1 + 1/c$ is associated with an unstable node.

Considering the division of the parameter space given in Fig. 3.1 we see that the origin is a saddle in regions I, II and III, a non-isolated steady state at γ, and an unstable node in region IV.

Inner Steady States

In case of an inner steady state the Jacobi matrix Eq. 3.11 becomes

$$J = \frac{1}{6}\begin{pmatrix} 3(r+w-2) & 6 \\ w^2 + 2(1-r)w - 2r - 8 + 6/c & 3(r-w+2) \end{pmatrix}$$

and

$$\begin{aligned} \tau &= r \\ \Delta &= \frac{1}{12}(-5w^2 + 4(4-r)w + (r+2)^2) - \frac{1}{c} \\ D &= \frac{1}{3}(5w^2 - 4\,(4-r)\,w + 2r^2 - 2r - 2) + \frac{12}{c}. \end{aligned} \tag{3.16}$$

Finding the regions where $\operatorname{sgn}(\Delta)$ differs we have to solve $\Delta = 0$, which implies

$$\begin{aligned} 0 &= \frac{1}{12}(-5w^2 + 4(4-r)w + (r+2)^2) - \frac{1}{c} \\ 0 &= \frac{1}{12}(-4w^2 + 4(4-r)w) \\ 0 &= w^2 - (4-r)w \\ 0 &= w(w - 4 + r) \end{aligned}$$

and therefore the following equations must hold

$$w = 4 - r \quad \text{or} \quad w = 0.$$

Since inner steady states only exist for $r \leq 4$ the inequality $w \geq 0$ holds and we get

$$\Delta = 0 \Leftrightarrow \begin{cases} c = \dfrac{1}{r-1} & r \leq 4 \\[2ex] c = \dfrac{12}{(r+2)^2} & \forall r. \end{cases}$$

Next we analyse the case $D = 0$. Starting from equation 3.16 we get

$$\begin{aligned} D &= \frac{1}{3}\left(5w^2 + 4(r-4)w + 2r^2 - 4r - 4 + \frac{12}{c}\right) \\ &= \frac{1}{3}\left(5w^2 + 4(r-4)w + 3r^2 - r^2 - 4r - 4 + \frac{12}{c}\right) \\ &= \frac{1}{3}\left(4w^2 + 4(r-4)w + 3r^2\right). \end{aligned}$$

This is a quadratic in w with solutions

$$w_{1,2} = \frac{1}{2}\left(4 - r \pm \sqrt{-2(r^2 + 4r - 8)}\right). \tag{3.17}$$

As $w \geq 0$ is demanded we have to make sure that the right side of Eq. 3.17 is real and positive. Finding the zeros of $r^2 + 4r - 8$

$$r_{1,2} = 2(\pm\sqrt{3} - 1),$$

it is obvious that the solutions of Eq. 3.17 for $0 < r \leq 2(\sqrt{3} - 1)$ are real. Next we have to show the positivity of 3.17. Since $r \leq 2(\sqrt{3} - 1)$ we get $r < 4$ and so positivity is given for the positive root. Therefore only the case of the negative root has to be analysed. But in this case we get the following equivalence

$$\begin{aligned} 4 - r - \sqrt{-2(r^2 + 4r - 8)} \geq 0 &\Leftrightarrow 4 - r \geq \sqrt{-2(r^2 + 4r - 8)} \\ &\Leftrightarrow 16 - 8r + r^2 \geq -2r^2 - 8r + 16 \\ &\Leftrightarrow r^2 \geq 0, \end{aligned}$$

which holds since $4 - r > 0$.

As we have found now the domain where equation Eq. 3.17 holds, we can start finding the solution curves explicitly. Replacing w by its definition yields to equation

$$\begin{aligned} \sqrt{(r+2)^2 - \frac{12}{c}} &= \frac{1}{2}\left(4 - r \pm \sqrt{-2(r^2 + 4r - 8)}\right) \\ 4\left((r+2)^2 - \frac{12}{c}\right) &= \left(4 - r \pm \sqrt{-2(r^2 + 4r - 8)}\right)^2 \\ 5r^2 + 32r - 16 - \frac{48}{c} &= \pm 2(4 - r)\sqrt{-2(r^2 + 4r - 8)}. \end{aligned}$$

Expressing c explicitly we get

$$
\begin{aligned}
c_{1,2} &= \frac{48}{5r^2+32r-16\pm 2(4-r)\sqrt{-2(r^2+4r-8)}} \\
c_{1,2} &= \frac{48\left(5r^2+32r-16\mp 2(4-r)\sqrt{-2(r^2+4r-8)}\right)}{33r^4+288r^3+672r^2-768} \\
c_{1,2} &= \frac{16\left(5r^2+32r-16\mp 2(4-r)\sqrt{-2(r^2+4r-8)}\right)}{(r+4)^2(11r^2+8r-16)}.
\end{aligned}
$$

These are rational functions continuously connected at $r = 2(\sqrt{3}-1)$ and can therefore be treated as one curve δ. As this curve is given by rational functions we have to find the singularities and the behaviour at these singularities. Setting $11r^2+8r-16=0$ gives the solutions

$$
-\frac{4}{11}\left(1\pm 2\sqrt{3}\right),
$$

as we are only interested in positive values, the only interesting solution is

$$
-\frac{4}{11}\left(1-2\sqrt{3}\right). \tag{3.18}
$$

Readily it can be seen that Eq. 3.18 is the only candidate for a singularity, because $(r+4)^2$ has no positive solution. Considering the two curves at this singularity it can easily be proven that in case of c_1 it is a real singularity, while in case c_2 the singularity can be lifted.

We have proved now, that the domain where δ exists is part of the domain where the inner steady states exist. Therefore δ is delimiting regions with different $\mathrm{sgn}(D)$ and therefore different stability properties, for the inner steady states. Evaluating D at points for the two different regions, we get $D>0$ inside region II and $D<0$ inside region III. Combining this result with Eq. 3.18 the inner steady states are unstable saddles in region III, unstable nodes in region II, and degenerated nodes on the curves δ. (See Tab. 3.2.)

Outer Steady States

Similar to the case of the inner steady states the Jacobi matrix Eq. 3.6 becomes

$$
J = \frac{1}{6}\begin{pmatrix} 3(r+w-2) & 6 \\ (w^2+2(1-r)w-2r-8)+6/c & 3(r+2+w) \end{pmatrix}
$$

and

$$\begin{aligned}
\tau &= r \\
\Delta &= \frac{1}{12}\left(-5w^2 - 4(4-r)w + (r+2)^2\right) - \frac{1}{c} \\
D &= \frac{1}{3}\left(5w^2 + 4(4-r)w + 2r^2 - 2r - 2\right) + \frac{1}{c}
\end{aligned}$$

Solving $\Delta = 0$ leads to

$$0 = \frac{1}{12}\left(-4w^2 + 4(r-4)w\right).$$

Therefore the following equations must hold

$$w = r - 4 \quad \text{or} \quad w = 0.$$

Since $w \geq 0$ has to be guaranteed we get

$$\Delta = 0 \Leftrightarrow \begin{cases} c \;=\; \dfrac{1}{r-1} & r \geq 4 \\[2ex] c \;=\; \dfrac{12}{(r+2)^2} & \forall r \end{cases}$$

Analogous to the case of the inner steady states the following equation for $D = 0$ holds.

$$\frac{1}{3}\left(4w^2 + 4(4-r)w + 3r^2\right) = 0$$

This is a quadratic in w with solutions

$$w_{1,2} = \frac{1}{2}\left(r - 4 \pm \sqrt{-2(r^2 + 4r - 8)}\right).$$

We have the same constraints on r as for the inner steady states. Therefore the right side is always negative, and the quadratic has no solution. This means D does not change sign for the outer steady states. Evaluating D for an arbitrary point shows $D > 0$. Summarizing the results of this subsection, the outer steady states are saddles in Regions II, III and IV as well as on the curves β and γ_1, and they are non isolated steady states at curves α and γ_2. (See Tab. 3.2.)

3.2.5 Regions of Stability

We next examine the steady states of the canonical system as functions of its two parameters r and c and determine their stability properties. As we can see in Fig. 3.1, the parameter space is divided into four regions, with different numbers of steady states and different stability properties. (See also Tab. 3.2.) The exact computations of the steady states and their properties for these regions can be found in Sections 3.2.3 and 3.2.4, respectively. The origin is always a steady state. There are in addition up to two pairs of steady states on either side of the origin. The two pairs are referred to as the inner and outer steady states based on their distance from the origin.

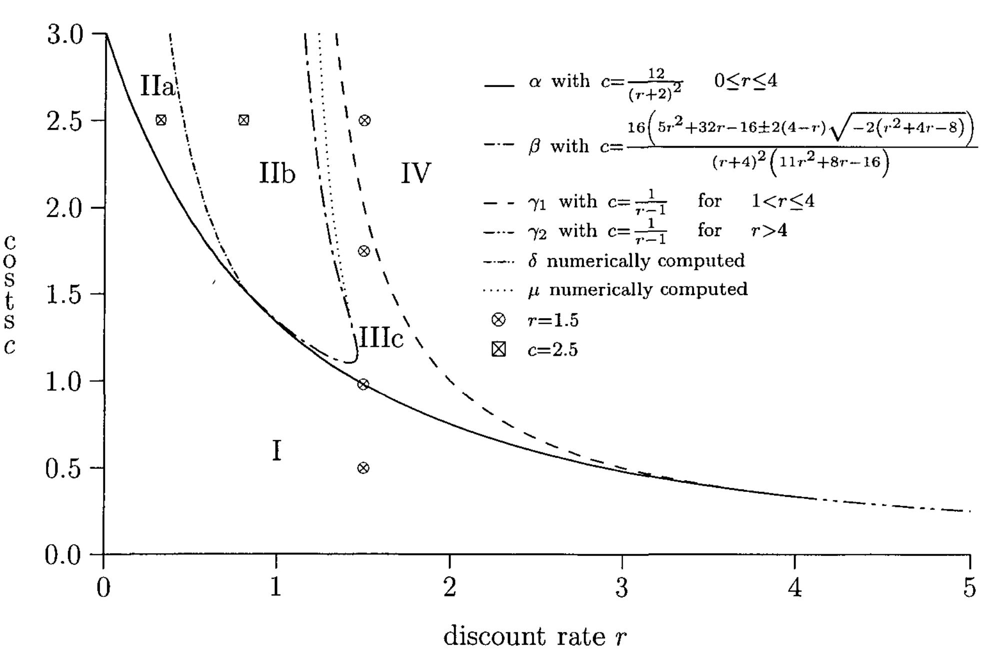

Figure 3.1: Regions of different stability and optimality divided by bifurcation lines α, β, γ and DNS bifurcation line δ and line μ. To avoid cluttering the figure Regions IIIa and IIIb are not marked in Fig. 3.1.
⊗ and ⊠ mark different positions of models depicted in Fig. 3.3 and Fig. 3.2.
⊠ mark different positions of models depicted in Fig. 3.2 and Fig. 3.4.

Region I

For parameters lying in Region I the only steady state is at the origin. This steady state is a saddle, and the region is delimited by the positive r and c axes and the curve (labeled α) defined by $c = 12/(r+2)^2$, for $r \leq 4$ and by $c = 1/(r-1)$ for $r \geq 4$.

Region II

In Region II there are five steady states. The origin and outer steady states are saddles, and the two inner steady states are unstable foci. The curve labeled β forms the boundary of Region II. It is defined by two rational functions whose exact formulae are found in Section 3.2.4. As will be seen shortly, Region II is split into two subregions with different behaviour of optimality.

Region III

Region III also has five steady states with saddles at the origin and two outer steady states, but the inner steady states have mutated into unstable nodes. Region III lies between the curves α, β and γ_1, where the latter is defined by $c = 1/(r-1)$ for $1 < r < 4$.

Region IV

The big difference between Region IV and the other regions is that the origin is an unstable node, not a saddle. No inner steady states exist, but the outer steady states remain saddles. Region IV lies above the curve γ, defined by $c = 1/(r-1)$ for $r > 1$.

3.2.6 Bifurcation Lines

These four regions are divided by bifurcation curves. (See Fig. 3.1.) Crossing these curves can mean a dramatic change in the system's behaviour. New steady states can emerge while others disappear or change their stability properties. As the possibility of such limiting cases is zero, they are of no vital importance for applications, but they nevertheless give insight into the mathematical formulations of radical changes in the model behaviour as parameters vary. At these lines, catastrophes can take place, and even models near such bifurcation lines can show some strange behaviour.

Bifurcation Line α

For models with parameters lying on this line, the origin is a saddle, and the inner and outer steady states coincide at non-isolated fixed points. When moving from Region I onto this line, the saddle at the origin in Region I trifurcates into one saddle and two non-isolated fixed points. Continuing by moving on into Region II, each non-isolated fixed point bifurcates further into a saddle and an unstable node.

Bifurcation Line β

This line separates region II, where the inner steady states are nodes, from region III, where the inner steady states are unstable foci. Lying on β they take on an intermediate state as degenerated nodes. Curve β lies above line α and they intersect at $r = 0$ and $c = 3$.

Bifurcation Line γ

Bifurcation line γ is split into γ_1 for $1 < r < 4$ and γ_2 for $r \geq 4$. On γ_1 the origin changes from a saddle to a non-isolated fixed point. This is the intermediate state between being a saddle in region III and an unstable node in region IV. The outer steady states are unchanged as saddles, while the inner steady states cease to exist at γ_1.

On γ_2 the origin is also a non-isolated fixed point, but no other steady states exist. To be exact the outer steady states coincide with the origin for $r > 4$, while for $r = 4$ all steady states coincide at the origin. So for this case no saddle exists and therefore the standard methods to find an optimal (extremal) solution are not applicable. Heuristically this strange behaviour can be understood by considering that γ_2 divides region I and IV which have very different optimal behaviour, as we will be elaborated in Section 3.3.

Bifurcation Line δ

Unlike to the bifurcation lines described so far, which split regions into parts with different stability properties, bifurcation line δ induces a change in the topology of the stable manifolds for the outer steady states and the origin. It is a so-called heteroclinic bifurcation line, where for parameters exactly on line δ the saddles are connected by heteroclinic paths. Such a bifurcation can indicate the existence of DNS points, as it does in our case (see, e. g. Wagener, 2003).

Bifurcation line δ was computed numerically by searching for heteroclinc connections. It separates Regions II and III into two parts with quite different optimal startegies as we will now elaborate.

3.2.7 Analysis of Local Optimality

In the following subsections the solutions for the Riccati differential equation Eq. 3.5 at the steady states and along extremal paths are analysed. As is shown in Section B.4 the solutions behaviour at steady states only depend on the steady states nature.

Origin

For $(x, u) = (0, 0)$ Eq. 3.5 reduces to

$$\dot{q} = \frac{q^2}{2c} + (r-2)q - 2.$$

Noticing that the origin is always a saddel or unstable node (see Tab. 3.2) it follows from the results of section B.4 that staying at the origin is always locally optimal.

Inner Steady States

As the inner steady states are given by

$$E_{2,3} = \left(\pm\sqrt{\frac{4-r-w}{6}}, \mp\sqrt{\frac{4-r-w}{6}} \left(\frac{r+2+w}{6} \right) \right),$$

where $w = \sqrt{(r+2)^2 - 12/c}$ Eq. 3.5 has to be written as

$$\dot{q} = \frac{q^2}{2c} + (2-w)^2 q - (2 - \frac{(4-r-w)(r+2+w)}{3}c).$$

In Region II the inner steady states are unstable foci, whereas they are unstable nodes in Region III (see Tab. 3.2). Therefore $E_{2,3}$ are only locally optimal at Region III.

Outer Steady States

As the outer steady states are given by

$$E_{4,5} = \left(\pm\sqrt{\frac{4-r+w}{6}}, \mp\sqrt{\frac{4-r+w}{6}} \left(\frac{r+2-w}{6} \right) \right),$$

where w is defined as before, the Riccati differential equation Eq. 3.5 is given by

$$\dot{q} = \frac{q^2}{2c} + (2+w)^2 q - \frac{1}{3}(6 - (4-r+w)(r+2-w)c).$$

Infering from Tab. 3.2 that the outer steady states are unstable nodes it is confirmed that persisting at the outer steady states is always locally optimal for all regions, where they exist.

Region I and IV

For Region I numerical integration of the corresponding Riccati differential equation for the only stabel manifold confirmed the local optimality of this solution.

The same holds true for Region IV where the local optimality for the two occuring stable manifolds has been established. Together with the result of Section 3.2.7 it has been proven that staying exactly at the origin is also locally optimal.

Region II and III

As well in Region II as in Region III we can distinguish two regions respectivly with different global optimal behaviour. Bifurcation line δ seperates these different regions. Whereas in Regions II/IIIa the stable manifold of the origin dominates the one of the outer steady states, we found a DNS threshold for Regions II/IIIb. Not surprisingly the local optimality of these solutions has been confirmed by numerical integration of the Riccati differential equation. But furthermore these numerical calculations showed that the stable manifolds of the outer steady states in Regions II/IIIa are also locally optimal. For Region IIa (Fig. 3.4a) this holds true als long as the stable manifold does not intersect the $\dot{x}$-isocline and a shock occurs (see Appendix C). For Region IIIa (Fig. 3.4b) the whole stable manifolds of the outer steady states up to the unstable nodes are locally optimal. These considerations correspond also to the results of 3.2.7, where we the local optimality of the unstable nodes has been shown, whereas the unstable foci even are not locally optimal.

At Region IIb the stable manifolds are locally optimal up to the point, where a shock occures on crossing the $\dot{x}$ isocline. For Region IIIb the local optimality of the extremal was confirmed.

Summarizing these results we have shown, that for Regions II and III, a whole interval exist, where different optimal solutions for (MoM) have been found. At least from a local point of view.

3.3 Interpretation of the Results

This section consists of three subsections. In the first subsection an interpretation of the global optimal strategies derived from the mathematical analysis of Section 3.2 is given. In the next subsection a more informal description of the change in optimal policies is given, when parameters vary. While in the last subsection the optimal strategies are reconsidered from a local point of view. In this section and the analogue sections for the other models for the interpretation of the occurence of DNS thresholds the property of continuous optimal policy is used. For critical remarks on the usage of this concept see Remark 5 in Appendix D.

3.3.1 Optimal Strategies

Having analysed the dynamic systems in terms of steady states and their properties, we next explore when various strategies are optimal. It turns out that there are essentially three strategies that may be optimal depending on the values of parameters r and c: (1) always move to the middle (origin), (2) (almost) always fall off to one side or the other, and (3) decide based on one's initial position.

The stability regions and bifurcation lines play an important role in defining when the various strategies are optimal, but the correspondence is not one for one. In particular, stability Region II, which is defined by two rational functions continuously connected at $r = 2(\sqrt{3} - 1)$, is divided into two subregions (Region IIa and Region IIb) by the bifurcation line δ. (See Section 3.2.4 for details.) So different strategies are optimal in different parts of a single stability region (namely Region II), and the same strategy may be optimal for different stability regions (e. g. , Regions I and IIa).

Strategy A: Always Move to the Middle

In stability Regions I, IIa and IIIa, it is always optimal to move to the origin, regardless of the starting position ($x(0)$). This makes intuitive sense because in these regions r and c are small. Hence the cost of exerting effort is modest (c small) and the low discount rate (r small) implies that the decision maker weighs the long-run future benefits of being in a position of moderation (at the origin) heavily relative to the short-run costs of exerting effort to get there.

Strategy B: (Almost) Always Fall Off to One Side or the Other

Stability Region IV represents the opposite case. If the starting position is exactly at the origin, it is optimal to stay there, balanced precariously between the attractions of aligning with the left or right positions. Otherwise, parameters r and c are so large that if the decision maker ever deviates even slightly from the origin, the decision maker is so short-sighted and the costs of control so great that the benefits of returning to the origin are not worth the effort. That does not mean that the optimal strategy is to be utterly passive. It is still optimal to exert some effort to slow the slide down the slippery slope, but not enough to alter the end result.

Strategy C: Move to the Middle if and only if One Starts Nearby

Regions IIb and IIIb/c present an intermediate case, whose prescription could be summarized "Maintain a position of moderation in the long run only if one initially holds a fairly moderate position". Or, as Polonius instructed Laertes, "To thine own self be true." If the decision maker's initial position is not too far from the origin, it is optimal to move back to the middle. But if the initial position is too far from the origin, moving to the origin is not worth the effort, although again some effort should be exerted to slow the slide. In between there are points of indifference, one on either side of the origin, from which the decision maker is equally happy moving left or right.

The character of the indifference points differs, however, in Regions IIb and IIIc. In Region IIIc where the inner steady states are unstable nodes and the stable manifolds do not intersect the $\dot{x}$-isocline, the indifference points occur at the inner (unstable) steady states, and the initial level of effort ($\nu^*(0)$) is only infinitesimally different whether one (arbitrarily) chooses to move left or right from that point.

In Region IIb/IIIb, where Region IIIb is separated by the lines β and μ, the indifference points do not necessarily correspond to the inner steady states. Furthermore, from the indifference point, if one chooses (arbitrarily) to move back to the origin the initial level of effort ($\nu^{\star}$) is noticeably greater (in absolute value) than is optimal if one chooses instead to move out to the outer steady states. In both cases, the indifference points are so-called DNS points (cf. Tragler, Caulkins, & Feichtinger, 2001).

3.3.2 Change in Optimal Strategy as Parameters Vary

This section examines in more detail how the optimal strategy varies as one of the two parameters in turn is increased.

Increasing c for Fixed Values of the Discount Rate r

Figure 3.3 shows phase portraits when $r = 1.5$ and the cost parameter c is $0.5, 12/3.5^2, 1.75$, and 2.5, respectively. When costs are low ($c = 0.5$; Fig. 3.3a) it is always optimal to move to the origin. As c increases further toward the bifurcation line α, the $\dot{x} = 0$ and $\dot{v} = 0$ isoclines begin to converge creating a bottleneck through which trajectories originating further outside have a hard time passing. This visually corresponds to the increasing cost of climbing up a slippery slope when one starts near the bottom.

When c reaches the bifurcation line ($c = 12/3.5^2$; Fig. 3.3b) the isoclines touch creating non-isolated fixed points. This is the critical case, where the origin is optimal for states starting inside the inner steady states and standard methods cannot be used to analyze cases outside this interval, because no saddles except the origin exist and therefore no extremal paths can be computed.

Increasing c beyond this bifurcation line splits the non-isolated fixed points into a saddle (outside steady state) and an unstable node (inside steady state), and the origin is only the optimal endpoint for initial conditions inside the indifference points, i. e. the inner steady states (Fig. 3.3c). Outside that range, the extreme positions (outer steady states) are optimal. As c increases further, the range of initial conditions for which it is optimal to converge to the origin shrinks until at bifurcation line γ, the inner steady states (thresholds) coincide with the origin, and the origin becomes a non-isolated fixed point, and the optimal strategy is to move out to the extremes unless one begins precisely on the origin.

Increasing c with r fixed at 0.5 gives slightly different results (figures not shown). The cost parameter can become very large before it is no longer optimal to always converge to the origin, and there is no critical change when crossing bifurcation curve α. From a heuristic point of view, this means the decision maker is far-sighted enough to want to reach the origin even when there are strong alternatives (the outer steady states exist). In mathematical terms the difference can be found in the relationship between the stable manifold and the emerging outer steady states. While for $r \geq 1$ these steady states lie on the stable manifold, that is not the case for $r < 1$. So driving to the origin remains always optimal until c reaches the DNS-bifurcation line δ. Numerical calculations show that there is no r small enough to make going to the origin universally optimal, without regard to both the cost parameter c and the initial position. Rather δ crosses the y axis. Extrapolation of the bifurcationline δ and direct calculations suggests this crossing point in about $c \approx 55$. (I.e., for high enough costs, the origin is never universally optimal.)

Increasing c with $r > 4$ gives a simple but quite different pattern. Strategy A (always going to the origin) is optimal for $c < \gamma$, and Strategy B (almost always not going to the origin) is optimal for $c \geq \gamma$. There is never an intermediate case (Strategy C) when $r > 4$. Thus, oddly, when one is very present-oriented ($r > 4$) one's initial state stops having any impact on the final outcome.

Increasing r for Fixed Values of the Cost Parameter c

When $c = 2.5$ and r is small it is always optimal to converge to the origin. That remains true until r reaches bifurcation line δ even though between lines α and δ two other sets of steady states emerge, because those steady states are bypassed by the optimal solution. (See Fig. 3.2a.) Between δ and β there are two DNS points, one on either side of the origin. For initial conditions between the DNS points it is optimal to converge to the origin; for initial conditions outside the DNS points, it is optimal to converge to the nearer of the two outer steady states. (See Fig. 3.2b.) Two things happen as r increases within these DNS regions (Region II and Region IIIb/c). First, the DNS points move inward. Second, the gap shrinks between the optimal initial control level when one moves right and the optimal initial control level when one moves left. That gap shrinks to zero when r reaches bifurcation line β that separates Region IIb, where the inner steady states are unstable foci, from Region IIIb/c, where they are unstable nodes. Decisive and dramatic actions that rapidly propel the decision maker toward the long run steady state are initially costly and, hence, are only recommended for far-sighted (smaller r) decision makers.

As r increases further in Region IIIb/c those inner steady states (unstable nodes) continue to converge until they collapse into the origin at bifurcation line γ_1. I.e., the decision maker becomes so present-oriented, that it is never worth working to get back to the origin or position of moderation.

When c is somewhat smaller (e. g. , $c = 1$) the sequence as r increases is similar except that crossing bifurcation line α takes one directly to the condition of two unstable foci (inner steady states) separating regions where it is optimal to converge to the origin from those where it is optimal to approach the outer steady states.

When c is very small such an intermediate region does not exist. To the left of γ it is always optimal to converge to the origin; to the right, for any initial position other than the origin, it is optimal to slide out to the outer steady states. Note that when control costs are extremely small, the γ curve occurs for very large values of the discount rate ($r > 4$) and so may not be practically relevant.

