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Volume Inequalities for Minkowski Valuations

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Abstract

In this thesis, new Orlicz–Brunn–Minkowski inequalities are established for rigid motion compatible Minkowski valuations of arbitrary degree. These extend classical log-concavity properties of intrinsic volumes and generalize seminal results of Lutwak and others. Two different approaches which refine previously employed techniques are explored. It is shown that both lead to the same class of Minkowski valuations for which these inequalities hold. This is a joint work with Lukas Parapatits, Franz Schuster and Manuel Weberndorfer.

The second focus of this thesis lies on the generalization of Lutwak's volume inequalities for polar projection bodies of all orders to polarizations of Minkowski valuations generated by o-symmetric zonoids of revolution. This is based on generalizations of the notions of centroid bodies and mixed projection bodies to such Minkowski valuations. A new integral representation is used to single out Lutwak's inequalities as the strongest among these families of inequalities, which in turn are related to a conjecture on affine quermassintegrals. In the dual setting, a generalization of Leng and Lu's volume inequalities for intersection bodies of all orders is proved. These results are related to Grinberg's inequalities for dual affine quermassintegrals.

Kurzfassung

In dieser Arbeit werden neue Orlicz-Brunn-Minkowski-Ungleichungen für Minkowski-Bewertungen beliebigen Grades, die mit Bewegungen verträglich sind, bewiesen. Diese Ungleichungen erweitern die klassische Log-Konkavität der intrinsischen Volumina auf allgemeinere Funktionale und verallgemeinern grundlegende Resultate von Lutwak und anderen. Zwei verschiedene Zugänge, die zuvor bekannte Techniken verfeinern, werden untersucht. Es wird gezeigt, dass beide auf dieselbe Klasse von Minkowski-Bewertungen führen, für welche diese Ungleichungen gelten. Dies sind Resultate aus einer gemeinsamen Arbeit mit Lukas Parapatits, Franz Schuster und Manuel Weberndorfer.

Der zweite Fokus dieser Arbeit liegt auf der Verallgemeinerung von Lutwaks Volumsungleichungen für polare Projektionenkörper beliebiger Ordnung auf Minkowski-Bewertungen, die von o-symmetrischen Rotationszonoiden erzeugt werden. Hierzu werden die Begriffe des Schwerpunktkörpers und der gemischten Projektionenkörper auf solche Minkowski-Bewertungen erweitert. Eine neue Integraldarstellung wird verwendet, um zu zeigen, dass unter diesen Ungleichungen Lutwaks Resultate die stärksten Ungleichungen darstellen. Diese Ungleichungen werden wiederum mit einer Vermutung über affine Quermaßintegrale in Beziehung gesetzt. In der dualen Theorie wird eine Verallgemeinerung von Lengs und Lus Ungleichungen für Schnittkörper beliebiger Ordnung bewiesen. Diese Resultate werden in Beziehung zu Grinbergs Ungleichungen für duale affine Quermaßintegrale gesetzt.

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Contents

Introduction

The problem of finding sharp bounds for the volume of projection bodies, given the volume of the original body, is a central quest in convex geometric analysis. In [Pet71], Petty conjectured a lower bound for the volume of the projection body. This problem is still unsolved but has sparked discoveries in neighboring areas. Among many other results, it has led to investigations of log-concavity properties of the volume of projection bodies and to the investigation of volume bounds for *polar* projection bodies.

The most famous log-concave functional in geometry is the volume on convex bodies, where the log-concavity is expressed by the classical Brunn-Minkowski inequality. This fact lies at the very heart of the Brunn–Minkowski theory and has sparked interest in the investigation of a variety of log-concave geometric functionals. In [Lut93a], Lutwak investigated log-concavity of a different family of functionals and established not only Brunn–Minkowski type inequalities for the volume of projection bodies, but for all the *intrinsic volumes* of projection bodies of arbitrary order.

In a different line of research, Lutwak [Lut84] discovered that an *affine* isoperimetric inequality of Petty [Pet71] for *polar* projection bodies is not only significantly stronger than the Euclidean isoperimetric inequality, but in fact an optimal version of this classical inequality. For the tremendous impact of Petty's inequality and its generalizations see, e.g., [Hab09b; Lut00b; Lut10b; Wan12; Zha99]. In [Lut85], Lutwak established a version of this inequality for projection bodies of all orders, the Lutwak–Petty projection inequalities. While all of these projection bodies are $\mathrm{SO}(n)$-equivariant, the projection body has a special place in affine geometry: Ludwig [Lud02; Lud05] characterized this operator as the unique continuous Minkowski valuation which is translation-invariant and $\mathrm{GL}(n)$-contravariant (see [Hab12; Lud03; Lud10a; Par14a; Par14b; Sch12a; Wan11] for related results).

Recently, translation-invariant and $\mathrm{SO}(n)$-equivariant Minkowski valuations have been investigated by a number of authors ([Kid06], [Sch07], [Sch10], [Sch15]). This has led to a number of new discoveries, extending previously known results for projection bodies. Various generalizations of the polar Petty projection inequality have since been obtained (see [Bör13; Hab09b; Lut00b; Lut10a] for extensions to the L_p and the Orlicz–Brunn–Minkowski theories and [Wan12] for the extension to sets of finite perimeter) and also Lutwak's Brunn–Minkowski type inequalities have been generalized in various directions [Par12; Sch06; Sch10]).

A theory for star bodies, dual to the Brunn–Minkowski theory for convex bodies, also has its origin in the work of Lutwak [Lut88a]. One of its central inequalities is the Busemann intersection inequality [Bus53], which relates the volume of a star body to the volume of its intersection body. The intersection body was first introduced by Lutwak in [Lut88b] and ever since a number of authors has contributed to the research on the duality between projection and intersection bodies (confer [Sch14] and [Gar95] for more details). Recently it was shown by Leng and Lu [Lu08] that inequalities analogous to the Busemann intersection inequality hold also for intersection bodies of all orders.

The aim of this thesis is to find a common generalization of all the previously known log-concavity properties of intrinsic volumes of Minkowski valuations, to extend the Lutwak–Petty projection inequalities to a certain class of Minkowski valuations and to extend Leng and Lu's result to a subclass of radial Minkowski valuations.

The thesis is structured as follows:

In Chapter 1, we recall the basic definitions from convex geometry and harmonic analysis, including the Radon transform and convolution on the sphere.

In Chapter 2, valuations, in particular Minkowski and radial Minkowski valuations, are defined. The most important examples, the projection and intersection bodies of all orders, are generalized.

In Chapter 3 we collect known results on log-concavity of Minkowski valuations. We show that a new *integration* operator on Minkowski valuations [Ale04a; Ber07a; Sch15] on one hand and a recent representation theorem for Minkowski valuations [Sch; Sch15] on the other hand lead to a natural class of Minkowski valuations which exhibit log-concavity properties. This is a joint work with Lukas Parapatits, Franz Schuster and Manuel Weberndorfer [Berb]. All the Brunn–Minkowski inequalities for Minkowski valuations established before turn out to be special cases of our new results. From new monotonicity properties of these Minkowski valuations, we are able to deduce a complete characterization of equality cases without any smoothness assumptions that were required before.

Moreover, all previously obtained and new Brunn–Minkowski inequalities for Minkowski valuations are shown to not only hold for Minkowski addition but for all *commutative* Orlicz–Minkowski additions (introduced in [Gar14]) of convex bodies. This includes, in particular, all L_p Minkowski additions.

In Chapter 4 we recall the Lutwak–Petty projection inequalities and Leng and Lu's intersection inequalities and establish generalizations to certain classes of Minkowski valuations and radial Minkowski valuations, respectively. To this end, we generalize notions and techniques of Lutwak [Lut85] and Haberl and Schuster [Hab]. These results will appear in [Bera].

CHAPTER 1

Background from convex geometry

1.1 Background material on convex bodies

We collect in this section some basic facts from convex geometry, in particular, on additions of convex bodies, inequalities for mixed volumes and their duals, and affine variants. As general reference for this material we recommend the book by Schneider [Sch14] and the article [Gar14].

