DISSERTATION

Toric Geometry and Mirror Symmetry in String Theory

ausgeführt zum Zwecke der Erlangung des akademischen Grades eines
Doktors der technischen Wissenschaften

unter der Anleitung von

Ao. Univ.-Prof. Dr. Maximilian Kreuzer
am Institut für Theoretische Physik (E 136)
der Technischen Universität Wien

eingereicht an der Technischen Universität Wien
Fakultät für Physik

von

Dipl.-Ing. Erwin Riegler

Matr. Nr: 9126155
Goldschlagstraße 36/31A
A-1150 Wien
Austria, Europe

Wien, Juli 2004

Kurzfassung

Eine der spannendsten Entwicklungen in der Stringtheorie in den letzten Jahren beschäftigt sich mit Dualitäten. Im Bereich der Stringkompaktifizierung spielt dabei vor allem Mirror Symmetrie eine wichtige Rolle, bei welcher Stringtheorien auf sogenannten Calabi–Yau (CY) Mannigfaltigkeiten, die ein Mirror–Paar bilden, zur selben Physik führen. Solche CY Mirror–Paare können in der torischen Geometrie durch vollständige Schnitte von Hyperflächen (CICYs) konstruiert werden, deren Information in Paaren von reflexiven Polyedern kodiert ist. Teil meiner Arbeit war es, ein C–Programmpaket zu erstellen, mit dem man diese Polyeder auf solche CICYs durchsuchen und deren Kohomologie (die sogenannten Hodge–Zahlen) berechnen kann. Die gewünschte Kodimension der CICY kann dem Programm als fixer, aber beliebiger, Parameter übergeben werden. Damit ist man nun in der Lage, die vollständige Kohomologie einer CICY mit beliebiger Dimension und Kodimension zu bestimmen.

Da sich noch niemand zuvor systematisch mit der Physik von torischen CICYs beschäftigt hat, berechne ich einige Beispiele und diskutiere die wesentlichen Unterschiede zum Hyperflächenfall, wie das Nichtschneiden von Divisoren, die zu Eckpunkten gehören. Aufgrund der hohen Dimensionen der torischen Räume spielt auch das Auflösen der Singularitäten eine entscheidende Rolle, und wird für die einfachsten auftretenden Fälle diskutiert. Spezielle Eigenschaften der CYs, so wie Faserungen und nichttriviale Fundamentalgruppen, werden in die torische Sprache übersetzt und diskutiert.

Im weiteren benütze ich die Mirror–Abbildung in einem unserer Beispiele, um damit explizit Weltflächen–Instantonen von einer torischen CICY zu berechnen. Diese Instantonen führen zu stringtheoretischen Korrekturen in der vierdimensionalen effektiven Supergravitation.

Acknowledgements

First of all I want to thank my supervisor Max Kreuzer for his dedicated support and encouragement. His trust in me was the foundation of this work.

I am especially grateful to my office mate Emanuel Scheidegger. With his enthusiasm for physics he brought new energy to this institute and I learned very much from him.

It is my great pleasure to thank all the members of the Theory Division, from the old guard up to the next generation. Especially I want to emphasize the following people: Herbert Balasin, with whom I had many interesting discussions during coffee breaks with topics starting at physics, mathematics, or linux, and ending at all the world and his brother (and we consumed a lot of coffee...). Robert Wimmer and Manfred Herbst, who began their thesis almost at the same time as I did. This autumn also Manfred and I will be at different departments to try our luck. I wish Manfred and Robert to bring out the best, and I am anxious to see where we will be in a few years. Johanna and Sebastian, the new shooting stars in our group.

Physics is more than a job, and certainly one has to give all to get ahead. But the most important issue I have learned in the last years is that there are other things in life which are more important. I want to thank Sylvia for wonderful eight and a half years. The last time was not easy, and next year will presumably be more difficult. But no matter what will come up in the future, our love will be strong enough to cope with things together.

I am appreciative to Michael. Unfortunately I had not enough time to continue our running units, but the next marathon comes for sure. I am much obliged to Wolfgang and Susanna. I spent a lot of nice hours with them, and I hope that our contact will not break away.

Contents

Chapter 1

Introduction

1.1 Motivations for Mirror Symmetry

Mirror symmetry is one of the most fascinating areas in string theory, and gives rise to extremely rich interactions between mathematics and physics [1–4]. Roughly speaking, the idea of mirror symmetry is that it relates identical string theories on topologically different manifolds. Unbroken $(N \geq 1)$ supersymmetry in four dimensions and conformal invariance enforce strong constraints on these spaces [5]. They have to be Ricci–flat and Kähler and are known as Calabi–Yau (CY) manifolds [6–8]. Now mirror symmetry is interesting for two reasons:

(i) It can be used to calculate gauge couplings and contributions to the superpotential in four dimensional supergravity arising from dimensional reduction of low energy effective actions in string compactifications.

(ii) It relates different types of string theories.

At very large distances, or low energies ($<< 10^{19}$GeV), the dimensions in the internal (the CY) part are hidden and the theory effectively looks four–dimensional. If one expands the (massless) fields appearing in such a ten–dimensional low energy effective action into an internal and a four–dimensional part, the Hamiltonian in the internal sector produces a mass term by the Kaluza–Klein effect [9]. Since the excitation spectrum is typically at 10^{19} GeV, we are mainly interested in the massless zero–modes. They also play an important role in algebraic geometry [10], and can be identified with the Cohomology of the CY space V: $H^{p,q}_{\bar{\partial}}(V)$ [11]. The dimensions of these spaces, h_{pq}, are called Hodge–numbers, and mirror symmetry interchanges h_{pq} with h_{n-pq} of the mirror CY. For a CY 3–fold, only h_{11} and h_{21} are independent, and correspond to Kähler and complex structure deformations, respectively [12]. These geometric parameters, which are called moduli, appear as charged matter fields in the four dimensional superpotential, where the coupling constants are equal to the threepoint functions [13, 14] in the topological string theory [15]. It turns out that there are two different ways to define the topological string, which are related by the mirror automorphism of the superconformal algebra [11]. The nice thing is that only in one case, called A–model, the Yukawa couplings receive corrections from worldsheet instantons, while in the other case, which is known as B–model, they are equal to the classical intersection numbers of the CY space [1,16]. Mirror symmetry relates these two models and, as mentioned in (i), can be used to calculate these non–perturbative corrections of the superpotential.

However, there are a lot of choices of different CY manifolds, each of which leads in general to different physics in four dimensions, and, even worse, each CY has a huge parameter space by itself. On top of this there are five different consistent superstring theories in ten dimensions, increasing the number of possible compactifications. One CY is as good as the other, and the problem that one has no preferred compactification is known as vacuum degeneracy problem. Unfortunately we do not have a good answer to this problem, but exciting progress has been made in understanding the relations between different types of string theories (see [17] and the references therein). The concept relating different theories is known as duality, and mirror symmetry is a duality relating type IIA and type IIB string theory on a mirror pair of CY manifolds.

1.2 Constructing mirror CYs

The first sizable sets of CY manifolds were constructed as complete intersections (CICY) in products of projective spaces [18, 19]. These manifolds have many complex structure deformations but only few Kähler moduli, which are inherited from the ambient space. The advantage of weighted ($W\mathbb{P}$), in contrast to ordinary, projective spaces, is that the resolution of the singularities contributes additional Kähler moduli, and thus provides a much more symmetric picture [20]. However, it turned out that mirror symmetry is only approximately realized in this class of models [21, 22]. $W\mathbb{P}$ spaces are a special class of toric varieties [23–28], and there is a remarkable construction of mirror pairs of CY hypersurfaces which was discovered by Batyrev [29]. In this construction mirror symmetry manifests itself in the duality of a pair of reflexive polytopes. A generalization to complete intersections was presented in [30, 31]. A special class of toric CICYs, which is per construction mirror symmetric, corresponds to nef–partitions of reflexive polytopes. Mirror symmetry at the cohomological level in this setup was proven in [32]. In this work the authors also gave an explicit formula for the string–theoretical hodge numbers, using the combinatorics of these polytopes. The development of the `nef`–code is the foundation of the present thesis, which enables us to discuss the specifics of CICYs corresponding to nef–partitions and their relevance in string theory. The code for this program is implemented in the PALP package [33]. First results are published in [34] and more, which are closely related with this thesis, will follow in [35].

1.3 Outline of the thesis

It is most convenient to formulate first quantized string theory in terms of a 2–dimensional quantum field theory on the world sheet swept out by the string. Chapter 2 of this thesis provides the necessary background for understanding the A– and B–twisted sector of the $N = (2, 2)$

topological field theory, which is crucial for type IIA and type IIB string compactifications on Calabi–Yau (CY) manifolds. We briefly discuss the calculation of the correlation functions and show why those of the A–model get corrections from worldsheet instantons, in contrast to those of the B–model. In the case of a non–linear sigma model on a mirror pair of CY manifolds they are related by the mirror map.

Toric geometry provides a powerful tool in constructing CYs. Instead of gluing together affine patches to a manifold, one can also intersect a certain number of hyperplanes in order to get a suitable space, which inherits the Kähler property from the ambient space. As already mentioned, mirror pairs of CY manifolds can be constructed in the toric setup using the duality of reflexive polytopes. In chapter 3 we give all the mathematics, which is needed to understand the geometry of toric ambient spaces. Since those spaces can also have singularities, we briefly discuss how one can resolve them using triangulations of polytopes. This will turn out to be very important when we determine the Mori cone of the CY, which is a certain section of the Mori cone of a smooth ambient space. We also give a review of the whole mirror–program for toric hypersurfaces and calculate an explicit example at the end of this chapter.

While the computation of the cohomology in the hypersurface case is rather simple, the case of complete intersections gets very complicated. Our `nef`–code is a very efficient tool which finds *all* nef–partitions in *arbitrary* codimensions of reflexive polytopes of *any* dimensions and calculates the *whole* cohomological data. With this program we are able to discuss a large number of examples, which is the main part of chapter 4. We found very interesting new effects. For example, we found out that even divisors corresponding to vertices of a polytope do not intersect the CY. Non–intersecting divisors correspond to projections of the Kähler–cone, which are dual to sections of the Mori cone, as mentioned above. Due to the high dimensions of the polytopes in contrast to hypersurfaces, the resolution of the singularities gets also more complicated. We discuss systematically the method of using lattice points which are at distance higher than one from the origin to resolve those singularities in order to get a smooth CY manifold. The appearance of singularities is also strongly related to quotients of lattices, which we use to construct CYs with non–vanishing fundamental classes.

Of particular interest are CY manifolds which are elliptic or K3 fibrations, where the latter can be used to construct models which admit Heterotic dual models on $K3 \times T2$. We discuss the appearance of fibrations in the toric setup, which boils down to the search for reflexive sections of polytopes. Our `cws`–program is very helpful for the construction of ambient spaces which admit fibrations, since it combines weights corresponding to fibrations of suitable weighted projective spaces.

At the end of this section we discuss different realizations of the first interesting pair of hodge numbers not appearing in the hypersurface case. To show that the manifolds are indeed isomorphic we calculate the triple intersection numbers. Adapting the techniques of the hypersurfaces case, we calculate also the Yukawa couplings of the mirror using the mirror map. Furthermore we also give an explicit rational transformation relating the complex structure moduli space of two different realizations of this model.

A listing of the available options can be found in appendix A. In appendix B we give a summary of the Hodge–data we have computed, and compare it to the hypersurface case and complete intersections in (products of) weighted projective (WP) spaces.

1.4 Outlook

There is an intimate relationship between Landau–Ginzburg (LG) orbifolds and CY manifolds [36, 37], which was the original motivation to create the code for the program `gen.x` (see appendix A.3). It transforms (if possible) Gorenstein cones arising from generalized CY–manifolds [30,38] into those coming from nef–partitions, and can be viewed as the toric version of the path integral procedure used in [36]. Unfortunately, our calculations are not sophisticated enough to be put into this thesis, but we mention them because there are a lot of newer developments in the area of super CYs [39], where generalized CYs may become important.

The idea of generalized CYs is that they act as higher dimensional geometrical replacements in the cases where no honest CYs are possible, for example, as mirrors of rigid CY manifolds ($h_{21} = 0$) [38]. Since we can associate a nef–partition to a lot of generalized CYs, and since we have for both a mirror construction [30, 32], it would be interesting to check if it is possible to perform the whole mirror program for a mirror pair of generalized CYs and compare it to that of the associated CICYs coming from a nef–partition.

From the point of view of super CYs the first question one should answer is the exact definition of a super CY, and its physical significance. In a next step one should try to give a clean definition of mirror symmetry for those spaces. In [40] the authors reformulated the mirror construction using T–duality [41] for super CYs, applying the ideas of [42] to relate a CICY to a fermionic bundle. However, this construction does not generalize the T–duality approach in [41]. The only exception they gave is the example of the mirror of the supermanifold $\mathbb{P}^{(3|3)}$. Since mirror symmetry using T–duality can be viewed as a non–compact version of the Batyrev mirror construction, it is natural to ask how super CYs can be described in the latter approach. Generalized CYs can be used to subserve as bodies of super CYs, and the known mirror construction for the former may become very useful. These considerations are very speculative, and in a first step the best will be just to play around with some examples until one finds a general structure.

From the CICY point of view it would be useful to find a more manageable formula for the Hodge–numbers, as it exists in the hypersurface case. This would have the advantage to get a better understanding of the assignment of divisors and certain lattice points in the polytope, and maybe one can say a priori, without analyzing the intersection ring, which of them not intersect the CY space.

Chapter 2

The physical background

In this chapter we give the physical background which is needed to understand the (twisted) topological string. Our starting point is the $N = (2, 2)$ superconformal algebra:

2.1 $N = (2, 2)$ SCFT

The $N = (2, 2)$ superconformal algebra is generated by the energy–momentum tensor T, the two weight $3/2$ supercurrents $\bar{G}, G$, and the $U(1)$ current J. They split into a left– and a rightmoving part:

$$\begin{array}{cc} T_+(z) & T_-(\bar{z}) \\ \overline{G}_+(z) \quad G_+(z) & \overline{G}_-(\bar{z}) \quad G_-(\bar{z}) \\ J_+(z) & J_-(\bar{z}). \end{array} \tag{2.1}$$

The algebra can either be defined by the OPE–expansions or in terms of (anti–) commutators of the modes:

$$\begin{array}{rcl} [L_m, L_n] &=& (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \\ {[J_m, J_n]} &=& \frac{c}{3}m\delta_{m,-n}, \\ {[L_n, J_m]} &=& -mJ_{m+n}, \\ {[L_n, \overline{G}_{m+a}]} &=& \left(\frac{n}{2}-(m+a)\right)\overline{G}_{m+n+a}, \quad [L_n, G_{m-a}] \;=\; \left(\frac{n}{2}-(m-a)\right)G_{m+n-a}, \\ {[J_n, \overline{G}_{m+a}]} &=& \overline{G}_{m+n+a}, \qquad\qquad\qquad\quad [J_n, G_{m+a}] \;=\; -G_{m+n-a}, \\ \{\overline{G}_{m+a}, G_{n-a}\} &=& 2L_{m+n} + (n-m+2a)J_{n+m} + \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m,-n}. \end{array} \tag{2.2}$$

If we omit $\pm$ the statements are valid for both the left and rightmoving, i.e. holomorphic and antiholomorphic, part of the algebra. Unitary (irreducible) representations of this algebra are those satisfying the hermicity conditions

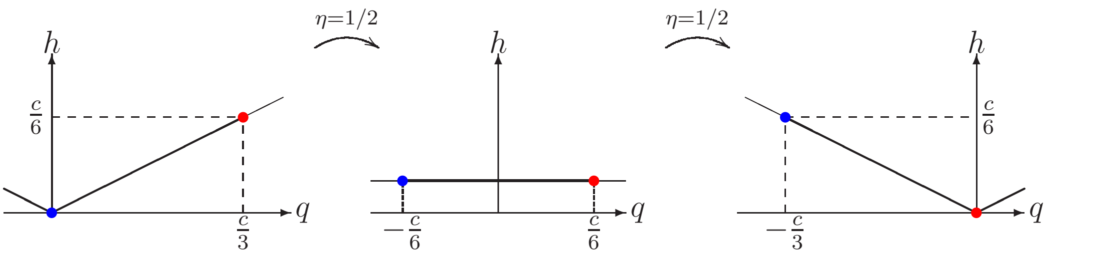
$$L_n^\dagger = L_{-n} \qquad J_n^\dagger = J_{-n} \qquad \overline{G}_s{}^\dagger = G_{-s}, \tag{2.3}$$

in a Hilbert space with positive definite norm. The parameter a takes values in $a \in [0,1)$, thus we have a whole family of algebras. It turns out that one can get from one member of this family to another by defining new generators which are just a suitable linear combination of the old ones. The fields in the $\eta = (a - 1/2) = 0$ sector are called Ramond (R), those in the $\eta = (a - 1/2) = 1/2$ sector are called Neveu-Schwarz (NS) fields. Fields transforming as tensors, i.e. primary fields, are in one–to–one correspondence to highest weight states, i.e. states which are annihilated by all modes with positive indices, which are per definition the annihilation operators.

There are two distinguished classes of states in the R and NS sector. A field in the R sector is called a R groundstate if it is annihilated by $\overline{G}_0$ and G_0. A field in the NS sector is called (anti) chiral if it is annihilated by $\overline{G}_{-1/2}$ $(G_{-1/2})$. It follows immediately from the algebra that a (anti) chiral primary field satisfies

$$h \leq \frac{c}{6} \qquad \text{and} \qquad h = \underset{(-)}{+} \frac{q}{2}, \tag{2.4}$$

where q is the $U(1)$ charge, h is the conformal weight, and c is the central charge of the algebra. From the additivity of the $U(1)$ charge it follows that the OPE of two chiral primary fields is again a chiral primary field (the same holds for the anti chiral fields), up to regular terms which vanish in the limit when the difference of the position of the two operators goes to zero. There is an important isomorphism, called spectral flow, interpolating between these sub–sectors:



chiral primary, $h = q/2$ R ground states, $h = c/24$ anti-chiral primary, $h = -q/2$

The Hamiltonian and the momentum are

$$H = L_0^+ + L_0^- \qquad \text{and} \qquad P = L_0^+ - L_0^-, \tag{2.5}$$

and the generators of the vector and of the axial R–symmetry are

$$F_V = J_0^+ + J_0^- \qquad \text{and} \qquad F_A = J_0^+ - J_0^-, \tag{2.6}$$

respectively.

If we define $\overline{Q}=\overline{G}_0$ and $Q=G_0$ we get the following (anti) commutator relations:

$$\begin{array}{rlrl}
 & Q_\pm^2=\overline{Q}_\pm^2=0, & & \\
 & \{Q_\pm,\overline{Q}_\pm\}=H\pm P, & & \\
\{\overline{Q}_+,\overline{Q}_-\}= & Z, & \{Q_+,Q_-\}= & Z^*, \\
\{Q_-,\overline{Q}_+\}= & \tilde{Z}, & \{Q_+,\overline{Q}_-\}= & \tilde{Z}^*, \\
[iM,Q_\pm] = & \mp Q_\pm, & [iM,\overline{Q}_\pm] = & \mp\overline{Q}_\pm, \\
[iM,Q_\pm] = & \mp Q_\pm, & [iM,\overline{Q}_\pm] = & \mp\overline{Q}_\pm, \\
[iF_V,Q_\pm]= & -iQ_\pm, & [iF_V,\overline{Q}_\pm]= & i\overline{Q}_\pm, \\
[iF_A,Q_\pm]= & \mp iQ_\pm, & [iF_A,\overline{Q}_\pm]= & \mp i\overline{Q}_\pm,
\end{array} \tag{2.7}$$

with $\overline{Q}_\pm=Q_\pm^\dagger$. In the following we assume that the central charges all vanish $(Z=\tilde{Z}=0)$. Let us now define the following operators:

$$\boxed{\begin{array}{ll} Q_A=\overline{Q}_+ + Q_-, & Q_A^\dagger=Q_+ + \overline{Q}_-, \\ Q_B=\overline{Q}_+ + \overline{Q}_-, & Q_B^\dagger=Q_+ + Q_-. \end{array}} \tag{2.8}$$

Setting $(Q,F)=(Q_A,F_A)$ or $(Q,F)=(Q_B,F_V)$ the operators obey

$$\boxed{\{Q,Q^\dagger\}=2H,\quad Q^2=0,\quad [F,Q]=Q.} \tag{2.9}$$

Thus the Hilbert space of states is a F–graded Q–complex:

$$\cdots\xrightarrow{Q}\mathcal{H}_n^{q-1}\xrightarrow{Q}\mathcal{H}_n^{q}\xrightarrow{Q}\cdots \tag{2.10}$$

at each energy level n. This complex is an exact sequence if $n>0$, i.e. the cohomology vanishes for $n>0$. At zero energy $Q\equiv 0$. Thus the space of SUSY ground states is characterized by the cohomology of the Q–operator. We note that F is not necessarily a conserved charge and the grading may not be a $\mathbb{Z}$–grading. However, the fermion number $(-1)^F$ is always conserved and thus we have at least a $\mathbb{Z}_2$–grading. SUSY ground states are in one–to–one correspondence with Q cohomology classes. There is an important automorphism, called mirror automorphism, interpolating between these two choices:

$$Q_A\Leftrightarrow Q_B,\qquad\qquad F_V\Leftrightarrow F_A,\qquad\qquad Z\Leftrightarrow\tilde{Z}. \tag{2.11}$$

An operator $\mathcal{O}$ is called

$$\boxed{\begin{array}{rcl} \text{chiral} & \Leftrightarrow & \{Q_B,\mathcal{O}\}=0, \\ \text{twisted chiral} & \Leftrightarrow & \{Q_A,\mathcal{O}\}=0. \end{array}} \tag{2.12}$$

Using the Jacobi Identity one easily finds for (twisted) chiral operators $\mathcal{O}$ that:

$$[(H+P),\mathcal{O}]=\{Q,[Q_+,\mathcal{O}]\}\quad\text{and}\quad[(H-P),\mathcal{O}]=\{Q,[\overset{(-)}{Q}_-,\mathcal{O}]\}, \tag{2.13}$$

with $\frac{i}{2}(\partial_{x_0} \pm \partial x_1)\mathcal{O} = [H \pm P, \mathcal{O}]$. Thus the worldsheet translation does not change the Q cohomology classes. After a Wick rotation $x_2 = ix_0$ and a change to complex coordinates $z = 1/2(x_1 + ix_2)$ we get

$$\partial_z \mathcal{O} = i[(H-P), \mathcal{O}] \quad \text{and} \quad \partial_{\bar{z}} \mathcal{O} = -i[(H+P), \mathcal{O}]. \tag{2.14}$$

Starting with a Q–closed field $\mathcal{O}^{(0)} = \mathcal{O}$ we find that:

$$d\mathcal{O}^{(0)} = 0, \qquad d\mathcal{O}^{(0)} = \{Q, \mathcal{O}^{(1)}\}, \qquad d\mathcal{O}^{(1)} = \{Q, \mathcal{O}^{(2)}\}, \qquad d\mathcal{O}^{(2)} = 0, \tag{2.15}$$

with

$$\mathcal{O}^{(1)} = idz[\overset{(-)}{Q}_-, \mathcal{O}] - id\bar{z}[Q_+, \mathcal{O}] \quad \text{and} \quad \mathcal{O}^{(2)} = dzd\bar{z}\{Q_+, [\overset{(-)}{Q}_-, \mathcal{O}]\}. \tag{2.16}$$

We can then construct the following Q–closed integrated vertex operators:

$$\int_\gamma \mathcal{O}^{(1)} \qquad \text{and} \qquad \int_\Sigma \mathcal{O}^{(2)}, \tag{2.17}$$

where γ is a closed 1–cycle and Σ is our worldsheet (without boundary). The operators of the second type can be used to deform the twisted theory. Twisting means that we combine the euclidian $SO(2) = U(1)$ rotation $M_E = iM$ with the $U(1)$ coming from F_V of F_A:

$$\boxed{\begin{array}{l} \text{A}-\text{twist}: \quad M'_E = M_E^A = M_E + F_V, \\ \text{B}-\text{twist}: \quad M'_E = M_E^B = M_E + F_A. \end{array}} \tag{2.18}$$

Twisting of the theory has some important consequences:

(i) It affects the spin of the supercharges. In particular, after a A (B) twist Q_A (Q_B) is a spin zero charge. The M_E–charges before and after the A (B) twist and the corresponding powers of the cotangent bundles are:

	Q_-	$\overline{Q}_+$	$\overline{Q}_-$	Q_+
M_E	1	-1	1	-1
M_E^A	0	0	2	-2
M_E^B	2	0	0	-2

	$\mathbb{C}$	$K^{1/2}$	$K^{-1/2}$	K	K^{-1}
M_E	0	1	-1	2	-2

(ii) The energy momentum tensor for the A (B) twist gets modified and is BRST exact:

$$\begin{array}{l} T_+^{\text{twisted}} = T_+ + \frac{1}{2}\partial J_+ = \frac{1}{2}\{\overline{Q}_+, G_+\} \\ T_-^{\text{twisted}} = T_- \pm \frac{1}{2}\partial J_- = \frac{1}{2}\{\overline{Q}_-, G_-\} \quad (\frac{1}{2}\{Q_-, \overline{G}_-\}. \end{array} \tag{2.19}$$

(iii) Because of (ii) correlation functions with only Q–closed operators inserted are independent of the metric on the worldsheet.

(iv) Deformations of the D–term can always be written as Q_A and Q_B exact terms. Twisted chiral and anti–chiral deformations are Q_B–exact, Chiral and anti–chiral deformations are Q_A–exact. Thus the B (A) model depends only holomorphically on (twisted) chiral deformations.

We will briefly discuss realizations of the $N = (2, 2)$ SCA:

2.2 The non–linear σ (nlσ) model and the Landau–Ginzburg (LG) model

2.2.1 The non–linear σ model

The Lagrangian of the σ– model is defined as

$$
\begin{aligned}
L &= -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{j}} + ig_{i\bar{j}}\overline{\psi}_-^{\bar{j}}(D_0+D_1)\psi_-^i + ig_{i\bar{j}}\overline{\psi}_+^{\bar{j}}(D_0-D_1)\psi_+^i \\
&\quad + R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\overline{\psi}_-^{\bar{j}}\overline{\psi}_+^{\bar{l}}) + g_{i\bar{j}}(F^i - \Gamma^i_{jk}\psi_+^j\psi_-^k)(\overline{F}^{\bar{j}} - \Gamma^{\bar{j}}_{\bar{k}\bar{l}}\overline{\psi}_-^{\bar{k}}\overline{\psi}_+^{\bar{j}}).
\end{aligned}
$$

We use the convention that Greek indices come from the worldsheet Σ, and Latin indices come from the target manifold V. If we want this action to be invariant under $N=(2,2)$ supersymmetry the target manifold, in which the fields $\{\phi^i\}$ embedding our worldsheet Σ take their values, has to be a Kähler manifold V with Kähler metric

$$g_{i\bar{j}} = \frac{\partial^2 K}{\partial\phi^i\partial\phi^{\bar{j}}}, \tag{2.20}$$

which can be written as the derivative of the Kählerpotential $K(\phi^i,\overline{\phi}^{\bar{i}})$. R is the curvature of $g_{i\bar{j}}$, and the covariant derivative is defined as

$$D_\mu\psi^i_\pm = \partial_\mu\psi^i_\pm + \partial_\mu\phi^j\Gamma^i_{jk}\psi^k_\pm. \tag{2.21}$$

The fermionic fields are sections of the bundles:

$$\boxed{\begin{array}{ll} \psi_+ \in \Gamma(K^{1/2}\otimes\phi^*T^{1,0}), & \psi_- \in \Gamma(K^{-1/2}\otimes\phi^*T^{1,0}), \\ \overline{\psi}_+ \in \Gamma(K^{1/2}\otimes\phi^*T^{0,1}), & \overline{\psi}_- \in \Gamma(K^{-1/2}\otimes\phi^*T^{0,1}), \end{array}} \tag{2.22}$$

where K is the cotangent bundle of the worldsheet Σ. We can also add a topological term to the action

$$\int_\Sigma \phi^*(B) \tag{2.23}$$

involving the antisymmetric B field $B \in H^2(V;\mathbb{R})/H^2(V,\mathbb{Z})$. In the superspace formalism, the Lagrangian of the non–linear σ–model can be written as a function of chiral superfields $\{\Phi^1,\ldots,\Phi^n\}$:

$$L = \int d^4\theta K(\Phi^i,\overline{\Phi}^{\bar{i}}). \tag{2.24}$$

2.2.2 LG models

If V has a holomorphic function $W(\phi^1,\ldots,\phi^n)$, one can deform the theory by adding the F–term to the Lagrangian:

$$\begin{aligned} L &= \int d^4\theta K(\Phi^i,\overline{\Phi}^{\bar{i}}) + \frac{1}{2}\left(\int d^2\theta W(\Phi^i) + c.c\right) \\ &= -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\overline{\phi}^{\bar{j}} + ig_{i\bar{j}}\overline{\psi}^{\bar{j}}_-(D_0+D_1)\psi^i_- + ig_{i\bar{j}}\overline{\psi}^{\bar{j}}_+(D_0-D_1)\psi^i_+ \\ &\quad +R_{i\bar{j}k\bar{l}}\psi^i_+\psi^k_-\overline{\psi}^{\bar{j}}_-\overline{\psi}^{\bar{l}}_+ - \frac{1}{4}g^{\bar{i}j}\partial_{\bar{i}}W\partial_jW - \frac{1}{2}D_i\partial_jW\psi^i_+\psi^j_- - \frac{1}{2}D_{\bar{i}}\partial_{\bar{j}}\overline{W}\overline{\psi}^{\bar{i}}_-\overline{\psi}^{\bar{j}}_+, \end{aligned}$$

where we have integrated out the F and $\overline{F}$:

$$F^i = \Gamma^i_{jk}\psi^j_+\psi^k_- - \frac{1}{2}g^{i\bar{l}}\partial_{\bar{l}}\overline{W}. \tag{2.25}$$

The SUSY variations of the fields are:

$$\boxed{\begin{array}{ll} \delta\phi^i = \epsilon_+\psi^i_- - \epsilon_-\psi^i_+, & \delta\overline{\phi}^{\bar{i}} = -\bar{\epsilon}_+\overline{\psi}^{\bar{i}}_- + \bar{\epsilon}_-\overline{\psi}^{\bar{i}}_+, \\ \delta\psi^i_+ = 2i\bar{\epsilon}_-\partial_+\phi^i + \epsilon_+F^i, & \delta\overline{\psi}^{\bar{i}}_+ = -2i\epsilon_-\partial_+\overline{\phi}^{\bar{i}} + \bar{\epsilon}_+\overline{F}^{\bar{i}}, \\ \delta\psi^i_- = -2i\bar{\epsilon}_+\partial_-\phi^i + \epsilon_-F^i, & \delta\overline{\psi}^{\bar{i}}_- = 2i\epsilon_+\partial_-\overline{\phi}^{\bar{i}} + \bar{\epsilon}_-\overline{F}^{\bar{i}}. \end{array}} \tag{2.26}$$

By the Nöther procedure one finds the supercurrents:

$$\begin{array}{ll} G^0_\pm = g_{i\bar{j}}(\partial_0\pm\partial_1)\overline{\phi}^{\bar{j}}\psi^i_\pm \mp \overline{\psi}^{\bar{i}}_\mp\partial_{\bar{i}}\overline{W}, & G^1_\pm = \mp g_{i\bar{j}}(\partial_0\pm\partial_1)\overline{\phi}^{\bar{j}}\psi^i_\pm - \overline{\psi}^{\bar{i}}_\mp\partial_{\bar{i}}\overline{W}, \\ \overline{G}^0_\pm = g_{\bar{i}j}\overline{\psi}^{\bar{i}}_\pm(\partial_0\pm\partial_1)\phi^j \pm \psi^i_\mp\partial_iW, & \overline{G}^1_\pm = \mp g_{\bar{i}j}\overline{\psi}^{\bar{i}}_\pm(\partial_0\pm\partial_1)\phi^j \pm \psi^i_\mp\partial_iW, \end{array} \tag{2.27}$$

with charges $Q^\pm = \int dx^1 G^0_\pm$ and $\overline{Q}^\pm = \int dx^1\overline{G}^0_\pm$.

2.2.3 R–symmetry

R–symmetry rotates the fermionic components as follows:

$$\boxed{\begin{array}{ll} U(1)_V : \psi_\pm \mapsto \mathrm{e}^{-i\alpha}\,\psi_\pm, & \overline{\psi}_\pm \mapsto \mathrm{e}^{i\alpha}\,\overline{\psi}_\pm, \\ U(1)_A : \psi_\pm \mapsto \mathrm{e}^{\pm i\beta}\,\psi_\pm, & \overline{\psi}_\pm \mapsto \mathrm{e}^{\mp i\beta}\,\overline{\psi}_\pm, \end{array}} \tag{2.28}$$

with charges $F_V = \frac{1}{2\pi}\int(\overline{\psi}_-\psi_- + \overline{\psi}_+\psi_+)dx^1$ and $F_A = \frac{1}{2\pi}\int(-\overline{\psi}_-\psi_- + \overline{\psi}_+\psi_+)dx^1$. On the supercharges R–symmetry acts in the same way as on the fermionic fields. Since the $d\theta^\pm$, $d\overline{\theta}^\pm$ are derivatives, R–symmetry acts on them with the opposite sign.

At the classical level $U(1)_A$ is always preserved. Since $d^2\theta = d\theta^- d\theta^+$ has a $U(1)_V$ charge of -2 this has to be compensated by the superpotential. If we assign for $U(1)_V$ ($U(1)_A$) a R–charge of q^i (0) to the fields Φ^i, the superpotential must be a quasi–homogeneous holomorphic function:

$$W(\lambda^{q^i}\Phi^i) = \lambda^2 W(\Phi^i). \tag{2.29}$$

At the quantum level, not only the action must be preserved under R–symmetry. If we want the path integral to be invariant, we should also examine the fermionic measure $D\Psi$. Let us do this for the euclidian two torus ($x_0 \mapsto ix_2$) in complex coordinates ($z = x_1 + ix_2,\ \bar{z} = x_1 - ix_2$). The anomaly comes from the zero modes of the fermionic kinetic term:

$$2ig_{i\bar{j}}\overline{\psi}_+^{\bar{j}}D_z\psi_+^i - 2ig_{i\bar{j}}\overline{\psi}_-^{\bar{j}}D_{\bar{z}}\psi_-^i, \tag{2.30}$$

which we have to insert to get a non–vanishing correlation function. The index theorem gives

$$\dim\ker D_{\bar{z}} - \dim\ker D_z = k = \int_\Sigma \phi^* c_1(T_V^{(1,0)}) = \langle c_1(V), \phi_*[\Sigma]\rangle, \tag{2.31}$$

and we get ($\#_0$ is the number of zero modes):

$$\#_0(\psi_- - \overline{\psi}_+) = \#_0(\overline{\psi}_+ - \psi_-) = k. \tag{2.32}$$

To get a non–vanishing correlation function we have to insert k ψ_- and k $\overline{\psi}_+$ fields. They have opposite $U(1)_V$ charge, but the $U(1)_A$ charge gives a contribution of $\mathrm{e}^{2ki\beta}$ and the axial R–symmetry is broken to $\mathbb{Z}_{2k}$. The integer k only depends on the cohomology class $\phi_*[\Sigma]$. For example, if we consider $\mathbb{P}^{N-1}$ the first Chern class is equal to N times the generator of $H^2(V,\mathbb{Z}) \cong \mathbb{Z}$ (each integer defines a line bundle) and $U(1)_A$ is broken to $\mathbb{Z}_{2N}$. $U(1)_A$ R–symmetry is preserved exactly when the first Chern class vanishes, i.e. when V is a CY manifold. To summarize, we give the following table [43]:

	nlσm CY	nlσm, $c_1(M) \neq 0$	LG on CY, W generic	LG on CY, W quasi–homogeneous
$U(1)_V$	$\circ$	$\circ$	$\times$	$\circ$
$U(1)_A$	$\circ$	$\times$	$\circ$	$\circ$

2.2.4 Twisting

So far we have assumed that our worldsheet is flat. String amplitudes are defined as the sum over all topologies and conformal classes of Riemann surfaces, and the starting point is the string amplitude F_g for a fixed but arbitrary curved Riemann surface Σ. SUSY variation of the action gives a term which vanishes only if the parameters $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ are covariantly constant. If Σ has non–vanishing curvature ($g \neq 1$) there is no covariant constant spinor. However, if we twist the theory we get one fermionic symmetry and can make use of the localization principle and deformation invariance.