3.3.3 Local Optimal Strategies

Whereas in Section 3.3.1 the global optimal strategies were analysed, this section gives insight to the interpretation of the concept of local optimal strategies. The idea behind this concept was treated in Section 2.1. The three global strategies have their correspondence to the local optimal strategies. This means three different strategies depending on the values of parameters r and c can be distinguished: (A) move to the boundary (± 1), (B) (almost) always fall off from the boundary to the origin or a state outside the boundaries, and (C) move either to the origin or the boundaries depending on one's initial position.

For the global situation multiple optimal solutions only occur at singular starting positions $x(0)$, the so-called DNS points. Whereas for the local situation entire intervals (finite and infinite) exist where the decision maker can choose between multiple local optimal decisions.

Again the bifurcation lines form the borders of different optimal strategies. But as in the global case there is no one for one correspondence between regions and strategies. To avoid repetition for the formulation of the optimal strategies, the local optimal strategies are only explained in contrast to global optimal strategies.

Strategy A: Always Move to the Middle

Moving always to the middle, regardless of ones starting position $x(0)$ was now proven to be locally optimal only for Region I, contrary to the global strategies, where this behaviour was also optimal for Region IIa.

Strategy B: (Almost) Always Fall Off to One Side or the Other

No difference between local and global optimality. This strategy is optimal for the whole Region IV.

Strategy C: Move to the Middle if and only if One Starts Nearby

Not surprisingly this is the most interesting case, as there have been multiple optimal solutions (DNS points) even for the global optimal case. In Region IIa and IIIa moving to the middle is always a local optimal strategy. But in addition to that this strategy is only unique for an interval in the state space between the outer steady states at -1 and 1. This interval is given by the crossing points of the $\dot{x}$-isocline and the extremals (see Appendix C) converging to the outer steady states (see Fig. 3.4a). Outside this interval moving to the outer steady states gives another local optimal solution. This makes intuitively sense, for starting positions $x(0)$ near the outer steady states one has to exert a great effort to move back to the middle, even if this is optimal in the long run, one can choose the second best possibility and fall off to the left or right side. Also for starting positions outside the outer steady states a great effort is necessary to move back to the middle. And although the decision maker is far sighted enough (r small) to know that this would be optimal in the long run, one can choose the easier way and end at the outer steady states.

We find a similar situation in Region IIb. But contrary to the case before, for starting positions $x(0)$ outside the interval $[-1, 1]$ falling off to one side is the only optimal choice. Only for starting positions in an interval around the $x-$value of the unstable foci at the inner steady states, the two strategies, moving to the middle or falling off to one side, are locally optimal (see Fig. 3.4b). Within the global sight of optimality there was only one initial position, namely the DNS point, where one had two choices. Now there exists an entire interval where different local optimal solutions have been proven.

At Region IIIb these intervals shrink to a singular point (unstable nodes at the inner steady states) and the global optimal behaviour is equal to the local optimal strategy.

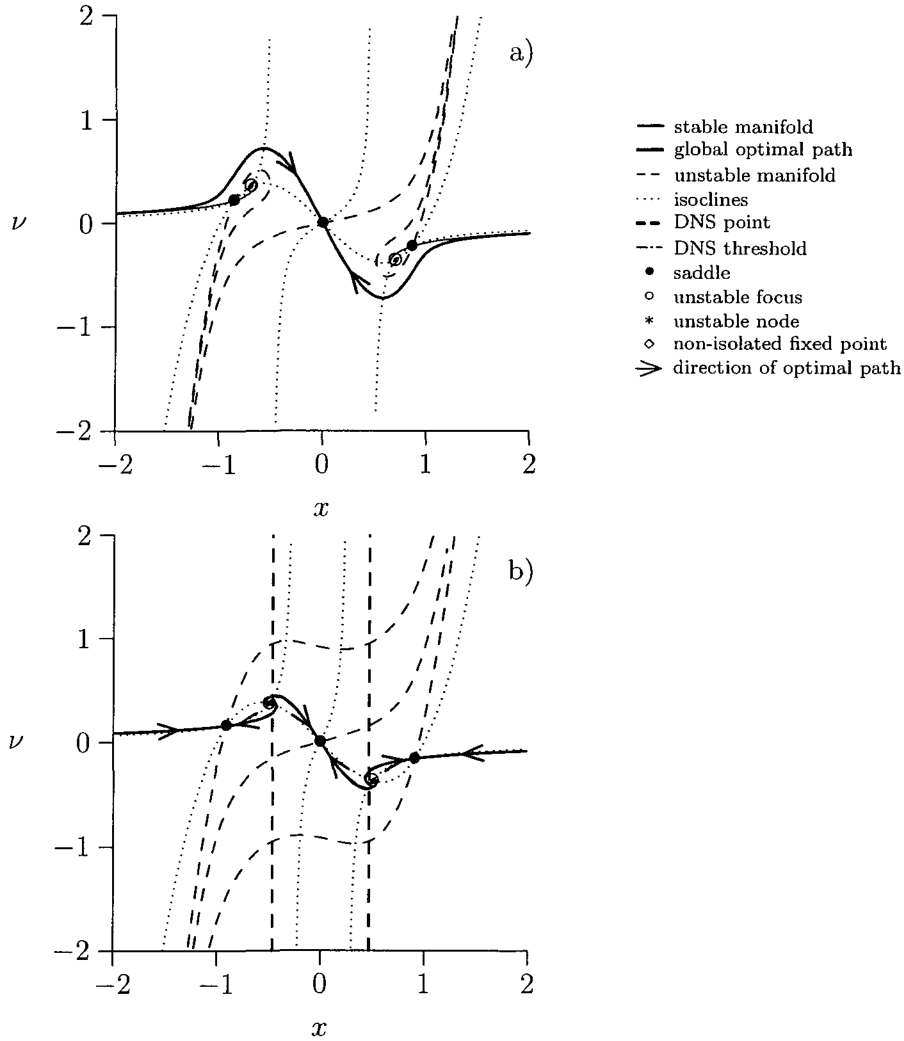

Figure 3.2: For constant cost $c = 2.5$ and different discount rates r the system dynamics is shown together with its optimal behaviour and direction. On the left (case a) $r = 0.32$ on the right (case b) $r = 0.8$.
The given caption is valid for all following figures describing the global optimal behaviour.

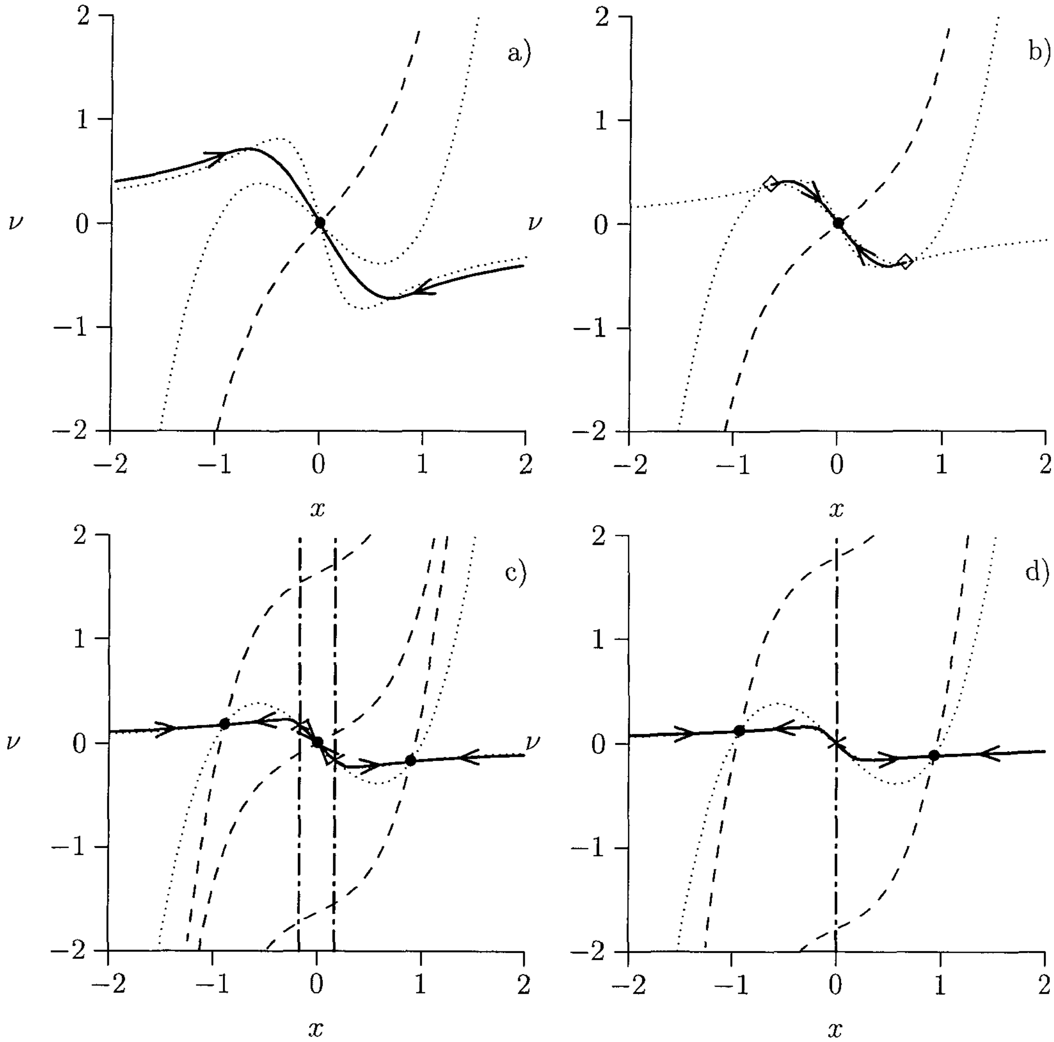

Figure 3.3: For constant discount rate $r = 1.5$ and different costs c the system dynamics is shown together with its optimal behaviour and direction, starting in the upper left and moving clockwise the four cost parameters are
a) $c = 0.5$ b) $c = 12/3.5^2$ c) $c = 1.75$ d) $c = 2.5$

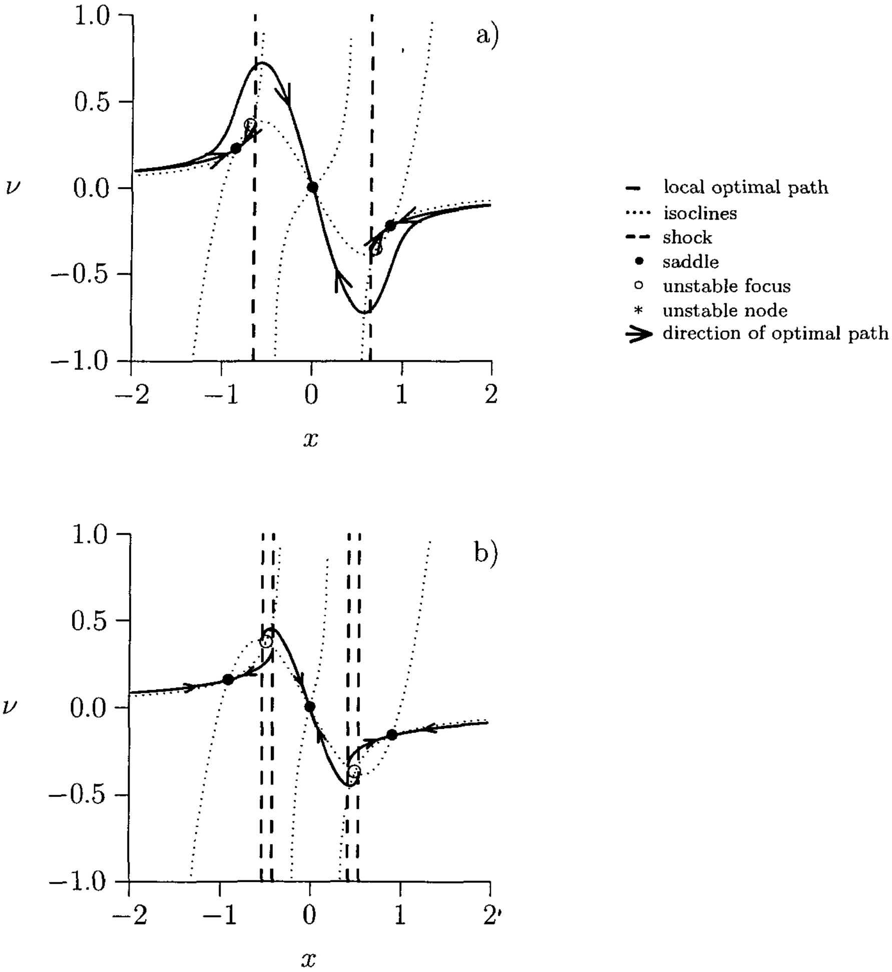

Figure 3.4: For constant cost $c = 2.5$ and different discount rates r the system dynamics is shown together with its local optimal behaviour and direction. On the left (case a) $r = 0.32$ on the right (case b) $r = 0.8$.
The given caption is valid for all following figures describing the local optimal behaviour.

Chapter 4

A Generalized Model of Moderation

This model introduces a generalized version of (MoM) presented in Chapter 3. On an interval the state dynamic and cost function are of the same shape as in (MoM). But as well the state dynamic as the cost function are periodically continued. As in (MoM) the optimal solutions are computed as functions of two extern parameters r the discounting rate and c a cost parameter. The optimal and dynamical behaviour of the solutions are classified for the whole parameter space. Analytical expressions of the occuring bifurcation lines allow the explicit identification of regions with different (local) optimal behaviour. Regions with different number of DNS points in the interval $[-\pi, \pi]$ have been identified and by periodicity we constructed a model with an infinite number of DNS points.

4.1 Motivation

In Chapter 3 the model (MoM) was motivated by the idea standing atop of a hill with "slippery" slopes on the left and right side. A quite natural generalization of this idea is to assume the existence of many hills along the x-axis. In an idealization an infinite number of hills ranging over the whole real line exist. In a further simplification all hills are assumed to be equally shaped.

Back to the interpretation of (MoM) a control variable u was introduced which stands for the effort to fight gravity to return atop if one slips down the steep slope. To measure the costs of exerting effort a costfunction was introduced. For our generalized model, having this interpretation in mind, we also define a cost function, but as we have an infinte number of equally shaped hills, this cost function is also periodic with the same period as the hills. The reason for that assumption is that costs for the exerting effort are assumed equal for every hill.

A simple model realizing this generalization, where the state variable x denoting the decision makers position, with $x = (2k+1)\pi$ where $k = k_0$ or $k = k_0 - 1$ and $k_0 \in \mathbb{Z}$ representing the positions of the two opposing sides (valleys) and $x = 2k\pi$ being the sought-for middle ground (top). There is tendency to be pulled down to the closest valley. This tendency could, however, be moderated by the exerting effort (control variable u) to adjust the state, but at a cost c. A simple functional form reflecting these properties would be

$$\begin{aligned} &\min_u \int_0^\infty e^{-rt}\left(1 - \cos x + cu^2\right) dt \\ &\text{s.t.} \quad \dot{x} = \sin x + u \\ &\text{and} \quad x(0) = x_0, \end{aligned} \tag{GMoM}$$

where $r > 0$ is a discount rate and parameter c governs the cost of adjusting one's position.

4.2 Analysis of the Model

This section consists of seven subsections. In the first two subsections the necessary and sufficient optimality conditions are formulated. After that the existence and properties of steady states are derived. In the following subsections the regions of stability and bifurcation lines are analysed. While in the last subsection the locally optimal solutions are summarized.

4.2.1 Necessary Optimality Conditions

For the special case (GMoM) the cost function g and state dynamic f are 2π-periodic functions

$$g(x) = 1 - \cos(x) \quad \text{and} \quad f(x) = \sin(x),$$

with derivatives

$$g'(x) = \sin(x) \quad \text{and} \quad f'(x) = \cos(x).$$

Following the usual optimal control analysis of Section 2.3 with the specified functional forms of f and g the current value Hamiltonian becomes

$$H(x, u, \lambda) = 1 - \cos(x) + cu^2 + \lambda\,(\sin(x) + u)$$

where λ denotes the costate variable in current value terms.
The state equation

$$\dot{x} = \sin x + u$$

and adjoint equation

$$\dot{\lambda} = r\lambda - H_x = \lambda\,(r - \cos(x)) - \sin(x).$$

or for control variable u

$$\dot{u} = u\,(r - \cos(x)) + \frac{\sin(x)}{2c},$$

yield the canonical system. The transversality condition Eq. 2.8 and the canonical system form the necessary optimality conditions for (GMoM).

Since the Hamiltonian H is not convex with respect to the state variable x, the usual (Mangasarian) sufficiency conditions are not satisfied.

4.2.2 Sufficient Optimality Conditions

For the special choice of the functions f and g we get the following Riccati differential equation derived in Section 2.3

$$\dot{p} = \frac{e^{rt}}{2c}p^2 - 2p\cos x - e^{-rt}(\cos x - \lambda \sin x),$$

or considering Eq. 2.11 for control variable u

$$\dot{p} = \frac{e^{rt}}{2c}p^2 - 2p\cos x - e^{-rt}(\cos x - \lambda \sin x).$$

For the usual transformation $q = e^{-rt}p$ these equations become

$$\dot{q} = \frac{q^2}{2c} + (r - 2\cos x)q - (\cos x - \lambda \sin x),$$

or in the state control space

$$\dot{q} = \frac{q^2}{2c} - (r - 2\cos x)q - (\cos x + 2cu \sin x). \tag{4.1}$$

4.2.3 Existence of Steady States

Considering the canonical system

$$\begin{aligned} \dot{x} &= \sin(x) + u \\ \dot{u} &= u\,(r - \cos(x)) + \tfrac{\sin(x)}{2c}. \end{aligned} \tag{4.2}$$

the steady states must satisfy

$$\begin{aligned} u &= -\sin(x) \\ u &= -\frac{\sin(x)}{2c\,(r - \cos(x))}. \end{aligned}$$

Setting these expressions for u equal to each other we get

$$\sin(x) = \frac{\sin(x)}{2c\,(r - \cos(x))}, \tag{4.3}$$

with the following solutions for x:

$$\begin{aligned} x_{5k+1} &= 2k\pi \quad k \in \mathbb{Z} \\ x_{5k+2,5k+3} &= 2k\pi \pm \arccos(r - \frac{1}{2c}) \quad k \in \mathbb{Z} \\ x_{5k+4} &= (2k+1)\pi \quad k \in \mathbb{Z} \end{aligned}$$

and therefore abbreviating $w = r - 1/2\,c^{-1}$ the formal solutions of the steady states are

Having in mind the interpretation of the states as top and valley we refer to steady states E_{5k+1} as top steady states, whereas the steady states E_{5k+4} are called valley steady states. As the steady states E_{5k+2} and E_{5k+3} lay between E_{5k+1} and E_{5k+4} they will be called inner steady states.

	$\hat{x}$	$\hat{u}$
E_{5k+1}	$2k\pi$	0
E_{5k+2}	$2k\pi + \arccos(w)$	$-\sqrt{1-w^2}$
E_{5k+3}	$2k\pi - \arccos(w)$	$\sqrt{1-w^2}$.
E_{5k+4}	$(2k+1)\pi$	0

Table 4.1: Steady states of the canonical system 4.2.

While E_{5k+1} and E_{5k+4} are global solutions for Eq. 4.3, whe have to determine the regions of existence for the steady states E_{5k+2} and E_{5k+3}. As they only depend on w we have to consider the case where $|w| \leq 1$ is fullfilled, implying

$$\left|r - \frac{1}{2c}\right| \leq 1 \Leftrightarrow \begin{cases} c \geq \dfrac{1}{2(r+1)} & \forall r \\ \\ c \leq \dfrac{1}{2(r-1)} & r > 1. \end{cases} \tag{4.4}$$

That is for c sufficing Eq. 4.4 the steady states E_{5k+2} and E_{5k+3} exist.

At least we determine the cases where these steady states coincide with other steady states.

Case 1: $w = 1 \Leftrightarrow c = 2/(r+1)$ the inner steady states coincide with the top steady states.

Case 2: $w = -1 \Leftrightarrow c = 2/(r-1)$ inner steady states and valley steady states coincide.

These results are summarized in Tab. 4.2.

4.2.4 Stability Properties

Knowing the regions of existence for the steady states, we now analyse their stability properties. The characterization of the steady state behaviour ensues from calculating the determinante, trace and discriminant of the Jacobi matrix J.

We get the common form of J, by linearizing the canonical system Eq. 4.2

$$J(x,u) = \begin{pmatrix} \cos(x) & 1 \\ u\sin(x) + \frac{\cos(x)}{2c} & r - \cos(x) \end{pmatrix} \tag{4.5}$$

calculating Δ, τ and D gives

$$\begin{aligned}
\tau &= r \\
\Delta &= \cos(x)r - \cos^2(x) - u\sin(x) - \frac{\cos(x)}{2c} \\
D &= r^2 - 4\cos(x)r + 4\cos^2(x) + 4v\sin(x) + 2\frac{\cos(x)}{c}
\end{aligned}$$

with

$$\begin{aligned}
\tau &\ \ldots\ \mathrm{tr}(J) \\
\Delta &\ \ldots\ \det(J) \\
D &\ \ldots\ \tau^2 - 4\Delta.
\end{aligned}$$

In the following paragraphs these formal results will be analysed for the different steady states.

Top Steady States

At the top steady states the Jacobi matrix Eq. 4.5 simplifies to

$$J = \begin{pmatrix} 1 & 1 \\ \frac{1}{2c} & r-1 \end{pmatrix}, \tag{4.6}$$

we get

$$\tau = r \tag{4.7}$$

$$\Delta = r - 1 - \frac{1}{2c} \tag{4.8}$$

$$D = r^2 - 4(r-1) + \frac{2}{c} \tag{4.9}$$

The stability properties are completely determined by the signs of the three parameters Δ, τ and D. As $\tau = r > 0$ always holds we only have to consider the occurrence of $\Delta = 0$ and $D = 0$. Solving these equations we get

$$\Delta = 0 \quad \Leftrightarrow \quad c = \frac{1}{2(r-1)} \tag{4.10}$$

$$D = 0 \quad \Leftrightarrow \quad c = -\frac{2}{(r-2)^2} < 0 \tag{4.11}$$

As $c > 0$ Eq. 4.11 is never fullfilled we only have to consider Eq. 4.10. Distinguishing the cases where the sign of Δ changes we get two regions in parameter space and a bifurcation line given by Eq. 4.10.

Case 1: $c > 0$ for $r \leq 1$ or $c < 1/2(r-1)$ for $r > 1 \Rightarrow D < 0$ characterizes a saddle.

Case 2: $c > 1/2(r-1)$ for $r > 1 \Rightarrow \Delta > 0 \wedge D > 0$ gives the case of an unstable node.

Valley Steady States

In case of valley steady states the Jacobi matrix Eq. 4.5 becomes

$$J = \begin{pmatrix} -1 & 1 \\ \frac{1}{2c} & r+1 \end{pmatrix}$$

and

$$\begin{aligned} \tau &= r \\ \Delta &= r + 1 - \frac{1}{2c} \\ D &= r^2 + 4(r+1) - \frac{2}{c}. \end{aligned} \tag{4.12}$$

Setting Δ and D to 0 we get the following equations

$$\Delta = 0 \quad \Leftrightarrow \quad c = \frac{1}{2(r+1)} \tag{4.13}$$

$$D = 0 \quad \Leftrightarrow \quad c = \frac{2}{(r+2)^2} \tag{4.14}$$

Combining the regions with different signs of Δ and D three distinctive regions are found.

Case 1: $c < \frac{2}{(r+2)^2} \Rightarrow \Delta > 0 \wedge D < 0$ characterizes unstable foci.

Case 2: $\frac{2}{(r+2)^2} < c < \frac{1}{2(r+1)} \Rightarrow \Delta > 0 \wedge D > 0$ and hencefor gives the case of unstable nodes.

Case 3: For $c > \frac{1}{2(r+1)} \Rightarrow \Delta < 0$ the valley steady states become saddles.

Inner Steady States

In case of inner steady states the Jacobi matrix Eq. 4.5 becomes

$$J = \begin{pmatrix} r - \frac{1}{2c} & 1 \\ -1 + r\left(r - \frac{1}{2c}\right) & \frac{1}{2c} \end{pmatrix}$$

and

$$\begin{aligned} \tau &= r \\ \Delta &= 1 - \left(r - \frac{1}{2c}\right)^2 \\ D &= r^2 - 4 + 4\left(r - \frac{1}{2c}\right)^2. \end{aligned} \tag{4.15}$$

Setting $\Delta = 0$ we find the solutions

$$\Delta = 0 \Leftrightarrow \begin{cases} c = \frac{1}{2(r-1)} \\ c = \frac{1}{2(r+1)} \end{cases}$$

Finding the solutions for $D = 0$ is straight forward finding the roots of the quadratic in c

$$D = 0 \quad \Leftrightarrow \quad c^2\left(5r^2 - 4\right) - 4rc + 1 = 0$$

becoming

$$c_{1,2} = \frac{4r \pm 2\sqrt{4 - r^2}}{2\left(5r^2 - 4\right)}. \tag{4.16}$$

These solutions c_1 and c_2 form two curves in the parameter space which are continuously connected at $r = 2$. As the polynom in the denominator of Eq. 4.16 has a positive real root at $r = \sqrt{4/5}$ the solutions have a singularity at this point, which can be lifted in the case of c_2, while it is a real singularity in the case of c_1.

Summarizing these considerations we can distinguish two different regions concerning the properties of the steady states, where Region IV denotes the enclosed region by the curves c_1 and c_2.

Case 1: For c lying in Region IV $\Delta > 0 \wedge D < 0$ entails that the inner steady states can be characterized as unstable foci.

Case 2: For c lying outside Region IV and $\frac{1}{2(r+1)} < c < \frac{1}{2(r-1)}$ we find $\Delta > 0 \wedge D > 0$ implying that the steady states are unstable nodes.

These results are summarized in Tab. 4.2.

4.2.5 Regions of Stability

In this subsection the different regions of stability for (GMoM) are considered. In Fig. 4.1 we can see that the parameter space is divided into five main regions with different stability properties of the occuring steady states. Furthermore in Fig. 4.2 and Fig. 4.3 characteristic phase portraits for these regions are depicted. In Tab. 4.2 the steady states and their stability properties are summarized.