Let $\mathcal{K}^n$ denote the space of convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff metric and let $\mathcal{K}^n_n$ denote the space of full-dimensional convex bodies. We will assume that $n \geq 3$ unless otherwise specified. By $\mathcal{K}^n_o$ we denote the set of convex bodies containing the origin and by $\mathcal{K}^n_{(o)}$ the subset of all $K \in \mathcal{K}^n$ containing the origin in their interiors. For a convex body $K \in \mathcal{K}^n$, the *support function* at a point $u \in S^{n-1}$ is defined by

$$h(K,u) = \max\{x \cdot u \,:\, x \in K\}.$$

This definition implies that $h(\vartheta K,u) = h(K,\vartheta^{-1}u)$ for every $u \in S^{n-1}$ and $\vartheta \in \mathrm{SO}(n)$. Since every twice continuously differentiable function on S^{n-1} is a difference of support functions (see, e.g., [Sch14, p. 49]), the subspace spanned by differences of support functions $\{h(K,\cdot) - h(L,\cdot) : K, L \in \mathcal{K}^n\}$ is dense in $C(S^{n-1})$. The *Steiner point* $s(K)$ of $K \in \mathcal{K}^n$ is defined by

$$s(K) = \frac{1}{\kappa_n} \int_{S^{n-1}} h(K,u)u \, du.$$

Here and in the following we use du to denote integration with respect to spherical Lebesgue measure and κ_m for the m-dimensional volume of the unit ball in $\mathbb{R}^m$.

For $K, L \in \mathcal{K}^n$ and $s, t \geq 0$, the support function of the *Minkowski combination*

$$s\,K + t\,L = \{sa + tb \colon a \in K, b \in L\}$$

is given by

$$h(s\,K + t\,L,\cdot) = s\,h(K,\cdot) + t\,h(L,\cdot).$$

By a classical result of Minkowski, the volume of a Minkowski linear combination $\lambda_1 K_1 + \cdots + \lambda_m K_m$, where $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, can be expressed as a homogeneous polynomial of degree n,

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{j_1,\ldots,j_n=1}^{m} V(K_{j_1},\ldots,K_{j_n})\lambda_{j_1}\cdots\lambda_{j_n}, \tag{1.1}$$

where the coefficients $V(K_{j_1},\ldots,K_{j_n})$, called *mixed volumes* of $K_{j_1},\ldots,K_{j_n}$, depend only on $K_{j_1},\ldots,K_{j_n}$ and are symmetric in their arguments. For $K, L \in \mathcal{K}^n$ and for $0 \le i \le n$, we denote the mixed volume with i copies of K and $n-i$ copies of L by $V(K[i],L[n-i])$. By $W_m(K,L)$ we denote the mixed volume $V(K[n-m-1],B[m],L)$ with $n-m-1$ copies of K and m copies of the Euclidean unit ball B. For $K, K_1,\ldots,K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1,\ldots,K_i)$, we write $V_i(K,\mathbf{C})$ instead of $V(K,\ldots,K,K_1,\ldots,K_i)$. The mixed volume with i copies of K and $n-i$ copies of B will be abbreviated by $W_{n-i}(K) = V(K[i],B[n-i])$ and is called the *quermassintegral of order* $n-i$. Sometimes a different normalization is used, namely the *ith intrinsic volume* $V_i(K)$ of K defined by

$$\kappa_{n-i}V_i(K) = \binom{n}{i} W_{n-i}(K).$$

A special case of (1.1) is the classical *Steiner formula* for the volume of the parallel set of K at distance $r > 0$,

$$V(K + rB) = \sum_{i=0}^{n} r^i \binom{n}{i} W_i(K) = \sum_{i=0}^{n} r^{n-i}\kappa_{n-i}V_i(K).$$

For $K_1,\ldots,K_{n-1} \in \mathcal{K}^n$, there is a uniquely determined finite Borel measure on S^{n-1}, the *mixed area measure* $S(K_1,\ldots,K_{n-1},\cdot)$, such that for every $K \in \mathcal{K}^n$,

$$V(K_1,\ldots,K_{n-1},K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)dS(K_1,\ldots,K_{n-1},u).$$

In the following, $S(\cdot) := S(B[n-1],\cdot)$ will again denote the spherical Lebesgue measure. Associated with a convex body $K \in \mathcal{K}^n$ is a family of Borel measures, denoted by $S_i(K,\cdot) := S(K[i],B[n-i-1],u)$, $0 \le i \le n-1$, on S^{n-1}, called the *area measures of order* i of K. They are uniquely determined by the property that

$$W_{n-1-i}(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)\,dS_i(K,u) \tag{1.2}$$

for all $L \in \mathcal{K}^n$. If $K \in \mathcal{K}^n_n$, then, by a theorem of Aleksandrov–Fenchel–Jessen (see, e.g., [Sch14, p. 449]), each of the measures $S_i(K,\cdot)$, $1 \le i \le n-1$, determines K up to translations. For $1 \le j \le n-1$ and $r > 0$, the area measures of lower order satisfy a Steiner type formula

$$S_j(K + rB,\cdot) = \sum_{i=0}^{j} r^{j-i}\binom{j}{i} S_i(K,\cdot).$$

A body $K \in \mathcal{K}^n$ is of class C^2_+ if the boundary of K is a C^2 submanifold of $\mathbb{R}^n$ with everywhere positive curvature. In this case, each measure $S_i(K,\cdot)$, $0 \le i \le n-1$, is absolutely continuous with respect to spherical Lebesgue measure and its density is (up to a constant) given by the ith elementary symmetric function of the principal radii of curvature of K.

The center of mass (*centroid*) of every area measure of a convex body is at the origin, that is, for every $K \in \mathcal{K}^n$ and all $i \in \{0,\dots,n-1\}$, we have

$$\int_{S^{n-1}} u \, dS_i(K,u) = o.$$

The set $\mathcal{S}_i$ of all area measures of order i of convex bodies in $\mathcal{K}^n$ is dense in the set of all non-negative finite Borel measures on S^{n-1} with centroid at the origin, endowed with the weak topology, if and only if $i = n-1$. However, $\mathcal{S}_i - \mathcal{S}_i$, $1 \le i \le n-1$, is dense in the set $\mathcal{M}_o(S^{n-1})$ of all *signed* finite Borel measures on S^{n-1} with centroid at the origin (see, e.g., [Sch14, p. 477]).

The Minkowski problem poses the question, which Borel measures on S^{n-1} can appear as area measures of order $n-1$ of a convex body with non-empty interior. The solution is given by Minkowski's existence theorem. It states that a Borel measure on S^{n-1} is the surface area of convex body with non-empty interior, which is unique up to translation, if and only if the measure has centroid at the origin and is not concentrated on any great subsphere of S^{n-1}. Since the mixed area measure $S(K_1,\dots,K_{n-1},\cdot)$ satisfies the assumptions of Minkowski's existence theorem, one can define for $K_1,\dots,K_{n-1} \in \mathcal{K}^n_n$ the associated *mixed body* $[K_1,\dots,K_{n-1}]$ by

$$S_{n-1}([K_1,\dots,K_{n-1}],\cdot) := S(K_1,\dots,K_{n-1},\cdot). \tag{1.3}$$

The convex body $[K_1,\dots,K_{n-1}]$ is uniquely determined up to translations. It was first introduced by Firey [Fir67] and later studied by Lutwak [Lut86b]. In the following we will abbreviate $[K]_i := [K[i], B[n-1-i]]$, for $0 \le i \le n-1$. These bodies satisfy the following inequality.

Lemma 1.1 [Lut86b] *For $K \in \mathcal{K}^n_n$ and $0 \le i \le n-1$,*

$$V([K]_i)^{n-1} \ge V(B)^{n-i-1} V(K)^i,$$

with equality if and only if K is a ball.

An important inequality for the volume of a convex body $K \in \mathcal{K}^n_n$ is the *Blaschke–Santaló inequality.* It states that

$$V(K)V(K^*) \le \kappa_n^2 \tag{1.4}$$

with equality if and only if K is an ellipsoid. Here κ_n denotes the volume of B in $\mathbb{R}^n$.

While the volume of a convex body is given by the integral representation

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h(K,u) dS_{n-1}(K,u),$$

there is a different and sometimes more convenient way to write it by introducing polar coordinates on the sphere, namely

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^n dS(u). \tag{1.5}$$

For mixed volumes and quermassintegrals respectively, there are two well-known inequalities which will be needed later on. Mixed volumes can be compared to the volume of the convex bodies by

$$V(K_1) \cdots V(K_n) \leq V^n(K_1, \dots, K_n), \tag{1.6}$$

where equality holds for $K_1, \dots, K_n \in \mathcal{K}_n^n$ if and only if they are pairwise homothetic (see e.g. [Sch14] or [Ale99]). For $0 \leq i < j \leq n-1$, the following inequality between quermassintegrals of different degree holds:

$$W_j(K)^{n-i} \geq \kappa_n^{j-i} W_i(K)^{n-j}. \tag{1.7}$$

A fundamental inequality for mixed volumes is the general *Minkowski inequality* (see [Sch14, p. 427]): If $2 \leq i \leq n$ and $K, L \in \mathcal{K}^n$ have dimension at least i, then

$$W_{n-i}(K,L)^i \geq W_{n-i}(K)^{i-1} W_{n-i}(L), \tag{1.8}$$

with equality if and only if K and L are homothetic.

