DISSERTATION

Discontinuous Optimal Policy Rules;

Accumulate or exploit? - starting from the same initial conditions both can be optimal.

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o.Univ.Prof. Dr. G. FEICHTINGER
Institut für Ökonometrie, Operations Research und Systemtheorie

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Fakultät für Technische Naturwissenschaften und Informatik

von
Dipl.-Ing. Josef L Haunschmied
Zentagasse 44/2/2
1050 Wien, Austria

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Haunschmied J L
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Kurzfassung

Nehmen wir an, daß ein Entscheidungstäger ein deterministisches dynamisches System derart zu kontrollieren versucht, sodaß der gemessene Output des Systems optimiert wird. Hat der Entscheidungsträger einen Kontrollpfad festgelegt, so hat das daraus resultierende Anfangswertproblem eine eindeutig festgelegte Lösung, aber unter Umständen kann es passieren, daß bei einem fix vorgegebenen Anfangszustand des Systems verschiedene optimale Kontrollpfade möglich sind, die zu verschiedenen optimalen Langzeitergebnissen führen. Die Unbestimmtheit (Nichteindeutigkeit) entspringt hier aus der Möglichkeit einen von mehreren optimalen Kontrollpfaden zu wählen. In dieser Arbeit behandle ich Mengen von (Anfangs)Zuständen, in denen derartige Unbestimmtheit auftritt.

In Modellen der nichtkonvexen Optimierung müssen Ergebnisse, welche die notwendigen Optimalitätsbedingungen erfüllen, nicht tatsächlich optimal sein. Zudem kann es sein, daß mehrere Langzeitergebnisse optimal sind, und über die Zeit gesehen hängt die Konvergenz zu dem einen oder anderen Langzeitergebnis vom Anfangszustands des Systems ab. Aufgrund von Stetigkeitsargumenten muß ein Schwellenwertverhalten auftreten, wo von der einen Seite dieser Schwelle es optimal ist, zu einem Langzeitergebnis zu konvergieren, während auf der anderen Seite dieser Schwelle es optimal ist, zu einem anderen Langzeitergebnis zu konvergieren. Der Rahmen dieser Arbeit ist es, zweidimensionale optimale Kontrollmodelle zu finden, wo optimale Gleichgewichtspunkte bzw. Grenzzyklen durch Schwellen getrennt werden. Ich wurde angehalten, solche Schwellen numerisch zu lokalisieren. Neben der bloßen Existenz ist die Beschaffenheit solcher Schwellenmengen von Interesse.

In der ökonomischen Literatur wird ein derartiges Schwellenverhalten oft mit " Skibaschwellen " oder " Skibapunkte " bezeichnet. Aufgrund einer Äußerung von Brock und Malliaris und wegen einer persönlichen Intervention von Kazuo Nishimura, haben wir (meine Arbeitsgruppe) uns entschlossen derartiges Schwellenverhalten als DNS (Dechert-Nishimura-Skiba)verhalten zu bezeichnen. Ein DNS-Punkt ist ein Punkt im Zustandsraum, wo es ausgehend von diesem optimal ist, zu zumindest zwei verschiedenen Attraktoren zu konvergieren. In den in dieser Arbeit untersuchten zweidimensionalen Modellen sind die DNS-Mengen Kurven in den zweidimensionalen Zustandsräumen.

Zuerst analysiere ich ein Modell der Kapitalakkumulation in einem optimalkontrolltheoretischen Rahmen, wo Kapitalstock und Investitionsrate als Zustandsvariablen sowie die Änderung der Investitionsrate als Kontrollvariable modelliert sind. Regulierungskosten treten für Investitionen aber auch bei Änderung der Investionsrate auf. Weiters modelliere ich externe Größeneffekte (network externalities) durch ein konvexes Segment in der Ertragsfunktion. Dies bewirkt die Existenz zweier optimaler Gleichgewichte, eines mit geringem und eines mit hohem Kapitalstock. Zu welchem der beiden Gleichgewichte es optimal ist zu konvergieren, ist abhängig von der ursprünglichen Kapitalausstattung und Investitionsrate. Ich berechne eine DNS-Kurve numerisch, und es gilt, daß, falls man in einem Punkt dieser Kurve startet, der Entscheidungsträger indifferent ist, ob er zu dem einem oder dem anderen Gleichgewicht konvergiert.

Heutzutage schreitet die Technologie rasch voran. Um wettbewerbsfähig zu bleiben, müssen Firmen auf neue technische Entwicklungen reagieren. Falls sie dies nicht tun, laufen sie Gefahr, daß ihre Produkte veraltet werden und die Kunden zur Konkurrenz abwandern. Ich bilde ein Modell zur optimalen dynamischen Technologieinvestition, in welchem das Basisniveau der allgemein eingesetzten Technologie konstant über die Zeit anwächst. Das Problem wird analysiert, indem ein zweidimensionales optimales Kontrollmodell entworfen wird. Es stellt sich heraus, daß im Zustandsraum eine DNS-Kurve ermittelt werden kann, welche verschiedene Langzeitinvestionspolitiken, wie keine Investition, konstant positive Investition oder eine zyklische Sequenz von Perioden mit keiner Investition und Perioden mit positiver Investition, trennt.

Im Modell der Kapitalakkumulation trennt eine DNS-Kurve zwei optimale Gleichgewichte, während im Modell der Technologieakkumulation eine DNS-Kurve ein optimales Gleichgewicht von einem Grenzzykel trennt. Speziell das Auftreten von Schwellenwertverhalten, das ein optimales Gleichgewicht und einen optimalen Grenzzyklus trennt, stellt eine wesentliche Neuerung in ökonomischen Modellen dar. Aber nicht nur die Existenz der DNS-Kurven sind ein besonderes Merkmal, sondern ihr Auftreten können - in beiden Modellen - intuitiv und ökonomisch interpretiert werden. Weitere besondere Merkmale sind das Auftreten von sub-kritischen Hopf-Bifokationen und Grenzpunkt-Bifokationen (wo ein stabiler und ein instabiler Grenzzykel zusammenfallen und verschwinden) im Modell der Technologieakkumulation.

Bei der Definition eines DNS-Punktes steht im Vordergrund, daß speziell Ökonomen verschiedene Langzeitergebnisse im Auge haben. Natürlich kann es auch passieren, daß verschiedene optimale Pfade zu demselben Langzeitergebnis führen; trotz eines eindeutigen Langzeitergebnis tritt ein Schwellenwertverhalten auf. Da ein DNS-Punkt diese Eigenschaft nicht wiederspiegeln kann, erweitere ich schlußendlich die Definition eines DNS-Punktes zu einer geringfügig allgemeineren Definition eines DB (Dechert-Brock)-Punktes, indem ich einen Vorschlag von Brock und Dechert, angeführt in einem nicht veröffentlichten Arbeitspapier aus dem Jahre 1983, aufgreife.

Ferner beschreibe ich kurz die verwendeten numerischen Algorithmen, die in der Berechnung der DNS-Kurven zur Anwendung gekommen sind. Die meisten verwendeten Algorithmen sind allgemein bekannt und deren Implementationen können mittels Kauf von Softwarepaketen wie MATHEMATICA oder MATLAB erworben werden. Einige der existierenden Algorithmen habe ich adaptiert, einige wenige neu entwickelt.

Summary

Let us assume that a single decision maker (central planner, management of a firm etc.) controls a deterministic dynamic system to optimize a measured output of this system. For a chosen control path the resulting initial value problem has a unique solution, but it may happen, that for given initial values different optimal controls lead to different long run outcomes. The indeterminacy arises here from the possibility to choose different optimal controls. In this work I deal with the subset of all states, starting from which such indeterminacies arise.

In a non-convex framework, outcomes fulfilling the necessary optimality conditions no longer have to be optimal. Further, it may happen that several long-run outcomes are optimal, and convergence to which outcome depends on the initial state. Due to continuity there has to be some threshold behavior, where from the one side of this barrier it is optimal to converge to one outcome whereas from the other side of this barrier it is optimal to converge to another outcome. The scope of this work is to find two-dimensional optimal control models with optimal steady states and limit cycles separated by thresholds. I got the directives to compute such thresholds numerically. Besides the pure existence the nature of the sets of thresholds has been of interest.

In economic literature, threshold levels are often called " Skiba thresholds " or " Skiba points ". On the basis of a statement by Brock and Malliaris and due to a personal intervention of Kazuo Nishimura, we (my working group) have reconsidered to denote threshold behavior by DNS (Dechert-Nishimura-Skiba). A DNS-point is a point in the state space, from which it is optimal to converge to at least two different attractors. In this work in the investigated two dimensional models, the DNS-sets are curves in the two dimensional state spaces.

I study a capital accumulation model in an optimal control theoretic framework, where the capital stock and the investment rate are modeled as state variables and the change in the investment rate as control. Adjustment costs are introduced for investment rate and its change. Moreover, I model network externalities by a convex segment in the revenue function, which implies the existence of two long-run optimal steady states, one with a low level and the other with a high level capital stock. It depends on the initial capital endowment and initial investment rate to which steady state it is optimal to converge. I numerically compute a DNS-curve, for which it holds that, when starting from a point on this curve, the decision-maker is indifferent between going to either one of these steady states.

Technology advances quickly these days. Therefore, in order to keep up with its economic environment a firm should respond to new technological developments. Otherwise, its products become old-fashioned and customers will tend to go to competitors. I establish the optimal technology investment decision within a dynamic model, in which the baseline technology level rises over time. The problem is analyzed by designing a two state optimal control model. It turns out that in the state space a DNS-curve can be determined that separates different long run outcomes, viz., zero investment, constant positive investment,

I would like to thank my parents who enabled me to succeed in my study. I would also like to thank Prof. Gustav Feichtinger, Prof. Richard Hartl (University of Vienna) and Prof. Peter Kort (Tilburg University) who supervised me in elaborating this work.

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Josef L Haunschmied

Contents

List of Figures

1.1 Indeterminacy

Costas Azariadis states in [3]: " But not all nicely behaved economies necessarily admit unique or stable stationary equilibria; uniqueness and/or stability in this sense are guaranteed if one restricts economic behavior with stronger assumptions like normality or gross substitutability in consumption, monotonicity or sufficient variability in factor reward, and the like. Some **non-uniqueness of equilibrium is unavoidable in dynamical economies** whose steady states are sinks and saddles. These states are associated with a well-defined equilibrium manifold which includes a continuum of convergent equilibrium sequences indexed on initial conditions. Economists typically dislike large equilibrium manifolds, and often refer to the limit points of such manifolds as indeterminate steady states. The Cauchy-Peano theorem and related results assure us that appropriately defined boundary-value problems have unique solutions and therefore that indeterminacy in economics arises in part from a shortage of initial conditions."

Assuming an economy with a deterministic structure for a given initial state no indeterminacy occurs, nevertheless if there exists a unique equilibrium or there are several. Different is the situation when the economy shows any randomness. Then in the case of multiple equilibria, even if the initial states are given, the final rest point of the economy is indeterminate. Azariades describes a model class as follows: " The economy described has a stationary deterministic structure of population, endowments, preferences, and production sets. Any randomness remaining in the solution comes from extraneous or psychological factors for which David Cass and Karl Shell [7] have invented the convenient name sunspots. The name is borrowed from Jevons [16] who used it for quite a different purpose he believed that solar activity influenced climate conditions and hence affected farm output." This model class, for instance used to explain financial markets, is based on an endogenous deterministic structure and exogenous random factors.

Again let us underlie a deterministic structure such as a dynamic system (see Appendix). But now a single decision maker (central planner, management of a firm etc.) controls this deterministic system to optimize a measured output of the system. For a chosen control path the resulting initial (or boundary) value problem has a unique solution, but it may happen, that for certain initial values different optimal controls lead to different long run outcomes. The indeterminacy arises here from the possibility to choose different optimal controls. In this work I deal with the subset of all states, starting from which such indeterminacies arise. (Numerical results show that in planar state models this indeterminacy set can be a one dimensional manifold - a curve.)

1.2 Threshold behavior

Generations of economists had esteemed convexity as the one and only truth. Non-increasing marginal returns and/or non-diminishing marginal costs had been seemed the only correct way to model economies. And this was certainly true, especially as long as the economies were resource based. With the uprise of technology based economies convex models became more and more inappropriate. For instance, network externality in information technology industry is one source of an economic reality showing increasing marginal returns.

However, in non-convex models necessary conditions are no longer sufficient for optimality and conventional marginal analysis fails. The analysis of general optimization problems is a lot more complicated than the analysis of convex optimization problems. This may be the reason that some economists with a fine tradition have dreaded to think on non-convex models. Nevertheless, a large number of economists started to investigate non-convex models. Doing that they faced among others the problem of multiple equilibria. In a non-convex framework equilibria fulfilling the necessary optimality conditions no longer have to be optimal. It has happened that several equilibria are optimal, and convergence to which equilibrium depends on the initial state. Due to continuity there has to be some threshold behavior, where from the one side of this barrier it is optimal to converge to one equilibrium whereas from the other side of this barrier it is optimal to converge to another equilibrium. It may happen that this threshold itself is optimal (an unstable optimal equilibrium, unstable limit cycle) or at this threshold there are more than one different ways behaving optimally.

Here, I want to quote two articles, one written in 1978, the other written in 1983, where the authors were one of the first researchers, who dealt with such thresholds. One article deals with a continuous, the other with a discrete optimal control problem exhibiting a threshold behavior.

In 1978, Skiba [25] investigated a one-sector problem of optimal growth theory over an infinite horizon, whereas he assumed a convex-concave production function. Qualitative analysis show that there are two optimal steady states: one with zero, another with a high level of capital endowment. In between there is a capital endowment level (Skiba called it " critical cut off point "), below which the marginal product of capital is so low, that accumulating capital takes too long to get to the point it pays off. Above this threshold level it is optimal to accumulate capital up to the steady state with a high level of capital endowment. With regard to this article in the literature similar threshold levels are often referred as "Skiba-tresholds " or " Skiba-points ". In 1981, Davidson and Harris [9] investigated a closely related problem of continuous time investment theory, where the systematically examined non-convexities. Assuming a convex-concave profit function for certain model configurations they proved the existence of Skiba-points.

In 1983, Dechert and Nishimura [10] characterized completely optimal growth paths in an aggregated (one-sector) model with convex-concave technology. Contrary to Skiba, they investigated a dicrete time model. Facilitating the decrete time Euler equation they proved that for certain parameter sets there exists a unique " critical level k_c " of the stock; below this critical level depleting the stock is optimal, whereas above this critical level it is optimal to accumulate capital up to a " Golden Rule " level $k* > k_c$.

1.2.1 Skiba thresholds

In my working group we are confident, that threshold behavior is practically more common than one may expect browsing existing literature on the occurrence of threshold behavior. Deissenberg et al. [11] state " ... multiple steady-states and thresholds (...) are fairly ubiquitous in dynamic economic models ". To support this statement they give a survey of examples of dynamic economic models with at least two optimal attractors, since different optimal attractors are a prerequisite to a threshold behavior. In the literature, different optimal attractors are often pointed out but the authors do not go into details concerning threshold behavior.

Although Skiba thresholds were introduced in more or less theoretical models in economics, they have by all means practical relevance, as the following two cases exemplify:

- In connection with treatment of illicit drug users a " cold turkey " stands for an abrupt reduction in drug consumption (think on the skin of a plucked turkey). Becker and Murphy [4] mention in their analysis of addiction that treating strongly addicted illicit drug users successfully requires a " cold turkey "; mathematically formulated a " cold turkey " treatment flags a discontinuous change in illicit drug consumption. Such a treatment is neccessary because there is a certain level of illicit drug consumption, beyond this threshold level an addicted person becomes inevitably strongly addicted in the long run. The only way to treat such a strongly addicted person successfully is to reduce instantaneously consumption of illicit drugs below this threshold level . Knowing this threshold level is substantial for a successful treatment. Gavrila et al. [14] confirmed this result based on a simple rational addiction model, where they showed the existence of a Skiba threshold separating cold turkey paths from trajectories to interior equilibria.

- In socio-economic literature, e.g. [17], " enforcement swamping " means that efforts or expenditures have a positive and number of offenders a negative effect on the effectiveness of law enforcement. Typical in modeling " enforcement swamping " is a ratio of a control and a state variable, like law enforcement efforts per offender. Transformation of the state variable leads to a model with a product of a control and a state variable. Such models, which are due to the product of a control and a state variable non-convex, often have multiple equlibria. For instance, Tragler et al. [28] use an enforcement swamping type model to investigate the last US-American cocaine epidemic in the eighties and the nineties. They facilitated this model to show that as long as a value of about 530.000 cocaine addicts is not exceeded, it would be rational to eradicate this epidemic in its early stages. Otherwise, an intertemporal trade-off between the social costs of cocaine use and law enforcement expenditures turns out to be optimal.

1.2.2 DNS-sets

As I mentioned above in the literature, threshold levels are often called " Skiba thresholds " or " Skiba points ". However, Brock and Malliaris write in [5] " To understand the Skiba - Dechert - Nishimura results ... ". On the basis of this fact and due to a personal intervention of Kazuo Nishimura, my working group has reconsidered to denote threshold behavior by DNS (Dechert-Nishimura-Skiba). It seems that in two-dimensional models this notation has been becoming popular (especially DNS-curve), whereas for thresholds in one dimensional models the notation " Skiba " persists. For the moment it is not clear whether Skiba or DNS, both or another notation will prevail. In this work I'll take the personally favored notation: A DNS (Dechert-Nishimura-Skiba)-point is a point in the state space, from which it is optimal to converge to at least two different attractors (or limit sets). " Limit sets " are sets that are composed of one or several attractors. Although the limit sets of the models which I scrutinize in this work consists of a single attractor (a steady state or a single limit cycle), it is reasonable to utilize the more general limit sets as one can see e.g. in [8]. Given limit sets - in order to verify the existence of DNS-points, it is important to know " from where it is possible " (region of convergence) and " from where it is optimal " (region of optimal convergence) to converge to which of these limit sets. A DNS-set separates different regions of optimal convergence.

1.3 Path-dependence

There are economists, who criticise that a DNS-set shows only a hairline case in the state space of no relevant economic meaning, because starting away from this set the final outcome is determined. Besides above arguments in treatment of illicit drug use, where the exact knowledge of the Skiba-point is vital for successful treatment I add here another argument for computing (knowing) DNS-sets.

Throughout the investigated model the evolution of the system is ergodic - ultimate outcomes are built in right from the start on the bases of initial state values, (profit) maximization, system dynamics and constraints. The model itself is deterministic and its long run outcomes are a priori well-defined. As in [1] W. Brian Arthur states: "It reduces history to the status of mere carrier - the deliverer of the inevitable."

Now, to make the models more realistic, let us add some unpredictable " small events" (cf. sunspots) outside the investigated structure. As far as the (initial) states of the model are far away from the DNS-curve such random small events do not have an unpredictable impact on the long run outcome - the evolution of the system remains ergodic.

However, when the (initial) states of the system are in the vicinity of a DNS-set such random small events can shift the system to the other side of the DNS-set resulting in a different long run outcome as the underlying deterministic model is predicting. Starting in the neighborhood of a DNS-set the evolution of the optimally controlled system is non-ergodic, the process becomes path dependent. Small indivisibilities, or chance meetings can become magnified by

" the wrong dynamics" to switch the system to a different outcome as predicted. As in [1] W. Brian Arthur states: "History becomes all-important." Knowing the DNS-set enables to give reasons for up to now inexplainable phenomenons, like the wondrous economic development of South Korea in comparison to some African countries, although South Korea started in the fifties from a weaker economic position (cf. Lucas' " Making a miracle " [21]).