A-twist (nlσm on a Kähler manifold)**:**

$$\boxed{\begin{array}{ll} \psi_+ \in \Gamma(K \otimes \phi^*T^{1,0}), & \psi_- \in \Gamma(\phi^*T^{1,0}), \\ \overline{\psi}_+ \in \Gamma(\phi^*T^{0,1}), & \overline{\psi}_- \in \Gamma(K^{-1} \otimes \phi^*T^{0,1}), \end{array}} \tag{2.33}$$

Thus, setting $\epsilon_- = \bar{\epsilon}_+ = 0$ and $\epsilon = \bar{\epsilon}_- = \epsilon_+$ and defining $\chi^i = \psi^i_-$, $\overline{\chi}^{\bar{i}} = \overline{\psi}^{\bar{i}}_+$, $\rho^{\bar{i}}_z = \overline{\psi}^{\bar{i}}_-$, and $\rho^i_{\bar{z}} = \psi^i_+$, the remaining SUSY variations $\delta = \epsilon Q_A$ are:

$$\boxed{\begin{array}{lll} \delta\phi^i = \epsilon\chi^i, & \delta\chi^i = 0, & \delta\rho^i_{\bar{z}} = 2i\epsilon\partial_{\bar{z}}\phi^i + \epsilon\Gamma^i_{jk}\rho^j_{\bar{z}}\chi^k, \\ \delta\overline{\phi}^{\bar{i}} = \epsilon\overline{\chi}^{\bar{i}}, & \delta\overline{\chi}^{\bar{i}} = 0, & \delta\rho^{\bar{i}}_z = -2i\epsilon\partial_z\overline{\phi}^{\bar{i}} + \epsilon\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\rho^{\bar{j}}_z\overline{\chi}^{\bar{k}}. \end{array}} \tag{2.34}$$

The physical operators $\mathcal{O}_\alpha$ are the Q_A cohomology classes and are easily seen to be in one–to–one correspondence with the de Rham cohomology classes $[\alpha] \in H^{(p,q)}_d(V)$ of $d = \partial + \bar{\partial}$ closed formes α:

$$\boxed{\mathcal{O}_\alpha := \alpha_{i_1\dots i_p\bar{j}_1\dots\bar{j}_q}\chi^{i_1}\dots\chi^{i_p}\overline{\chi}^{\bar{j}_1}\dots\overline{\chi}^{\bar{j}_q} \quad \delta\mathcal{O}_\alpha = \epsilon\mathcal{O}_{d\alpha}.} \tag{2.35}$$

These operators have $U(1)_V$ ($U(1)_A$) charge $q_V = -p_i + q_i$ ($q_A = p_i + q_i$). For the correlators we get the following selection rules: The vector R–charge is still not anomalous, so $\sum_{i=1}^s p_i = \sum_{i=1}^s q_i$ must hold for s insertions. For $U(1)_A$ we have the following mismatch between zero modes:

$$\#_0(\chi - \rho) = 2k, \tag{2.36}$$

where k is the index of the Dolbeault operators ∂ and $\bar{\partial}$ on the worldsheet (the χ's are now scalar fields on the worldsheet). Using the Riemann Roch theorem we get

$$k = \int_\Sigma c_1(V) + \dim V(1-g) = \langle c_1(V), [\phi_*\Sigma]\rangle + \dim V(1-g), \tag{2.37}$$

and we thus get the following selection rule:

$$\boxed{\sum_{i=1}^s p_i = \sum_{i=1}^s q_i = \langle c_1(V), \phi_*[\Sigma]\rangle + \dim V(1-g).} \tag{2.38}$$

In what follows we assume that $k \geq 0$ and that there are no ρ zero–modes. Localization tells us that we only have to look at Q_A fixed points, which obey $\partial_{\bar{z}}\phi^i = 0$. At these fixpoints we can write the bosonical action as

$$S_b = \int_\Sigma \phi^*(\omega - iB) = \langle \phi_*[\Sigma], (\omega - iB)\rangle, \tag{2.39}$$

where ω is the Kähler form and we have also added the B–field. The map $\phi : \Sigma \to \beta$ has to be holomorphic (localization), and for a fixed cycle $\beta = [\phi_*\Sigma]$ infinitesimal deformations correspond to holomorphic vector fields lying in $H^0_{\bar{\partial}}(\phi^*T_V)$. This space, denoted by $\mathcal{M}_\Sigma(V,\beta)$, is precisely the space of the χ^i's and has dimension (because of our assumption) equal to k. The measure on this moduli space comes from the insertions of the operators $\mathcal{O}_i$ at the points $x_i \in \Sigma$. Each $\mathcal{O}_i$ corresponds to a class $\omega_i \in H^*_d(V)$. Since we want to have a form on the moduli space and not on V we use the pull–back of the evaluation map at x_i:

$$\begin{array}{rcl} \mathrm{ev}_i : \mathcal{M}_\Sigma(V,\beta) & \to & V, \\ \phi & \mapsto & \phi(x_i). \end{array}$$

The correlation function is then

$$\langle\mathcal{O}_1\ldots\mathcal{O}_s\rangle = \sum_{\beta=\phi_*[\Sigma]} \mathrm{e}^{-\langle\beta,(\omega-iB)\rangle} \underbrace{\int_{\mathcal{M}_\Sigma(V,\beta)} \mathrm{ev}_1^*\,\omega_1\wedge\cdots\wedge\mathrm{ev}_s^*\,\omega_s}_{n_{\beta,D_1,\ldots,D_r}}\,. \tag{2.40}$$

The ω_i are Poincare dual to divisors D_i and can be chosen to have delta function support on D_i. Each integral above is only non–vanishing if $\phi(x_i)\in D_i(i=1,\ldots,s)$ and the numbers $n_{\beta,D_1,\ldots,D_r}$ count all holomorphic mappings such that $\phi_*[\Sigma]=\beta$ and $\phi(x_i)\in D_i(i=1,\ldots,s)$. Integration of the Kähler form on any cycle is non–negative and zero precisely when Σ is mapped to a point ($\beta=0$). In the large volume limit the $\beta=0$ contribution is dominant and the moduli space is V itself. The mappings ev_i are the identities in this case and the integrals of the $\beta=0$ contribution are just the classical intersection numbers (we will use this later to normalize our Yukawa couplings). Because of our assumption the genus has to be zero, $g=0$.

Of particular interest are the $g=0$ threepoint functions, where we assume that V is a CY manifold:

$$\boxed{K_{(123)}=\langle\mathcal{O}_{\alpha_1}\mathcal{O}_{\alpha_2}\mathcal{O}_{\alpha_3}\rangle_{g=0}.} \tag{2.41}$$

From the selection rule we get that $\alpha_i\in H_d^{(1,1)}(V)$, which correspond to Kähler deformations. One may wonder if there are any non–vanishing correlators with more than three insertions. The additional insertions must have $U(1)_A$ charge equal to zero. There are indeed such operators, namely the integrated operators $\mathcal{O}^{(2)}$ which can be used to deform the action: $\delta S=\sum_i t_i\int_S\mathcal{O}_i^{(2)}$. Correlators with additional insertions can be constructed by differentiation of the threepoint functions with respect to t_i:

$$\frac{\delta}{\delta t_i}\langle\mathcal{O}_j\mathcal{O}_k\mathcal{O}_l\rangle_{g=0}=\langle\mathcal{O}_j\mathcal{O}_k\mathcal{O}_l\int_\Sigma\mathcal{O}^{(2)}\rangle_{g=0}. \tag{2.42}$$

Conformal invariance implies that it does not matter if we exchange i with j, k, or l. Thus we get

$$\partial_l K_{(ijk)}(t)=\partial_i K_{(ljk)}(t), \tag{2.43}$$

which is known as the WDVV equation. Together with the symmetry in permutation of the indices, it follows that the K can be integrated:

$$K(t)=\partial_i\partial_j\partial_k F_0. \tag{2.44}$$

F_0 is the genus zero partition function.
B–twist (compact CY or LG model on a non–compact CY)**:**

$$\boxed{\begin{array}{ll}\psi_+\in\Gamma(K\otimes\phi^*T^{1,0}), & \psi_-\in\Gamma(K^{-1}\otimes\phi^*T^{1,0}),\\ \overline{\psi}_+\in\Gamma(\phi^*T^{0,1}), & \overline{\psi}_-\in\Gamma(\phi^*T^{0,1}).\end{array}} \tag{2.45}$$

Now we set $\epsilon_+ = e_- = 0$ and $\epsilon = \bar{\epsilon}_- = \bar{\epsilon}_+$. We define the fermionic fields: $\theta_i = g_{i\bar{j}}(\overline{\psi}^{\bar{j}}_- - \overline{\psi}^{\bar{j}}_+)$, $\eta^{\bar{j}} = \overline{\psi}^{\bar{j}}_- + \overline{\psi}^{\bar{j}}_+$, $\rho^i_{\bar{z}} = \psi^i_+$, and $\rho^i_z = \psi^i_-$. The SUSY variations $\delta = \epsilon Q_B$ are:

$$\boxed{\delta\phi^i = 0, \quad \delta\theta_j = -\epsilon\partial_j W, \quad \delta\overline{\phi}^{\bar{i}} = \epsilon\eta^{\bar{i}}, \quad \delta\eta^{\bar{i}} = 0, \quad \delta\rho^i_\mu = 2\epsilon J^\nu_\mu \partial_\nu \phi^i.} \tag{2.46}$$

The Q_B–cohomology is now the space of holomorphic functions f_i in ϕ^i modulo $\partial_j W$. Q_B–fixed points have to obey:

$$\delta\phi = 0 \qquad\qquad \partial_i W = 0. \tag{2.47}$$

If we assume that there is only a finite number N of isolated critical points y_i, which are non–degenerate, the path integral decomposes into a sum

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle = \sum_{i=1}^{N} \langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle \mid_{y_i} . \tag{2.48}$$

Each summand can be computed by the quadratic approximation around the fixpoints $\phi_i(x) = y_i$. The bosonic and fermionic determinants from the constant modes cancel, and the contribution from the non–constant modes comes from the quadratic approximation of the superpotential. $\overline{\phi}^{\bar{i}}$, ϕ^i, and $\overline{\psi}^{\bar{i}}_\pm$ are scalars on the worldsheet and each of them has one constant mode. $\psi^i_\pm$ are (co–) vectors and thus each of them has g constant modes. In summary we end up with:

$$\langle \mathcal{O}_1 \dots \mathcal{O}_N \rangle_g = \sum_{i=1}^{N} f_1(y_i) \dots f_s(y_i)(\det \partial_i \partial_j W)^{g-1}(y_i). \tag{2.49}$$

If $W = 0$ and V is a compact CY the physical operators $\mathcal{O}_\omega$, which are the Q_B cohomology classes, can be identified with the $\bar{\partial}$ cohomology classes $[\omega]$ in $H^{(0,p)}(V, \wedge^q T_V)$:

$$\boxed{\mathcal{O}_\omega := \omega^{i_1 \dots i_q}_{\bar{j}_1 \dots \bar{j}_p} \eta^{\bar{j}_1} \dots \eta^{\bar{j}_p} \theta_{j_1} \dots \theta_{j_q}, \quad \delta\mathcal{O}_\omega = \epsilon\mathcal{O}_{\bar{\partial}\omega}.} \tag{2.50}$$

The correlation functions again have to fulfill selection rules. Each $\omega_i \in H^{(0,p)}(V, \wedge^q T_V)$ has $U(1)_V$ ($U(1)_A$) charge $q_V = -p_i + q_i$ ($q_A = p_i + q_i$). Since the target space is a CY, the anomaly of the axial charge is $2 \dim V(1-g)$. We thus have the selection rule:

$$\boxed{\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = \dim V(1-g).} \tag{2.51}$$

The only non–vanishing correlators are at $g = 0$ and $g = 1$.

Localization tells us that $d\phi = 0$, i.e. the maps ϕ are constant maps. The whole worldsheet Σ is mapped onto a point in V. The path integral reduces to an integral over V. In order to get a (n, n) form (for $g = 0$) we have to contract the operator with the holomorphic n–form Ω:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_s \rangle = \int_V \langle \omega_1 \wedge \cdots \wedge \omega_s, \Omega \rangle \wedge \Omega. \tag{2.52}$$

In the case of a CY threefold, the threepoint function of operators corresponding to the Beltrami differentials $\mu_1, \mu_2, \mu_3 \in H^1(V.T_V)$ is:

$$\boxed{\overline{K}_{(123)} \;=\; \langle \mathcal{O}_{\mu_1} \mathcal{O}_{\mu_2} \mathcal{O}_{\mu_3} \rangle_{g=0} \;=\; \int_V \mu_1^i \wedge \mu_2^j \wedge \mu_3^k \Omega_{ijk} \wedge \Omega,} \tag{2.53}$$

which will turn out to be the third–order derivative of the prepotential $\mathcal{F}$.

2.2.5 Moduli of a CY

Infinitesimal deformations of the metric on a CY manifold split into two types. Kähler deformations correspond to elements in $H_d^{1,1}(V, \mathbb{C})$, while deformations of the complex structure correspond to elements in $H_{\bar{\partial}}^{0,1}(V, T_V)$. We will briefly discuss the second type of these deformations.

The almost complex structure J satisfies $J^2 = -1$ and can be written in local complex coordinates $\{z_a, \bar{z}_{\bar{a}}\}$ as $J_b^a = i\delta_b^a$ and $J_{\bar{b}}^{\bar{a}} = i\delta_{\bar{b}}^{\bar{a}}$. Now we send $J \mapsto J' = J + \epsilon$. The new J' must still square to -1. Linearizing this gives the condition $J\epsilon + \epsilon J = 0$ and ϵ must be of the form

$$\epsilon = \epsilon_a^{\bar{a}} dz^a \partial_{\bar{z}_{\bar{a}}} + \epsilon_{\bar{a}}^a d\bar{z}^{\bar{a}} \partial_{z_a} = \epsilon_A + \epsilon_H. \tag{2.54}$$

Integrability means that the Lie bracket of two holomorphic vectorfields must again be a holomorphic vectorfield. Defining the projectors

$$P_\pm = \frac{1}{2} (1 \mp iJ), \tag{2.55}$$

we get the condition

$$P_-[P_+X, P_+Y] = 0 \tag{2.56}$$

for all vector fields X, Y. Sending $J \longrightarrow J + \epsilon$ transforms the projectors as $P_\pm \longrightarrow P_\pm \mp i/2\epsilon$. The integrability constraint applied to the vectorfields $\partial_a = \partial_{z_a}, \partial_b = \partial_{z_b}$ then gives:

$$\begin{aligned} 0 &= \left(P_- + \frac{i}{2}\epsilon\right) \; [(P_+ - \frac{i}{2}\epsilon)\partial_a, \; (P_+ - \frac{i}{2}\epsilon)\partial_b] \\ &= \left(P_- + \frac{i}{2}\epsilon\right) \; [\partial_a - \frac{i}{2}\epsilon_A(\partial_a), \; \partial_b - \frac{i}{2}\epsilon_A(\partial_b)] \\ &\overset{\text{lin}}{=} -\frac{i}{2} P_- \left(\partial_a \left(\epsilon_A(\partial_b)\right) - \partial_b \left(\epsilon_A(\partial_a)\right)\right) = -\frac{1}{2}(\partial_a \epsilon_b^{\bar{a}} - \partial_b \epsilon_a^{\bar{a}})\partial_{\bar{a}}. \end{aligned}$$

Thus $\partial_a \epsilon_A = 0$. The condition that the Lie bracket of two anti–holomorphic vectorfields is an anti–holomorphic vectorfield gives $\partial_{\bar{a}} \epsilon_H = 0$. Exact terms are easily seen to correspond to infinitesimal change of coordinates without changing the complex structure. We conclude that infinitesimal deformations of the complex structure correspond to cohomology groups in

$$H^1_{\bar{\partial}}(T_V) = H^1(T_V), \tag{2.57}$$

where in the last step we have used the Cech–Dolbeaut isomorphism.

In different complex structures, the decomposition of the tangent (cotangent) bundle into holomorphic and anti–holomorphic parts may be different. Thus, the cohomology class representing the $(n, 0)$ form on a CY manifold changes over the moduli space of complex structures. All cohomology 3–form classes form a bundle over the moduli space, the hodge bundle $\mathcal{H}$, and the CY 3–form is a section of this bundle. The bundle $\mathcal{H}$ can be given a flat connection, called Gauss–Manin connexion. The fibers of $\mathcal{H}$ are $H^3(V; \mathbb{C})$ for CY manifolds V. We can define a metric on $\mathcal{H}$ by

$$(\theta, \eta) = i \int \theta \wedge \bar{\eta} \qquad\qquad \forall \eta, \theta \in H^3(V; \mathbb{C}). \tag{2.58}$$

This metric is hermitian since $(\theta, \eta) = (\eta, \theta)^*$, which implies that we can find a symplectic basis of real integer three–forms $\{\alpha_a, \beta^a\}$ $a = 1, \ldots, h^3/2$ such that

$$(\alpha_a, \beta^b) = i\delta_a^b \qquad (\alpha_a, \alpha_b) = 0 \qquad (\beta^a, \beta^b) = 0, \tag{2.59}$$

with dual basis $\{A^a, B_a\}$ $a = 1, \ldots, h^3/2$ in $H_3(V; \mathbb{Z})$. The holomorphic 3–form can be expanded in this basis:

$$\Omega = z^a \alpha_a + i w_b \beta^b \qquad \text{with} \qquad z^a = \int_{A^a} \Omega, \qquad i w_b = \int_{B_b} \Omega. \tag{2.60}$$

It can be shown that both, z^a and w_b, define local projective coordinates on the moduli space [44].

Since the CY 3–form is unique up to scale, it defines a complex line bundle in the Hodge bundle. A natural metric in this line bundle is

$$h = \|\Omega\|^2 = (\Omega, \Omega) = i \int \Omega \wedge \bar{\Omega}. \tag{2.61}$$

If z is a (local) coordinate vector on the moduli space and $f(z)$ a holomorphic function, then $\Omega \to \mathrm{e}^{f(z)}\, \Omega$ defines the same projective section, but $h \to h\, \mathrm{e}^{f + \bar{f}}$. Thus the quantity

$$K = -\log\left(\|\Omega\|^2\right) \tag{2.62}$$

transforms as a Kähler potential, $K \to K - f - \bar{f}$, and we can define a metric on the moduli space by

$$g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K, \tag{2.63}$$

called the Weil–Petersson metric. Since the variation $\partial_a\Omega$ gives a $(3,0)$ and a $(2,1)$ part $k_a\Omega + \mu_a$, the variation $\bar{\partial}_{\bar{a}}\bar{\Omega}$ gives a $(0,3)$ and a $(1,2)$ part $\bar{k}_{\bar{a}}\Omega + \bar{\mu}_{\bar{a}}$, and $\partial_a\bar{\Omega} = \bar{\partial}_{\bar{a}}\Omega = 0$, it is easy to check that

$$g_{a\bar{b}} = \frac{i\int \mu_a \wedge \bar{\mu}_{\bar{a}}}{\|\Omega\|^2}. \tag{2.64}$$

Important identities are $\int \Omega\partial_c\Omega = \int \Omega\partial_c\partial_d\Omega = 0$, which will also turn out to be useful for us in calculating relations between the different Yukawa couplings. From the first we get immediately that

$$(z^a\alpha_a + iw_b\beta^b, \alpha_c + i\partial_c w_d\beta^d) = w_c - z^a\partial_c w_a = 0. \tag{2.65}$$

Setting

$$\mathcal{F} = z^a w_a, \tag{2.66}$$

we see that $w_c = 1/2\partial_c\mathcal{F}$. Summing with z^c on both sides we get $z^c\partial_c\mathcal{F} = 2\mathcal{F}$, so $\mathcal{F}$ is homogeneous of degree 2. $\mathcal{F}$ is called the prepotential. From the prepotential one can compute the Kähler potential and all the couplings. The Kähler potential is given by the formula:

$$h = \mathrm{e}^{-K} = i\int \Omega \wedge \bar{\Omega} = i(\bar{z}^{\bar{a}}\partial_a\mathcal{F} - z^a\bar{\partial}_{\bar{a}}\mathcal{F}). \tag{2.67}$$

The Yukawa couplings are:

$$\overline{K}_{abc} = \langle \mathcal{O}_{\mu_a}\mathcal{O}_{\mu_b}\mathcal{O}_{\mu_c}\rangle_{g=0} = \int_V \mu_a^i \wedge \mu_b^j \wedge \mu_c^k \Omega \wedge \Omega_{ijk} = \partial_a\partial_b\partial_c\mathcal{F}, \tag{2.68}$$

where μ_a corresponds to the $(2,1)$–part of $\partial_a\Omega$. Note that the Yukawa couplings are only given up to multiplication with a non–zero complex number. The idea of mirror symmetry is now that, for a mirror pair V, V^* of CY manifolds, the Yukawa couplings corresponding to Kähler deformations on V can then be expressed in terms of the Yukawa couplings corresponding to deformations of the complex structure on the mirror V^* using the mirror map. It is the task of the next chapter to give all the mathematical background to understand how this works.

Chapter 3

The mathematical background

3.1 Basics of toric geometry

Let M and $N = \mathrm{Hom}(M, \mathbb{Z})$ be dual free abelian groups of rank d, $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ the real scalar extension of M and N, respectively, and $\langle *, * \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural pairing. Eventually we have to enlarge the lattices by $\mathbb{Z}^r$. Then we write $\bar{N} = N \oplus \mathbb{Z}^r$ and $\bar{M} = M \oplus \mathbb{Z}^r$ with real scalar extensions $\bar{N}_{\mathbb{R}}$ and $\bar{M}_{\mathbb{R}}$ $(\dim \bar{M}_{\mathbb{R}} = \dim \bar{N}_{\mathbb{R}} = \bar{d})$. The M–lattice can be regarded as the space of exponents of Laurant **m**onomials. More precisely, the M-lattice can be identified with the character group of all group homomorphisms from the algebraic torus $\mathbb{C}^{d^*}$ to $\mathbb{C}^*$. Objects of $N_{\mathbb{R}}$ will usually be marked with a $*$, vertices in the M (N)–lattice will be named by ρ (ρ^*). The basic objects in these spaces are cones and polytopes:

Definition 3.1.1 A subset $\sigma \subset M_{\mathbb{R}}$ is called a $k = \dim(\mathrm{Span}(\sigma))$–dimensional rational convex polyhedral cone if there exists a finite set $\{\rho_1, \ldots, \rho_n\} \subset M$ such that

$$\sigma = \{\lambda_1 \rho_1 + \cdots + \lambda_n \rho_n \in M_{\mathbb{R}} : \lambda_i \in \mathbb{R}_{\geq},\ i = 1, \ldots, n\}. \tag{3.1}$$

If $n = k$ the cone is called simplicial. If $\mathrm{Vol}(\{\rho_1, \ldots, \rho_n\}) = 1$ we call the cone unimodular. The dual cone $\sigma^* \subset N_{\mathbb{R}}$ is defined as

$$\sigma^* = \{z^* \in N_{\mathbb{R}} : \langle z, z^* \rangle \geq 0 \quad \forall z \in \sigma\}. \tag{3.2}$$

It can be shown that σ^* is also rational ([23]), i.e. σ^* is generated by lattice points.

An important class of cones are Gorenstein cones:

Definition 3.1.2 A $\bar{d}$–dimensional rational convex polyhedral cone $C \subset \bar{M}_{\mathbb{R}}$ is called Gorenstein if there exists a lattice point $n_C \in \bar{N}$ in the dual lattice such that $\langle \rho, n_C \rangle = 1$ for all generators of C. The polytope $\Delta_C = \{z \in C : \langle z, n \rangle = 1\}$ is called the support of C. A Gorenstein cone is called reflexive if the dual cone $C^* \subset \bar{N}_{\mathbb{R}}$ is also a Gorenstein cone. In this case, the integer $\langle m_C^*, n_C \rangle$ is called the index of C (C^*).

Definition 3.1.3 Let $A = \{\rho_1, \ldots, \rho_n\} \subset M$ be a finite set of lattice points. The convex hull

$$\Delta = \mathrm{Conv}\,(A) \tag{3.3}$$

is called a lattice polytope $\Delta \subset M_{\mathbb{R}}$. If the origin 0 is in an interior point of Δ, the dual polytope $\Delta^* \subset N_{\mathbb{R}}$ is defined as:

$$\Delta^* = \{z^* \in N_{\mathbb{R}} : \langle z, z^* \rangle \geq -1 \;\forall\; z \in \Delta\}. \tag{3.4}$$

Δ is called reflexive if both, Δ and Δ^*, are lattice polytopes.

3.1.1 Definition of a toric variety

Toric ambient spaces X_{Σ^*} are defined in terms of a fan Σ^*, which is a collection of rational polyhedral cones $\sigma^* \in \Sigma^*$ containing all faces and intersections of its elements [23–25, 27]. X_{Σ^*} is compact if and only if $\cup_{\sigma^* \in \Sigma^*} = N_{\mathbb{R}}$. We are mainly interested in the case where Σ^* consists of the cones over the faces of a reflexive polytope $\Delta^* \subset N_{\mathbb{R}}$. The origin is associated to the 0–dimensional cone $\{0\}$.

The toric variety is smooth if and only if each cone of maximal dimension in the fan is simplicial and unimodular, i.e. the generators of each cone of maximal dimension generate the lattice. Singularities at codimension 4 of such an ambient space are irrelevant for a sufficiently generic choice of equation for a Calabi–Yau 3–fold. Higher–dimensional singularities have to be resolved by a subdivision of the fan. Sometimes it is necessary to add points which are contained in $(r\Delta^* \setminus (r-1)\Delta^*) \cap N$ in order to get a smooth ambient space. This will turn out to be important to get the right Mori cone, and we will address this issue in more detail in Section 4.2. In the rest of this subsection we will give a definition of a toric variety in terms of homogeneous coordinates, which is one of the simplest ways to define these spaces. Since we are also interested in resolving singularities, we briefly discuss the main arguments used for triangulating polytopes. Note that we have been a little sloppy in the definition of our toric ambient space: we also need the lattice in which this polytope lives and which triangulation we use to resolve the singularities. Thus we should better write down a triple $(\Delta^*, N, \mathcal{T}(A, \Delta^*))$ to get all the information we need to define our variety (the set $A \subset \Delta^* \cap N$ is a set of lattice points containing the vertices of Δ^*, and the vertices of each simplex in the triangulation $\mathcal{T}$ are all contained in A).

Definition using homogeneous coordinates

Let Σ^* be a complete fan and denote by $\Sigma^*(l)$ the set of l–dimensional cones in Σ^*. In particular $\Sigma^*(1) = \{\rho_1^*, \ldots, \rho_n^*\}$ is the set of generators of the fan Σ^*. We have the following exact sequence of groups:

$$\boxed{0 \longrightarrow M \xrightarrow{\alpha} \bigoplus_{\rho^* \in \Sigma^*(1)} \mathbb{Z} \cdot D_{\rho^*} \xrightarrow{\beta} A_{n-1}(X) \longrightarrow 0,} \tag{3.5}$$

with $\alpha : m \mapsto \sum_{\rho^*} \langle m, \rho^* \rangle D_{\rho^*}$ and $\beta : (a_\rho) \mapsto [\sum_{\rho^*} a_{\rho^*} D_{\rho^*}]$. Each $\rho^* \in \Sigma^*(1)$ corresponds to a torus invariant Weil divisor D_ρ, and each $m \in M$ gives a character χ^m. Regarding every character as a rational function, the map α gives an embedding of the rational functions into the group of $T_{N_\mathbb{R}}$–invariant Weil divisors. The map β maps the $T_{N_\mathbb{R}}$–invariant Weil divisors onto the Chow group, which is the group of Weil divisors modulo rational equivalence. Inside the Chow group sits the Picard group $\mathrm{Pic}(X)$, which is the group of Cartier divisors modulo rational equivalence. A Weil divisor $D = \sum_{\rho^*} a_{\rho^*} D_{\rho^*}$ is Cartier if and only if there is an Σ^*–piecewise linear integral function $\Phi_D : N_\mathbb{R} \to \mathbb{R}$ such that $\Phi_D(\rho^*) = a_{\rho^*} \ \forall \rho^* \in \Sigma^*$. On cones of maximal dimension σ we have $\Phi_D \mid_\sigma = m_\sigma \mid_\sigma$ for some unique $m_\sigma \in M$. There are two important cases [2] [1]:

(i) D is generated by global sections $\Leftrightarrow \langle m_\sigma, \rho^* \rangle \geq -a_{\rho^*}$ whenever $\rho^* \notin \sigma$,

(ii) D is ample $\Leftrightarrow \langle m_\sigma, \rho^* \rangle > -a_{\rho^*}$ whenever $\rho^* \notin \sigma$.

In the first case Φ is called (upper) convex, and in the second case Φ is called strictly (upper) convex. Every Cartier divisor D defines a polytope $\Delta_D \subset M_\mathbb{R}$ as

$$\Delta_D = \{m \in M_\mathbb{R} : \langle m, \rho^* \rangle \geq -a_{\rho^*}\}. \tag{3.6}$$

Condition (i) is precisely that this polytope is the convex hull of the $\{m_\sigma\}$. Condition (ii) requires in addition that $m_\sigma \neq m_t$ for all different cones of maximal dimension $\sigma \neq \tau$. There is a one–to–one correspondence between lattice points of Δ_D and monomials in $S_{[D]}$:

$$\boxed{m \in \Delta \cap M \quad \Leftrightarrow \quad \sum_{\rho^* \in \Sigma^*(1)} {x_{\rho^*}}^{\langle m, \rho^* \rangle + a_{\rho^*}} \in S_{[D]}.} \tag{3.7}$$

Note that the Chow group in the exact sequence (3.5) may have torsion. When X is smooth, then $\mathrm{Pic}(X) = A_{n-1}(X)$. If Σ^* is simplicial then the Picard group has finite index in the Chow group. The Picard group is always torsions free when the fan is complete, while $A_{n-1}(X)$ may have torsion, even when Σ^* is simplicial [2].

If we apply $\mathrm{Hom}(\_, \mathbb{C}^*)$ to the exact sequence (3.5) we get another exact sequence:

$$0 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma^*(1)} \longrightarrow T_{N_\mathbb{R}} \longrightarrow 0. \tag{3.8}$$

In particular

$$\boxed{G = \mathrm{Hom}(A_{n-1}(X), \mathbb{C}^*).} \tag{3.9}$$

Thus G is isomorphic to $(\mathbb{C}^*)^{k-d}$ times some finite cyclic group, which is present precisely when A_{n-1} has torsion. The embedding of G into $(\mathbb{C}^*)^{\Sigma^*(1)}$ extends to an action of G on $\mathbb{C}^{\Sigma^*(1)}$, where $a = (a_{\rho^*}) \in \mathbb{C}^{\Sigma^*(1)}$ and $g \in G$ map to

$$g \cdot a = (g([D_\rho])a_{\rho^*}). \tag{3.10}$$

[1]For a detailed discussion of semiample divisors see [45]

Now consider the polynomial ring

$$S = \mathbb{C}[x_{\rho^*} : \rho^* \in \Sigma^*(1)], \tag{3.11}$$

where every monomial $x_D = \prod_{\rho^*} x^{a_{\rho^*}} \in S$ corresponds to an effective torus–invariant divisor $D = \sum_{\rho^* \in \Sigma^*(1)} a_{\rho^*} D_{\rho^*}$. The exact sequence (3.5) induces a grading of this ring by defining

$$\boxed{\deg x_D = [D] \in A_{n-1}(X),} \tag{3.12}$$

which decomposes S into a direct sum:

$$\boxed{S = \bigoplus_{\alpha \in A_{n-1}} S_\alpha, \qquad S_\alpha = \mathrm{Span}(\{x_D : \deg x_D = \alpha\}).} \tag{3.13}$$

The toric variety X_{Σ^*} is then defined as the categorical quotient

$$\boxed{X_{\Sigma^*} = \left(\mathbb{C}^{\Sigma^*(1)} \setminus Z(\Sigma^*)\right)/G,} \tag{3.14}$$

where $Z = V(B)$ and B is the G–invariant ideal defined as

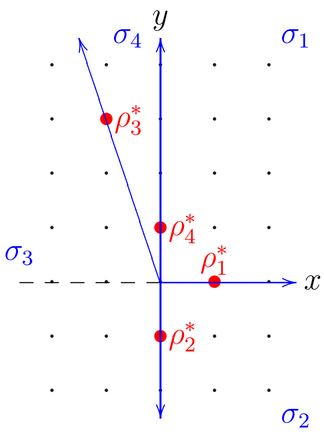
$$\boxed{B = \{\prod_{\rho^* \notin \sigma} x_{\rho^*} : \sigma \in \Sigma^*(d)\}.} \tag{3.15}$$

This quotient is geometrical if and only if the fan Σ^* is simplicial [26]. In this case we really have homogeneous coordinates $z = (z_{\rho^*})$. Up to torsion the group G acts as

$$(z_1 : \ldots : z_n) \sim (\lambda_{(a)}^{q_1^{(a)}} z_1 : \ldots : \lambda_{(a)}^{q_n^{(a)}} z_n), \qquad a = 1, \ldots, h = k - d, \tag{3.16}$$

where $\lambda_a \in \mathbb{C}^*$ $(a = 1, \ldots, h)$ and the h vectors $(q_i^{(a)})$ are generators of the linear relations $\sum q_i^{(a)} \rho_i^* = 0$. If L is a matrix such that the rows yield a basis of relations of the vertices $\Sigma^*(1)$, the map β can be identified with L.

Example 3.1.4 As an example, we will show how the Hirzebruch surface F_a is constructed from a fan $\Sigma \subset N_{\mathbb{R}}$:



$$\begin{tikzcd} U_{\sigma_4} = \mathrm{Spec}(\mathbb{C}[x^{-1}, x^a y]) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[x, y]) \arrow[d, leftrightarrow] \\ U_{\sigma_3} = \mathrm{Spec}(\mathbb{C}[x^{-1}, x^{-a} y^{-1}]) \arrow[r, leftrightarrow] & U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[x, y^{-1}]). \end{tikzcd}$$

The vertices generating the rays are $\rho_1^* = e_x$, $\rho_2^* = -e_y$, $\rho_3^* = -e_x + ae_y$, and $\rho_4^* = e_y$ with $a \in \mathbb{Z}_{\geq}$. The diagram on the right hand side shows how the different patches are glued together. The exponents of the monomials are the generators of the dual cones. The horizontal patching gives two copies of $\mathbb{C} \times \mathbb{P}^1$. performing also the vertical patching gives a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$.