There are also affine invariant analogs to quermassintegrals. For $0 < k < n$ and $K \in \mathcal{K}_n^n$, the *affine quermassintegral* is defined by

$$A_{n-k}(K) := \frac{\kappa_n}{\kappa_k} \left(\int_{G(n,k)} V_k(K|E)^{-n} d\nu_k(E) \right)^{-1/n}.$$

We also set $A_0(K) := V(K)$ and $A_n(K) = \kappa_n$. The affine quermassintegrals were first defined by Lutwak in [Lut84]. They are $SL(n)$ invariant, translation invariant (affine invariant) and satisfy Brunn–Minkowski type inequalities. Lutwak also conjectured in [Lut88a] a volume bound for these expressions:

Conjecture 1.2 [Lut88a] *For* $0 \leq i<n$ *and* $K \in K_n^n$*,*

$$\kappa_n^i V(K)^{n-i} \leq A_i(K)^n.$$

Of these inequalities, only the cases $i = n-1$ and $i = 1$ are known to be true; they follow from the Blaschke–Santaló inequality and the Petty projection inequality, respectively.

Another affine invariant functional that we will be concerned with is the *affine surface area*. For $K \in \mathcal{K}_n^n$ with positive continuous curvature function $f(K, \cdot)$, it is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K,u)^{\frac{n}{n+1}} dS(u).$$

The affine surface area satisfies the affine isoperimetric inequality, namely

$$\Omega(K)^{n+1} \leq n^{n+1}\kappa_n^2 V(K)^{n-1},$$

with equality if and only if K is an ellipsoid (cf. [Sch14] for further references).

1.2 Background material on star bodies

A *star body* is a compact star-shaped set (with respect to the origin) with positive continuous radial function. The set of all star bodies in $\mathbb{R}^n$ is denoted by $\mathcal{S}_o^n$, and we endow this space with the radial metric. The *radial function* of $K \in \mathcal{S}_o^n$ is defined by

$$\rho(K,u) = \max\{\lambda > 0\colon \lambda u \in K\}, \quad u \in S^{n-1}.$$

The i-radial combination of two star bodies $K, L \in \mathcal{S}_o^n$ for $i = 1, \dots, n-1$ is the star body whose radial function satisfies

$$\rho^i(K \tilde{+}_i L,\cdot) = \rho^i(K,\cdot) + \rho^i(L,\cdot),$$

where $\tilde{+}_1$ is called *radial addition* and $\tilde{+}_{n-1}$ is called *radial Blaschke addition*. For $K_1, \dots K_m \in \mathcal{S}_o^n$ and $\lambda_1, \dots, \lambda_m \geq 0$, the *radial combination* $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m$ is defined by

$$\rho(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m, \cdot) = \lambda_1 \rho(K_1, \cdot) + \cdots \lambda_m \rho(K_m, \cdot). \tag{1.9}$$

There is again a polynomial expansion

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \dots \lambda_{i_n} \tilde{V}(K_{i_1}, \dots, K_{i_n}),$$

where the function $\tilde{V}\colon (\mathcal{S}_o^n)^n \to \mathbb{R}$ is the *dual mixed volume* given explicitely by

$$\tilde{V}(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1, u) \cdots \rho(K_n,u) du.$$

We will abbreviate $\tilde{V}_i(K,L) = \tilde{V}(K[n-i], L[i])$. The special case $\tilde{W}_i(K) = \tilde{V}_i(K,B)$ is called the *ith dual quermassintegral*. As was shown by Lutwak [Lut79], these quantities satisfy the integral representation

$$\tilde{W}_{n-k}(K) = \frac{\kappa_n}{\kappa_k} \int_{G(n,k)} V_k(K \cap E) d\nu_k(E).$$

We will also write $\tilde{v}$ and $\tilde{w}$ for the corresponding functionals on $\mathbb{R}^{n-1}$.

The *dual affine quermassintegrals* are defined for $K \in \mathcal{S}_o^n$ and $0 < k < n$ by

$$\tilde{A}_{n-k}(K) := \frac{\kappa_n}{\kappa_k} \left(\int_{G(n,k)} V_k(K \cap E)^n d\nu_k(E) \right)^{\frac{1}{n}}.$$

We also set $\tilde{A}_0(K) = V(K)$ and $\tilde{A}_n(K) = \kappa_n$. They were first proposed by Lutwak for full-dimensional convex bodies and later extended by Gardner to bounded Borel sets [Gar07]. We will in the following only be concerned with star bodies. From Jensen's inequality it follows that

$$\tilde{W}_i(K) \leq \tilde{A}_i(K).$$

The dual affine quermassintegrals also satisfy

$$\kappa_n^{i-1} \tilde{A}_{n-i}(K)^n \leq \kappa_n^{n-1} V(K)^i \tag{1.10}$$

for $1 < i < n$. This was shown by Busemann and Straus in [Bus60] and independently by Grinberg in [Gri91] for $K \in \mathcal{K}_n^n$ and was later extended by Gardner [Gar07]. Grinberg also proved that these functionals are invariant under volume-preserving linear transformations. The special case for $i = n-1$ of this inequality is the Busemann intersection inequality (cf. Section 4.2).

1.3 Radon transforms and convolutions

We recall basic facts about Radon transforms on Grassmannians and convolutions of functions on S^{n-1}, where we mainly follow [Sch15] and [Sch07].

In the following, $G(n,i)$ will denote the manifold of i-dimensional subspaces of $\mathbb{R}^n$, the *i-Grassmannian*. For $1 \leq i \neq j \leq n-1$ and $F \in G(n,j)$, we denote by $G(n,i)^F$ the submanifold of $G(n,i)$ consisting of all $E \in G(n,i)$ that contain (respectively are contained in, depending on i and j) F. The *Radon transform* $R_{i,j} \colon L^2(G(n,i)) \to L^2(G(n,j))$ is defined by

$$(R_{i,j}f)(F) = \int_{G(n,i)^F} f(E) d\nu_i^F(E),$$

where ν_i^F is the unique invariant probability measure on $G(n,i)^F$. The Radon transform is a continuous linear operator and $R_{i,j}$ is the adjoint of $R_{j,i}$, in the sense that

$$\int_{Gr_{(n,j)}} (R_{i,j}f)(F) g(F) d\nu_j(F) = \int_{Gr_{(n,i)}} f(E)(R_{j,i}g)(E) d\nu_i(E) \tag{1.11}$$

for $f \in L^2(G(n,i))$ and $g \in L^2(G(n,j))$. For $f \in L^2(G(n,i))$, we write $f^\perp$ for the function given by $f^\perp(E) = f(E^\perp)$. With this notation, we have

$$(R_{i,j}f)^\perp = R_{n-i,n-j} f^\perp. \tag{1.12}$$

For $1 \leq i < j < k \leq n-1$ we also have that

$$R_{i,k} = R_{j,k} \circ R_{i,j} \quad \text{and} \quad R_{k,i} = R_{j,i} \circ R_{k,j}.$$

Next we recall the definition of the convolution of functions in $C(S^{n-1})$. Let $\mathcal{M}(S^{n-1})$ denote the space of signed Borel measures on the sphere and $\mathcal{M}_+(S^{n-1})$ the subset of non-negative measures. We will be particularly interested in $\mathrm{SO}(n-1)$-invariant measures, where $\mathrm{SO}(n-1)$ is the subgroup of $\mathrm{SO}(n)$ stabilizing an arbitrary but fixed pole $\bar{e} \in S^{n-1}$.