1.4 Exploited Models

At the beginning of this work threshold behavior separating different optimal limit sets has been hardly applied to models, which were higher than one dimensional; little was known on threshold behavior in higher dimensions. Existence of two different optimal limit sets suffices for threshold behavior, because assuming there are at least two optimal limit sets there has to exist " some " set separating these limit sets. The scope of this work was to find two dimensional optimal control models with at least two steady states separated by thresholds. As it was supposed that it is a very tough job to compute a set of thresholds separating different optimal limit sets analytically, I got the directives to compute such thresholds numerically. Besides the pure existence the nature of the sets of thresholds has been of interest. In this work in the investigated two dimensional models, the sets of thresholds are one dimensional manifolds (curves) in the two dimensional state space.

In Chapter 2 I study a capital accumulation model in an optimal control theoretic framework, where the capital stock and the investment rate are modeled as state variables and the change in the investment rate as control. Adjustment costs are introduced for investment rate and its change. Moreover, I model network externalities by a convex segment in the revenue function, which implies the existence of two long-run optimal steady states, one with a low level and the other with a high level capital stock. It depends on the initial capital endowment and initial investment rate to which steady state it is optimal to converge. I numerically compute a DNS-curve, for which it holds that, when starting from a point on this curve, the decision-maker is indifferent between going to either one of these steady states. The negative slope of the DNS-curve indicates that there is a trade-off between the initial capital endowment and initial investment rate.

Technology advances quickly these days. Therefore, in order to keep up with its economic environment a firm should respond to new technological developments. Otherwise, its products become old-fashioned and customers will tend to go to competitors. In the Chapter 3 I establish the optimal technology investment decision within a dynamic model, in which the baseline technology level rises over time. The problem is analyzed by designing a two state optimal control model. It turns out that in the state space a DNS-curve can be determined that separates different long run outcomes, viz., zero investment, constant positive investment, or a cyclical sequence of zero and positive investment. For example, there is one scenario in which holds that on one side of the DNS-curve a steady state and a stable limit cycle coexist where the regions of optimal convergence are separated by an unstable limit cycle lying in the interior of the stable one.

In Chapter 2 a DNS-curve separates two limit sets, which are steady states, whereas in Chapter 3 a DNS-curve separates a steady state and a limit cycle. In both models, the existence of DNS-curves are not only the features of the investigated models, but also their occurrence and shapes can be suggestively and economically interpreted. Other interesting features of Chapter 3 are a sub-critical Hopf-bifurcation and a limit point bifurcation, where a stable and an unstable limit cycle coincide and disappear.

In case of multiple optimal long run outcomes, obviously economists care about under which circumstances, one or the other long run outcome will finally be chosen and where the threshold is. Different long run outcomes were the leading point, when Skiba uses " cut-off point ", and other authors redefine it as a Skiba-point. In this sense I formulate the definition of a DNS-set in the Appendix of this chapter.

Economists prefer a unique long run outcome, but it is not all-important, if there are one or several optimal paths leading to this long run outcome. Of course, it happens that different optimal paths lead to the same long run outcome; there exists threshold behavior but the long run outcome is unique. In Chapter 4 I reconceive the definition of a DNS-set and I seize a suggestion of Brock and Dechert given in an unpublished working paper written in 1983. Maybe Brock and Dechert were discouraged by the abstract, technical characterization of a threshold behavior, so that they finally decided not to publish. None the less their characterization of threshold behavior generalize my definition of a DNS-set and it applies to threshold behavior paired with a unique long run outcome.

Additionally, in Chapter 4 I describe briefly the used numerical algorithms in order to compute the DNS-curves described in Chapter 2 and 3. Most of the algorithms are pretty well-known and implementations can be purchased by packages like MATHEMATICA or MATLAB. Some existing algorithms I have adapted and a few algorithms I have developed.

1.5 Appendix

Systems

Sinha [26] considers a system as " a collection of objects arranged in an ordered form, which is, in some sense, purpose or goal directed ". Mays and Tung [22] characterize a system by

- a system boundary which is a rule that determines whether an element is to be considered as a part of the system or environment,
- a statement of input and output interactions with the environment, and
- statements of interrelationships between the system elements, inputs and outputs.

Throughout this work I consider a dynamic, deterministic, continuous-time and lumped parameter system; the system structure is given explicit representation as a vector $x = (x_1, \ldots, x_n)$. In this n dimensional continuous-time deterministic state variable model the n state variables $x_i(t)$ are functions of time t.

Optimal Control Problems

Athans and Falb [2] specify the essential elements of the optimal control problem as

- a system to be controlled,
- a desired output of the system,
- a set of admissible inputs or controls, and
- a performance index (or cost functional) which measures the effectiveness of a given control action.

There are several ways to formulate an optimal control model, like Bolza, Mayer or Lagrange type. In a mathematical point of view, these different types are equivalent (see e.g. [13]). A simplified model, which I use throughout this work, of a mathematical continuous-time optimal control model is to optimize the objective functional

$$J(u) = \max_{u \in U} \int_0^T e^{-rt} F(x(t), u(t))\, dt + \phi(x(T), T) \qquad (1.1)$$

subject to the system dynamics

$$\dot{x} = f(x(t), u(t)), \quad x(0) = x_0. \qquad (1.2)$$

where $u = (u_1(t), \ldots u_m(t))$ is the control variable and $U \subset \mathbb{R}^n$ is closed but not necessarily bounded. In what follows OCM refers to this optimal control model. In case of $T = \infty$, one allude to an infinite, otherwise to a finite planning horizon. The integrand F and the system dynamics f are assumed to be C^1 in x and continuous in u. Similarly, the scrap value function ϕ is continuously differentiable in x and T; it disappears in case of an infinite planning horizon.

Definition 1 *Michel, [23]) defines a trajectory $(x(t), u(t)), 0 < t < T$ to be admissible if*

1. *$x(t)$ is a solution of the system dynamics (1.2)*
2. *with a piecewise continous control $u(t) \in U$ on $0 < t < T$, and*
3. *if the integral in (1.1) converges.*

Notation 2 *A control function $u(t), t \in [0, T]$ is admissible of OCM, if there exists a solution $x(t)$ of the system dynamics with control $u(t)$, whereupon $(x(t), u(t))$is admissible. I denote $x(t)$ to be corresponding to $u(t)$.*

Definition 3 *The (optimal) value of the problem OCM is*

$$J^* = \sup_{u \text{ admissible}} J(u).$$

Any admissible control $u^*(t)$ fulfilling

$$J^* = J(u^*)$$

is called an optimal control (path or trajectory), and the corresponding $x^*(t)$ an optimal (state) path or trajectory. An optimal solution is a pair (x^*, u^*) of an optimal state and an optimal control path.

Pontryagin's Maximum Principle

Similar to Langrange or Karusch-Kuhn-Tucker theory in static optimization (e.g. [15], [20]), a solution of a dynamic optimization problem has to fulfill a set of necessary conditions. The extensive literature on these necessary conditions is summarized under the traditional expression (Pontryagin's) Maximum Principle (see e.g. [12], [6], [13], [18],[19], or the original article [24]).

Before turning to the Maximum Principle I define the (current-value) Hamiltonian Function of the optimal control problem OCM:

$$H(x, u, \lambda_0, \lambda) := \lambda_0 F(x, u) + \sum_{i=1}^{n} \lambda_i f_i(x, u),$$

or in a more compact form

$$H(x, u, \lambda_0, \lambda) := \lambda_0 F(x, u) + \lambda' f(x, u).$$

The n-dimensional vector $\lambda' = (\lambda_1, \ldots, \lambda_n)$ is called co-state (or adjoint) vector, λ_i co-state (or adjoint) variable. In case of infinite planning horizon, the Maximum Principle applied on problem OCM can be written as follows:

Theorem 4 *(Michel, [23]) If u^* is an optimal control and x^* the corresponding optimal (state) path of the optimal control problem (1.1) and (1.2), then there exists a constant $\lambda_0 > 0$ and a continuous and piecewise continuously differentiable adjoint vector $\lambda(t)$, so that $(\lambda_0, \lambda(t)) \neq 0 \; \forall t > 0$,*

$$\dot{\lambda}(t) = r\lambda(t) - H_x(x^*(t), u^*(t), \lambda_0, \lambda(t)), \tag{1.3}$$

and for every t where $u^*(t)$ is continuous

$$u^*(t) \quad \in \quad Argmax_{u \in U}\, H\left(x^*(t), u, \lambda_0, \lambda(t)\right) \,. \tag{1.4}$$

In addition,

$$\lim_{t \to \infty} e^{-r\,t}\, H\left(x^*(t), u^*(t), \lambda_0, \lambda(t)\right) \quad = \quad 0 \tag{1.5}$$

$$e^{-r\,t}\, H\left(x^*(t), u^*(t), \lambda_0, \lambda(t)\right) = \quad r\,\lambda_0 \int_t^{\infty} e^{-r\,\tau}\, F\left(x^*(\tau), u^*(\tau)\right)\, d\tau. \tag{1.6}$$

Let us call (1.4) the " effective maximum principle "; in general, the dynamic equation (1.3) are known as the co-state dynamics and equation (1.5) as transversality condition. Note, condition (1.6) makes it quite simply to compute the optimal (objective function) value, if $\lambda_0 > 0$:

Corollary 5 *Provided that $\lambda_0 > 0$ and the co-state variables $\lambda(t)$for $t = 0$ are known, the optimal value of OCM is:*

$$J\left(u^*\right) \quad = \quad \frac{1}{r\,\lambda_0}\, H\left(x^*(0), u^*(0), \lambda_0, \lambda(0)\right). \tag{1.7}$$

Hence, calculating the values λ_0 and A(0) *is the job; but generally a difficult one! If* AO > 0 *is ensured, one can omit λ_0 by normalizing the Hamiltonian with $\frac{1}{\lambda_0}$ (and consequently normalizing λ) .*

In case of an open planning horizon and $\phi = 0.$, Theorem 4 remains valid (but ∞ has to be replaced by T*, the optimal planning horizon).

Finally, the following theorem does not surprise:

Theorem 6 *// the maximized Hamiltonian*

$$\max_{u \in U} H(x, u, \lambda_0, \mathrm{A})$$

is for any given (Ao, $\lambda(t)$) *concave in x, then the conditions in Theorem 4 are sufficient.*

Limit sets

Influenced by Strogatz' [27] definition of an attractor of a dynamic system I define:

Definition 7 *An attractor of OCM with an open planning horizon is a closed subset A of the state-costate space with the following properties:*

1. *A is an invariant set: any trajectory $(x(t), \lambda(t))$ that starts in A and fulfills the first order necessary conditions of Prontryagin's maximum principle (1.4), (1.3) and (1.5) stays in A for $0 < t < T^*$,*

2. *A attracts an open set of initial conditions: there is an open set $\mathcal{N}$ containing A such that if* $(x(0), \lambda(0))$ is *element of* $\mathcal{N}$*,then the distance form* $(x(t), \lambda(t))$ *to A tends to zero as* $t \rightarrow T^*$. *The largest such* $\mathcal{N}$ *is called the basin of attraction of A.*

3. *A is minimal: there is no proper subset of A that satisfies conditions 1 and 2.*

Definition 8 */ define an attractor A of OCM optimal, if any trajectory that starts in A is an optimal solution.*

Definition 9 */ define a limit set of OCM to be a unification of attractors of OCM. An optimal limit set consists only of optimal attractors.*

Definition 10 */ define the region of convergence of a limit set L of OCM to be the unification of the projection into the state space of the basins of attraction of all attractors L consists of.*

Whereas the basin of attraction is a subset of the state-costate space, the region of convergence is a subset of the state space.

Definition 11 */ define the optimal region of convergence of an optimal limit set L of OCM to be the largest possible subset of the region of convergence of L, where there exists an optimal solution* $(x^*(t), u^*(t))$ *of OCM with initial values in this region (i.e.* x*(0) *is element of this region.)*

Definition 12 *A DNS (Dechert-Nishimura-Skiba)-pointis a point in the state space, which belongs to the regions of optimal convergence of at least two (pairwise) disjoint optimal limit sets. The* DNS-set is *the set of all DNS-points.*

Value Function

Above mentioned OCM depends exogenously on the initial values x_0; pointing to this fact I denote $J(u; x_0)$, and I define:

Definition 13 *The optimal value function of OCM is defined as*

$$V(x_0) := \sup_{u \text{ admissible}} J(u; x_0).$$

In this work I use value function in a more general way. Consider any limit set of OCM; if x_0 is an element of the region of convergence of this limit set, I am interested in the value(s) of the objective functional of OCM, when I take admissible controls, which ensure that the system converges to this limit set. Being more accurate the supremum (maximum) of these values quickens my interest.

Definition 14 *For any initial value* x_0, *which is an element of the region of convergence of* L, *the value function of the limit set L is defined as*

$$V_L(x_0) := \sup_{u \text{ is admissible for } L} J(u; x_0),$$

By u is admissible for L I mean, that u is an admissible control, which ensures that the corresponding $(x(t), \lambda(t))$ *converges for t* $\rightarrow T^*$ *to the limit set* L. *If the initial value* x_0 *is not element of the region of convergence of* L, $V_L(x_0)$ *equals per definitionem* $-\infty$.

Theorem 15 *Assume that OCM has a finite number of limit sets* L_l, $l = 1, \ldots, \ell$, *and assume that each optimal solution* (x^*, u^*) *converges for t* $\rightarrow$ T *to one of these limit sets. Then it holds:*

$$V(x_0) = \max_{l=1,\ldots,\ell} V_{L_l}(x_0).$$

Proof. Obvious. ■

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Chapter 2

Capital Accumulation Model

2.1 Introduction

As in Chapter 1 mentioned, most of the contributions with threshold behavior consider a dynamic optimization problem with a one-dimensional structure. Intuition suggests that extending this feature to the class of optimal control models with two state variables will lead to the occurrence of a set of DNS–points. In the literature contributions that deal with this topic are scarce. Brock and Dechert [1] prove the existence of a DNS-set, but the exact location of it is not determined. In this chapter, I determine numerically a DNS-set in a two-dimensional capital accumulation model.

In the literature, the capital accumulation framework has been extensively analyzed assuming a constant or decreasing returns to scale technology and adjustment costs of investment; representative I refer to Eisner's, Gould's and Lucas' articles, [7, 9, 13]. One specific feature of the model of this chapter is that - besides the prevalent adjustment costs associated with investment - additional costs are included for changes in the investment rate.

Another specific feature in the used model is that the formulation admits the existence of network externalities. By a network externality it is meant that the value of a good increases with the number of users, cf. [6]. An example would be that it is beneficial to the users of a certain software package, if this package is used by many other people. Similarly, for the user of some GSM network the value increases with the number of users because of the significantly reduced rates within the network. In the model this is reflected by an inverse demand function, which increases for intermediate quantities produced. This implies that the firm's revenue function is convex in a segment of intermediate capital stock values, whereas the firm's revenue function is concave for small and large capital stock values outside this segment. In a similar manner Davidson and Harris [3] as well as Dechert [4] use a revenue function, which contains a convex segment. In contrast to this, Barucci [2] studies the classical framework with the exception that the revenue function is strictly convex throughout.

Given certain functional forms and parameter sets the introduced model features two optimal steady states. It depends on the firm's initial capital endowment and investment rate to which steady state it is optimal to converge to. In the present paper, the numerically computed DNS-set turns out to be a one-dimensional curve.

The organization of this chapter is as follows. The next section specifies the model. In Section 2.3, the model is analyzed and economic intuitions of the mathematical results are provided in Section 2.4.

2.2 Model Formulation

Consider a firm that needs capital goods to produce commodities, which are sold on an output market. The more capital goods the firm owns the more commodities can be sold and thus the more revenue is obtained. Of course, in case that the firm has some market power the output price decreases with the number of goods that are sold, which implies that decreasing returns to scale will be present, especially if the capital stock is sufficiently large. On the other hand scale economies can cause increasing returns to scale. To analyze the effect of this on optimal firm behavior it is assumed that there exists an interval of capital stock values for which there are increasing returns to scale.

Denoting revenue by R and capital stock by K[1], it is imposed that $R(K)$ is a positive, twice continuously differentiable, and increasing function with one convex segment for intermediate values of the capital stock; cf. [3] - Figure 2b. This convex segment arises due to the fact that for these values of the capital stock the firm's production technology exhibits increasing returns to scale.

An alternative explanation for the convex segment in the revenue function is that the inverse demand function is locally increasing, which arises due to a network externality. A network externality implies that the value of a good increases with the number of users of this good. An example would be that it is beneficial to the users of a certain word processing program, if this package is used by many other people, because, e.g., it makes it easier to exchange files. Similarly, for the user of some cellular phone network the value increases with the number of users because of the significantly reduced rates within the network. In the model this is reflected by an inverse demand function which decreases for small and large quantities produced, but has an increasing segment for intermediate quantities produced. A relevant specification is

$$P(Q) = \frac{\gamma_1}{\sqrt{Q}} - \frac{\gamma_3}{1 + \gamma_2 Q^4}, \tag{2.1}$$

where P denotes the price and Q is the quantity produced. Figure 2.1 depicts the graph of this inverse demand function.
When a linear production function is assumed, $Q(K) — \beta K$, then the firm's revenue function $R(K) = P(Q(K))Q(K)$ equals

$$R(K) = k_1 \sqrt{K} - k_3 \frac{K}{1 + k_2 K^4}, \tag{2.2}$$

with $k_1 = \gamma_1 \sqrt{\beta}, k_2 = \gamma_2 \beta^4$ and $k_3 — \gamma_3 \beta$. Figure 2.2 depicts the graph of the firm's revenue function. In the segment between $K \approx 1.2$ and $K \sim 3.3$, the network externality leads to a convex segment of the revenue function.

The firm increases its capital stock by investing, where the investment rate is denoted by I. Besides the purchase costs, the cost of investments $c(I)$ also consists of adjustment costs which are assumed to be strictly convex, i.e. $c(0) = 0$ and $c'' > 0$. For the numerical analysis below I adopt the quadratic specification imposed by, e.g., Barucci [2]:

$$c(I) = c_1 I + \frac{c_2}{2} I^2. \tag{2.3}$$

As usual, capital stock increases with investments and decreases with depreciation. Assuming a constant depreciation rate $6 > 0$, the following state

[1] Since the model is dynamic, all variables are functions of time, i.e. $K = K(t)$. For notational convenience, in what follows this dependence of variables on time is not explicitly denoted.