Character of Steady States at:

Region	Top Steady S.	Valley Steady S.	Inner Steady S.	# of DNS in$[-\pi\ \pi)$
I	saddle	unstable focus	–	1p
IIa	saddle	unstable node	–	1p
IIb	saddle	unstable node	–	1t
IIIa	saddle	saddle	unstable node	1p
IIIb	saddle	saddle	unstable node	2t
IVa	saddle	saddle	unstable focus	1p
IVb	saddle	saddle	unstable focus	2p
V	unstable node	saddle	–	1t

Table 4.2: Number and properties of steady states. See Fig. 4.1 for definitions of regions and bifurcation curves.
t denotes a DNS threshold.
p denotes a DNS point.

Region I

Bifurcation line α given by the explicit formula $c = \frac{2}{(r+2)^2}$ and the r and c axes form the boundary of Region I. In this case the top steady states are saddles, wheras the valley steady states at $(2k+1)\pi,\ k \in \mathbb{Z}$ are unstable foci.

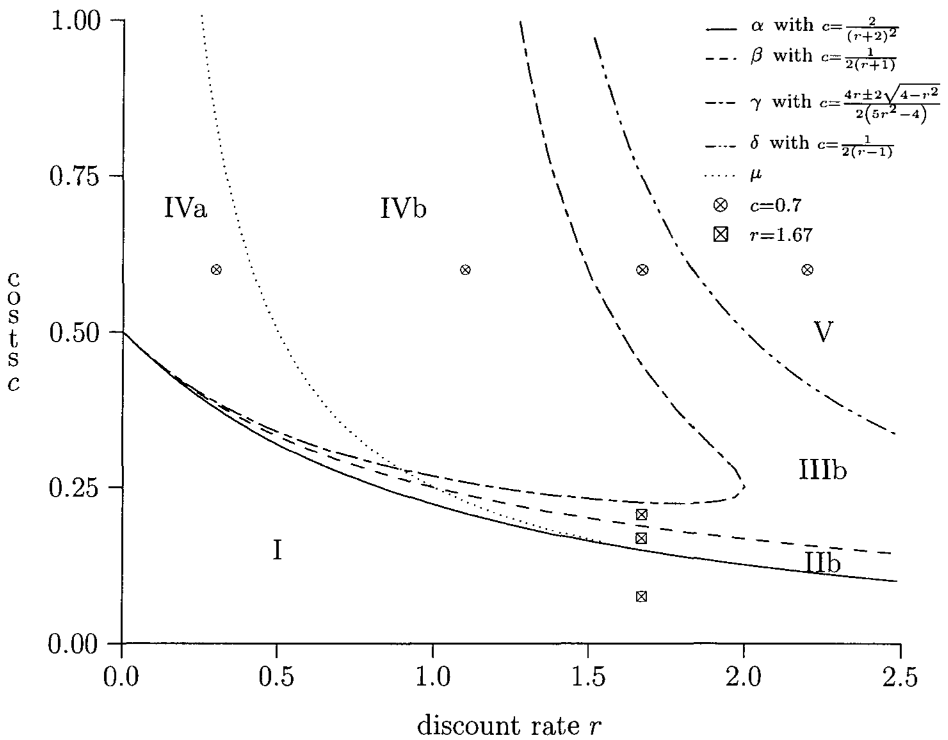

Figure 4.1: Regions of different stability and optimality divided by bifurcation lines $\alpha, \beta, \gamma, \delta$ and the heteroclinic bifurcation line μ.
Regions II and III are separated into two parts Region IIa and IIb as well as IIIa and IIIb by μ . To avoid cluttering the figure Regions IIa and IIIa are not marked in Fig. 4.1.
⊗ and ⊠ indicate different parameter sets for models depicted in Fig. 4.2 and Fig. 4.3.
⊗ mark different positions of models depicted in Fig. 4.2 and Fig. 4.4.
⊠ mark different positions of models depicted in Fig. 4.3 and Fig. 4.4.

Region II

Region II lying between bifurcation line α and bifurcation line β with $c = \frac{1}{2(r+1)}$ differs from Region I insofar as the valley steady states at $(2k+1)\pi,\ k \in \mathbb{Z}$ are now unstable nodes. The top steady states remain saddels.

Region III

The three bifurcation lines labled as β, γ and δ delimit Region III. The functional form for γ is $c = \frac{4r \pm 2\sqrt{4-r^2}}{2(5r^2-4)}$, whereas δ is given by $c = \frac{1}{2(r-1)}$. Inside this region all steady states exist, where the top steady states still remain saddles, the valley steady states now become saddles, and the inner steady states are further on saddles.

Region IV

Region IV is enclosed by bifurcation line γ and all steady states exist inside this region. Where the top steady states and valley steady states have the same properties as in Region III, the inner steady states change to unstable foci.

Region V

The last Region V lies above bifurcation line δ and for this region the inner steady states cease to exist. Furthermore that the top steady states are now unstable nodes, whereas the valley steady states are saddles.

4.2.6 Bifurcation Lines

These five regions are divided by bifurcation lines (see Fig. 4.1), which are now studied into more detail.

Bifurcation Line α

Crossing bifurcation line α from Region I to Region II leaves the saddels at the top steady states unchanged, whereas the unstable nodes at the valley steady states mutate from unstable foci to unstable nodes. Exactly at the curve α the valley steady states become degenerated nodes.

Bifurcation Line β

Moving from Region II to Region III we observe a saddle node bifurcation. Where the unstable nodes at $2k\pi,\ k \in \mathbb{Z}$ bifurcate into a saddle, which replaces the unstable node and an unstable node at the inner steady states, which come to existence. The saddles at the top steady states do not change.

Bifurcation Line γ

Bifurcation line γ is given by $c = \frac{4r \pm 2\sqrt{4-r^2}}{2(5r^2-4)}$. Crossing curve γ changes only the inner steady states form unstable nodes to unstable foci. Leaving the valley steady states and the top steady states unchanged.

Bifurcation Line δ

On crossing bifurcation line δ a saddle node bifurcation takes place. Where the inner steady states (unstable nodes) and top steady states (saddles) coincide exactly at the curve δ and the inner steady states cease to exist at Region V and the top steady states change their stability properties and become unstable nodes.

Bifurcation Line μ

Bifurcation line μ has different meanings in the various regionsof its occurence. In Region IV and V a heteroclinc bifurcation takes place on crossing curve μ and can therefor be called a heteroclinc bifurcation line. In Region II it seperates Region IIa, where the stable manifolds approaching the nodes at $(2k+1)\pi,\ k \in \mathbb{Z}$ (for reversed time) have an overlapping interval in common in projection onto the state space and Region IIb, where there is no such overlapping interval. As there exist no method describing the phenomena of heteroclinic bifurcations and the existene of overlapping intervals analytically, this bifurcation line has been computed numerically.

4.2.7 Analysis of Local Optimality

In the following sections we analyse the local optimal behaviour of the solutions for Eq. 4.2.2 when the extremals are the steady states. Furthermore the local optimal behaviour for the extremal of the different regions are described.

Top Steady States

For $(x, u) = (2k\pi, 0),\ k \in \mathbb{Z}$ Eq. 4.1 reduces to

$$\dot{q} = \frac{q^2}{2c} + (2 + r)q + 4.$$

As the top steady states are saddles or unstable nodes (see Tab. 4.2) it is always locally optimal to persist at these steady states.

Valley Steady States

In the case of valley steady states the Riccati differential equation is of the form

$$\dot{q} = \frac{q^2}{2c} + (r - 4)q - 8.$$

In Region I, the valley steady states are unstable foci and hencefor it is even not locally optimal to stay there. In all other regions the steady states are saddels or unstable nodes and therefore become local optimal solutions (see Tab. 4.2).

Inner Steady States

As the inner steady states are given by

$$E_{5k+2,5k+3} = \left(w, w(1 - w^2)\right),$$

where $w = \frac{\sqrt{3}}{3}\sqrt{1 + r - \frac{2}{c}}$, the Riccati differential equation is given by

$$\dot{q} = \frac{q^2}{2c} - (2(3w^2 - 1) - r)q - (12w^2 - 4 + -12cw^2(1 - w^2)).$$

Looking at Tab. 4.2 it can be seen that staying at the inner steady states is locally optimal for Region III (unstable nodes), whereas it is not for Region IV (unstable foci).

Region I and II

For Region I and Region II numerical calculations have shown the local optimality of the stable manifolds of the top steady states up to some threshold

$\pm x_c + 2k\pi,\ k \in \mathbb{Z}$. For Regions I and IIa this threshold corresponds to a shock, which occurs on crossing the $\dot{x}$- isocline. As can be seen in Fig. 4.4a/b in the interval $(-x_c + 2k\pi, x_c + 2k\pi),\ k \in \mathbb{Z}$ the extremals converging to the top steady states at the left and on the right are locally optimal. For Region IIb this intervall reduces to the state value of the unstable nodes at the valley steady states. Therefore the entire stable manifolds of the top steady states are local optimal but no overlapping interval, where two of them are locally optimal, exist.

It has to be noted that for the entire Region II the valley steady states are locally optimal.

Region III and IV

In Region III and Region IV as well the stable manifolds of the top steady states as the stable manifolds of the valley steady states have been numerically proven to be locally optimal at least up to some thresholds, given by the occuring shocks of the corresponding Riccati differential equation. Two different cases can be distinguished (see Fig. 4.4c and Fig. 4.4d). For Regions IIIa and IVa intervals enclosing the top steady states exist, where it is locally optimal to move either to the top steady states or the adjacent valley steady states. Furthermore there exist small regions outside these intervals, where it is not optimal to move to the top steady states but to move to one of the adjacent valley steady states. Outside these regions it is only optimal to move to the nearest valley steady state.

For Region IIIb no interval exist where three states are locally optimal in the long run. But there exist small intervals where the extremals converging to the top steady states or the nearest valley steady state is locally optimal.

Region V

The stable manifolds of the valley steady states have been proven to be locally optimal but no overlapping interval, where two extremals are together locally optimal, exist.

4.3 Interpretation of the Results

This section consists of three subsections, where in the first subsection the optimal strategies are analysed. In the next subsection the change of optimal strategies are studied, when the external parameters r and c are increased. In the last subsection the local optimal strategies compared to the global optimal strategies are treated.

4.3.1 Optimal Strategies

After having analysed the different regions of stability we shed light on the connection between the stability properties of the steady states and the resulting optimal strategies. We can distinguish between three different optimal strategies: (A) always move to the nearest top, (B) always fall off to an adjacent valley and (C) move to the top or fall off depending on one's initial starting position.

Although we can relate different regions to optimal strategies this correspondence is not one-to-one. There are regions with the same stability properties but different optimal strategies (e. g. Region IVa/b) or regions where the same strategy is optimal but with different stability properties (e. g. Region I and IVa). The heteroclinic bifurcation line μ plays a crucial role for these differences.

Strategy A: Always Move to the Nearest Top

In stability Regions I, II, IIIa and IVa it is always optimal to move to the nearest top, regardless of the starting position ($x(0)$). In all of these regions except Region IIb we have DNS points at the valley steady states, where one can choose to move to the left or right top steady state. This makes sense as r and c are small and as the state dynamics and cost function are periodic and moving to the left or right is symmetric.

In Region IIb the optimal policy is continuous at the valley steady states and hencefor the state values of the valley steady states become DNS thresholds.

Strategy B: (Almost) Always Fall Off to Adjacent Valleys

In Region V we find the opposite to strategy A. In this case it is never optimal to move to the top. Only in the hairline case when one starts exactly at the top it is optimal to stay there forever. For every other starting position it is not worth the effort to move back, but falls off to the nearest valley steady state. In this case the top steady states are DNS thresholds.

Strategy C: Move to the Top if and only if One Starts Nearby

Strategy C is the case inbetween the foregoing strategies A and B. It is related to Region IIIb and IVb. For this strategy the initial starting position $x(0)$ is crucial. There exist a threshold separating regions in the state space with different optimal behaviour. If one starts between the valley steady states and the threshold it is optimal to move to the valley steady states, whereas it is optimal to move to the top if one starts between the threshold and the top. For Region IVb the threshold is a real DNS point, whereas in Region IIIb starting at a valley steady state means staying there forever and hencefor these points are DNS thresholds.

4.3.2 Change in Optimal Strategy as Parameters Vary

This section examines in more detail how the optimal strategy varies as one of the two parameters in turn is increased.

Increasing c for Fixed Values of the Discount Rate r

Figure 4.2 shows the phase portraits for fixed $r = 1.7$ and varying c. If c is small ($c < 1/3.7^2$) than moving to the top steady states is always optimal (see Fig. 4.2a). Increasing c above the threshold $c = 2/3.7^2$ does not change the optimal behaviour except for the hairline case, without any practical interest, that one starts exactly at the x-state of one of the boundary steady states. As the optimal policy is continuous at these steady states, one has to stay at this steady state, if one starts there (see Fig. 4.2b). Letting c increase furthermore above $c = 1/2.7$ a saddle node bifurcation takes place and the valley steady states bifurcate into a saddle and an unstable node respectively. Now the costs c are thus high, that moving to the top steady states is not always optimal. For initial starting positions $x(0)$ between the valley steady states and the x-value of the unstable node at the inner steady states the optimal policy is moving back to the valley steady states (see Fig. 4.2c). Only for starting positions between two successive unstable nodes it is optimal to move to the top steady states. As the optimal policy is continuous at the unstable nodes, it has to be considered that one has no choice if starting exactly at the x-value of the unstable nodes. One has to remain there forever. Increasing c further on the unstable nodes loose this status and the optimal policy becomes discontinuous (see Fig. 4.2d). This is the classical case of a DNS point. Where the decision maker is indifferent choosing to move to the nearest valley steady state or back to the top steady states. Both strategies are equally optimal. The interval, where it is optimal to move to the top steady states shrinks with increasing c until bifurcation line δ is crossed. There a saddel node bifurcation takes place and the saddles at the top steady states are replaced by unstable nodes. This is the case, where falling off to one side or the other is always optimal (see Fig. 4.3d) except when starting exactly at the top steady states.

For fixed but smaller r it has to be noted, that the optimal policy becomes discontinuous at the valley steady states on crossing bifurcation line α, contrary to the case described above. Furthermore the optimal policy, of moving always back to the top steady states does not change even c is increased further on and one crosses bifurcation line β. Only when c is increased above bifurcation line μ DNS points emerge and moving to the valley steady states becomes optimal too if one starts nearby. If $r < 1$ this optimal policy does not change with increasing c. Indeed the interval, where it is optimal to move to the top steady states, shrinks but remain of finit length.

Increasing r for Fixed Values of the Cost Parameter c

Figure 4.3 shows phase portraits for different discount rates r if the costs c are held fixed at 0.6. If the discount rate r is small, that is if future costs play an important role for the current decision, it is always optimal to move to the nearest top steady states (see Fig. 4.3a). If one starts exactly at the a valley steady state one is indifferent to choose the top steady state lying on the left or lying on the right. Therefore the valley steady states are DNS points. If r is increased and the decision maker is getting more myopic, the gap at the valley steady states for the effort moving to the left or right top steady states shrinks until it coincide exactly at the heteroclinic bifurcation line μ (see Fig. 4.3b). Increasing r further leads to a bifurcation of the DNS points at the valley steady states. For starting values near the valley steady states, inside an interval given by the DNS points to the left an rigth side of the valley steady states, it is now optimal to move back to the valley steady states. Whereas outside this interval it is optimal to move to the nearest top steady state (see Fig. 4.3c). Letting r increase further on the interval, where it is optimal to move to the top steady states shrinks, until the inner steady states coincide with the top steady states (r lying on bifurcation line δ). This collision leaves unstable nodes at the top steady states behind, and now falling off to one or the other side is the only optimal choice.

This kind of optimal behaviour does not change even for smaller or larger costs c.

4.3.3 Local Optimal Strategies

Having analysed the global optimal strategies of the model from a descriptive (Sec. 4.3.1) and a dynamic (Sec. 4.3.2) point of view, we now consider the concept of local optimality. Within this concept we detect an even finer substructure of optimal strategies. This structure has also been observed for the global strategies. But within global strategies this details have only been hairline cases, like continuous or discontinuous optimal policies at unstable nodes, without importance for real applications. Within the locally optimal framework some effects can now be found along entire intervals.

Nevertheless the main strategies remain unchanged, which are: (A) always move to the nearest top, (B) always fall off to the adjacent valley and (C) move to the top or fall off.

But now some regions, e. g. Region IVa, do not longer belong to the same strategy as they have done using global optimality. Others remain completely unchanged within the two concepts (e. g. Region V).

Strategy A: Always Move to a Top

Contrary to global strategy A, where it was always optimal to move to the nearest top in Region I and IIa, this is only true for local strategies near a top. Because starting inside an interval enclosing the x-value of an valley steady state, one has the choice to move to a top steady state at the left side or at the right side (see Fig. 4.4a). Both strategies are locally optimal. Note that the unstable nodes at the valley steady states of Region IIa are also local optimal strategies (see Fig. 4.4b).

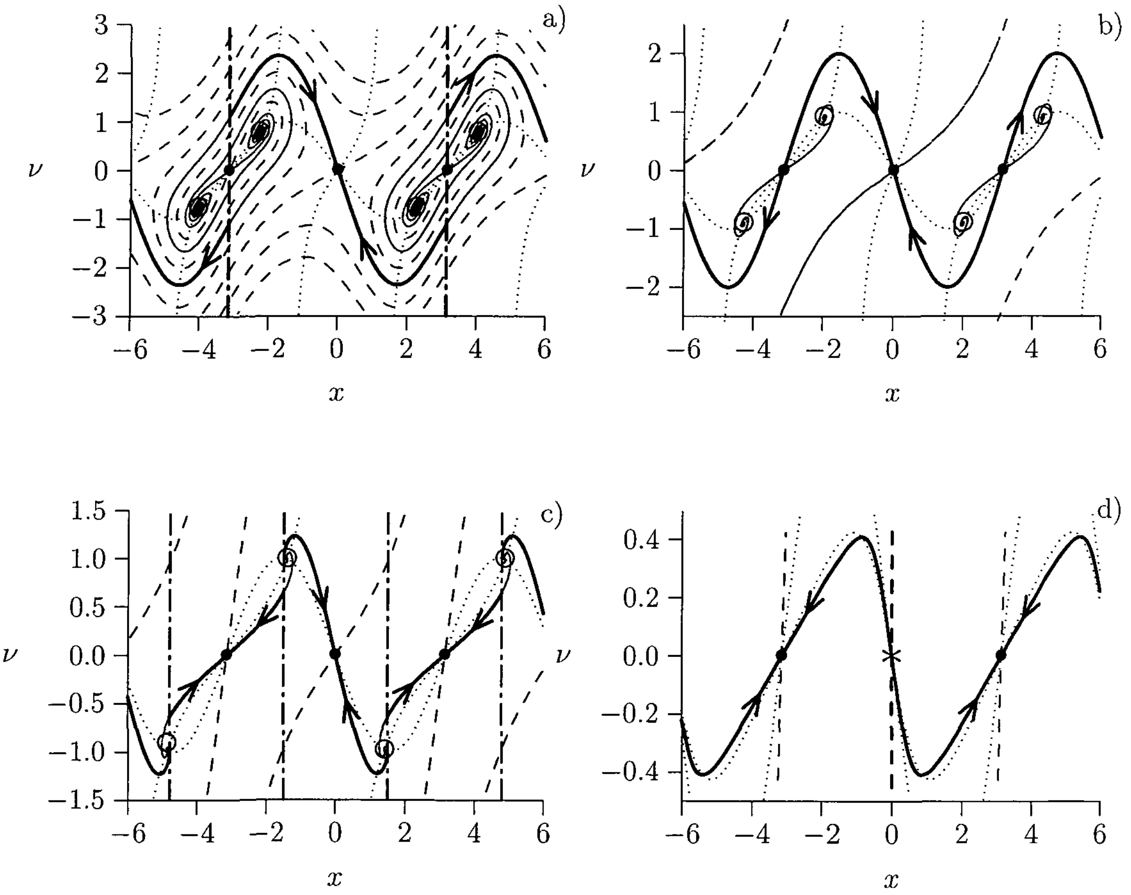

Figure 4.2: For constant cost $c = 0.6$ and different discount rates r the system dynamics is shown together with its optimal behaviour and direction, starting in the upper left and moving clockwise the four cost parameters are
a) $r = 0.2$ b) $r = 0.4$ c) $r = 1.67$ d) $r = 2.2$

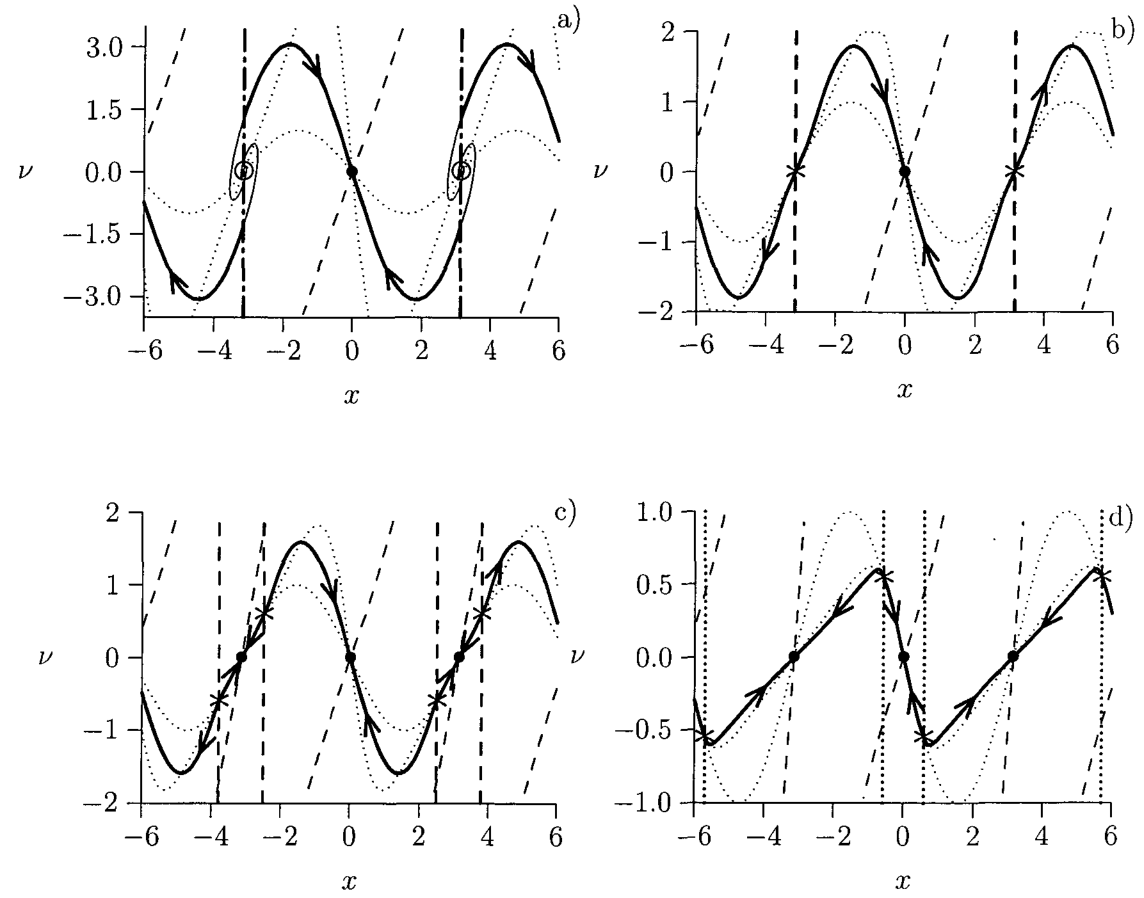

Figure 4.3: For constant discount rate $r = 1.7$ and different costs c the system dynamics is shown together with its optimal behaviour and direction, starting in the upper left and moving clockwise the four cost parameters are
a) $c = 0.08$ b) $c = 0.17$ c) $c = 0.2$ d) $c = 0.6$

Strategy B: (Almost) Always Fall Off to One Side or the Other

No change to the global optimal strategy.

Strategy C: Move to A Top Or Fall Off

As this strategy is more detailed for locally optimal strategies we distinguish two substrategies.

Substrategy C'

In Region IIIa and IVa the local optimal strategy for staying at the unstable node of Region IIa, is extended to an entire interval I_1, bounded by the shocks of the corresponding Riccati differential equation along the extremal paths leading to the valley steady states (see Fig. 4.4c). Moreover an interval I_2 containing I_1, bounded by the shocks appearing along the extremal paths leading to the top steady states, exists, where it is locally optimal to move to the top steady states on the left or right side. Therefore inside intervals I_1 one has three possible choices, each of which are locally optimal, namely moving to the valley steady state or moving to the left or to the right top steady state.

Substrategy C": Move to the Nearest Top or Fall Off

In Region IVb and IIIb the phase portrait change qualtitativly and neither interval I_1 nor interval I_2 exist in this case. Nevertheless an interval enclosing the x-value of the inner steady states can be observed, which is bounded from the left and right by shocks occuring along extremals leading to the valley steady states (see Fig. 4.4d). Starting inside this interval one can choose between moving to the nearest top or moving to the nearest valley steady state. Remaining only Region IIIc, where this interval of multiple locally optimal solutions is reduced to a single point at the inner steady states. In this case no multiple optimal choices exist. If one starts left of the inner steady states moving to the nearest valley steady state is optimal, otherwise moving to the top is optimal.