For the construction using homogeneous coordinates we need a basis of the relations between the vertices:

$$\begin{aligned} \rho_2^* + \rho_4^* &= 0, \\ \rho_1^* + k\rho_2^* + \rho_3^* &= 0, \end{aligned}$$

which will give us the two scaling relations after assigning to every vertex ρ_i^* a coordinate z_i. Since the sets $\{\rho_1^*, \rho_3^*\}$ and $\{\rho_2^*, \rho_4^*\}$ are not contained in any cone we get for the exceptional set $Z = \{z_1, z_3\} \cup \{z_2, z_4\}$. Thus our toric variety is

$$\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \setminus Z : (z_1, z_2, z_3, z_4) \sim (\mu z_1, \lambda\mu^k z_2, \mu z_3, \lambda z_4)\ \forall\ \lambda, \mu \in \mathbb{C}^*\}. \tag{3.17}$$

The fact that we have a fibration of two projective spaces is because the two relations do not act independently (for $k \neq 0$). The divisor $D = \sum_{\rho^*} D_{\rho^*}$ in our example is:

ample	F_0,F_1
generated by global sections	F_2

If $k > 2$, Δ_D is no longer the convex hull of the m_σ, which are equal to Φ_D on cones of maximal dimension σ. For $k \leq 2$ the first chern class of F_k is that of the line bundle associated to the divisor D. For both, F_0 and F_1, D is ample, which implies that the first Chern class is positive. For F_2 D is no longer ample, but is still generated by global sections. Thus the first chern class is still non–negative for F_2. In the next section we define the Kähler cone. There we will see that ample means that the divisor is in the Kähler cone, and generated by global sections means that it is in the closure of the Kähler cone. Varieties with positive first Chern class are called Fano, those with non–negative first Chern class are called Nef.

3.1.2 The Kähler and the Mori cone

Let $\Sigma^* \subset N_{\mathbb{R}}$ be a simplicial fan and denote by $A^+_{n-1} \otimes \mathbb{R}$ the cone generated by the divisor classes $[D_{\rho^*}]$. Then the Kähler cone sits inside $A^+_{n-1} \otimes \mathbb{R}$ and is defined as follows. Let $a = [\sum_{\rho^* \in \Sigma^*(1)} a_{\rho^*} D_{\rho^*}] \in A^+_{n-1} \otimes \mathbb{R}$. Since Σ^* is simplicial, (a_{ρ^*}) corresponds to a Σ^*–piecewise linear integral function Φ_a on $N_{\mathbb{R}}$ defined by $\Phi_a(\rho^*) = a_{\rho^*}\ \forall\ \rho^* \in \Sigma^*(1)$ (Note that Φ_a is only defined modulo rational equivalence). Then one defines the cone

$$\boxed{\mathrm{cpl}\,(\Sigma^*) = \{a \in A^+_{n-1} \otimes \mathbb{R} : a \text{ is convex.}\}} \tag{3.18}$$

For a simplicial projective variety $X = X_{\Sigma^*}$ the Kähler cone is the interior of the cone $\mathrm{cpl}\,(\Sigma^*)$, which is the cone of all strictly convex functions.

For a simplicial fan Σ^* these functions can be found using primitive collections of vertices:

Definition 3.1.5 Let Σ^* be a simplicial fan. A primitive collection P is a subset $P \subset \Sigma^*(1)$ with the property that P is not the set of generators of a cone in Σ^* while any proper subset of P is.

Proposition 3.1.6 [46] On a projective simplicial toric variety Φ_a coming from $[\sum_{\rho^* \in \Sigma^*(1)} a_{\rho^*} D_{\rho^*}] \in A^+_{n-1} \otimes \mathbb{R}$ is strictly convex if and only if for any primitive collection $P = \{\rho^*_1, \ldots, \rho^*_s\} \subset \Sigma^*(1)$ we have

$$\Phi_a(\rho^*_1 + \cdots + \rho^*_s) > \Phi_a(\rho^*_1) + \cdots + \Phi_a(\rho^*_s). \tag{3.19}$$

The Mori cone $M(X_{\Sigma^*})$ of a simplicial and complete variety X_{Σ^*} is the cone of effective 1-cycles. It turns out that the Mori cone is dual to the Kähler cone.

Toric varieties via symplectic reduction

There is another very important approach to construct toric varieties via symplectic reduction. It is strongly related to the gauged linear sigma model and we will discuss it briefly. For us, the most important example is $\mathbb{C}^k$ with symplectic form $\omega = \sum_{i=1}^k dx_i \wedge dy_i$. The special feature of $\mathbb{C}^k$ is that the action of the group S^k_1 on $\mathbb{C}^k$ is symplectic, which means that the ω is invariant under this action. The Lie algebra of this group is $\mathbb{R}^k$, and the vector field corresponding to the flow of an element $\lambda \in \mathbb{R}^k$ in $\mathbb{C}^k$ is

$$X_\lambda = \sum_{i=1}^r \lambda_i \left(-y_i \frac{\partial}{\partial_{x_i}} + x_i \frac{\partial}{\partial_{y_i}} \right). \tag{3.20}$$

This action is Hamiltonian, which means that for each X_λ we can find a function f_λ such that $\omega(X_\lambda, \_) = df_\lambda$: defining the moment map as

$$\mu : \mathbb{C}^k \to (\mathbb{R}^k)^* \quad z \mapsto \frac{1}{2}(|z_1|^2, \ldots, |z_k|^2) \tag{3.21}$$

gives $f_\lambda = \lambda \circ \mu$. Now suppose that we have a simplicial fan $\Sigma^* \subset N_{\mathbb{R}}$. The maximal compact subgroup $G_{\mathbb{R}}$ of G in the exact sequence (3.8) is

$$G_{\mathbb{R}} = \mathrm{Hom}(A_{n-1}(X), S^1),$$

with Lie algebra $g_{\mathbb{R}} = \mathrm{Hom}(A_{n-1}(X), \mathbb{R})$. Thus $g^*_{\mathbb{R}} = A_{n-1}(X) \otimes \mathbb{R}$ and we get the moment map as:

$$\mu_{\Sigma^*} : \mathbb{C}^r \xrightarrow{\mu} \mathbb{R}^k \xrightarrow{\beta} A_{n-1}(X) \otimes \mathbb{R}, \tag{3.22}$$

where β is defined by tensoring (3.5) with $\mathbb{R}$. If $a \in A_{n-1}(X) \otimes \mathbb{R}$ is Kähler, then one defines

$$\tilde{X} = \mu^{-1}_{\Sigma^*}(a)/G_{\mathbb{R}}. \tag{3.23}$$

One can think of μ_{Σ^*} as fixing the size and the quotient by $G_{\mathbb{R}}$ is needed to get a non–degenerate symplectic form. Since this group can have finite stabilizers, the quotient has the natural structure of an orbifold. If X_{Σ^*} is simplicial and projective, there is an orbifold diffeomorphism between X_{Σ^*} and X [2].

A simple example is the construction of $\mathbb{P}^n$ out of $\mathbb{C}^{n+1}$: first one fixes the size by setting $f(z) = 1/2\|z\|^2 = r > 0$ and gets a sphere S^{2n+1} with radius r. The induced symplectic form is degenerate. The null–vectorfield is precisely X_f. Thus we have to mod out the S_1 corresponding to the flow of X_f.

One important thing is that both, $G_{\mathbb{R}}$ and μ_{Σ^*}, in the definition (3.23) of $\tilde{X}$ only depend on $\Sigma^*(1)$. If $\Sigma^{*\prime}$ is another simplicial projective fan such that $\Sigma^*(1)' \subset \Sigma^*(1)$ one can show that there is the following orbifold diffeomorphism [2]:

$$\mu^{-1}_{\Sigma^*(1)}(a)/G(\Sigma^*(1))_{\mathbb{R}} \sim X_{\Sigma^{*\prime}}, \tag{3.24}$$

where a is in the interior of $\mathrm{cpl}\,(\Sigma^{*\prime}) \subset A(\Sigma^*(1))$. Thus the moment map $\mu_{\Sigma^*(1)}$ can be used to construct not just one toric variety, but all projective simplicial toric varieties $X_{\Sigma^{*\prime}}$ with $\Sigma^*(1)' \subset \Sigma^*(1)$. In the language of gauged linear sigma models [37] $a \in A(\Sigma^*(1))$ is a parameter, and one gets different physical theories depending where a lies. If $a \in \mathrm{cpl}\,(\Sigma^*(1)')$ the theory involves the toric variety $X_{\Sigma^{*\prime}}$, while if it lies outside, one gets LG theories or Hybrid models.

3.1.3 Triangulations and the secondary fan

Definition 3.1.7 Let $\Delta^* = \mathrm{Conv}\,(A) \subset \mathbb{R}^{d-1}$ be a $(d-1)$–dimensional polytope. A triangulation $\mathcal{T}$ of (Δ^*, A) is a triangulation of Δ^* into simplices with vertices in A. A triangulation $\mathcal{T}$ of (Δ^*, A) is called coherent if there exists a strictly convex $\mathcal{T}$–piecewise–linear function whose domains of linearity are precisely the (maximal) simplices of $\mathcal{T}$.

Every such $\mathcal{T}$–piecewise linear function is uniquely defined by it's values on the vertices of the maximal simplices of $\mathcal{T}$. Thus we get a surjective (we do not require every element of A to appear as a vertex of a simplex) linear map $\psi \mapsto g_{\psi,\mathcal{T}}$ from linear maps on $\mathbb{R}^A$ on the space of $\mathcal{T}$–piecewise linear functions on Δ^*.

Definition 3.1.8 Let $C(\mathcal{T})$ be the cone of functions $\psi : A \to \mathbb{R}$ with the property that

(i) $g_{\psi,\mathcal{T}} : \Delta^* \to \mathbb{R}$ is convex.

(ii) for any $z^* \in A$ which is not a vertex of any simplex $\mathcal{T}$ we have $g_{\psi,\mathcal{T}}(z^*) \geq \psi(z^*)$.

Clearly a Triangulation $\mathcal{T}$ is coherent (3.1.7) if and only if the interior of $C(\mathcal{T})$ is non empty.

Proposition 3.1.9 [47] For fixed (Δ^*, A), the cones $C(\mathcal{T})$ for all the triangulations of (Δ^*, A) together with all faces of these cones form a complete fan in $\mathbb{R}^A$. This fan is called secondary fan of A.

Definition 3.1.10 Let again (Δ^*, A) $(\dim \Delta^* = d - 1)$ be as above, and fix a translation invariant volume form $\mathrm{Vol}\,(\_)$ on $\mathbb{R}^{d-1}$. A characteristic function on a triangulation of (Δ^*, A) is a function $A \to \mathbb{R}$ defined as

$$\varphi_{\mathcal{T}}(\rho^*) = \sum_{\substack{\sigma\,:\,\rho^* \in \mathrm{Vert}\,(\sigma)\,,\\ \dim\sigma = d-1}} \mathrm{Vol}\,(\sigma)\,. \tag{3.25}$$

The secondary polytope $\Sigma(A)$ is defined as the convex hull of the set $\{\varphi_{\mathcal{T}}\} \subset \mathbb{R}^A$, where $\mathcal{T}$ runs over all triangulations $\mathcal{T}$ of (Δ^*, A). For any $\varphi_{\mathcal{T}}$ we can define the normal cone $N_{\varphi_{\mathcal{T}}}\Sigma(A)$ as [2]

$$N_{\varphi_{\mathcal{T}}}\Sigma(A) = \{z \in \mathbb{R}^A : \langle z, \varphi_{\mathcal{T}}\rangle \geq \langle z, z^*\rangle \;\forall\; z^* \in \Sigma(A)\}. \tag{3.26}$$

$\varphi_{\mathcal{T}}$ is a vertex if and only if the interior of $N_{\varphi_{\mathcal{T}}}\Sigma(A)$ is not empty.

Proposition 3.1.11 [47] The secondary polytope $\Sigma(A)$ has the following properties:

(i) $\dim \Sigma(A) = k - d$, where $k = \#A$.

(ii) The vertices of $\Sigma(A)$ are in one–to–one correspondence with the coherent triangulations of (Δ^*, A).

(iii) For any triangulation $\mathcal{T}$ the normal cone $N_{\varphi_{\mathcal{T}}}\Sigma(A)$ coincides with the cone $C(\mathcal{T}) \subset \mathbb{R}^A$ from definition (3.1.8).

Definition 3.1.12 Let L_A be the set of all affine relations between the elements of A, i.e.

$$L_A = \{(l_{z^*}) \in \mathbb{R}^A : \sum_{z^* \in A} l_{z^*} z^* = 0, \sum_{z^* \in A} l_{z^*} = 0\}. \tag{3.27}$$

Let b_{z^*} be the images of the canonical basis vectors under the projection $\mathbb{R}^{A^*} \to L_A^*$, which is dual to the injection $L_A \to \mathbb{R}^A$. The set $B = \{b_{z^*} : z^* \in A\}$ is called the Gale transform [47,48] of A.

[2] we identify the space $\mathbb{R}^A$ with it's dual by defining the scalar product $(z, w) = \sum_{v^* \in A} \langle z, v^*\rangle\langle w, v^*\rangle$.

In matrix notation the rows of B correspond to a basis of the affine relations, and the columns are the b_{z^*}. Now let $\tilde{A} = \{z^* \times 1 : z^* \in A\}$ and define the following projection on the basis $\{e_{z^*}\}$ of $\mathbb{R}^A$:

$$\begin{aligned} \pi : \mathbb{R}^A &\rightarrow \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}, \\ e_{z^*} &\mapsto (z^* \times 1) \in \tilde{A}. \end{aligned}$$

Clearly, $\ker \pi = L_A$. Thus for any linear form ψ on $\mathbb{R}^A$ the following statements are equivalent:

$$\sum_{z^* \in A} \psi_{z^*} b_{z^*} = 0 \Longleftrightarrow \psi \perp \ker \pi \Longleftrightarrow \psi = \tilde{g} \circ \pi \;\; (\tilde{g} \in \mathbb{R}^{d^*}) \Longleftrightarrow \psi = g \circ \pi, \tag{3.28}$$

where in the last step g is an affine–linear function on $\mathbb{R}^{d-1}$ defined as $g(z^*) = \tilde{g}(z^*, 1) \; \forall \; z^* \in A$. As an immediate consequence, we get:

Proposition 3.1.13 [47]

(i) Let I be a subset of A. Then the set of forms $\{b_{z^*} : z^* \in I\}$ form a basis of L_A^* if and only if the set $A \setminus I$ is affine independent, i.e. corresponds to a $(d-1)$–dimensional simplex of Δ^*.

(ii) the convex hull of B contains the origin $0 \in L_A^*$ in its interior.

Thus any simplex σ of Δ^* defines a cone $C_\sigma \subset L_A^*$ which is generated by the forms $\{b_{z^*} : z^* \in I\}$. For any cone C we denote by C° the relative interior of this cone. The interiors C_σ° of the cones C_σ are called dual chambers. Now there is the following bijective correspondence between coherent triangulations $\mathcal{T}$ of (Δ^*, A) and dual chambers:

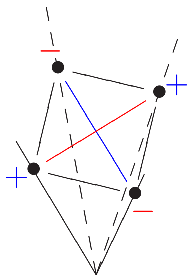
Proposition 3.1.14 [47] Let π be the projection $\mathbb{R}^{A^*} \rightarrow L_A^*$, which is dual to the injection $L_A \rightarrow \mathbb{R}^A$. Then for any coherent triangulation $\mathcal{T}$ the corresponding dual chamber has the form $(-1)\pi(C^\circ(\mathcal{T}))$, where $C(\mathcal{T})$ is the cone of definition $3.1.8$. The closure of this chamber coincides with the intersection $\cap_{\sigma \in \mathcal{T}} C_\sigma$. By a slightly abuse of notation, we will call the fan generated by the closure of this chambers also secondary fan.

Definition 3.1.15 [47]) A circuit is a collection Z of points in an affine space such that any proper subset $Z' \subset Z$ is affinely independent but Z itself is not, *i.e.* the points satisfy

$$\sum_{\rho_m^* \in Z} c_m \rho_m^* = 0 \qquad \sum_{\rho_m^* \in Z} c_m = 0 \qquad c_m \in \mathbb{Z} \setminus \{0\} \; \forall \; \rho_m^* \in Z. \tag{3.29}$$

In other words, we can obtain a circuit by adding just one point in general position to the set of vertices of a simplex. The convex hull of a circuit has precisely the two triangulations $T_{\pm} = \{\mathrm{Conv}\,(Z \setminus \{\rho_m^*\})\,|\rho_m^* \in Z_{\pm}\}$ where $Z_{\pm} = \{\rho_m^* \in Z | c_m \lessgtr 0\}$. As an easy consequence, we note that $|c_m| = \mathrm{Vol}\,(Z \setminus \{\rho_m^*\})$.

To summarize, we have the following important result: There is a one–to–one correspondence between simplices of Δ^* with vertices in A and cones of maximal dimension in the (projected) secondary fan. Each cone of maximal dimension $(-1)\pi(C^\circ(\mathcal{T}))$ corresponds to a coherent triangulations $\mathcal{T}$. The simplices of this triangulation are in one to one correspondence with all cones containing $(-1)\pi(C^\circ(\mathcal{T}))$. Contiguous coherent triangulations are related by circuits.



2 possible triangulations of a circuit of 4 points in $\mathbb{R}^3$.

3.2 Special Geometry and Mirror–symmetry

In this section we review how to calculate the Yukawa couplings for a toric hypersurface by deforming the complex structure. We also define the mirror map, which we use in order to calculate the worldsheet instantons in the A–model of the mirror. At the end of this section we discuss everything explicitly in one example.

3.2.1 Picard–Fuchs (PF) equations and Yukawa couplings (YCs)

We discuss two methods of getting the PF equations: the first is the Griffiths–Dwork (GD) method, which works (theoretically) for any ample hypersurface V in a d–dimensional simplicial projective toric variety and leads to a complete set of operators. The disadvantage of this method is that it uses Gröbner basis techniques for the Jacobian ideal.

The second method uses the $\mathcal{A}$–system associated to the polytope Δ. This is a very fast way to get a set of PF operators. However, the problem is that one has to reduce this system in order to get generators of the PF ideal. Also the system needs not be complete, so that one has to find additional operators which annihilate the periods.

The (GD) method

The GD method can be used to calculate the PF system for any ample hypersurface V in an d–dimensional simplicial projective toric variety X [2, 29]. Let Σ^* be the fan in $N_{\mathbb{R}}$ defining X_{Σ^*}. The hypersurface V is defined by the equation $f = 0$ for a general element $f \in S_\beta$, with $\deg f = \beta \in A_{n-1}(X)$ such that β is the first chern class of an ample bundle. We first define a homogeneous volume form of degree $\beta_0 = \sum_{i=1}^k [D_i]$ on the toric ambient space:

$$\Omega_0 = \sum_{|I|=k} \det(e_I)\, \widehat{x_I} dX_I, \qquad dX_I = dx_{i_1} \wedge \cdots \wedge dx_{i_n}, \qquad \widehat{x_I} = \prod_{i \notin I} x_i, \tag{3.30}$$

where I runs over all subsets $I \subset \{1, \ldots, k\}$ with $\#I = d$ and $\det(e_I) = \det\left(\langle e_a, \rho^*_{i_b}\rangle_{1\le a,b\le d}\right)$ is used to make the definition of Ω_0 independent of the ordering of the vertices $\{\rho^* \in \Sigma^*\}$. n–forms on X_{Σ^*} with poles along V are then of the form

$$\frac{P\Omega_0}{f^k}, \qquad \text{with} \qquad \deg P = k\beta - \beta_0. \tag{3.31}$$

To reduce the pole order, one needs for each $1 \le i \le k$ a $(d-1)$–form Ω_i of degree $\beta_0 - \beta_i$ ($\beta_i = \deg[D_i]$) whose exact form will not concern us. Important for us is the formula

$$d\left(\frac{P\Omega_i}{f^k}\right) = \frac{\left(f\frac{\partial P}{\partial x_i} - kP\frac{\partial f}{\partial x_i}\right)\Omega_0}{f^{k+1}}, \tag{3.32}$$

which can be used to reduce the pole order modulo exact forms. One defines the Jacobian ideal by

$$J(f) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}\rangle, \tag{3.33}$$

and considers also the ideal quotient $J_1(f)$:

$$J_1(f) = \langle \frac{x_1 \partial f}{\partial x_1}, \ldots, \frac{x_k \partial f}{\partial x_k} \rangle : x_1 \ldots x_n. \tag{3.34}$$

Then one can show that there are the following isomorphisms between the primitive cohomology and the $(k\beta - \beta_0)$–part of $S/J(f)$ and $S/J_1(f)$, respectively [2]:

$$\begin{aligned} (S/J(f))_{k\beta-\beta_0} &\simeq PH^{n-k,k-1} \quad \forall k \neq (d/2)+1 \\ (S/J_1(f))_{k\beta-\beta_0} &\simeq PH^{n-k,k-1} \quad \forall k. \end{aligned}$$

Note that J_1 is only needed if d is even (the reason for using the modified ideal $J_1(f)$ has its origin in the non–vanishing of $H^d(X) = H^{d/2,d/2}$ of X when d is even. In this case the kernel of the residue map is not empty for $k = n/2 + 1$ [2]). Now we can perform the following procedure:

- choose a basis $\{\omega_i = P_i \Omega_f^{k_i} \ (i = 1, \ldots, h = h^{d-1} - 1)\}$ of the primitive cohomology of the hypersurface V, i.e. of $S/J(f)$ or $S/J_1(f)$;

- repeatedly differentiate $\omega = \Omega_0/f$ with respect to the moduli $\{z_i\}$ to get $h+1$ sections (including Ω) of the Hodge–bundle $\mathcal{H}$. Each of these terms can be written as a linear combination of the basis plus a form η, where the numerator of the coefficient functions of η lies in the ideal $J(f)$ ($J_1(f)$). This can be done most easiest by using Gröbner basis techniques. Then one can use the formula (3.32) to reduce the pole ordering of η modulo exact forms.

- The result can again be expressed as a linear combination of our basis plus a form η_1, where the numerator of the coefficient functions of η_1 is again in the ideal. Since we have $h+1$ sections and $\mathcal{H}$ has rank h, iterating this procedure leads to a relation between the different derivatives of Ω.

PF equations and A–systems

Definition 3.2.1 Let $\mathcal{A} = \{\bar{m}_1, \ldots, \bar{m}_k\} \subset \bar{M} \cong \mathbb{Z}^{d+1}$ be a collection of $k > d+1$ points lying in an integral affine hyperplane and fix a vector $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{C}^{d+1}$. Introduce a set of variables $\{\lambda_{\bar{m}}\}$ and define $\delta_{\bar{m}} = \lambda_{\bar{m}} \partial_{\bar{m}} = \lambda_{\bar{m}} \partial / \partial_{\lambda_{\bar{m}}}$. Then the $\mathcal{A}$–system is the system of differential equations, defined by the following two types of operators:

$$\boxed{\begin{aligned} Z_j &= \left(\sum_{\bar{m} \in \mathcal{A}} \bar{m}_j \delta_{\bar{m}} \right) - \beta_j \quad (j = 1, \ldots, d+1) \\ \Box_l &= \prod_{l_{\bar{m}} > 0} \partial_{\bar{m}}^{l_{\bar{m}}} - \prod_{l_{\bar{m}} < 0} \partial_{\bar{m}}^{-l_{\bar{m}}} \quad l \in \Lambda, \end{aligned}}$$

where Λ is the lattice of relations among the elements of $\mathcal{A}$.

Now let $\Delta \subset M$ be a reflexive polytope. Then the sections of the anticanonical bundle can be identified with the Laurant polynomials

$$f = \sum_m \lambda_m t^m = \lambda_0 + \sum_{m \neq 0} \lambda_m t^m \in L(\Delta), \tag{3.35}$$

where $L(\Delta)$ denotes the set of Laurant polynomials with exponents in Δ. Then define

$$\boxed{\mathcal{A} = (\Delta \cap M) \times \{1\} \subset \bar{M} = M \times \mathbb{Z},} \tag{3.36}$$

and $\beta = (0, \ldots, 0, -1)$. The lattice of relations Λ of $\mathcal{A}$ is precisely the lattice of affine relations of $\Delta \cap M$, i.e.

$$\sum_{\bar{m} \in \mathcal{A}} l_{\bar{m}} = 0 \, \forall \, l \in \Lambda \qquad \text{and} \qquad \sum_{\bar{m} \in \mathcal{A}} l_{\bar{m}} m = 0. \tag{3.37}$$

Defining the d–form

$$\boxed{\omega = \frac{1}{f}\eta, \quad \eta = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d},} \tag{3.38}$$

it follows immediately form (3.37) that ω is annihilated by the operators $\Box_l$ for all $l \in \Lambda$. From the formula

$$d\left(\frac{1}{f}\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{\widehat{dt_j}}{t_j} \wedge \cdots \wedge \frac{dt_d}{t_d}\right) = \frac{(-1)^k}{f^2}\left(\sum_m \lambda_m m_j t^m\right)\eta, \tag{3.39}$$

one can easily show that also the operators $Z_j (j = 1, \ldots, d+1)$ annihilate ω. The form ω is defined on $T \setminus \{f = 0\}$. However, we want PF equations on V and not just on the affine part. This can be done by relating the torus coordinates t_i to the homogeneous coordinates x_i (use (3.7) with $a_i = 1 (i = 1, \ldots, k)$):

$$t_j = \prod_{i=1}^{k} x_i^{\langle e_j, \rho_i^* \rangle + 1}. \tag{3.40}$$

Under this substitution we get

$$\boxed{\Omega_0 = x_1 \ldots x_k \eta, \quad \omega = \frac{\Omega_0}{f(x_1, \ldots, x_k)}.} \tag{3.41}$$

Note that the action of

$$T \times \mathbb{C}^*, \tag{3.42}$$

where T acts on itself (on the torus coordinates) and $\mathbb{C}^*$ acts by multiplying f, leads to isomorphic hypersurfaces. Thus we have an action of $T \times \mathbb{C}^*$ on $L(\Delta)$ and the quotient defines the moduli space. We now choose a basis $l^{(a)}$ $(a = 1, \ldots, h$ on the lattice Λ of relations and define the following functions:

$$\boxed{z_a = \prod_{\bar m \in \mathcal{A}} \lambda_{\bar m}^{l^{(a)}_{\bar m}}\,.} \tag{3.43}$$

Then the z_a are invariant under the action of $T \times \mathbb{C}^*$ and define local coordinates in the quotient of the moduli space near the point of maximally unipotent monodromy. It remains to express the PF operators in the $\{z_a\}$–coordinates. The following identity is very useful when one pushes the $\lambda_{\bar m}$ through the $\delta_{\bar m}$:

$$\delta_{\bar m}\lambda^p_{\bar m} = \lambda^p_{\bar m}(\delta_{\bar m} + p). \tag{3.44}$$

The last thing we have to do is to transform ω in equation (3.41) into an expression that is invariant under our $\mathbb{C}^*$ action. This can be done be considering the form

$$\boxed{\tilde\omega = \lambda_0 \omega} \tag{3.45}$$

and multiplying our PF system with λ_0^{-1} from the right:

$$\boxed{\begin{array}{ll} Z_j \rightarrow Z_j\lambda_0^{-1} & (j = 1, \ldots, d+1) \\ \Box_l \rightarrow \Box_l\lambda_0^{-1} & l \in \Lambda. \end{array}} \tag{3.46}$$

Yukawa couplings

Once we have the PF–system, it is very easy to compute the YCs from equation (2.53). $\overline{K}_{i_1 \ldots i_3}$. First we define the quantities

$$\boxed{\overline{K}_{i_1\ldots i_k} = \int_V \Omega \wedge \nabla_{\delta_{i_1}} \ldots \nabla_{\delta_{i_k}}\Omega \quad i_1 \leq \cdots \leq i_k,} \tag{3.47}$$

which are zero for $k \leq 2$ due to Griffiths transversality. If $k = 3$ we get the YCs. Repeated Differentiation of the $k \leq 2$ case leads to the following identities:

$$\begin{aligned} \overline{K}_{(i_1 i_1 i_1 i_1)} &= 2\delta_{i_1}\overline{K}_{(i_1 i_1 i_1)} \\ \overline{K}_{(i_1 i_1 i_1 i_2)} &= \tfrac{3}{2}\delta_{i_1}\overline{K}_{(i_1 i_1 i_2)} + \tfrac{1}{2}\delta_{i_2}\overline{K}_{(i_1 i_1 i_1)} \\ \overline{K}_{(i_1 i_1 i_2 i_2)} &= \delta_{i_1}\overline{K}_{(i_1 i_2 i_2)} + \delta_{i_2}\overline{K}_{(i_1 i_1 i_2)} \\ \overline{K}_{(i_1 i_1 i_2 i_3)} &= \delta_{i_1}\overline{K}_{(i_1 i_2 i_3)} + \tfrac{1}{2}\delta_{i_2}\overline{K}_{(i_1 i_1 i_3)} + \tfrac{1}{2}\delta_{i_3}\overline{K}_{(i_1 i_1 i_2)} \\ \overline{K}_{(i_1 i_2 i_3 i_4)} &= \tfrac{1}{2}\delta_{i_1}\overline{K}_{(i_2 i_3 i_4)} + \tfrac{1}{2}\delta_{i_2}\overline{K}_{(i_1 i_3 i_4)} + \tfrac{1}{2}\delta_{i_3}\overline{K}_{(i_1 i_2 i_4)} + \tfrac{1}{2}\delta_{i_4}\overline{K}_{(i_1 i_2 i_3)}\,. \end{aligned} \tag{3.48}$$

The YCs can then be computed as follows:

(i) multiply all PF operators with logarithmic derivatives $\{\delta_i\}$ to get a set of differential operators of third order which annihilate $\int_V \Omega \wedge \Omega$. Due to Griffiths transversality we only need to keep terms which are third derivatives. The resulting equations can be used to express all YCs as rational functions of a single one, say $\overline{K}_{(111)}$.

(ii) multiply all PF operators with logarithmic derivatives $\{\delta_i\}$ to get a set of forth order differential operators acting on $\int_V \Omega \wedge \Omega$. Again, we only need to keep terms which are derivatives of order higher than two. The identities (3.48) can then be used to replace the quantities $\overline{K}_{(ijkl)}$ in terms of derivatives of the YCs. Expressing all $\overline{K}_{(ijk)} \neq \overline{K}_{(111)}$ in terms of $\overline{K}_{(111)}$ gives a system of linear differential equations for $\overline{K}_{(111)}$.

The Mirror map

We use a definition of the mirror map which takes advantage of the toric data and defines mirror symmetry for toric hypersurfaces [2]. As usual, let $\Delta \subset M_{\mathbb{R}}$ and $\Delta^* \subset N_{\mathbb{R}}$ be a pair of reflexive polytopes corresponding to the (family of) toric hypersurfaces $V \subset X_{\Sigma^*}$ and $V^* \subset X_\Sigma$, respectively. The idea is to find a local isomorphism between a subspace of the large complex structure moduli space of V and a subspace of the Kähler moduli space of V^*.

To give a precise definition on these subspaces we need the following definition:

Definition 3.2.2 Given a reflexive polytope $\Delta \subset M_{\mathbb{R}}$, a fan Σ^* is a simplified projective subdivision if it has the following properties:

(i) Σ^* refines the normal fan of Δ.

(ii) $\Sigma^*(1)$ consists of all lattice points of Δ^* except the interior point $\{0\}$ and interior points of facets of Δ^*.

(iii) X_{Σ^*} is projective and simplicial.

The difference to maximal projective subdivisions used for Σ and Σ^* in the Batyrev mirror construction is that we throw away all points lying in the interior of facets, which correspond to divisors that do not intersect the CY hypersurface. It will turn out that the mirror map is completely fixed by a smooth cone $\sigma \subset \mathrm{cpl}\,(\Sigma)$. The subspaces of the moduli spaces are defined as follows (for details see [2]):

Definition 3.2.3 Let Σ correspond to a simplified projective subdivision of the normal fan of Δ^*. The simplified moduli space $\mathcal{M}_{simp}(V)$ of the CY hypersurface V is defined as the Chow quotient [49]:

$$\overline{\mathcal{M}}_{simp}(V) = \mathbb{P}(L(\Sigma(1) \cup 0))//T, \tag{3.49}$$

where T is the torus action from (3.42).

There is a conjecture [50] which proposes that each of the cones σ gives a maximal unipotent boundary point [3] of $\mathcal{M}_{simp}$.

[3] For an exact definition of the point of maximal unipodent monodromy see, for example, [2]. For us, this point is a special point in the moduli space such that there are one holomorphic and $h_{2,1}$ logarithmic–holomorphic solutions of the PF–system.

Definition 3.2.4 Let Σ^* corresponding to a simplified projective subdivision of the normal fan of Δ. The toric Kähler moduli space $\overline{\mathcal{KM}}_{toric}(V)$ is defined as follows. First note that we exclude interior points of facets of Δ^*. Thus the restriction map $H^2(X_{\Sigma^*}) \to H^2(V)$ is injective and we have an obvious inclusion $\mathrm{cpl}\,(\Sigma^*) \subset \overline{K(V)}_{toric} = \overline{H^2_{toric}(V) \cap K(V)}$ (K is the Kähler cone of X_{Σ^*}). Now there is another conjecture which asserts that the toric moduli space is, up to an action of a finite group, the union $\cup_{\Sigma^{*\prime}} \mathrm{cpl}\,(\Sigma^{*\prime})$, where the union runs over all fans which differ from Σ^* by a sequence of trivial flips. In order to get a smooth moduli space we use a refinement given by the unimodular cones $\{\sigma\}$.