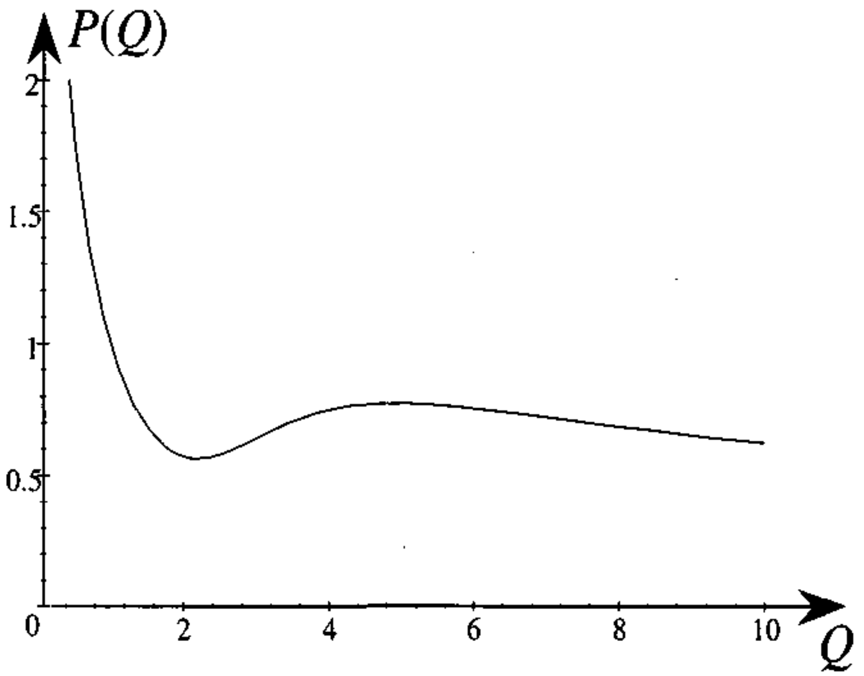

Figure 2.1: The inverse demand function $P(Q)$ for 71 = 2, $\gamma_2 = \frac{3}{256}$, and 73 = 1. In the segment between $Q \approx 2.2$ and $Q \approx 4.9$ network externalities lead to an increasing shape.

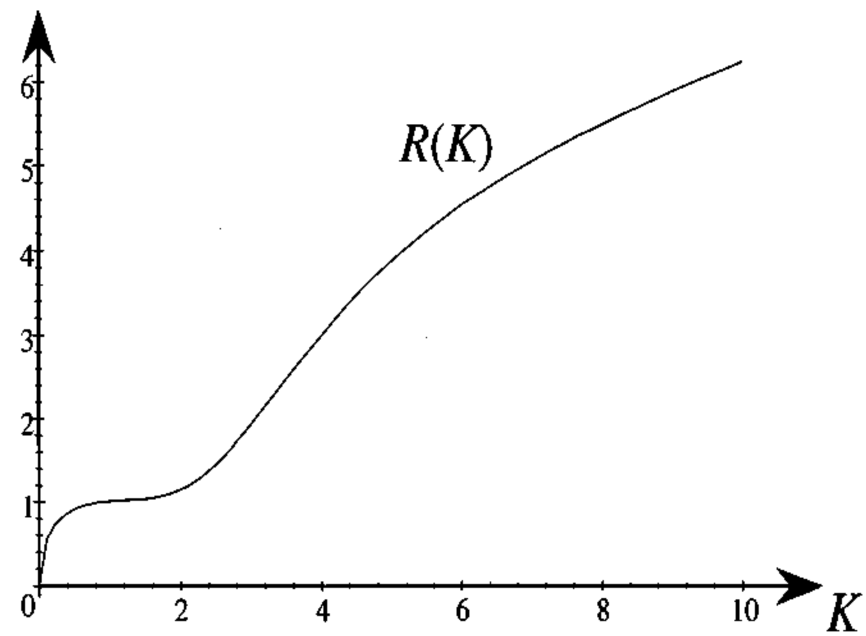

Figure 2.2: The firm's revenue function with the inverse demand function *P (Q)* from Figure 1 and ß = 1 (i.e. k_1 — 2, $k_2 = \frac{3}{256}$, and $k_3 = 1$).

equation for capital stock arises:

$$\dot{K} = I - \delta K. \tag{2.4}$$

The concept of investment adjustment costs is refined here by explicitly modeling that changes in the investment rate are costly, see also [12]. Such costs can arise in cases, where an organization is used to a certain rate of investment, so that it has to reorganize when changes in this investment rate occur. Representing the change of investment by v, these costs equal $g(v)$. Assuming a quadratic cost function is quite common in the literature, compare inter alia [15, 16]. Even in the case of non-symmetric, piecewise linear costs with a kink at the level zero a quadratic approximation is reasonable, see [11]. Here I also adopt the quadratic form. Admittedly it simplifies the calculations, but it is not essential:

$$g(v) = \frac{\alpha}{2} v^2,$$

where

$$\dot{I} = v. \tag{2.5}$$

In order to include (2.5) in the optimization problem, the investment I will be modeled as a state variable.

The firm's objective is to maximize the discounted cash flow stream over an infinite planning horizon. Collecting the revenue function and both types of adjustment costs described above, and assuming a constant discount rate ρ, I arrive at the following expression for the objective functional:

$$\max_v \int_0^\infty e^{-\rho t} \left[R(K) - c(I) - \frac{\alpha}{2} v^2 \right] dt. \tag{2.6}$$

The optimal control model now consists of the expressions (2.4)-(2.6). It has two state variables, K and I, and one control variable, v. Summarizing, the following model is obtained:

$$\max_v \quad \int_0^\infty e^{-\rho t} \left[R(K) - c(I) - \frac{\alpha}{2} v^2 \right] dt,$$

$$\text{s.t.} \quad \dot{K} = I - \delta K,$$
$$\dot{I} = v.$$

When $\alpha = 0$, the control v is costless and adjustment of I can be done instantaneously. Thus, v can be deleted and the problem becomes an optimal control model with state variable K and control variable I; the model reduces to the classical capital accumulation models. In case of linear or concave $R(K)$ and convex $c(I)$ the basic framework arises, which inter alia Gould and Lucas [13, 9] analyzed extensively. Dechert [4] proved the existence of a Skiba-point for this reduced model using a convex-concave $R(K)$, while Davidson and Harris [3] studied the implications of convex segments in $R(K)$ and concave segments in $c(I)$. Barucci [2] considers convex quadratic functions for both $R(K)$ and $c(I)$.

2.3 Analysis of the Model

This section consists of four subsections. First, in Subsection 2.3.1 the optimality conditions are listed, after which in Subsection 2.3.2 the stability behavior around the steady states is studied. In Subsection 2.3.3 the stable manifolds are analyzed, while in Subsection 2.3.4 the DNS-curve is determined.

2.3.1 Optimality Conditions

To apply Pontryagin's Maximum Principle I start out by stating the current value Hamiltonian:

$$H = R(K) - c(I) - \frac{\alpha}{2}v^2 + \lambda_1 (I - 6K) + \lambda_2 v.$$

Maximization of the Hamiltonian with respect to the control variable v gives:

$$v = \frac{1}{a}\lambda_2. \tag{2.7}$$

Further application of Pontryagin's Maximum Principle and taking into account (2.4) and (2.5), lead to the following dynamic system:

$$\begin{aligned} \dot{K} &= I - \delta K, \\ \dot{I} &= v = \tfrac{1}{\alpha}\lambda_2, \\ \dot{\lambda}_1 &= (p + \delta)\lambda_1 - R'(K), \\ \ddot{A}2 &= \rho\lambda_2 - \lambda_1 + c'(I). \end{aligned} \tag{2.8}$$

2.3.2 Stability Analysis

From $\dot{K}$ — $\dot{I}$ = 0 it is straightforward to see why $\lambda_2 = 0$ and J = $8K$ is required for a steady state. Additional to these conditions the equations $\dot{\lambda}_1 = \dot{\lambda}_2 = 0$ imply that steady state values also have to satisfy

$$\lambda_1 = c'(\delta K) = \frac{1}{(\rho + \delta)} R'(K), \tag{2.9}$$

which is illustrated in Figure 2.3.

Here it can be seen that in the case of a convex cost function $c(I)$ at most one steady state exists, if the revenue function $R(K)$ is concave, as it is the case in the basic capital accumulation models. However, due to the convex segment for intermediate values of the capital stock, the considered revenue function has a concave-convex-concave shape. Given the functional forms in (2.2) and (2.3), it is easily seen from (2.9) that depending upon the choice of the parameters for the functions $R(K)$ and $c(I)$ there are at most three steady states (see also Figure 2.3).

The investigation continuous by the case of three different steady states $\mathcal{K}_i = (K_i, I_i, v_i, \lambda_{1i}, \lambda_{2i}), i = 1, 2, 3$, where $\mathcal{K}_1$ corresponds to the steady state with the smallest capital stock, and the steady state with the largest capital stock value is denoted by $\mathcal{K}_3$.

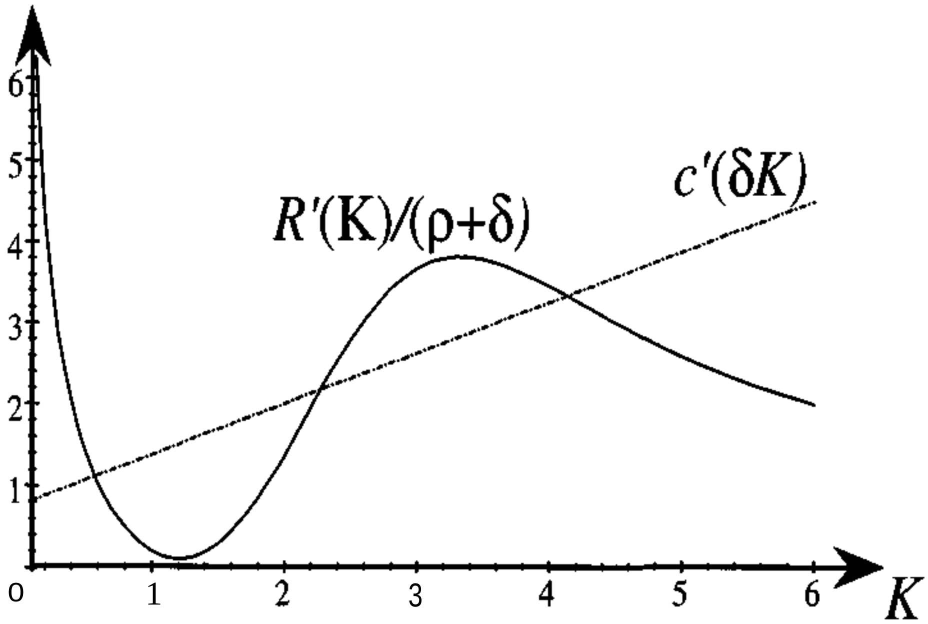

Figure 2.3: Three steady states satisfy $R'(K)/(\rho + 6) = c'(\delta K)$. The points of intersection of $R'(K)/(\rho + \delta)$ (solid) and $c'(\delta K)$ (dotted) are the steady states.

Proposition 16 *In the case of the existence of three steady states, the stability analysis gives the following results:*

- *The stable invariant manifold of the steady state $\mathcal{K}_2$ is one dimensional.*
- *The stable invariant manifolds of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$ are two dimensional.*
- *The steady state $\mathcal{K}_i$, $i = 1,3$, is a saddle point with transient oscillations, if*

$$[\alpha\,\delta\,(\rho+\delta) - c_2]^2 + 4R''(K_i)\alpha < 0,$$

otherwise it is a saddle point with real eigenvalues.

Proof. Analysis of the stability behavior of the dynamic system around the steady states follows the lines of [5], p. 96. To do this, it is necessary to compute the Jacobian (for reasons of readability I do not yet substitute $R(K)$ given in specification (2.2)).

$$\det J(K) = \begin{vmatrix} -S & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} \\ -R''(K) & 0 & P+\delta & 0 \\ 0 & c_2 & -1 & p \end{vmatrix} = \frac{1}{a}\left[\delta\,(\rho+\delta)\,c_2 - R''(K)\right].$$

It is easily seen that the Jacobian is positive in the first and the third steady state, $\mathcal{K}_1$ and $\mathcal{K}_3$, while it is negative in the second steady state, $\mathcal{K}_2$. This implies that $\mathcal{K}_2$ is unstable, except for a one-dimensional manifold, cf. Feichtinger et al. [8] - Figure 1.

To determine the signs of the eigenvalues, besides the Jacobian, I have to compute a specific quantity K as defined by [5], p. 96:

$$K = \begin{vmatrix} -\delta & 0 \\ -R''(K) & \rho+\delta \end{vmatrix} + \begin{vmatrix} 0 & \frac{1}{\alpha} \\ c_2 & p \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = -5(\rho + 8) - \frac{c_2}{a} < 0.$$

Given the fact that the Jacobian is positive and K is negative in $\mathcal{K}_1$ and $\mathcal{K}_3$, it is obvious that both steady states are saddle points. Now, when

$$\frac{\kappa^2}{4} - \det J(K) = \frac{1}{4\alpha^2}\left[(\delta(\rho+\delta)\alpha - c_2)^2 + 4R''(K)\alpha\right] > 0,$$

then the steady state is a saddle point with real eigenvalues. Otherwise a saddle point convergence occurs with transient (damped) oscillations. •

The stable invariant manifold of $\mathcal{K}_2$ is one-dimensional (it would be practically impossible to have initial states in $\mathcal{K}_2's$ region of convergence, which is a sparse subset of the state plane (curve in the state plane with Lebesgue measure zero)). This result leads us to the conclusion that only $\mathcal{K}_1$ and $\mathcal{K}_3$ are the candidates for steady states with regions of optimal convergence, which have a positive measure.

The aim is to numerically determine the regions of optimal convergence of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$, respectively, and the DNS-set, which separates these regions. To do so the parameter values are fixed as follows (note that it is not really necessary to choose eight different parameter values; see Appendix (a)):

$$\rho = \frac{1}{25};\ k_1 = 2;\ k_2 = \frac{3}{250};\ k_3 = 1;\ c_1 = \frac{3}{4};\ c_2 = \frac{5}{2};\ \alpha = 12;\ \delta = \frac{1}{4}. \tag{2.10}$$

	K	I	v	λ_1	λ_2
$\mathcal{K}_1$	0.6	0.1	0.0	1.1	0.0
$\mathcal{K}_2$	2.3	0.6	0.0	2.2	0.0
$\mathcal{K}_3$	4.1	1.0	0.0	3.3	0.0

Table 2.1: Numerical values of the steady states

The numerical values of the steady states are listed in Table 2.1. For the following considerations it will suffice to retain the K-I-v - values of the first and of the third steady state, $\mathcal{K}_1 = \{0.58, 0.15, 0.0\}$ and $\mathcal{K}_3 = \{4.1, 1.0, 0.0\}$. These steady states are saddle points with transient oscillations.

2.3.3 Stable Invariant Manifolds

Figure 2.4 and 2.5 visualize the stable invariant manifolds of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$. More precisely, these figures show projections of higher dimensional surfaces into the three dimensional state, namely the state-control space (K, I, v). In the following discussion of the figures I omit the phrase "...projection of...into the (K, I, v)space".

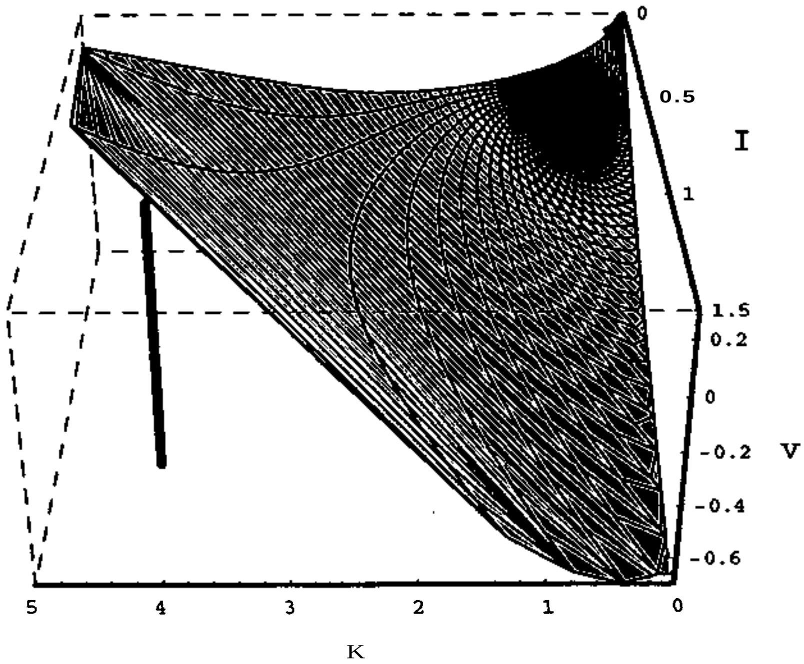

Figure 2.4: Stable invariant manifold of the steady state $\mathcal{K}_1$.

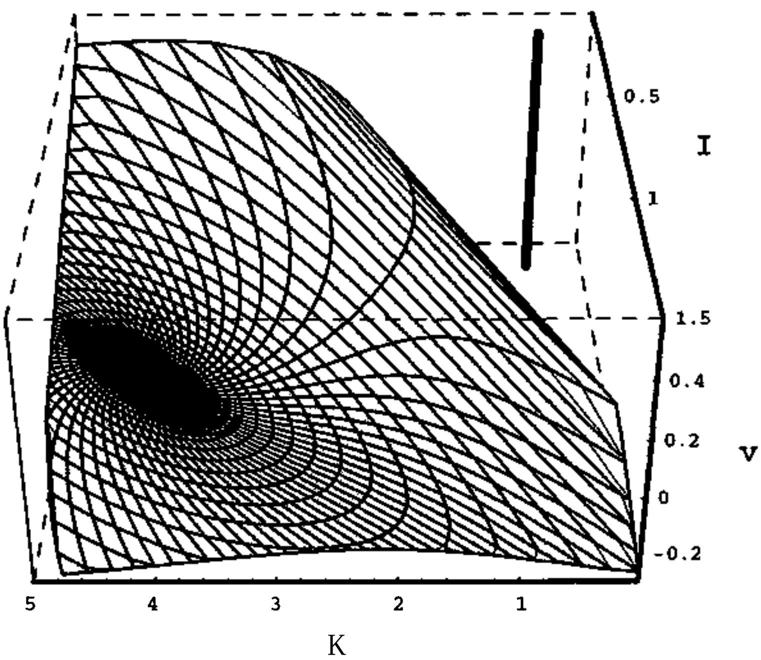

Figure 2.5: Stable invariant manifold of the steady state $\mathcal{K}_3$.

I determine these surfaces by computing the eigenvectors of the linearization of the canonical system in the steady states. Proposition 1 states that steady states $\mathcal{K}_1$ and $\mathcal{K}_3$ have exactly one conjugate complex pair of eigenvalues with negative real parts, respectively. I take the real and imaginary part of one of the eigenvectors, which correspond to eigenvalues with negative real parts. Which eigenvector is used, is not significant, because the eigenvectors are conjugate complex. These two vectors (real and imaginary part of one of the eigenvectors) define a linear approximation of the stable invariant manifold locally at the steady state, see e.g. [10]. With the aid of these two vectors I define an ellipse, which has the steady state as center and which is situated in an ε-ball around the steady state; to get this ellipse normalize these two vectors and multiply one by $\varepsilon\ cos\,\phi$ and the other by $\varepsilon\ sin\,\phi$, ϕ G $[0, 2\pi]$.

It is evident that this ellipse is situated in the linear approximation of the stable invariant manifold. To compute an approximation of the stable invariant manifold I chose starting points on this ellipse and moved back in time (i.e. by multiplying the r.h.s. of the state equations by -1) The quality of this approximation depends on the value of e and on the smoothness of the manifold in the the vicinity of the steady state.