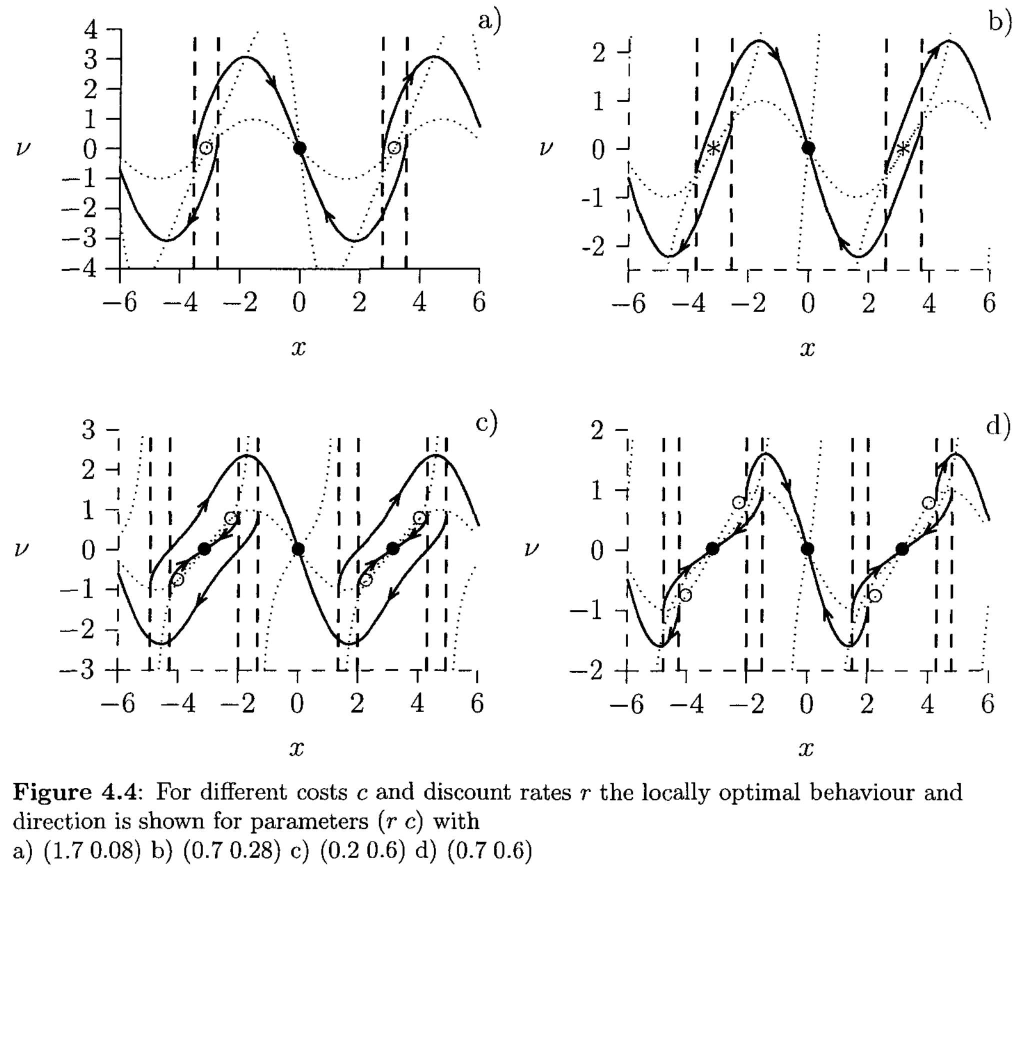

Figure 4.4: For different costs c and discount rates r the locally optimal behaviour and direction is shown for parameters (r c) with
a) (1.7 0.08) b) (0.7 0.28) c) (0.2 0.6) d) (0.7 0.6)

Chapter 5

A Model of Bridge Building

A simple optimal control model is introduced, where "bridge building" positions are rewarded. The optimal solutions can be classified in regards of the two extern parameters, (1) costs for the control staying at such an exposed position and (2) the discount rate. A complete analytical description of the bifurcation lines in parameter space is derived, which seperates regions with different optimal behaviour. These are resisting the influence from inner and outer forces, always fall off from the boundaries or decide based on ones's initial state. This latter case gives rise to the emergence of DNS points and thresholds respectively describing optimal solution strategies. Furthermore the bifurcation from a single DNS point into two DNS points has been analysed in parameter space. Within the concept of local optimality regions with up to three different local opti,al strategies have been identified. All these strategies have a funded interpretation within the limits of the model.

5.1 Motivation

Connectors that tie together disparate objects are often under stress, but they are crucially important, whether the objects are physical or social. Welding joints and metal fasteners are common failure points in mechanical structures, and they are appropriately the focus of design effort. Social networks can be similar.

People who straddle two groups or organizations may be pulled in competing directions, but they can also exploit their position to control information flows and create value. For example, they can market their home organization to the outside world and serve as a conduit for ideas and information flowing into the organization from the outside. It takes effort to maintain connections with diverse audiences; it is usually easier to establish relationships with people who are similar and to maintain relationships with people one sees routinely than to do so with outsiders. Yet people who manage to be bridge builders are rewarded for their special position.

Social network researchers have found that being "in the middle" of an organization (known as "centrality") confers advantages, including power (Krackhardt, 1990; Brass & Burkhardt, 1992). They distinguish among (at least) three types of centrality (Freeman, 1979). "Degree centrality" refers to the number of people to whom one is connected. "Closeness" is self-explanatory; individuals connected to many others by relatively direct paths, with few intermediaries, score high on closeness measures. "Betweenness" centrality refers to the extent to which an actor falls between pairs of other actors on the shortest paths (geodesics) connecting them.

The focus here is on betweenness. Its value is intuitive. If one person is the sole connection between two others, that intermediary has unique bargaining power with respect to any beneficial exchanges among those so connected. In effect, he or she has a monopoly over brokerage services between those powers. When the intermediary connects not just individuals but distinct groups each with multiple internal connections but with no overlap between groups the bridging person benefits all the more.

This idea is at least as old as Medieval Venice profiting by connecting Western Europe with the Orient, but in modern social network theory it is closely associated with Mark Granovetter's classic (1973) article on "The Strength of Weak Ties." Subsequent authors (e. g. , Burt, 1992) have argued that the key is not that the ties are weak, but that they be nonredundant "information bridges" that overcome "structural holes" in the organizational network. "Information benefits are expected to travel over all bridges, strong or weak. ... The task for a strategic player building an efficient-effective network is to focus resources on the maintenance of bridge ties." (Burt, 1992, p.75)

This paper introduces a very simple model that reflects the challenges and benefits of building bridges by standing at the edge of one's home organization and reaching out to the external world. It describes when various professional strategies are preferred as a function of one's level of patience (discount rate) and the cost of adjusting one's social position. For many sets of parameter values, the solution is characterized by DNS points and thresholds. In particular, as the rate of time preference varies, a single DNS point bifurcates into two, a phenomenon not previously observed in applied model of this sort.

The Model

Consider an individual who exerts effort to create value for an organization. The organization is abstracted as a ball of unit radius. The effort-minimizing path is for individuals within the organization to interact with and build relationships with others within the organization and for people outside the organization to interact with and build relationships with others outside the organization. So, in the absence of conscious effort, people outside the organization will tend to lose touch with what the organization is doing and people inside the organization will become more and more inwardly focused (cf, DeGroot, 1974).

In such circumstances, organizations can become too incestuous, recycling ideas that were "invented here" and overlooking developments in the wider world. So it can be valuable for some people to stand at the "edge" of the organization, connected to it but also strategically positioned as a bridge between the organization and the outside world.

For simplicity, assume the individual optimising his or her position vis a vis the organization does so along a single dimension x. Generalizations to multiple dimensions would be of interest, but even this one-dimensional case proves insightful. Let the origin denote the "center of gravity" of the organization on this dimension and -1 and 1 denote its boundaries. As a further simplification, we will consider here a case in which the organization is symmetric about its center of gravity, but asymmetric cases could also be considered.

Our model of the natural evolution of social interaction is that the boundary is unstable. People within the organization gravitate toward its center. Those outside it are drawn toward other organizations and activities. Thus, the uncontrolled state dynamics might be take a form such as:

$$\dot{x} = x^3 - x,$$

so $\dot{x} = 0$ at both boundaries, the state converges toward the origin for $|x| < 1$, and it diverges for $|x| > 1$.

The individual can modify this trajectory in either direction by exerting some effort, denoted by the control variable u. It is conventional to assume that costs are a convex function of effort, and we will assume a quadratic dependence for simplicity. Ideally the individual would like to stand on the boundary between the organization and the outside world. We presume no distinctive benefit to being on the left-hand boundary vs. the right-hand boundary, but do assume that it is better for an individual to be a little too "close" to his or her own organization than a little too far. That is, we are imagining a situation in which the individual is an "employee" or otherwise receives compensation from the organization that is centered at the origin, so the individual is better off being "inside" the organization's boundary rather than a similar distance outside the boundary. Perhaps the simplest cost function satisfying these considerations is $(x^2 - 1)^2$, so our overall optimisation problem becomes

$$\begin{aligned} &\min_u \int e^{-rt} \left(\left(x^2 - 1\right)^2 + cu^2 \right) dt \\ &\text{s.t.} \quad \dot{x} = x^3 - x + u \\ &\text{and} \quad x(0) = x_0, \end{aligned} \tag{MoBB}$$

where r is the individual's discount rate, and c is a positive constant reflecting the cost of adjusting one's position.

5.2 Analysis of the Model

Analogous to the analysis of the preceding models this section is parted into the following subsections. First the usual necessary optimality conditions and the Riccati differential equation as a sufficient second order condition are stated. After that the models steady states are analysed. In the sequel the regions of stability seperated by the bifurcation lines are summarized. Whereas in the last subsection the second order conditions are studied for extremal paths.

5.2.1 Necessary Optimality Conditions

To solve model (MoBB) we follow the outline given in Section 2.3 for the special functions f and g given by

$$g(x) = \left(x^2 - 1\right)^2 \quad \text{and} \quad f(x) = x^3 - x,$$

and derivatives

$$g'(x) = 4x\left(x^2 - 1\right) \quad \text{and} \quad f'(x) = 3x^2 - 1.$$

Substituting these functions in Eq. 2.3 of Section 2.3 the current value Hamiltonian is

$$H(x, u, \lambda) = \left(x^2 - 1\right)^2 + cu^2 + \lambda(x^3 - x + u),$$

where λ denotes the costate variable in current value terms.

Following the derivations of Section 2.3 it can be seen that the adjoint equation is written as

$$\dot{\lambda} = r\lambda - H_x = \lambda\left(1 - 3x^2 + r\right) - 4x\left(x^2 - 1\right).$$

while for the the differential equation of the control variable u we get

$$\dot{u} = u\left(1 - 3x^2 + r\right) + \frac{2x}{c}\left(x^2 - 1\right).$$

Equation 5.2.1 or Eq. 5.2.1 and state dynamic

$$\dot{x} = x^3 - x + u$$

yield the canonical system. Together with the transversality condition Eq. 2.8 this canonical system give the necessary conditions for the optimal control problem (MoBB).

Since the Hamiltonian is not convex with respect to the state variable, the usual (Mangasarian) sufficiency conditions are not satisfied.

5.2.2 Sufficient Optimality Conditions

As was stated in Section 2.3 the second order condition demands the solvability of a Riccati differential equation Eq. 2.11. Substituting f and g in Eq. 2.11 the Riccati differential equation is given as

$$\dot{p} = \frac{e^{rt}}{2c}p^2 - 2(3x^2 - 1)p - e^{-rt}(12x^2 - 4 + 6\lambda x).$$

When using the control variable u this equation can be written as

$$\dot{p} = \frac{e^{rt}}{2c}p^2 - (2(3x^2 - 1) - r)p - e^{-rt}(12x^2 - 4 - 12cux).$$

Or by setting $q = e^{-rt}p$ we get

$$\dot{q} = \frac{q^2}{2c} - (2(3x^2-1) - r)q - (12x^2 - 4 - +6\lambda x),$$

and for control variable u

$$\dot{q} = \frac{q^2}{2c} - (2(3x^2-1) - r)q - (12x^2 - 4 - 12cux). \tag{5.1}$$

5.2.3 Existence of Steady States

Considering the dynamical system

$$\begin{aligned} \dot{x} &= x^3 - x + u \\ \dot{u} &= u\,(1 - 3x^2 + r) + \frac{2x}{c}\,(x^2 - 1)\,. \end{aligned} \tag{5.2}$$

the steady states must satisfy

$$\begin{aligned} u &= x - x^3 \\ u &= \frac{2x\,(1-x^2)}{c\,(1 - 3x^2 + r)}. \end{aligned}$$

Setting these expressions for u equal to each other we get the following solutions for x:

$$\begin{aligned} x_1 &= 0 \\ x_{2,3} &= \pm 1 \\ x_{4,5} &= \pm\sqrt{\frac{1 + r - \frac{2}{c}}{3}}, \end{aligned}$$

and therefore abbreviating $w = \sqrt{\frac{1+r-\frac{2}{c}}{3}}$ the formal solutions of the steady states are

Having in mind the interpretation of -1 and 1 as the boundary states, we refer to E_2 and E_3 as the boundary steady states. While E_1-E_3 are global solutions for Eq. 5.3, whe have to determine the regions of existence for the

	$\hat{x}$	$\hat{u}$
E_1	0	0
E_2	1	0
E_3	-1	0
E_4	w	$w\,(1-w^2)$
E_5	$-w$	$-w\,(1-w^2)$.

Table 5.1: Steady states of the canonical system 5.2.

steady states E_4 and E_5. As they only depend on w we have to consider the case where w is real, implying

$$\begin{aligned} 1+r-\frac{2}{c} &\geq 0 \\ c &\geq \frac{2}{r+1}. \end{aligned} \tag{5.3}$$

That is for c sufficing Eq. 5.3 the steady states E_4 and E_5 exist.

At least we determine the cases where these steady states coincide with the other steady states.

Case 1: $w = 0 \Leftrightarrow c = 2/(r+1)$ the equilibira coincide with the origin.

Case 2: $w = 1 \Leftrightarrow c = 2/(r-2)$ E_2, E_4 and E_3, E_5 respectively coincide. Beneath this curve the value of the state variable of E_4 and E_5 are smaller than 1, while above this curve the steady states are lying outside the boundary steady states.

These results are summarized in Tab. 5.2. g

5.2.4 Stability Properties

Knowing the number of steady states for the different regions, we analyse now their stability properties. The characterization of the steady state behaviour ensues from calculating the determinante, trace and discriminant of the Jacobi matrix J.

We get the common form of J, by linearizing the system of differential equations Eq. 5.2

$$J(x,u) = \begin{pmatrix} 3x^2-1 & 1 \\ -6xu + \frac{2}{c}\,(3x^2-1) & 1-3x^2+r \end{pmatrix} \tag{5.4}$$

calculating Δ, τ and D gives

$$\begin{aligned}
\tau &= r \\
\Delta &= \left(3x^2-1\right)\left(1-3x^2+r-\frac{2}{c}\right)+6xu \\
D &= r^2-4\left(\left(3x^2-1\right)\left(1-3x^2+r-\frac{2}{c}\right)+6xu\right)
\end{aligned}$$

with

$$\begin{aligned}
\tau &\ \ldots\ \mathrm{tr}(J) \\
\Delta &\ \ldots\ \det(J) \\
D &\ \ldots\ \tau^2-4\Delta.
\end{aligned}$$

In the following subsections these formal results will be analysed for the different steady states.

Origin

At the origin the Jacobi matrix Eq. 5.4 simplifies to

$$J(0,0)=\begin{pmatrix} -1 & 1 \\ -2/c & r+1 \end{pmatrix}, \tag{5.5}$$

and so we get

$$\begin{aligned}
\tau &= r &(5.6)\\
\Delta &= -r-1+2/c &(5.7)\\
D &= r^2+4r+4-8/c &(5.8)
\end{aligned}$$

The stability properties are completely determined by the signs of the three parameters Δ, τ and D. As $\tau = r > 0$ always holds we only have to consider the occurrence of $\Delta = 0$ and $D = 0$. Solving these equations we get

$$\begin{aligned}
\Delta = 0 &\Leftrightarrow c = \frac{2}{r+1} \\
D = 0 &\Leftrightarrow c = \frac{8}{(r+2)^2}
\end{aligned}$$

Distinguishing the cases where the sign of Δ and D change whe get five regions in parameter space.

Case 1: $c < 8/(r+2)^2 \Rightarrow \Delta > 0 \wedge D < 0$ characterizes an unstable spiral.

Case 2: $c = 8/(r+2)^2 \Rightarrow \Delta > 0 \wedge D = 0$ gives the limiting case of a degenerated node at the origin.

Case 3: $8/(r+2)^2 < c < 2/(r+1) \Rightarrow \Delta > 0 D > 0$ is associated with an unstable node.

Case 4: $c = 2/(r+1) \Rightarrow \Delta = 0 \wedge D > 0$ implies the critical case of a non-isolated fixed point.

Case 5: While for $c > 2/(r+1) \Rightarrow \Delta < 0 \wedge D > 0$ the origin is a saddle.

Boundary Steady States

In case of a boundary steady states the Jacobi matrix Eq. 5.4 becomes

$$J = \begin{pmatrix} 2 & 1 \\ 4/c & r-2 \end{pmatrix},$$

and

$$\begin{aligned} \tau &= r \\ \Delta &= 2r - 4 - \frac{4}{c} \\ D &= r^2 - 8r + 16 + \frac{16}{c}. \end{aligned} \tag{5.9}$$

As Eq. 5.9 has no real root for $c > 0$, D does not change sign and hence $D > 0$ in the whole parameter space. Therefore only the regions where $\mathrm{sgn}(\Delta)$ differs have to be considered.

But $\Delta = 0 \Leftrightarrow c = \frac{2}{r+1}$, so we can distinguish three different regions

Case 1: $c < 2/(r+2) \Leftrightarrow \Delta < 0$ characterizes a saddle.

Case 2: $c = 2/(r+2) \Leftrightarrow \Delta = 0$ gives the critical case of non-isolated fixed points.

Case 3: $c > 2/(r+2) \Leftrightarrow \Delta > 0$ is associated with an unstable node.

Steady States E_4 and E_5

In case of steady states E_4 and E_5 the Jacobi matrix Eq. 5.4 becomes

$$J = \begin{pmatrix} 3w^2 - 1 & 1 \\ -6w^2\left(1 - w^2\right) + \frac{2}{c}\left(3w^2 - 1\right) & r - 3w^2 + 1 \end{pmatrix}$$

and

$$\begin{aligned} \tau &= r \\ \Delta &= \left(3w^2 - 1\right)\left(1 - 3w^2 + r - \frac{2}{c}\right) + 6w^2\left(1 - w^2\right) \qquad (5.10) \\ D &= r^2 - 4\left(3w^2 - 1\right)\left(1 - 3w^2 + r - \frac{2}{c}\right) + 24w^2\left(1 - w^2\right). \end{aligned}$$

Resubstituting w in Eq. 5.10 the factor $1 - 3w^2 + r - \frac{2}{c} = 0$ and therefore Δ and D reduce to

$$\begin{aligned} \Delta &= 6w^2\left(1 - w^2\right) \\ D &= r^2 - 24w^2\left(1 - w^2\right) \end{aligned}$$

Setting $\Delta = 0$ we find the solutions

$$\Delta = 0 \Leftrightarrow \begin{cases} c = \dfrac{2}{r+1} & \forall r \\ c = \dfrac{2}{(r-2)} & r > 2. \end{cases}$$

Finding the solutions for $D = 0$ is straight forward.

$$\begin{aligned} D = 0 &\Leftrightarrow r^2 - 24w^2\left(1 - w^2\right) = 0 \\ &\Leftrightarrow w^4 - w^2 + \frac{r^2}{24} = 0 \\ &\Leftrightarrow w^2 = \frac{1}{2} \pm \sqrt{\frac{6 - r^2}{24}} \\ &\Leftrightarrow \frac{r + 1 - 2/c}{3} = \frac{1}{2} \pm \sqrt{\frac{6 - r^2}{24}} \end{aligned}$$

For notational simplicity we set $k = 2/c$ and get

$$2r - 1 - 2k = \pm 6\sqrt{\frac{6 - r^2}{24}}$$

$$\begin{aligned}(2r-1-2k)^2 &= \frac{3}{2}\left(6-r^2\right)\\ k^2+k(1-2r) &= -\frac{11r^2-8r-16}{8}\end{aligned}$$

a quadratic in k and after resubstituting $2/c$ for k we find the solutions

$$c_{1,2}=\frac{4\left(4r-2\pm\sqrt{6\left(6-r^2\right)}\right)}{11r^2-8r-16}. \tag{5.11}$$

These solutions c_1 and c_2 form a curve in the parameter space which are continuously connected at $r=\sqrt{6}$. As the polynomial in the denominator of Eq. 5.11 has a positive real root at $r=\frac{4}{11}\left(1+2\sqrt{3}\right)$ the solutions have a singularity at this point, that can be lifted in the case of c_2, while it is a real singularity in the case of c_1.

Summarizing these considerations we can distinguish six different regions, for the properties of the steady states.

Case 1: $c=2/(1+r) \Rightarrow \Delta=0 \wedge D>0$ entails that E_4 and E_5 coincide with the origin and can be characterized as non-isolated fixed points.

Case 2: $2/(1+r)<c<2/(r-2)$ and c not inside Region III then the inequality $\Delta>0 \wedge D>0$ implies that the steady states are unstable nodes.

Case 3: $c \in \gamma$ gives the limiting case of a degenerated node.

Case 4: If c lies in Region III then $\Delta>0 \wedge D<0$ is associated to the case of unstable foci.

Case 5: $c=2/(r-2) \Rightarrow \Delta=0 \wedge D>0$ let the steady states coincide with the boundary steady states and have the properties of non-isolated fixed points

Case 6: $c>2/(r-2) \Rightarrow \Delta<0 \wedge D>0$ gives the case of saddles.

These results are summarized at Tab. 5.2.

5.2.5 Regions of Stability and Bifurcations

We next examine the steady states of the canonical system as functions of its two parameters r and c and determine their stability properties. As we can see in Fig. 5.1, the parameter space is divided into five main regions, with different numbers of steady states and different stability properties. (See also Tab. 5.2.) The origin and the boundary states ± 1 are always steady states. One additional pair of steady states can emerge. These additional steady states are between the origin and the boundary steady states when the boundary steady states are saddles; otherwise they are outside the boundary steady states.

These five regions are divided by bifurcation lines. (See Fig. 5.1.) Crossing these curves can mean a change in the system's dynamic or optimal behaviour. New steady states can emerge while others disappear or change their stability properties. As the possibility of such limiting cases is zero, they are of no vital importance for applications, but they nevertheless give insight into the mathematical formulations of radical changes in the model behaviour as parameters vary. While finding explicit formulae for the bifurcation lines was not surprising it was furthermore also possible to formulate the explicit solution for the heteroclinic bifurcation line μ. Whereas changes in the model's dynamic behaviour can take place at the other bifurcation lines, a change in the model's optimal behaviour is given at the continuous policy bifurcation line ν and the heteroclinic bifurcation line μ.

Region I

For parameters lying in Region I the only steady states are at the origin and at the boundaries ± 1. While the origin is an unstable focus the boundaries are saddles, and the region is delimited by the positive r and c axes and the curve (labeled α) defined by $c = 8/(r+2)^2$.

Region II

The only difference between Region I and Region II is the nature of the steady state at the origin, which in this case is an unstable node. As can be expected, there is an intermediate state of the origin at bifurcation line α, where the origin becomes a degenerate node.

Region II is bounded by the bifurcation line α and β, defined by $c = 2/(r+1)$.

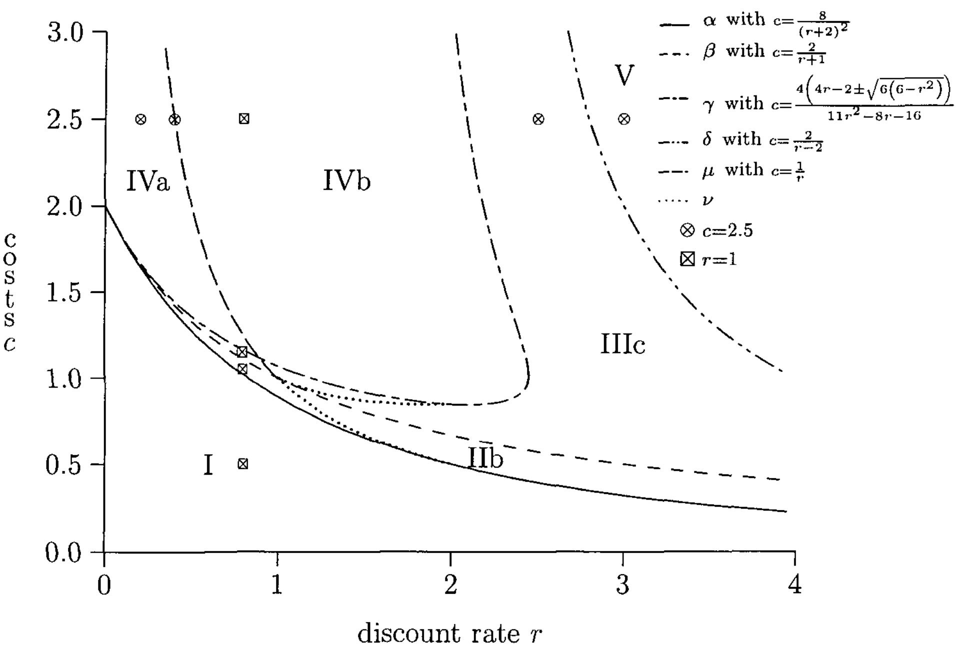

Figure 5.1: Regions of different stability and optimality divided by bifurcation lines $\alpha, \beta, \gamma, \delta$ the heteroclinic bifurcation line μ and line ν.
Region III is separated into three parts Region IIIa-IIIc by μ and the line ν. The line ν in Region II, divides Region II into two parts Region IIa and Region IIb. To avoid cluttering the figure Regions IIa, IIIa and IIIb are not marked in Fig. 5.1.
⊗ and ⊠ indicate different parameter sets for models depicted in Fig. 5.2 and Fig. 5.3.
⊗ mark different positions of models depicted in Fig. 5.2 and Fig. 5.4.
⊠ mark different positions of models depicted in Fig. 5.3 and Fig. 5.4.

Character of Steady States at:

Region/Curve	Origin	Boundary S. S.	E_4 and E_5	# of DNS
I	unstable focus	saddle	–	1p
II a	unstable node	saddle	–	1p
II b	unstable node	saddle	–	1t
III a	saddle	saddle	unstable node	1p
III b	saddle	saddle	unstable node	2p
III c	saddle	saddle	unstable node	2t
IV a	saddle	saddle	unstable focus	1p
IV b	saddle	saddle	unstable focus	2p
V	saddle	unstable node	saddle	2t

Table 5.2: Number and properties of steady states. See Fig. 5.1 for definitions of regions and bifurcation curves.
t denotes a DNS threshold.
p denotes a DNS point.