The mirror map is now an isomorphism between $\overline{\mathcal{KM}}_{toric}(V^*)$ and $\mathcal{M}_{simp}(V)$. Note that we have to take the mirror in definition (3.2.4). We choose local coordinates $q_1, \ldots, q_r$ around the large radius point of the complexified Kähler moduli space using the map

$$(D^*)^h \to (H^2_{toric}(V^*, \mathbb{R}) + iK_{toric}(V^*))/\,\mathrm{im}(H_{toric}(V^*, \mathbb{Z})), \tag{3.50}$$

sending $(\mathrm{e}^{2\pi i t_1}, \ldots, \mathrm{e}^{2\pi i t_h})$ to $\left[\sum_{a=1}^h t_a T_a\right]$. Here, $(D^*)^h$ is the punctured polydisk, and T_a are the generators of σ. This map extends to an immersion of the whole polydisk, where the origin corresponds to the large radius limit point. Thus, we will use the coordinates

$$q_a = \mathrm{e}^{2\pi i t_a} \quad (a = 1, \ldots, h) \tag{3.51}$$

to define the mirror map. As coordinates around the point of maximal unipotent monodromy we can choose the $\{z_a\}$ defined in (3.43). The mirror map can then be defined as [2]

$$\boxed{q_a = (-1)^{l_0^{(a)}} z_a \exp\left(2\pi i \tilde{y}_a / y_0\right),} \tag{3.52}$$

with y_0 being the unique holomorphic solution of the PF equations at the point of maximally unipotent monodromy, and the $\{y_a\ (a = 1, \ldots, h)\}$ being solutions of the PF equations that are of the form:

$$y_a = y_0 \log\left((-1)^{l_0^{(a)}} z_a\right) + \tilde{y}_a, \quad a = 1, \ldots, h, \tag{3.53}$$

where $\tilde{y}_a$ is holomorphic at the origin with $y_a(0) = 0$. Thus, it remains to find the quantities $\{y_0, y_1, \ldots, y_h\}$. Let $\tilde{\omega}$ be the volume form on $X_{\Sigma^*} - V$ from equation (3.45) with its residue on V being the holomorphic $(n-1)$ form Ω. We also have a cycle $\gamma \subset T \subset X_{\Sigma^*}$ defined by $|t_1| = \cdots = |t_d| = 1$ corresponding to a cohomology class $g \in H^{d-1}(V)$. Thus we have a pairing

$$\boxed{\langle g, \Omega \rangle = \frac{1}{(2\pi i)^d} \int_\gamma \frac{\lambda_0}{f} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d},} \tag{3.54}$$

where Ω is the residue of $\tilde{\omega}$ from equation (3.45). Plugging the expansion

$$\frac{\lambda_0}{f} = \frac{1}{1 - \sum_{m \in \Sigma(1)} \lambda_m (\lambda_0)^{-1} t^m} = \sum_{K=0}^{\infty} \left(\sum_{m \in \Sigma(1)} \lambda_m (\lambda_0)^{-1} t^m \right)^K \tag{3.55}$$

into (3.54) and performing the integration gives

$$\langle g, \Omega \rangle = \sum_{l_1,\ldots,l_s} \frac{(-l_0)!}{l_1! \ldots l_s!} (-\lambda_0)^{l_0} \lambda_1^{l_1} \ldots \lambda_s^{l_s}, \tag{3.56}$$

where the sum runs over all relations $l = (l_0; l_1, \ldots, l_s) \in \Lambda$ of $\Sigma \times \{1\}$ with $l_i \geq 0$ $(i = 1, \ldots, s = \#\Sigma(1))$. Switching to the coordinates $\{z_a\}$ defined in (3.43) coming from a basis

$$l^{(a)} = (l_0^{(a)}; l_1^{(a)}, \ldots, l_s^{(a)}) \qquad a = 1, \ldots, h, \tag{3.57}$$

and writing each l as $l = n_1 l^{(1)} + \cdots + n_h l^{(h)}$ gives:

$$\boxed{\langle g, \Omega \rangle = \sum_{n_1,\ldots,n_h} \frac{\left(-\sum_{a=1}^h n_a l_0^{(a)}\right)!}{\prod_{j=1}^s \left(\sum_{a=1}^h n_a l_j^{(a)}\right)!} \left((-1)^{l_0^{(1)}} z_1\right)^{n_1} \ldots \left((-1)^{l_0^{(h)}} z_h\right)^{n_h},} \tag{3.58}$$

where the sum is over all $n_1, \ldots, n_h$ such that $\sum_{a=1}^h n_a l_j^{(a)} \geq 0$ $(j = 1, \ldots, s)$. Now the crucial point is that $l_i \geq 0$ $(i = 1, \ldots, s = \#\Sigma(1))$, which implies that l is in the cone dual to the cone spanned by effective divisor classes, $\langle [D_j] \rangle$. Dualizing $\sigma \subset \mathrm{cpl}\,(\Sigma) \subset \langle [D_j] \rangle$ gives $l \in \sigma^* = \langle l^{(a)} \rangle$. Thus the integers $\{n_a\}$ are all non–negative. It is also not hard to see that the series (3.58) converges [2], and we can set

$$y_0(z_1, \ldots, z_h) = \langle g, \Omega \rangle. \tag{3.59}$$

Next we calculate the y_a using the method of Frobenius. The idea is to replace the integers n_a by $n_a + \rho_a$, where ρ_a is a real parameter. This means that we have to replace the factorials:

$$\left(\sum_{a=1}^h n_a l_j^{(a)}\right)! \to \Gamma\left(\sum_{a=1}^h (n_a + \rho_a) l_j^{(a)} + 1\right), \tag{3.60}$$

and that we also have to shift the exponents of $(-1)^{l_0^{(a)}} z_a$ $(a = 1, \ldots, h)$. The key result is that the quantities

$$y_a = (\partial_{\rho_a} y_0(z, \rho))\,|_{\rho=0}, \quad (a = 1, \ldots, h) \tag{3.61}$$

have the desired property, which follows from the fact that $[\Box_l, \partial_{\rho_a}] = 0$ and that the action of ∂_{ρ_a} on powers of $(-1)^{l_0^{(a)}} z_a$ $(a = 1, \ldots, h)$ generates a factor $\log\left((-1)^{l_0^{(a)}} z_a\right)$. The following two formulas are needed when performing the calculation:

$$\frac{d}{ds}\Gamma(s)\,|_{s=n+1} = \Gamma(n+1)\left(-\gamma + \sum_{j=1}^n \frac{1}{j}\right), \tag{3.62}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{3.63}$$

where s is a real parameter, n is a positive integer, and γ is Euler's constant.

Let $\{V, V^*\}$ be a mirror pair, and denote by K ($\overline{K}$) the A- (B-) model threepoint correlators on V^* (V). The large complex structure coordinates for V, (z_a), are related to the large radius coordiantes on V^*, (q_a). By mirror symmetry, the correlators are related by:

$$\boxed{K_{(abc)} = \langle \mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c \rangle = \int_V \tilde{\Omega} \wedge \nabla_{\delta_a} \nabla_{\delta_b} \nabla_{\delta_c} \tilde{\Omega}.} \tag{3.64}$$

Here, $\delta_a = 2\pi i q_a \partial/\partial_{q_a}$ and $\tilde{\Omega} = \Omega/y_0$ is the normalized 3–form on V. In order to evaluate the rhs, it remains to insert the mirror map. The $K_{(ijk)}$ are then given up to a multiplication by a non–vanishing complex constant, which can be determined by calculating the intersection numbers in the large radius limit. The final form is:

$$\boxed{K_{(abc)} = \kappa_{abc} + \sum_{d_a, d_b, d_c \geq 0} n^{(0)}_{abc} d_a d_b d_c \frac{\prod_a q_a^{d_a}}{1 - \prod_a q_a^{d_a}},} \tag{3.65}$$

where κ_{abc} are the classical intersection numbers, q_a are the coordinates defined in (3.51), and the sum counts the worldsheet instantons.

A simple example

In this subsection we will show how all this works explicitly in the case of the mirror of a degree 8 hypersurface in $\mathbb{P}(1,1,2,2,2)$. The polytope $\Delta^{*\prime} \subset N_{\mathbb{R}}$ defining the weighted projective space is defined as the convex hull of the vertices

$$\rho_0^* = -e_1^* - 2e_2^* - 2e_3^* - 2e_4^*, \qquad \rho_1^* = e_1^*, \qquad \rho_2^* = e_2^*, \qquad \rho_3^* = e_3^*, \qquad \rho_4^* = e_4^*. \tag{3.66}$$

The cone generated by $\{\rho_0^*, \rho_1^*\}$ is not unimodular. To resolve the singularity, we can add the vertex $\rho_5^* = 1/2(\rho_0^* + \rho_1^*)$. We denote the polytope with the extra vertex by Δ^*, and the fan with cones over the faces of the polytope Δ^* ($\Delta^{*\prime}$) by Σ^* ($\Sigma^{*\prime}$).

The Kähler cone

In this simple example it is very easy to get the Kähler cones using the primitive collections of definition (3.1.5). For Σ^* they are:

$$\{\rho_0^*, \rho_1^*\} \quad \text{and} \quad \{\rho_2^*, \ldots, \rho_5^*\}. \tag{3.67}$$

Using the two relations

$$2\rho_5^* = \rho_0^* + \rho_1^* \quad \text{and} \quad \rho_2^* + \rho_3^* + \rho_4^* + \rho_5^* = 0, \tag{3.68}$$

we see that a support function ψ is strictly convex if and only if

$$\begin{aligned} 2\psi(\rho_5^*) = \psi(\rho_0^* + \rho_1^*) &\;>\; \psi(\rho_0^*) + \psi(\rho_1^*), \\ 0 = \psi(\rho_2^* + \cdots + \rho_5^*) &\;>\; \psi(\rho_2^*) + \cdots + \psi(\rho_5^*). \end{aligned}$$

Setting $\psi(\rho_i^*) = -a_i$ we have to interpret these inequalities in terms of the Chow group. Under the map β of the exact sequence (3.5) $(a_0, \ldots, a_5)$ is mapped to $(s,t) = (a_2 + a_3 + a_4 + a_5, -2a_5 + a_0 + a_1)$. In these coordinates the Kähler cone is the interior of the cone

$$\mathrm{cpl}\,(\Sigma^*) = \{(s,t) \in A(\Sigma^*) : s, t \geq 0\}. \tag{3.69}$$

The fan $\Sigma^{*\prime}$ has only one primitive collection: $\{\rho_0^*, \ldots, \rho_4^*\}$. Defining again $\psi(\rho_i^*) = -a_i$ and using the relation $\rho_0^* + \rho_1^* + 2\rho_2^* + 2\rho_3^* + 2\rho_4^* = 0$ we find that

$$a_0 + a_1 + 2a_2 + 2a_3 + 2a_4 > 0. \tag{3.70}$$

Thus the Kähler cone is the interior of the cone [4]

$$\mathrm{cpl}\left(\Sigma^{*\prime}\right) = \{(s,t) \in A(\Sigma^*) : s \geq 0, 2s + t \geq 0\}. \tag{3.71}$$

Griffiths–Dwork on the mirror

In order to find the mirror family we have to construct the dual polytope $\Delta' \subset M_{\mathbb{R}}$ [5] which turns out to be the convex hull of the vertices

$$\begin{pmatrix} \rho_0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 \\ -1 & 7 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & -1 \\ -1 & -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & -1 & 3 \end{pmatrix}. \tag{3.72}$$

This time the group G defining the quotient contains torsion, which can be seen by the fact that $\rho_1 + \cdots + \rho_4 = 0 \mod 4$. The lattice points in $\Delta^{*\prime}$ are in one to one correspondence to the set of monomials having the same grading as the divisor $\beta_0 = [D_{\rho_0}] + \cdots + [D_{\rho_4}]$, i.e. the monomials are

$$\prod_{\rho \in \mathrm{Vert}(\Delta')} x_\rho^{\langle \rho, z^* \rangle + 1} \quad z^* \in \Delta^{*\prime} \cap N, \tag{3.73}$$

which are just all $\mathbb{Z}_4$–invariant monomials of degree eight on $\mathbb{P}(1,1,2,2,2)$. We can write the equation of the anticanonical hypersurface in the form

$$f = z_2 x_1^8 + x_2^8 + z_1 x_3^4 + x_4^4 + x_5^4 + x_1 x_2 x_3 x_4 x_5 + x_1^4 x_2^4, \tag{3.74}$$

where the choice of the parameters (z_1, z_2) is such that $z_1 = z_2 = 0$ is a point of maximal unipotent monodromy. Defining $\delta_{z_i} = z_i d/dz_i$ we can choose the basis

$$\begin{aligned} &\omega_1 = \frac{\Omega_0}{f}, \quad \omega_2 = \delta_{z_1}\omega_1 = -\frac{z_1 x_3^4 \Omega_0}{f^2}, \quad \omega_3 = \delta_{z_2}\omega_1 = -\frac{z_2 x_1^8 \Omega_0}{f^2}, \\ &\omega_4 = \delta_{z_1^2}\omega_1 = \omega_2 + 2\frac{2z_1^2 x_3^8 \Omega_0}{f^3}, \quad \omega_5 = \delta_{z_1}\delta_{z_2}\omega_1 = \frac{2z_1 z_2 x_1^8 x_3^4 \Omega_0}{f^3}, \quad \omega_6 = \delta_{z_1}^2 \delta_{z_2}\omega_1 = \omega_5 - \frac{6z_1^2 z_2 x_1^8 x_3^8 \Omega_0}{f^4}. \end{aligned} \tag{3.75}$$

[4] Note that we consider $\Sigma^{*\prime}$ instead of Σ^*, but $\Sigma^*(1)$ is fixed.

[5] Since we now construct the mirror CY V^*, the meaning of the lattices has changed: The ambient space is now defined by the polytope in the M–lattice, and the exponents of the monomials are from the N–lattice.

The PF equations are then obtained by expressing various $\delta_{z_i}^j \omega_1$ terms in this basis. For example, we can use

$$\delta_{z_2}^2 \omega_1 = \omega_3 + \frac{2 z_2^2 x_1^{16} \Omega_0}{f^3}. \tag{3.76}$$

Using Gröbner basis techniques, we get the following relations between x_1^{16}, x_3^8, and $x_1^8 x_3^4$ (the latter two are the monomials appearing in our basis $\{\omega_i\}$ over f^3):

$$z_2(1-4z_2)x_1^{16} - z_1^2 x_3^8 + 4 z_1 z_2 x_1^8 x_3^4 = \frac{-1}{16} \sum_{i=1}^{3} A_i x_i \frac{\partial f}{\partial x_i} \in J(f) = \langle \frac{\partial f}{\partial x_i} \rangle, \tag{3.77}$$

with

$$\begin{aligned} A_1 &= 2(4z_2 - 1)x_1^8 + x_1 x_2 x_3 x_4 x_5 - 4 x_1^4 x_2^4, \\ A_2 &= 2 x_1^8, \\ A_3 &= 4 z_1 x_3^4 - 16 z_2 x_1^8 - x_1 x_2 x_3 x_4 x_5. \end{aligned}$$

Inserting this into equation (3.76) gives

$$(1-4z_2)\delta_{z_2}^2 \omega_1 - z_2(\delta_{z_1}^2 - \delta_{z_1} - 4\delta_{z_1}\delta_{z_2} + 2\delta_{z_2})\omega_1 = (1-6z_2)\delta_{z_2}\omega_1 - \frac{2z_2}{16 f^3} \sum_{i=1}^{3} A_i x_i \frac{\partial f}{\partial x_i} \Omega_0. \tag{3.78}$$

Using (3.32) the right hand side can be written modulo exact forms as:

$$(1-6z_2)\delta_{z_2}\omega_1 - \frac{2z_2}{16 f^3} \sum_{i=1}^{3} A_i x_i \frac{\partial f}{\partial x_i} \Omega_0 \sim (1-6z_2)\delta_{z_2}\omega_1 - \frac{z_2}{16 f^2} \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (A_i x_i)\, \Omega_0 =$$
$$-\frac{5z_2}{2}\delta_{z_2}\omega_1 + \frac{10 z_2}{8 f^2}(x_1^4 x_2^4 - z_1 x_3^4)\Omega_0.$$

Applying again Gröbner basis techniques we can write the right hand side as

$$\begin{aligned} -\frac{5z_2}{2}\delta_{z_2}\omega_1 + \frac{10 z_2}{8 f^2}(x_1^4 x_2^4 - z_1 x_3^4)\Omega_0 &= -\frac{5z_2}{2}\delta_{z_2}\omega_1 + \frac{5 z_2}{8 f^2}\left(-4 z_2 x_1^8 - \frac{x_3}{2}\frac{\partial f}{\partial_{x_3}} + \frac{x_1}{2}\frac{\partial f}{\partial_{x_1}}\right)\Omega_0 \\ &= \frac{5 z_2}{16 f^2}\left(\frac{x_1 \partial f}{\partial_{x_1}} - \frac{x_3 \partial f}{\partial_{x_3}}\right)\Omega_0 \sim 0, \end{aligned}$$

where in the last step again (3.32) was used. Thus we end up with the Picard Fuchs equation:

$$(1-4z_2)\delta_{z_2}^2 \omega_1 - z_2(\delta_{z_1}^2 - \delta_{z_1} - 4\delta_{z_1}\delta_{z_2} + 2\delta_{z_2})\omega_1 = 0. \tag{3.79}$$

PF equations using the A–system of the mirror

We now compute the PF equations using the A–system of the polytope (3.72). Homogenizing $f \in L(\Delta^*)$ gives

$$f = \lambda_1 x_1^8 + \lambda_2 x_2^8 + \lambda_3 x_3^4 + \lambda_4 x_4^4 + \lambda_5 x_5^4 + \lambda_6 x_1 x_2 x_3 x_4 x_5 + x_1^4 x_2^4, \tag{3.80}$$

which is the same as equation (3.74). A basis of relations for the A–system is given by:

$$l^{(1)} = (0,0,1,1,1,1,-4) \qquad \text{and} \qquad l^{(2)} = (1,1,0,0,0,-2,0). \tag{3.81}$$

Thus we get the following T–invariant moduli coordinates:

$$z_1 = \frac{\lambda_3\lambda_4\lambda_5\lambda_6}{\lambda_0^4} \qquad \text{and} \qquad z_2 = \frac{\lambda_1\lambda_2}{\lambda_6^2}. \tag{3.82}$$

From the two relations (3.81) we get two PF operators:

$$\begin{aligned} \lambda_0\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6\Box_{l_1}\lambda_0^{-1} &= \lambda_0\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6(\partial_2\partial_3\partial_4\partial_5\partial_6 - \partial_0^4)\lambda_0^{-1} = 0, \\ \lambda_0\lambda_1\lambda_2\Box_{l_2}\lambda_0^{-1} &= \lambda_0\lambda_1\lambda_2(\partial_1\partial_2 - \partial_6^2)\lambda_0^{-1} = 0. \end{aligned}$$

Using identity (3.44), and the identities

$$\delta_1 = \delta_2 = \delta_{z_2}, \qquad \delta_3 = \delta_4 = \delta_5 = \delta_{z_1}, \qquad \delta_6 = \delta_{z_1} - 2\delta_{z_2}, \qquad \delta_0 = -4\delta_{z_1}, \tag{3.83}$$

we end up with the following PF operators in the (z_1, z_2) coordinates:

$$\begin{aligned} \delta_{z_1}^3(\delta_{z_1} - 2\delta_{z_2}) - 4z_1\delta_{z_1}(4\delta_{z_1}+1)(4\delta_{z_1}+2)(4\delta_{z_1}+3) &= 0, \\ \delta_{z_2}^2 - z_2(\delta_{z_1} - 2\delta_{z_2})(\delta_{z_1} - 2\delta_{z_2} - 1) = 0. \end{aligned}$$

The first operator can be reduced (dividing it by δ_{z_1} from the left) and we have two operators of rank 3 and 2. Hence the space of solutions has dimension of at most 6, which is the Betti number b_3 of the mirror. Thus we have found a complete system of PF operators.

YCs

Having a complete set of PF operators we can now compute the YCs. After some algebra the PF operators take the form:

$$0 = (1-4z_2)\delta_{z_2}^2 - z_2(\delta_{z_1}^2 - 4\delta_{z_1}\delta_{z_2} - \delta_{z_1} + 2\delta_{z_2}), \tag{3.84}$$
$$0 = (1-256z_1)\delta_{z_1}^3 - 2\delta_{z_1}^2\delta_{z_2} - 8z_1(48\delta_{z_1}^2 + 22\delta_{z_1} + 3). \tag{3.85}$$

Multiplying the first equation with δ_{z_1} or δ_{z_2} gives two third order equations. Thus we have three independent equations of order three relating our couplings $\overline{K}_{(111)}, \overline{K}_{(112)}, \overline{K}_{(122)}$ and $\overline{K}_{(222)}$:

$$\begin{aligned} \overline{K}_{(112)} &= \frac{(1-256z_1)\overline{K}_{(111)}}{2}, \\ \overline{K}_{(122)} &= \frac{z_2(-1+512z_1)\overline{K}_{(111)}}{1-4z_2}, \\ \overline{K}_{(222)} &= \frac{z_2(1-256z_1+4z_2-3072z_1z_2)\overline{K}_{(111)}}{2(1-4z_2)^2}, \end{aligned} \tag{3.86}$$

where we again have used Griffiths transversality.

Multiplying the first PF operator with $\delta_{z_1}^2, \delta_{z_2}^2$ or $\delta_{z_1}\delta_{z_2}$ and the second with δ_{z_1} or δ_{z_2} gives five differential equations of order four. Again derivatives with degree lower than three can be thrown away. Inserting the identities (3.48) and using (3.86) to express all YCs in terms of $\overline{K}_{(111)}$ gives us a system of linear differential equations for $\overline{K}_{(111)}$. The calculation is straight forward and can be found, for example, in [2]. The resulting couplings are:

$$\boxed{\begin{array}{ll} \overline{K}_{(111)} = \frac{c}{\Delta_1\Delta_3^3}, & \overline{K}_{(112)} = \frac{c(1-256z_1)}{2\Delta_1\Delta_3^2\Delta_4}, \\ \overline{K}_{(122)} = \frac{c(512z_1-1)}{\Delta_1\Delta_2\Delta_3\Delta_4}, & \overline{K}_{(222)} = \frac{c(1-256z_1+4z_2-3072z_1z_2)}{2\Delta_1\Delta_2^2\Delta_4^2}, \end{array}} \tag{3.87}$$

with the discriminants:

$$\Delta_1 = (1-512z_1+65536z_1^2-262144z_1^2z_2), \quad \Delta_2=(1-4z_2), \quad \Delta_3=z_1, \quad \Delta_4=z_2. \tag{3.88}$$

The mirror map

We now compute the mirror map in our example. Using the basis of relations from equation 3.81 we get

$$(l_1,\dots,l_6,l_0) = n_1(0,0,1,1,1,1,-4)+n_2(1,1,0,0,0,-2,0), \tag{3.89}$$

i.e. $l_1 = l_2 = n_1$, $l_3 = l_4 = l_5 = n_2$, $l_6 = n_1 - 2n_2$ and $l_0 = -4n_1$, and $l_i \geq 0$ implies $n_1 \geq 2n_2$. Thus the series in equation (3.58) is given by

$$y_0(z_1,z_2) = \sum_{n_1\geq 2n_2\geq 0} \frac{(4n_1)!}{(n_1!)^3(n_2!)^2(n_1-2n_2)!} z_1^{n_1} z_2^{n_2}. \tag{3.90}$$

The subtle point is that $n_1 \geq 2n_2$ and that we really have

$$y_0(z_1,z_2) = \sum_{n_1,n_2\geq 0} \frac{\Gamma(4n_1+1)}{\Gamma(n_1+1)^3\Gamma(n_2+1)^2\Gamma(n_1-2n_2+1)} z_1^{n_1} z_2^{n_2}, \tag{3.91}$$

which agrees with the foregoing series because the Γ function has simple poles at $\{0,-1,-2,\dots\}$. We have to calculate

$$y_1 = \partial_{\rho_1} y_0 \mid_{\rho_1=\rho_2=0} \qquad \text{and} \qquad y_2 = \partial_{\rho_2} y_0 \mid_{\rho_1=\rho_2=0}. \tag{3.92}$$

Derivation of y_1: Differentiation of $z_1^{n_1+\rho_1}$ gives the part $y_0(z_1,z_2)\log(z_1)$. The derivatives of the coefficients are:

$$\begin{aligned}
\tfrac{d}{d\rho_1}\Gamma(4n_1+4\rho_1+1)\mid_{\rho_1=0} &= 4(4n_1)!\left(-g+\sum_{j=1}^{4n_1}\tfrac{1}{j}\right) && \text{for} \quad n_1,n_2\geq 0,\\
\tfrac{d}{d\rho_1}\tfrac{1}{\Gamma(n_1+1)^3}\mid_{\rho_1=0} &= \tfrac{-3}{(n_1!)^3}\left(-g+\sum_{j=1}^{n_1}\tfrac{1}{j}\right) && \text{for} \quad n_1,n_2\geq 0,\\
\tfrac{d}{d\rho_1}\tfrac{1}{\Gamma(-2n_2+n_1+\rho_1+1)}\mid_{\rho_1=0} &= \tfrac{-1}{(n_1-2n_2)!}\left(-g+\sum_{j=1}^{n_1-2n_2}\tfrac{1}{j}\right) && \text{for} \quad n_1-2n_2\geq 0,\\
\tfrac{d}{d\rho_1}\tfrac{1}{\Gamma(-2n_2+n_1+\rho_1+1)}\mid_{\rho_1=0} &= \tfrac{d}{d\rho_1}\left(\tfrac{\sin(\pi(2n_2-n_1-\rho_1))}{\pi}\Gamma(2n_2-n_1-\rho_1)\right)\mid_{\rho_1=0}\\
&= (-1)^{n_1+1}((2n_2-n_1-1)!) && \text{for} \quad n_1-2n_2<0.
\end{aligned}$$

Altogether we find:

$$\boxed{\begin{aligned}
y_1(z_1,z_2) &= y_0(z_1,z_2)\log(z_1)+\sum_{n_1<2n_2}\tfrac{(4n_1)!(2n_2-n_1-1)}{(n_1!)^3(n_2!)^2(n_1-2n_2)!}(-1)^{n_1+1}z_1^{n_1}z_2^{n_2}\\
&+\sum_{n_1\geq 2n_2}\tfrac{(4n_1)!}{(n_1!)^3(n_2!)^2(n_1-2n_2)!}z_1^{n_1}z_2^{n_2}\left(4\sum_{j=1}^{4n_1}\tfrac{1}{j}-3\sum_{j=1}^{n_1}\tfrac{1}{j}-\sum_{j=1}^{n_1-2n_2}\tfrac{1}{j}\right)z_1^{n_1}z_2^{n_2}.
\end{aligned}} \tag{3.93}$$

If we write y_1 as $y_1=y_0(z_1,z_2)\log(z_1)+\tilde{y}_1$, the mirror map for q_1 reads as:

$$\boxed{q_1=z_1\exp(\tilde{y}_1/y_0)}. \tag{3.94}$$

Derivation of y_2: Differentiation of $z_2^{n_2+\rho_2}$ gives the part $y_0(z_1,z_2)\log(z_2)$ The derivatives of the coefficients are:

$$\begin{aligned}
\tfrac{d}{d\rho_2}\tfrac{1}{\Gamma(n_2+\rho_2)^2+1}\mid_{\rho_2=0} &= \tfrac{-2}{(n_2!)^2}\left(-g+\sum_{j=1}^{n_2}\tfrac{1}{j}\right) && \text{for} \quad n_1,n_2\geq 0,\\
\tfrac{d}{d\rho_2}\tfrac{1}{\Gamma(n_1-2n_2-2\rho_2+1)}\mid_{\rho_2=0} &= \tfrac{2}{(n_1-2n_2)!}\left(-g+\sum_{j=1}^{n_1-2n_2}\tfrac{1}{j}\right) && \text{for} \quad n_1-2n_2\geq 0,\\
\tfrac{d}{d\rho_2}\tfrac{1}{\Gamma(n_1-2n_2-2\rho_2+1)}\mid_{\rho_2=0} &= \tfrac{d}{d\rho_2}\left(\tfrac{\sin(\pi(2n_2+2\rho_2-n_1))}{\pi}\Gamma(2n_2+2\rho_2-n_1)\right)\mid_{\rho_2=0}\\
&= 2(-1)^{n_1}(2n_2-n_1-1)! && \text{for} \quad n_1-2n_2<0.
\end{aligned}$$

Altogether we find:

$$\boxed{\begin{aligned}
y_2(z_1,z_2) &= y_0(z_1,z_2)\log(z_2)+2\sum_{n_1<2n_2}\tfrac{(4n_1)!(2n_2-n_1-1)}{(n_1!)^3(n_2!)^2(n_1-2n_2)!}(-1)^{n_1}z_1^{n_1}z_2^{n_2}\\
&+2\sum_{n_1\geq 2n_2}\tfrac{(4n_1)!}{(n_1!)^3(n_2!)^2(n_1-2n_2)!}z_1^{n_1}z_2^{n_2}\left(\sum_{j=1}^{n_1-2n_2}\tfrac{1}{j}-\sum_{j=1}^{n_2}\tfrac{1}{j}\right)z_1^{n_1}z_2^{n_2}.
\end{aligned}} \tag{3.95}$$

If we write y_2 as $y_2=y_0(z_1,z_2)\log(z_2)+\tilde{y}_2$, the mirror map for q_2 reads as:

$$\boxed{q_2=z_2\exp(\tilde{y}_2/y_0)}. \tag{3.96}$$

Kähler YCs on V

We are now ready to compute the Kähler threepoint functions on V using the mirror map. The Kähler classes modulo rational equivalence are:

$$L = D_0 \sim D_1 \qquad H = D_2 \sim D_3 \sim D_4 \qquad D_5 \sim D_2 - 2D_0, \tag{3.97}$$

where each D_i corresponds to the vertex ρ_i^* $(i = 0, \ldots, 6)$ of Δ^*. From the primitive collections (3.67) we get in addition the relations:

$$D_0 D_1 = 0 \qquad \text{and} \qquad D_2 D_3 D_4 D_5 = 0. \tag{3.98}$$

Since $V \in \mid \sum_i D_i \mid = \mid 4D_3 \mid$, we get, for the classical triple intersection numbers: $\langle H, H, H \rangle_{\rm CL}$:

$$\begin{array}{rcccccccc} 4HH^3 & \sim & D_2D_3D_4D_2 & \sim & 2D_3D_4D_5 + 8D_0D_2D_3D_5 & \sim & 8\,\mathrm{Vol}\,(\rho_0^*, \rho_2^*, \rho_3^*, \rho_4^*) & = & 8, \\ 4H^3L & \sim & 4D_0D_2D_3D_4 & \sim & 4\,\mathrm{Vol}\,(\rho_0^*, \rho_2^*, \rho_3^*, \rho_4^*) & = & 4, & & \\ 4HL^3 & \sim & 4D_0^2D_1D_3 & = & 0, & & & & \\ 4H^2L^2 & \sim & 4D_0D_1D_3^2 & = & 0. & & & & \end{array} \tag{3.99}$$

The mirror conjecture predicts that

$$\boxed{\langle H, H, H \rangle \;=\; K_{(111)} \;=\; \int_{V^*} \tilde{\Omega} \wedge \nabla_{\delta_1}\nabla_{\delta_1}\nabla_{\delta_1}\tilde{\Omega} \;=\; \langle H, H, H \rangle_{\rm CL} + \ldots,} \tag{3.100}$$

where $\delta_1 = 2\pi i q_1 \frac{\partial}{\partial_1}$, $\tilde{\Omega} = \Omega / y_0$, and $q_i = q_i(z_1, z_2)$ $(i = 1, 2)$ via the mirror map. Using the chain rule we get:

$$\begin{aligned} \langle H, H, H \rangle \;=\; \frac{(2\pi i)^3}{y_0^2} \Bigg[& \left(\frac{q_1}{z_1} \frac{\partial_{z_1}}{\partial_{q_1}} \right)^3 \overline{K}_{(111)} + 3 \left(\frac{q_1}{z_1} \frac{\partial_{z_1}}{\partial_{q_1}} \right)^2 \left(\frac{q_1}{z_2} \frac{\partial_{z_2}}{\partial_{q_1}} \right) \overline{K}_{(112)} + \\ & 3 \left(\frac{q_1}{z_1} \frac{\partial_{z_1}}{\partial_{q_1}} \right) \left(\frac{q_1}{z_2} \frac{\partial_{z_2}}{\partial_{q_1}} \right)^2 \overline{K}_{(122)} + \left(\frac{q_1}{z_2} \frac{\partial_{z_2}}{\partial_{q_1}} \right)^3 \overline{K}_{(222)} \Bigg]. \end{aligned} \tag{3.101}$$

It remains to set the constant c in (3.87) equal to $c = \frac{8}{(2\pi i)^3}$ in order to get the classical intersection numbers as leading term. The first terms of the expansion are:

$$\boxed{\begin{aligned} \langle H, H, H \rangle &= 8 + 640 \tfrac{q_1}{1 - q_1} + 10032 \tfrac{2^3 q_1^2}{1 - q_1^2} + 640 \tfrac{q_1 q_2}{1 - q_1 q_2} + O(q^3) \\ &= 8 + 640 q_1 + 80896 q_1^2 + 640 q_1 q_2 + O(q^3), \\ \langle H, H, L \rangle &= 4 + 640 q_1 q_2 + O(q^3), \\ \langle H, L, L \rangle &= 640 q_1 q_2 + O(q^3), \\ \langle L, L, L \rangle &= 4 q_2 + 4 q_2^2 + 640 q_1 q_2 + O(q^3), \end{aligned}} \tag{3.102}$$

where we have also added the expansions of the other couplings [51] for completeness. Because of our normalization, the constant terms in the expansions are the classical intersection numbers from equation (3.99). Comparing the result with equation (3.65) gives the instanton number $n(a,b) = n_{ah+bl}$, where $ah + bl$ is an element of $H_2(V,\mathbb{Z})$, and $\{h,l\}$ is the basis dual to $\{H,L\}$.

Chapter 4

The geometry of toric CICY's

4.1 Complete intersections in toric varieties

Batyrev showed that a generic section of the anticanonical bundle of $\mathbb{P}_\Delta = X_{\Sigma^*}$ defines a Calabi–Yau hypersurface if Δ is reflexive, which means, by definition, that Δ and its dual Δ^* are both lattice polytopes. Mirror symmetry corresponds to the exchange of Δ and Δ^* [29]. The generalization of this construction to complete intersections of codimension $r > 1$ (CICYs) was obtained by Batyrev and Borisov [31] who introduced the notion of a nef–partition.

Let $\Delta \subset M_\mathbb{R}, \Delta^* \subset N_\mathbb{R}$ be a dual pair of d–dimensional reflexive polytopes. Denote by $\Sigma[\Delta^*] \subset N_\mathbb{R}$ the fan over faces of Δ^* and let $X_{\Sigma[\Delta^*]}$ be the toric variety corresponding to the fan $\Sigma[\Delta^*]$.

Definition 4.1.1 A partition $E = E_1 \cup \cdots \cup E_r$ of the set of vertices of Δ^* into disjoint subsets $E_1, \ldots, E_r$ is called a nef–partition if there exist r integral convex $\Sigma[\Delta^*]$– piecewise linear support functions $\varphi_i : N_\mathbb{R} \to \mathbb{R}$ $(i = 1, \ldots, r)$ such that

$$\varphi_i(\rho^*) = \begin{cases} 1 & \text{if } \rho^* \in E_i, \\ 0 & \text{otherwise.} \end{cases}$$

Each φ_i corresponds to a semi–ample Cartier divisor $D_i = \sum_{\rho^* \in E_i} D^*_\rho$ on X, where D^*_ρ is the irreducible component of $X \setminus T$ corresponding to the vertex $\rho^* \in E_i$, and $V = D_1 \cap \cdots \cap D_r$ defines a family of Calabi–Yau complete intersections. Moreover, each φ_i corresponds to a lattice polyhedron Δ_i defined as

$$\Delta_i = \{x \in M_\mathbb{R} : \langle x, y\rangle \geq -\varphi_i(y) \; \forall \; y \in N_\mathbb{R}\} \tag{4.1}$$

supporting global sections of the semi–ample invertible sheaf $\mathcal{O}(D_i)$. Since the knowledge of the decomposition $E = E_1 \cup \cdots \cup E_r$ is equivalent to that of the set of supporting polyhedra $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$, this data is often also called a nef–partition. We note that the polytopes

$$\nabla_i = \mathrm{Conv}\,(\{0\} \cup E_i) \subset N_\mathbb{R}$$

define also a nef–partition $\Pi^*(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$ with $\nabla = \Delta_1 + \cdots + \Delta_r$. In summary we have the following relations between all the polyhedra in the different spaces:

$$\begin{array}{|c|c|} \hline M_{\mathbb{R}} & N_{\mathbb{R}} \\ \hline \Delta = \Delta_1 + \cdots + \Delta_r & \Delta^* = \mathrm{Conv}\,(\nabla_1, \ldots, \nabla_r) \\ \nabla^* = \mathrm{Conv}\,(\Delta_1, \ldots, \Delta_r) & \nabla = \nabla_1 + \cdots + \nabla_r \\ \hline \end{array} \tag{4.2}$$

This yields a pair (V, V^*) of Calabi–Yau varieties, where $V \subset X_{\Sigma^*}$ and $V^* \subset X_\Sigma$ are mirror to each other.