Figure 2.4 illustrates the shape of the stable manifold of the steady state $\mathcal{K}_1$. The depicted trajectories converge to the steady state $\mathcal{K}_1$ located in the upper-right corner of the figure. The lines between the trajectories are drawn to form polygons so as to display the shape of the surface. I outline the position of the steady state $\mathcal{K}_3$ by a bold face bar (instead of a dot or circle), which crosses orthogonally the state plane at the steady state values of $\mathcal{K}_3$. I decided to choose this bold face bar to visualize the position of the steady state $\mathcal{K}_3$ relative to the stable manifold of the steady state $\mathcal{K}_1$. Now, it is easily seen that the stable manifold does not include steady state $\mathcal{K}_3$, since the stable manifold folds back far away before crossing the bold face bar.

In Figure 2.5 the shape of the stable manifold of the steady state $\mathcal{K}_3$ is plotted. The depicted trajectories converge to this steady state, which is located in the lower-right corner of the figure. I outline the position of the other steady state, now $\mathcal{K}_1$, by a bold face bar, which crosses orthogonally the state plane at the steady state values of $\mathcal{K}_1$. Here again, the stable manifold does not include the other steady state.

2.3.4 The DNS-curve

Emanating from the initial states I need to find a trajectory that converges to a steady state. This is equivalent to complementing the given initial state values with values for the co-state and/or control variables in a way that the resulting initial point lies on the stable manifold of that steady state, where the trajectory should converges to. Then the value of the objective functional corresponding to this trajectory can be determined by dividing the Hamiltonian by the discount factor, see Appendix of Chapter 1 or [14].

So it is important to compute the stable invariant manifolds first. In Figures 2.4 and 2.5, it can be seen that both stable invariant manifolds corresponding to $\mathcal{K}_1$ and $\mathcal{K}_3$ do not include the other long run equilibrium points $\mathcal{K}_3$ and $\mathcal{K}_1$, respectively. Hence, in this particular example it can never *be* optimal to start from one saddle point and converge to the other. Under the assumption that optimal trajectories will converge to one of the two steady states, it is obvious that there must be a division of the state space into two different regions of optimal convergence of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$.

Next, I compute the set in the state space that separates these regions of optimal convergence. Right on this DNS-set the firm is indifferent between converging to either one of the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$, since the values of the objective functional of the two resulting trajectories are exactly the same. Here I solely rely on numerical evidence that this set is actually a curve, a DNS-curve. It is left to be proven that this DNS-set is in fact a one-dimensional manifold.

Apparently this "indifference" curve is situated in an area of the state space, where it is possible (for a given set of initial state values) to choose a trajectory starting with these initial states and converging to the steady state $\mathcal{K}_1$ and to choose another trajectory starting with these initial state and converging to the steady state $\mathcal{K}_3$.

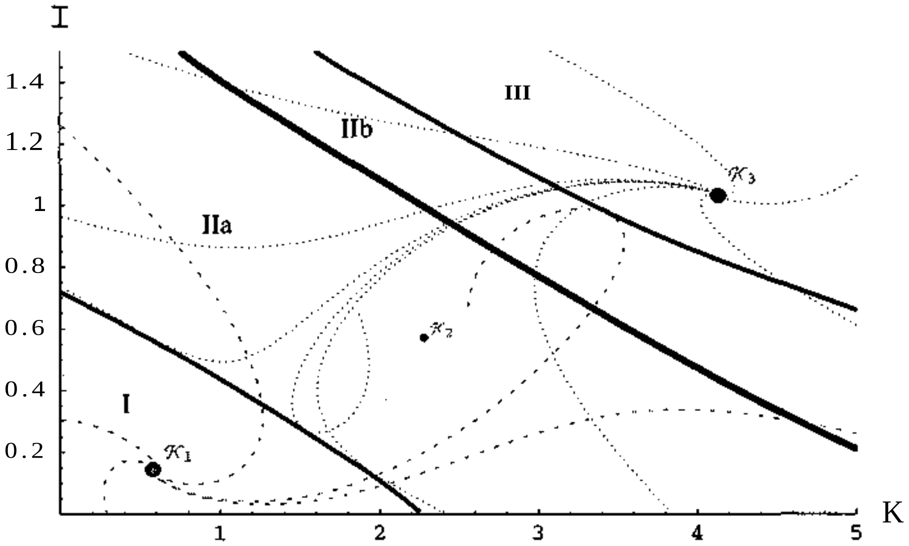

Figure 2.6: Regions of (optimal) convergence and DNS-curve. The bold line separating Regions IIa and IIb is the DNS-curve. The typical motions are illustrated by dashed (stable invariant manifold of the steady state $\mathcal{K}_1$) and by dotted curves (stable invariant manifold of the steady state $\mathcal{K}_3$).

As one can see from Figure 2.6 the state space *(K, I)* splits into three regions:

Region I: for initial values in this region there exist no trajectories converging to the steady state $\mathcal{K}_3$, and thus $\mathcal{K}_1$ is the only candidate for the long run optimal equilibrium.

Region IIa + IIb: for initial values in this region there exist trajectories converging to the steady state $\mathcal{K}_3$, and trajectories converging to the steady state $\mathcal{K}_1$. Here both trajectories are candidates for representing the optimal solution.

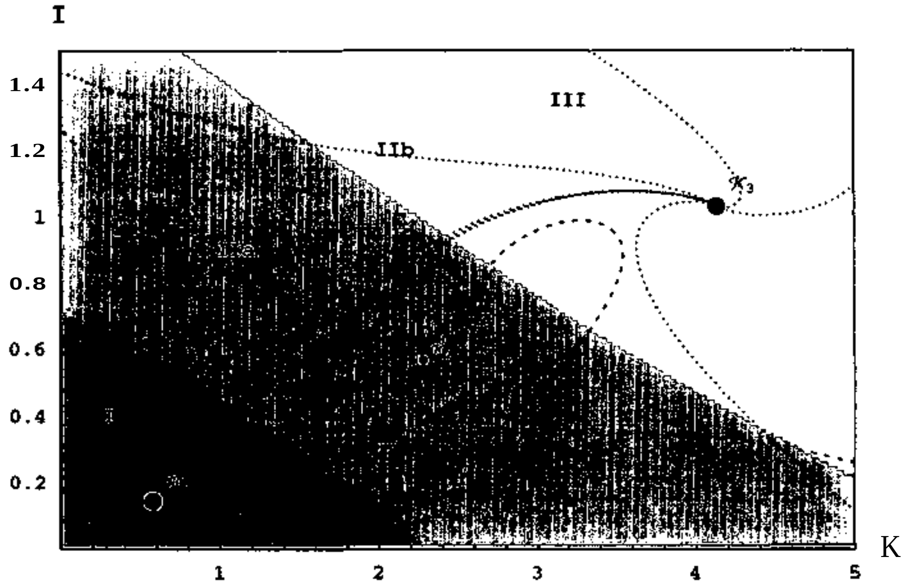

Figure 2.7: Similar to Figure 2.6, but now colored.

Region III: for initial values in this region there exist no trajectories converging to the steady state $\mathcal{K}_1$, and thus $\mathcal{K}_3$ is the only candidate for the long run optimal equilibrium.

In order to specify the regions mentioned, for each unit square of the state space used in the scaling of Figure 2.7 I put a lattice of an array of 80 times 80 rectangular cells. Numerically, for each cell of the lattice I try to find trajectories starting inside the cell and converging to the steady states $\mathcal{K}_1$ and $\mathcal{K}_3$. This is done indirectly by considering many starting points on the boundary of the above mentioned ellipses in ε-neighborhoods of $\mathcal{K}_1$ and $\mathcal{K}_3$ and going back in time. I mark the cells of the lattice that are only reached by those trajectories starting from $\mathcal{K}_1$ as belonging to Region I, the cells that are only reached by those trajectories starting from $\mathcal{K}_3$ as belonging to Region III, and the cells which are reached by those trajectories starting from $\mathcal{K}_1$, as well as those starting from $\mathcal{K}_3$ as belonging to Region II with different colors.

For each cell of the lattice in the Region II, I compare the values of the objective functionals for the trajectories converging to the steady state $\mathcal{K}_1$ and for the trajectories converging to the steady state $\mathcal{K}_3$. It turns out that for the cells in the Region IIa (see Figure 2.6), it is better (in the sense of maximizing the objective functional) to converge to the steady state $\mathcal{K}_1$. On the other hand, for the cells in the Region IIb, it is better to converge to the steady state $\mathcal{K}_3$. The region of optimal convergence of the steady state $\mathcal{K}_1$ consists of the Regions I and IIa, the region of optimal convergence of the steady state $\mathcal{K}_3$ consists of the Regions IIb and III, whereas the boundary between the Regions IIa and IIb is the DNS–curve. It so happens that the unstable steady state $\mathcal{K}_2$ is situated in the interior of the Region IIa. For some other features of the solution the interested reader can see Appendix (b).

2.4 Economic Interpretation and Concluding Remarks

For the parameter values concerned (see (2.10)) the optimal solution consists of two steady states to which it can be optimal to converge in the long run. In the steady state with the larger capital stock the revenue is larger, but on the other hand more replacement investment has to be undertaken to remain in this steady state, which implies that the adjustment costs are larger too.

Both steady states have their own region of optimal convergence, whereas a DNS-curve forms the boundary between these regions. This DNS-curve consists of all points in the state space for which converging to each of the steady states leads to exactly the same value of the objective. Converging to the larger steady state requires an increase of the investment rate, while investments have to decrease in case the firm starts to approach the smaller steady state. This implies that exactly on this curve the firm's policy function is discontinuous: the control $v(K,I)$ is positive on the trajectory that approaches the larger steady state, while $v(K,I)$ is negative on the trajectory that will converge to the smaller steady state. Because the stable manifolds of the equilibria $\mathcal{K}_1$ and thus $\mathcal{K}_3$ are smooth, it implicitly follows that the policy function $v(K,I)$ is continuous and smooth everywhere with the exception of the DNS-curve.

It thus depends on the initial levels of the capital stock and the investment rate to which steady state it is optimal to converge to. From Figure 2.6 it can be concluded that the DNS-curve is decreasing in the $(K,I)-$plane. From an economic point of view this can be explained as follows. In Figure 2.3 one identifies that for capital stock values between, say, 0.6 and 2.0, marginal revenue is low compared to marginal investment costs. Therefore, if the firm starts out with a low capital stock value and sufficiently low investment rate it is not profitable to enter a growth phase that passes this interval of capital stock values. This implies that convergence to the lower steady state is optimal for low values of investment and capital stock. Convergence to the larger steady state is optimal for sufficiently large initial values of capital stock and investment. This is especially caused by the fact that (1) changing the investment rate is costly (also in a negative direction), and (2) marginal revenue is large compared to marginal investment costs for capital stock values around 3 (see Figure 3).

The situation in Figure 6 occurs for a relatively large value of a, which is the parameter in the adjustment cost function for the rate of change of investment. For lower values of a the costs of $\dot{I}$ are less, so that vertical motions in the state space are less costly. Then it will turn out that only one of the stable steady states will remain optimal so that the DNS-curve disappears. For this model with the parameters chosen, numerical experiments show that reducing a leads to an upward movement of the DNS-curve, so that for a larger domain of initial values of capital stock and investment it becomes optimal to converge to the lower steady state. If a is sufficiently low, it will never be optimal to end up in the larger steady state. This can be explained by the fact that (for the parameters chosen) in the steady state $\mathcal{K}_1$ the per capita revenue is relatively high compared to the other equilibrium, which can be see in Figure 2.2.

2.4.1 Concluding Remarks

This chapter considers two main features. First, and most important, in a two state variable optimal control model with two optimal limit sets, the location of a DNS-curve is numerically determined and the economic intuition is provided. The DNS-curve connects all points in the state space on which the decision maker is indifferent concerning to which of the two long run steady states to converge to. While Brock and Dechert [1] prove the existence of a DNS-curve, this contribution is the first one in which a DNS-curve is computed.

Second, it contributes to the literature of capital accumulation models. Especially the effects of increasing returns to scale for an intermediate interval of capital stock values are investigated. As such the paper extends the analysis of Davidson and Harris [3] to a two dimensional framework. Furthermore the concept of adjustment costs is refined by making changes in the investment rate costly.

2.5 Appendix

(a) In fact the parameter space of the analysed problem is only 5-dimensional. This can be seen by considering the following dimensionless transformation:

$$\tau = \delta t,\ k_2' = \sqrt[4]{k_2},\ \bar{\rho} = \frac{\rho}{\delta},\ a_1 = \frac{c_3}{\kappa_1\sqrt{k_2'}},\ a_2 = \frac{c_1\delta}{\kappa_1\sqrt{k_2'}},\ a_3 = \frac{c_2\delta^2}{\kappa_1 k_2'\sqrt{k_2'}}\quad a_4 = \frac{\alpha\delta^4}{k_1 k_2'\sqrt{k_2'}}\ .$$

With the transformed model variables

$$x(\tau) = k_2' K\left(\frac{\tau}{\delta}\right),\ y(\tau) = \frac{k_2'}{\delta} I\left(\frac{\tau}{\delta}\right),\ u(\tau) = \frac{k_2'}{\delta^2} v\left(\frac{\tau}{\delta}\right),$$

the problem can be written as:

$$\max_u \int_0^\infty e^{-\bar{\rho}\tau}\left[\sqrt{x} - a_1\frac{x}{1+x^4} - a_2 y - \frac{a_3}{2}y^2 - \frac{a_4}{2}u^2\right] d\tau,$$

$$\text{s.t.}\quad \dot{x} = y - x,$$
$$\dot{y} = u.$$

With the specification (2.10) the transformed parameters are:

$$k_2' = 0.329\ \bar{\rho} = 0.160,\ a_1 = 0.872,\ a_2 = 0.163,\ a_3 = 0.414,\ a_4 = 0.124.$$

(b) For this particular numerical example both stable invariant manifolds do not include the other steady state. For this model this property is necessary for the existence of a DNS–curve, because the maximized Hamiltonian is strictly convex in the co-state variables λ_1 and λ_2, and for any fixed states K, I it attains its minimum at the $\dot{K} = \dot{I} = 0$ surface. Since the steady states essentially lie on this surface, it follows that, if a stable invariant manifold included the other steady state, it would never be optimal to remain in that steady state (motivation: the objective functional can be determined by dividing the Hamiltonian by the discount factor).

Regardless of the fact that the stable invariant manifold folds in this example, optimal homoclinic cycles are not possible. Since a homoclinic cycle has to pass the fold of the stable invariant manifold, its projection into the state space traverses Region II and has to cross the DNS-curve before it converges. However, I have computed the stable invariant manifolds up to the folds, and can be seen that the DNS-curve splits Region II into two different regions of optimal convergence (Regions IIa and IIb). Then the existence of a homoclinic cycle would contradict the Principle of Optimality.

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Chapter 3

Technology Investment and Customer Attraction

3.1 Introduction

Technology advances quickly these days and for firms it is important to keep up with the latest developments since customers like to buy the most modern products. The firm has a certain technology level which is reflected in the technological content of its products. In general it holds that the greater the technological content or the more modern a product, the more attractive it is to customers. Since an important determinant of the firm's sales is their degree of competitiveness; not the absolute technological content of its products is important, but its content relative to some baseline level. In this chapter by the firm's technology level or the technological content of a product, the technological level or content relative to the baseline level is meant. Due to general technological progress this baseline level increases over time. This implies that the firm's relative standing will decrease over time, if it refrains from technology investments.

However, a certain trade off may exist because rapid technology changes may distract some customers from buying. In other words, a percentage of sales volume is lost because of the presence of some negative effects due to technology investment, which I subsume under **adverse investment effects.** For instance, a new software package may cause compatibility problems or forces the customers to go up a learning curve (adverse learning effect). Customers may defer to buy a new version of product, because especially in case of rapid technological developments there is always the risk that a better version appears on the market soon after the date of the present purchase (adverse update effect). There are some customers who are attached to a particular product. Replacing this particular product by another one with a higher technology content may results in losing of these customers to competitors (adverse new product effect).

Firms can raise their technology level by adopting new technologies or by performing R&D itself. It is clear that these technology investments are not very effective when there is almost no know-how present within the firm. In this case the firm's current technological level is low, and will remain low unless a very substantial effort is made to increase the technological content. Increasing the technological content is also a difficult task if the firm is already producing the most modern products; in such a case very advanced R&D activities are required to increase the technological content of a top-quality product (see also [2]). By now it is clear that the effectivity of technology investments depends on the present technological content of the firm's products. To increase the technology level of the firm by a given amount, more effort is needed either when the technology level is low or high compared to intermediate levels of technology.

In this chapter I present a model, in which the these characteristics

(i) technology investments increase the firm's technology level which increases the amount of customers and thus sales,

(ii) some customers are distracted by technological changes so that the sales volume decreases with technology investments, and

(iii) the effectiveness of technology investments depends on the firm's technology level.

are incorporated. The model is an optimal control problem with two state variables, the amount of customers and the firm's technology level, and one control variable, which is the firm's technology investment rate.

The solution has very interesting characteristics. If the initial technology level is low, it turns out that it is not optimal to increase the technology level by technology investments. The reason is that investments are not very effective due to a lack of know-how within the firm. The implication is that technology level and sales volume stagnate and the firm will quit its operations in the long run. In a contrary manner, for a sufficiently high initial technology level it is worthwhile for the firm to converge to a long run optimal limit set, which can either be a steady state or a stable limit cycle.

For a newcomer or a ruined firm facing a lack of know-how there is an exogenous technology boost necessary, to bring the know-how in the firm above a critical level, because only above this level it is efficient to stay in the market in the long run. In a plane with the sales volume on the horizontal axis and technology level on the vertical axis, a curve (called DNS-curve) marks this critical level (i.e. the critical level depends on the current sales volume of the firm). On this DNS-curve the firm is indifferent between two long run optimal outcomes (stagnation versus revival).

Above the DNS-curve oscillatory behavior of efficient paths is caused by adverse investment effects (model characteristic (ii)). When this negative effect of technological changes on the sales volume is low, then converging to the interior steady state is optimal, but happens in a less damped oscillatory way the larger this effect is. At a certain level of this effect a limit point (blue sky) bifurcation occurs, which means that a semi-stable limit cycle arises ("out of the blue sky"). The implication for long run behavior is that inside the limit cycle convergence to the steady state is still optimal, while outside the limit cycle convergence to the limit cycle itself takes place. Beyond the blue sky bifurcation level of the adverse investment effects the semi-stable limit cycle splits to a stable and an unstable limit cycle, where the unstable limit cycle is situated entirely within the stable one. Concerning long run behavior it now holds that within the unstable limit cycle convergence to the steady state occurs, while outside the unstable limit cycle, a path-efficient firm ends up at the stable limit cycle. Increasing the adverse investment effects, results in a larger size of the stable limit cycle and a reduction in size of the unstable limit cycle, and eventually the unstable limit cycle and the steady state coincide. Exactly when this happens a Hopf bifurcation arises, implying that above the DNS-curve always convergence to a stable limit cycle is efficient. This stable limit cycle still exists and increases in size when the adverse investment effects increase even further, and eventually it turns out that the firm will refrain completely from technology investments, because due to pronounced adverse investment effects too many customers are distracted.

This chapter is organized as follows: in Section 3.2 the model is presented, while in Section 3.3 Pontryagin's maximum principle is applied to establish the necessary conditions for an optimal solution. For different scenarios the optimal solution is presented in Section 3.4, while an economic interpretation is provided in Section 3.5.

3.2 A Model of Technology Investment and Customer Attraction

By $T(t)$ I denote the product's technological content relative to some baseline level at time t. In general it holds that the greater the technological content or the more modern the product, the more attractive it is to customers. It is assumed that in this economy technology develops with a constant rate 6 (> 0). The state variable $T(t)$ is also a measure for the level of know-how within the firm at time t.