Region III

Region III has five steady states. The unstable node at the origin trifurcates into a saddle at the origin and two unstable nodes, while the boundary steady states remain saddles. Moving from Region II to Region III, bifurcation line β has to be crossed. As the origin trifurcates into a saddle and two unstable nodes we have the important case of a saddle-node bifurcation. As an intermediate state the origin becomes a non-isolated fixed point. Its importance is indicated by the change in the optimal solution strategy as will be investigated in the next section.

Region III lies between the curves β, δ and γ, where the exact formulae for the latter is derived in Appendix 5.2.4.

Region IV

Moving to Region IV the steady states lying between the origin and the boundary steady states, mutate from unstable nodes (Region III) to unstable foci. The properties of the other steady states remain unchanged. The bifurcation line γ forms the boundary of this region, and the inner steady states become degenerate nodes when crossing this curve.

Region V

While the saddle at the origin does not change in Region V, the boundary steady states become unstable nodes and the other pair of equlibria, now outside the boundary, change from unstable nodes to saddles. The curve labeled δ defined by $c = 2/(r-1)$ delimits Region V.

At curve δ a substantial change in the boundary steady states takes place. Below this curve they were saddles. As parameter c increases, approaching δ from below, the steady states lying inside the boundary approach the boundary steady states, until they collide with the outer steady states at the bifurcation line δ. The collision of the steady states produces non-isolated fixed points at the boundary. Moving on into Region V leaves unstable nodes behind at the boundary, whereas the former inner steady states lie now outside the boundary and become saddles. This change has a considerable effect on the optimal solution. (See section "Optimal Strategies").

Bifurcation Lines μ and ν

While the bifurcation lines mentioned so far separate regions with different properties and/or number of steady states the heteroclinic bifurcation line μ and continuous policy line ν lie inside such regions. A heteroclinic bifurcation occurs, when two steady states previously not connected by any orbit are now connected by a so called heteroclinic orbit (see cf. Guckenheimer & Holmes, 1983). Investigating such heteroclinic bifurcations is very interesting as they may give rise to DNS thresholds (see Wagener, 2003), which is the case for our model. Crossing bifurcation line μ produces a dramatic change for the global optimal policy. Line ν separates regions with only a slightly different local optimal behaviour, viz the continuity of the optimal policy at the occuring unstable nodes, that is regions with DNS points from regions with DNS thresholds as defined in Section 2.1. In contrast, when crossing the part of ν lying in Region II, the optimal behaviour at the origin changes, while for the part lying in Region III the inner steady states are affected.

5.2.6 Analysis of Local Optimality

In the following sections we analyse the behaviour of the solutions for the Riccati differential equation Eq. 5.1 at the steady states and the extremals of the different regions.

Origin

For $(x, u) = (0, 0)$ Eq. 5.1 reduces to

$$\dot{q} = \frac{q^2}{2c} + (2 + r)q + 4. \tag{5.12}$$

Finding the region in parameter space, where this differential equation has a bounded solution reduces to find the region, where the origin becomes an unstable node or saddle. Considering Tab. 5.2 we see that the origin is a local optimal solution for parameters in every region except region I, where it becomes an unstable focus.

Boundary Steady States

In the case of boundary steady states the Riccati differential equation Eq. 5.1 is of the form

$$\dot{q} = \frac{q^2}{2c} + (r - 4)q - 8. \tag{5.13}$$

Furthermore the boundary steady states are saddles or unstable nodes respectively therefore staying at the boundary steady states is always locally optimal.

Steady States E_4 and E_5

As the steady states E_4 and E_5 are given by

$$E_{4,5} = \left(\pm w, \pm w(1 - w^2)\right),$$

where $w = \frac{\sqrt{3}}{3}\sqrt{1 + r - \frac{2}{c}}$, the Riccati differential equation Eq. 5.1 is given by

$$\dot{q} = \frac{q^2}{2c} - (2(3w^2 - 1) - r)p - (12w^2 - 4 + -12cw^2(1 - w^2)). \tag{5.14}$$

Inspecting Tab. 5.2 we see that persisting in steady state E_4 or E_5 is only locally optimal for parameters in Region III or V, whereas it is even not locally optimal to stay there in Region IV.

Region I and II

As well as for Region I as for Region II there exist an entire interval around state 0 where the stable manifolds emerging from the unstable steady states at the origin have been numerically proven to be locally optimal (see Fig. 5.4b and Fig. 5.4c). But whereas in region I the origin is an unstable focus and therefore not locally optimal, it is an unstable node in region II and hencefor staying there is locally optimal (see Appendix B.3).

Region IV and III

In Regions IV and III a quite interesting phenomenon occurs. While in Regions IV/IIIa three different local optimal solutions exist on an entire interval around state 0. These are the three stable manifolds for the origin and the outer steady states (see Fig. 5.4a). In Regions IV/IIIb only two separated intervals around the state values of the inner steady states (unstable foci/nodes) have been proven to possess the property that two different locally optimal solutions (Fig. 5.4c) exist. In the hairline case, where parameters r and c lie exactly on bifurcation line μ the two separated intervals are connected and on the entire interval two different locally optimal solutions have been found. This kind of behaviour is the same for Regions IVa/b and IIIa/b. The difference concerns the inner steady states, where they are locally optimal solutions if they are unstable nodes in Region III and they are not locally optimal for Region IV. Furthermore in Region IIIc the stable manifolds have been proven to be locally optimal and connected by the unstable nodes, which are also locally optimal.

Region V

For region V all occuring stable manifolds and steady states have been at least numerically proven to be locally optimal.

5.3 Interpretation of the Results

This section consists of three subsections, where the mathematical results are interpreted in a more informal manner. In the first subsection the optimal strategies are worked out from the mathematically analysed optimal behaviour. In the next subsection the connection between the change of optimal strategies and varying the parameter values is formulated. In the last subsection the problem of optimal strategies is once again considered but now from the point of view of local optimality.

5.3.1 Optimal Strategies

Having analysed the dynamic systems in terms of steady states and their properties, we next explore when various strategies are optimal. It turns out that there are essentially three strategies that may be optimal depending on the values of parameters r and c: (A) move to the boundary (± 1), (B) (almost) always fall off from the boundary to the origin or a state outside the boundaries, and (C) move either to the origin or the boundaries depending on one's initial position.

The stability regions and bifurcation lines play an important role in defining when the various strategies are optimal. Bifurcation line μ in particular separates regions with different optimal behaviour but the same dynamic behaviour. In particular μ separates stability Region IV into two subregions, with that falling on the left of μ denoted IVa, respectively, and that to the right denoted IVb. In addition Region III is subdivided into three subregions by μ and ν, with Region IIIa to the left of line μ, Region IIIb between μ and line ν, and Region IIIc to the right of ν.(See Section 5.2.4 for details.) So different strategies are optimal in different parts of a single stability region (namely Regions II, III and IV), and the same strategy may be optimal for different stability regions (e. g. , Regions I, IIa, IIIa and IVa).

Strategy A: Always Move to the Boundary

In stability Regions I, IIa, IIIa and IVa, it is always optimal to move to one of the boundary states ± 1, depending only on the sign of the initial starting position x_0. That is for $x_0 > 0$ it is optimal to tend to state 1, while state -1 is the long run optimal state for $x_0 < 0$. Only in the case when starting exactly at $x_0 = 0$ both options, moving to the left or right boundary, are equally optimal, whereas staying at the origin would be more expensive and hence be suboptimal. Therefore the origin is a so called DNS point (cf. Tragler et al., 2001). Note that, in contrast to the situation when substrategy A' pertains, an infinitesimally small deviation from the starting position $x_0 = 0$ leads to a finite change in the optimal initial level of effort $u_0^\star$, that is to say the optimal policy is discontinuous at $x_0 = 0$.

Moving to the boundaries makes intuitive sense because in these regions parameters r and/or c are small. Clearly the boundary points are the most advantageous points. If the discount rate is small (decision maker is far sighted) and/or the cost of adjusting one's position is low enough, it is always worth investing the effort needed to reach one of these advantageous points.

Substrategy A': Continuous Policy at the Origin

Models with parameters r and c lying either exactly at μ or in Region IIb show slightly different optimal behaviour at $x_0 = 0$. Moving to the boundary continues to be optimal for starting positions $x_0 \neq 0$, but if $x_0 = 0$ it is optimal to remain at the origin. Moreover, deviating an infinitesimal distance from a starting position at $x_0 = 0$ leads only to an infinitesimally small change in the optimal initial level of effort $v_0^\star$. That is, the optimal policy is continuous at $x_0 = 0$ and hencefor a DNS threshold. This stands in contrast to strategy A, where we observed a discontinuous optimal policy (DNS point). The continuity of optimal policy means moving to one of the boundaries optimally from $x_0 \sim 0$ involves only an infinitesimally small exertion of effort and the cost for such a policy differs only infinitesimally from that of staying at the origin.

Strategy B: (Almost) Always Fall Off from the Boundary

Stability Region V represents the opposite case to strategy A. If the starting position is exactly at the boundary, it is optimal to stay there. Otherwise, parameters r and c are large enough that if the decision maker deviates even from the boundary, the decision maker is short-sighted to such a degree and the costs of control are so high that the benefits of returning to the boundary are not worth the effort. So for starting positions inside the boundary ($|x_0| < 1$) it is optimal to tend to the origin, while for starting positions outside the boundary it is optimal to move to the steady states outside the boundaries.

The optimal policy at the boundary steady states is continuous and therefore starting exactly at this steady states means staying put (DNS threshold). Therefore the long run optimal behaviour is sensitive to the exact initial starting position near the boundary steady states.

Strategy C: Move to the Origin if One Starts Nearby; Otherwise move to the boundary

Regions IIIb and IVb present an intermediate case to some extent. If the decision maker's initial position is inside the boundary and close to the origin, then it is optimal to move to the origin. But if the initial position is inside the boundaries ($|x_0| < 1$) but farther from the origin, it is optimal to move out to a boundary, specifically the closer one. In between there are points of indifference, one on either side of the origin, from which the decision maker is equally happy moving left or right. Note that the optimal policy is discontinuous at these points of indifference (especially for Region IIIb, where the inner steady states are unstable nodes) in contrast to substrategy C'. Therefore these points of indifference are DNS points. If one starts outside the boundaries returning to the boundaries is always worth the effort, presumably because of the heavy penalty in the objective function for being far from the boundary, and the cubic term in the state dynamics that tends to drive states that are outside the boundary further away from the boundary at an ever increasing rate.

Substrategy C': Continuous Policy at the Inner Steady States

Models with parameters r and c lying in Region IIIc show a slightly different optimal behaviour at the inner steady states (unstable nodes). In contrast to strategy C the optimal policy is continuous at these steady states. Hence staying at these points become optimal and they are no longer DNS points but DNS thresholds.

5.3.2 Change in Optimal Strategy as Parameters Vary

This section examines in more detail how the optimal strategy varies as one of the two parameters in turn is increased.

Increasing r for Fixed Values of the Cost Parameter c

Figure 5.2 shows how the optimal solution changes for a given cost parameter c as the discount rate parameter r increases, i.e., as the decision maker gets more and more myopic. In particular, Figures 5.2a-d show phase portraits when $c = 2.5$ and the discount rate parameter r is 0.2, 0.4, 2.5 and 3.5 respecgtively.

When $c = 2.5$ and r is small it is always optimal to converge to a boundary, specifically the closer one. If one starts at the origin moving left or right generates the same costs while remaining at the origin is more expensive. Therefore a decision has to be made arbitrarily between moving to the left or right boundary. (See Fig. 5.2a). This statement holds true for every parameter r and c lying below bifurcation lines α or μ. Increasing r leads the stable manifolds of the boundary steady states at the u-axis to approach the origin, until the origin lies precisely on the stable manifolds for $r = 1/2.5$. This is the case of a heteroclinic orbit. For this hairline case the origin becomes a point with continuous optimal policy at $x_0 = 0$. (See Fig. 5.2b). If r increases further the DNS threshold at the origin bifurcates into two DNS thresholds. Starting between these thresholds moving to the origin is optimal, while converging to the boundary is optimal for initial starting positions outside the DNS thresholds. This optimal behaviour does not change upon crossing bifurcation line γ, however the steady states inside the boundaries change from unstable foci to unstable nodes and the inner steady states become points with a continuous policy function. (See Figs. 5.2c and 5.3d) I.e. to the left of μ (in Region IVb) a decision maker - starting at the inner steady states - should choose (arbitrarily) to move left or right. To the right of μ (in Region IIIc) a decision maker should stay put. Letting r grow further, the equlibria inside the boundaries move towards the boundary steady states. Reaching bifurcation line δ these steady states coincide leaving non-isolated fixed points behind at ± 1. At this limiting case the optimal behaviour can only be analysed with standard methods for initial starting positions inside the boundaries, where the optimal paths converge to the boundary steady states. For discounting rates r above bifurcation line δ the optimal behaviour changes dramatically. As the boundary steady states become unstable nodes, it is never optimal to converge to these steady states. Instead the origin becomes optimal for every state starting inside the boundary, while outside the boundaries the optimal solution paths converge to steady states with absolute state values greater than 1. (See Fig. 5.2d.)

For $c < 1$ two other cases can occur as r increases. Letting e. g. $c = 0.8$, a small r implies optimal paths converging to the boundaries and $x_0 = 0$ becoming a DNS threshold where the costs for moving to the left or to the right steady states are the same. Crossing bifurcation line α the optimal behaviour at the origin remains unchanged until bifurcation line ν is reached. As long as we are in Region II there is no change in the global optimal behaviour as described in the section "Regions of Stability" for bifurcation line ν. But for starting positions at $x_0 = 0$ the possible optimal solutions change. Below bifurcation line ν it is optimal to exert a finite initial effort u_0^* to move to the boundary. That means if the decision maker is farsighted enough he or she accepts a higher initial effort and moves to one of the boundaries. Getting more myopic the decision maker no longer invests in a high initial effort, and staying at the origin becomes optimal. This is the case when the Policy function at the origin becomes continuous as described in the section "Optimal Strategies".

Increasing r further and crossing bifurcation line β the unstable node at the origin trifurcates into two unstable nodes at the inner steady states and a saddle at the origin. For our choice of c we get a continuous policy function at the unstable nodes. Therefore if one starts near the origin, one returns to the origin. For all initial starting positions outside the unstable nodes moving to the boundary steady states is worth the effort. While starting exactly at the unstable nodes, means staying put.

Increasing c for Fixed Values of the Discount Rate r

Figure 5.3 shows phase portraits when $r = 0.8$ and the cost parameter c is $0.5, 1.05, 1.15$, and 2.5, respectively. When costs are low ($c = 0.5$; Fig. 5.3a) starting left (right) from the origin it is optimal to move to the left (right) boundary, while if starting at $x_0 = 0$ the decision maker can choose arbitrarily between moving to the left or right boundary. As c increases further towards the bifurcation line ν, the initial level of effort u_0^* for starting positions deviating only slightly from $x_0 = 0$, shrinks to 0, while the optimal behaviour remains the same as before. This is clear from the decision maker's point of view as the effort is getting more and more costly. (See Fig. 5.3b.) Crossing μ the stable manifolds coincide at the origin and the difference in the initial level effort is 0. Therefore the optimal policy becomes continuous at the origin and we achieve the hairlinecase where staying at the origin becomes optimal. Crossing μ the region where moving towards the origin is optimal expands and two DNS thresholds appear. To the left (right) of the left (right) DNS threshold it is optimal to converge to the left (right) steady state, while between the DNS thresholds moving to the origin becoms optimal. Now the costs are so high that it is not worth moving to the boundaries when starting near the origin. (See Fig. 5.3d.)

The cases described before characterize the behaviour for all models with $r \leq 1$, while two slight differences take place for $r > 1$. Whereas μ make up the limiting case for the emergence of two DNS thresholds for $r \leq 1$ the continuation of μ (part of ν lying in region II) only divides regions with different local optimal behaviour at the origin for $r > 1$ as described in the section "Regions of Stability". Increasing the costs c leads to a shrinking gap between the initial level effort $u_0^\star$ for starting positions on either side of $x_0 = 0$. Nevertheless costs are low enough to be worth the effort of moving to one of the boundaries, even if one starts at $x_0 = 0$. As the costs get higher (above ν) moving to the closer boundary is only infinitesimally more expensive then staying at the origin and therefor staying at the origin becomes optimal. If one starts near the origin it is optimal to stay near the origin for a while and not move away too quickly. The higher the costs are increasing the longer the duration for staying near the origin. Crossing bifurcation line β the movement near the origin comes to a stillstand. This is the case where the origin becomes a non-isolated fixedpoint. Raising the costs above the bifurcation line β the movement near the origin is reversed and moving to the origin becomes optimal when starting nearby. As a consequence, two starting positions x_0 become points of indifference between moving to the origin and moving to the boundary. Augmenting the costs further moves these points of indifference out toward the outer steady states. But for $r \leq 2$ the outer steady states still remain optimal at least for starting positions with $|x_0| \geq 1$ and in a shrinking neighbourhood inside the boundaries. If $r > 2$ the situation changes and crossing bifurcation line δ the boundary steady states are only optimal in the case of starting exactly at ± 1. For every other case the origin is optimal, starting inside the boundaries $|x_0| < 1$, and steady states outside the boundaries become optimal for starting positions meeting $|x_0| > 1$.

5.3.3 Local Optimal Strategies

As we have seen for the models analysed so far, there exists a strong connection between the local optimal behaviour and the various regions separated

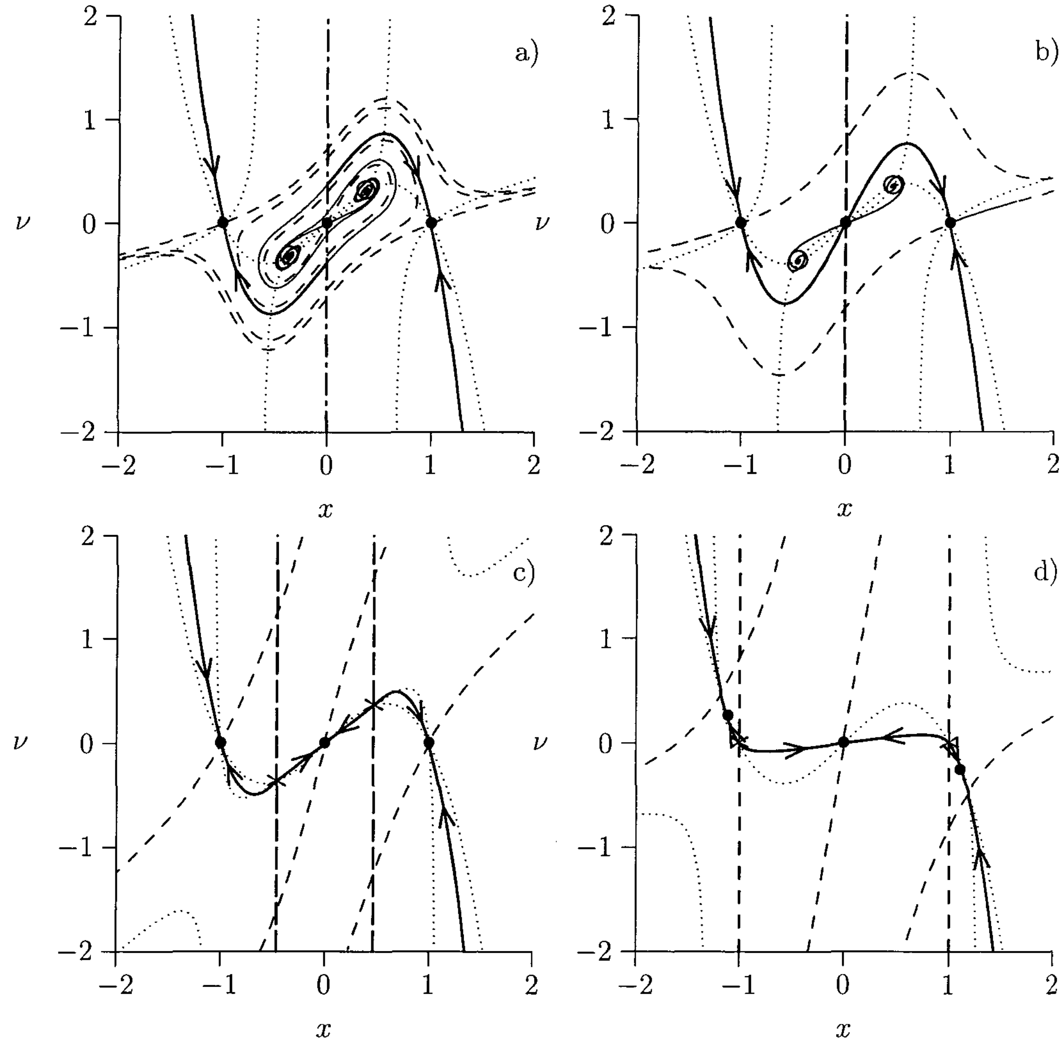

Figure 5.2: For constant cost $c = 2.5$ and different discount rates r the system dynamics is shown together with its optimal behaviour and direction, starting in the upper left and moving clockwise the four cost parameters are
a) $r = 0.2$ b) $r = 0.4$ c) $r = 2.5$ d) $r = 3.5$

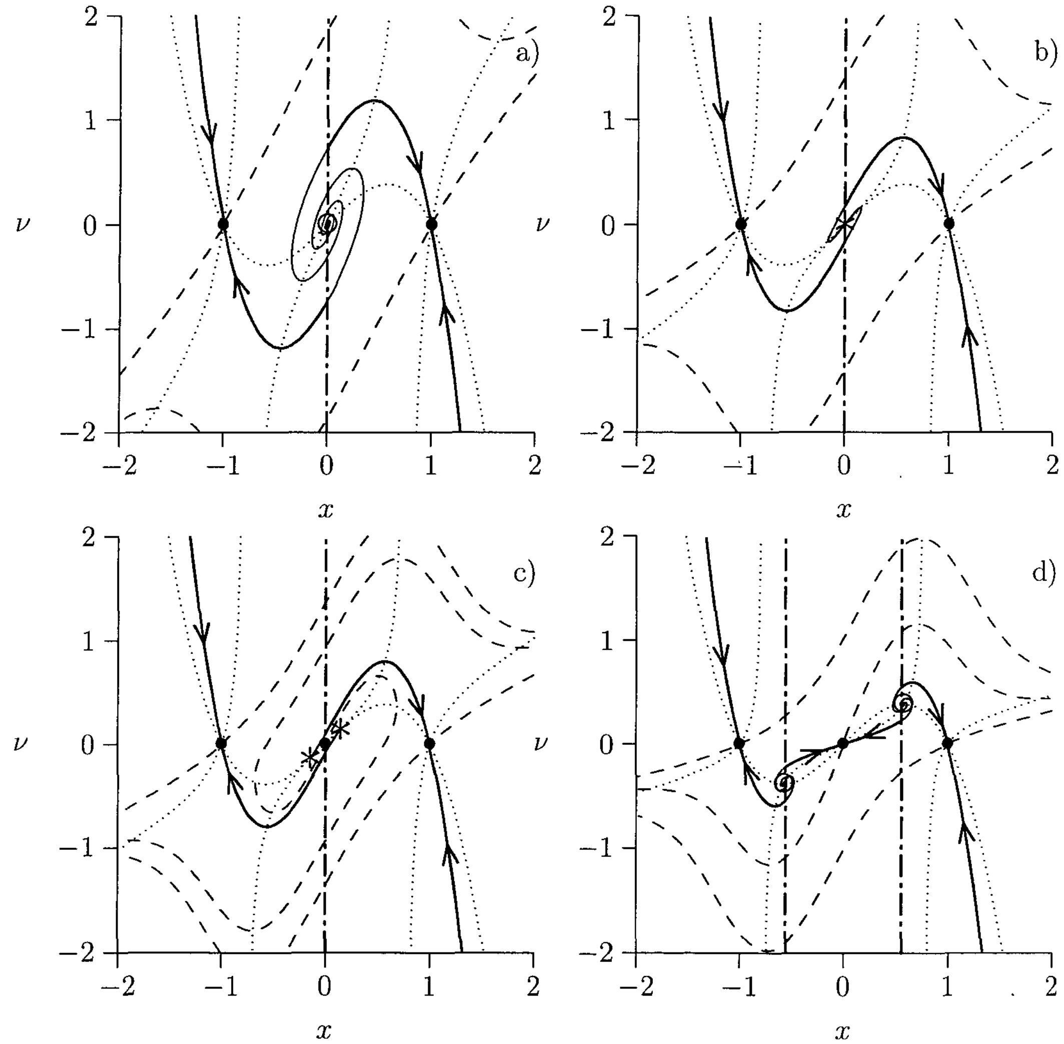

Figure 5.3: For constant discount rate $r = 0.8$ and different costs c the system dynamics is shown together with its optimal behaviour and direction, starting in the upper left and moving clockwise the four cost parameters are
a) $c = 0.5$ b) $c = 1.05$ c) $c = 1.15$ d) $c = 2.5$

by the bifurcation lines. The term bifurcation line is used here in a wider sense. Because it includes also the case, where regions with a different kind of behaviour near an unstable node are distinguished (e. g. bifurcation line ν in Fig. 5.4). As it was possible to find all bifurcation lines at least numerically every possible local optimal strategy has been assigned to an unique region.

Once again the main strategies retain the same but with a more differentiated sub-structure. Recapitulating the main optimal strategies we find: (A) move to the boundary (± 1), (B) (almost) always fall off from the boundary to the origin or a state outside the boundaries, and (C) move either to the origin or the boundaries depending on one's initial position.

Note that the dependence on one's starting position for strategy (C) has not this strict interpretation like for the global optimality. That is there exists an entire interval where all three long run states ± 1 and 0 can become locally optimal. Furthermore whereas Region IIa and IVa were assigned to Strategy (A) underlying a global concept, these regions are now assigned to Strategy (C).