We call $\mu \in \mathcal{M}(S^{n-1})$ *zonal* if $\theta\mu := \mu \circ \theta^{-1} = \mu$ for every $\theta \in \mathrm{SO}(n-1)$. The set of zonal measures will be denoted by $\mathcal{M}(S^{n-1}, \bar{e})$. Since S^{n-1} is diffeomorphic to the homogeneous space $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, there is a natural identification between $C(S^{n-1})$ and right-$\mathrm{SO}(n-1)$-invariant functions in $C(\mathrm{SO}(n))$, by setting

$$\breve{f}(\theta) = f(\theta\bar{e}), \quad f \in C(S^{n-1}).$$

Conversely, it is possible to define for $f \in C(\mathrm{SO}(n))$ a continuous function $\hat{f}$ on S^{n-1} by

$$\hat{f}(\eta\bar{e}) = \int_{\mathrm{SO}(n-1)} f(\eta\theta)d\theta.$$

A *zonal* function $f \in C(\mathrm{SO}(n))$ satisfies that $f = \breve{\hat{f}}$. There is also an identification between $\mathcal{M}(S^{n-1})$ and right-$\mathrm{SO}(n-1)$-invariant measures in $\mathcal{M}(\mathrm{SO}(n))$, by setting

$$\langle \breve{\mu}, f \rangle = \langle \mu, \hat{f} \rangle.$$

Since $\mathrm{SO}(n)$ is a compact Lie group, the *convolution* $\sigma * \tau$ of signed measures σ, τ on $\mathrm{SO}(n)$ can be defined by

$$\int_{\mathrm{SO}(n)} f(\vartheta)\, d(\sigma * \tau)(\vartheta) = \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} f(\eta\theta)\, d\sigma(\eta)\, d\tau(\theta), \qquad f \in C(\mathrm{SO}(n)).$$

For $f \in C(\mathrm{SO}(n))$ and $\mu \in \mathcal{M}(\mathrm{SO}(n))$, the convolutions $f * \mu$ and $\mu * f$ are defined by

$$(f * \mu)(\eta) = \int_{\mathrm{SO}(n)} f(\eta\vartheta^{-1})d\mu(\vartheta), \quad (\mu * f)(\eta) = \int_{\mathrm{SO}(n)} \vartheta f(\eta)d\mu(\vartheta).$$

Using the identification of $C(S^{n-1})$ and $\mathcal{M}(S^{n-1})$ with right-$\mathrm{SO}(n-1)$-invariant functions and measures on $\mathrm{SO}(n)$, respectively, the convolution of measures on $\mathrm{SO}(n)$ induces a convolution product of functions and measures on S^{n-1}. We denote the set of continuous zonal functions on S^{n-1} by $C(S^{n-1},\bar{e})$. Clearly, zonal functions depend only on the value of $u \cdot \bar{e}$. The convolution of $f \in C(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1})$ is then defined via the convolution of $\breve{f}$ and $\breve{\mu}$ on $\mathrm{SO}(n)$ by

$$(f * \mu)(\eta\bar{e}) = \int_{\mathrm{SO}(n)} f(\eta\theta^{-1}\bar{e})d\breve{\mu}(\theta).$$

For $\mu \in \mathcal{M}(S^{n-1})$ and $f \in C(S^{n-1})$, we introduce the canonical pairing by

$$\langle \mu, f \rangle = \int_{S^{n-1}} f(v)d\mu(v).$$

There is a simpler form of the spherical convolution of zonal measures. If we write a point $u \in S^{n-1}$ in the form $u = \eta\bar{e}$ for some $\eta \in \mathrm{SO}(n)$, then for $f \in C(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1}, \bar{e})$

the convolution is given by

$$(f * \mu)(\eta\bar{e}) = \int_{S^{n-1}} f(\eta v) d\mu(v),$$

where $f * \mu$ is again a function on the sphere. This definition is independent of the choice of η. For $f \in C(S^{n-1})$ and $\mu \in \mathcal{M}(S^{n-1}, \bar{e})$, the convolution on the sphere induces an SO(n)-equivariant operation, that is

$$(\theta f) * \mu = \theta(f * \mu). \tag{1.13}$$

For $\sigma \in \mathcal{M}(S^{n-1})$, $f \in C(S^{n-1},\bar{e})$, and $\eta \in \mathrm{SO}(n)$, it is easy to check that

$$(\sigma * f)(\eta\bar{e}) = \int_{S^{n-1}} f(\eta^{-1}u)\, d\sigma(u). \tag{1.14}$$

By (1.14), we have for every $\vartheta \in \mathrm{SO}(n)$, that

$$(\vartheta\sigma) * f = \vartheta(\sigma * f),$$

where $\vartheta\sigma$ is the image measure of σ under the rotation $\vartheta \in \mathrm{SO}(n)$. It is also not difficult to check from (1.14) that the convolution of zonal functions and measures is Abelian.

The convolution of two measures $\mu, \nu \in \mathcal{M}(S^{n-1})$ is determined by (see, e.g., [Sch07])

$$\langle \mu * \nu, f\rangle = \langle \mu, f * \nu\rangle.$$

It is a commutative operation for $\mu, \nu \in \mathcal{M}(S^{n-1}, \bar{e})$. We will in the following make use of the fact that convolution on $\mathcal{M}(S^{n-1})$ is associative, in particular

$$(f * \mu) * \nu = f * (\mu * \nu). \tag{1.15}$$

Another property of spherical convolution which is going to be critical for us is the fact that the convolution is self-adjoint; in particular, we have for all $\sigma, \tau \in \mathcal{M}(S^{n-1})$ and every $f \in C(S^{n-1},\bar{e})$

$$\int_{S^{n-1}} (\sigma * f)(u)\, d\tau(u) = \int_{S^{n-1}} (\tau * f)(u)\, d\sigma(u). \tag{1.16}$$

There is a connection between the Radon transform and convolution on the sphere. For $f \in C(S^{n-1})$, the *spherical Radon transform* $Rf \in C(S^{n-1})$ is defined by

$$(Rf)(u) = \frac{1}{n-1} \int f(v) d\lambda_{S^{n-1}\cap\bar{e}^{\perp}}(v),$$

where $\lambda_{S^{n-1}\cap\bar{e}^{\perp}}$ denotes the invariant measure concentrated on $S^{n-1} \cap \bar{e}^{\perp}$ with total mass κ_{n-1}. With the help of the convolution above, this can be written as

$$Rf = \frac{1}{n-1} f * \lambda_{S^{n-1}\cap\bar{e}^{\perp}}.$$

1.4 Spherical harmonics and distributions

In this section we collect facts about spherical harmonics, in particular on the series expansion of distributions on the sphere. We also recall C. Berg's functions used in his solution of the Christoffel problem, since they are closely related to the action of the Hard Lefschetz integration operator on Minkowski valuations. In the final part of this section, we give a new proof of the bijectivity of integral transforms involving C. Berg's functions. For the background material we refer the reader to [Sch14, Chapter 8.3], [Gro96], and [Mor98].

We write Δ_S for the *Laplacian* (or Laplace–Beltrami operator) on S^{n-1}. For functions $f, g \in C^2(S^{n-1})$, we have

$$\int_{S^{n-1}} f(u)\, \Delta_S g(u)\, du = \int_{S^{n-1}} g(u)\, \Delta_S f(u)\, du.$$

The finite-dimensional vector space of *spherical harmonics* of dimension n and degree k will be denoted by $\mathcal{H}_k^n$ and we write $N(n,k)$ for its dimension. Spherical harmonics are eigenfunctions of Δ_S, more precisely, for $Y_k \in \mathcal{H}_k^n$,

$$\Delta_S Y_k = -k(k+n-2)\, Y_k. \tag{1.17}$$

Let $L^2(S^{n-1})$ denote the Hilbert space of square-integrable functions on S^{n-1} with the usual inner product $(\,\cdot\,,\cdot\,)$. The spaces $\mathcal{H}_k^n$ are pairwise orthogonal with respect to this inner product. If $\{Y_{k,1}, \dots, Y_{k,N(n,k)}\}$ is an orthonormal basis of $\mathcal{H}_k^n$, then the collection $\{Y_{k,1}, \dots, Y_{k,N(n,k)} : k \in \mathbb{N}\}$ is a complete orthogonal system in $L^2(S^{n-1})$, that is, the Fourier series

$$f \sim \sum_{k=0}^{\infty} \pi_k f \tag{1.18}$$

converges to f in the L^2 norm for every $f \in L^2(S^{n-1})$. Here, we use $\pi_k : L^2(S^{n-1}) \to \mathcal{H}_k^n$ to denote the orthogonal projection. Since the *Legendre polynomial* $P_k^n \in C([-1,1])$ of dimension n and degree k satisfies

$$\sum_{i=1}^{N(n,k)} Y_{k,i}(u)\, Y_{k,i}(v) = \frac{N(n,k)}{\omega_n} P_k^n(u \cdot v),$$

where ω_m denotes the surface area of the m-dimensional unit ball, we have

$$(\pi_k f)(v) = \sum_{i=1}^{N(n,k)} (f, Y_{k,i}) Y_{k,i}(v) = \frac{N(n,k)}{\omega_n} \int_{S^{n-1}} f(u)\, P_k^n(u \cdot v)\, du. \tag{1.19}$$

The subspace of zonal functions in $\mathcal{H}_k^n$ is 1-dimensional for every $k \in \mathbb{N}$ and spanned by the function $u \mapsto P_k^n(u \cdot \bar{e})$. Since the spaces $\mathcal{H}_k^n$ are invariant under the natural action of $\mathrm{SO}(n)$, the functions $u \mapsto P_k^n(u \cdot v)$, for fixed $v \in S^{n-1}$, are elements of $\mathcal{H}_k^n$. The orthogonality of the spaces $\mathcal{H}_k^n$ is reflected by the fact that the Legendre polynomials P_k^n form a complete orthogonal system with respect to the inner product $[\cdot,\cdot]_n$ on $C([-1,1])$

defined by

$$[p,q]_n = \int_{-1}^{1} p(t)\, q(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt.$$