The firm has the possibility to undertake investments, / (t), in order to keep up with the technological development or even increase its own technological content relative to the baseline level. It is assumed that the effectiveness of technology investment depends on the technology level. Let us denote the rate of effectiveness of technology investment by $h(T)$. The firm will have a higher technology level when it invests, i.e. / > 0, than when it refrains from investment, which implies that $h(T) > 0$.

A firm lacking know-how can purchase/license standard technology, and then try to adapt and improve it, but this would neither substantially increase the technology content of it products relative to the competitors nor the effectiveness of further investments. Hence, if the firm has almost no know-how, I assume that technology investments will not be very effective, and additionally that it is hard to raise the effectiveness of technology investment in the near future. On the other hand, it is certainly true that it is difficult to raise the technology level further when it is already large. The extreme case is a firm already producing most modern products. Then in order to increase the technology level even further it needs to develop new technologies on its own account, which is in general quite expensive. Only top specialists can help on raising the firm's technology level, but it requires a lot of money to attract these people. Consequently, the technology increase per unit of investment is low.

Translating these observations in terms of the function $h(T)$ I conclude that the investment effectiveness function h must increase for low values of T, while it is decreasing for high values of T; I model h unimodal with a peak for a medium value of T. Furthermore, if it is taken into account that technology investments will be not that effective, if T is small (almost no know-how), and that $h(T)$ is non-negative also for very large technology levels, it can be concluded, that a convex-concave-convex-bell shape for h is reasonable (cf. Figure 3.1). In this way I arrive at the following specification:

$$\lim_{T\to\infty} h(T) = 0, \quad h(T) > 0$$

$$h'(T)\begin{pmatrix}>\\<\end{pmatrix}0 \text{ for } T\begin{pmatrix}<\\>\end{pmatrix}T_{\max}, \quad h'(0) \geq 0, \tag{3.1}$$

$$h''(T) > 0 \text{ for } T < T_1 \text{ and T} > T_2,$$
$$h''(T) < 0 \text{ for } T_1 < T < T_2,$$

where obviously for the inflection points T_1 and T_2 it holds that

$$T_1 < T_{\max} < T_2.$$

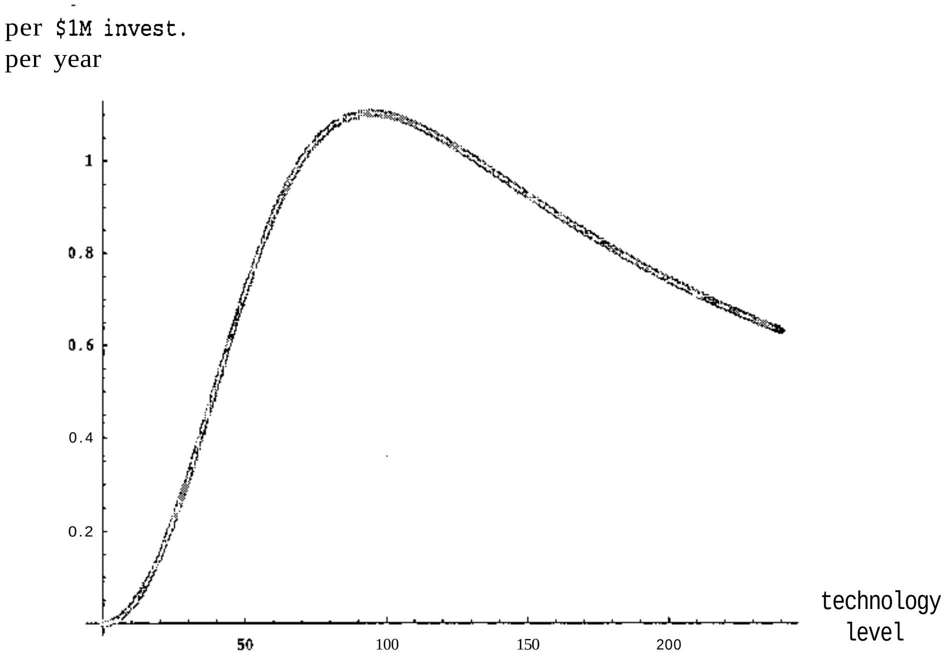


Figure 3.1: A generic shape of the effectiveness of technology investment $h(T)$ on the basis of the existing know-how in the firm.

The state equation for the firm's technological content relative to the baseline level is (I omit the time argument *t;* $\dot{T}$ is the derivative of T with respect to time t)

$$t = h(T)I - \delta T, \quad \text{T}(0) = T_0 > 0. \tag{3.2}$$

By *C (t)* I denote the firm's sales volume at time t. The function $f(T)$ describes, how much sales volume the firm would attract in a long run, if it held its technology level at T forever. I assume /(O) = 0, of course $f'(T) > 0$ (because the better the technology, the higher sales volume the firm would have), and non-increasing returns to technology level, $f''(T) < 0$. One could imagine that the rate of change in the number of sales positively depends on the difference between the current sales volume (C) and the number the current level of technology warrants ($f(T)$). The speed at which customers adjust to the technology level of the firm's products is denoted by the positive factor 7.

Customers favor products with a larger technological content, but they may get frustrated by rapid changes in the technology. In fact, I assume adverse investment effects, which have a negative impact on sales volume. Although or simply because the firm is investing a lot in improving the technology, some customers will tend to go to competitors, refrain from buying or simply wait longer with their purchase. Three examples of adverse investment effects are:

Adverse learning effect: If new versions of products appear soon after each other customers have to get used all the time to handling these new versions. Obvious examples are the new generations of software: people can be very reluctant to change the software they use. The older version still works in the same way as if it was new, while changing to a new (more powerful) version leads to an, at least, temporary drop in effectiveness since the user has to learn how to work with it and there may also be compatibility problems. This is one of the reasons for the empirical fact that new technologies are often adopted on a large scale only after a prolonged period of time, as inter alia Mansfield [5] as well as Chari/Hopenhayn [1] have reported.

Adverse update effect: Another reason for rapid technology changes having a negative effect on the sales volume is that, if a firm has a reputation that new versions of its products follow up each other soon, customers may refrain from buying a product so that they can wait for a more modern version to appear on the market.

Adverse new product effect: Also the demand for low cost daily products like shampoo, tooth paste, vanishing cream, etc., can be affected by rapid technological changes. For instance, on a spam page on the World Wide Web it can be read that "Infused with xyz's legendary age-delaying ingredients and the **best of modern technology,** xyz unite to transform the present - and future - of your skin". A firm has to improve the technology standard for e.g. a moisturizer, to stay in or even to expand its market position. Before they opt for surgery, customers demand creams with higher and higher technology level. On the other hand there are customers who are attached to a particular version of the product and they like it due to several reasons. At the moment that this version is replaced by a newer one, and thus not available in the shop anymore, such a customer may switch to a product from a competitive firm.

When I denote these adverse investment effects by a parameter k, I get the following dynamic equation for the sales volume:

$$\dot{C} = \gamma [f(T) - C] - kCI, \; C(0) = C_0 > 0. \tag{3.3}$$

Technology investments are irreversible, so that

$$I > 0. \tag{3.4}$$

Finally, the firm invests in such a way that the discounted profit flow is maximized. Profits decrease with investment costs, while they increase with revenue, where it is assumed that revenue per sale ($a > 0$) is constant . Investment costs are the costs associated with carrying out technology investments. They can consist of purchasing new machines by which more modern products can be fabricated, laboratory development of product improvements, etc. Investment costs are denoted by a convexly increasing function $a(I)$ ($a(0) = 0, a' > 0, a'' > 0$). The convex shape reflects the fact that there are diminishing returns to effort at any time. Hence, the firm's objective functional is given by

$$\max_I \int_0^\infty e^{-rt}[\alpha C - a(I)]dt, \tag{3.5}$$

where r is the constant discount rate.

Now the optimal control model consists of the objective (3.5) subject to the state equations (3.2) and (3.3), and the non-negativity constraint (3.4).

3.3 Analysis

In this section I present the necessary optimality conditions and I analyze the possible occurrence of steady states.

3.3.1 Necessary Optimality Conditions

First I specify the current value Hamiltonian for the optimal control model (3.2) - (3.5):

$$H = \alpha C - a(I) + \lambda_1 \left(\gamma [f(T) - C] - k C I\right) + \lambda_2 \left(h(T)I - \delta T\right),$$

Applying Pontryagin's maximum principle, i.e. maximizing the Hamiltonian w.r.t. $I > 0$, I derive for the control variable I that

$$\begin{cases} I = 0 & a'(0) + \lambda_1 k C > \lambda_2 h(T) \\ a'(I) + \lambda_1 k C = \lambda_2 h(T) & a'(0) + \lambda_1 k C < \lambda_2 h(T). \end{cases} \tag{3.6}$$

Expression (3.6) shows that the firm invests such that marginal costs, consisting of the marginal investment costs, $a'(I)$, and the negative effect of marginal investment on the sales volume arising from adverse investment effects, $\lambda_1 k C$, equals marginal revenue. The latter consists of the increase of the technology level due to marginal investment, $h(T)$, which is valued by the shadow price of technology, λ_2. The firm refrains from investment when marginal costs, valued for zero investment, $I = 0$, exceeds marginal revenue.

According to the maximum principle an optimal path has to fulfill the following dynamic system (combined with the algebraic equation (3.6)):

$$\dot{C} = \gamma [f(T) - C] - k C I, \tag{3.7}$$

$$\dot{T} = h(T)I - \delta T, \tag{3.8}$$

$$\dot{\lambda}_1 = [r + \gamma + k I]\lambda_1 - \alpha, \tag{3.9}$$

$$\dot{\lambda}_2 = [r + \delta - h'(T) I]\lambda_2 - \gamma f'(T) \lambda_1. \tag{3.10}$$

3.3.2 Steady States

Prom the algebraic-dynamic system (3.6) - (3.10) it can be obtained that there is a trivial steady state at the origin of the state-control space,

$$C = T = I = 0, \quad \lambda_1 = \frac{\alpha}{r+\gamma} \quad \lambda_2 = \frac{\gamma}{r+\delta}\frac{\alpha}{r+\gamma} f'(0), \tag{3.11}$$

when I assume that under assumption of zero technology level the marginal costs of investments at zero investment are greater equal the marginal revenue of investment. In other words, I postulate $h(0)$ small enough, that

$$a'(0) \geq \frac{\alpha}{r+\gamma}\frac{\gamma}{r+\delta} f'(0)\, h(0) \tag{3.12}$$

is fulfilled. In what follows I narrow effectiveness of technology investment down to an assumption which is sufficient that inequality (3.12) is fulfilled:

Assumption 17 *I postulate that the average effectiveness of technology investment converges to zero when the technology level T decreases to zero, i.e.* $\lim_{T \to 0+} \frac{h(T)}{T} = 0$.

Straightforward calculations show that the steady state (3.11) is saddle point stable in the 4-dimensional state co-state space and appears as a stable node in the state space.

Concerning the possible interior steady states (i.e. T > 0) it can be obtained from (3.7) - (3.10) that they have to satisfy:

$$I = \frac{\delta T}{h(T)}, \tag{3.13}$$

$$C = \frac{\gamma f(T)}{\gamma + k I}, \tag{3.14}$$

$$\lambda_1 = \frac{\alpha}{r+\gamma+k I}, \tag{3.15}$$

$$\lambda_2 = \frac{\gamma f'(T)}{r+\delta-h'(T) I}\lambda_1. \tag{3.16}$$

Furthermore, from (3.6) and (3.13) - (3.16) it can be derived that at a steady state it must hold that

$$G(T) = a'(I) + \frac{\alpha}{r+\gamma+k I}\left(k C - \frac{\gamma f'(T)}{r+\delta\left[1-\frac{h'(T)}{h(T)}T\right]} h(T)\right) = 0. \tag{3.17}$$

Assumption 18 *I postulate that the "technology investment effectiveness elasticity"* $\frac{h'(T)}{h(T)}T$ *decreases monotonically from a positive value for T = 0 to a negative value for* $T \to \infty$.

Proposition 19 *Under the Assumptions 17 and 18 we get at most three interior steady states.*

The function G decreases from infinity for $T \to 0+$ to a local minimum and increases to infinity for $T \to \infty$. Depending on the minimum value of G, there are null, one or two roots of G. If there exists a zero in the denominator $r + \delta \left[1 - \frac{h'(T)}{h(T)}T\right]^{-1}$, which can only happen at a T larger than the minimum point, G "jumps" from +00 to $-\infty$ (asymptotic pole), i.e. there exists another root right to the pole. Depending on the the minimum value of G, and the existence of a pole, the equation $G(T) = 0$ either has null, one, two or three solutions.

3.4 Numerical Analysis

The next step is to investigate the dimensions of the stable invariant manifolds of the steady states. Due to the complexity of the analysis I have to rely on numerical tools. The numerical computations were done by MATHEMATICA (Version 4.1.0.0, Mathematica is registered trademark by Wolfram Research) using a precision of 22 digits. Bifurcation analysis was done by CONTENT, which is an environment designed for investigating the properties of dynamical systems. Its core was developed by Y.A. Kuznetsov and V.V. Levitin at the Centrum voor Wiskunde en Informatica (CWI), Amsterdam, The Netherlands, ftp.cwi.nl/pub/CONTENT.

3.4.1 Functional specifications

To start my numerical analysis I first need to specify the functions occurring in the model. For the investment costs I choose the following function:

$$a(I) = a_1 I + a_2 I^2. \tag{3.18}$$

For the function / (T) I simply take

$$f(T) = f_1 T. \tag{3.19}$$

A functional form for $h(T)$ that satisfies the requirements in (3.1), while also its elasticity decreases monotonically from a positive value for $T = 0$ to a negative value for $T \to$ oo, is

$$h(T) = \frac{h_1 T^2}{1 + \left(\frac{T}{h_2}\right)^3}. \tag{3.20}$$

Substitution of these specifications in the optimal control model formed by (3.2) - (3.5) leads to the conclusion that the model contains ten parameters. The Appendix shows that this model can be transformed to a model with only five parameters. This facilitates numerical analysis. Moreover, this makes the results more general since a particular specification of these five parameter values coincides with many combinations of parameter values of the original model.

The physical dimension of C, T and / are sales in million, technology units, and investment in million \$, respectively; the unit of time is one year. With the parameter values

$$r = 5\%, \quad 6 \approx 4.17\%, \quad a = \$2, \tag{3.21}$$

$$7 = 0.0625, \quad f_1 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{3} \tag{3.22}$$

$$h_1 = \frac{1}{2700}, \quad h_2 = 75, \quad k = 0.00852, \tag{3.23}$$

the function G in expression (3.17) has the shape depicted in Figure 3.2. From

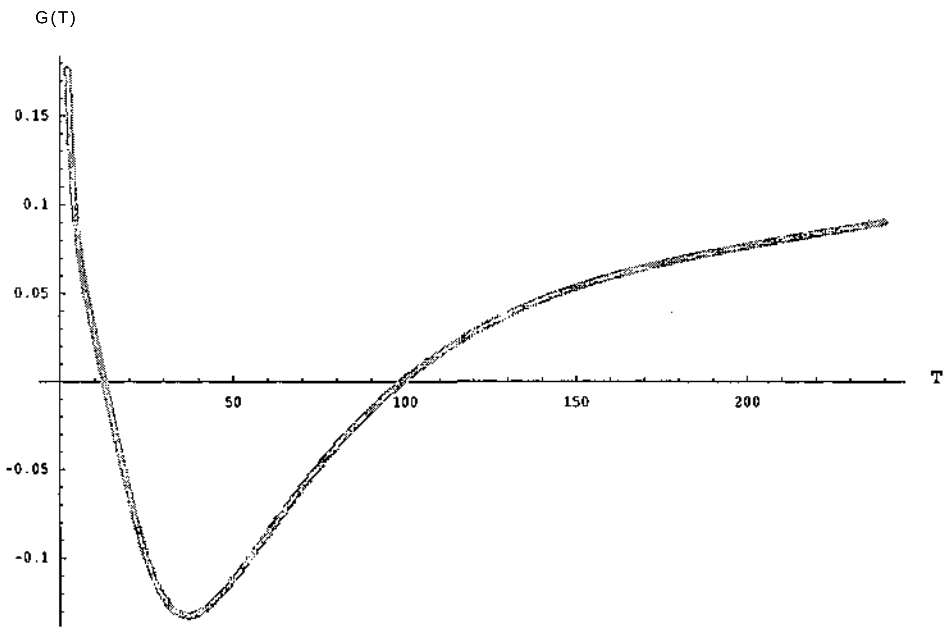


Figure 3.2: For the chosen parameter values the equation $G(T) = 0$ has two roots.

Figure 3.2 it can be concluded that two steady states with positive technology level exist. Varying parameters, numerical calculations prove that the smaller one is always unstable while the larger one can be saddle point stable or lies in the interior of a saddle point stable limit cycle. In order to investigate long run outcomes I am interested in the stable invariant manifold of the saddle point stable steady state or in the stable invariant manifold of the limit cycle. In the projection of these manifolds into the state space, the saddle point stability changes to stability. Therefore, I refer to these long run outcomes as " stable steady state " and " stable limit cycle ". Numerical experiments suggest that this result seems to be robust in the sense that it also holds for other parameter sets.

Besides one non-optimal, unstable steady state, I face two optimal steady states (including the origin). The question arises from what starting values convergence to which steady state is optimal, whether there exists a DNS-set (i.e. whether there exist initial state values, from which the optimally controlled system is indifferent to converge to one or the other steady state), and where this DNS-set is located. The situation is even more interesting since around the positive stable steady state a (saddle point) stable and an unstable limit cycle can occur simultaneously. For a particular parameter set the unstable limit cycle and the stable steady state merge and result in an unstable steady state.

In the next section, for the moment I disregard the limit set at the origin, so that I restrict myself to the analysis of the dynamic behavior around the interior stable steady state.

3.4.2 Bifurcation Analysis

One of the key parameters of this model is fc, since it measures the adverse investment effects. To be more precise, k equals the loss of sales volume due to each \$1M investment per year (cf. $-kCI$ in (3.3)). The system features different types of optimal long run outcomes: a steady state solution at the origin, a steady state solution with a positive technology level, and an undamped oscillation (limit cycle). Varying k the model shows all possible combinations of these different limit sets. Therefore, I choose k to be the bifurcation parameter. Depending on the value of this parameter four regimes can be distinguished.