Given the nef–partition $\Delta^* = \mathrm{Conv}\,(\nabla_1, \ldots, \nabla_r)$, let $\lambda_m \in \mathbb{C}$ be the coefficients of the Laurent polynomials $f_1, \ldots, f_r$,

$$f_l(t) = 1 - \sum_{m \in \Delta_l \cap M} \lambda_m t^m \in L(\Delta_l) \qquad (l = 1, \ldots, r). \tag{4.3}$$

The simultaneous vanishing of these polynomials f_l then defines the complete intersection Calabi–Yau manifold $V \subset X_{\Sigma^*}$, or in other words, the Δ_l are the Newton polyhedra for f_l. Similarly, the ∇_l are the Newton polyhedra for V^*.

4.1.1 The Cayley trick

To compute the cohomology of a complete intersection, we can use the Cayley–trick. The affine case is treated in [52]: let $\{f_1, \ldots, f_r\}$ be a system of nondegenerate Laurent polynomials. They define a complete intersection $Y = Z_1 \cap \cdots \cap Z_r$ in the d–dimensional algebraic torus T. Now add a set of Lagrange multipliers $\{\lambda_1, \ldots, \lambda_r\}$ and consider the hypersurface $Z_F \subset T \times \mathbb{C}^r$ defined by the equation

$$F(\lambda, z) = \lambda_1 f_1 + \cdots + \lambda_r f_r - 1 = 0. \tag{4.4}$$

Restrict the projection $\pi : T \times \mathbb{C}^r \to T$ to Z_F. If a point z belongs to Y equation (4.4) obviously has no solution and the fiber $\pi^{-1}(z)$ is empty. On the other hand if $z \notin Y$ the fiber is an affine linear subspace $\mathbb{C}^{r-1} \subset C^r$. Using the fact that the E–polynomial [52]

$$E(u, v) = \sum_{p,q} h^{p,q} (-1)^{p+q} u^p v^q \tag{4.5}$$

of a bundle that is locally trivial in the Zariski topology is the product of the E–polynomial of the base space and that of the fiber we get:

$$E(Z_F; u, v) = E(T \setminus Y; u, v)\; E(\mathbb{C}^{r-1}; u, v). \tag{4.6}$$

Now $E(\mathbb{C}^{r-1}; u, v) = (uv - 1)^{(r-1)}$, $E(T; u, v) = (uv)^d$, and $E(T \setminus Y; u, v) = E(T; u, v) - E(Y; u, v)$, so we end up with

$$E(Y; u, v) = (uv)^d - E(Z_F; u, v)(uv - 1)^{(1-r)}. \tag{4.7}$$

This result was generalized to relate a complete intersection $V = D_1 \cap \cdots \cap D_r$ in a d–dimensional toric variety X to a hypersurface $\mathcal{Y}$ in the projective bundle $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r))$ over X [30,32]: let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the natural projection. Then $\pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) = \mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r)$ and every global section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ corresponds to global sections $(s_1, \ldots, s_r)$ of $\mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r)$. Again one can show that

$$\mathbb{P}(\mathcal{E}) - \mathcal{Y} \to X - D_1 \cap \cdots \cap D_r \tag{4.8}$$

is a locally trivial $\mathbb{C}^{r-1}$ bundle in the Zariski topology and hence

$$H_c^i(X - D_1 \cap \cdots \cap D_r) \simeq H_c^{i+2(r-1)}(\mathbb{P}(\mathcal{E}) - \mathcal{Y}). \tag{4.9}$$

Now we extend the lattices M, N to $\bar{M} = \mathbb{Z}^r \oplus M$, $\bar{N} = \mathbb{Z}^r \oplus N$, respectively, and define the $\bar{d} = d + r$–dimensional reflexive Gorenstein cone C of index r corresponding to the complete intersection as [1]

$$C = \{(\lambda_1, \ldots, \lambda_r, \lambda_1 x_1 + \cdots + \lambda_r x_r) \in \bar{M}_{\mathbb{R}} : \lambda_i \in \mathbb{R}_{\geq}, x_i \in \Delta_i,\ i = 1, \ldots r\}. \tag{4.10}$$

The support of C can be identified with the set of global sections of $\mathcal{O}(D_1) \oplus \cdots \oplus \mathcal{O}(D_r)$. The string–theoretic Hodge numbers of the complete intersection V are then related via the Cayley trick to those of $\mathrm{Proj}(\mathbb{C}[C \cap \bar{M}])$ and can be computed from the combinatorial data of the reflexive pair of Gorenstein cones C, C^* [32, 34]:

$$\boxed{E(V; u, v) = \sum (-1)^{p+q}\, h^{pq}\, u^p\, v^q = \sum_{I=[x,y]} \frac{(-)^{\rho_x} t^{\rho_y}}{(uv)^r} S(C_x, \frac{v}{u}) S(C_y^*, uv) B_I(u^{-1}, v).} \tag{4.11}$$

In this formula x, y label faces C_x^* of dimension ρ_x of C^* and C_x denotes the dual face of the dual cone C. The interval $I = [x, y]$ labels all cones that are faces of C_y^* containing C_x^*. The polynomials $B_I(u, v)$ encode the combinatorics of the face lattice [32]. The polynomials $S(C_x^*, u) = (1-u)^{\rho_x} \sum_{n \geq 0} u^n l_n(C_x^*)$ of degree $\rho_x - 1$ are related to the numbers $l_n(C_x^*)$ of lattice points at degree n in C_x^* and hence to the Ehrhart polynomial [53] of the Gorenstein polytope generating C_x [32].

In the case of hypersurfaces, $r = 1$, it is known [29] that the Picard number can also be computed with the formula

$$h^{11} = l(\Delta^*) - 1 - d - \sum_{\mathrm{codim}(\Theta^*)=1} l^*(\Theta^*) + \sum_{\mathrm{codim}(\Theta^*)=2} l^*(\Theta^*) l^*(\Theta) \tag{4.12}$$

where Θ and Θ^* are faces Δ and Δ^*, respectively. $l(\Theta)$ is the number of lattice points of a face Θ, and $l^*(\Theta)$ the number of its interior points. This formula has a simple interpretation (see also [45]): The principal contributions come from the toric divisors $D_i = \{z_i = 0\}$

[1] n_C is defined uniquely by the conditions $\langle e_i, n_C \rangle = 1$ for the generators of e_i of $\mathbb{Z}^r$ and $\langle x, n_C \rangle = 0\ \forall\, x \in M_{\mathbb{R}}$. The dual Gorenstein cone C^* is given by the dual partition $\Pi^*(\nabla) = \{\nabla_1, \ldots, \nabla_r\}$.

that correspond to points in Δ^* different from the origin. There are d linear relations among these divisors. The first sum corresponds to the subtraction of interior points of facets. The corresponding divisors of the ambient space do not intersect a generic Calabi–Yau hypersurface. Lastly, the bilinear terms in the second sum can be understood as multiplicities of toric divisors so that their presence indicates that only a subspace of the cohomology (i.e. the Kähler moduli space) can be analyzed with toric methods.

Unfortunately, the general formula (4.11) does not lend itself to a similar interpretation in any known way.[2] As for hypersurfaces, interior points of facets of Δ^* never contribute divisors to a toric Calabi–Yau, but explicit computations of intersection rings show that for complete intersections it may happen that even for vertices ρ_i the corresponding divisor D_i does not intersect $[V] = \bigcap_{l=1}^{r} D_{0,l}$, where

$$D_{0,l} = \sum_{i \in E_l} D_i \tag{4.13}$$

corresponds to the l^{th} equation of the Calabi–Yau variety V. A simple example is the blowup by a non–intersecting divisor of the degree (3,4) CICY in $\mathbb{P}^5_{111112}$ with Hodge numbers (1,79) that is discussed in [34]. It is very important to find a more explicit formula for the Picard number of a complete intersection that allows for an interpretation in terms of the multiplicities of divisors D_i after restriction to the Calabi–Yau. There is a good chance that such a formula exists: While intersection numbers depend on the triangulation of the fan we observed in many examples that these multiplicities are independent of the triangulation and thus should depend only on the combinatorics of the data of the polytope that enter (4.11).

4.2 Resolution of singularities

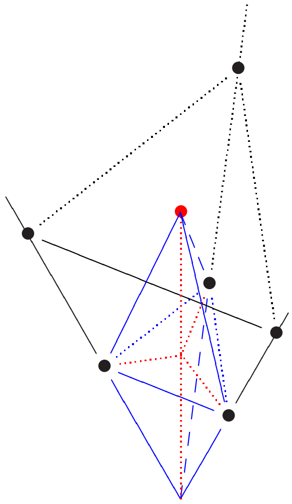
It is known that the points on the polyhedron Δ^* are in general not sufficient to give a smooth ambient space [54, 55]. In addition, one may have to take into account certain points in degree greater than one, i.e. points in $(k\Delta^* \backslash (k-1)\Delta^*) \cap N$ ($k > 1$), in order to resolve all singularities. In this section, we give a general discussion of these points, in particular in the context of constructing smooth complete intersection CY spaces.

Recall that a toric ambient space is smooth if its fan is simplicial and unimodular, i.e. if all its cones are simplicial and unimodular. Suppose we have added all the lattice points in the polyhedron Δ^* and determined one of the possible star triangulations of this set of points.

A star triangulation is a triangulation T for which all simplices contain the origin of Δ^*. In other words, each simplex $\sigma \in T$ determines a pointed cone C_σ, i.e. a cone over a facet of Δ^* whose apex is the origin. If there is a simplex, say σ, with $\mathrm{Vol}\,(\sigma) > 1$, then the corresponding coordinate patch $U_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M]$ of the toric variety will be singular.

[2] There are, however, more suggestive formulas for h^{11} in special cases [31].

The general procedure to resolve this singularity is to subdivide σ by adding lattice points on the polyhedron. But since we already have added all these points, the next best thing one can do is to add points at distance two, i.e. points in $C_\sigma \cap 2\Delta^* \cap N$. We then replace the simplex σ by a double pyramid $\bar{\sigma}$ over the corresponding facet of Δ^* whose apices are the origin and the extra point, and with $\mathrm{Vol}\,(\bar{\sigma}) = 2\,\mathrm{Vol}\,(\sigma)$. We then have to find a triangulation of $\bar{\sigma}$ such that all simplices have unit volume. If such a point were to lie in the interior of C_σ, triangulating $\bar{\sigma}$ would divide it into d simplices of integral volume.



Idea of adding one extra points $\bullet$ at layer two in order to triangulate the simplex $\bar{\sigma} \subset N'_{\mathbb{R}}$ with $\mathrm{Vol}\,(\bar{\sigma}) = 2\,\mathrm{Vol}\,(\sigma) > 2$ (this is only necessary if $d > 3$).

This is not always possible, and if it is not, the corresponding point will lie in a face of $2\Delta^*$ whose codimension is at least one. In this case, there must be at least one other singular simplex, say σ', adjacent to σ, such that there is a $\tau = \sigma \cap \sigma' \neq \emptyset$. The existence of σ' is independent of the star triangulation of Δ^*. We can get an upper bound on the codimension of τ as follows: Any two–dimensional simplex without additional points has volume one, therefore three–dimensional simplices in a star triangulation of Δ^* have the same property, due to the Gorenstein condition. Hence codim $\tau < d - 3$.

Before we can analyze this situation in more detail, we have to find these singular cones. At this point it is helpful to recall the notion of a circuit (3.1.15), which gives us an easy way to find simplices of the triangulation of Δ^* whose volume is larger than one, and have to be subdivided: they correspond to simplices with $|c_m| > 1$.

We will discuss the case $d = 5$ in more detail since it is most relevant to the examples in the later sections. If $\mathrm{Vol}\,(\sigma) < 5$, then we cannot divide $\bar{\sigma}$ into $d = 5$ integral parts. Therefore we already know that the extra point will lie precisely in codimension one, and that there will be another singular simplex. Let $\{\rho_1^*, \ldots, \rho_4^*\}$ be the set of vertices generating the four dimensional simplex τ. Then the extra point ρ_r^* lying in layer k must be of the form:

$$p = \sum_1^4 a_i \rho_i^*, \qquad \sum_1^4 a_i = k, \qquad a_i > 0 \;\; (i = 1, \ldots, 4). \tag{4.14}$$

If $k = 2$, i.e. the extrapoint is in the interiour of 2τ, we can discuss the following situations:

Case $k = 2$ and $\mathrm{Vol}\,(\sigma)) = 2$:

We can use the extra point $\rho_1^* = \frac{1}{2}\sum_{i=1}^4 \rho_i^*$.

Case $k = 2$ and $\mathrm{Vol}\,(\sigma)) = 3$:

In this situation we have to add two points at layer two. This can be seen as follows: The generators of τ are:

$$\begin{pmatrix} \rho_1^* & \rho_2^* & \rho_3^* & \rho_4^* \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

That this choice is unique (up to a basis transformation) follows from the Gorenstein condition (the entries of the first line are all equal to one), and the fact that in three dimensions there is only one simplex without any extra points and volume three. Condition (4.14) with $r = 2$ gives pricisely two interiour points at layer two:

$$\rho_{\mathrm{r},q}^* = \frac{1}{3}\left(q(\rho_1^* + \rho_2^*) + (3-q)(\rho_3^* + \rho_4^*)\right) \quad q = 1, 2.$$

Now we have to triangulate $(\bar{\tau}, A)$, with $A = \{\rho_1^*, \ldots, \rho_4^*, \rho_{\mathrm{r},1}^*, \rho_{\mathrm{r},1}^*, 0\}$. We can choose the following basis of affine relations:

$$L = \begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -2 \\ 1 & 1 & -1 & -1 & -3 & 3 & 0 \end{pmatrix}.$$

The columns $\{b_1, \ldots, b_7\}$ of this matrix are the generators of the secondary fan. It has five cones of maximal dimensions, corresponding to the five coherent triangulations, and five rays correspondig to the circuits. There is one unique star triangulation related to the cone $\langle b_1, b_3\rangle$. According to proposition (3.1.14) we find eight simplices in this triangulation dual to the chambers:

$$\begin{array}{llll} \langle b_1, b_3\rangle, & \langle b_1, b_4\rangle, & \langle b_2, b_3\rangle, & \langle b_2, b_4\rangle, \\ \langle b_1, b_5\rangle, & \langle b_2, b_5\rangle, & \langle b_3, b_6\rangle, & \langle b_4, b_6\rangle. \end{array}$$

Since all coefficients in the circuits relating two of these simplices are plus or minus one we see that they all have the same volume. Thus $\bar{\tau}$ has volume eight and is devided into eight simplices, each having volume one.

Case $k = 2$ and $\mathrm{Vol}\,(\sigma)) > 3$:

The cases where the volume V is greater than three can be treated in a similar way. Generically, these three dimensional simplices are of the form:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & V & 0 \end{pmatrix}.$$

However, there are dimensions where other joices are possible. For example in dimension five we have in addition:

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix}.$$

If $V = 7$ in turns out that there are even two non–standard simplices. If V is greater than five the extra point can also lie in the interior of σ and we have to study triangulations of four dimensional simplices. A more detailed discussion of $V \in 4, 5$ can be found in [35].

It is clear that all the points in the polyhedron Δ^* must be affine linear combinations of the vertices belonging to one set E_r. This follows immediatly from the properties of the integral piecewise linear functions used in the definition of the nef–partition given in section 4. Points in the Gorenstein cone belonging to layer $r > 1$ are in general *not* a sum of r lattice points of layer two. However, since the D_r are lattice points, the values of the piecewise linear functions used in the definition (4.1.1) of a nef–partition are integral. Therefore, (4.13) becomes

$$D_{0,l} = \sum_{\rho_i^* \in E_l} D_i + \sum_q c_{\mathrm{r},q,l} D_{\mathrm{r},q}, \tag{4.15}$$

where the second sum is over the extra points, and the coefficients are integers and satisfy $\sum_{l=1}^{r} c_{\mathrm{r},q,l} = k$. For example, in the case $d = 5$ and $r = 2$, i.e. CY threefolds, and $\mathrm{Vol}\,(\sigma) = 2$ only an even number of the ρ_i^* can belong to one E_l, and therefore $(c_{\mathrm{r},1}, c_{\mathrm{r},2})$ is either $(2, 0)$, $(1, 1)$, or $(0, 2)$ (we dropped the index $q = 1$).

One last point concerns the toric quotients of section 4.3. If the variety is a quotient, the volumina of all the simplices will be multiples of the index $N' : N_0$. In this case it turns out that we can work on the covering space and need only to resolve the singularities there.

4.3 Free quotients

We now come to the discussion of toric CY spaces with non–trivial fundamental groups. We mainly restrict our attention to the situation where they arise from free quotients coming from group actions that correspond to lattice quotients. We call such quotients *toric*. Let N' be the sublattice of N that is generated by the lattice vectors in $\Sigma^*(1)$, and let $\Sigma^{*\prime}$ be the fan obtained from Σ^* by relating everything to the lattice N'. Then $X_{\Sigma^*} = X_{\Sigma^{*\prime}}/G$ is a quotient of $X_{\Sigma^{*\prime}}$ by a finite abelian group G isomorphic to N/N' that acts by multiplication with phases ω^{α_i} on the homogeneous coordinates z_i. Here ω is a $|G|^{\text{th}}$ root of unity. We will denote such group actions by $(\alpha_1, \ldots, \alpha_n)/|G|$, where $\sum_{i=1}^{n} \alpha_i = 0 \mod |G|$.

To see how this comes about we note that the ring of regular functions on an affine coordinate patch U_σ of X_{Σ^*} is spanned by the monomials $\prod z_i^{<m,\rho_i>}$, where $m \in \sigma^\vee \cap M$ is a vector in the dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$. If we change from the lattice N' to the finer lattice N then we have to exclude all monomials corresponding to vectors $m \in M'$ that do not belong to the sublattice $M \subset M'$. Thus there are no more functions available to distinguish points in $X_{\Sigma^{*\prime}}$ that live on orbits of G (in turn, this can be used to define G). The quotient $X_{\Sigma^*} = X_{\Sigma^{*\prime}}/G$ is never free for a toric variety [25]. If, however, a (Calabi–Yau) hypersurface or complete intersection does not intersect the set of fixed points then we get a manifold with nontrivial fundamental group π_1 isomorphic to G. This is the case for CY 3-folds if the refinement of the lattice does not lead to additional lattice points of Δ^*. For a given pair of reflexive polytopes, the dual of the lattice M' that is spanned by the vertices of Δ is the finest lattice N' with respect to which the polytope is reflexive. If the lattice N_0 that is spanned by the vertices of Δ^* is a proper sublattice of the lattice N' then any subgroup of N'/N_0 corresponds to a different choice of the N lattice and hence to a different toric CY hypersurface:

$$\begin{tikzcd}
\langle\{\rho_i^*\}\rangle \arrow[r, equal] & N_0 \arrow[r, hook] \arrow[d] & N \arrow[r, hook] \arrow[d] & N' \arrow[d, "\mathrm{Hom}(_,\mathbb{C}^*)"] & \\
& M_0 & M \arrow[l, hook'] & M' \arrow[l, hook'] \arrow[r, equal] & \langle\{\rho_i\}\rangle.
\end{tikzcd}$$

There is thus only a finite number of lattices that have to be checked to find all toric free quotients.

Some well–known examples are the free $\mathbb{Z}_5$ quotient of the quintic and the free $\mathbb{Z}_3$ quotient of the CY hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$. For both cases cyclic permutation of the coordinates of the projective spaces defines another free group action of the same order that commutes with the toric quotient, leading to Euler numbers 8 and 18, respectively. These free "double quotients" are, however, not toric in the sense that the resulting manifold is not a CICY in a toric variety. Analyzing the complete list of 473'800'776 reflexive polytopes for CY hypersurfaces [56–58] one finds 14 more examples of toric free quotients [33]: The elliptically fibered $\mathbb{Z}_3$ quotient of the degree 9 surface in $\mathbb{P}^4_{11133}$, whose group action on the homogeneous coordinates is given by the phases $(1, 2, 1, 2, 0)/3$, and 13 elliptic K3 fibrations where the lattice quotient has index 2. Among the latter there is the $\mathbb{Z}_2$ quotient of $(\mathbb{P}^1)^4$ with phases $(0, \frac{1}{2})$ on each factor which admits an additional $\mathbb{Z}_2$ freely acting on the CY hypersurface by simultaneous exchange of the coordinates of all $\mathbb{P}^1$ factors. Models of a similar type are currently studied because of their promising phenomenlogical properties [59, 60].

The condition that $\Delta^* \cap N$ and $\Delta^* \cap N'$ coincide is sufficient for a free quotient of a CICY with dimension up to 3 because the singularities of a maximal crepant resolution are at codimension 4 and can be avoided by a generic choice of the defining equations. It is, however, not necessary: Divisors corresponding to interior points of facets of Δ^* do not intersect the CY and hence do not kill the fundamental group if they are generated by a refinement of the N lattice.

An analysis of the complete lists of reflexive polytopes with up to 4 dimensions shows that the only case where the weaker condition is relevant for hypersurfaces is that of the torus. In that case a free $\mathbb{Z}_3$ quotient in $\mathbb{P}^2$, and a free $\mathbb{Z}_2$ in $\mathbb{P}^2_{112}$ and in $\mathbb{P}^1 \times \mathbb{P}^1$ can be realized torically in the obvious way. Taking a product with a (toric) K3 this can be used to construct CICYs at codimension 2 with first Betti number $b_1 = 2$. The resulting Hodge numbers are $h^{11} = 13$ for the $\mathbb{Z}_2$ quotients and $h^{11} = 9$ in case of $\mathbb{Z}_3$; the simplest examples are $\mathbb{P}^2 \times \mathbb{P}^3_{1122}$ with group action $(0, 1, 2; 0, 1, 0, 2)/3$ for $\mathbb{Z}_3$, and $\mathbb{P}^2_{112} \times \mathbb{P}^3$ with phases $(0, 1, 1; 0, 1, 0, 1)/2$ or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$ with phases $(0, 1; 0, 1; 0, 1, 0, 1)/2$ for $\mathbb{Z}_2$. These Hodge numbers have also been obtained for Landau–Ginzburg orbifold models [61], where the values $h^{11} = 3, 5, 9, 13$ were found for $h^{01} = 1$ [58, 61]. One might expect that the models with 3 and 5 would require a larger order of the free group action. It was shown, however, in [61] that the 1–forms in Landau–Ginzburg models can only arise if the model factors into $T^2 \times \mathrm{K3}$. Moreover, the twist that reduces the Picard number must act with unit determinant on the T^2 and hence must agree with the free $\mathbb{Z}_3$ on the elliptic curve in $\mathbb{P}^2$ or with the free $\mathbb{Z}_2$ on $\mathbb{P}^2_{112}$ that we already know from the other examples. Only the K3s can be realized in different ways. Indeed, analyzing the lists of [61] we found several realizations of these Hodge numbers with K3s that correspond the generalized Calabi–Yau varieties in the sense of [30, 38] like the cubic in $\mathbb{P}^5$, where a $\mathbb{Z}_3$–twist with phases $(1,1,1,0,0,0)/3$ and a free action on the torus leads to $h^{11} = h^{12} = 3$, and the degree 12 hypersurface in $\mathbb{P}^5_{334446}$, where a $\mathbb{Z}_2$ with phases $(0,0,0,0,1,1)/2$ and a free action on the torus in $\mathbb{P}_{112}$ leads to $h^{11} = h^{12} = 5$. For all of these models a mirror construction is available both in the CFT framework [62, 63] and, more geometrically, in terms of reflexive Gorenstein cones [30].

There are many more examples of toric free quotients for CICYs with codimension $r > 1$, some of which will play a role later on. In that case a group may act freely under even weaker conditions because even divisors that correspond to vertices may not intersect the CY. But with the present state of the art this requires a case by case analysis of the intersection ring. For Calabi–Yau 4–folds, on the other hand, the above criterion for a free quotient is no longer sufficient because the codimensions of the singularities in the ambient space may be too small to avoid them by an appropriate choice of the hypersurface equations. There are many examples where this happens.[3]

4.4 Fibrations

We will now discuss some properties of fibrations. Again, we restrict ourselves to the situation where the combinatorial data of the polytopes contain the relevant information. For (non–toric) K3 surfaces and Calabi–Yau 3–folds there exists a criterion by Oguiso for the existence of elliptic and K3 fibrations in terms of intersection numbers [64]. We will state it in Section 4.8.1. Like the latter, fibration properties thus depend on the triangulation, or in other words, on the choice of the phase in the extended Kähler moduli space.

For toric Calabi–Yau spaces there is, however, a more direct way to search for fibrations that manifest themselves in the geometry of the polytope and to single out appropriate triangulations [57, 65–67]. These fibrations descend from toric morphisms of the ambient space [23, 68]: Let Σ and Σ_b be fans in N and N_b, respectively, and let $\phi : N \to N_b$ be a lattice homomorphism that induces a map of fans $\phi : \Sigma \to \Sigma_b$ such that for each cone $\sigma \in \Sigma$ there is a cone $\sigma_b \in \Sigma_b$

[3] Free quotients can be constructed easily with PALP [33]. To find all candidates among quotients of the sextic 4–fold one can use the following commands (`zbin.aux` is an auxiliary file storing polytopes on sublattices)

```
$ cws.x -w5 6 6 -r | class.x -f -sv -po zbin.aux
$ class.x -b -pi zbin.aux | class.x -sp -f
```

and obtains the N–lattice polytopes in a basis where the lattice quotient is diagonal.

```
$ class.x -b -pi zbin.aux | class.x -sp -f|cws.x -N -f | poly.x -fg
```

displays weight systems and Hodge data. This yields 6 candidates for free $\mathbb{Z}_3$ quotients. They are, however, all singular, as the Euler numbers are bigger than 1/3 of those of the respective covering spaces.

that contains the image of σ [4]. Then there is a T–equivariant morphism $\tilde{\phi} : V_\Sigma \to V_{\Sigma_b}$ and the lattice N_f for the fibers is the kernel of ϕ in N.

$$\begin{tikzcd} 0 \arrow[r] & N_f \arrow[r, "i"] & N \arrow[r, "\phi"] & N_b \arrow[r] & 0 \arrow[d, "\mathrm{Hom}(\_,\mathbb{C}^*)"] \\ 0 & M_f \arrow[l] & M \arrow[l, "\pi"'] & M_b \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

For our construction of a fibered Calabi–Yau variety we require the existence of a reflexive section $\Delta^*_f \subset \Delta^*$ of the polytope $\Delta^* \subset N_\mathbb{R}$. The toric morphism ϕ is then given by the projection along the linear space spanned by Δ^*_f and N_b is defined as the image of N in the quotient space $N_\mathbb{R}/\langle\Delta^*_f\rangle_\mathbb{R}$. In order to guarantee the existence of the projection we choose a triangulation of Δ^*_f and then extend it to a triangulation of Δ^*. For each such choice we can interpret the homogeneous coordinates that correspond to rays in Δ^*_f as coordinates of the fiber and the others as parameters of the equations and hence as moduli of the fiber space. Reflexivity of the fiber polytope Δ^*_f ensures that the fiber also is a CICY because a nef–partition of Δ^* automatically induces a nef–partition of Δ^*_f. This follows immediately from the definition by restriction of the convex piecewise linear functions defining the partition to $(N_f)_\mathbb{R}$.

For hypersurfaces the geometry of the resulting fibration has been worked out in detail in [67]. The codimension r_f of the fiber generically coincides with the codimension r of the fibered space also for complete intersections. For $r > 1$ it may happen, however, that Σ^1_f does not intersect one (or more) of the E_l's, in which case the codimension decreases. An example of that type is the model

$$\mathbb{P}\begin{pmatrix} 2\,2\,2\,4\,1\,1\,0 \\ 0\,0\,0\,4\,1\,1\,2 \end{pmatrix} \left[\begin{array}{c|c} 8 & 4 \\ 8 & 0 \end{array}\right] / \mathbb{Z}_2 : \; 1\,1\,0\,1\,1\,0\,0 \tag{4.16}$$

with $h^{11} = 3$ and $h^{12} = 43$, which is a free $\mathbb{Z}_2$ quotient of a blowup of $\mathbb{P}^5_{222411}$ with the position of the additional vertex $\rho^*_7 = -\frac{1}{2}(4\rho^*_4 + \rho^*_5 + \rho^*_6)$ given by the second linear relation (the bottom line in the parenthesis). (This notation will be explained in more detail in the next section.) This polytope has one nef–partition with $E_1 = \{\rho^*_3, \rho^*_4, \rho^*_5, \rho^*_6, \rho^*_7\}$ and $E_2 = \{\rho^*_1, \rho^*_2\}$. The corresponding bidegrees are seperated by a vertical line in the bracket (they are given by the sums of the gradings of the homogeneous coordinates that correspond to the vertices that belong to E_l). The codimension two fiber polytope Δ^*_f that is spanned by $\rho^*_4, \ldots, \rho^*_7$ has all of its vertices in E_1 so that we obtain a K3 fibration with the generic fiber being a degree 8 hypersurface in $\mathbb{P}^3_{4112}$ instead of an elliptic codimension two fiber that would naively be expected.

In our searches for fiber spaces with certain properties we mostly restricted attention to the generic case where $r = r_f$. We also analyzed the intersection numbers of many spaces and found no example where a fibration has no toric realization, provided that the possible change of the codimension is taken into account. There are, however, cases where the fibration does

[4]For example, the projection $\mathbb{Z}^2 \to \mathbb{Z}$ sending $(x, y) \mapsto x$ for the Hirzebruch surface F_a (example 3.1.4) fulfills this condition if and only if $a = 0$.

not lift to a toric morphism of the ambient space. An example is the polytope (4.40) which will be discussed in section 4.8.2.

4.5 The (2,30) example

In this section we will discuss a set of examples of CICYs in detail. We will exhibit their non–trivial fundamental group, their fibrations, and their partitions.

In our analysis of the geometry and of applications of complete intersections it was natural to start with models with a small number h^{11} of Kähler moduli. In this realm it is quite likely that our lists of toric CICYs is fairly complete, at least for codimension two. Among the one–parameter models we found only two new Hodge numbers, namely $h^{12} = 25$ for the free $\mathbb{Z}_3$ quotient of the degree (3,3) CICY in $\mathbb{P}^5$ and $h^{12} = 37$ for the free $\mathbb{Z}_2$ quotient of the degree (4,4) CICY in $\mathbb{P}^5_{111122}$. The Picard–Fuchs equations of the respective universal covers were both analysed in [69].

We therefore turn to the list of 2–parameter examples, the first of which have Hodge numbers (2,30). They will serve as our main examples in this and the next section. [5] In the appendix we compile a brief overview of toric CICYs with small h^{11}.

There are three different polytopes which allow for codimension two complete intersections with Hodge numbers (2,30). These have eight or nine vertices and no additional boundary points. In a convenient basis the coordinates of the vertices of the first of these polytopes are given by the column vectors

$$\Delta^*_{(A)}\ \{\rho^*_i\} = \left\{\begin{array}{rrrrrrrr} 1 & 0 & 0 & 0 & -2 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \end{array}\right\}. \qquad (4.17)$$

If the number of vertices is close to the simplex case it is most economical to describe a polytope in a coordinate independent way by the linear relations among the vertices. This data is sufficient if the lattice N is generated by the vertices. Otherwise it has to be supplemented by an abelian group action that defines the lattice. The toric variety corresponding to the polytope $\Delta^*_{(A)}$, generalizing the notation $\mathbb{P}^n_w$ or $\mathbb{P}^n(w)$ in the simplex case, is thus

$$\mathbb{P}_{\Delta^*_{(A)}} = \mathbb{P}\begin{pmatrix} 2\,1\,1\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix} \ /\,\mathbb{Z}_2:\ 1\,1\,1\,0\,0\,1\,0\,0. \qquad (4.18)$$

The lines in the parenthesis indicate the linear relations among the vertices. The first two tell us that the toric variety corresponds to a product space $\mathbb{P}^4_{21111} \times \mathbb{P}^1$, while the third linear relation $\rho^*_1 + \rho^*_8 = 0$ amounts to a blow–up of $\mathbb{P}^4_{21111}$ by the last vertex ρ^*_8. Finally, the group action

[5] The first hypersurfaces example (2,29) is the free $\mathbb{Z}_3$ quotient of the degree (3,3) hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$.

indicates that the lattice N is not generated by the vertices alone. It requires, as an additional generator, the lattice point $\frac{1}{2}(\rho_1^* + \rho_2^* + \rho_3^* + \rho_6^*)$, i.e. the linear combination of vertices that corresponds to the phases of the $\mathbb{Z}_2$ action on the homogeneous coordinates.

The coordinates displayed in eq. (4.17), with the last line divisible by two for *all* lattice points, shows that the CICY is a free quotient. The group action can be recovered by finding an integer linear combination ρ^* of the column vectors with coefficients in $\frac{1}{2}\mathbb{Z}$ whose last coordinate is odd (thus refining the lattice). The resulting generator for the $\mathbb{Z}_2$ action is unique only up to linear combinations with the weight vectors modulo 2, which corresonds to a different choice $\rho^* \to \rho^* + \Delta\rho^*$ with $\Delta\rho^* = \frac{1}{2}(\rho_2^* + \rho_3^* + \rho_4^* + \rho_5^*) = -\rho_1^*$, $\frac{1}{2}(\rho_6^* + \rho_7^*) = 0$ or $\frac{1}{2}(\rho_1^* + \rho_8^*) = 0$.

Next, we have to look at the possible nef–partitions for $\Delta^*_{(A)}$. It turns out that, up to symmetries, [6] there is a unique nef–partition, given by $E_1 = \{\rho_1^*, \rho_2^*, \rho_4^*, \rho_8^*\}$ and $E_2 = \{\rho_3^*, \rho_5^*, \rho_6^*, \rho_7^*\}$ with the Hodge numbers (2,30). This leads to the partitioning $6 = 4 + 2$, $2 = 2 + 0$ and $2 = 2 + 0$ of the total degrees of the complete intersection V into multidegrees. We will augment the previous notation by a bracket indicating these multidegrees and write

$$V_{(A)} = \mathbb{P}\begin{pmatrix} 2\,1\,1\,1\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 2 & 0 \\ 2 & 0 \end{array}\right] \;/\,\mathbb{Z}_2 :\; 1\,1\,1\,0\,0\,1\,0\,0. \tag{4.19}$$

In general these degrees do not specify the partition uniquely. We observed, however, in all examples that equal multidegrees of different partitions always lead to the same Hodge numbers. The $\mathbb{Z}_2$ quotient now indicates that the lattice M is replaced by the sublattice corresponding to monomials that are invariant under the given phase symmetry. Since this quotient does not lead to additional lattice points in $\Delta^*_{(A)}$ the corresponding group action is free on the CICY, i.e. $\pi_1(V) = \mathbb{Z}_2$.