From the orthogonality property of the Legendre polynomials and (1.19), it is not difficult to show that any function $\phi \in L^2([-1,1])$ (or, equivalently, any zonal $g \in L^2(S^{n-1})$) admits a series expansion

$$\phi \sim \sum_{k=0}^{\infty} \frac{N(n,k)}{\omega_n}\, a_k^n[\phi]\, P_k^n, \tag{1.20}$$

where

$$a_k^n[\phi] = \omega_{n-1} \int_{-1}^{1} \phi(t)\, P_k^n(t)\, (1-t^2)^{\frac{n-3}{2}}\, dt = \omega_{n-1}\, [P_k^n,\phi]_n\,. \tag{1.21}$$

For the explicit calculation of integrals of the form (1.21), the following *formula of Rodrigues* for the Legendre polynomials is often very useful:

$$P_k^n(t) = \frac{(-1)^k}{2^k \left(\frac{n-1}{2}\right)_k} (1-t^2)^{-\frac{n-3}{2}} \frac{d^k}{dt^k} (1-t^2)^{\frac{n-3}{2}+k}, \tag{1.22}$$

where, for $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, $(\alpha)_k$ abbreviates the product $\alpha(\alpha+1)\cdots(\alpha+k-1)$. Using (1.22) one can show that the derivatives of Legendre polynomials are again Legendre polynomials. For $l \geq k$, we have

$$\frac{d^k}{dt^k} P_l^n(t) = 2^k \left(\frac{n}{2}\right)_k \frac{N(n+2k,l-k)}{N(n,l)}\, P_{l-k}^{n+2k}. \tag{1.23}$$

Next we recall the *Gegenbauer polynomials* , which can be defined for $\alpha > 0$ by means of the generating function

$$\frac{1}{(1+r^2-2rt)^{\alpha}} = \sum_{k=0}^{\infty} C_k^{\alpha}(t)\, r^n.$$

For $n \geq 3$, their relation to Legendre polynomials can be expressed by

$$C_k^{(n-2)/2} = \binom{n+k-3}{n-3} P_k^n. \tag{1.24}$$

For the following well-known auxiliary result about the spherical harmonic expansion of smooth functions, see, e.g., [Mor98, p. 36].

Lemma 1.3. *If $f \in C^\infty(S^{n-1})$, then the sequence $\|\pi_k f\|_\infty$, $k \in \mathbb{N}$, is* rapidly decreasing*; that is, for any $m \in \mathbb{N}$, we have $\sup\{k^m \|\pi_k f\|_\infty : k \in \mathbb{N}\} < \infty$. Conversely, if $Y_k \in \mathcal{H}_k^n$, $k \in \mathbb{N}$, is a sequence of spherical harmonics such that $\|Y_k\|_\infty$ is rapidly decreasing, then the function*

$$f(u) = \sum_{k=0}^{\infty} Y_k(u), \qquad u \in S^{n-1},$$

is C^∞ and $\pi_k f = Y_k$ for every $k \in \mathbb{N}$.

For $f \in C^\infty(S^{n-1})$ and $m \in \mathbb{N}$, define

$$(-\varDelta_S)^{\frac{m}{2}} f = \sum_{k=0}^{\infty} (k(k+n-2))^{\frac{m}{2}} \pi_k f.$$

Note that, by Lemma 1.3, $(-\varDelta_S)^{\frac{m}{2}} f \in C^\infty(S^{n-1})$.

If we endow the vector space $C^\infty(S^{n-1})$ with the topology defined by the family of seminorms $\|(-\varDelta_S)^{\frac{m}{2}} f\|_\infty$, $m \in \mathbb{N}$, then $C^\infty(S^{n-1})$ becomes a Fréchet space. Moreover, the spherical harmonic expansion (1.18) of any $f \in C^\infty(S^{n-1})$ converges to f in this topology.

A *distribution* on S^{n-1} is a continuous linear functional on $C^\infty(S^{n-1})$. We write $C^{-\infty}(S^{n-1})$ for the space of distributions on S^{n-1} equipped with the topology of weak convergence and use $\langle \cdot , \cdot \rangle$ to denote the canonical bilinear pairing on $C^\infty(S^{n-1}) \times C^{-\infty}(S^{n-1})$.

A (signed) measure σ on S^{n-1} defines a distribution T_σ by

$$\langle f, T_\sigma \rangle = \int_{S^{n-1}} f(u)\, d\sigma(u), \qquad f \in C^\infty(S^{n-1}).$$

Using the continuous linear injection $\sigma \mapsto T_\sigma$, we can regard $\mathcal{M}(S^{n-1})$ as a subspace of $C^{-\infty}(S^{n-1})$. In the same way, the spaces $C^\infty(S^{n-1})$, $C(S^{n-1})$, and $L^2(S^{n-1})$ can be viewed as subspaces of $C^{-\infty}(S^{n-1})$, and we have

$$C^\infty(S^{n-1}) \subseteq C(S^{n-1}) \subseteq L^2(S^{n-1}) \subseteq \mathcal{M}(S^{n-1}) \subseteq C^{-\infty}(S^{n-1}). \tag{1.25}$$

Since $\pi_k : L^2(S^{n-1}) \to \mathcal{H}_k^n$ is self-adjoint, that is, $(\pi_k f, g) = (f, \pi_k g)$ for all $f, g \in L^2(S^{n-1})$ and $k \in \mathbb{N}$, it is consistent to define the *k-spherical harmonic component* $\pi_k T$ of $T \in C^{-\infty}(S^{n-1})$ as the distribution given by

$$\langle f, \pi_k T \rangle = \langle \pi_k f, T \rangle, \qquad f \in C^\infty(S^{n-1}).$$

Lemma 1.4 [Mor98, p. 38] *If $T \in C^{-\infty}(S^{n-1})$, then $\pi_k T \in \mathcal{H}_k^n$ for every $k \in \mathbb{N}$ and the sequence $\|\pi_k T\|_\infty$, $k \in \mathbb{N}$, is* slowly increasing, *that is, there exist $C > 0$ and $j \in \mathbb{N}$ such that $\|\pi_k T\|_\infty \leq C(1 + k^j)$ for every $k \in \mathbb{N}$.*
Conversely, if $Y_k \in \mathcal{H}_k^n$, $k \in \mathbb{N}$, is a sequence of spherical harmonics such that $\|Y_k\|_\infty$ is slowly increasing, then

$$\langle g, T \rangle = \sum_{k=0}^{\infty} \int_{S^{n-1}} g(u) Y_k(u)\, du, \qquad g \in C^\infty(S^{n-1}),$$

defines a distribution $T \in C^{-\infty}(S^{n-1})$ for which $\pi_k T = Y_k$ for every $k \in \mathbb{N}$.

We can also extend the Laplacian to distributions $T \in C^{-\infty}(S^{n-1})$, by defining $\varDelta_S T$ as the distribution given by

$$\langle f, \varDelta_S T \rangle = \langle \varDelta_S f, T \rangle, \qquad f \in C^\infty(S^{n-1}).$$

Note that, by (1.25), $\varDelta_S$ can now also act on continuous functions on S^{n-1}. This is of particular importance for us, since the support function $h(K,\cdot)$ and the first-order area measure $S_1(K,\cdot)$ of a convex body $K \in \mathcal{K}^n$ are related by

$$\square_n h(K,\cdot) = S_1(K,\cdot), \tag{1.26}$$

where $\square_n$ is the differential operator given by

$$\square_n h = h + \frac{1}{n-1}\Delta_S h.$$

From the definition of $\square_n$ and (1.17), we see that for $f \in C^\infty(S^{n-1})$ the spherical harmonic expansion of $\square_n f$ is given by

$$\square_n f \sim \sum_{k=0}^{\infty} \frac{(1-k)(k+n-1)}{n-1}\pi_k f. \tag{1.27}$$

Thus, the kernel of the linear operator $\square_n : C^\infty(S^{n-1}) \to C^\infty(S^{n-1})$ is given by $\mathcal{H}_1^n$ and consists precisely of the restrictions of linear functions on $\mathbb{R}^n$ to S^{n-1}. Let $C_\mathrm{o}^\infty(S^{n-1})$ denote the Fréchet subspace of $C^\infty(S^{n-1})$ given by

$$C_\mathrm{o}^\infty(S^{n-1}) = \{f \in C^\infty(S^{n-1}) : \pi_1 f = 0\}$$

and define $C_\mathrm{o}^{-\infty}(S^{n-1})$ analogously.