In the following paragraphs only the differences between global and local optimal solutions are worked out. Arguments which are equal for both concepts are omitted.

Strategy A: Always Move to the Boundary

This is the main strategy for Region I and II. Nevertheless small deviations have to be considered.

For Regions I and IIa exist an interval $I_c = (-x_c, x_c)$ (see Fig. 5.4b), such that starting inside this interval one can choose to move to -1 or 1. The value x_c is calculated as the state value of at the conjugate point of the Riccati differential equation along the extremals converging to the boundary steady states. Both decisions are locally optimal. If one starts outside the interval I_c it is optimal to move to the nearest boundary. This can be interpreted quite well, as in some situations it may make sense to choose the suboptimal strategy if the initial effort for the optimal strategy is that large and one considers costs not modeled explicitly (see Section 2.1).

Although neglectable for real application it can be noticed, that while in Region I the steady state at the origin is even not locally optimal, it becomes locally optimal in Region IIa (see Fig. 5.4c). This can be interpreted as the first appereance of a new local optimal strategy, which is expanded to an interval as will be seen for strategy (C).

At last Region IIb remains, where the global optimal strategy coincides with the local strategy. In this case the costs c are small enough and the decision maker is far sighted enough to move to the boundaries, but c and r are too high to let a sub optimal choice even become locally optimal.

Strategy B: (Almost) Always Fall Off from the Boundary

Global and local optimal strategies are the same. It has been proven that staying at the outer steady states is locally optimal.

Strategy C: Move to the Origin or Move to the Boundary

All the strategies considered in this paragraph have in common that it is locally optimal to move to one of the boundaries or to move to the origin.

For Region IIIa and IVa all three strategies are locally optimal. That is for initial positions inside an interval $I_c = (-x_c, x_c)$, where x_c is calculated as before (see Fig. 5.4a), moving to one of the boundaries ± 1 or moving to the middle is locally optimal. Outside this interval there is a small range of initial states where moving to one of the boundaries but not moving to the middle is locally optimal. Not till one starts outside this range moving to the nearest boundary is the only local optimal choice.

In Region IIIb an IVb the interval I_c is splitted into two disjunct intervals I_1 and I_2 (see Fig. 5.4d), where the boundaries of the intervals are given by the state values of the conjugate points for the extremal paths converging to the boundary or to the middle. Inside these intervals one has the choice to move to the middle or move to the nearest boundary. For starting positions between these intervals the only choice is to move to the middle. Whereas outside the intervals only moving to the nearest boundary is optimal. One subtle difference between Region IIIb and IVb, respectively IIIa and IVa has to be considered, while the inner steady states are even not locally optimal for Region IV, they become locally optimal at Region III.

For Region IIIc global optimal strategies and local optimal strategies are the same.

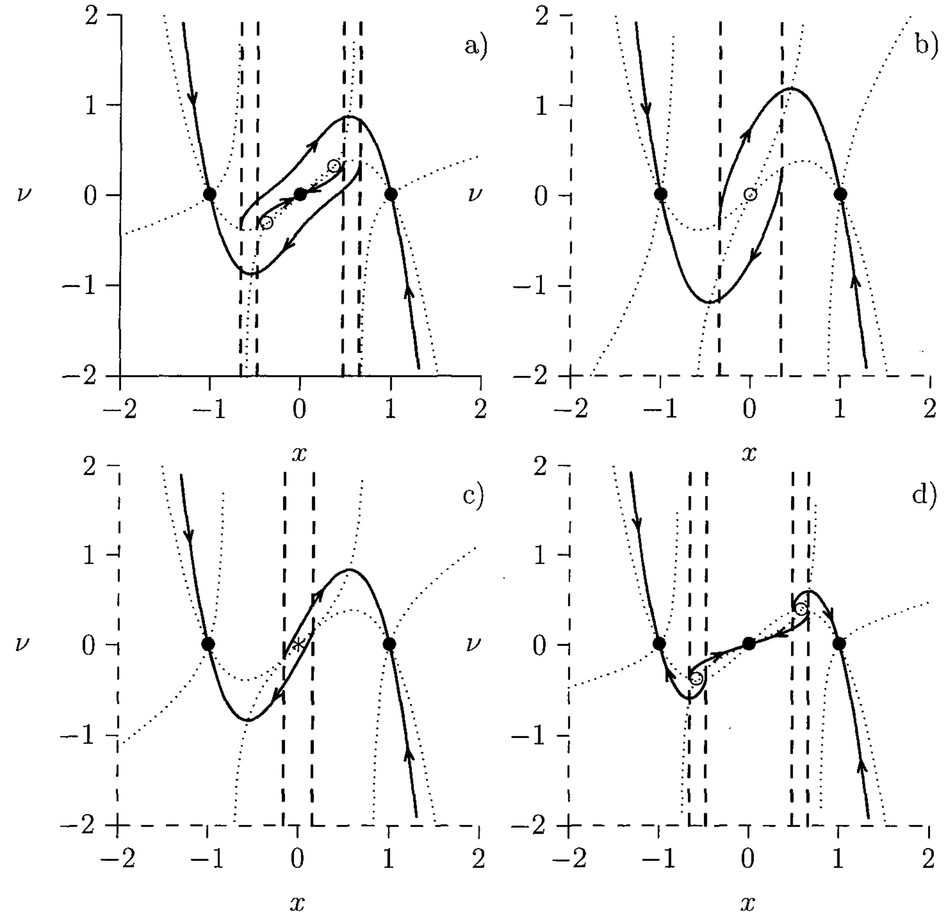

Figure 5.4: For different discount rates r and different costs c the system dynamics is shown together with its locally optimal behaviour and direction, starting in the upper left and moving clockwise the four pairs of parameter (r, c) are
a) $(0.2, 2.5)$ b) $(0.8, 0.5$ c) $(0.8, 1.05)$ d) $(0.8, 2.5)$

Conclusion

Even though this "model of moderation" is very simple (one state, one co-state, and just two parameters), from a mathematical point of view very interesting features emerge. Despite the model's simplicity, the existence of multiple DNS points has been shown. Moreover, we found a region in parameter space split by a DNS bifurcation line. Even though the number of steady states and their properties are the same throughout this region, DNS points exist on one side of this bifurcation line but not on the other. Furthermore, precisely because of the model's simplicity, analytical expressions can be written for this and the model's other various bifurcation lines and associated regions.

The model's optimal solutions have consistent and sensible interpretations, even in the critical extreme cases, and a variety of intriguing extensions can be envisioned. To begin with, other functional forms for the state dynamics could be investigated representing different "curvatures" of the "slippery slope". Also, the model is essentially separable at the origin because trajectories involving positive state values never become negative and vice versa. Hence, there is no reason why investigations need to be restricted to cases that are symmetric about the origin. Furthermore, in this paper the one-dimensional state space can be thought of as reflecting the cross-section of a hill. That is appropriate when the "issue space" in question is one-dimensional with two opposing camps. Sometimes, however, the neutral position is not intermediate between just two alternatives but rather is central relative to a large number of alternatives that can not neatly be arrayed along a line. Hence, two-dimensional versions of the model could be of interest, and their investigation might yield closed two-dimensional DNS thresholds, which to the best of our knowledge have never before been discovered in applied models.

In this model, positioning oneself as a bridge between one's own organization and the outside world yields benefits but also takes effort. Whether moving to a bridging position is worthwhile depends on how costly it is to alter one's position and on how far sighted one is. Individuals who are sufficiently myopic and for whom such movement is sufficiently painful should not bother. Those who are sufficiently far-sighted and/or flexible should always become bridges. For others, the optimal strategy depends on whether one is initially close to or far from being such an organizational bridge.

Because of the relative simplicity of this model, the model's structure and resulting optimal behaviour could be fully characterized in the parameter space. In particular, it was possible to find explicit solutions for every bifurcation line, including the heteroclinic bifurcation at the origin. Furthermore the lines where the optimal policy becomes continuous at the relevant unstable nodes were numerically calculated.

This solution yielded quite a number of mathematically interesting structures. Even though it is a one state model, varying a single parameter generates instances of zero, one, or two DNS thresholds and even instances in which a single DNS threshold trifurcates into two DNS thresholds and a saddle point. More generally we found regions with the same number and properties of steady states but different optimal behaviour, divided by a heteroclinic bifurcation line, and regions where the optimal solution was sensitive to the exact starting position

This simple model may have interesting extensions. One would replace the one-dimensional (one-state) model of the organization with a two-dimensional or even n-dimensional model. The unit circle or unit sphere, respectively, could still denote the boundaries of the organization. The state dynamics could still be taken such that people within the organization gravitate toward its center at the origin whereas those outside the boundary are drawn further away. And the cost function could still reflect the ideal of staying at the boundary or some selected points along that boundary. Considering such a model, one could expect to find two-dimensional DNS curves and a DNS point at the origin with an arbitrarily large number of alternative optimal strategies available.

Another variation would recognize that people can be members of more than one organization simultaneously, so the objective function could be the sum of distances from the centers of several different organizations. Again, in a two- or higher-dimensional model, this might likewise yield rather complex and interesting solutions.

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Appendix A

Local Optimality

This chapter consists of five sections, where in the first section the first order necessary conditions are restated and some fundamental definitions are given. In the next section the corresponding accessory problem is introduced together with the necessary second order conditions. In the third section the Riccati differential equaiton is derived from the concept of conjugate points. Whereas the next section adresses the relation between conjugate points as defined in (Zeidan, 1994) and shocks as treated in (Caroff & Frankowska, 1996). The last section considers sufficient second order conditions.

At the beginning I want to give some general remarks concerning this chapter. First I have to admit that this approach is thought for as an attempt to give a compact sketch of the basic ideas, therefore it lacks the mathematical strength in formulating all technical details. Furthermore all the theorems can be stated for models with path constraints, but were of no interest for this thesis and therefor omitted. For a rigorous mathematical treatment I refer to (Zeidan, 1994; Caroff & Frankowska, 1996; Maurer & Pickenhain, 1995).

The underlying problem is modeled as

$$\begin{aligned}
&V(x,u) = \int_0^T g(x,u,t)dt && \text{(A.1)} \\
&\min_{u\in\mathbb{R}^m} V(x,u) && && \text{(F)} \\
&\text{s.t.} \quad \dot{x} = f(x,u,t) && \text{(A.2)} \\
&\text{and} \quad x(0) = x_0,\ x(T) = x_T, && \text{(A.3)}
\end{aligned}$$

where $x : [0,T] \to \mathbb{R}^n$ is absolutely continuous (AC), $u : [0,T] \to \mathbb{R}^m$ is piecewise continuous, and it is assumed that $f, g \in C^2$, where it has to be noted that this assumption can be weakend, but sufficies for our purpose.

In order not to overload notation the functional argument t is omitted whenever there is no ambiguity to expect. Morevover most of the conditions asserted have to hold only almost everywhere which is not stated explicitly.

A.1 Necessary Conditions and Definitions

First some definitions are introduced

Definition 1 A pair (x, u) is admissible for (F) if $x \in$ AC, u is piecewise continuous and the constraints Eqs. A.2-A.3 are satisfied by (x, u).

Definition 2 An admissible pair $(\hat{x}, \hat{u})$ is a weak local minimum for (F) if for some $\varepsilon > 0$, $(\hat{x}, \hat{u})$ minimizes $J(x, u)$ over all admissible pairs (x, u) satisfying

$$\| x - \hat{x} \|_\infty < \varepsilon \quad \text{and} \quad \| u - \hat{u} \|_\infty < \varepsilon.$$

Now Pontryagin's minimum priniple for problem (F) can be proven as (see, e.g., Zeidan, 1994 Theorem 3.1)

Theorem 1 (Pontryagin's Principle (1962)) *Let $(\hat{x}, \hat{u})$ be an weak local minimum for (F). Then a constant $\lambda_0 \geq 0$ and an AC (costate) function $\lambda(t) \in \mathbb{R}^n$ exist, such that*

$$(\lambda_0, \lambda(t)) \neq (0,0) \quad \forall t \tag{A.4}$$

$$H_u(\hat{x}, \hat{u}, \lambda_0, \lambda, t) = 0 \tag{A.5}$$

$$\dot{\lambda} = -H_x(\hat{x}, \hat{u}, \lambda_0, \lambda, t) \tag{A.6}$$

where

$$H(x, u, \lambda_0, \lambda, t) = \lambda_0 g(x, u, t) + \lambda f(x, u, t),$$

is the usual Hamiltonian. □

Definition 3 An admissible pair $(\hat{x}, \hat{u})$ for (F) satisfying the necessary conditions A.4-A.6 is an extremal.

Definition 4 An extremal $(\hat{x}, \hat{u})$ is normal if $\lambda_0 > 0$ holds.

Furthermore a definiton strengthening the usual Legendre-Clebsh condition $H_{uu} \geq 0$ is given

Definition 5 We say that the strengthend Legendre-Clebsh condition is satisfied at an extremal $(\hat{x}, \hat{u})$ if for some $\alpha > 0$ we have

$$\hat{H}_{uu} \geq \alpha \mathrm{I}_m. \tag{A.7}$$

A.2 Accessory Problem

In accordance with the second order conditions in the calculus of variation (see, e.g., Sagan, 1969) and using the calculus of Fréchet derivatives (see, e.g., Warga, 1972; Gilbert & Bernstein, 1983) the so called accessory problem can be stated. Denoting the Fréchet derivative by δ it can be shown that for the second derivative of V (see Eq. A.1)

$$\delta^2(V) = \frac{1}{2}\int_0^T \left(\eta^* \hat{H}_{xx}\eta + 2\eta^* \hat{H}_{xu} w + w^* \hat{H}_{uu} w\right) dt =: V_2(\eta, w) \tag{A.8}$$

holds, with $\eta \in \mathrm{AC}$ and w piecewise continuous, satisfying

$$\begin{aligned} \dot{\eta} &= \hat{f}_x \eta + \hat{f}_u w \\ x \qquad \eta(0) &= 0, \ \eta(T) = 0, \end{aligned}$$

where $*$ denotes matrix transposition.

Now the accessory problem (AP) corresponding to (F) can be stated

$$\left.\begin{array}{lll} \min\limits_{w \in \mathrm{L}^\infty} & V_2(\eta, w) & \\ \text{s.t.} & \dot{\eta} = \hat{f}_x \eta + \hat{f}_u w & \text{(A.9)} \\ \text{and} & \eta(0) = 0, \ \eta(T) = 0. & \text{(A.10)} \end{array}\right. \qquad \text{(AP)}$$

Remark 1 In context with necessary second order conditions the term strong normality has to be introduced, which insures normality of both problem (F) and the accessory problem (AP) and uniqueness of the costate ψ. As strong normality is more a technical term, at least at this level of consideration, it is omitted here and the interested reader is refered to (Zeidan, 1994). Nevertheless the term is used in stating the theorems.

Using the Pontryagin's principle the necessary conditions for an extremal of (AP) becomes

Proposition 1 *Let $(\hat{\eta}, \hat{w})$ be an extremal for (AP) then the following conditions have to be satisfied*

$$\hat{H}_{ux}\eta + \hat{H}_{uu}w + \hat{f}_u^*\psi = 0 \tag{A.11}$$

$$\dot{\psi} = -\left(\hat{H}_{xx}\eta + \hat{H}_{xu}w + \hat{f}_x^*\psi\right) \tag{A.12}$$

□

PROOF For the proof one only has to consider the Hamiltonian $\hat{\mathcal{H}}$ of (AP), which becomes

$$\hat{\mathcal{H}} = \psi_0 \frac{1}{2}\left(\eta^* \hat{H}_{xx}\eta + 2\eta^* \hat{H}_{xu}w + w^* \hat{H}_{uu}w\right) + \psi^*\left(\hat{f}_x\eta + \hat{f}_u w\right). \tag{A.13}$$

As is explained in Remark 1 ψ_0 can be set to 1.

For minimizing $\hat{\mathcal{H}}$ in respect to w we set $\hat{\mathcal{H}}_w = 0$ yielding

$$\hat{H}_{ux}\eta + \hat{H}_{uu}w + \hat{f}_u^*\psi = 0,$$

and therefore Eq. A.11 follows. Whereas differentiating $\hat{\mathcal{H}}$ in respect to η and considering

$$\begin{aligned}\dot{\psi} &= -\hat{\mathcal{H}}_\eta \\ &= -\left(\hat{H}_{xx}\eta + \hat{H}_{xu}w + \hat{f}_x^*\psi\right),\end{aligned}$$

yields Eq. A.12. ■

Considering problem (AP) one would expect that if $(\hat{x}, \hat{u})$ is a minimal solution of (F) $(\eta, w) = (0, 0)$ is a minimizer of (AP), what can be confirmed by the following Theorem.

Theorem 2 (Zeidan, 1994 Theorem 3.2) *If $(\hat{x}, \hat{u})$ is a strongly normal weak local minimum of (F) then $V_2(\eta, w) \geq 0$ and the minimum value of (AP) is zero.* □

A.3 Conjugate Points and Riccati Equation

Analogous to the calculus of variation the definition of a conjugate point (to T) along an extremal $(\hat{x}, \hat{u})$ can be given

Definition 6 A point $c \in (0, T)$ is conjugate to T along $(\hat{x}, \hat{u})$ if there exists a nonzero $(\eta, \psi, w) \in AC \times AC \times \mathrm{L}^{\infty}[0, T]$ satisfying Eqs. A.9-A.12 and $\eta(c) = 0$.

This definition of a conjugate point can be rephrased in terms of the solution of a linear system of differential equations in (η, ψ), which will lead directly to a Riccati differential equation, therefore the following Proposition has to be stated

Proposition 2 (Zeidan, 1994 Proposition 5.1) *Assuming that the strengthened Legendre-Clebsh condition holds, then c is conjugate to T if and only if $(\eta, \psi) \neq 0$ and*

$$\begin{aligned} \dot{\eta} &= \left(\hat{f}_x - \hat{f}_u \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) \eta - \hat{f}_u \hat{H}_{uu}^{-1} \hat{f}_u^* \psi && \text{(A.14)} \\ -\dot{\psi} &= \left(\hat{H}_{xx} - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) \eta + \left(\hat{f}_x^* - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{f}_u^*\right) \psi && \text{(A.15)} \end{aligned}$$

with

$$\eta(T) = 0, \qquad \eta(c) = 0,$$

and

$$w = \hat{H}_{uu}^{-1} \left(\hat{H}_{ux} \eta + \hat{f}_u^* \psi\right).$$

□

PROOF It is only a rough sketch, details can be found in (Zeidan & Zezza, 1988; Zeidan, 1994). Considering Eq. A.11 we get

$$\begin{aligned} 0 &= \hat{H}_{ux} \eta + \hat{H}_{uu} w + \hat{f}_u^* \psi \quad \text{hencefor} \\ w &= \hat{H}_{uu}^{-1} \left(\hat{H}_{ux} \eta + \hat{f}_u^* \psi\right) \quad \text{holds.} \end{aligned}$$

Substituting this formula for w in Eq. A.9 and Eq. A.12 yields Eq. A.14 and Eq. A.15. ■

Now a necessary condition involving the conjugate point can be stated

Theorem 3 (Zeidan, 1994 Theorem 5.1) *Let* $(\hat{x}, \hat{u})$ *be a weak local minimum of (F). Assume that* $(\hat{x}, \hat{u})$ *is strongly normal on* $[0, T]$*, and the strenghtend Legendre-Clebsh condition holds then there is no point in* $(0, T)$ *conjugate to* T. □

Associating the following matrix system to the linear differential system Eqs. A.14-A.15

$$\dot{X} = \left(\hat{f}_x - \hat{f}_u \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) X - \hat{f}_u \hat{H}_{uu}^{-1} \hat{f}_u^* \Psi \tag{A.16}$$

$$-\dot{\Psi} = \left(\hat{H}_{xx} - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) X + \left(\hat{f}_x^* - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{f}_u^*\right) \Psi, \tag{A.17}$$

with boundary conditions

$$X(0) = 0,$$

leads to the following Corollary.

Corollary 1 (Zeidan, 1994 Corollary 5.2) *Under the conditions of Th. 3, there exists a Lipschitz continuous matrix function* R *satisfying on* $(0, T)$ *the equation*

$$\dot{R} + \hat{f}_x^* R + R \hat{f}_x + \hat{H}_{xx} - \left(R \hat{f}_u + \hat{H}_{xu}\right) \hat{H}_{uu}^{-1} \left(\hat{f}_u^* R + \hat{H}_{ux}\right) = 0. \tag{A.18}$$ □

PROOF (A short sketch) Define on $(0, T)$

$$R = \Psi X^{-1},$$

where (X, Ψ) solves Eqs. A.16-A.17. under strong normality assumptions it can be shown that R is symmetric and X^{-1} exists. What remains to be shown is that R is a solution of Eq. A.18, but for this we only have to calculate the time derivative of R

$$\begin{aligned}
\dot{R} &= \dot{\Psi} X^{-1} - \Psi X^{-1} \dot{X} X^{-1} \\
&= -\left(\hat{H}_{xx} - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) X X^{-1} - \left(\hat{f}_x^* - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{f}_u^*\right) \Psi X^{-1} \\
&\quad + \Psi X^{-1} \left(\hat{f}_x - \hat{f}_u \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) X X^{-1} + \hat{f}_u \hat{H}_{uu}^{-1} \hat{f}_u^* \Psi X^{-1} \\
&= -\left(\hat{H}_{xx} - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) - \left(\hat{f}_x^* - \hat{H}_{xu} \hat{H}_{uu}^{-1} \hat{f}_u^*\right) R \\
&\quad + R \left(\hat{f}_x - \hat{f}_u \hat{H}_{uu}^{-1} \hat{H}_{ux}\right) + R \hat{f}_u \hat{H}_{uu}^{-1} \hat{f}_u^* R \\
&= -\hat{f}_x^* R - R \hat{f}_x - \hat{H}_{xx} + \left(R \hat{f}_u + \hat{H}_{xu}\right) \hat{H}_{uu}^{-1} \left(\hat{f}_u^* R + \hat{H}_{ux}\right).
\end{aligned}$$ ■

A.4 Conjugate Point and Shock

As the term shock has different meaning in economic applications it has to be pointed out, that shock in this context is understood as a critical point, where the solution of a differential equation becomes unbounded. This term arises from physical applications, where e. g. shock waves of fluids or gases are considered.

But at the beginning a somewhat different definition of shock, as it can be found in (Caroff & Frankowska, 1996), is given. Therefore we consider the canonical system of (F)

$$\begin{aligned} \dot{x} &= H_\lambda \\ \dot{\lambda} &= -H_x, \end{aligned} \tag{CS}$$

with boundary conditions $x(0) = x_0, \lambda(T) = \lambda_T$. Then a shock is defined as

Definition 7 The system (F) has a shock at time $c \in (0, T)$ if there exist two solutions $(x_i, p_i), i = 1, 2$, of (CS) such that

$$x_1(c) = x_2(c) \quad \text{and} \quad \lambda_1(c) \neq \lambda_2(c)$$

In (Caroff & Frankowska, 1996 Theorem 2.3 and Theorem 5.1) it has been proven that this definition of shock is equivalent (under convenient assumptions like the existence of an optimal solution and the strengthended Legendre-Clebsh condition) to the statement, that the Riccati differential equation Eq. A.18 has an unbounded solution in c.

Therefore we can give a new definiton of a shock at time c,

Definition 8 Let the pair $(\hat{x}, \hat{u})$ be an extremal for (F). Then (F) has a shock at time c with

$$c = \inf_{t \in (0,T)} \{R \text{ is bounded on } [t, T]\}, \tag{A.19}$$

where R is a solution for the Riccati differential equation Eq. A.18 along the path $(\hat{x}, \hat{u})$.

Having in mind the results of Corollary 1 the relation to a conjugate point can be analysed. Noting that if c is conjugate to T, then $\det(X(c)) = 0$

(Def. 6) and considering Corollary 1 we know that X is continuous and as Ψ can be assumed bounded,

$$\lim_{t\to c+} \det(R(t)) = \lim_{t\to c+} \det(\Psi(t)) \det\left(X^{-1}(t)\right) = \infty,$$

therefore R is unbounded at c satisfying Def. 8. Hencefor it has been shown that every conjugate point is a shock.

A.5 Sufficient Second Order Conditions

Within the concept of local optimality sufficient conditions can be given. These sufficient conditions are stated as second order conditions and are related to the Riccati differential equation Eq. A.18 in matrix form. In the sequel I give a short summary of the main results of sufficient second order conditions as far as they are important for this thesis. For further details see e. g. (Zeidan, 1994), (Caroff & Frankowska, 1996) or (Maurer & Pickenhain, 1995).

Now the following theorem of sufficiency results for weak and strong local minimality of (F) can be stated

Theorem 4 (Zeidan, 1994) *Let $(\hat{x}, \hat{u})$ be an normal extremal. Suppose in addition that*

1. *$\hat{H}_{uu}$ satisfies the strenghtend Legendre-Clebsch condition*
2. *there exists a symmetric bounded matrix function R satisfying on $[0, T]$ the Riccati matrix differential equation*

$$\dot{R} + \hat{f}_x^* R + R\hat{f}_x + \hat{H}_{xx} - \left(R\hat{f}_u + \hat{H}_{xu}\right) \hat{H}_{uu}^{-1} \left(\hat{f}_u^* R + \hat{H}_{ux}\right) = 0 \quad \text{(A.20)}$$

holds. Then $(\hat{x}, \hat{u})$ is a weak local minimum for (F). □

Considering Def. 8 then Condition 2 of Th. 4 can be stated as the non-existence of a shock (see, e.g., Caroff & Frankowska, 1996).

A.5.1 Restating the Riccati Differential Equation

The Riccati differential equation Eq. A.20 can be written in a more compact form for state x and costate λ (see, e.g., Caroff & Frankowska, 1996) as

$$\dot{R} + H^{\circ}_{\lambda x}R + RH^{\circ}_{x\lambda} + RH^{\circ}_{\lambda\lambda}R + H^{\circ}_{xx} = 0, \tag{A.21}$$

where H° is the minimized Hamiltonian H.

In the sequel I use the Riccati matrix differential equation in the representation of Eq. A.21, therefore the equivalence of both representation has to be shown.