In the present example the ambient space is a product space with a $\mathbb{P}^1$ factor. The CICY is, nevertheless, a nontrivial K3–fibration over $\mathbb{P}^1$ because the coefficients of the equations defining the K3 fiber V_f depend on the coordinates of the base. The K3 family containing the generic fibers is obtained by dropping the second line and the columns corresponding to ρ_6^* and ρ_7^*,

$$V_f = \mathbb{P}\begin{pmatrix} 2\,1\,1\,1\,1\,0 \\ 1\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 2 & 0 \end{array}\right]. \tag{4.20}$$

Note that the $\mathbb{Z}_2$ quotient does not change the fiber lattice because it also acts on $\mathbb{P}^1$, effectively dividing the base by 2. Over the two fixed–points on the base we obtain, however, an Enriques fiber. (Since $K3$ only admits free $\mathbb{Z}_2$ quotients and since the group action on the base $\mathbb{P}^1$ always has fixed points, a free quotient of a $K3$ fibration can only have order 2.) The induced nef–partition is obtained by dropping ρ_6^* and ρ_7^* from E_2. It does, of course, lead to the bi–degrees of the divisors given in (4.20).

The lines in the parenthesis of our notation for toric varieties, as in (4.18), (4.19), or (4.20), generate the cone of non–negative linear relations among the points in N. We will often call

[6] The symmetry group has order 16 and is generated by the transpositions $\rho_2^* \leftrightarrow \rho_3^*$, $\rho_4^* \leftrightarrow \rho_5^*$, $\rho_6^* \leftrightarrow \rho_7^*$ and by the exchange $(\rho_2^*, \rho_3^*) \leftrightarrow (\rho_4^*, \rho_5^*)$.

them *weight vectors*. The definition of the polytope only requires the linear relations among the vertices (and possibly the group action defining a sublattice). For the discussion of fibrations and other geometrical data it is, however, often convenient to include the linear relations among all lattice points of Δ^*. When the partitioning of the total degree $d = \sum w_i$ of a weight vector by the nef–partition is specified as $(-d_1, -d_2; w_1, \ldots, w_N)$ with $d = d_1 + d_2$ we may also call them *charge vectors* because this is the data that characterizes part of the gauged linear sigma model realization of these geometries [37]. Note that for the definition of the polytope and for the degree data of the nef–partition, it is sufficient to give the charge vectors that correspond to the linear relations among the vertices. A more redundant description may, nevertheless, be useful to make fibrations or non–free lattice quotients visible. A complete definition of the model, on the other hand, may require a resolution of singularities through triangulations and the inclusion of additional points. In the weighted projective case the (single) weight vector coincides with the generator of the Mori cone of the ambient space. In general, however, the Mori cone will be larger than the cone that is spanned by the charge vectors.

The other realizations of the (2,30) model are

$$V_{(B)} = \mathbb{P}\begin{pmatrix} 2\,1\,1\,1\,1\,0\,0\,0 \\ 2\,2\,1\,0\,1\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 4 & 4 \\ 2 & 0 \end{array}\right] \;/\,\mathbb{Z}_2 :\; 1\,1\,1\,0\,0\,1\,0\,0 \tag{4.21}$$

with nef–partitions $E_1 \cup E_2 = \{\rho_1^*, \rho_3^*, \rho_5^*, \rho_8^*\} \cup \{\rho_2^*, \rho_4^*, \rho_6^*, \rho_7^*\}$, $\{\rho_1^*, \rho_3^*, \rho_4^*, \rho_7^*, \rho_8^*\} \cup \{\rho_2^*, \rho_5^*, \rho_6^*\}$, or $\{\rho_1^*, \rho_2^*, \rho_4^*, \rho_8^*\} \cup \{\rho_3^*, \rho_5^*, \rho_6^*, \rho_7^*\}$ and

$$V_{(C)} = \mathbb{P}\begin{pmatrix} 2\,1\,1\,1\,1\,0\,0\,0\,0 \\ 2\,2\,1\,0\,1\,1\,1\,0\,0 \\ 1\,0\,0\,0\,0\,0\,0\,1\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 4 & 4 \\ 2 & 0 \\ 0 & 2 \end{array}\right] \;/\,\mathbb{Z}_2 :\; 1\,1\,1\,0\,0\,1\,0\,0\,0 \tag{4.22}$$

with partitions $E_1 \cup E_2 = \{\rho_1^*, \rho_3^*, \rho_4^*, \rho_7^*, \rho_8^*\} \cup \{\rho_2^*, \rho_5^*, \rho_6^*, \rho_9^*\}$ or $\{\rho_1^*, \rho_3^*, \rho_5^*, \rho_8^*\} \cup \{\rho_2^*, \rho_4^*, \rho_6^*, \rho_7^*, \rho_9^*\}$. The polytope $\Delta^*_{(B)}$ for $V_{(B)}$ has the same "K3 fiber" polytope as $\Delta^*_{(A)}$, but the two points above and below the fiber–hyperplane are shifted along the fiber as can be seen by the non–zero entries $(2, 2, 1, 0, 1)$ in the second line, below the weights of the fiber. The ambient space looks, at first sight, like a non–trivial fibration over the $\mathbb{P}^1$ with homogeneous coordinates $(x_6 : x_7)$. This is, however, not true because the line $\overline{\rho_6^* \rho_7^*}$ now intersects the fiber hyperplane outside the convex hull of the other lattice points. This line thus becomes an edge of any star triangulation of $\Delta^*_{(B)}$ so that the points in the intersection $D_6 \cap D_7$, which have homogeneous coordinates $x_6 = x_7 = 0$, have no image in the base $\mathbb{P}^1$.

We will see in the next section that $V_{(A)}$ and $V_{(B)}$ are nevertheless diffeomorphic and that their Picard–Fuchs equations are related by a change of variables. In particular, also $V_{(B)}$ is a K3 fibration. This is only possible if $D_6 \cap D_7$ does not intersect the CICY (as is indeed the case).

The polytope $\Delta^*_{(C)}$ is similar to $\Delta^*_{(B)}$ except for an additional blowup of the fiber polytope with an exceptional divisor D_9 that, as we will see in section 4.8.2, does not intersect the CY threefold. The additional point does, however, make the K3 fibration manifest, because $\Delta^*_{(C),f}$ is again reflexive.

In the next section we will discuss in more detail how different partitions or different polytopes leading to CICYs with the same topological data may be related.

4.5.1 Construction the nef–partition out of the Newton polytopes

It is instructive to see how the nef–partition for $\Delta^*_{(A)}$ can be obtained from the Newton polytopes of the degree (4,0) and (2,2) polynomials in the double cover of the ambient space. In order to arrive at the polytope (4.17) we observe that the ambient space of the double cover is closely related to the product space $\mathbb{P}^4_{21111} \times \mathbb{P}^1$. We thus start with the Newton polytope $\hat{\Delta}$ of a degree (6,2) equation in that space. It has 10 vertices corresponding to the monomials

$$x_0^3 y_j^2,\ x_i^6 y_j^2 \text{ with } 1 \le i \le 4,\ 0 \le j \le 1,$$

where x_i and y_j are the homogeneous coordinates in $\mathbb{P}^4_{21111}$ and $\mathbb{P}^1$, respectively. The degree (4,0) and (2,2) polynomials correspond to the Newton polytopes

$$\hat{\Delta}_1 = \langle x_0^2, x_i^4 \rangle,\ \hat{\Delta}_2 = \langle x_0 y_j^2, x_i^2 y_j^2 \rangle$$

so that $\hat{\Delta} = \hat{\Delta}_1 + \hat{\Delta}_2$. The $\mathbb{Z}_2$ quotient acting with signs $(---++,-+)$ kills the two vertices $x_0^3 y_j^2$ and generates 9 additional ones:

$$x_0^3 y_0 y_1, x_0^2 x_i^2 y_j^2 \text{ with } 1 \le i \le 4,\ 0 \le j \le 1. \tag{4.23}$$

The resulting polyhedron is not reflexive, has 196 points, 17 vertices and 9 facets, but can be made reflexive by dropping the vertex $x_0^3 y_0 y_1$. This yields a polyhedron $\Delta_{(A)}$ with 195 points and 16 vertices that possesses a nef–partition with Hodge numbers $h^{11} = 2$ and $h^{21} = 30$ (up to automorphisms there is only one additional nef–partition whose Hodge numbers are $h^{11} = 4$ and $h^{21} = 44$). The dual polyhedron $\Delta^*_{(A)}$ has 9 points and (in an appropriate basis) the 8 vertices given in eq. (4.17). The linear relations are $2\rho_1^* + \rho_2^* + \rho_3^* + \rho_4^* + \rho_5^* = 0 = \rho_6^* + \rho_7^*$ and the facet equation corresponding to the last vertex $\rho_8^* = -\rho_1^*$ is the one that eliminates $x_0^3 y_0 y_1$ and makes $\Delta_{(A)}$ reflexive.

The nef–partition of $V_{(A)}$ is now constructed from the Newton polyhedra $\hat{\Delta}_i$ as follows: In order to get $\Delta_{(A)} = \Delta_1 + \Delta_2$ we drop the point $x_0 y_0 y_1$ (which becomes a vertex on the sublattice) from $\hat{\Delta}_2$ and obtain

$$\Delta_1 = \langle x_0^2, x_i^4 \rangle,\ \Delta_2 = \langle x_i^2 y_j^2 \rangle.$$

With $v_0 = x_0^2$, $v_i = x_i^4$ and $w_{ij} = x_i^2 y_j^2$ we thus find $\Delta_{(A)} = \Delta_1 + \Delta_2 = \langle v_0 w_{ij}, v_i w_{ij} \rangle$ for the decomposition of the 8+8=16 vertices of $\Delta_{(A)}$. Shifting Δ_1 (Δ_2) by subtracting the exponent vectors of $x_0 x_1 x_3 x_4$ $(x_2 y_0 y_1)$ and dropping the redundant exponents of x_4 and y_1 we obtain the vertex–matrices

$$\Delta_1^\sigma = \left\{ \begin{array}{rrrrr} 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & -1 \\ 0 & 4 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right\}, \quad \Delta_2^\sigma = \left\{ \begin{array}{rrrrrrrr} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{array} \right\}.$$

with $\sigma_1^T = (1,1,0,1,0)$, $\sigma_2^T = (0,0,1,0,1)$ and $\Delta_l^\sigma \sim \Delta_l - \sigma_l$. The shifted Newton polytopes Δ_i^σ can be separated by a hyperplane, $\langle \rho, D_1^\sigma\rangle \le 0 \le \langle \rho, D_2^\sigma\rangle$, with $\rho = (2,1,0,1,0)$. The points of D_1^σ and those of D_2^σ on that hyperplane have no common non–zero coordinates, which implies that $\Delta_1^\sigma \cap \Delta_2^\sigma = \{0\}$ and thus establishes the nef–property. (Up to symmetries of $\hat{\Delta}$ there is only one other choice of integral shift vectors σ_i with $\Delta_1^\sigma \cap \Delta_2^\sigma = \{0\}$ and $\sigma_1 + \sigma_2 = (1,\ldots,1)$, which leads to the same nef–partition). Converting to the basis

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 2 \end{pmatrix}$$

of the $\mathbb{Z}_2$ quotient of the original lattice we find the nef–partition $\Delta_l^{(A)} = B^{-1}\Delta_l^\sigma$,

$$E_1^* = \left\{ \begin{array}{rrrrr} 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & -1 \\ 0 & 4 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & -1 & 1 \end{array} \right\}, \quad E_2^* = \left\{ \begin{array}{rrrrrrrr} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 \end{array} \right\},$$

which is dual to the partition

$$E_1 = \left\{ \begin{array}{rrrr} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right\}, \quad E_2 = \left\{ \begin{array}{rrrr} 0 & -2 & 1 & -1 \\ 0 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{array} \right\}, \tag{4.24}$$

of the convex hull $\Delta^*_{(A)} = \langle \nabla_1, \nabla_2 \rangle = \langle E_1, E_2 \rangle$.

4.6 The geometry of toric CICYs

It is well–known that the same Hodge numbers can come from different polyhedra and even at different codimensions, so it is important to identify constructions that actually give equivalent CYs. First note that any hypersurface or complete intersection can be reconstructed at higher codimension: Just multiply with an interval $[-1,1]$ and take the corresponding trivial nef–partition. [7] A less trivial redundancy is due to partitions where one of the Δ_i consists of a single vertex, say ρ_1^*: In that case the nef condition implies that the projection of Δ^* along ρ_1^* is reflexive. Moreover, the CY is given by the intersection of the toric divisor D_1 with the remaining divisor(s) defined by the partition of the vertices. Since D_1 can only intersect the toric divisors that correspond to points bounding the reflexive projection along V_1 we conclude that we can construct the same CY variety in the ambient space that is given by that projection of Δ^*. In this section we discuss for some examples how CICYs coming from different polytopes and/or from different nef–partitions can be related.

[7] In the formulas for the Hodge numbers [32] this leads to a doubling because a quadratic equation in $\mathbb{P}^1$ is solved by two points.

4.7 Equivalence of different nef–partitions

As our example for different nef–partitions we have chosen one of the nine polytopes in our list of models with Hodge numbers (2,44),

$$\mathbb{P}\begin{pmatrix} 4\,2\,2\,2\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 8 & 4 \\ 2 & 0 \end{array}\right] \;/\,\mathbb{Z}_2:\; 1\,1\,1\,0\,1\,0, \tag{4.25}$$

and its double cover, which we will relate to two different hypersurface polytopes. The polytope Δ of the quotient and its dual Δ^* have $(P_V, {P^*}_{V^*}) = (232_{10}, 9_7)$ points P and vertices V. The polytope for the double cover has $(461_{10}, 9_7)$, so that the $\mathbb{Z}_2$ quotient is free (the number P^* of points in Δ^* does not change). This model is again a K3 fibration, but this time the fiber polytope has an additional vertex, namely $\rho_8^* = \frac{1}{2}(\rho_5^* + \rho_6^*)$, which is a lattice point on an edge of Δ^*. Anticipating the structure of the nef–partitions, the corresponding relation $2\rho_1^* + \rho_2^* + \rho_3^* + \rho_4^* + \rho_8^* = 0$ shows that the fiber is again V_f in (4.20). Up to permutation symmetries this polyhedron only admits two nef–partitions, $\{\rho_1^*, \rho_4^*, \rho_5^*, \rho_6^*, \rho_7^*\} \cup \{\rho_2^*, \rho_4^*\}$ and $\{\rho_1^*, \rho_3^*, \rho_4^*, \rho_7^*\} \cup \{\rho_2^*, \rho_5^*, \rho_6^*\}$. Both lead to the same partitioning of the degrees, $12 = 8 + 4$ and $2 = 2 + 0$, as indicated above. Note that non–vertices always belong to one of the ∇_l's of a partition. [8] In our example this implies that ρ_5^* and ρ_6^* always have to belong to the same ∇_l. More cases with Hodge numbers (2,44) will be discussed in Section 4.8.3.

Since the reflexivity constraint on $\nabla_1 + \nabla_2$ is weaker on a sublattice all of our partitions must lift to the double cover, but additional ones can show up. Indeed, up to automorphisms, we find a total of nine nef–partitions with four different degrees and three different sets of Hodge numbers:

$$\mathbb{P}\begin{pmatrix} 4\,2\,2\,2\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 8 & 4 \\ 2 & 0 \end{array}\right]^{2,86}_{-168} \qquad \begin{array}{l} \{\rho_1^*, \rho_2^*, \rho_5^*, \rho_6^*, \rho_7^*\} \cup \{\rho_3^*, \rho_4^*\} \\ \{\rho_1^*, \rho_2^*, \rho_3^*, \rho_7^*\} \cup \{\rho_4^*, \rho_5^*, \rho_6^*\}, \end{array} \tag{4.26}$$

$$\mathbb{P}\begin{pmatrix} 4\,2\,2\,2\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 8 & 4 \\ 1 & 1 \end{array}\right]^{2,86}_{-168} \qquad \begin{array}{l} \{\rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*, \rho_6^*, \rho_7^*\} \cup \{\rho_1^*\} \\ \{\rho_1^*, \rho_2^*, \rho_3^*\} \cup \{\rho_4^*, \rho_5^*, \rho_6^*, \rho_7^*\} \\ \{\rho_1^*, \rho_2^*, \rho_5^*, \rho_6^*\} \cup \{\rho_3^*, \rho_4^*, \rho_7^*\}, \end{array} \tag{4.27}$$

$$\mathbb{P}\begin{pmatrix} 4\,2\,2\,2\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 6 & 6 \\ 1 & 1 \end{array}\right]^{3,69}_{-132} \qquad \begin{array}{l} \{\rho_1^*, \rho_5^*, \rho_6^*\} \cup \{\rho_2^*, \rho_3^*, \rho_4^*, \rho_7^*\} \\ \{\rho_1^*, \rho_2^*\} \cup \{\rho_3^*, \rho_4^*, , \rho_5^*, \rho_6^*\rho_7^*\}, \end{array} \tag{4.28}$$

$$\mathbb{P}\begin{pmatrix} 4\,2\,2\,2\,1\,1\,0 \\ 1\,0\,0\,0\,0\,0\,1 \end{pmatrix} \left[\begin{array}{c|c} 10 & 2 \\ 2 & 0 \end{array}\right]^{3,99}_{-192} \qquad \begin{array}{l} \{\rho_1^*, \rho_3^*, \rho_4^*, \rho_5^*, \rho_6^*, \rho_7^*\} \cup \{\rho_2^*\} \\ \{\rho_1^*, \rho_2^*, \rho_3^*, \rho_4^*, \rho_7^*\} \cup \{\rho_5^*, \rho_6^*\}. \end{array} \tag{4.29}$$

For the double cover we thus find two trivial partitions, for which we can construct the corresponding hypersurfaces: It is quite easy to work this out in terms of the weight data: A projection along ρ_1^*, as required by the third partition in (4.27), just amounts to dropping that vertex from the linear relations. Since $\rho_7^* = -\rho_1^*$ the last vertex is projected onto the origin and we find the weighted projective space $\mathbb{P}^4_{22211}[8]$, whose degree 8 hypersurface indeed has Hodge

[8] The piecewise linear functions ψ_l defining the nef–partition are integral on lattice points with values 0 or 1 on the vertices. The facets of Δ^* thus cannot contain lattice points with other values.

data (2,86). For the trivial partition in (4.29) we project along ρ_2^* and find the CY hypersurface $\mathbb{P}\begin{pmatrix}4\,2\,2\,1\,1\,0\\1\,0\,0\,0\,0\,1\end{pmatrix}\begin{bmatrix}10\\2\end{bmatrix}$, again with the expected Hodge numbers (3,99).

We also observe that the partitioning indeed fixes the Hodge numbers, i.e. equal charge vectors always lead to the same spectrum. This could be expected because for weighted projective intersections the degrees contain all information. How this result comes about in the toric context will be seen explicitly in the examples below. The converse is, however, not true: The first two partitions with Euler number $\chi = -168$ are the ones that survive the $\mathbb{Z}_2$ quotient, but there is now a different realization of the same Hodge numbers. The difference in the charge vectors is, however, only due to the contribution of ρ_7^* and it turns out that the corresponding divisor D_7 does not intersect the CICY. All spaces with equal Hodge numbers turn out to be topologically equivalent so that there do not seem to be any phase boundaries associated with a transition among the respective partitions.

4.8 Equivalence of different polyhedra: the (2,30) model

We now take the main example introduced in section 4.5 and show that the three different models $V_{(A)}$, $V_{(B)}$, and $V_{(C)}$ are topologically equivalent.

4.8.1 The first realisation of the (2,30) model

We first discuss the model $V_{(A)}$ in detail. Recall that the intersection ring of the complete intersection CY is obtained as the quotient of the intersection ring of the ambient toric variety by the ideal generated by the linear relations among the points, and by the Stanley–Reisner ideal. The latter is obtained from the primitive collections which are collections of vertices which do not form a cone but any proper subset forms a cone [46]. In this example these primitive collections can easily be seen in the geometry of the polyhedron $\Delta^*_{(A)}$. As discussed in section 4.5, the section in the lattice N that corresponds to the K3 fiber is a blowup of $\mathbb{P}^4_{21111}$ by the vertex $\rho_8^* = -\rho_1^*$, hence it is a double pyramid over the tetrahedron $\langle\rho_2^*, \rho_3^*, \rho_4^*, \rho_5^*\rangle$. This implies the relations $D_1 \cdot D_8 = 0 = D_2 \cdot D_3 \cdot D_4 \cdot D_5$ because the respective vertices never can belong to the same cone of any (triangulated) fan over the polyhedron. The complete polyhedron is a double pyramid over that 4–dimensional double pyramid. As $\rho_6^* + \rho_7^* = 0$ this leads to the additional relation $D_6 \cdot D_7 = 0$, which completes the generators of the Stanley–Reisner ideal. The polyhedron is simplicial and has the 16 facets $\widehat{18}\widehat{2345}\widehat{67}$, where a hat above a sequence of numbers indicates to take all simplices that arise by dropping one of the respective vertices. The linear equivalences (up to principal divisors) follow from the lines of (4.17) and we find altogether

$$D_1 \sim 2D_5 + D_8, \qquad D_2 \sim D_3 \sim D_4 \sim D_5, \qquad D_6 \sim D_7, \tag{4.30}$$

$$D_1 \cdot D_8 = 0, \qquad D_2 \cdot D_3 \cdot D_4 \cdot D_5 = 0, \qquad D_6 \cdot D_7 = 0. \tag{4.31}$$

According to the nef–partition (4.24) the complete intersection $V_{(A)}$ is given by $D_{0,1} \cdot D_{0,2}$ (cf. (4.13)) with

$$\begin{aligned} D_{0,1} &= D_1 + D_2 + D_4 + D_8 \sim 2D_1, \\ D_{0,2} &= D_3 + D_5 + D_6 + D_7 \sim 2D_2 + 2D_6, \end{aligned} \tag{4.32}$$

so that D_8 does not intersect the Calabi–Yau and the Kähler moduli correspond to the volumes of, for example, D_3 and D_6.

The unique star triangulation of the simplicial polytope (4.17) fixes the toric intersection numbers and therefore the Mori cone of the ambient space $\mathbb{P}_{\Delta^*_{(A)}}$. We determine this Mori cone as

$$\begin{aligned} \hat{l}^{(1)} &= (0,0,0,0,0,1,1,\;\;\;0), \\ \tilde{\hat{l}}^{(2)} &= (0,1,1,1,1,0,0,-2), \\ \hat{l}^{(3)} &= (1,0,0,0,0,0,0,\;\;\;1)\,. \end{aligned} \tag{4.33}$$

As mentioned above the divisor D_8 of $\mathbb{P}_{\Delta^*_{(A)}}$ does not intersect the complete intersection. All other divisors descend to divisors on the hypersurface. The entries in the Mori vectors are the intersections of the curves $c^{(a)}$, which have positive volume inside the Kähler cone, with the corresponding divisor. The Mori vectors of the complete intersection must therefore have a zero in the eighth entry. Besides $\hat{l}^{(1)}$ there is the unique minimal length combination $l^{(2)} = \tilde{\hat{l}}^{(2)} + 2\hat{l}^{(3)} = (2,1,1,1,1,0,0,0)$ with this property. We drop the eighth entry and add the negative value of the intersection of the $c^{(a)}$ with $D_{0,1}$ and $D_{0,2}$ as the first entries (these numbers correspond to the negative degrees in the charge vectors). Thus we get the Mori vectors for the complete intersection $V_{(A)}$

$$\begin{aligned} l^{(1)} &= (-4,-2;2,1,1,1,1,0,0), \\ l^{(2)} &= (\;\;\;0,-2;0,0,0,0,0,1,1)\,. \end{aligned} \tag{4.34}$$

(in the present example they coincide with the charge vectors because, dropping the non–intersecting vertex, the ambient space is a product of weighted projective spaces, but generically the Mori vectors span a larger cone). This way of summarizing the Mori generators becomes particularly useful, when we discuss the Picard–Fuchs system.

It is convenient to summarize all the relevant information for $V_{(A)}$ in the following table:

$$\begin{array}{cccccccc|ccc} & & & & & & & & c^{(1)} & c^{(2)} & \\ D_{0,1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & \\ D_{0,2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & \\ D_1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 2H \\ D_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & H \\ D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & H \\ D_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & H \\ D_5 & 0 & 1 & -2 & -1 & -1 & -1 & 0 & 1 & 0 & H \\ D_6 & 0 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 1 & L \\ D_7 & 1 & 0 & -1 & -1 & -1 & 0 & -2 & 0 & 1 & L \end{array} \tag{4.35}$$

On the left–hand side of the vertical line we have listed from top to bottom the points ρ_i^* of the polyhedron Δ^*, where the first two entries refer to the nef–partition $E_l, l = 1,2$, and the next five entries are their coordinates in $N = \mathbb{Z}^5$. Together they form the coordinates of the generators of the 7–dimensional Gorenstein cone $\Gamma(\Delta^*)$ that was defined below (4.11). To each point ρ_i^* we have associated the corresponding divisor D_i. The first two rows, i.e. $D_{0,1}$ and $D_{0,2}$ correspond to the interior point appearing once in either partition. The two columns labeled by $c^{(a)}$ on the right–hand side of the vertical line denote the Mori generators. Its entries are the intersection numbers of the restrictions of the divisors D_i to V with the curves $c^{(a)}$. The data only refer to the CY manifold, i.e. we dropped the non–intersecting divisors and the curves that do not descend to the complete intersection.

Let us denote the independent divisors of the complete intersection by H and L. In our example we can choose $H = D_3D_{0,1}D_{0,2}$ and $L = D_6D_{0,1}D_{0,2}$, as is indicated on the right in (4.35). The classical intersection numbers are defined as

$$\kappa_{a,b,c} = \int_V J_a \wedge J_b \wedge J_c = D_a \cap D_b \cap D_c, \tag{4.36}$$

where $J_a \in H^2(V,\mathbb{Z})$ and $D_a \in H_4(V,\mathbb{Z})$. Here, J_1 and J_2 are the Kähler forms dual to H and L, respectively. They can be easily evaluated from the intersections on $\mathbb{P}_{\Delta_{(A)}}$, *e.g.* $D_1D_2D_3D_4D_6 = \mathrm{Vol}\,(\langle\rho_1^*,\rho_2^*,\rho_3^*,\rho_4^*,\rho_6^*\rangle) = 2$, and the relations in (4.30) and (4.31). The fact that we are dealing with a free $\mathbb{Z}_2$ quotient is accounted for by a division by 4. Therefore, the intersection numbers are

$$\begin{aligned} \kappa_{1,1,1} = \kappa_{1,1,2} = 2, \\ \kappa_{1,2,2} = \kappa_{2,2,2} = 0\ . \end{aligned} \tag{4.37}$$

According to Oguiso [64], a CY threefold admits a $K3$–fibration if there exists an effective divisor L such that

$$\boxed{L \cdot c \geq 0 \text{ for all curves } c \quad L^2 \cdot D = 0 \text{ for all divisors } D.} \tag{4.38}$$

Therefore, we conclude from (4.37) the geometry of the CY space $V_{(A)}$ is a fibration with J_1 the Kähler class of the fiber and J_2 the Kähler class of the base. This is in agreement with the purely combinatorial argument in (4.20). Normally, one expects in such cases the fiber to be K3 and $\int_V \mathcal{H}_2 J_2 = 24$ [64, 70, 71]. That is because the integral of J_2 over the base $\mathbb{P}^1$ gives 1, the rest of the integral extends over the fiber and yields $\int_{\mathrm{K3}} \mathcal{H}_2 = 24$. Instead one has here

$$\mathcal{H}_2 L = \int_V \mathcal{H}_2 J_2 = 12, \qquad \mathcal{H}_2 H = \int_V \mathcal{H}_2 J_1 = 20\ , \tag{4.39}$$

which indicates that the $\mathbb{Z}_2$ quotient has divided the volume of the $\mathbb{P}^1$ by two. Indeed, we know that the model is a free $\mathbb{Z}_2$ quotient of an ordinary K3 fibration, which might be represented as the complete intersection of degree $(4,0)$ and $(2,2)$ in $\mathbb{P}^4_{21111} \times \mathbb{P}^1$ with Euler number $\chi = -112$ and the same $l^{(a)}$ vectors as in (4.34). Now recall our extensive discussion of the properties of $V_{(A)}$ in section 4.5. With the homogeneous coordinates x_i and y_j, respectively, the polynomials f_1 and f_2 are

$$\begin{aligned} f_1 &= x_0^2 + x_1^4 + x_2^4 + x_3^4 + x_4^2 + \ldots \\ f_2 &= x_0^2(y_0^2 + y_0 y_1 + y_2^2) + \ldots \end{aligned}$$

For any choice $(y_0 : y_1)$ in $\mathbb{P}^1$ the fiber is the complete intersection K3 V_f. The $\mathbb{Z}_2$ acts on the coordinates by $(x_0, x_1, x_2, x_3, x_4; y_0, y_1) \to (-x_0, x_1, -x_2, x_3, -x_4; y_0, -y_1)$. Hence the $\mathbb{P}^1$ gets folded with fixpoints $(1 : 0)$ and $(0 : 1)$, which explains the division of $\int_V \mathcal{H}_2 J_a$ by 2, while the $\mathbb{Z}_2$-action leads to an Enriques fiber over the fixed points in the base. Note that, therefore, only $\mathbb{Z}_2$ quotients can act freely on K3 fibered Calabi–Yau manifolds.

4.8.2 Other realisations of the (2,30) model

One difficulty with complete intersection realisations is a high redundancy in the description of a given family of Calabi–Yau spaces. For example, we know from section 4.5 that there are two more polyhedra with three and two nef–partitions respectively, which lead to complete intersections with Hodge numbers $(2, 30)$. In addition, these polyhedra admit two and five different star triangulations respectively, which could potentially lead to different large volume phases of the families.

To settle the question about equivalences we will follow the topological classification of real six manifolds by C.T.C. Wall [72]. Specialized to Calabi–Yau manifolds it states that two manifolds V and V' are of the same topological type if, beside the Hodge numbers, the triple intersection numbers $\int_V J_a \wedge J_b \wedge J_c$ and the integrals $\int_V \mathcal{H}_2 J_a$ are the same in a suitable basis of J_a.

We find no counterexample to the statement that families with a large volume limit of the same topological type have the same Gromov–Witten invariants, but the toric mirror may have a different natural parametrisation of the complex structure variables, which leads to different Picard–Fuchs equations and mirror maps.

We first consider the second realisation of the Hodge numbers (2,30), $V_{(B)}$. In this case the dual polyhedron $\Delta^*_{(B)}$ has vertices

$$\Delta^*_{(B)}\ \{\rho^*_i\} = \left\{ \begin{array}{rrrrrrrr} 1 & 0 & 0 & 0 & -2 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \end{array} \right\} \tag{4.40}$$

and admits the three nef–partitions given below eq. (4.21). In order to find the triangulations we observe that the shift of ρ^*_7, as compared to its position in $\Delta^*_{(A)}$, moves the intersection point of the line $\overline{\rho^*_6\rho^*_7}$ with the fiber hyperplane through the facet $[1345]$ of Δ^*_f till it reaches the linear span of $[3458]$ outside of that facet of the fiber polytope. This kills the 4 simplices $[\widehat{1}8345\widehat{6}7]$ of $\Delta^*_{(A)}$ and replaces them by the 3 triangles $[1\widehat{345}67]$ and by the 16th facet $[345678]$. The vertices of the non–simplicial facet form a circuit, $\rho^*_3 + 2\rho^*_4 + \rho^*_5 = \rho^*_6 + \rho^*_7 + 2\rho^*_8$, for which we introduce the short hand notation $\langle 3_1 4_2 5_1 | 6_1 7_1 8_2 \rangle$. It indicates the labels m of the involved vertices ρ^*_m and, as subscripts, their coefficients c_m in the linear relation (3.1.15). Therefore, it can be triangulated in the two different ways: $[\widehat{345}678]$ or $[345\widehat{678}]$, so that we find two different triangulations with 18 simplices:

$$
\begin{aligned}
T_1 &= \{[\widehat{18}2\widehat{345}\widehat{67}], [\widehat{18}\widehat{345}67]\}, \\
T_2 &= \{[\widehat{18}2\widehat{345}\widehat{67}], [1\widehat{345}67], [345\widehat{678}]\}.
\end{aligned}
\tag{4.41}
$$

Since we deal with a free $\mathbb{Z}_2$ quotient, the volume of each simplex is divisible by two. Because of the coefficient 2 of ρ_4^* and ρ_8^* in the circuit the simplices that do not contain one of these vertices, i.e. $[35678] \in T_1$ and $[34567] \in T_2$, have volume 4. We thus need to resolve the singularities of the ambient space by adding points at higher degree, following the general discussion given in Section 4.2. From that we expect another simplex of volume 4, sharing a facet, which has to be the same for both triangulations. The only possibility is $[13567]$, which indeed has volume 4. The additional point in degree two, which resolves all singularities is $\rho_{\mathrm{r}}^* = \frac{1}{2}(\rho_3^* + \rho_5^* + \rho_6^* + \rho_7^*)$. The corresponding triangulations are

$$
\begin{aligned}
T_1 &= \{[\widehat{18}2\widehat{345}\widehat{67}], [\widehat{18}\widehat{35}467], [\widehat{18}\widehat{3567r}]\}, \\
T_2 &= \{[\widehat{18}2\widehat{345}\widehat{67}], [1467\widehat{35}], [3458\widehat{67}], [\widehat{14}\widehat{3567r}]\}.
\end{aligned}
\tag{4.42}
$$

The linear relations are

$$
\begin{aligned}
&D_6 \sim D_7, \qquad D_1 \sim 2D_5 + D_8 + D_{\mathrm{r}}, \\
&D_3 \ \sim \ D_5 \ \sim \ D_4 + D_6 \ \sim \ D_2 - D_6 - D_{\mathrm{r}}.
\end{aligned}
\tag{4.43}
$$

For the first two nef–partitions in (4.21), the face $\langle 3, 5, 6, 7 \rangle$ belongs to both sets of vertices, therefore (4.15) becomes for the first nef–partition

$$
\begin{aligned}
D_{0,1} &= D_1 + D_3 + D_5 + D_8 + D_{\mathrm{r}} \sim 2D_1, \\
D_{0,2} &= D_2 + D_4 + D_6 + D_7 + D_{\mathrm{r}} \sim 2D_2.
\end{aligned}
\tag{4.44}
$$

The second nef–partition is analogous and yields the same result. For the third one, however, this face lies entirely in the second set of vertices, so that

$$
\begin{aligned}
D_{0,1} &= D_1 + D_2 + D_4 + D_8 && \sim 2D_1, \\
D_{0,2} &= D_3 + D_5 + D_6 + D_7 + 2D_{\mathrm{r}} && \sim 2D_2.
\end{aligned}
\tag{4.45}
$$

yields again the same result. Since the triangulations are in general independent of the nef–partition, we can discuss them for a single nef–partition, say, the first one. The Stanley–Reisner ideal, of course, always contains the generator $D_1 \cdot D_8$ because antipodal points can never belong to the same simplex. The divisor D_8 thus never intersects the Calabi–Yau manifold $V_{(B)}$, whose first defining equation is a section of $\mathcal{O}(D_{0,1}) = \mathcal{O}(2D_1)$. Furthermore, since ρ_2^* and ρ_{r}^* never belong to the same simplex, the divisor D_{r} coming from the blow–up of the ambient space never intersects $V_{(B)}$ because its second defining equation is a section of $\mathcal{O}(D_{0,2}) = \mathcal{O}(2D_2)$. Otherwise it depends on the triangulation, and we find, using similar arguments, from (4.42)

$$
\begin{aligned}
\mathcal{I}_{SR}(T_1) &= \{D_1D_8, D_2D_6D_7, D_3D_4D_5, \\
&\quad\ D_3D_5D_6D_7, D_2D_{\mathrm{r}}, D_4D_{\mathrm{r}}\}, \\
\mathcal{I}_{SR}(T_2) &= \{D_1D_8, D_2D_6D_7, D_6D_7D_8, D_1D_3D_4D_5, D_2D_3D_4D_5, \\
&\quad\ D_3D_5D_6D_7, D_2D_{\mathrm{r}}, D_8D_{\mathrm{r}}, D_1D_4D_{\mathrm{r}}\}.
\end{aligned}
\tag{4.46}
$$

Note that the first lines in (4.46) correspond to the Stanley–Reisner ideal of the unresolved toric variety $\mathbb{P}_{\Delta^*_{(B)}}$. Next, we determine the Mori cone of the resolved ambient space and find

$$
\begin{aligned}
\hat{l}_{T_1} &= \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 1 \end{pmatrix}, \\
\hat{l}_{T_2} &= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & -2 \end{pmatrix}.
\end{aligned}
$$

Since D_8 and D_{r} do not intersect the Calabi–Yau space, we consider the linear combinations of the vectors above for which the eighth and the ninth entry vanishes, and adding the intersections of $-D_{0,l}$ with $c^{(a)}$ as before, we get for both triangulations the two Mori generators

$$
\begin{aligned}
\tilde{l}^{(1)} &= (-4, \quad 0; 2, 0, 1, \quad 2, 1, -1, -1, 0, 0), \\
l^{(2)} &= (\quad 0, -2; 0, 1, 0, -1, 0, \quad 1, \quad 1, 0, 0)\,.
\end{aligned}
\tag{4.47}
$$

Now we have to check that the curves which bound the corresponding Kähler cones in $\mathbb{P}_{\Delta^*_{(B)}}$ descend to the Calabi–Yau space $V_{(B)}$. As mentioned above these curves have intersection $c^{(a)} \cdot D_i = l_i^{(a)}$. In particular $c^{(1)}$ has negative intersection with both D_6 and D_7. Negative intersection numbers indicate that the curves are actually contained in the corresponding divisors. Since by (4.46), $D_6 D_7 = 0$ on the Calabi–Yau space, we conclude that $c^{(1)}$ does not descend to $V_{(B)}$. For this reason the Mori cone must become smaller and the Kähler cone becomes bigger due to the absence of the bounding curve in $V_{(B)}$. The minimal positive integer linear combination of $\tilde{l}^{(1)}$ and $l^{(2)}$ without two negative entries is $l^{(1)} = \tilde{l}^{(1)} + l^{(2)}$

$$
\begin{aligned}
l^{(1)} &= (-4, -2; 2, 1, 1, \quad 1, 1, 0, 0, 0, 0), \\
l^{(2)} &= (\quad 0, -2; 0, 1, 0, -1, 0, 1, 1, 0, 0)\,.
\end{aligned}
\tag{4.48}
$$

We can then pick $H = D_3 D_{0,1} D_{0,2}$ and $L = D_6 D_{0,1} D_{0,2}$ and observe exactly the same classical intersections as in (4.37) and (4.39). According to the theorem of Wall the Calabi–Yau manifolds are then of the same topological type. It turns out that the world sheet instanton numbers on both Calabi–Yau spaces are the same. However, as we will see in section 4.9.1, the slightly different vectors $l^{(i)}$ lead to a different parametrization of the complex structure moduli space. Since these arguments used only the triangulations, which are independent of the nef–partition, they show that the other two nef–partitions lead to the same topological type of the Calabi–Yau space.