Since the linear operator $\square_n : C_\mathrm{o}^\infty(S^{n-1}) \to C_\mathrm{o}^\infty(S^{n-1})$ is an isomorphism, it is a natural problem to find an (explicit) inversion formula. This was accomplished by C. Berg [Ber69] in the late 1960s and, due to (1.26), is closely related to his solution of the classical Christoffel problem which consists in finding necessary and sufficient conditions for a Borel measure on S^{n-1} to be the first-order area measure of a convex body.

In order to describe C. Berg's inversion formula for $\square_n$, let us recall the *Funk–Hecke theorem*: If $\phi \in C([-1,1])$ and F_ϕ is the integral transform on $\mathcal{M}(S^{n-1})$ defined by

$$(\mathrm{F}_\phi\sigma)(u) = \int_{S^{n-1}} \phi(u\cdot v)\, d\sigma(v), \qquad u \in S^{n-1},$$

then the spherical harmonic expansion of $\mathrm{F}_\phi\sigma \in C(S^{n-1})$ is given by

$$\mathrm{F}_\phi\sigma \sim \sum_{k=0}^{\infty} a_k^n[\phi]\, \pi_k\sigma, \tag{1.28}$$

where the numbers $a_k^n[\phi]$ are given by (1.21) and are called the *multipliers* of F_ϕ. From the obvious identification of zonal functions on S^{n-1} with functions on $[-1,1]$, (1.14), and the Funk–Hecke theorem, it follows that for $f \in C(S^{n-1},\bar{e})$, there are $a_k^n[f] \in \mathbb{R}$ such that the spherical harmonic expansion of $\sigma * f \in C(S^{n-1})$ is given by

$$\sigma * f \sim \sum_{k=0}^{\infty} a_k^n[f]\, \pi_k \sigma.$$

Hence, convolution from the right induces a multiplier transformation.

Using the theory of subharmonic functions on S^{n-1}, C. Berg proved that for every $n \geq 2$ there exists a uniquely determined C^∞ function g_n on $(-1,1)$ such that the zonal function $u \mapsto g_n(u \cdot \bar{e})$ is in $L^1(S^{n-1})$ and

$$a_1^n[g_n] = 0, \qquad a_k^n[g_n] = \frac{n-1}{(1-k)(k+n-1)}, \quad k \neq 1. \tag{1.29}$$

For later reference, we just state here that

$$g_2(t) = \frac{1}{2\pi}\left((\pi - \arccos t)(1-t^2)^{\frac{1}{2}} - \frac{t}{2}\right) \tag{1.30}$$

and

$$g_3(t) = \frac{1}{2\pi}\left(1 + t\ln(1-t) + \left(\frac{4}{3} - \ln 2\right)t\right). \tag{1.31}$$

We note that, by (1.29), our normalization of the g_n differs from C. Berg's original one. It follows from (1.27), (1.28), and (1.29) that

$$f(u) = \int_{S^{n-1}} g_n(u \cdot v)(\Box_n f)(v)\, dv, \qquad u \in S^{n-1},$$

for every $f \in C_\mathrm{o}^\infty(S^{n-1})$, which is the desired inversion formula. However, for our purposes we need the following more general fact.

Theorem 1.5. *For every $n \geq 2$ and $2 \leq j \leq n$, the integral transform $\mathrm{F}_{g_j} : C_\mathrm{o}^\infty(S^{n-1}) \to C_\mathrm{o}^\infty(S^{n-1})$ given by*

$$(\mathrm{F}_{g_j} f)(u) = \int_{S^{n-1}} g_j(u \cdot v) f(v)\, dv, \qquad u \in S^{n-1},$$

is an isomorphism.

Theorem 1.5 follows, for example, from a recent result of Goodey and Weil [Goo14, Theorem 4.3]. However, we give a different and more elementary proof below that also yields additional information required after the proof of Theorem 3.3. For this, note that, by Lemma 1.3, it is sufficient to show that the multipliers $a_k^n[g_j]$ are non-zero for $k \neq 1$ and that they are slowly increasing. Therefore, Theorem 1.5 is a direct consequence of the following.

Theorem 1.6. *For $n \geq 2$, $2 \leq j \leq n$, and $k \neq 1$, we have*

$$a_k^n[g_j] = -\frac{\pi^{\frac{n-j}{2}}(j-1)}{4} \frac{\Gamma\left(\frac{n-j+2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{j+k-1}{2}\right)}{\Gamma\left(\frac{n-j+k+1}{2}\right)\Gamma\left(\frac{n+k+1}{2}\right)}.$$

Proof. For $n \geq 2$, $d \geq 0$, and $k \neq 1$, by (1.21), we have to determine

$$a_k^{n,d} := a_k^{n+d}[g_n] = \omega_{n+d-1}\left[P_k^{n+d}, g_n\right]_{n+d}, \tag{1.32}$$

where we know from (1.29) that

$$a_k^{n,0} = \frac{n-1}{(1-k)(n-1+k)}. \tag{1.33}$$

We start with the case $d = 1$. By (1.20) and (1.29), we have

$$g_n \sim \sum_{l=0}^{\infty} \frac{N(n,l)}{\omega_n} a_l^{n,0} P_l^n,$$

where the sum converges in the topology induced by $[\,\cdot\,,\cdot\,]_n$, which implies convergence in the topology induced by $[\,\cdot\,,\cdot\,]_{n+1}$. Consequently,

$$a_k^{n,1} = \omega_n \left[P_k^{n+1}, g_n\right]_{n+1} = \sum_{l=0}^{\infty} N(n,l)\, a_l^{n,0} \left[P_k^{n+1}, P_l^n\right]_{n+1}. \tag{1.34}$$

Since Legendre polynomials of degree k are even if k is even and odd otherwise, we may assume that k and l have the same parity. Since $\left[P_k^{n+1}, P_l^n\right]_{n+1}$ vanishes for $l < k$ (see the next calculation), let $l \geq k$ and put

$$\beta := \frac{n-2}{2}.$$

If $\beta + k \geq \frac{1}{2}$, that is, $(n,k) \neq (2,0)$, then it follows from (1.22), integration by parts, (1.23), and (1.24) that

$$\begin{aligned}
\left[P_k^{n+1}, P_l^n\right]_{n+1} &= \frac{(-1)^k}{2^k(\beta+1)_k} \int_{-1}^{1} \left(\frac{d^k}{dt^k}(1-t^2)^{\beta+k}\right) P_l^n(t)\, dt \\
&= \frac{1}{2^k(\beta+1)_k} \int_{-1}^{1} (1-t^2)^{\beta+k} \left(\frac{d^k}{dt^k} P_l^n(t)\right) dt \\
&= \frac{N(n+2k, l-k)}{N(n,l)} \int_{-1}^{1} (1-t^2)^{\beta+k}\, P_{l-k}^{n+2k}(t)\, dt \\
&= \frac{\beta+l}{N(n,l)(\beta+k)} \int_{-1}^{1} (1-t^2)^{\beta+k}\, C_{l-k}^{\beta+k}(t)\, dt.
\end{aligned}$$

For $\alpha \in \frac{1}{2}\mathbb{N}$ and even m, we have (cf. [Goo92, p. 424])

$$c_m^\alpha = \int_{-1}^{1} (1-t^2)^\alpha \, C_m^\alpha(t)\, dt = -\frac{\alpha\, 4^{\alpha+\frac{1}{2}}\, m!\, \Gamma\left(\frac{m}{2}+\alpha+1\right)^2}{(m-1)\left(\frac{m}{2}+\alpha\right)(m+2\alpha+1)!\,\Gamma\left(\frac{m}{2}+1\right)^2}.$$

Plugging this into (1.34) and changing the summation index yields

$$a_k^{n,1} = \sum_{l=0}^{\infty} \frac{\beta+k+2l}{\beta+k}\, a_{k+2l}^{n,0}\, c_{2l}^{\beta+k} = \left(\beta+\frac{1}{2}\right) 4^{\beta+k+1} \sum_{l=0}^{\infty} q(\beta,k,l),$$

where

$$q(\beta,k,l) = \frac{2l\,(2l-2)!\,(\beta+k+2l)\,\Gamma(\beta+k+l+1)^2}{(k+2l-1)\,(2\beta+k+2l+1)\,(\beta+k+l)\,(2\beta+2k+2l+1)!\,\Gamma(l+1)^2}.$$

Using Zeilberger's algorithm (see, e.g., [Pet96]), we find that q satisfies the following recurrence relation:

$$A(\beta,k)q(\beta+1,k,l) + B(\beta,k)q(\beta,k,l) = q(\beta,k,l+1)C(\beta,k,l+1) - q(\beta,k,l)C(\beta,k,l),$$

where

$$A(\beta,k) = 4(2\beta+k+4), \quad B(\beta,k) = -(2\beta+k+1), \quad C(\beta,k,l) = -\frac{l(k+2l-1)}{\beta+k+2l}.$$