Lemma 1 *Let $H^{\circ}(x, \hat{u}, \lambda, t) = \min_{u \in U} H(x, u, \lambda, t)$ be the minimized Hamiltonian, such that H takes its minimum for $\hat{u}$ with $H_u(x, \hat{u}, \lambda, t) = 0$* □

PROOF Applying the implicite function theorem we get

$$\begin{aligned} &H^{\circ}(x, \lambda, t) = H(x, \hat{u}(x, \lambda), \lambda, t) \\ &\text{with} \quad H_u(x, \hat{u}, \lambda, t) = 0. \end{aligned} \tag{A.22}$$

The derivatives of u_x and u_λ can therefore be calculated as

$$u_x = -H_{uu}^{-1}H_{ux} \tag{A.23}$$

$$u_\lambda = -H_{uu}^{-1}H_{u\lambda} = -H_{uu}^{-1}f_u. \tag{A.24}$$

Next we have to calculate the second order partial derivatives of H° in respect to x and λ.

Using Eqs. A.22-A.24 and omitting the functional arguments for notational clearness we get

$$\begin{aligned} H^{\circ}_x &= g_x + g_u u_x + \lambda(f_x + f_u u_x) \\ H^{\circ}_\lambda &= g_u u_\lambda + f + \lambda f_u u_\lambda \end{aligned}$$

and

$$\begin{aligned} H^{\circ}_{x\lambda} &= g_{xu}u_\lambda + g_{uu}u_\lambda u_x + g_u u_{x\lambda} + f_x + f_u u_x + \lambda(f_{xu}u_\lambda + f_{uu}u_\lambda u_x + f_u u_{x\lambda}) \\ &= H_{xu}u_\lambda + H_{uu}u_\lambda u_x + H_u u_{x\lambda} + f_x + f_u u_x \\ &= -H_{xu}H_{uu}^{-1}f_u - f_u u_x + f_x + f_u u_x \end{aligned} \tag{A.25}$$

$$
\begin{aligned}
&= -H_{xu}H_{uu}^{-1}f_u + f_x \\
H^{\circ}_{\lambda x} &= g_{ux}u_\lambda + g_u u_{\lambda x} + f_x + \lambda(f_{ux}u_\lambda + f_u u_{\lambda x}) \\
&= H_{ux}u_\lambda + H_u u_{\lambda x} + f_x \\
&= -H_{ux}H_{uu}^{-1}f_u + f_x \\
H^{\circ}_{\lambda\lambda} &= g_{uu}u_\lambda u_\lambda + g_u u_{\lambda\lambda} + f_u u_\lambda + f_u u_\lambda + \lambda(f_{uu}u_\lambda u_\lambda + f_u u_{\lambda\lambda}) \\
&= H_{uu}u_\lambda u_\lambda + H_u u_{\lambda\lambda} + 2f_u u_\lambda \\
&= -f_u u_\lambda + 2f_u u_\lambda \\
&= -f_u H_{uu}^{-1} f_u \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.26) \\
H^{\circ}_{xx} &= g_{xx} + g_{xu}u_x + g_{ux}u_x + g_{uu}u_x u_x + g_u u_{xx} + \\
&\quad \lambda(f_{xx} + f_{xu}u_x + f_{ux}u_x + f_{uu}u_x u_x + f_u u_{xx}) \\
&= H_{xx} + H_{xu}u_x + H_{ux}u_x + H_{uu}u_x u_x + H_u u_{xx} \\
&= H_{xx} - H_{xu}H_{uu}^{-1}H_{ux} + H_{ux}u_x - H_{ux}u_x \\
&= H_{xx} - H_{xu}H_{uu}^{-1}H_{ux},
\end{aligned}
$$

Substituting this formulas in Eq. A.21 we get

$$
\begin{aligned}
\dot{R} + \left(-H_{ux}H_{uu}^{-1}f_u + f_x\right)R + R\left(-H_{xu}H_{uu}^{-1}f_u + f_x\right) + \\
R\left(-f_u H_{uu}^{-1} f_u\right) + H_{xx} - H_{xu}H_{uu}^{-1}H_{ux} = 0 \\
\dot{R} + f_x R + R f_x + H_{xx} - (R f_u + H_{xu})\, H_{uu}^{-1}\,(f_u R + H_{ux}) = 0,
\end{aligned}
$$

what had to be shown. ■

Appendix B

Riccati Equation in Economic Models

This chapter consists of four sections, where the first section contains an extension of the sufficient second order Th. 4, for infinite time horizon problems. In the next section the Riccati differential equation is introduced for current value terms. In the third section the solutions of the Riccati differential equation are studied at steady states. While in the last section the solutions behaviour at steady states are related to the corresponding Jacobian for the associated canonical system of the optimal control model.

Furthermore I restate the class of models this thesis is dealing with and which is introduced in Sec.2.2. This general class of models is given as

$$\begin{aligned} &\min_u \int_0^\infty e^{-rt}\left(g(x)+cu^2\right)dt \\ &\text{s. t.} \quad \dot{x} = f(x) + u \\ &\text{and} \quad x(0) = x_0, \end{aligned} \tag{GM}$$

which is an infinite horizon problem, where x denotes the state variable, u the control variable and $r > 0$ is a discounting rate, while $c > 0$ is an exogenous parameter.

For notational simplicity we denote the Hamiltonian evaluated along an extremal $(\hat{x}, \hat{u})$ by H instead of $\hat{H}$.

B.1 Sufficiency for Infinite Time

As we are interested in the infinite time horizon problem (GM) the following slightly modified version of theorem Th. 4 can be stated. It is assumed that

1. the optimal pair $(\hat{x}, \hat{u})$ converges to a stationary point (x^*, u^*), where such a convergence is often met in autonomous control problems (see e. g. Feichtinger & Hartl, 1986).
2. $f, g \in C^2$.

Then the theorem can be stated as

Theorem 5 *Let $(\hat{x}, \hat{u})$ be a normal extremal of (GM), sufficing conditions 1) and 2) of Th. 4 and conditions 1) and 2) stated before. Then $(\hat{x}, \hat{u})$ is a weak local minimum for the infinite time horizon problem (GM).* □

PROOF First we note that (x^*, u^*) is a saddle (see e. g. Wagener, 2003) and $(\hat{x}, \hat{u})$ a saddle path. Furthermore (x^*, u^*) itself is a locally optimal solution. Therefore we have to consider Eq. A.26, Eq. B.17 (see Sec. B.4) and the characterization of a saddle, which has to satisfy the following condition (see Sec. B.4

$$(r - 2H^\circ_{x\lambda})^2 - 4H^\circ_{xx}H^\circ_{\lambda\lambda} > 0. \tag{B.1}$$

From Eq. A.26 we conclude that $H^\circ_{\lambda\lambda} < 0$ as the Legendre-Clebsh condition $H_{uu} > 0$ holds. As Eq. B.1 has to be positiv this yields $H^\circ_{xx} > 0$, which is a sufficient condition for a local minimum (see e. g. Feichtinger & Hartl, 1986, Theorem 2.4).

Now we have proven the local optimality of (x^*, u^*), we can conclude the local optimality of the stable manifold in a small neighbourhood of the saddle. This can be seen on considering that $f, g \in C^2$ and hencefor $H^\circ_{xx} > 0$ holds also in a sufficient small neighbourhood around (x^*, u^*).

Let $\tilde{x}$ denote a point on the part of the stable path, which has been proven to be locally optimal, and T_0 the time, reaching $\tilde{x}$ from $\hat{x}(0)$. But as $(\hat{x}, \hat{u})$ is locally optimal for every finite time under end constraint $x(T) = \hat{x}(t) = \tilde{x}$ we have shown the local optimality for the whole stable path and hencefor for the infinite time problem (GM). ■

B.2 Current Value Riccati Equation

In this section an optimal control problem of the following type, which often occurs in economic applications, is given

$$\left.\begin{array}{lll} \min\limits_{\nu} \displaystyle\int_0^\infty e^{-rt} g(x,u,t)dt & & \\ \text{s.t.} \quad \dot{x} = f(x,u,t) & & (\text{B.2}) \\ \text{and} \quad x(0) = x_0, & & (\text{B.3}) \end{array}\right\} \quad (\text{E})$$

where f, g, x and u satisfy the usual conditions.

To avoid confusion I have to make some notational remarks. In the last section H denoted the usual Hamiltonian as is used for optimal control problems of type (F). In economic models like those of type (E) the same letter H denotes the current value Hamiltonian. To properly distinguish between these two types of Hamiltonian. $\tilde{H}$ denotes the present value Hamiltonian while H is used for the current value Hamiltonian. Given this notational convenience the following relations can be stated

$$\begin{aligned} H &= g(x,u,t) + \lambda f(x,u,t) \\ \tilde{H} &= e^{-rt} g(x,u,t) + \tilde{\lambda} f(x,u,t) = He^{-rt} \quad \text{with} \\ \tilde{\lambda} &= \lambda e^{-rt}. \end{aligned} \tag{B.4}$$

Lemma 2 *For the current value Hamiltonian H the Riccati differential equation becomes*

$$\dot{R} + H^\circ_{\lambda x} R + R H^\circ_{x\lambda} + e^{rt} R H^\circ_{\lambda\lambda} R + e^{-rt} H^\circ_{xx} = 0, \tag{B.5}$$

where H° is the minimized current value Hamiltonian. □

PROOF As the factor e^{-rt} does not depend on u, $\tilde{H}^\circ = H^\circ e^{-rt}$ holds.

Using Eq. B.4 the following equations hold

$$\frac{\partial \lambda}{\partial \tilde{\lambda}} = e^{rt} \tag{B.6}$$

$$\frac{\partial \lambda^2}{\partial \tilde{\lambda}^2} = 0 \tag{B.7}$$

Hencefor

$$\begin{aligned}
\tilde{H}_{\tilde{\lambda}\tilde{\lambda}} &= \frac{\partial}{\partial\tilde{\lambda}}\frac{\partial\tilde{H}}{\partial\tilde{\lambda}} \\
&= e^{-rt}\frac{\partial}{\partial\tilde{\lambda}}\frac{\partial H}{\partial\lambda}\frac{\partial\lambda}{\partial\tilde{\lambda}} \\
&= \frac{\partial^2 H}{\partial\lambda^2}\frac{\partial\lambda}{\partial\tilde{\lambda}} \\
&= e^{rt}H_{\lambda\lambda} \\
\tilde{H}_{x\tilde{\lambda}} &= H_{x\lambda} \\
\tilde{H}_{xx} &= e^{-rt}H_{xx}
\end{aligned}$$

Substituting into Eq. A.21 leads to

$$\dot{R} + H^{\circ}_{\lambda x}R + RH^{\circ}_{x\lambda} + e^{rt}RH^{\circ}_{\lambda\lambda}R + e^{-rt}H^{\circ}_{xx} = 0,$$

what had to be shown. ■

B.3 Local Optimality at Steady States

In general the Riccati differential equation cannot be solved analytically. Nevertheless in the special one state case of staying exactly at a steady state (x^*, u^*), which is always an extremal, one can find an explicit solution of Eq. B.5. Although this seems only a trivial degenerated case of marginal interest, it gives insight to the locally optimal strategies as analysed in Sections 3.2.7, 4.2.7 and 5.2.6.

Setting $a = H^{\circ}_{\lambda\lambda}(x^*, u^*)$, $b = 2H^{\circ}_{x\lambda}(x^*, u^*)$ and $c = H^{\circ}_{xx}(x^*, u^*)$ and substituting in Eq. B.5 the Riccati differential equation is reduced to

$$\dot{p} = e^{rt}ap^2 + bp + e^{-rt}c, \tag{B.8}$$

with $a, b, c \in \mathbb{R}$.

Without loss of generality it can be assumed that $a \geq 0$. In case of $a < 0$ we substitute p by $-p$ which yield

$$\dot{p} = -e^{rt}ap^2 + bp - e^{-rt}c,$$

and setting $\tilde{a} = -a$ and $\tilde{c} = -c$ we get Eq. B.8. In the following part we analyse the possible explicit solutions for this kind of Riccati differential equation. Furthermore Eq. B.8 can be simplified by setting $q = pe^{rt}$, which leads to $(\dot{q} - rq)e^{-rt} = \dot{p}$. Hencefor Eq. B.8 is replaced by

$$\dot{q} = aq^2 + (b + r)q + c. \tag{B.9}$$

Rewriting this equation we obtain

$$\frac{dq}{aq^2 + (b + r)q + c} = dt,$$

which can be solved analytically.

The solution of this differential equation depends on the factorization of the quadratic polynomial $aq^2 + (b + r)q + c$. The possibly complex roots of this quadratic equation are given by

$$q_{1,2} = \frac{-(b + r) \pm \sqrt{(b + r)^2 - 4ac}}{2a} = \gamma_1 \pm \gamma_2 \tag{B.10}$$

with $\gamma_1 = \frac{-(b+r)}{2a}$ and $\gamma_2 = \frac{\sqrt{(b+r)^2 - 4ac}}{2a}$. The crucial part for the solutions behaviour depends on the discriminant $D = (b + r)^2 - 4ac$. Depending on the sign of D three cases can be distinguished.

B.3.1 Positive Discriminant D

For $(b+r)^2 - 4ac > 0$ we have two different real valued solutions and Eq. B.9 is solved by

$$\begin{aligned}
\dot{q} &= a(q - \gamma_1 - \gamma_2)(q - \gamma_1 + \gamma_2) \\
\frac{dq}{a(q - \gamma_1 - \gamma_2)(q - \gamma_1 + \gamma_2)} &= dt \\
\frac{1}{2a\gamma_2} \ln \frac{q - \gamma_1 - \gamma_2}{q - \gamma_1 + \gamma_2} &= t + C
\end{aligned}$$

Under boundary condition $q(T) = q_T$ the integration constant C can be specified by

$$C = \frac{1}{2a\gamma_2} \ln \frac{q_T - \gamma_1 - \gamma_2}{q_T - \gamma_1 + \gamma_2} - T.$$

Abbreviating $\Gamma = \frac{q_T - \gamma_1 - \gamma_2}{q_T - \gamma_1 + \gamma_2}$ we find the explicit solution

$$\begin{aligned} \frac{q - \gamma_1 - \gamma_2}{q - \gamma_1 + \gamma_2} &= \Gamma e^{(t-T)2a\gamma_2} \\ q(1 - \Gamma e^{(t-T)2a\gamma_2}) &= \gamma_1 + \gamma_2 + (\gamma_2 - \gamma_1)e^{(t-T)2a\gamma_2}\Gamma \\ q &= \frac{(\gamma_2 - \gamma_1)\left(1 + \Gamma e^{(t-T)2a\gamma_2}\right) + 2\gamma_1}{1 - \Gamma e^{(t-T)2a\gamma_2}} \end{aligned}$$

Resubstitution of p for q leads to

$$p = \frac{(\gamma_2 - \gamma_1)\left(1 + \Gamma e^{(t-T)2a\gamma_2}\right) + 2\gamma_1}{1 - \Gamma e^{(t-T)2a\gamma_2} e^{-rt}}. \tag{B.11}$$

The solution of this equation is bounded as long as the denominator of Eq. B.11 is unequal zero and hencefor

$$t \neq \frac{1}{2a\gamma_2} \ln \frac{1}{\Gamma} + T.$$

As $t \in [0, T]$ and $\gamma_2, a > 0$ by definition we only have to prove that $\ln \frac{1}{\Gamma} > 0$. This condition is accomplished for $\Gamma > 1$. This can always be achieved by choosing the boundary condition p_T such that $p_T - \gamma_1 + \gamma_2 < 0$.

B.3.2 Zero Discriminant D

For $(b+r)^2 - 4ac = 0$ the quadratic equation has a double root at $\gamma_1 = \frac{-(b+r)}{2a}$ and therefor Eq. B.9 reduces to

$$\dot{q} = a(q - \gamma_1)^2,$$

which is solved by

$$\begin{aligned} \dot{q} &= a(q - \gamma_1)^2 \\ \frac{dq}{a(q - \gamma_1)^2} &= dt \\ -\frac{1}{a(q - \gamma_1)} &= t + C, \end{aligned} \tag{B.12}$$

for the boundary condition $q(T) = q_T$ the integration constant can be calculated as

$$C = \frac{1}{a\Gamma} - T,$$

with $\Gamma = q_T + \gamma_1$.

Equivalence transformation of Eq. B.12 and substitution of the C leads to

$$\begin{aligned} -\frac{1}{a(q-\gamma_1)} &= t - T + \frac{1}{a\Gamma} \\ q &= -\frac{\Gamma}{a\Gamma(t-T)+1} + \gamma_1 \\ q &= \frac{\gamma_1\left(a\Gamma\left(t-T\right)+1\right)-\Gamma}{a\Gamma\left(t-T\right)+1}, \end{aligned}$$

and after resubstituting $q = e^{-rt}p$ we get the explicit solution

$$p = \frac{\hat{p}\left(a\Gamma\left(t-T\right)+1\right)-\Gamma}{a\Gamma\left(t-T\right)+1}e^{-rt}, \tag{B.13}$$

with $\hat{p} = \gamma_1 e^{rt}$.

This solution is bounded as long as the denominator does not become zero, which is characterized by

$$t \neq T - \frac{1}{a\Gamma}.$$

Noticing that $t \in [0, T]$ this is accomplished for $a\Gamma < 0$. This can always be guaranteed as we are free in choosing the boundary condition p_T.

B.3.3 Negative Discriminant D

In the case of a negative discriminant $(b+r)^2 - 4ac < 0$ the quadratic equation cannot be factorized to real linear factors. Therefore Eq. B.9 has to be written as

$$\begin{aligned} \dot{q} &= aq^2 + (b+r)q + c \\ \frac{dq}{aq^2+(b+r)q+c} &= dt \\ \frac{2}{\sqrt{4ac-(b+r)^2}} \arctan \frac{2aq+b+r}{\sqrt{4ac-(b+r)^2}} &= t + C. \end{aligned} \tag{B.14}$$

For the boundary condition $q(T) = q_T$ and setting $D = \sqrt{4ac - (b+r)^2}$ the integration constant is equal to

$$C = \frac{2}{D} \arctan \frac{2aq_T + b + r}{D} - T,$$

and hencefor the explicit solution for Eq. B.14 is given by

$$\begin{aligned}
\arctan \frac{2aq + b + r}{D} &= (t - T)\frac{2}{D} + \arctan \frac{2aq_T + b + r}{D} \\
\frac{2aq + b + r}{D} &= \tan\left((t - T)\frac{2}{D} + \arctan \frac{2aq_T + b + r}{D}\right) \\
\frac{2aq + b + r}{D} &= \frac{\tan\left((t-T)\frac{2}{D}\right) + \frac{2aq_T+b+r}{D}}{1 - \frac{2aq_T+b+r}{D}\tan\left((t-T)\frac{2}{D}\right)} \\
2aq &= \frac{\tan\left((t-T)\frac{2}{D}\right)(D^2 + (b+r)^2 + (b+r)2aq_T) + 2Daq_T}{D - \frac{2aq_T+b+r}{D}\tan\left((t-T)\frac{2}{D}\right)}
\end{aligned}$$

Considering that $D^2 + (b+r)^2 = 4ac$ we get

$$q = \frac{\tan\left((t-T)\frac{2}{D}\right)(2 + (b+r)q_T) + Dq_T}{D - \frac{2aq_T+b+r}{D}\tan\left((t-T)\frac{2}{D}\right)}.$$

At last resubstituting $q = e^{-rt}p$ the solution can be written as

$$p = \frac{\tan\left((t-T)\frac{2}{D}\right)(2 + (b+r)p_T) + Dp_T}{(D - 2ap_T + b + r)\tan\left((t-T)\frac{2}{D}\right)} e^{-rt}, \tag{B.15}$$

where $p_T = e^{rT}q_T$.

As we are interested in bounded solutions of Eq. B.15 we have to investigate the behaviour at the poles of Eq. B.15. Setting the denominator equal to zero we get the following equation

$$D - (2ap_T + b + r)\tan\left((t-T)\frac{2}{D}\right) = 0,$$

which can be solved for t

$$t = T + \frac{2}{D}\left(\arctan\left(\frac{D}{b + r + 2ap_T}\right)\right).$$

Taking into account that the tangens function is periodic with periodicity π this equation hold for every t with

$$t = T + \frac{2}{D}\left(\arctan\left(\frac{D}{b+r+2ap_T}\right) - k\pi\right) \qquad k \in \mathbb{N}$$

Choosing now k_0 as the minimal integer such that

$$\frac{2}{D}\left(\arctan\left(\frac{D}{b+r+2ap_T}\right) - k_0\pi\right) < 0$$

holds for $0 \leq t \leq T$ we can conclude that if

$$T \geq \left|\frac{2}{D}\left(\arctan\left(\frac{D}{b+r+2ap_T}\right) - k_0\pi\right)\right|$$

the solution of Eq. B.15 goes to infinity in finite time. Therefore no bounded solution of Eq. B.15 with $T = \infty$ exists.

B.4 Local Optimality and Jacobian

After having found a characterization for the boundedness of the solution of the Riccati differential equation for one state models staying at a steady state, we can analyse this characterization in respect to the models Jacobian J. The Jacobian J can be written in terms of the Hamiltonian H as

$$J = \begin{pmatrix} H_{x\lambda} & -H_{xx} \\ H_{\lambda\lambda} & r - H_{x\lambda} \end{pmatrix}. \tag{B.16}$$

To classify the different types of steady states, we have to determine the trace τ, determinante Δ and discriminant D, which can be calculated from Eq. B.16 as

$$\begin{aligned} \tau &= r \\ \Delta &= H_{x\lambda}(r - H_{x\lambda}) + H_{xx}H_{\lambda\lambda} \\ D' &= r^2 - 4H_{x\lambda}(r - H_{x\lambda}) - 4H_{xx}H_{\lambda\lambda}. \end{aligned}$$

Rewriting D we get

$$\begin{aligned} D' &= r^2 - 4H_{x\lambda}r + 4H_{x\lambda}^2) - 4H_{xx}H_{\lambda\lambda} \\ &= (r - 2H_{x\lambda})^2 - 4H_{xx}H_{\lambda\lambda}. \end{aligned}$$

For the one dimesional case Eq. B.5 simplifies to

$$\dot{R} = -e^{rt}H_{\lambda\lambda}R^2 - 2H_{x\lambda}R - e^{-rt}H_{xx}$$

and in current value notation

$$\dot{Q} = -H_{\lambda\lambda}Q^2 - (2H_{x\lambda} - r)Q - H_{xx}.$$

In the last section B.3 it has been shown that the crucial term for the boundedness of the solution for this differential equation is the term

$$D = (b + r)^2 - 4ac,$$

which becomes

$$D = (r - 2H_{x\lambda})^2 - 4H_{xx}H_{\lambda\lambda} = D', \tag{B.17}$$

in terms of the Hamiltonian. D' and hencefor D discriminate between unstable foci and unstable nodes or saddles, combining this with the results of the last Section B.3 we have shown, that for infinte time T the only possible candidates for local optimal solutions at steady states are unstable nodes and saddles.

Appendix C

Numerical Methods

In this chapter the numerical methods and its theoretical foundations are described as they were used to analyse the different models. The first section states the Theorem (Michel, 1982), which provides the theoretical tools to find the extremals and the corresponding objective value function. These extremal will be identified as the stable manifolds of saddles, hencefor in the next section an algorithm is presented for calculating stable manifolds at least for the one dimensional case. In the third section this algorithm is used to show how to find DNS points and thresholds respectively.

C.1 Necessary Conditions for Extremals

The crucial point for analysing an optimal control problem is of course finding the extremals $(\bar{x}, \bar{u})$ and the calculation of the objective function along these paths. For models of class (GM) the following theorem gives the decesive hint for finding these extremals:

Theorem 6 (Michel, 1982) *Given a control problem of the form:*

$$\begin{aligned} & V(x,u) = \int_0^\infty e^{-rt} g(x,u)dt \\ & \min_{u \in U} V(x,u) \\ & s.t. \quad \dot{x} = f(x,u) \\ & and \quad x(0) = x_0. \end{aligned} \tag{I}$$

Let the pair $(\hat{x}, \hat{u})$ be an optimal solution, then there exist a constant scalar λ_0 and a continuous costate variable λ, such that the following conditions are satisfied:

$$\lim_{t\to\infty} e^{-rt}H(\hat{x}, \hat{u}, \lambda_0, \lambda) = 0 \tag{C.1}$$

$$e^{-rt}H(\hat{x}, \hat{u}, \lambda_0, \lambda) = r\lambda_0 \int_t^\infty e^{-rs}g(x, u)ds, \tag{C.2}$$

where H is the current value Hamiltonian.
If additionally $g \geq 0$ for all admissible (x, u) and for t large enough 0 lies inside the set

$$E(\hat{x}) = \{u \in U : f(\hat{x}, u)\},$$

then λ satisfies the transversality condition

$$\lim_{t\to\infty} e^{-rt}\lambda(t) = 0. \tag{C.3}$$ □

As for models of class (GM) all assumptions for Th. 6 are satisfied the stated conditions are necessary condtions for an optimal path $(\hat{x}, \hat{u})$. Furthermore it was shown in Section 2.3 that $\lambda_0 = 1$ holds true.

From the Pontryagins minimum priciple a canonical system (CS) of differential equations (see Section 2.3) was derived. Considering transversality condition Eq. C.3 it can easily be seen that the stable manifolds M^s of saddles (x^*, u^*)

$$M^s(x^*, u^*) = \{(x, u) : \lim_{t\to\infty} \phi^t(x, u) = (x^*, u^*)\},$$

where ϕ^t is the flow associated to (CS), are candidates for optimal paths.