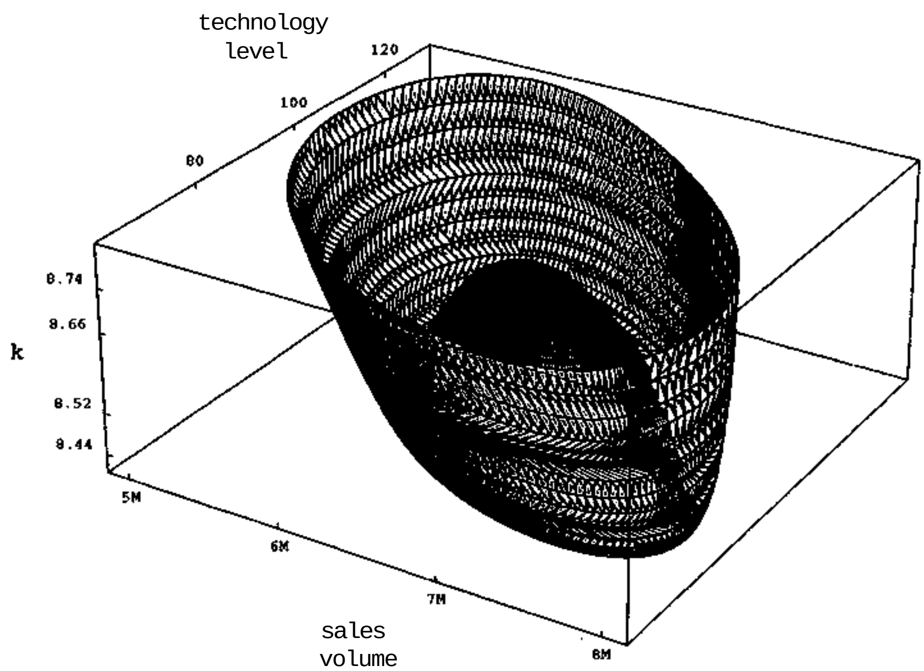


Figure 3.3: Shape and amplitude of stable and unstable limit cycles (parameter k in tenth percentage points).

Regime I

When the loss of sales volume due to $1M investment per year is less than approx. 844 sales per 100K sales ($\approx$ 0.844%, i.e. k = 0.00844), investment, sales volume, and technology level converge to their steady state values. When the loss is a lot less than 0.844%, convergence is monotonically, otherwise damped oscillatory.

Regime II

I obtain that limit cycle behavior occurs for a loss greater than $\approx$ 0.844%. Figure 3.3 shows how the shape and the amplitude of stable and unstable limit cycles projected into the state plane (C, T) depend on the bifurcation parameter fc; parameter k varies between 0.84% and 0.88%. For a given value of k the intersection of the big "basket" with the state plane $(C,$ T) gives the stable limit cycle while the intersection of the smaller interior " basket" (which is placed upside down in the big basket) gives the unstable limit cycle.

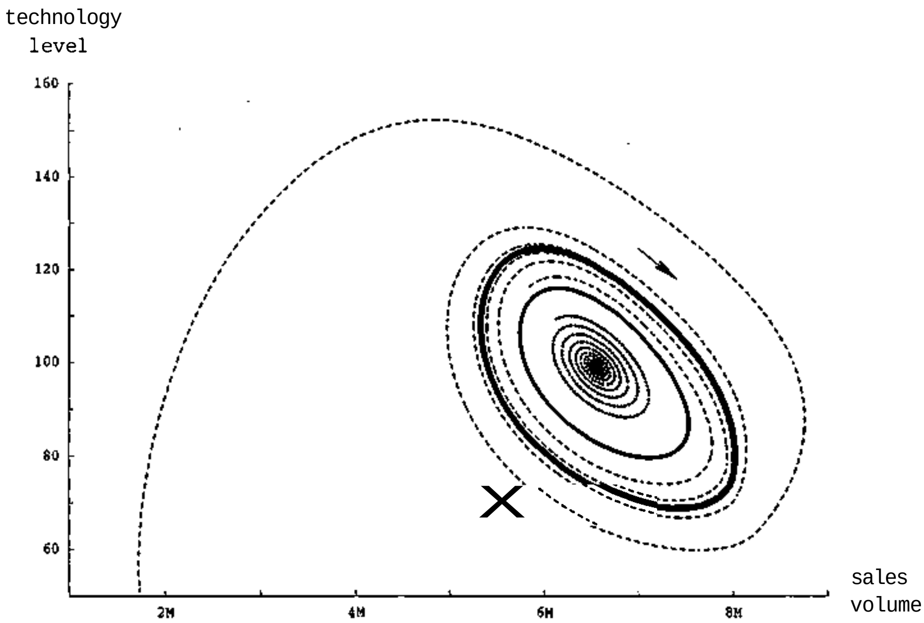


Figure 3.4: The oscillatory behavior around the stable steady state for a parameter value k = 0.00852.

Regime **IIa** At $k \sim$ 0.00844 there occurs a " limit point " (" blue sky ") bifurcation (cf. e.g. [3], [4], [6]). The stable and the unstable limit cycle coincide and result in just one semi-stable limit cycle (being attracting from outside and repelling inside). Inside this semi-stable cycle convergence to the steady state occurs (inside and outside refers to the projection into the state space).

For a loss of sales volume in between *fa* 0.844% and 0.866% there are two limit cycles around the stable steady state: the inner one being unstable and the outer one being (saddle point) stable. Inside the inner cycle convergence to the stable steady state occurs, whereas outside the outer cycle and in between the inner and outer cycle convergence to the outer (stable) limit cycle takes place. In Figure 3.4 the (damped) oscillatory behavior around the stable steady state is shown.

The movement of all trajectories is clockwise, which will be economically interpreted in Section 3.5.2. The amplitude of the unstable (repelling) limit cycle decreases in size, when k increases, while the amplitude of the stable (attracting) limit cycle increases in size. At k *fa* 0.00866 a sub-critical Hopf bifurcation (cf. e.g. [3], [4]) arises. Then the smaller (unstable) limit cycle collapses to the steady state.

Regime IIb For a loss in between 0.866% and 1.302% only the larger (stable) limit cycle exists and the steady state is now unstable. Increasing k results in a growing amplitude of the limit cycle. For initial levels of the state variables located in- or outside the limit cycle it is optimal to converge to the limit cycle.

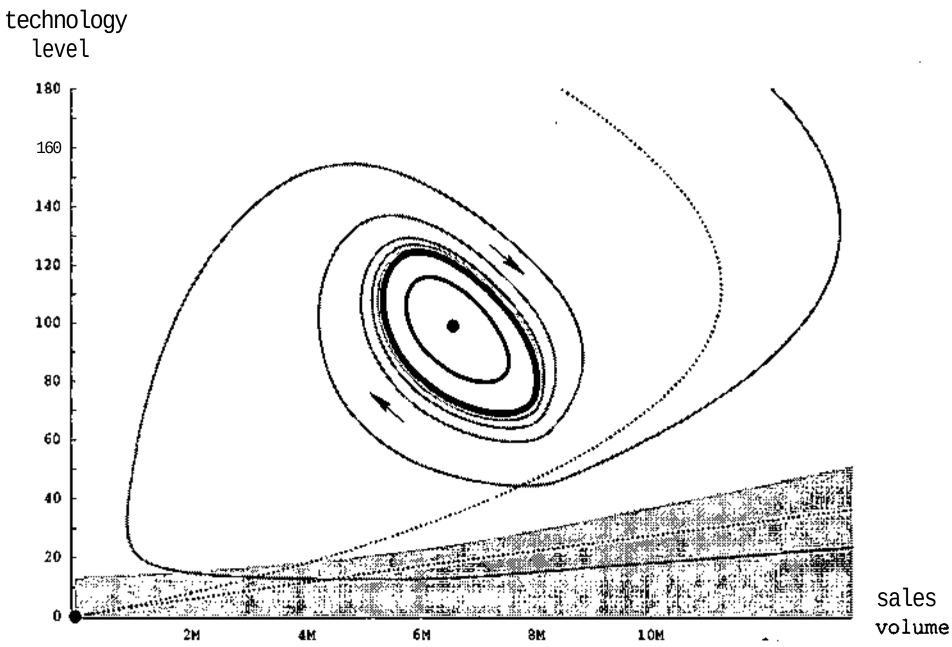


Figure 3.5: Three optimal long run outcomes and their regions of optimal convergence for k = 0.00852 : (1) gray-shaded area for the origin, (3) interior area of the unstable limit cycle (limit cycle depicted with the thinner line) for the interior steady state and (2) remaining white area for the stable limit cycle (limit cycle depicted with the thicker line).

Regime III

When the loss of sales volume due to $1M investment per year is greater than $\sim$ 1.3%, in the long run it is not optimal to sustain a positive stock of customers. For a more detailed description I refer to the next section.

3.4.3 DNS–curves

In this section I discuss numerically computed DNS-sets, which turn out to be one-dimensional curves.

So far I have only considered initial values in the neighborhood of the interior (stable) steady state and the limit cycles. Another candidate for an optimal long run outcome is the limit set at the origin; cf. (3.11). Expanding Figure 3.4, Figure 3.5 shows candidate trajectories converging to the origin represented as dotted curves. Additionally, the candidates converging to the stable limit cycle are represented as solid curves. Apparently, there is a region where it is possible to converge to either the origin or to the stable limit cycle. In this region, the objective function has to be evaluated for both of these solution candidates in order to find out to which region of optimal convergence this initial point belongs.

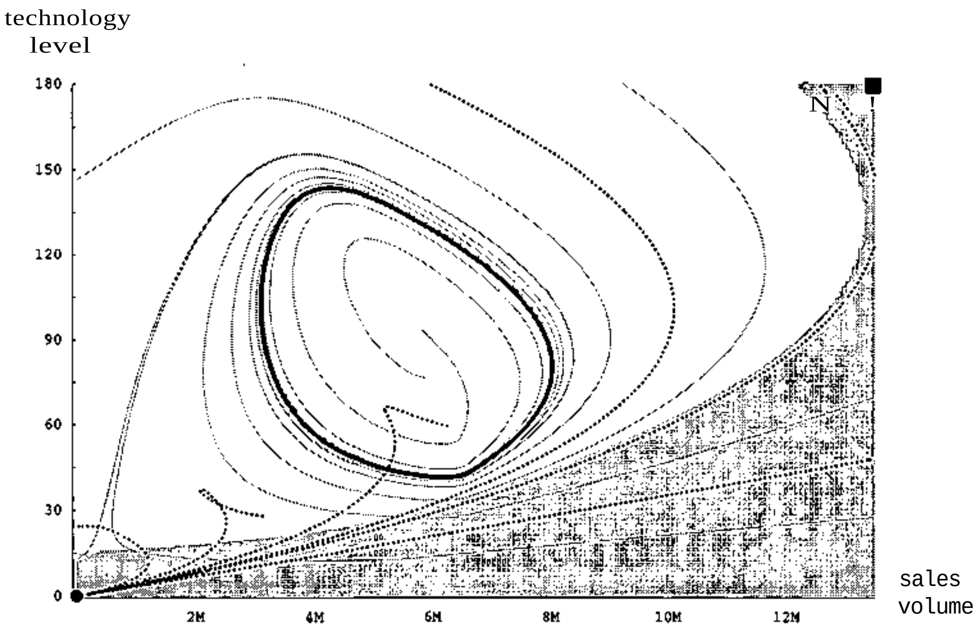


Figure 3.6: This figure illustrates two optimal long run outcomes and their regions of optimal convergence for k = 0.011: dotted lines in the gray-shaded area show paths of optimal long run zero investment policies (origin), solid lines in the white region show paths with an optimal long run cyclical sequence of zero and positive investment periods (limit cycle).

Figure 3.5 illustrates three optimal long run outcomes and their regions of optimal convergence for a loss of sales of 0.852%:

1. dotted lines in the gray-shaded area show paths of optimal long run zero investment policies (steady state at the origin),

2. solid lines in the white region show paths with an optimal long run cyclical sequence of zero and positive investment periods (limit cycle depicted by the thicker line), and

3. in the interior of the limit cycle, which is depicted by the thinner line, in the long run a constant investment policy is optimal (steady state with approx. 6.2M sales per year and technology level of approx. 100).

In the Figures 3.5, 3.6, and 3.7, the DNS-set is the borderline between the gray-shaded and the white region. There is no doubt that there the DNS-sets are curves. The procedure chosen to numerically determine the DNS-curve (and the regions of optimal convergence) is extensively explained in Chapter 2.

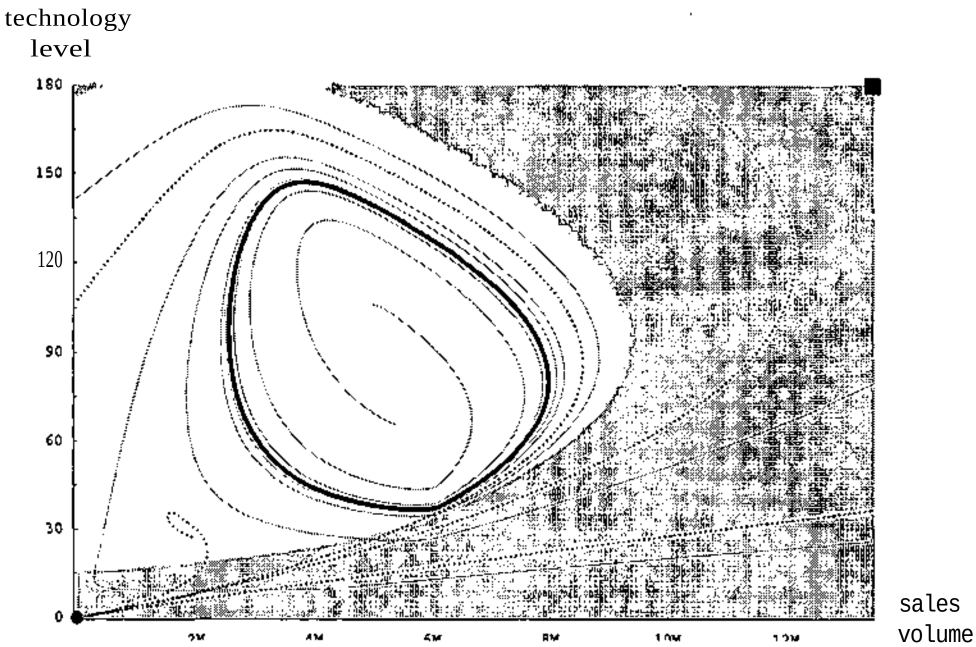


Figure 3.7: Similar to Figure 3.6 but now computed for parameter value k = 0.012. With the chosen precision of 22 digits the accuracy of the computed value function for k = 0.012 was enlarged to 6 digits. As one can see here the borderline between the gray-shaded and white region is fringed. This is why because in case of parameter values $k \approx 0.013$, the value functions of the two different optimal long run outcomes differ even in mediate distances to the DNS-curve (where the value function of the limit cycle and the value function of the steady state at the origin intersect) about values less than 10^{-6}.

Increasing k reduces the amplitude of the unstable limit cycle, which finally disappears completely, when k is equal or higher than ≈ 0.00866 (cf. Regime IIa and IIb). Figure 3.6 is computed for a loss of sales volume of 1.1%. For every initial point located on and above the DNS-curve convergence to the stable limit cycle is optimal, whereas for every initial point located on and below the DNS-curve convergence to the origin is optimal.

Increasing k further increases the amplitude of the stable limit cycle. As one can see in Figure 3.7 the DNS-curve folds back for larger losses. Also, as one can see in Figure 3.5, 3.6, and 3.7, the DNS-curves moves upwards, and for larger k they get closer to the stable limit cycle. For parameter values of k near 0.013 I faced noticeable numerical instabilities, when I tried to compute the regions of optimal convergence and DNS-curves. These numerical instabilities are not astonishing knowing that for (a numerically computed) parameter value k — 0.0130194 the DNS-curve and the limit cycle coincide. In other words, starting from any initial value inside the limit cycle convergence to the limit cycle is optimal, whereas for any initial value outside the limit cycle convergence to the origin is optimal.

In case of a loss higher than $\approx$ 1.3%, it is always optimal to converge to the origin. (Trajectories that converge to the limit cycle still fulfill the necessary optimality conditions, but a numerical analysis shows that this limit cycle is no longer path optimal.)

3.5 Economic Interpretation

In this section I provide the economic intuition for my results.

As mentioned in the Introduction of this chapter, the main model characteristics are the following:

(i) technology investments increase the firm's technology level which increases the amount of customers and thus sales,

(ii) customers are distracted by technological changes so resulting in a loss of sales volume because of technology investments (adverse investment effect), and

(iii) the effectiveness of technology investments depends on the firm's technology level: it is difficult to build up technology when the technology level is small (absence of know-how makes building up the technology difficult) or large (when the firm is already producing the most modern products advanced R&D is needed to increase technology level even further). Hence technology investments are most effective for intermediate values of the technology level.

Numerical analysis shows three different kinds of optimal long run behavior: on the one hand moving out of the market (see section 3.5.1), on the other hand staying in the market with constant sales volume, or persistent oscillation with periods of positive and periods of zero technology investment (see section 3.5.2). Consequently, the state plane - sales volume is on the abscissa and technology level on the Ordinate - partitions to regions where in the long run a path efficient firm

- follows a zero investment policy (=moves out) or
- follows a constant positive investment policy (=constant sales volume) or at least a cyclical sequence of zero - positive investment policy (=persistent oscillation).

Generically, these two regions overlap and I call this overlapping area DNS (Dechert-Nishimura-Skiba)-curve (see section 3.5.3). Numerical results show that this area is indeed a curve.

3.5.1 Moving out

For a firm with a lack of know-how investing in technology is not effective. Hence, if the technology level is low, i.e. below the DNS-curve, an efficient firm does not invest sufficiently to prevent that sales volume and technology level finally erode to zero.

Moreover, especially if the adverse investment effects are high and if the firm's sales are booming, investment causes a serious absolute loss of customers. Therefore it is less attractive to invest. But low or even zero investment results in a decline in technology level, which in turn results in a decline in sales volume. This pattern continues until sales volume is sufficient small, so that the absolute loss of sales due to investment is reasonably negligible. But then it could be the case that the firm's know-how is not high enough for investments to be sufficiently effective. This would imply that the firm refrains from investing further on. Consequently, a firm facing pronounced adverse investment effects can stagnate and in the long run move out of the market, in spite of having been in business with booming sales and a relatively high technology level.

3.5.2 Staying in the market

If the initial technology level is above the DNS-curve, in the long run the system converges to a limit cycle on which the movement is clockwise - an efficient firm follows persistent oscillatory paths on which sales volume lags behind technology level. Let us have a closer look at this undamped oscillation (see also Figures 3.8 - 3.11):

Technology Boost Low sales volume implies that not too many customers are lost when investment is increased dramatically. The difference to the case of being below the DNS-curve is that now there is sufficient know-how within the firm for technology investment being effective. Intensifying investment causes a boost in the technology level, and proximate sales volume starts to rise (years 0-12).

Booming Sales At the end of this technology boost further technology growth is too expensive mainly due to the diminishing effectiveness of investment. Hence investment is lowered, which implies that the technology level starts to decline. As the technology level is high, sales volume still increases (years 12-30).