If we let $Q(\beta,k) = \sum_{l=0}^{\infty} q(\beta,k,l)$, then we obtain

$$Q(\beta+1,k) = \frac{2\beta+k+1}{4(2\beta+k+4)}\, Q(\beta,k)$$

or, in terms of the multipliers,

$$a_k^{n+2,1} = \frac{(n+1)\,(n+k-1)}{(n-1)\,(n+k+2)}\, a_k^{n,1}. \tag{1.35}$$

The function q also satisfies the recurrence relation

$$D(\beta,k)q(\beta,k+2,l) + E(\beta,k)q(\beta,k,l) = q(\beta,k,l+1)F(\beta,k,l+1) - q(\beta,k,l)F(\beta,k,l),$$

where

$$D(\beta,k) = 16(k+2)(2\beta+k+4), \quad E(\beta,k) = -(k-1)(2\beta+k+1)$$

and

$$F(\beta,k,l) = -\frac{1}{(\beta+k+2l)(2\beta+2k+2l+3)} \sum_{i=1}^{3} l^i\, p_i(\beta,k),$$

with polynomials p_1, p_2, p_3 given by

$$\begin{aligned}
p_1(\beta,k) &= 8\beta^3 + 16k\beta^2 + 20\beta^2 + 12k^2\beta + 26k\beta + 10\beta + 4k^3 + 9k^2 + 4k + 3,\\
p_2(\beta,k) &= 16\beta^2 + 24k\beta + 32\beta + 12k^2 + 24k + 4,\\
p_3(\beta,k) &= 8\beta + 8k + 12.
\end{aligned}$$

Summing again over all l, we arrive at

$$Q(\beta,k+2) = \frac{(k-1)\,(2\beta+k+1)}{16(k+2)\,(2\beta+k+4)}\,Q(\beta,k).$$

In terms of the multipliers, this means

$$a_{k+2}^{n,1} = \frac{(k-1)\,(n+k-1)}{(k+2)\,(n+k+2)}\,a_k^{n,1}. \tag{1.36}$$

In order to solve (1.35) and (1.36), we need four initial values of $a_k^{n,1}$. We also have to calculate $a_0^{2,1}$, which was not covered by the above arguments. Using (1.30), (1.31), and (1.32), elementary integration yields

$$a_0^{2,1} = \frac{\pi^2}{4}, \quad a_2^{2,1} = -\frac{\pi^2}{32}, \quad a_3^{2,1} = -\frac{4}{45}, \quad a_0^{3,1} = \frac{2\pi}{3}, \quad a_3^{3,1} = -\frac{\pi}{24}. \tag{1.37}$$

This leads to the sequence

$$a_k^{n,1} = -\frac{\pi}{8}(n-1)\frac{\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{n+k-1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)\Gamma\left(\frac{n+k+2}{2}\right)}, \tag{1.38}$$

which satisfies (1.35), (1.36), and has the initial values (1.37).

Now let $d \geq 0$ be arbitrary. For $l \geq 2$, the Legendre polynomials satisfy the recurrence relation (see, e.g., [Gro96, Lemma 3.3.10])

$$(n+d+2l-2)(n+d-1)P_l^{n+d} = (n+d+l-2)(n+d+l-1)P_l^{n+d+2} - (l-1)lP_{l-2}^{n+d+2}.$$

From this and the fact that for all $m \geq 1$ $P_0^m(t) = 1$ and $P_1^m(t) = t$, we obtain

$$\begin{aligned}
a_k^{n,d+2} &= \omega_{n+d+1}\left[P_k^{n+d+2}, g_n\right]_{n+d+2}\\
&= \omega_{n+d+1}\sum_{l=0}^{\infty}\frac{N(n+d,l)}{\omega_{n+d}}\,a_l^{n,d}\left[P_k^{n+d+2}, P_l^{n+d}\right]_{n+d+2}\\
&= \frac{2\pi}{n+d+2k}\left(a_k^{n,d} - a_{k+2}^{n,d}\right).
\end{aligned}$$

Finally, the sequence which solves this recurrence relation and has the initial values (1.33) and (1.38) is given by

$$a_k^{n,d} = -\frac{\pi^{\frac{d}{2}}(n-1)}{4}\frac{\Gamma\left(\frac{d+2}{2}\right)\Gamma\left(\frac{k-1}{2}\right)\Gamma\left(\frac{n+k-1}{2}\right)}{\Gamma\left(\frac{d+k+1}{2}\right)\Gamma\left(\frac{n+d+k+1}{2}\right)}, \qquad k \neq 1.$$

□

We end this section with the following important definition, given rise to by Theorem 1.5.

Definition 1.7. *For $2 \leq j \leq n$, let $\Box_j : C_{\mathrm{o}}^{\infty}(S^{n-1}) \to C_{\mathrm{o}}^{\infty}(S^{n-1})$ denote the linear operator which is inverse to the integral transform F_{g_j}.*

1.5 L_p and Orlicz addition

On the set $\mathcal{K}_{\mathrm{o}}^n$ of convex bodies containing the origin, Firey [Fir62] introduced in the 1960s a more general way of combining convex sets. For $K, L \in \mathcal{K}_{\mathrm{o}}^n$, $s,t \geq 0$, and $1 \leq p < \infty$, the L_p *Minkowski combination* $s \cdot K +_p t \cdot L$ is defined by

$$h(s \cdot K +_p t \cdot L,\cdot)^p = s\, h(K,\cdot)^p + t\, h(L,\cdot)^p.$$

Initiated by Lutwak [Lut93b; Lut96], in the last two decades an entire L_p theory of convex bodies was developed which represents a powerful extension of the classical Brunn–Minkowski theory (see, e.g., [Hab09b; Lut00a; Lut00b; Par14a; Par14b; Sch12b; Web13]).

In [Lut93b], Lutwak introduced the L_p *mixed volume* $V_p(K,L)$ for $K, L \in \mathcal{K}_{(o)}^n$ by

$$\frac{n}{p}V_p(K,L) = \lim_{r\to 0^+}\frac{V(K +_p r \cdot_p L) - V(K)}{r}.$$

Moreover, Lutwak proved that there exists for each $K \in \mathcal{K}_{(o)}^n$ a positive Borel measure $S_{\mathbf{p}}(K,\cdot)$ on S^{n-1}, the L_p *surface area measure* , such that

$$V_p(K,L) = \frac{1}{n}\int_{S^{n-1}} h(L,u)^p dS_{\mathbf{p}}(K,u) \tag{1.39}$$

for each $Q \in \mathcal{K}_{(o)}^n$. We note that $S_{\mathbf{1}}(K,\cdot)$ is just the surface area measure $S_{n-1}(K,\cdot)$. In [Lut93b], it was shown that there is an L_p analog of the classical Brunn–Minkowski inequality, stating that for K, $L \in \mathcal{K}_{(o)}^n$,

$$V_p(K,L) \geq V(K)^{(n-p)/n}V(L)^{p/n},$$

with equality if and only if K and L are dilates.

We recall the definition of the L_p *dual mixed volume* $\tilde{V}_{-p}(K,L)$ by using its integral representation

$$\tilde{V}_{-p}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho_L^{-p} dS(u). \tag{1.40}$$

This L_p dual mixed volume satisfies the inequality

$$\tilde{V}_{-p}(K,L) \geq V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{1.41}$$

with equality if and only if K and L are dilates. For further information on V_p and V_{-p}, confer [Lut00b].

A still more recent extension of the Brunn–Minkowski theory goes back to two articles of Lutwak, Yang, and Zhang [Lut10a; Lut10b] and Haberl, Lutwak, Yang, and Zhang [Hab10]. While these articles form the starting point of an emerging Orlicz–Brunn–Minkowski theory that generalizes the L_p theory of convex bodies in the same way that Orlicz spaces generalize L_p spaces, the fundamental notion of an Orlicz Minkowski combination of convex bodies was introduced later by Gardner, Hug, and Weil [Gar14].

Let Θ_1 be the set of convex functions $\varphi : [0,\infty) \to [0,\infty)$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$. For $K, L \in \mathcal{K}^n_{\mathrm{o}}$, $s, t \geq 0$, and $\varphi, \psi \in \Theta_1$, the *Orlicz Minkowski combination* $+_{\varphi,\psi}(K,L,s,t)$ is defined by

$$h(+_{\varphi,\psi}(K,L,s,t),u) = \inf\left\{\alpha > 0 : s\,\varphi\left(\frac{h(K,u)}{\alpha}\right) + t\,\psi\left(\frac{h(L,u)}{\alpha}\right) \leq 1\right\}$$

for $u \in S^{n-1}$. The notation $+_{\varphi,\psi}(K,L,s,t)$ is necessitated by the fact that it is not possible in general to isolate an Orlicz scalar multiplication. We note that for $\varphi(t) = \psi(t) = t^p$, $p \geq 1$, the Orlicz Minkowski combination $+_{\varphi,\psi}(K,L,s,t)$ equals the L_p Minkowski combination $s \cdot K +_p t \cdot L$.