Furthermore Th. 6 provides together with Eq. C.2 a formula to calculate the objective function $V(x, u) = \int_0^\infty e^{-rt}g(x, u)dt$ without integrating along the extremal $(\bar{x}, \bar{u})$. For if $t = 0$ in Eq. C.1 this equation reduces to $H(\bar{x}(0), \bar{u}(0), \lambda_0, \lambda(0)) = r\lambda_0 \int_0^\infty e^{-rs}g(\bar{x}, \bar{u})ds$. Noting that along an extremal path the minimized Hamiltonian $H^\circ(\bar{x}, \bar{u}) = H(\bar{x}, \bar{u}, \bar{\lambda})$ and hencefor we get for $\lambda_0 = 1$

$$V(\bar{x}(0), \bar{u}(0)) = \frac{1}{r}H^\circ(\bar{x}(0), \bar{u}(0)) \tag{C.4}$$

Having these theoretical results in mind all we have to do is finding the steady states of the canonical system picking out the saddles, getting the corresponding stable manifolds and in light of Eq. C.4 calculating the objective function values.

C.2 Calculating the Stable Manifolds

The theoretical background for the calculation of the stable manifold of a saddle (x^*, u^*) is given by the Local Stable Manifold Theorem, which is stated here for the sake of completeness. A comprehensive treatment of this topic can be found e. g. in (Kuznetsov, 1998).

Let the dynamical sytem (in our case the canonical system) be denoted by

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n, \tag{C.5}$$

where f is differentiable. Let x^* be a hyperbolic steady state of saddle type. Furthermore the stable manifold M^s is defined by

$$M^s(x^*) = \{x : \lim_{t\to\infty} \phi^t(x) = x^*\}, \tag{C.6}$$

where ϕ^t is the flow associated to Eq. C.5. If n_- denotes the number of eigenvalues with negative real part of the Jacobian $J(x^*)$ then the following theorem can be stated

Theorem 7 (Local Stable Manifold) *Let x^* be a hyperbolic steady state. Then the intersection of $M^s(x^*)$ with a sufficiently small neighborhood of x^* contain smooth submanifolds $M^s_{\text{loc}}(x^*)$ of dimension n_-.*

Moreover, $M^s_{\text{loc}}(x^*)$ is tangent at x^* to T^s, where T^s is the generalized eigenspace corresponding t othe union of all eigenvalues μ of J with $Re\mu <$ 0. □

Applied to our two dimensional canonical system this theorem states that in a small neighborhood of a saddle (x^*, u^*), the local stable manifold can be written as

$$M^s_{\text{loc}}(x^*, u^*) = \{(x, u) : (x, u) = (x^*, u^*) + \kappa e_s, \mid \kappa \mid < \varepsilon\},$$

where e_s is the eigenvector for the eigenvalue μ with $\mathrm{Re}\mu < 0$ and ε small.

Noting that starting on the stable manifold one will end for some finite time in a small neighbourhood of (x^*, u^*) and therefore in $M^s_{\text{loc}}(x^*, u^*)$. Hencefor starting at a point inside $M^s_{\text{loc}}(x^*, u^*)$ and following the resulting flow in reversed time gives the stable manifold. That is we solve the canonical system for an initial starting point (x_0, u_0) with $(x_0, u_0) = (x^*, u^*) + \kappa_0 e_s$, where κ_0 is small enough, in reversed time. This provides us a numerical algorithm, where we only have to specify a small κ_0 and starting a differential equation solver, as provided for example in MATLAB©, for the initial starting point (x_0, u_0) as specified before for negative time. Doing this for κ_0 and $-\kappa_0$ we get the two branches of the stable manifold, connecting them at (x^*, u^*) the stable manifold is computed. A good presentation of this algorithm and how to choose κ_0 properly is give in (Parker & Chua, 1989).

Having now calculated the stable manifold, we obtain the objective function values $V(x, u)$ for every point of the stable manifold by using equation Eq. C.4.

Although this algorithm is also applicable to the higher dimensional case it has some drawbacks, like a strong dependence of the flows velocity starting from different initial positions and the problem of reconstructing a higher dimensional manifold from onedimensional paths. To overcome these problems other algorithms have been tested. One algorithm, which does in principle not depend on the manifolds dimension is presented in (Krauskopf & Osinga, 1998) and (Krauskopf & Osinga, 1999). But as it lacks an efficient algorithm sorting the detected points it is only usable for two dimensional manifolds. A completely different concept is given by (Dellnitz & Hohmann, 1997), where the manifolds are calculated using a set theoretical approach. For these algorithms exist a MATLAB© toolbox GAIO, for more information see http://www-math.uni-paderborn.de/ agdellnitz/gaio. The applicability of these algorithms to optimal contrrol problems will be tested in the near future.

C.3 Finding DNS Points and Thresholds

DNS points are defined as points where multiple optimal solutions exist (see Section 2.1). This means that the objective function values for different extremals have to be the same at this point. But as we have identified the extremals as the stable manifolds all we have to do is to find crossing points for the corresponding objective value functions. As the objective value functions are provided by the algorithm described in the last section it is no great effort to extend this algorithm for finding DNS points.

Actually my algorithm does not search for crossing points directly. Moreover every objective value function is numerated by the index (arbitrarily numeration of the occuring stable manifolds) of its corresponding stable manifold. A possible DNS point is now detected if the number of the objective value function changes on searching for the optimal path (see Fig. C.1a/b). Within this algorithm one do not only receive DNS points but also DNS thresholds (see Fig. C.1c). Because although there is no crossing of the objective value functions, there is a change of optimal solution. By construction different optimal solutions correspond to different stable manifolds and this is what my algorithm is actually searching for.

C.4 Finding Shocks

Given Def. 8 of a shock and reminding of Th. 4 it has been shown that an extremal $(\hat{x}, \hat{u})$ is locally optimal as long as no shock occurs. Furthermore if a shock occurs the solution of the Riccati differential equation becomes unbounded. In the proof of Th. 5 the infinite time horizon problem was approximated by a finite time problem. For numerical calculations the method of this proof can be used to find shocks. Therefore time T has to be chosen to come close enough to the steady state (saddle), then the Riccati differential equation is integrated numerically along $(\hat{x}, \hat{u})$ for time T and while the integration process the norm of the solution is screened. Exceeding a given threshold a shock is assumed and the integration process is stopped.

This method is straight forward but depends critically on the integration time T. As the integration process near a shock is ill conditioned the detection of a shock can fail. Therefore one is interested in another criterion when a shock can occur.

At least for the one dimensional case a (heuristic) argument for the existence of a shock can be given. Although no mathematical proof has been found so far.

Considering Def. 7 of a shock, searching for shocks reduces to find critical points, where at least two solutions of the canonical system of problem (GM) exist. These critical points can be identified as the crossing points of the extremal $(\hat{x}, \hat{u})$ and the $\dot{x}$-isocline. The argument for this conjecture works the following way. Denoting this crossing point as (x_c, λ_c) we conclude that there exist two solutions $(x_c, \lambda_c \pm d\lambda)$ (where $d\lambda$ is infinitesimally small) of the canconical system, with the same initial state x_c but different costates $\lambda_1 = \lambda_c + d\lambda$ and $\lambda_2 = \lambda_c - d\lambda$, which suffices Def. 7. Although only a heuristic argument we have found an confirmation for what has been proven numerically (see Fig. C.1d).

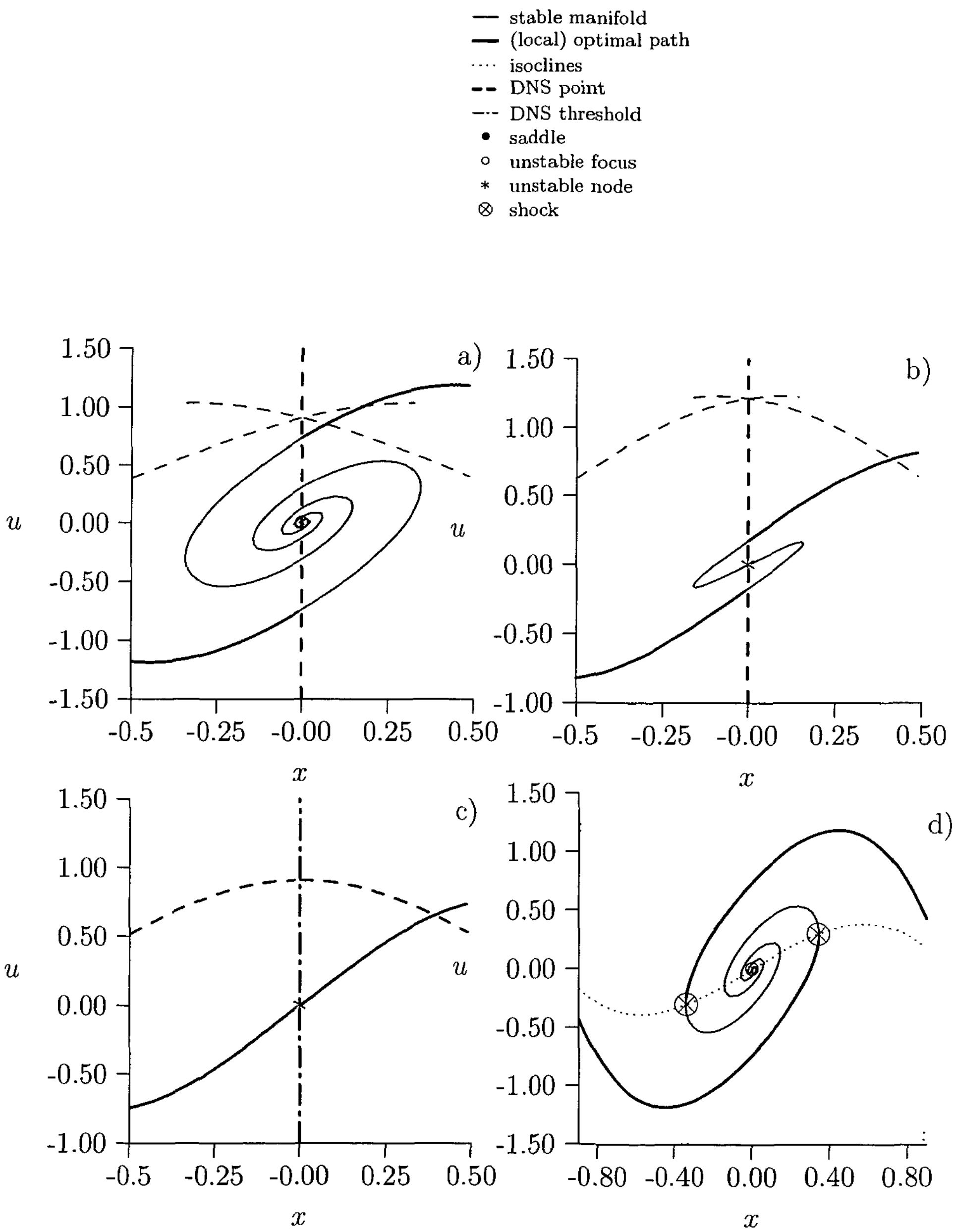

Figure C.1: Depicts the case of a DNS point at an a) unstable focus b) an unstable node, c) a DNS threshold and d) shocks along locally optimal paths.

Appendix D

The Clash of Definitions

When Skiba (1978) and Dechert and Nishimura (1983) explored the phenomenon of points of indifference, they considered a special class of economic models formulated as optimal control problems, namely the classical one-sector problem of optimal growth with a non-convex production function. There was no need to investigate this phenomenon in a very general and extended context. Relating on these studies the term Skiba point and DNS point respectively was introduced for points of indifference, where the objective value of different paths was equal and hencefor the decision maker indifferent which to choose. The most important case, where such a behaviour can be expected in the one dimensional case is a focus lying between two steady states of saddle type. But while this constellation does not necessarily imply the occurence of points of indifference (see e. g. Region IIa in MoM or Region IVa in GMoM and MoBB) this property can also occur in case of an unstable node (see e. g. IIIb in MoM and MoBB), lying between two saddles. Aside from the possibility that there exists an unique optimal solution (see e. g. IIIa in MoM and MoBB), one has to distinguish a different optimal behaviour for models with an unstable node. In most instances the optimal paths are continuously connected at the unstable node and the steady state itself becomes a global optimal solution (see e. g. IIIc in MoM and MoBB). A point of indifference can only occur if there exists an overlap near the unstable node for the projection of two extremals into the state space. So far no analytical argument can be given for the occurence of such an overlap, therefore one has to do numerical calculations for detecting points of indifference near an unstable node (see Appendix C). Refering to points of indifference as DNS points and having the three different cases in mind one is in favour defining a DNS point in the following way, where it is assumed that the extremals considered in the following definition are the only possible candidates for optimal paths.

Definition 1 A point in the state space is called a DNS point if there exist at least two different extremals with the same objective value and a discontinuity of the control in this point.

But this definition is problematic, because continuity and discontinuity of the control is not inherent to the formulation of the optimal control problem and depends on the underlying norm.

Considering Def. 1 more closely it can be seen that the added property of discontinuity for the control is neither necessary to distinguish the two possible cases at an unstable node nor even adapted to the the property of a point of indifference as will be seen in the following example.

This example is a one-dimensional modification of a two-dimensional problem stated by Vladimir Veliov, but as he pointed out it is rather different from what he actually meant. The problem stated here is more pathologic than the original one, as the solution set is not closed. Nevertheless it is very interesting as it allows to show the limitations of the hitherto given definition of a DNS point. Assuming the subsequent problem

$$\begin{aligned}
&\min_u \int_0^\infty e^{-t}\left(x^2\left(1-u^2\right)^2 u^2\right)dt \\
&\text{s.t.}\quad \dot{x} = kx \quad 0<k<1 \quad \text{(D.1)} \\
&\text{and}\quad x(0)=x_0,
\end{aligned} \tag{P}$$

it can be seen immediately that the minimal costs are 0 and therefore in case of $x_0 = 0$, the control u can be choosen arbitrarily, while $x_0 \neq 0$ yields $u = 0, \pm 1$. On the other hand we can also use the principle of Pontryagin to state the necessary conditions for problem (P). Hence the Hamiltonian H is given by

$$H = x^2\left(1-u^2\right)^2 u^2 + \lambda k x,$$

The minimum condition for u leads to the equations

$$\begin{aligned}
H_u &= x^2\left(6u^5 - 8u^3 + 2u\right) \\
H_{uu} &= x^2\left(30u^4 - 24u^2 + 2\right)
\end{aligned}$$

which proofs our assertion that u is arbitrary for $x_0 = 0$ and has the minimum solutions $u = 0, \pm 1$ in case of $x_0 \neq 0$.

Using $H_x = 2x\left(1-u^2\right)^2 u^2 + \lambda k$ and the property $\left(1-u^2\right)^2 u^2 = 0$ for a minimal u the costate equation is given by

$$\dot{\lambda} = \lambda(1-k). \tag{D.2}$$

As the canonical system of (P) defined by Eqs. D.1-D.2 is decoupled, the explicit solutions for x and λ can be written as

$$\begin{aligned} x(t) &= x_0 e^{kt} \\ \lambda(t) &= \lambda_0 e^{(1-k)t}. \end{aligned} \tag{D.3}$$

Since the transversality condition for the admissible k is satisfied for every initial λ_0 no further necessary conditions can be stated for extremals. In Fig. D.1a the phase portraits of possible extremals for different initial starting positions x_0 and λ_0 are shown. But as we can see in Fig. D.1a or deduce from Eq. D.3 the projection into the state space remains unaffected by the choice of r. Furthermore the extremals provide no information how to choose an optimal control $u(t)$. Since $u(t)$ hast to be absolutely continuous only almost everywhere, we can build up any optimal control as a step function with the possible values ± 1 and 0. Valid realisations are shown in Fig. D.1b. Trying now to apply Def. 1 to problem (P) we get into troubles. First the usage of the term extremals in Def. 1 is not well adapted for problem (P) since the costate dynamic is of no importance for the optimal behaviour. Only the projection into the state space is of interest and determinate the optimal solutions behaviour. Second if we assume the difference in the controls as critical demanding a discontinuity at the initial control we would fail to notice a DNS point. Consider for example two controls with the same value for the timespan $[0, t_0]$, but differing in some interval for $t > t_0$. These would give two different optimal solutions starting from the same initial position x_0 but they are not discontinuous at the initial controls. Therefore the condition of discontinuity in Def. 1 has to be replaced by another criterion.

As this example has shown the critical term in our definition of a DNS point are the "different extremals" or "different optimal solutions". Thus "different" has to be specified and the definition will depend on what cases we want to distinguish. If different controls leading to the same optimal behaviour of the state variable are wanted to be differentiated, the controls have to be considered. If only the projection into the state space counts the controls will not be considered in the definition of a DNS point. One could say that this example is degenerated and under stronger constraints such

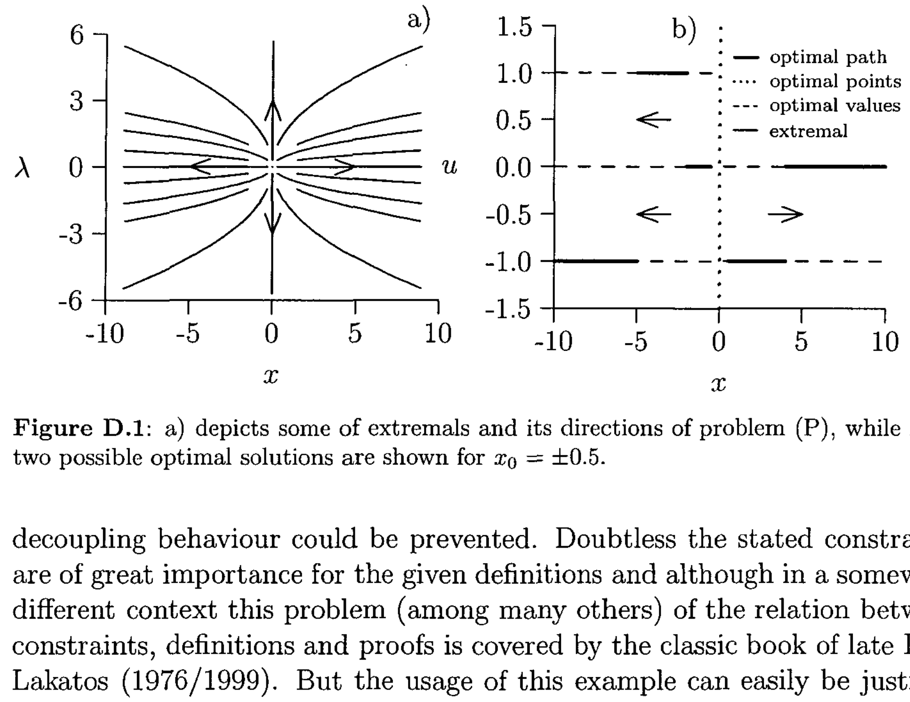

Figure D.1: a) depicts some of extremals and its directions of problem (P), while in d) two possible optimal solutions are shown for $x_0 = \pm 0.5$.

decoupling behaviour could be prevented. Doubtless the stated constraints are of great importance for the given definitions and although in a somewhat different context this problem (among many others) of the relation between constraints, definitions and proofs is covered by the classic book of late Imre Lakatos (1976/1999). But the usage of this example can easily be justified even in the case of rather "normal" applied models. If we consider the slightly different version of state dynamic Eq. D.3

$$x(t) = x_0 e^{kt} + \rho u, \tag{D.4}$$

the decoupling behaviour is abolished. Now it could happen that ρ becomes zero or almost zero, which leads to the considered problem (P). In this case one has to decide if this "degenerated" model is meaningful in the given context or not. But nevertheless the definition of a DNS point has to account for this problem.

Summing up these considerations the following definitions can be given, where the model under consideration is given by

$$\begin{aligned} &\min_u \int_0^T g(x,u)dt \\ &\text{s.t.} \quad \dot{x} = f(x,u) \\ &\text{and} \quad x(0) = x_0, \end{aligned} \tag{A}$$

Definition 2 A point x_0 in the state space is called a DNS point if there exist at least two optimal solutions $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$ for model (A), with $\|x_1(\cdot) - x_2(\cdot)\| > 0$.

Remark 2 From Def. 2 it is clear that the optimal solutions $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$ have the same objective value at $t = 0$.

Remark 3 These definitions are stated without refering to the dimension of the considered model. Hencefor in higher dimensions sets of DNS points could be classified as DNS curves or DNS manifolds. Nevertheless questions concerning properties, like e. g. continuity of DNS curves/manifolds, arise in higher dimensions which have no counterpart in the one-dimensional case.

Remark 4 In the light of Def. 2 one is only interested in deviations of the state variable x. Therefore applying Def. 2 to problem (P) there exist no DNS point, as for every initial poisition x_0 the corresponding optimal state variable is uniquely determined by $x(t) = x_0 e^{kt}$.

An alternative definition can be stated

Definition 3 A point x_0 in the state space is called a DNS point if there exist at least two optimal solutions for model (A) where the following condition holds $\| (x_1(\cdot), u_1(\cdot)) - (x_2(\cdot), u_2(\cdot)) \| > 0$.

Remark 2 clearly remains valid for Def. 3.

Remark 5 Although it has been shown that the property of (dis)continuous opimal control is not well adpated to give a rigour mathematical definition of a DNS point and in the following for a DNS threshold it is used in the descriptions of the optimal policies for the models of this thesis. This can be done as it has a meaningful interpretation and coincide with the given definitions for the considered models.

Remark 6 Applying Def. 3 to problem (P) every point of the state space is a DNS point, as for every initial position x_0 at least two different controls $u_1(t)$ and $u_2(t)$ can be specified (see e. g. Fig. D.1b).

For the majoritiy of "normal" cases Def. 2 and Def. 3 will describe an identic set of DNS points. Nevertheless they can produce very different results as was shown within model (P).

In the following we are concentrating on the threshold property of a DNS point. This is motivated by the example of an unstable node with continuous optimal paths at the steady state. For this special case the unstable node can not be classified as DNS point, noticing that the only optimal solution when starting at such an unstable node is to stay put. Nevertheless it has the property of a threshold as it seperates regions in the state space with different optimal behaviour. This property is of general interest as it indicates critical points where slight differences in the starting positions can lead to very different optimal behaviour. In the light of the preceding considerations the following definitions can be given.

Definition 4 A point x_0 in the state space is called a DNS threshold if there exist $\varepsilon > 0$ such that for every open interval I around x_0 there exist at least two optimal solutions $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$ for model (A) with $x_1(0), x_2(0) = x_2 \in I$ and $\|x_1(\cdot) - x_2(\cdot)\| > \varepsilon$

Remark 7 Applying Def. 4 to problem (P) only the origin is a DNS threshold.

Remark 8 Principally this definition could be generalized for higher dimensions on using the term neighbourhood instead of interval. I rwstricted this defintion to the one-dimensional case as for higher dimensions this is an open discussion.

Analogue to Def. 3 a second definition for a DNS threshold can be given

Definition 5 A point x_0 in the state space is called a DNS threshold if there exist $\varepsilon > 0$ such that for every open interval I around x_0 there exist at least two optimal solutions $(x_1(t), u_1(t))$ and $(x_2(t), u_2(t))$ for model (A) with $x_1(0), x_2(0) = x_2 \in I$ and $\| (x_1(\cdot), u_1(\cdot)) - (x_2(\cdot), u_2(\cdot)) \| > \varepsilon$.

Remark 9 Applying Def. 5 to problem (P) every point in the state space is a DNS threshold.

Remark 10 The given Defs. 2-5 are in the line of a definition given by Vladimir Veliov, who defined a mapping $S : X \to O$, where X is the state space and O the set of optimal solutions, with $S(x_0) := \{\text{optimal solutions with } x(0) = x_0\}$. If this mapping is multivalued at x_0 we have the case of a DNS point at x_0 (Def. 3), whereas a DNS threshold is equal to discontinuity of S at x_0 (Def. 5). Definitions Def. 2 and 4 strengthen the argument considering only the optimal behaviour in the state space as is motivated by problem (P).

Remark 11 In all of these definitions the property of beeing a DNS point or threshold critically dependes on the used norm at least in the case of infinite time. But as this consideration once more is motivated by Vladimir Veliov, who is working on a paper treating this topic among other things in a more general framework, I don't go into further details.

I restricted these definitions to the one-dimensional case, as finding the right formulation is an open discussion. For higher dimensions the possible behaviour is more complex (see e. g. Haunschmied et al., 2003) and as a mathematical definition should also embrace extreme cases such as the one given in model (P), it is not an easy task to generalize the stated concepts. Nevertheless the intuition behind these ideas is clear. It concerns the existence of points where the mathematical theory alone gives no hint which optimal solution the decision maker has ideally to choose. But this is not only of theoretical interest, the existence of points of indifference can entail serious consequences e. g. in physical applications. Near such a point measurement errors can lead to a chattering behaviour of the calculated optimal control. For another example in an environmental model see Wagener (2003) where it is of great importance for the decision maker to realize the existence of a DNS point and to know the precisie position.

Summing up this example sheds light on the struggle on expressions in the tension of formal rigour and heuristic intuition. Outside the formal strenght of mathematical formulation a term obtains the power of its meaning not only from the exactness, clearness and generality but also from the ideas it provides to explain the underlying phenomenon. At the best mathematical rigour and explanatory intuition coincide, but in most cases one has to decide where to lay stress upon more on formal strength or more on intuitive explanation. A serious discussion on this topic has to clarify among other things this point.

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ser Aufzählung nicht fehlen. Die Auswirkungen meiner Reisen nach Kalaallit Nunaat auf mein Denken lassen sich in Worte kaum fassen und die Idee und Motivation zu dieser Doktorarbeit haben ihren Ursprung zum Teil auch in meinen Erfahrungen mit diesem gewaltigen Land. Um diese Beschreibung nicht abstrakt stehen zu lassen möchte ich dies rückbinden an Namen und damit Menschen die ich mit Grönland auf immer verbinde. Da seien zunächst die Inuit Abel, Hans, Dorthe und Tobias genannt, weiters Robert Peroni, dem ich den Aufenthalt und die Arbeit im Roten Haus verdanke, seiner Freundin Inge und den MitarbeiterInnen und FreundInnen Moni, Mo, Claudia und Frank. Die Faszination Ostgrönlands wird mich auf ein Leben lang an dieses Land und seine Menschen binden. Mit dem Boot durchs Eis von Kulusuk nach Tasiilaq zu fahren ist jedesmal die Fahrt in meine zweite Heimat.

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Curriculum Vitae

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Publications

Caulkins, J. P., Feichtinger, G., Graß, D., Tragler, G. (2004a) A model of moderation: finding Skiba points on a slippery slope. to appear in *CEJOR*

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