The third model with Hodge numbers $(2, 30)$ has the dual polyhedron

$$
\Delta^*_{(C)}\ \{\rho^*_i\} = \left\{ \begin{matrix} 1 & 0 & 0 & 0 & -2 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \end{matrix} \right\}
\tag{4.49}
$$

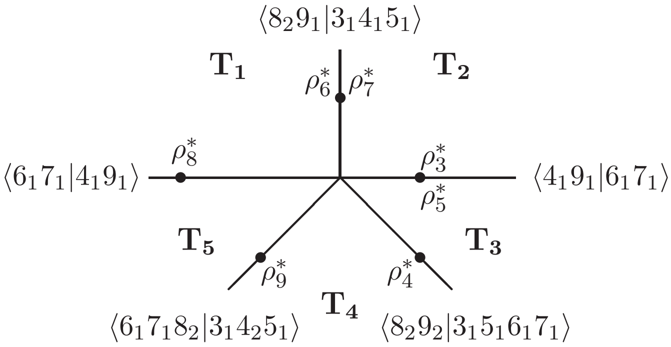


Figure 4.1: Secondary fan of the facet $[3456789]$ of $\Delta^*_{(C)}$ with circuits relating its triangulations.

and admits the two nef–partitions given below (4.22). The star triangulations of $\Delta^*_{(C)}$ again contain the twelve simplices $[\hat{1}8\widehat{234}5\widehat{67}]$ of $\Delta^*_{(A)}$, but now there are 5 additional facets, namely the two simplices $[135\widehat{67}9]$, and the facets $[14\widehat{35}679]$ and [3456789]. The circuit $\rho_6^* + \rho_7^* = \rho_4^* + \rho_9^*$ shows that the additional vertex ρ_9^* restores the possibility of a fibered ambient space. Namely, if we triangulate this circuit as $[4\widehat{67}9]$, we avoid the edge $\overline{\rho_6^*\rho_7^*}$. The triangulations of $\Delta^*_{(C)}$ that are consistent with the fibration are easily found by triangulating the reflexive section $\Delta^*_{(C),f}$, which is spanned by $\rho_1^*, \ldots, \rho_5^*, \rho_8^*, \rho_9^*$. Its single non–simplicial facet, $[34589]$, yields a circuit $\langle 3_1 4_1 5_1 | 8_2 9_1 \rangle$. One triangulation, $[\widehat{345}89]$, leads to a regular ambient space while the other contains the simplex $[3459]$ of volume 2. In the 4–dimensional ambient space of the fiber we expect a resolution of the singularity by a point in degree 2 in the interior of the cone. Indeed, $\rho_{\mathrm{r}}^* = \frac{1}{2}(\rho_3^* + \rho_4^* + \rho_5^* + \rho_9^*)$ is a lattice point (which actually is identical to the point ρ_{r}^* that resolved the singularities in the previous example). Extending these subdivisions to a triangulation of the complete polytope we thus obtain the first two triangulations T_2 and T_1 below. T_2 is regular while T_1 requires the subdivision of the two triangles $[3459\widehat{67}]$ through ρ_{r}^*.

The complete set of triangulations can be found by constructing the secondary fan 3.1.14 for the facet [3456789]: Let A denote the matrix consisting of the coordinates of the respective vertices $\rho_3^*, \ldots, \rho_9^*$ and compute its Gale transform (see definition 3.1.12):

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & -2 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}, \tag{4.50}$$

which is the transpose of its kernel, i.e. $AB^T = 0$, where we have chosen the two circuits $\langle 3_1 4_1 5_1 | 8_2 9_1 \rangle$ and $\langle 6_1 7_1 | 4_1 9_1 \rangle$ as generators of the kernel. The rays of the secondary fan are generated by the column vectors of B, which we label by the respective vertices of $\Delta^*_{(C)}$. The triangulations T_i can then be read off as indicated in the figure of the secundary fan 4.1.

As an example consider T_4, where these complements are $\{48, 39, 49, 59\}$, yielding the triangulation $\{[35679], [\widehat{345}678]\}$. Adjacent triangulations are connected by (bistellar flips for) circuits involving, on either side, vertices that can form a strictly convex cone with the ray that separates the corresponding phases (see Proposition 2.12 in chapter 7 of [47]).

A triangulation of $[3456789]$ induces a triangulation of the other two non–simplicial facets $[14\hat{3}5679]$. Writing the triangulations as a union of the simplices of the big facet $[3456789]$, simplices of the induced triangulations of the circuits $[14\hat{3}5679]$, and simplical facets of $\Delta^*_{(C)}$, respectively, we obtain:

$$\begin{aligned}
T_1 &= \{[345\hat{6}7\hat{8}9]\} && \cup \{[14\hat{3}5\hat{6}79]\} \cup \{[\hat{1}82\widehat{34}5\hat{6}7], [135\hat{6}79]\},\\
T_2 &= \{[\widehat{345}\hat{6}789]\} && \cup \{[14\hat{3}5\hat{6}79]\} \cup \{[\hat{1}82\widehat{34}5\hat{6}7], [135\hat{6}79]\},\\
T_3 &= \{[\hat{3}5\hat{4}9678], [35\hat{6}789]\} && \cup \{[1\hat{4}9\hat{3}567]\} \cup \{[\hat{1}82\widehat{34}5\hat{6}7], [135\hat{6}79]\},\\
T_4 &= \{[\widehat{345}678], [35679]\} && \cup \{[1\hat{4}9\hat{3}567]\} \cup \{[\hat{1}82\widehat{34}5\hat{6}7], [135\hat{6}79]\},\\
T_5 &= \{[345\widehat{678}], [35679]\} && \cup \{[1\hat{4}9\hat{3}567]\} \cup \{[\hat{1}82\widehat{34}5\hat{6}7], [135\hat{6}79]\}.
\end{aligned}$$

T_2 and T_3 have 24 regular simplices. The triangulations T_1, T_4 and T_5 have 22 simplices, two of which have volume 2. Inspection of the coefficients in the circuits connecting the phases shows that these are $[3459\hat{6}7]$, $[35674\hat{9}]$ and $[35674\hat{9}]$, respectively, so that the refinement induced by adding

$$\rho^*_{\rm r} = \tfrac{1}{2}(\rho^*_3 + \rho^*_4 + \rho^*_5 + \rho^*_9) = \tfrac{1}{2}(\rho^*_3 + \rho^*_5 + \rho^*_6 + \rho^*_7) \tag{4.51}$$

resolves the singularities in all cases. Note that star triangulations are refinements of the polyhedral subdivision induced by the cones over the facets of Δ^*. Figure 4.1 is therefore a face of the complete secondary fan that describes all triangulations of $\Delta^*_{(C)}$ (see Theorem 2.4 in chapter 7 of [47]). We list here the data for the ambient space only for two triangulations, T_2 and T_1. For T_2 we do not need ρ^*_ρ Therefore, the linear relations are

$$D_6 \sim D_7, \qquad D_1 \sim 2D_5 + D_8, \qquad D_2 - D_6 - D_9 \sim D_3 \sim D_4 + D_6 \sim D_5, \tag{4.52}$$

and the Stanley-Reisner ideal is

$$I_{SR}(T_2) = \{D_1D_8, D_2D_9, D_6D_7, D_3D_4D_5\}. \tag{4.53}$$

The Mori generators associated to the triangulation T_2 are

$$\hat{l}_{T_2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \end{pmatrix}. \tag{4.54}$$

The complete intersection $V_{(C)}$ for the first nef–partition below (4.22) is defined by

$$\begin{aligned} D_{0,1} &= D_1 + D_3 + D_4 + D_7 + D_8, \\ D_{0,2} &= D_2 + D_5 + D_6 + D_9. \end{aligned} \tag{4.55}$$

For T_1 the linear relation involve ρ^*_ρ

$$D_6 \sim D_7, \quad D_1 \sim 2D_5 + D_8 + D_\rho, \quad D_2 - D_6 - D_9 - D_\rho \sim D_3 \sim D_4 + D_6 \sim D_5, \tag{4.56}$$

and so does the Stanley-Reisner ideal

$$\begin{aligned} I_{SR}(T_1) \;&=\; \{D_1D_8, D_1D_\rho, D_2D_9, D_2D_\rho, D_6D_7, D_8D_9, D_8D_\rho, D_1D_3D_4D_5, && (4.57)\\ &\quad D_2D_3D_4D_5, D_3D_4D_5D_9\}. && (4.58)\end{aligned}$$

The Mori generators become accordingly

$$\hat{l}_{T_1} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 \end{pmatrix}. \tag{4.59}$$

The complete intersection for the first nef–partition below (4.22) is defined by

$$\begin{aligned} D_{0,1} &= D_1 + D_3 + D_4 + D_7 + D_8 + D_\rho, \\ D_{0,2} &= D_2 + D_5 + D_6 + D_9 + D_\rho. \end{aligned} \tag{4.60}$$

We find for all triangulations (and all partitions) that D_8 and D_9 do not intersect the Calabi-Yau space $V_{(C)}$. Taking linear combinations for which the corresponding components of the Mori vectors vanish and going to a basis where curves on V bound the Kähler cone yields again

$$\begin{aligned} l^{(1)} &= (-4, -2; 2, 1, 1, \quad 1, 1, 0, 0, 0, 0, 0, 0), \\ l^{(2)} &= (\quad 0, -2; 0, 1, 0, -1, 0, 1, 1, 1, 0, 0, 0)\,. \end{aligned} \tag{4.61}$$

With $H = D_3D_{0,1}D_{0,2}$ and $L = D_6D_{0,1}D_{0,2}$ we find the same intersections as (4.37) and (4.39). The same conclusions arise for the other star triangulations.

To summarize all representations of the (2,30) model, be it different polyhedra, different nef–partitions, or different triangulations, are equivalent. Some of them exhibit however different parametrisations of the complex moduli space of the mirror.

4.8.3 A selection of other models

In this section we will present a few more codimension two complete intersection CY manifolds, but without going into so much detail as in the last section. The selection contains manifolds with the next smallest Hodge number after (2,30): These are (2,36) and (2,44). The latter is particularly interesting since it has realizations as both a simply connected space and a free $\mathbb{Z}_2$ quotient.

In Section 4.7 we mentioned that there are nine polyhedra admitting nef–partitions giving

Hodge numbers (2,44). The CICYs obtained from these polyhedra are

$$V_1(272_{12}, 8_7) \quad \sim \quad \mathbb{P}\begin{pmatrix} 2 & 1 & 1 & \mathbf{1} & 1 & 0 & 0 \\ 0 & 1 & 1 & \mathbf{3} & 3 & 2 & 2 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 6 & 6 \end{array}\right], \tag{4.62}$$

$$V_2(294_{13}, 9_8) \quad \sim \quad \mathbb{P}\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 4 & 6 \\ 0 & 2 \end{array}\right], \tag{4.63}$$

$$V_3(298_{16}, 9_8) \quad \sim \quad \mathbb{P}\begin{pmatrix} 2 & 1 & 1 & 3 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & \mathbf{3} & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & \mathbf{0} & 0 & 0 & 1 \end{pmatrix} \left[\begin{array}{c|c} 6 & 2 \\ 6 & 6 \\ 2 & 0 \end{array}\right], \tag{4.64}$$

$$V_4(232_{10}, 9_7) \quad \sim \quad \mathbb{P}\begin{pmatrix} 2 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 & 0 \\ 4 & \mathbf{2} & 2 & 2 & 0 & 1 & 1 & 0 \\ 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 8 & 4 \\ 2 & 0 \end{array}\right] \;/\, \mathbb{Z}_2 : \; 1\,1\,0\,1\,0\,1\,0\,0, \tag{4.65}$$

$$V_5(232_{12}, 9_8) \quad \sim \quad \mathbb{P}\begin{pmatrix} 2 & 1 & \mathbf{1} & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & \mathbf{1} & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \left[\begin{array}{c|c} 4 & 2 \\ 4 & 2 \\ 2 & 0 \end{array}\right] \;/\, \mathbb{Z}_2 : \; 1\,1\,0\,1\,0\,1\,0\,0. \tag{4.66}$$

V_4 is the same variety as the one given in (4.25). The other 4 polytopes yield blow-ups of the ambient spaces of V_1, V_3, V_4 and V_5, respectively. These blow-ups are obtained by adding a vertex in the definition of the ambient spaces in (4.62) to (4.66) an additional column $(0,0,0)^T$ and an additional row. This row has zeros everywhere except for a one at both the entries corresponding to the vertex in bold face and the new vertex. The linear relation corresponding to this row describes the $\mathbb{P}^1$ resulting from the blow-up. Only the blow-up of the ambient space of V_3 descends to the complete intersection.

Some of the nine polytopes admit several nef–partitions and/or several triangulations. Similar to the $(2,30)$ example, it turns out that for a given polytope there is only one topologically inequivalent manifold. This justifies the notation in (4.62) to (4.66). A representative will be given below.

V_2, V_4, V_5, as well as the blow-up of V_3, have reflexive hyperplane sections of codimension one, and hence admit K3 fibrations (or a $\mathbb{Z}_2$ quotient thereof in the cases of V_4 or V_5). In addition V_3 itself also admits a K3 fibration, which however does not come from a toric morphism in the ambient space. In fact, V_3 and its blow-up are related in the same manner as $V_{(B)}$ and $V_{(C)}$ discussed in Section 4.5.

Two out of the nine polytopes have lattice points which are not vertices. These are related to the ambient spaces of V_4 and its blowup. In fact, there is only one such lattice point, namely $\rho_5^* = \frac{1}{2}(\rho_6^* + \rho_7^*)$. This can be seen by subtracting twice the (redundant) first weight vector from the second one. We included that point to make the K3 fiber of V_4 visible, although it is redundant for the characterization of the polytope. Finally, it turns out that V_4 and V_5 are topologically equivalent, for the same reason as $V_{(A)}$ and $V_{(B)}$ in Section 4.8 were equivalent, cf. also the discussion below (4.22). Note that V_4 and V_5 are free $\mathbb{Z}_2$ quotients.

We summarize the reduced data for V_1 to V_4 in the same way as we did for $V_{(A)}$ in (4.35), together with the intersection numbers and the linear forms:

$$V_1: \quad \begin{array}{c|ccccccc|cc} & & & & & & & & c^{(1)} & c^{(2)} \\ D_{0,1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ D_{0,2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ D_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ D_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ D_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ D_5 & 0 & 1 & -2 & -1 & -1 & -1 & 0 & 1 & 0 \\ D_6 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ D_7 & 0 & 1 & 3 & 1 & 1 & 0 & -1 & 0 & 1 \end{array}$$

$$\kappa_{111} = 1, \qquad \kappa_{112} = 3, \qquad \kappa_{122} = 7, \qquad \kappa_{222} = 11,$$
$$\mathcal{H}_2 J_1 = 22, \qquad \mathcal{H}_2 J_2 = 50.$$

$$V_2: \quad \begin{array}{c|ccccccc|cc} & & & & & & & & c^{(1)} & c^{(2)} \\ D_{0,1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ D_{0,2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ D_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ D_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ D_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ D_5 & 0 & 1 & -2 & -1 & -1 & -1 & 0 & 1 & 0 \\ D_6 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ D_7 & 0 & 1 & 3 & 1 & 1 & 1 & -1 & 0 & 1 \\ D_8 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}$$

$$\kappa_{111} = 2, \qquad \kappa_{112} = 4, \qquad \kappa_{122} = 0, \qquad \kappa_{222} = 0,$$
$$\mathcal{H}_2 J_1 = 32, \qquad \mathcal{H}_2 J_2 = 24.$$

$$V_3: \quad \begin{array}{c|ccccccc|cc} & & & & & & & & c^{(1)} & c^{(2)} \\ D_{0,1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ D_{0,2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ D_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\ D_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ D_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ D_5 & 0 & 1 & -2 & -1 & -1 & -3 & 0 & 1 & 0 \\ D_6 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ D_7 & 0 & 1 & 3 & 1 & 1 & 3 & -1 & 0 & 1 \\ D_8 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{array}$$

$$\kappa_{111} = 1, \qquad \kappa_{112} = 2, \qquad \kappa_{122} = 0, \qquad \kappa_{222} = 0,$$
$$\mathcal{H}_2 J_1 = 22, \qquad \mathcal{H}_2 J_2 = 24.$$

$$
V_4 : \quad \begin{array}{c|ccccccc|cc}
 & & & & & & & & c^{(1)} & c^{(2)} \\
D_{0,1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\
D_{0,2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\
D_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 & 0 \\
D_2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
D_3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
D_4 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
D_5 & 0 & 1 & -2 & -1 & -1 & -1 & 0 & 1 & 0 \\
D_6 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 0 & 1 \\
D_7 & 0 & 1 & -5 & -3 & -2 & -3 & -2 & 0 & 1 \\
D_8 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\kappa_{111} = 4, \qquad \kappa_{112} = 2, \qquad \kappa_{122} = 0, \qquad \kappa_{222} = 0,
$$
$$
\mathcal{H}_2 J_1 = 28, \qquad \mathcal{H}_2 J_2 = 12.
$$

There are three polyhedra yielding nef–partitions with Hodge numbers (2,36). The first polyhedron admits two star triangulations, one with 12 and one with 10 simplices, both of which are not unimodular. We have to add four and five points in degree two, respectively. After going through the procedure explained in detail in Section 4.8.2 we can describe the (reduced) data as follows

$$
\begin{array}{c|ccccccc|cc}
 & & & & & & & & c^{(1)} & c^{(2)} \\
D_{0,1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\
D_{0,2} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\
D_1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
D_2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 2 \\
D_3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
D_4 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
D_5 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \\
D_6 & 1 & 0 & -1 & -3 & -3 & -4 & -4 & 0 & 1 \\
D_7 & 0 & 1 & 0 & 0 & -1 & -1 & -1 & 2 & -2
\end{array}
$$

All divisors of the ambient space in degree one descend to the Calabi-Yau threefold and the flop in the ambient space is also realized in the complete intersection CY. The family contains two large volume limits of different topological type. The intersections numbers are

$$
\kappa_{111} = 2, \qquad \kappa_{112} = 2, \qquad \kappa_{122} = 2, \qquad \kappa_{222} = 1,
$$
$$
\mathcal{H}_2 \cdot J_1 = 20, \qquad \mathcal{H}_2 \cdot J_2 = 22.
$$

4.9 Periods and Picard–Fuchs equations for toric CICYs

We start our discussion with the construction of the fundamental period of a toric CICY. A natural generalization of (3.54) is:

$$
\boxed{\langle g, \Omega \rangle = \frac{1}{(2\pi i)^d} \int_\gamma \left(\prod_{l=1}^{r} \frac{\lambda_0^{(l)}}{f^{(l)}} \right) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_d}{t_d},} \tag{4.67}
$$

with the Laurant polynomials

$$f^{(l)} = \sum_{m \in \Delta_l^0} \lambda_m t^m = \lambda_{0,l} + \sum_{m \in \Delta_l^0 \setminus \{0\}} \lambda_m t^m \in L(\Delta_l) \qquad (l = 1, \ldots, r). \tag{4.68}$$

Here, Δ_l^0 is a list of the 'relevant' lattice points of Δ_l. In the hypersurface case we introduced the notion of a projective subdivisions (Definition 3.2.2) to exclute all interior points of facets of Δ_l. Now even divisors corresponding to vertices do not intersect, so we have to exclude additional points, which can be found after performing a careful analysis of the intersection ring. We assume that the first element of the Δ_l^0 always corresponds to the origin. We now use the expansions

$$\frac{\lambda_{0,l}}{f^{(l)}} = \frac{1}{1 - \sum\limits_{m \in \Delta_l^0 \setminus \{0\}} \lambda_m (-\lambda_{0,l})^{-1} t^m} = \sum_{K^{(l)}=0}^{\infty} \left(\sum_{m \in \Delta_l^0 \setminus \{0\}} \lambda_m (-\lambda_{0,l})^{-1} t^m \right)^{K^{(l)}}, \tag{4.69}$$

and do further expansion of the expressions $(\ldots)^{K^{(l)}}$, where the powers l_m of the monomials t^m are partitions of $K^{(l)}$:

$$-l_{0,l} := K^{(l)} = \sum_{m \in \Delta_l^0 \setminus \{0\}} l_m. \tag{4.70}$$

It is clear that the integral in equation (4.67) gets a non–zero contribution if and only if the vectors

$$\boxed{l = (l_{0,1}, \ldots, l_{0,r}; l_1, \ldots, l_s) \quad \text{with} \quad s + r = \sum_{l=1}^{r} \#\Delta_l^0 \quad \text{and} \quad l_i \in \mathbb{Z}_{\geq}} \tag{4.71}$$

are relations of $\Delta_C^0 \cap \bar{M}$, the set of relavant points of the support $\Delta_C \cap \bar{M}$, where $C \subset \bar{M}_{\mathbb{R}}$ is the Gorenstein cone of the nef–partition $\Pi(\Delta) = \{\Delta_1, \ldots, \Delta_r\}$ ($\Delta_C^0 \cap \bar{M}$ is constructed by the lattice points of $\Delta_l^0 \times e_l$ $(l = 1, \ldots, r)$). Thus we get:

$$\langle g, \Omega \rangle = \sum_{l_1, \ldots, l_s} \frac{(-l_{0,1})! \ldots (-l_{0,r})!}{l_1! \ldots l_s!} (-\lambda_{0,1})^{l_{0,1}} \ldots (-\lambda_{0,r})^{l_{0,r}} \lambda_1^{l_1} \ldots \lambda_s^{l_s}, \tag{4.72}$$

where the sum runs over all relations l of the form (4.71). If we define the $\mathcal{A}$ as $\mathcal{A} = \Delta_C^0$, it is clear that the operators from the $\mathcal{A}$–system in definition 3.2.1, after multiplying them with $(\lambda_{0,1} \ldots \lambda_{0,r})^{-1}$, annihilate the fundamental period (4.72). Again, we choose a basis

$$\boxed{l^{(a)} = (l_{0,1}^{(a)}, \ldots, l_{0,r}^{(a)}; l_1^{(a)}, \ldots, l_s^{(a)}), \quad \text{for } a = 1, \ldots, h} \tag{4.73}$$

for the Mori generators and introduce torus invariant (z)–coordinates:

$$\boxed{z_a = \prod_{l=1}^{r} {\lambda_{0,l}}^{l_{0,l}^{(a)}} \prod_{i=1}^{s} {\lambda_i}^{l_i^{(a)}}.} \tag{4.74}$$

Writing each l as $\sum_{a=1}^{h} n_a l^{(a)}$, we end up with:

$$\boxed{\langle g,\Omega\rangle = \sum_{n_1,\ldots,n_h}\left[\frac{\prod_{l=1}^{r}\left(-\sum_{a=1}^{h} n_a l_{0,l}^{(a)}\right)!}{\prod_{j=1}^{s}\left(\sum_{a=1}^{h} n_a l_j^{(a)}\right)!}\prod_{a=1}^{h}\left((-1)^{\sum_{l=1}^{r} l_{0,l}^{(a)}} z_a\right)^{n_a}\right].} \qquad (4.75)$$

Note that we have to restrict the sum over the integers n_a to those linear combinations of the relations that are of the form (4.71). Once we have determined the fundamental period (4.75) and the PF operators, we can determine the YCs and the mirror map in the same way as for toric hypersurfaces. We give an explicit example of a toric CICY:

4.9.1 Periods, Picard–Fuchs equations, and instanton numbers of the (2,30) model

V_A from section 4.8.1

We start with the first realization of the (2,30) model. Since we want to compute the periods of the mirror, we have to swtich to the N–lattice. In section 4.8.1 we found out that D_8 (corresponding to ρ_8^*) does not intersect the CY. The polytope Δ^* has only vertices, so the relevant points are:

$$\nabla_1^0 = \{0,\rho_1^*,\rho_2^*,\rho_4^*\} \qquad \text{and} \qquad \nabla_2^0 = \{0,\rho_3^*,\rho_5^*,\rho_6^*,\rho_7^*\}. \qquad (4.76)$$

Thus the polytope $\Delta_{C^*}^0 \cap \bar{M}$ is:

$$\Delta_{C^*}^0 \cap \bar{M} = \left\{\begin{array}{ccccccccc} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{array}\right\}, \qquad (4.77)$$

with a basis of relations (4.34):

$$\boxed{\begin{array}{cll} & & \quad\overset{l}{\longrightarrow}\quad \overset{i}{\longrightarrow} \\ a\downarrow & l^{(1)} & = (-4,-2;2,1,1,1,1,0,0) \\ & l^{(2)} & = (\ \ \,0,-2;0,0,0,0,0,1,1). \\ & & \qquad\underset{\bar{m}}{\longrightarrow} \end{array}}$$

Equation (4.75) gives the fundamental period:

$$y_0 \;=\; \langle g,\Omega\rangle \;=\; \sum_{n_1,n_2\geq 0} z_1^{n_1} z_2^{n_2}\frac{(4n_1)!(2(n_1+n_2))!}{(n_1!)^4(2n_1)!(n_2!)^2}, \qquad (4.78)$$

with

$$z_1 = \frac{\lambda_1^2\lambda_2\lambda_3\lambda_4\lambda_5}{\lambda_{0,1}^4\lambda_{0,2}^2} \qquad \text{and} \qquad z_2 = \frac{\lambda_6\lambda_7}{\lambda_{0,2}^2}. \tag{4.79}$$

The logarithmic derivatives are related by:

$$\begin{gathered}\delta_1 = 2\delta_{z_1} \quad \delta_2 = \delta_{z_1} \quad \delta_3 = \delta_{z_1} \quad \delta_4 = \delta_{z_1} \quad \delta_5 = \delta_{z_1} \quad \delta_6 = \delta_{z_1} \quad \delta_7 = \delta_{z_1} \\ \delta_{0,1} = -4\delta_{z_1} \quad \delta_{0,2} = -2\delta_{z_1} - 2\delta_{z_2}.\end{gathered} \tag{4.80}$$

Now recall the definition of the $\Box_l$–operators:

$$\Box_l \;=\; \prod_{l_{\bar m}>0} \partial_{\bar m}^{l_{\bar m}} - \prod_{l_{\bar m}<0} \partial_{\bar m}^{-l_{\bar m}} \quad l \in \Lambda \tag{4.81}$$

of an $\mathcal{A}$–system (see definition 3.2.1), which have to be rescaled as in the hypersurface case (equation (3.46)). In our example they are:

$$\Box_1 \;=\; \left(\partial_1^2\partial_2\partial_3\partial_4\partial_5 - \partial_{0,1}^4\partial_{0,2}^2\right)(\lambda_{0,1}\lambda_{0,2})^{-1}, \tag{4.82}$$

$$\Box_2 \;=\; \left(\partial_6\partial_7 - \partial_{0,2}^2\right)(\lambda_{0,1}\lambda_{0,2})^{-1}. \tag{4.83}$$

Multiplying $\Box_1$ with $z_1\lambda_{0,1}^5\lambda_{0,2}^3$ and $\Box_2$ with $z_2\lambda_{0,1}\lambda_{0,2}^3$ from the left gives:

$$\begin{aligned}\Box_1 &= (\delta_1^2 - \delta_1)\delta_2\delta_3\delta_4\delta_5 - z_1\lambda_{0,1}^5\lambda_{0,2}^3(\lambda_{0,1}^{-1}\delta_{0,1})^4(\lambda_{0,2}^{-1}\delta_{0,2})^2(\lambda_{0,1}\lambda_{0,2})^{-1}, \\ \Box_2 &= \delta_6\delta_7 - z_2\lambda_{0,2}^3(\lambda_{0,2}^{-1}\delta_{0,2})^2\lambda_{0,2}^{-1}.\end{aligned} \tag{4.84}$$

After inserting form (4.80) for the logarithmic derivatives and repeated application of (3.44) we end up with:

$$\begin{aligned}\Box_1 &= (4\delta_{z_1}^2 - 2\delta_{z_1})\delta_{z_1}^4 - z_1(4\delta_{z_1}+4)(4\delta_{z_1}+3)(4\delta_{z_1}+2)(4\delta_{z_1}+1) \\ &\qquad (2\delta_{z_1}+2\delta_{z_2}+1)(2\delta_{z_1}+2\delta_{z_2}+2), \\ \Box_2 &= \delta_{z_2}^2 - z_2(2\delta_{z_1}+2\delta_{z_2}+1)(2\delta_{z_1}+2\delta_{z_2}+2).\end{aligned} \tag{4.85}$$

In order to reduce this system it is useful to push the z_is through the derivatives. This can be done by using the identity

$$z_i\delta_{z_j} = (\delta_{z_j} - \delta_{ij})z_i, \tag{4.86}$$

and we end up with:

$$\begin{aligned}\Box_1 &= 2(2\delta_{z_1}^2 - \delta_{z_1})\delta_{z_1}^4 - 16\delta_{z_1}(4\delta_{z_1}-1)(2\delta_{z_1}-1)(4\delta_{z_1}-3)(2\delta_{z_1}+2\delta_{z_2}-1)(\delta_{z_1}+\delta_{z_2})z_1, \\ \Box_2 &= \delta_{z_2}^2 - (2\delta_{z_1}+2\delta_{z_2}-1)(\delta_{z_1}+\delta_{z_2})z_2.\end{aligned} \tag{4.87}$$

These operators can be reduced by defining the operators $\tilde\Box_1$ and $\tilde\Box_2$:

$$\tilde\Box_1 \;=\; \frac{\Box_1 - 2\delta_{z_1}(2\delta_{z_1}-1)\delta_{z_1}^2\Box_2}{-2\delta_{z_1}(2\delta_{z_1}-1)(\delta_{z_1}+\delta_{z_1})}, \tag{4.88}$$

$$\tilde\Box_2 \;=\; \Box_2, \tag{4.89}$$

and a complete set of PF operators is given by:

$$\boxed{\begin{array}{rcl}\tilde{\Box}_1 &=& \delta_{z_1}^2(\delta_{z_2}-\delta_{z_1})+(2\delta_{z_1}+2\delta_{z_2}-1)(8(4\delta_{z_1}-1)(4\delta_{z_1}-3)z_1-2\delta_{z_1}^2 z_2)\\ \tilde{\Box}_2 &=& \delta_{z_2}^2-(2\delta_{z_1}+2\delta_{z_2}-1)(\delta_{z_1}+\delta_{z_2})z_2.\end{array}} \tag{4.90}$$

The YCs are now easily obtained from the system (4.90) in the same way as for a toric hypersurface:

$$\boxed{\begin{array}{ll}\overline{K}_{(111)}=2\,\dfrac{1+a-b}{a^3\Delta_1}, & \overline{K}_{(112)}=2\,\dfrac{1-a+b}{a^2b\Delta_1},\\ \overline{K}_{(122)}=2\,\dfrac{3-a+b}{ab\Delta_1\Delta_2}, & \overline{K}_{(222)}=2\,\dfrac{1-a+b(6+b-a)}{b^2\Delta_1\Delta_2^2},\end{array}} \tag{4.91}$$

where we performed a linear transformation on the coordinates (z_i):

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 256 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}. \tag{4.92}$$

Δ_i are the discriminants:

$$\boxed{\Delta_1=(1-a)^2-b(2+2a-b),\quad \Delta_2=1-b.} \tag{4.93}$$

The two lorarithmic solutions can be obtained from the Frobenius method (3.60). The calculation is straight foward and even simplyer as in the example at the end of chapter 3, because wo do not perform a distinction of cases. Inserting the mirror map into equation (3.64) gives the instanton corrected YCs of the mirror. For the present example, the integral expansion of the instanton contribution (3.65) with respect to (d_1, d_2) yields:

d_1 d_2 = 0	1	2	3	4	5	6
0	8					
1 384	1088	384				
2 4688	117088	247680	117088	4688		
3 146816	12092928	84309504	148640576	84309504	12092928	146816
4 5462064	1205851824	20072874752	86051357872	135328662848	86051357872	20072874752

V_C from section 4.8.2

As we mentioned before the large complex structure variables defined by the polyhedron can differ even for topologically equivalent families. In particular, the Picard-Fuchs equations for the variables $(\tilde{z}_1, \tilde{z}_2)$ defined by (4.61) are formally different from (4.90), namely

$$\boxed{\begin{array}{l}\Box_1=\delta_{\tilde{z}_1}^2(\delta_{\tilde{z}_1}-\delta_{\tilde{z}_2})-8(4\delta_{\tilde{z}_1}-3)(4\delta_{\tilde{z}_1}-1)(2\delta_{\tilde{z}_1}+2\delta_{\tilde{z}_2}-1)\tilde{z}_1\\ \Box_2=\delta_{\tilde{z}_2}^2-2(\delta_{\tilde{z}_1}-\delta_{\tilde{z}_2}+1)(2\delta_{\tilde{z}_1}+2\delta_{\tilde{z}_2}-1)\tilde{z}_2.\end{array}} \tag{4.94}$$

The derivation is the same as in section 4.9.1 with the Mori vectors (4.61):

$$\boxed{\begin{array}{ccl} & & \qquad\overset{l}{\longrightarrow}\quad \overset{i}{\xrightarrow{\hspace{6em}}} \\ a\big\downarrow & l^{(1)} & = (-4,-2;2,1,1,\quad 1,1,0,0) \\ & l^{(2)} & = (\quad 0,-2;0,1,0,-1,0,1,1). \\ & & \qquad\quad\underset{\bar m}{\xrightarrow{\hspace{9em}}} \end{array}}$$

The triple intersections of this model are:

$$\boxed{\begin{array}{ll} \overline{K}_{(111)} = 2\,\dfrac{1+\tilde a}{\tilde a^3\tilde\Delta_1}, & \overline{K}_{(112)} = 2\,\dfrac{1-\tilde a}{\tilde a^2\tilde b\tilde\Delta_1} \\ \overline{K}_{(122)} = 2\,\dfrac{1+\tilde a}{\tilde a\tilde b\tilde\Delta_1\tilde\Delta_2}, & \overline{K}_{(222)} = 2\,\dfrac{1-\tilde a}{\tilde b^2\tilde\Delta_1\tilde\Delta_2} \end{array}} \tag{4.95}$$

with

$$\boxed{\tilde\Delta_1 = (1-\tilde a)^2 - 4\tilde a\tilde b, \quad \tilde\Delta_2 = 1+\tilde b.} \tag{4.96}$$

The coordinates $(\tilde a, \tilde b)$ are related to $(\tilde z_1, \tilde z_2)$ by the same transformation as in equation (4.92). As the A-models are topologically equivalent we should find a rational transformation of variables preserving the large complex structure limit $(z_1 = 0, z_2 = 0) \mapsto (\tilde z_1 = 0, \tilde z_2 = 0)$ and identifying (4.90) with (4.94). To find this transformation we make the following ansatz:

$$\frac{\Delta_1(a,b)}{P(a,b)} = \tilde\Delta_1(\tilde a,\tilde b) \qquad \text{and} \qquad \frac{\Delta_1(a,b)}{P(a,b)} = \tilde\Delta_1(\tilde a,\tilde b), \tag{4.97}$$

where $P(a,b)$ is some rational function. Solving for $\tilde a$ and $\tilde b$ gives:

$$\begin{aligned} \tilde b &= \frac{-1-P+b}{P}, \\ \tilde a &= \frac{-4-2P+4b \pm 2\sqrt{4(-1+b)^2 - P(-3+a^2+2b+b^2-2a(1+b))}}{2P}. \end{aligned}$$

This transformation is rational if and only if we can get rid of the squareroot. Setting

$$P = -(b-1)^2 \tag{4.98}$$

completes the square in the expression under the squareroot and we find (for the plus sign in (4.98)):

$$\boxed{\begin{pmatrix} \tilde a \\ \tilde b \end{pmatrix} = \begin{pmatrix} \frac{1}{1-b} & 0 \\ 0 & \frac{1}{1-b} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.} \tag{4.99}$$

This phenomenon is not special to complete intersections, it can also occur for hypersurfaces. For example, the degree 12 hypersurface in the resolved weighted projective space $\mathbb{P}^4_{1,2,2,3,4}$ and

$$\mathbb{P}\begin{pmatrix} 2\,1\,1\,1\,1\,1\,0 \\ 1\,2\,1\,1\,1\,0\,1 \end{pmatrix}\begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

yield topologically equivalent Calabi-Yau hypersurfaces with $h^{1,1} = 2$ and $h^{2,1} = 74$, but different Mori cones. The corresponding polyhedra correspond to different blowups of the same simplex associated to $\mathbb{P}^4_{1,1,1,1,2}$, and the Picard-Fuchs equations as well as the triple intersections can be mapped onto each other in a similar way. As an other example, the well-studied two-parameter model $\mathbb{P}^4_{1,1,2,2,6}[12]$ also has a cousin which is

$$\mathbb{P}\begin{pmatrix} 4\,1\,1\,1\,1\,1\,0 \\ 1\,4\,1\,1\,1\,0\,1 \end{pmatrix}\begin{bmatrix} 8 \\ 8 \end{bmatrix}.$$

It is easy to read off from the weight vectors that both polyhedra come from two different blow-ups of $\mathbb{P}^4_{1,1,1,1,4}$. We observe that in all these examples the number of vertices and points of Δ is the same, while a single vertex is added to the simplex Δ^*.