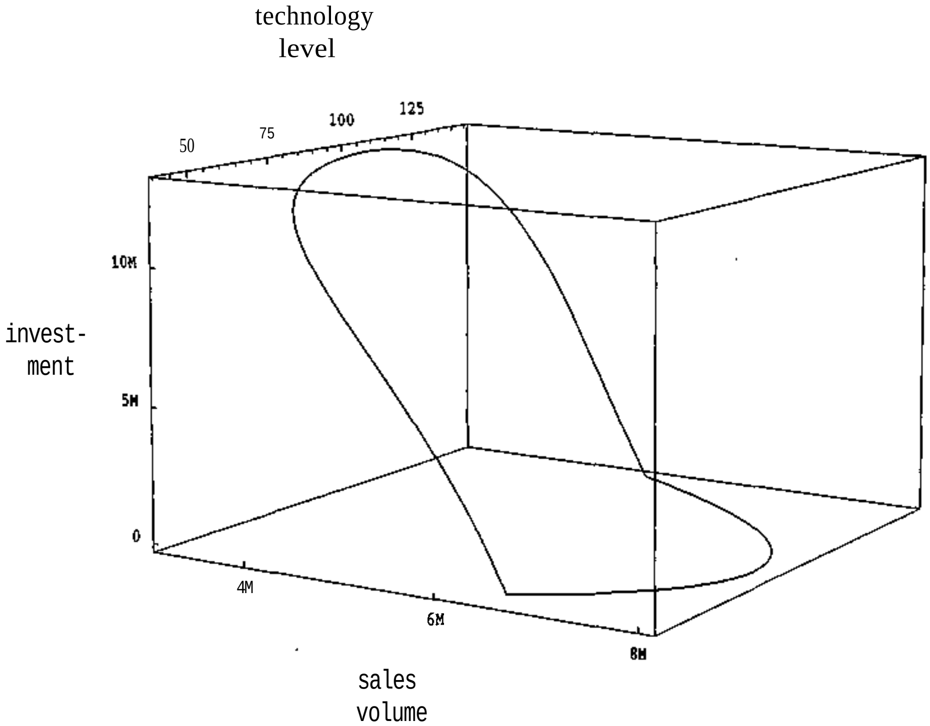


Figure 3.8: Optimal limit cycle in the state-control space for k = 0.0852.

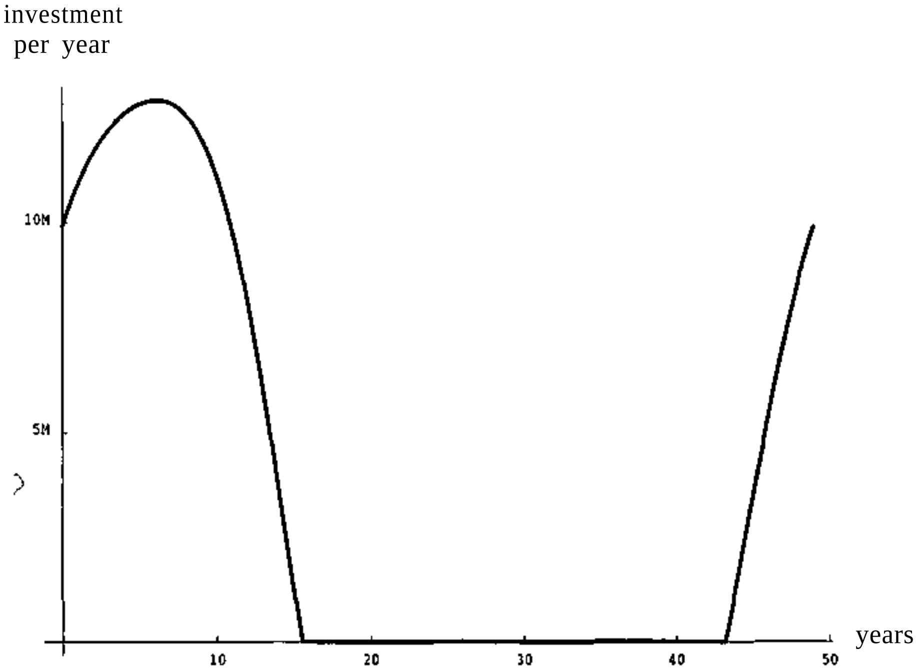


Figure 3.9: Investment time path of the optimal limit cycle for k = 0.0852.

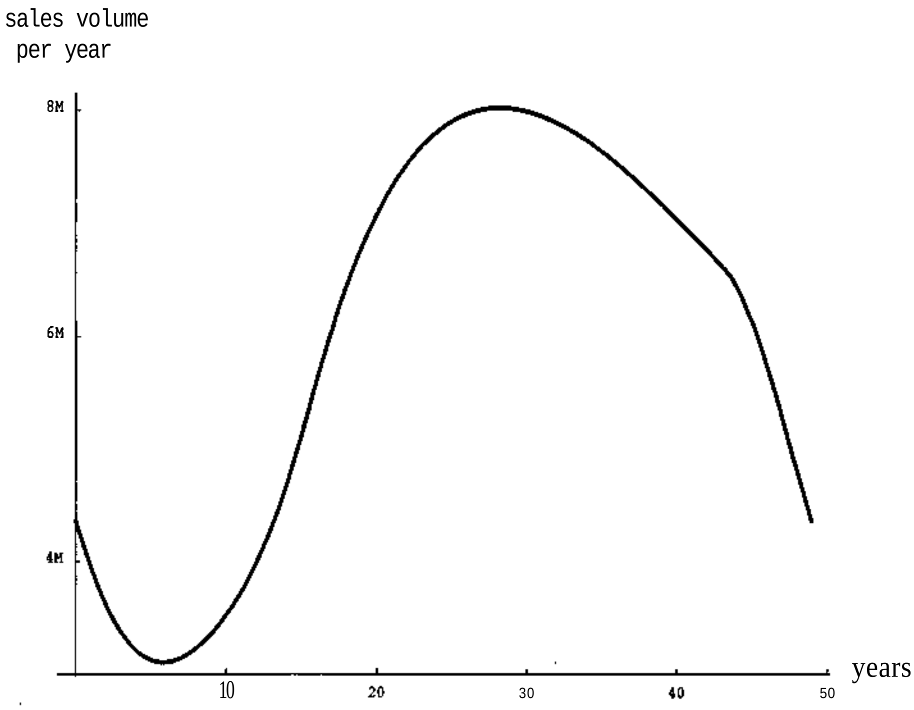


Figure 3.10: Sales volume time path of the optimal limit cycle for k = 0.0852.

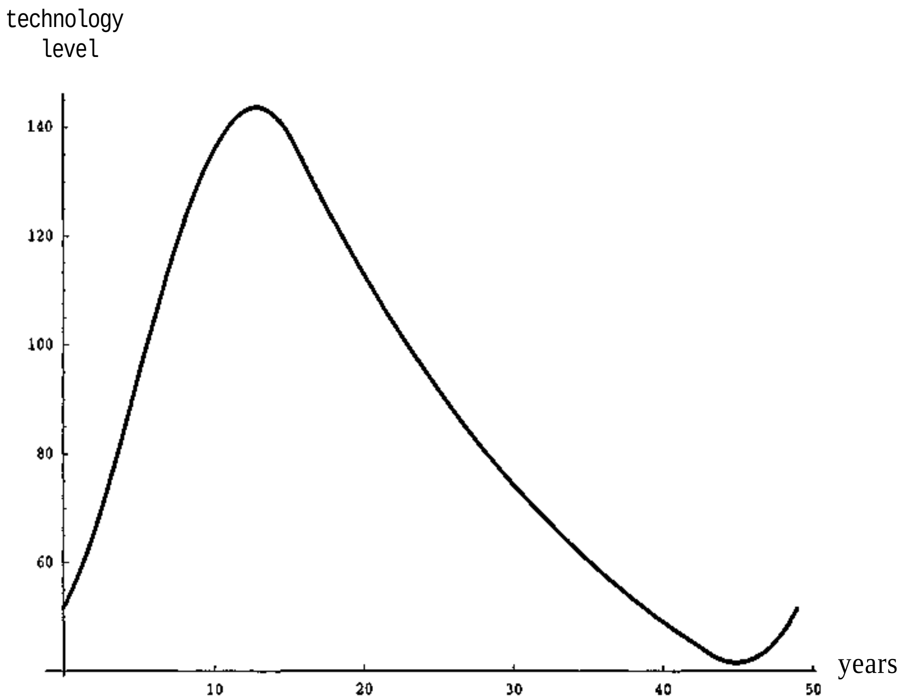


Figure 3.11: Technology level time path of the optimal limit cycle for k = 0.0852.

Don't bother customers The firm does not invest to stop declining technology because it does not want to antagonize a large stock of customers. However, due to zero investment the technology level erodes (years 30-45).

Hit the road When the technology level becomes dramatically low the firm starts up to invest moderately (cf. the "kink" in the right lower part of the limit cycles in Figures 3.6 and 3.7 or year 45 in Figures 3.8 - 3.11). As both the low technology level and investment distracts customers, in a short period (compared with the other three phases of the oscillation) the sales volume becomes that low that it is now time to boost the technology level again (years 45-50).

Remark: The period of the cycle with about 50 years is too longish to allow a serious economic interpretation. However, accelerating the adjustment parameter 7 in (3.3) more than ten times - (7 = 0.9) - reduces the period of the limit cycle to eight months. All the mentioned features perpetuate, but the limit point value of parameter k increases to 13.4%, the Hopf bifurcation value to 14.7% and the DNS-curve coincides with the limit cycle for k *a* little bit less than 50%. There is some trade-off between adjustment of actual sales volume C and long-run sales volume $f(T)$ on the one hand and the negative impact of adverse investment effects on the other hand. If sales volume adjust on changes in technology level quicker, oscillatory behavior mitigates adverse investment effects more effective.

Apparently, oscillation is induced mainly by adverse investment effects. Thus it can be expected that oscillatory behavior is especially present when the loss of sales volume due to investment - the parameter k in the term $k\,C\,I$ in the state equation for C - is high. In order to investigate this fact I analyzed the sensitivity of the solution w.r.t. to changes of the adverse investment effects (see additionally Figure 3.12):

- In the presence of relatively small adverse investment effects, the firm converges after a transition period to a constant (positive) investment policy. For negligible small adverse investment effects convergence is monotonic, for slightly larger but still small adverse invesment effects convergence is damped oscillatory, viz., higher investment for smaller sales volume and smaller investment for higher sales volume (cf. Regime I).

- Increasing k, at a certain value ("out of the blue sky") an optimal semi-stable limit cycle comes into existence. This implies that convergence to the steady state only takes place from within the limit cycle. From outside, convergence to the limit cycle is optimal thus preserving oscillatory behavior forever.

- Increasing k further the semi-stable limit cycle splits to a stable and an unstable limit cycle. Both are optimal. Convergence to the (larger) stable limit cycle is optimal when starting outside the (smaller) unstable limit cycle, whereas inside the unstable limit cycle convergence to the steady state takes place.

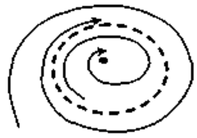
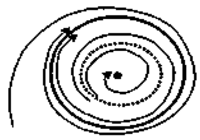
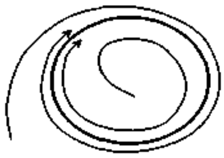
k		Figure
< 0.844%		-
0.844%		-
0.844% – 0.866%		3.4 and 3.5
0.866% – 1.302%		3.6 and 3.7
> 1.302%	Stagnation to the origin	-

Figure 3.12: The impact of adverse investment effects on the oscillatory behavior of the efficient paths.

- In the presence of relatively medium adverse learning effects in the long run the unstable limit cycle vanishes and convergence to the steady state never occurs. This implies that all trajectories eventually end up at the stable limit cycle; the firm applies an undamped oscillatory investment policy - technology boost in case of small sales volume and no investment occurs in case of booming sales. With this behavior the firm alleviates the adverse investment effects.

- In case of a relatively high adverse learning effects even undamped oscillatory investment policy does not suffice to compensate for these effects. Hence, the firm refrains from investing, stagnates and finally moves out of the market.

3.5.3 DNS-curve

A necessary condition for existence of the DNS-curve is the occurrence of effect (iii). As already stated the reason is that for a low technology level the effectiveness of technology investment is too low for building up the firm.

The shape of the DNS-curve is in general upward sloping. This is caused by adverse investment effects implying that the number of customers has a negative effect on the profitability of investment.

For a large sales volume (and when the adverse investment effects are pronounced) the DNS-curve is not only upward sloping, but in fact it folds back to north-west; see e.g. Figure 3.7. The reason for the latter is that if the technology level is very high (and - or although - the sales volume is moderate), one need not invest in technology to let sales volume increase. Then, eventually sales volume becomes so large that (because of the absolute loss of sales volume) it is also not reasonable to invest, because this would mainly erode the current sales. In the end the firm's technology level is so small that investment in technology is not sufficiently effective to build up the firm.

Path-dependence

Throughout the investigated model the evolution of the firm is ergodic - ultimate outcomes are built in right from the start on the bases of initial state values, profit maximization and the system dynamics. The model itself is deterministic and its long run outcomes are a priori well-defined.

Now, to make this model more realistic, let us add some unpredictable "small events" outside the investigated structure. As far as the (initial) states of the model are far away from the DNS-curve such random small events do not have an unpredictable impact on the long run outcome - the evolution of the firm remains ergodic.

Even a firm exactly following the path-efficient, cyclical sequence of zero - positive investment policy (cf. cycle in Figure 3.7) runs the risk of a non ergodic evolution, when the DNS-curve approaches too closely to the limit cycle due to an increase the adverse investment effects. As long as the firm traverses the "technology boost", the "booming sales" and the "don't bother customers" phases there is no risk, that small events shift the firm to the other side of the DNS-curve - to a different, not predicted long run outcome (here moving out of the market). The firms medium-term future is predictable. However, when the sales volume has already begun to erode, when the technology level has declined to a value just little bit higher to the critical mass of know-how so that there is a serious chance to build up the firm, a small event - such as a fatal car accident of a leading technician - can have a tremendous effect on the long run outcome. Here, the firm has periods of an ergodic (medium-term predictable) and periods of a non-ergodic evolution.

3.6 Conclusion and Extension

In this chapter I presented an investment control model, where investment in technology is controlled by a firm in order to maximize profit. A higher technology level of the product has a positive effect on sales. The effectiveness of investment, however, depends on the existing know-how in the firm, and, additionally, investment in technology has negative side effects on sales volume (adverse investment effects).

The investigated model class exhibits different types of optimal long run behavior. If the initial technology level is too low, it is not optimal to increase the technology level by investments. The reason is that technology investments are not very effective due to a lack of know-how within the firm. The implication is that in the long run the firm will cease its operations. For a sufficiently high initial technology level, in the long run it is worthwhile for the firm to converge to a constant or undamped oscillatory investment policy. Depending on the initial endowment the firm applies one of these types.

For certain initial endowments in the long run the firm is indifferent between zero investment on the one hand and constant positive or a least a cyclical zero and positive investment policy on the other hand. In the plane with sales volume on the horizontal axis and technology level on the vertical axis an upward sloping DNS-curve separates these two regions of optimal convergence (different long run behavior). The adverse investment effects lead to an upward sloping DNS-curve since now the profitability of investment depends on the number of customers: the more customers this firm has the more can be distracted by technology changes.

Optimal oscillatory investment policy has its seeds in the model characteristic adverse investment effects: if these effects are negligible low, convergence to a constant investment policy is optimal, otherwise in the long run a cyclical sequence of zero and positive investment periods reduces the negative impact of the adverse investment effects. When the adverse investment effects are relatively high, the firm only remains in the market, when its initial sales volume and technology level allow to establish the optimal long run sequence of zero - positive investment policy without a long transition phase, otherwise the firm will finally quit business. In case of pronounced adverse investment effects the firm ceases business in any case.

Recapitulating there are three main features:

- Coexistence of an optimal steady state (origin) and an optimal limit cycle separated by a DNS-curve.
- A DNS-curve coincides with a limit cycle. After the collision only the steady state at the origin (zero investment) is optimal.
- An unstable limit cycle separating an optimal limit cycle and an interior steady state.

3.7 Appendix

The parameter reduction is achieved by the following linear time scaling transformation:

$$\tau = \delta t. \tag{3.24}$$

Then I reformulate the state and control variables as follows:

$$\bar{C}(\tau) = \frac{\delta}{\gamma f_1 h_2} C\left(\frac{\tau}{\delta}\right), \tag{3.25}$$

$$\bar{T}(\tau) = \frac{1}{h_2} T\left(\frac{\tau}{\delta}\right), \tag{3.26}$$

and

$$\bar{I}(\tau) = \frac{h_1 h_2}{\delta} I\left(\frac{\tau}{\delta}\right). \tag{3.27}$$

Furthermore, I define the parameters:

$$\begin{aligned} \rho &= \frac{r}{\delta}, \\ \beta_1 &= \frac{\gamma}{\delta}, \\ \beta_2 &= \frac{a_1 \delta}{\alpha f_1 h_1 h_2^2}, \\ \beta_3 &= \frac{a_2 \delta^2}{\alpha f_1 h_1^2 h_2^3}, \\ \beta_4 &= \frac{k}{h_1 h_2}. \end{aligned} \tag{3.28}$$

Now, substitution of (3.24)-(3.28) into the original optimal control model (3.5), (3.2), (3.3), and (3.4), including the functional forms (3.18)-(3.20), gives the following model:

$$\max_{\bar{I}} \frac{\alpha f_1 h_2}{\delta} \int_0^\infty e^{-\rho\tau} \left[\beta \bar{C} - \beta \bar{I} - \frac{\beta_3}{2} \bar{I}^2\right] d\tau, \tag{3.29}$$

subject to

$$\dot{\bar{C}} = \bar{T} - \beta_1 \bar{C} - \beta_4 \bar{C} \bar{I}, \tag{3.30}$$

$$\dot{\bar{T}} = \tilde{h}(\bar{T}) \bar{I} - \bar{T}, \tag{3.31}$$

$$\bar{I} > 0, \tag{3.32}$$

in which

$$\tilde{h}(T) = \frac{T^2}{1 + T^3}. \tag{3.33}$$

This model provides the basis for my numerical analysis. For the parameters (3.21) - (3.23) I get

$$\rho = 1.2, \quad \beta_1 = 1.5, \quad \beta_2 = 0.01, \quad \beta_3 = 0.005, \quad \beta_4 = 0.36 \tag{3.34}$$

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Chapter 4

Mathematical and Computational Aspects

4.1 An alternative approach

The following considerations are going back on an unpublished working paper by Brock and Dechert [2]. To be honest, in 1983 Brock and Dechert already describe a model with DNS-curves. There their configuration is simple; they take a model with two control variables, which separates to two subproblems both similar to Skiba's example in 1978:

$$V(x_0) = \max \int_0^\infty e^{-rt} \left[U\left(F(x_1) - \dot{x}_1\right) + U\left(F(x_2) - \dot{x}_2\right)\right] dt$$

$$(x_1(0), x_2(0)) = x_0 = (x_{10}, x_{20}).$$

Based on the one-dimensional case, obviously for certain values of the discount factor r the model has four steady states and two DNS-curves, which intersect and at the point of intersection there are four different optimal trajectory, each of them converging to one of the steady states. Additionally Brock and Dechert show, that their example does not depend on the separability of the two coordinates. By adding a sufficiently small non-separable perturbation term to the objective functional, the structure of the solution remains unchanged. Accordingly, an easy way to construct models with DNS-curves is by combining two one state model with Skiba-points. However, they construct a DNS-curve and prove the existence in a more general case, but they have never computed such a curve.

Besides the existence of a DNS-curve Brock and Dechert tried to characterize formally DNS-sets (albeit they did not use the notation DNS). However, they based their statements on the calculus of variations. Here I intend to restate their results in the optimal control approach.

4.1.1 Brock and Dechert's approach

Remember the model stated in the Appendix of Chapter 1, which I refer to by OCM

$$J(u; x_0) = \sup_{u \in U} \int_0^T e^{-rt} F(x(t), u(t)) dt \tag{4.1}$$

subject to the system dynamics

$$\dot{x} = f(x(t), u(t)) \quad x(0) = x_0, \tag{4.2}$$

where I consider an open planning horizon and scrap value identically zero.