Appendix A

Computer programs

The program `nef.x` calculates nef–partitions of arbitrary codimension for reflexive polytops and computes the cohomological data. The software `cws.x` creates weight systems and is also able to combine them. This is very useful for constructing certain toric ambient spaces, especially fibrations. `gen.x` is under development and transforms Gorenstein polytopes for a generalized CY into those arising from a nef–partition. `nef.x` and `cws.x` are already implemented into the PALP package [33] for testing.

A.1 The program nef.x

A.1.1 help listing for nef.x

$ nef.x -h

```
This is ‘nef.x’:  calculates hodge numbers of nef-partitions.
Usage:   nef.x -<options>
Options:
  -h        print this information.
  -f or -   use as filter; otherwise parameters denote I/O files.
  -N        starting-poly is in N-lattice (default is M).
  -H        gives full list of hodge numbers.
  -Lv       prints L vector of vertices (in N-lattice).
  -Lp       prints L vector of points (in N-lattice).
  -p        prints only partitions, no Hodge numbers.
  -D        calculates also direct products.
  -P        calculates also projections.
  -t        full time info.
  -cCODIM   codimension (default = 2).
  -Fcodim   fibrations up to codim (default = 2).
Input:    degrees and weights ‘d1 w11 w12 ... d2 w21 w22 ...’
          or ‘d np’ or ‘np d’ (d=Dimension, np=#[points]) and
              (after newline) np*d coordinates.
Output:   as specified by options.
```

A.1.2 extended (experimental) options for nef.x

$ nef.x -x

```
This is extended help for ‘‘./nef.x’’:
   -y        print poly/CWS in M lattice if it has nef-partitions.
   -S        information about #points calculated in S-poly.
   -T        checks Serre-duality.
   -s        don’t remove symmetric nef-partitions.
   -n        prints poly only if it has nef-partitions.
   -v        prints vertices and #points of starting-poly in one
             line. With the following option the output is limited
             by #points:
       -uPOINTS  ... upper limit of #points (default = POINT_Nmax).
       -lPOINTS  ... lower limit of #points (default = 0).
   -m        starts with [d  w1 w2 ... wk d=d_1 d_2 (Minkowski sum).
   -R        prints vertices of starting-poly if it is
             not reflexive.
   -V        prints vertices of poly (in N-lattice).
   -Q        only direct products (up to lattice quotient).
   -gNUMBER  prints points of Gorenstein poly in N-lattice.
   -dNUMBER  prints points of Gorenstein poly in M-lattice.
         If NUMBER = 0 ... no            0/1 info.
         If NUMBER = 1 ... no redundant  0/1 info (=default).
         If NUMBER = 2 ... full          0/1 info.
   -M        print VPM of gore polytopes.
```

A.2 The program cws.x

A.2.1 help listing for cws.x

$ cws.x -h

```
This is ‘cws.x’: create weight systems and combined weight systems.
Usage:   cws.x -<options>;     the first option must be ‘w’, ‘c’,
‘i’,or ‘h’.
Options:
   -h        print this information.
   -w# [L H] make IP weight systems for #-dimensional polytopes.
             For #>4 the lowest and highest degrees L<=H are required.
       -r/-t make reflexive/transversal weight systems (optional).

   -c#       make combined weight systems for #-dimensional polytopes.
             For #<=4 all relevant combinations are made by default,
             otherwise the following option is required:
       -n[#] followed by the names wf_1 ... wf_# of weight files
             currently #=2,3 are implemented.
        [-t] followed by # numbers n_i specifies the CWS-type, i.e.
             the numbers n_i of weights to be selected from wf_i.
             Currently all cases with n_i<=2 are implemented.

    -i       compute the polytope data M:p v [F:f] N:p [v] for all IP
             CWS, where p and v denote the numbers of lattice points
             and vertices of a dual pair of IP polytopes; an entry
             F:f and no v for N indicates a non-reflexive ‘dual pair’.
        -f   use as filter; otherwise parameters denote I/O files
```

A.2.2 extended (experimental) options for cws.x

$ cws.x -x

```
This is ‘cws.x’: -x gives undocumented extensions:
   -ip    printf PolyPointList.
   -id    printf dual PolyPointList.
   -N     make CWS for PPL in N lattice.
   -p#    [infile1] [infile2] makes cartesian product
             of Vertices. # dimensions are identified.
```

A.3 The program gen.x

A.3.1 help listing for gen.x

$ gen.x -h

```
This is 'gen.x':  computing splits of generalized CYs.
Usage:   gen.x [-<Option-string>] [in-file [out-file]]
Options (concatenate any number of them into <Option-string>):
  h  print this information.  | p  print input (gen) poly.
  f  use as filter.           | l  print input (gen) poly
  s  no hodge calculation.    |    + weights.
  L  print split poly + nef   | d  print dual (M) input
     info + L info.           |    (gen) poly.
  P  print split poly.        | t  show transformations for
  r  print index x input      |    split.
     poly.


Input:    degrees and weights 'd1 w11 w12 ... d2 w21 w22 ...'
          or 'd np' or 'np d' (d=Dimension, np=#[points]) and
              (after newline) np*d coordinates
Output:   as specified by options
```

A.3.2 extended (experimental) options for gen.x

$ gen.x -x

```
Testing options:
  x  print this information.  | c[r] (needs points in
  W  print weight if poly     |       N-lattice as input):
     has full dimension.      |      print poly if it has diff.
  r  print r x input-poly     |      splits.
     (r=index).               |    s   print poly if it splits.
  M  print pairing matrix     |    m   print poly if it has mult.
     of gen. polys.           |        splits.
                              |    a   print all different splits.
```

Appendix B

Cohomological results

B.1 Toric Calabi-Yau spaces with small Picard numbers

In this appendix we compile the Hodge data with $h^{11} \leq 3$ that have been obtained for weighted projective spaces and more general toric ambient spaces. The results are complete for hypersurfaces and they are probably (at least almost) complete for codimension 2:

H:(1,h^{12})	weighted projective	toric
hypersurfaces:	101 103 145 149	21
codimension 2:	61 73 79 89 129	25 37

H:(2,h^{12})	weighted projective	toric
hypersurfaces:	74 86 95 106 122 128 132 272	29 38 83 84 90 92 102 116 120 144
codimension 2:	62 68 (83 84 90)	30 36 44 50 54 56 58 59 60 64 66 70 72 76 77 78 80 82 100 112

H:(3,h^{12})	weighted projective	toric
hypersurfaces:	66 69 75 87 99 103 105 123 131 165 195 231 243	43 45 51 57 59 63 65 67 71 72 73 76 77 78 79 81 83 84 85 89 91 93 95 107 111 115 119 127 141
codimension 2:	47 55 61 87 (45 51 57 67 71 77 81 83 89 91 93 111)	23 24 27 29 31 33 35 37 39 41 42 44 48 49 50 52 53 54 56 58 60 62 64 68 70 80 101 113

Complete intersections in products of projective spaces were enumerated completely for arbitrary codimension many years ago [73]. The relevant Hodge data from [74] are

65 73 89 101 for $h^{11} = 1$,

46 **47** 50 **55** 56 58 59 62 64 66 68 72 76 77 83 86 for $h^{11} = 2$,

27 31 33 35 **36** 37 **38** 39 **40** 41 43-45 **46** 47-61 63 66 69 72 75 for $h^{11} = 3$.

Bold-face numbers are those values of h^{12} that do not occur in the above tables. As a check for the completeness of our results we used the lists that are available at [75] to verify that the missing values indeed require codimension 3 or more. Possible representations of minimal codimension are

$$\mathbb{P}^7[2\ 2\ 2\ 2] \equiv \mathbb{P}\,(1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)\,[2\ 2\ 2\ 2]^{1,65}_{-128}$$

i.e. 4 quartics in $\mathbb{P}^7$ for the example with Picard number 1, and

$$\begin{array}{c}\mathbb{P}^2\\ \mathbb{P}^5\end{array}\begin{bmatrix}2\ 1\ 0\ 0\\ 1\ 1\ 2\ 2\end{bmatrix}^{2,46}_{-88}, \qquad \begin{array}{c}\mathbb{P}^2\\ \mathbb{P}^4\end{array}\begin{bmatrix}2\ 1\ 0\\ 1\ 1\ 3\end{bmatrix}^{2,47}_{-90}, \qquad \begin{array}{c}\mathbb{P}^3\\ \mathbb{P}^3\end{array}\begin{bmatrix}2\ 1\ 1\\ 1\ 2\ 1\end{bmatrix}^{2,55}_{-106},$$

$$\begin{array}{c}\mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^2\end{array}\begin{bmatrix}2\ 1\ 0\\ 1\ 1\ 1\\ 0\ 1\ 2\end{bmatrix}^{3,36}_{-66}, \qquad \begin{array}{c}\mathbb{P}^2\\ \mathbb{P}^2\\ \mathbb{P}^3\end{array}\begin{bmatrix}2\ 1\ 0\ 0\\ 1\ 0\ 1\ 1\\ 0\ 1\ 2\ 1\end{bmatrix}^{3,38}_{-70}, \qquad \begin{array}{c}\mathbb{P}^1\\ \mathbb{P}^2\\ \mathbb{P}^3\end{array}\begin{bmatrix}1\ 1\ 0\\ 1\ 0\ 2\\ 1\ 2\ 1\end{bmatrix}^{3,40}_{-74}, \qquad \begin{array}{c}\mathbb{P}^1\\ \mathbb{P}^2\\ \mathbb{P}^3\end{array}\begin{bmatrix}1\ 1\ 0\\ 2\ 0\ 1\\ 1\ 2\ 1\end{bmatrix}^{3,46}_{-86}$$

for $h^{11} = 2$ and $h^{11} = 3$, respectively.

B.2 Free quotients of elliptic K3 fibrations

In this appendix we present the Hodge data and some polytopes that we found for free $\mathbb{Z}_2$ quotients of elliptic K3 fibrations.

Among all Calabi–Yau hypersurfaces in toric varieties there are 16 polytopes that correspond to free quotients. This can be compatible with a K3 fibration only for $\mathbb{Z}_2$ quotients because the action on the base $\mathbb{P}^1$ always has fixed points and the K3 fibers only admit a free $\mathbb{Z}_2$ action. The well-known example of the free $\mathbb{Z}_5$ quotient of the quintic has no fibration. The two $\mathbb{Z}_3$ examples are elliptic. The remaining 13 polytopes have fundamental group $\mathbb{Z}_2$ and are elliptic-K3 fibrations. The Hodge data of these manifolds and of their double covers are

h^{11} h^{12} [χ]	double cover	h^{11} h^{12} [χ]	double cover	h^{11} h^{12} [χ]	double cover
3 43 [-80]	3 83 [-160]	4 28 [-48]	4 52 [-96]	5 29 [-48]	7 55 [-96]
3 59 [-112]	3 115 [-224]	4 36 [-64]	4 68 [-128]		
3 75 [-144]	4 148 [-288]	4 44 [-80]	5 85 [-160]		

At codimension 2 we found 72 polytopes with nef partitions and elliptic K3 structure that coresond to a free quotient. In 3 cases the lattice quotient actually corresponds to a $\mathbb{Z}_4$ quotient, but only the $\mathbb{Z}_2$ refinement of the lattice is compatible with the nef partition.

h^{11} h^{12} [χ]	double cover	h^{11} h^{12} [χ]	double cover	h^{11} h^{12} [χ]	double cover
3 23 [-40]	3 43 [-80]	4 22 [-36]	5 41 [-72]	5 25 [-40]	7 47 [-80]
3 27 [-48]	3 51 [-96]	4 24 [-40]	5 45 [-80]	5 27 [-44]	8 52 [-88]
3 29 [-52]	3 55 [-104]	4 26 [-44]	6 50 [-88]	5 29 [-48]	6 54 [-96]
3 31 [-56]	3 59 [-112]	4 36 [-64]	6 70 [-128]	5 33 [-56]	7 63 [-112]
3 33 [-60]	4 64 [-120]	4 42 [-76]	5 41 [-152]	5 35 [-60]	7 67 [-120]
3 39 [-72]	4 76 [-144]	4 58 [-108]	6 114 [-216]	5 41 [-72]	8 80 [-144]
6 24 [-36]	10 46 [-72]	7 19 [-24]	11,35 [-48]	8 14 [-12]	13 25 [-24]
6 26 [-40]	9 49 [-80]				

The hypersurface with Hodge data (3,43) and the complete intersection (4,36) are discussed in Section 4.8.3.

A surprise in view of the Heterotic-Type II anomaly conditions [76], [77] is the small number of hypermultiplets even for small numbers (like $h^{11} = 3$) of vectors. Some of the corresponding polytopes are

$$\mathbb{P}^{3,23}_{-40}\begin{pmatrix}1\,1\,1\,1\,0\,0\,0\,0\\0\,0\,0\,0\,1\,1\,0\,0\\0\,0\,0\,0\,0\,0\,1\,1\end{pmatrix}\left[\begin{array}{c|c}2&2\\2&0\\0&2\end{array}\right]\ /\,\mathbb{Z}_2 : 1\,1\,1\,0\,1\,0\,1\,0,$$

$$\mathbb{P}^{3,27}_{-48}\,(2\;2\;1\;1\;1\;1)\,[4\,|\,4]\ /\,\mathbb{Z}_4 : 0\,2\,2\,1\,3\,0,$$

$$\mathbb{P}^{3,29}_{-52}\begin{pmatrix}2\,2\,1\,0\,2\,1\,0\\0\,0\,0\,1\,0\,0\,1\end{pmatrix}\left[\begin{array}{c|c}4&4\\2&0\end{array}\right]\ /\,\mathbb{Z}_2 : 1\,0\,1\,1\,1\,0\,0,$$

$$\mathbb{P}^{3,31}_{-56}\begin{pmatrix}0\,2\,1\,1\,0\,2\,2\\1\,1\,0\,0\,1\,1\,0\end{pmatrix}\left[\begin{array}{c|c}4&4\\2&2\end{array}\right]\ /\,\mathbb{Z}_2 : 1\,1\,1\,0\,0\,0\,1,$$

$$\mathbb{P}^{3,33}_{-60}\begin{pmatrix}2\,2\,1\,2\,0\,1\,0\,0\\1\,1\,0\,2\,1\,0\,1\,0\\0\,0\,0\,1\,0\,0\,0\,1\end{pmatrix}\left[\begin{array}{c|c}4&4\\4&2\\2&0\end{array}\right]\ /\,\mathbb{Z}_2 : 1\,0\,1\,1\,1\,0\,0\,0,$$

$$\mathbb{P}^{3,39}_{-72}\begin{pmatrix}0\,2\,1\,4\,2\,2\,1\,0\\1\,1\,0\,1\,0\,1\,0\,0\\0\,0\,0\,1\,0\,0\,0\,1\end{pmatrix}\left[\begin{array}{c|c}8&4\\2&2\\2&0\end{array}\right]\ /\,\mathbb{Z}_2 : 1\,1\,1\,0\,1\,0\,0\,0,$$

$$\mathbb{P}^{4,22}_{-36}\begin{pmatrix}2\,2\,1\,0\,1\,2\,0\,0\\0\,0\,1\,2\,1\,2\,2\,0\\0\,0\,0\,1\,0\,0\,0\,1\end{pmatrix}\left[\begin{array}{c|c}4&4\\4&4\\2&0\end{array}\right]\ /\,\mathbb{Z}_2 : 0\,1\,1\,1\,0\,1\,0\,0,$$

$$\mathbb{P}^{4,24}_{-40}\begin{pmatrix}2\,2\,1\,0\,0\,2\,1\,0\\1\,1\,0\,1\,1\,0\,0\,0\\0\,0\,0\,1\,0\,0\,0\,1\end{pmatrix}\left[\begin{array}{c|c}4&4\\2&2\\2&0\end{array}\right]\ /\,\mathbb{Z}_2 : 1\,0\,1\,1\,0\,1\,0\,0,$$

$$\mathbb{P}^{4,26}_{-44}\begin{pmatrix}2\,2\,1\,1\,0\,0\,0\,2\\1\,1\,0\,0\,0\,1\,1\,0\\0\,0\,0\,0\,1\,0\,0\,1\end{pmatrix}\left[\begin{array}{c|c}4&4\\2&2\\2&0\end{array}\right]\ /\,\mathbb{Z}_2 : 0\,1\,1\,0\,1\,1\,0\,0.$$

Bibliography

[1] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Mirror symmetry*, vol. 1 of *Clay Mathematics Monographs*. American Mathematical Society, Clay Mathematics Institute, 2003.

[2] D. A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, vol. 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[3] S. T. Yau, *Mirror symmetry I*. AMS/IP Studies in Advanced Mathematics. Amer. Math. Soc., Providence, RI, 1998.

[4] B. Greene and S. T. Yau, *Mirror symmetry II*. AMS/IP Studies in Advanced Mathematics. Amer. Math. Soc., Providence, RI, 1997.

[5] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum configurations for superstrings," *Nucl. Phys.* **B258** (1985) 46–74.

[6] E. Calabi, "On kaehler manifolds with vanishing canonical class," *Princeton Math. Series* **12** (1957) 78 – 89.

[7] S.-T. Yau, "Calabi's conjecture and some new results in algebraic geometry," *Proc. Natl. Acad. Sci. USA* **74** (1977) 1798 – 1799.

[8] T. Hübsch, *Calabi-Yau Manifolds. A Bestiary for Physicists*. World Scientific, Singapore, 1992.

[9] T. Kaluza, "On the problem of unity in physics," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1921** (1921) 966–972.

[10] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. Wiley Classics Library. John Wiley & Sons Ltd., New York, NY, 1994.

[11] W. Lerche, C. Vafa, and N. P. Warner, "Chiral rings in n=2 superconformal theories," *Nucl. Phys.* **B324** (1989) 427.

[12] K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, vol. 283 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1986.

[13] A. Strominger, "Yukawa couplings in superstring compactification," *Phys. Rev. Lett.* **55** (1985) 2547.

[14] J. Knapp, "Aspects of superstring compactifications," *unpublished*.

[15] E. Witten, "Topological sigma models," *Commun. Math. Phys.* **118** (1988) 411 – 449.

[16] E. Witten, "Mirror manifolds and topological field theory," in *Essays on mirror manifolds*, S.-T. Yau, ed., pp. 120 – 159. International Press, Cambridge, MA, 1991. hep-th/9112056. Mirror Symmetry, topological field theory.

[17] A. Sen, "Unification of string dualities," *Nucl. Phys. Proc. Suppl.* **58** (1997) 5, hep-th/9609176.

[18] P. Green and T. Hubsch, "Calabi-yau manifolds as complete intersections in products of complex projective spaces," *Commun. Math. Phys.* **109** (1987) 99.

[19] P. Candelas, A. M. Dale, C. A. Lutken, and R. Schimmrigk, "Complete intersection Calabi-Yau manifolds," *Nucl. Phys.* **B298** (1988) 493.

[20] P. Candelas, M. Lynker, and R. Schimmrigk, "Calabi-Yau manifolds in weighted p(4)," *Nucl. Phys.* **B341** (1990) 383–402.

[21] M. Kreuzer and H. Skarke, "No mirror symmetry in landau-ginzburg spectra!," *Nucl. Phys.* **B388** (1992) 113–130, hep-th/9205004.

[22] A. Klemm and R. Schimmrigk, "Landau-ginzburg string vacua," *Nucl. Phys.* **B411** (1994) 559 – 583, hep-th/9204060.

[23] W. Fulton, *Intersection Theory*, vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer, Berlin, 2nd ed. ed., 1998.

[24] T. Oda, *Convex Bodies and Algebraic Geometry. An Introduction to the Theory of Toric Varieties*, vol. 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, 1988.

[25] V. Danilov, "Geometry of toric varieties," *Russ. Math. Surv.* **33** (1978) 97 – 154.

[26] D. A. Cox, "The homogeneous coordinate ring of a toric variety," *J. Algebr. Geom.* **4** (1995) 17 – 50.

[27] D. A. Cox, "Recent Developments in Toric Geometry," in *Algebraic geometry. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995*, J. K. et al, ed., no. 62 in Proc. Symp. Pure Math., pp. 389 – 436, AMS. Providence, RI, 1997. alg-geom/9606016.

[28] H. Skarke, "String dualities and toric geometry: An introduction," hep-th/9806059.

[29] V. V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties," *J. Algebr. Geom.* **3** (1994) 493 – 535, `alg-geom/9310003`.

[30] V. V. Batyrev and L. A. Borisov, "Dual Cones and Mirror Symmetry for Generalized Calabi-Yau Manifolds," `alg-geom/9402002`.

[31] V. V. Batyrev and L. A. Borisov, "On Calabi-Yau complete intersections in toric varieties," `alg-geom/9412017`.

[32] V. V. Batyrev and L. A. Borisov, "Mirror duality and string-theoretic hodge numbers," `alg-geom/9509009`.

[33] M. Kreuzer and H. Skarke, "Palp: A package for analyzing lattice polytopes with applications to toric geometry," `math.na/0204356`.

[34] M. Kreuzer, E. Riegler, and D. A. Sahakyan, "Toric complete intersections and weighted projective space," `math.ag/0103214`.

[35] E. R. A. Klemm, M. Kreuzer and E. Scheidegger, "Complete intersection Calabi-Yau spaces in toric varieties and string duality," *close to be submitted*.

[36] B. R. Greene, C. Vafa, and N. P. Warner, "Calabi-Yau manifolds and renormalization group flows," *Nucl. Phys.* **B324** (1989) 371.

[37] E. Witten, "Phases of n=2 theories in two-dimensions," *Nucl. Phys.* **B403** (1993) 159 – 222, `hep-th/9301042`.

[38] P. Candelas, E. Derrick, and L. Parkes, "Generalized Calabi-Yau manifolds and the mirror of a rigid manifold," *Nucl. Phys.* **B407** (1993) 115–154, `hep-th/9304045`.

[39] S. Sethi, "Supermanifolds, rigid manifolds and mirror symmetry," *Nucl. Phys.* **B430** (1994) 31–50, `hep-th/9404186`.

[40] M. Aganagic and C. Vafa, "Mirror symmetry and supermanifolds," `hep-th/0403192`.

[41] K. Hori and C. Vafa, "Mirror symmetry," `hep-th/0002222`.

[42] A. Schwarz, "Sigma models having supermanifolds as target spaces," *Lett. Math. Phys.* **38** (1996) 91–96, `hep-th/9506070`.

[43] K. Hori, "Trieste lectures on mirror symmetry,". Prepared for ICTP Spring School on Superstrings and Related Matters, Trieste, Italy, 18-26 Mar 2002.

[44] A. Strominger, "Special geometry," *Commun. Math. Phys.* **133** (1990) 163 – 180.

[45] A. R. Mavlyutov, "Semiample hypersurfaces in toric varieties," *Duke Math. J.* **101** (2000) 85–116, `math.AG/9812163`.

[46] V. Batyrev, "On the classification of smooth projective toric varieties," *Tohoku Math. J., II. Ser.* **43** (1991) 569 – 585.

[47] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*. Birkhäuser, Boston, 1994.

[48] B. S. L.J. Billera, P. Filliman, "Constructions and complexity of secondary polytopes," *Adv. in Math.* **83** (1990) 155.

[49] Kapranov, Sturmfels, and Zeleviski, "Quotients of toric varieties," *Mathematische Annalen* **290** (1991) 643–655.

[50] P. Berglund and S. Katz, "Mirror symmetry for hypersurfaces in weighted projective space and topological couplings," *Nucl. Phys.* **B420** (1994) 289 – 314, `hep-th/9311014`.

[51] S. Hosono, A. Klemm, and S. Theisen, "Lectures on mirror symmetry," in *Integrable models and strings*, A. A. et al, ed., vol. 436 of *Lect. Notes Phys.*, pp. 235 – 280. Springer-Verlag, Berlin, 1994. `hep-th/9403096`.

[52] V. Danilov and Khovanskiĭ, "Newton polyhedra and an algorithm for computing hodge-deligne numbers," *Math. USSR-Izv.* **29** (1987) 279–298.

[53] D. I. Dais, "Enumerative combinatorics of invariants of certain complex threefolds with trivial canonical bundle," *MPI for mathematics preprint series* (1994), no. 64, 1 – 98.

[54] S. Hosono, A. Klemm, S. Theisen, and S. T. Yau, "Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces," *Commun. Math. Phys.* **167** (1995) 301 – 350, `hep-th/9308122`.

[55] S. Hosono, B. H. Lian, and S. T. Yau, "Gkz-generalized hypergeometric systems in mirror symmetry of Calabi-Yau hypersurfaces," *Commun. Math. Phys.* **182** (1996) 535 – 578, `alg-geom/9511001`.

[56] M. Kreuzer and H. Skarke, "Complete classification of reflexive polyhedra in four dimensions," `hep-th/0002240`.

[57] M. Kreuzer and H. Skarke, "Reflexive polyhedra, weights and toric Calabi-Yau fibrations," *Rev. Math. Phys.* **14** (2002) 343–374, `math.ag/0001106`.

[58] `http://hep.itp.tuwien.ac.at/~kreuzer/CY.html`.

[59] B. A. Ovrut, T. Pantev, and R. Reinbacher, "Torus-fibered Calabi-Yau threefolds with non-trivial fundamental group," *JHEP* **05** (2003) 040, `hep-th/0212221`.

[60] R. Donagi, B. A. Ovrut, T. Pantev, and R. Reinbacher, "Su(4) instantons on Calabi-Yau threefolds with z(2) x z(2) fundamental group," *JHEP* **01** (2004) 022, `hep-th/0307273`.

[61] M. Kreuzer and H. Skarke, "All abelian symmetries of landau-ginzburg potentials," *Nucl. Phys.* **B405** (1993) 305–325, `hep-th/9211047`.

[62] P. Berglund and T. Hubsch, "A generalized construction of mirror manifolds," *Nucl. Phys.* **B393** (1993) 377–391, `hep-th/9201014`.

[63] M. Kreuzer and H. Skarke, "Orbifolds with discrete torsion and mirror symmetry," *Phys. Lett.* **B357** (1995) 81 – 88, `hep-th/9505120`.

[64] K. Oguiso, "On Algebraic Fiber Space Structures on a Calabi-Yau 3-fold," *Int. J. Math.* **4** (1993) 439 – 465.

[65] A. C. Avram, M. Kreuzer, M. Mandelberg, and H. Skarke, "Searching for k3 fibrations," *Nucl. Phys.* **B494** (1997) 567 – 589, `hep-th/9610154`.

[66] M. Kreuzer and H. Skarke, "Calabi-Yau 4-folds and toric fibrations," *J. Geom. Phys.* **26** (1998) 272–290, `hep-th/9701175`.

[67] Y. Hu, C.-H. Liu, and S.-T. Yau, "Toric morphisms and fibrations of toric Calabi-Yau hypersurfaces," *Adv. Theor. Math. Phys.* **6** (2003) 457–505, `math.ag/0010082`.

[68] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*. Springer, New York, 1996.

[69] A. Klemm and S. Theisen, "Mirror maps and instanton sums for complete intersections in weighted projective space," *Mod. Phys. Lett.* **A9** (1994) 1807 – 1818, `hep-th/9304034`.

[70] A. Klemm, W. Lerche, and P. Mayr, "K3 fibrations and heterotic type ii string duality," *Phys. Lett.* **B357** (1995) 313 – 322, `hep-th/9506112`.

[71] P. S. Aspinwall and J. Louis, "On the ubiquity of k3 fibrations in string duality," *Phys. Lett.* **B369** (1996) 233–242, `hep-th/9510234`.

[72] C. T. C. Wall, "Classification problems in differential topology. v: On certain 6-manifolds," *Invent. Math.* **1** (1966) 355 – 374.

[73] P. Candelas, A. M. Dale, C. A. Lutken, and R. Schimmrigk, "Complete intersection Calabi-Yau manifolds," *Nucl. Phys.* **B298** (1988) 493.

[74] P. S. Green, T. Hubsch, and C. A. Lutken, "All hodge numbers of all complete intersection calabi-yau manifolds," *Class. Quant. Grav.* **6** (1989) 105–124.

[75] `http://thew02.physik.uni-bonn.de/˜netah/cy.html`.

[76] P. Candelas and A. Font, "Duality between the webs of heterotic and type ii vacua," *Nucl. Phys.* **B511** (1998) 295 – 325, `hep-th/9603170`.

[77] G. Aldazabal, A. Font, L. E. Ibanez, and A. M. Uranga, "New branches of string compactifications and their f-theory duals," *Nucl. Phys.* **B492** (1997) 119–151, `hep-th/9607121`.

Curriculum Vitæ

Surname	Riegler	**Current Address**	
First Name	Erwin Herbert	Institute of Theoretical Physics	
Date of Birth	January 7, 1972	Vienna University of Technology	
Place of Birth	Neunkirchen, Austria	Wiedner Hauptstr. 8-10	
Sex	Male	A-1040 Vienna, Austria	
Nationality	Austria		
Marital Status	Unmarried	**Phone/Fax**	+43–1–58801 13623 / 13699
Languages	German, English	**E-mail**	riegler@hep.itp.tuwien.ac.at

University Education

04/01	Start of doctoral studies at the Institute of Theoretical Physics, Vienna University of Technology; supervisor: Prof. Maximilian Kreuzer
03/01	Diploma “Diplom–Ingenieur” in Technical Physics
02/99 – 03/01	Diploma thesis: “Algebraic Toric Varieties and Complete Intersections” in Technical Phyics; supervisor: Prof. Maximilian Kreuzer
03/94	Change to the studies of Technical Physics and Technical Mathematics
10/91 – 02/94	Undergraduate studies in electrical engineering at the Vienna University of Technology

Work Experience

04/01 –	Fellow of the FWF-project “D-branes, Gepner-Punkte und String-Geometrie” and “Nichtkommutative Strukturen in der offenen Stringtheorie”, Austrian Science Foundation, Project Nr. P14639 and P15584, respectively
10/99 – 10/00	Civilian service at the elderly care-center “CS Pflege- u. Sozialzentrum”
07/98 – 09/98	Collaboration at the center of technology at the Plansee Werke in Tirol
	Financial support for my studies was also provided by several jobs during the summerbreaks and by system administration work at the department of Theoretical Physics

Hobbies and Other Interests

Strong interests in running, hiking, and rock climbing. Other hobbies are musics, movies, computer algebra and the operating system Linux.