For a given initial value x_0 an optimal solution $(x^*(t), u^*(t))$ must satisfy certain necessary conditions, which includes the existence of a continuous and piecewise continuously differentiable co-state variable $\lambda(t)$ fulfilling co-state dynamics

$$\dot{\lambda}(t) = \left(r - f_x\left(x^*(t), u^*(t)\right)\right) \lambda(t) - F_x\left(x^*(t), u^*(t)\right), \tag{4.3}$$

the effective maximum-principle

$$u^*(t) \in Argmax_{u \in U} H(x^*(t), u, \lambda(t)), \tag{4.4}$$

and the transversality conditions

$$\lim_{t \to T^*} e^{-rt} H(x^*(t), u^*(t), \lambda(t)) = 0, \tag{4.5}$$

whereas I assume that $\lambda_0 > 0$; thus without loss of generality $\lambda_0 = 1$ and in the Hamiltonian λ_0 can be omitted.

I pool all solutions fulfilling the F.O.N.C. (First Order Necessary Conditions) and a transversality condition in a set denoted by Ω :

Definition 20 *I define $(x(t), u(t), \lambda(t), T)$ to be element of $\Omega(x_0)$, if*

1. *$(x(t), u(t))$ is admissible of OCM(x_0) with time horizon T (confer Definition 1),*
2. *$\lambda(t)$ is a solution of the co-state dynamics 4.3 with $x(t)$ and $u(t)$ (on the interval $[0, T]$), and*
3. *the transversality conditions*

$$\lim_{t \to T} e^{-rt} H(x(t), u(t), \lambda(t)) = 0,$$

Next I pool all co-state values initializing a solution fulfilling the F.O.N.C. and transversality condition in a set denoted by Γ :

Definition 21

$$\Gamma(x_0) := \{\lambda(0) \mid (x(t), u(t), \lambda(t)) \in \Omega(x_0)\}.$$

Lemma 22 *The maximized Hamiltonian is defined to be*

$$H^0(x, \lambda) := \sup_{u \in U} H(x, u, \lambda).$$

When $(x(t), u(t), \lambda(t), T) \in \Omega(x_0)$, the value of the objective functional can easily be computed by evaluating the maximized Hamiltionian

$$H^0(x_0, \lambda(0)) = r \int_0^T e^{-rt} F(x(t), u(t))\, dt. \tag{4.6}$$

Proof. See Michel [6]. •

Corollary 23 *Assume the existence of an optimal solution (x^*, u^*), with $x^* = x_0$. Then*

$$r V(x_0) = \max_{\lambda \in \Gamma(x_0)} H^0(x_0, \lambda) \tag{4.7}$$

Finally I pool all initial co-state conditions constituting an optimal solution in a set denoted by Γ^* :

Definition 24

$$\Gamma^*(x_0) \quad := \quad \{\lambda \in \Gamma(x_0) \mid r\,V(x_0) = H^0(x_0, \lambda)\} \,. \tag{4.8}$$

If an optimal solution does not exist, the set $\Gamma(x_0)$ is empty, otherwise the cardinality of $\Gamma^*(x_0)$ is greater equal 1, i.e. $\#\Gamma^*(x_0) > 1$.

Definition 25 *I define the DB-set (Dechert-Brock-set) to be*

$$\mathbb{X} := \{x_0 \mid \#\Gamma^*(x_0) > 1\}$$

Lemma 26 *The DNS-set of 0CM is a subset of the DB-set of OCM.*

Proof. Obvious. •

4.2 Numerical aspects

First of all let me assume that optimal solutions exist, i.e. $\#\Gamma^*(x_0) > 0 \; \forall x_0$ feasible. For a given x_0, it would be straightforward to compute the solutions in OCM(XO) by applying Corollary 23, if I knew $\Gamma(x_0)$, which is a subset of the co-state space and a superset of $\Gamma^*(x_0)$. Given a co-state A, it is not target-oriented to solve the initial value problem 4.2, 4.3 combined with 4.4 and the initial values $(x(0), \lambda(0)) = (x_0, \lambda)$. At least a numerical algorithm would fail to answer the question, whether A belongs to $\Gamma(x_0)$, due to arithmetical and methodological errors, because verifying the transversality condition 4.5 raise difficulties (especially when T* equals infinity). Hence, my idea was to turn round and carry through the computation from finish to start.

4.2.1 Steady states and limit cylces

Besides unbounded evolvement (for instance a balance growth path) optimal solutions converge to attractors like steady states and limit cycles. In the models which I have investigated in the Chapters 2 and 3 unbounded optimal solution can be excluded due to increasing marginal costs and diminishing returns to scale. Moreover, I assume that there are no optimal strange attractors.

The first step for the investigated models has been to locate attractors fulfilling the F.O.N.C., particularly the state and the co-state dynamics, which is a first order differential equation system in explicit form:

$$\dot{y}(t) \quad = \quad G\,(y(t))\,, \tag{4.9}$$

where we assume G three times continuously differentiable (C^3 function). There are many textbooks covering the topic of dynamic systems; representatively, I want to cite [3] and [4].

A system being in a steady state means that it remains in this state forever. The momentum y is in a steady state zero, thus steady states of the system corresponds to the roots y of G in the mathematical model. Solving the non-linear equations

$$G(y) \quad = \quad 0$$

symbolically and numerically I have used MATHEMATICA's *Solve* or *FindRoot*, respectively.[1] I have used *Solve* to reduce the number of equations, the remaining equation (for the investigated models I have been able to reduce them to a single equation) I have solved numerically by *FindRoot*.

A cycle of (4.9) is a non-steady-state solution of this dynamical system, where the system starting at any state of this cycle returns after finite time to the starting point. The minimal possible return time is the period of the cycle, and does not depend on from which point of the cycle the system starts. A limit cycle is an isolated cycle. Unfortunately, in general for the detection of limit cycles an algorithm does not exist. One way to trace down limit cycles is to test steady states in the parameter space for Hopf-Bifurcations. At a supercritical Hopf-bifurcation a stable limit cycle, at a subcritical Hopf-bifurcation an unstable limit cycle (normally paired with a stable one) bifurcates. This modus operandi I employed to find limit cycles for the model in Chapter 3. The numerical bifurcation analysis I have done by Kuznetsov's package CONTENT. Further numerical tests for Hopf-Bifurcation, both for the model in Chapter 2 and Chapter 3, have not provided evidence for further limit cycles. In fairness, this cannot be considered as a stringent (numerical) proof for the absence of further limit cylces.

4.2.2 Stability

Steady states and limit cycles are classified by their stability characteristics.

Steady State

To check the stability of a steady state y, the right hand side of the dynamic system (4.9) has to be displayed in a power series:

$$y \quad = \quad G(y) = \quad G_y(\bar{y})(y-\bar{y}) + \frac{1}{2}(y-\bar{y})' G_{yy}(\bar{y})(y-\bar{y}) \ldots \tag{4.10}$$

The at y locally linearized system

$$y \quad = \quad G_y(\bar{y})(y-\bar{y}) \tag{4.11}$$

governs the stability of the steady state y. More precisely, the sign of the real part of the eigenvalues of the Jacobian Matrix $G_y(\bar{y})$ defines the stability of the steady state. The variable n_- denotes the number of eigenvalues with negative real part, n_+ the number with positive real part, and no the number of those, which are purely imaginary; I exclude steady states with an eigenvalue zero. Steady states with no = 0 are called hyperbolic. Hopf bifurcations occur at non-hyperbolic steady states. A hyperbolic steady state is stable, when $n_+ = 0$,

[1] I will neither describe nor specify the numerical algorithms implemented by Wolfram Research in MATHEMATICA, but only mention the used commands. I refer the interested reader to MATHEMATICA manuals and to pertinent literature. The precise description on the numerical algorithms one can find in any textbook on numerical mathematics, like [9]. Similar I will proceed with the numerical algorithms implemented in CONTENT, a package for bifurcation analysis. The interested reader finds the relevant theory in [4].

unstable, when $n_- = 0$, and saddle point stable, when $n_-n_+ \neq 0$. I have done the numerical computation of the eigenvalues of the Jacobian matrix G_y by the MATHEMATICA's *Eigensystem.*

Limit cycles

It is impossible to classify the stability of a limit cycle by just looking at the right hand side of the dynamical system (4.9). But I have to take a point at the limit cycle and I have to introduce a cross section to the cycle at this point. The cross section is a smooth hyperspace of codimension 1 (i.e. one dimension less than the dimension of y) intersecting the limit cycle at a nonzero angle. A solution of (4.9) starting on this hyperspace sufficiently close to the limit cycle, returns to this hyperspace; in this way a mapping, called Poincare map associated with the limit cycle, is defined. The intersection of the hyperspace with the limit cycle is a fix point of the Poincare map. The Jacobian matrix of the Poincare map at the fix point is called monodromy matrix. Obviously one eigenvalue (multiplier) equals 1, the other eigenvalues (multipliers) define the stability of the Poincare map in this fix point, which match the stability of the limit cycle. I have solved numerically the so-called variational equations about the limit cycle with MATHEMATICA's *NDSolve* in order to get the numerical values of the monodromy matrix. Subsequently I computed the multipliers of the Poincare map using *Eigensystem.*

4.2.3 Invariant Manifolds

Knowing the stability of the steady states and limit cycles I continue to compute their regions of convergence. To do this I have to compute the stable invariant manifolds of this limit sets.

Theorem 27 *Let y_0 be a hyperbolic steady state, $(n_0 = 0, n_-, n_+)$. A sufficiently small neighborhood of y_0 contains a smooth invariant manifold $W^s_{loc}(y_0)$, and a smooth (with respect to the reverse problem $\dot{y} = -G(y)$) invariant manifold $W^u_{loc}(y_0)$ of dimension n_- and n_+, respectively.*

$W^s_{loc}(y_0)$ is tangent at y_0 to T^s, and $W^u_{loc}(y_0)$ to T^u, where T^s is the generalized eigenspace corresponding to the union of all eigenvalues of the Jacobian $G_y(y_0)$ with negative real part and T^u is the generalized eigenspace corresponding to the union of all eigenvalues of the Jacobian $G_y(y_0)$ with positive real part.

Proof. See [7]. •

Continuing $W^u_{loc}(y_0)$ along solutions of $\dot{y} = G(y)$ defines the unstable invariant manifold $W^u(y_0)$, and continuing $W^s_{loc}(y_0)$ along solutions of $\dot{y} = -G(y)$ defines the stable invariant manifold $W^s(y_0)$. The manifold W^s and W^u have the same smoothness properties as G. Stable and unstable invariant manifold $W^{u,s}(LC)$ of a limit cycle LC are defined in a similar manner by using the Poincare map.

Backwards computation

Bringing the solution of OCM to mind, only the stable invariant manifold of an attractor of the system is of interest. Let me consider a saddle point stable steady state $(\bar{x}, \bar{\lambda})$ of the dynamics system (4.2)-(4.3) and the algebraic equation (4.4). For a given initial state x_0 computing a path on the stable invariant $W^s(\bar{x},\bar{\lambda})$ equals solving a boundary value problem on an infinite interval, where the solution stays closely to the steady state almost everywhere of the interval (cf. [10]).

The first step making this problem numerically tractable is to approximate by boundary value problems on a finite interval. Using the local linearisation T^s of W^s one gets boundary values closely to the steady state, which gives an excellent approximation of W^s. However, the length of the integration interval T is at this particular time of the solution procedure unknown. In order to get values (x_T,λ_T) in the local linearisation T^s, I utilize the n_- (general) eigenvectors corresponding to the eigenvalues with negative real part of the Jacobian of the state-costate dynamics evaluated in the steady state. In case of complex eigenvalues I take both the real and the imaginary part of the (general) eigenvectors. These vectors affixed at the steady state $(\bar{x}, \bar{\lambda})$ define T^s. E.g. in case of a two-dimensional stable manifold, I use these two vectors to define a closed curve in neighborhood of the steady state: when ζ_1 and ζ_2 define T^s, I define a closed curve

$$\mathcal{C}(\phi) = (x, \bar{\lambda}) + \varepsilon\Big(c(\mu_1)\,\zeta_1\,\cos(\phi) + c(\mu_2)\,\zeta_2\,\sin(\phi)\Big)\ \phi \in [-\pi, \pi], \quad (4.12)$$

which collapses to the steady state for $\varepsilon \to 0$. The weights $c(\mu_i)$ depend on the eigenvalues μ_i.

A solution of (4.2)-(4.3) starting at x_0 and arriving at $(x_T, \lambda_T) = \mathcal{C}(\tilde{\phi}) \in T^s$ is extremely sensitive on small perturbations in its initial values $(x_0,\lambda(0))$, because in the neighborhood of the steady state the influence of the unstable invariant manifold W^u dominates. Therefore, one favorably reverses time (mathematically formulated, one multiplies the right hand side of (4.2) and (4.3) by -1). The numerical condition of the algorithm improves significantly (as long as the solution does not close in on another attractor of OCM).

Numerically I favored a one-step shooting algorithm to solve this boundary value problem. Multi-step shooting or collocation algorithms are possible. I have decided to use the standard MATHEMATICA algorithms and to avoid implementations if it has been possible. MATHEMATICA does not supply non-linear boundary value problem solvers; for a one-step shooting algorithm *NDSolve* suffices to solve the initial value problem. To make the one-step shooting algorithm reliable I have used another feature of MATHEMATICA: the possibility to employ easily higher precisions. Using a precision of 22 digits even long range shots have been become possible and reasonable. The limitation is that higher precision causes longer computation time.

In a similar way the computation of the stable invariant manifold of a saddle point stable limit cycle works. In a Poincare map of the limit cycle I find a local linearisation of the intersection of the invariant manifolds with the Poincare map by investigating the eigenvalues and -vectors of the monodromy matrix. In the

Poincare map this linearisation yields a point p_1 close to the fixed point $(\bar{x}, \bar{\lambda})$. A one step reverse Poincare mapping yield another point p_0, where the convex combination of these two points

$$\mathcal{C}(\phi) = \left(\phi p_0 + (1-\phi) p_1\right) \qquad \phi \in [0,1], \tag{4.13}$$

gives excellent initial values close the stable invariant manifold of the limit cycle. These initial values I use for backward shooting to hit a given state x_0.

Power series method

A mathematically more profound method is to approximate the stable invariant manifold by power series. Whereas in the linearisation and shooting method only first order derivatives are used (confer the Jacobian or monodromy matrix), I have employed higher order derivatives of the right hand side of the dynamic system at a steady state (or a fixed point of the Poincare map) in order to get a power series expansion of the stable invariant manifold. The higher the order of the used derivatives, the higher the order of the power series expansion, the higher is the accuracy of this method ([8], [3], [4]).

Especially, in steady states higher order derivatives can be analytically (symbolically) computed. Hence, values of higher order derivatives for a given steady state bear only arithmetic (and no methological) errors. However, in a two state (and hence a two co-states) problem there are 16 first order derivatives, 64 second order derivatives and so on. Hence the arithmetic errors accumulate and sweep (at least partly) off the information provided by higher order derivatives in order to compute a more accurate approximation of the stable invariant manifold. Nevertheless in mediate distances to the steady states the power series expansion of the manifold was sufficiently accurate.

In case of a limit cycle, it was not so easy. At least, I do not know, how to compute analytically higher order derivatives at the fixed point in the Poincare map. Hence, I had to compute this derivatives numerically; this means besides arithmetic errors metholocial errors are added. I implemented the computation of the second and third order derivatives at the fixed point in the Poincare map limiting the methodological error in the difference quotients to the order $O(|h|^3)$ for $h \to 0$. A lot of computational time was used, but the accuracy of the power series expansion was not satisfactory.

As the power series expansion of invariant manifolds was not the main goal of my work for the moment I have not followed up this matter.

4.2.4 DNS-sets

A DNS-set separating two disjunct limit sets is a subset of the intersection of the regions of convergence of the limit sets. The region of convergence of a limit set is the projection (into the state space) of the unification of the stable invariant manifolds of all attractors the limit set consists of. Computing the stable invariant manifolds of the attractors is the same as computing the regions of convergence. Moreover, having the stable invariant manifold computed means also that for a given initial state *XQ* the with respect to this manifold corresponding co-state A is known. When x_0 belongs to regions of convergence of several limit sets, the question to which region(s) of optimal convergence x_0 belongs is then easy to answer by applying Corollary 23.

The lattice-method to compute DNS-sets

However, keeping the complete information of a manifold in memory is problematic, even if I reduce the information to a quasi infinite data structure (like interpolation points, which define an approximation of this (hyper)surface). On the other hand, solving separately for each investigated x_0 the boundary value problem with the boundaries $(x(-\mathcal{T}), \lambda(-\mathcal{T})) = \mathcal{C}(\phi_{x_0})$ and $x(0) = x_0$ is costly; particularly locating ϕ_{x_0} is time consuming. What's more, for each state on the path between $(x_0, \lambda(0))$ and $\mathcal{C}(\phi_{x_0})$ all the needed information about the manifold is computed.

On a reasonable sector of the (two-dimensional) state space I put a lattice of an array of rectangular cells. For each cell of the lattice and for each considered limit set, I save the highest observed objective value. I solve a lot of initial value problems of the time reversed problem with initial values $(x(-\mathcal{T}), \lambda(-\mathcal{T})) = \mathcal{C}(\phi)$, where ϕ varies in $[-\pi, \pi]$ or $(0, 1]$, respectively. Along the computed trajectories I check the traversed cells, if the up to now highest observed objective value can be raised. This procedure I apply to all limit sets. Then, for each cell one can decide to which region of optimal convergence this cell belongs to by comparing the observed objective values. Finally, I mark the cells of the lattice that belong different regions of optimal convergence by different colors. The boundary of the different colored regions marks the DNS-set.

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Curriculum Vitae

Personalia

Name: Josef Leopold Haunschmied

Geburtsdatum und -ort: 22.6.1963 in Linz an der Donau

Staatsbürgerschaft: Österreich

Familienstand: Long-term relationship

Ausbildung

Studium "Technische Mathematik" mit Studienzweig "Wirtschaftsmathematik" an der Technischen Universität Wien.

Jänner 1995: Dipl.-Ing. (mit Auszeichnung)

Nichtwissenschaftliche Tätigkeiten

Bundesheer (Oberleutnant der Reserve), EDV, Internatsaufsicht, Landwirtschaft.

Wissenschaftliche Karriere

Mai 1993 - März 1995: Studienassistent am Institut für Ökonometrie, OR und Systemtheorie an der Technischen Universität Wien.

März 1995 - März 2001: Vertragsassistent (Teilzeit, 50%) am Institut für Ökonometrie, OR und Systemtheorie an der Technischen Universität Wien.

August 1999 - Jänner 2000: Projektmitarbeit (Teilzeit, 50%) an dem ITF und WWFF Projekt namens "Virtueller Technologiepark - Ressourcenschonendes Bauen".

Jänner 2000 - Februar 2001: Projektmitarbeit (Teilzeit, 50%) an dem FWF Projekt "P11711-OEK" namens "Dynamic Law Enforcement".

März 2001 - ?: Vertragsassistent (Vollzeit) am Institut für Ökonometrie, OR und Systemtheorie an der Technischen Universität Wien.

In den Studienjahren 1994-1999 wissenschaftlicher Koordinator des Institutes für das Erasmus-Studentenaustauschprogramm (etwa 40 Studentenaustausche).

Lokaler Koordinator (1997-1999) des Tempus Tacis Projektes T_JEB-10278-96.
Lokaler Koordinator (1999-2002) des Tempus Phare Proj. AC_JEB-13425-98.