For $s = t = 1$, we write $K +_{\varphi,\psi} L$ instead of $+_{\varphi,\psi}(K,L,1,1)$ and call this the *Orlicz Minkowski sum* of K and L. In fact, Gardner, Hug, and Weil defined a more general Orlicz addition but proved (see [Gar14, Theorem 5.5]) that their definition leads (essentially) to the Orlicz Minkowski addition as defined here *and* the L_∞ Minkowski addition obtained as the Hausdorff limit of the L_p Minkowski addition, that is, for $K, L \in \mathcal{K}^n_{\mathrm{o}}$,

$$K +_\infty L = \lim_{p\to\infty} K +_p L = \operatorname{conv}(K \cup L).$$

While all L_p Minkowski additions are commutative, in general, the Orlicz Minkowski addition of convex bodies is not. A classification of those Orlicz additions which are commutative was obtained by Gardner, Hug, and Weil.

Theorem 1.8 [Gar14] *Let $\varphi, \psi \in \Theta_1$. The addition $+_{\varphi,\psi} : \mathcal{K}^n_{\mathrm{o}} \times \mathcal{K}^n_{\mathrm{o}} \to \mathcal{K}^n_{\mathrm{o}}$ is commutative if and only if there exists $\phi \in \Theta_1$ such that $+_{\varphi,\psi} = +_{\phi,\phi}$.*

In the following, we will only be interested in commutative Orlicz additions. For $K, L \in \mathcal{K}^n_{\mathrm{o}}$, $\varphi \in \Theta_1$, and $\lambda \in (0,1)$ we use $K +_{\varphi,\lambda} L$ to denote the *Orlicz Minkowski convex combination* $+_{\varphi,\varphi}(K, L,(1-\lambda),\lambda)$. More explicitly,

$$h(K +_{\varphi,\lambda} L,u) = \inf\left\{\alpha > 0 \colon (1-\lambda)\varphi\left(\frac{h(K,u)}{\alpha}\right) + \lambda\varphi\left(\frac{h(L,u)}{\alpha}\right) \leq 1\right\}$$

for $u \in S^{n-1}$. For the proof of Theorem 3.6, we need the following simple fact.

Lemma 1.9. *If $\varphi \in \Theta_1$ and $K, L \in \mathcal{K}_{\mathrm{o}}^n$, then for all $\lambda \in (0,1)$,*

$$K +_{\varphi,\lambda} L \supseteq (1-\lambda)K + \lambda L. \tag{1.42}$$

Proof. For $u \in S^{n-1}$ choose $t > h(K +_{\varphi,\lambda} L,u)$. Then, by the convexity of φ and the definition of $K +_{\varphi,\lambda} L$, we have

$$\varphi\left(\frac{(1-\lambda)h(K,u) + \lambda h(L,u)}{t}\right) \le (1-\lambda)\varphi\left(\frac{h(K,u)}{t}\right) + \lambda\varphi\left(\frac{h(L,u)}{t}\right) \le 1.$$

Since every $\varphi \in \Theta_1$ is increasing and satisfies $\varphi(1) = 1$, we conclude that

$$(1-\lambda)h(K,u) + \lambda h(L,u) \le t.$$

Now, letting t approach $h(K +_{\varphi,\lambda} L,u)$, we obtain the desired inclusion (1.42). □

CHAPTER 2

Minkowski valuations and their generalizations

2.1 Scalar-valued valuations and generalized valuations

In the following, we recall several results on translation-invariant (scalar- and convex-body-valued) valuations, in particular, the product structure on smooth valuations and the Alesker–Poincaré duality. We also discuss basic properties of the Hard Lefschetz operators and a new isomorphism between generalized valuations of degree 1 and generalized functions on the sphere. At the end of this section, we state a recent representation theorem for Minkowski valuations intertwining rigid motions and define the class of Minkowski valuations for which we can establish log-concavity properties.

A map μ defined on convex bodies in $\mathbb{R}^n$ and taking values in an Abelian semigroup A is called a *valuation* or *additive* if

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

whenever $K \cup L$ is convex. If G is a group of affine transformations on $\mathbb{R}^n$, a valuation μ is called G-*invariant* if $\mu(gK) = \mu(K)$ for all $K \in \mathcal{K}^n$ and $g \in G$.

The theory of *scalar*-valued valuations has long played a prominent role in convex geometry (see, e.g., [Had57; Kla97] for the history of scalar valuations and [Ale01; Ber11; Fu06; Hab14; Lud10b; Par13; Wan14] for more recent results), but other semigroups have also been considered. For *vector*-valued valuations, there is a well-lnown result that the Steiner point map is the unique vector-valued, rigid-motion-equivariant and continuous valuation on $\mathcal{K}^n$ (see e.g., [Sch14, p. 363]).

Let **Val** denote the vector space of continuous translation-invariant scalar-valued valuations. The structure theory of translation-invariant valuations has its starting point in a classical result of McMullen [McM77], who showed that

$$\mathbf{Val} = \bigoplus_{0 \le i \le n} \mathbf{Val}_i^+ \oplus \mathbf{Val}_i^-, \tag{2.1}$$

where $\mathbf{Val}_i^+ \subseteq \mathbf{Val}$ denotes the subspace of *even* valuations (homogeneous) of degree i, and $\mathbf{Val}_i^-$ denotes the subspace of *odd* valuations of degree i. The space **Val** becomes a

Banach space when endowed with the norm

$$\|\mu\| = \sup\{|\mu(K)| : K \subseteq B\}.$$

The general linear group $\mathrm{GL}(n)$ acts on the Banach space **Val** in a natural way: For every $A \in \mathrm{GL}(n)$ and $\mu \in \mathbf{Val}$,

$$(A \cdot \mu)(K) = \mu(A^{-1}K), \qquad K \in \mathcal{K}^n.$$

Note that the subspaces $\mathbf{Val}_i^{\pm}$ are invariant under this $\mathrm{GL}(n)$-action. In fact, a deep result of Alesker [Ale01], known as the Irreducibility Theorem, states that these subspaces are also irreducible:

Theorem 2.1. (Alesker [Ale01]) *The natural representation of* $\mathrm{GL}(n)$ *on* $\mathbf{Val}_i^{\pm}$ *is irreducible for any* $i \in \{0, \ldots, n\}$.

It follows from Theorem 2.1 that any $\mathrm{GL}(n)$-invariant subspace of translation-invariant continuous valuations (of a given degree i and parity) is already dense in $\mathbf{Val}_i^{\pm}$.

Definition 2.2. *A valuation* $\mu \in \mathbf{Val}$ *is called smooth if the map* $\mathrm{GL}(n) \to \mathbf{Val}$ *defined by* $A \mapsto A \cdot \mu$ *is infinitely differentiable.*

The subspace of smooth translation-invariant valuations is denoted by $\mathbf{Val}^{\infty}$, and we write $\mathbf{Val}_i^{\pm,\infty}$ for smooth valuations in $\mathbf{Val}_i^{\pm}$. It is well known (cf. [Wal88, p. 32]) that $\mathbf{Val}_i^{\pm,\infty}$ is a dense $\mathrm{GL}(n)$-invariant subspace of $\mathbf{Val}_i^{\pm}$. Moreover, $\mathbf{Val}^{\infty}$ carries a natural Fréchet space topology, called Gårding topology (see [Wal88, p. 33]), which is stronger than the topology induced from **Val**. Finally, we note that the representation of $\mathrm{GL}(n)$ on $\mathbf{Val}^{\infty}$ is continuous.

Examples:

(a) If $L \in \mathcal{K}^n$ is strictly convex with smooth boundary, then

$$\mu_L : \mathcal{K}^n \to \mathbb{R}, \qquad \mu_L(K) = V_n(K + L),$$

is a smooth valuation.

(b) If $f \in C_{\mathrm{o}}^{\infty}(S^{n-1})$ and $0 \leq i \leq n-1$, then $\nu_{i,f} : \mathcal{K}^n \to \mathbb{R}$, defined by

$$\nu_{i,f}(K) = \int_{S^{n-1}} f(u)\, dS_i(K,u), \tag{2.2}$$

is a smooth valuation in $\mathbf{Val}_i^{\infty}$.

Before we turn to generalized valuations, we recall the definition of the Alesker product of smooth translation-invariant valuations.

Theorem 2.3 [Ale04b] *There exists a bilinear product*

$$\mathbf{Val}^{\infty} \times \mathbf{Val}^{\infty} \to \mathbf{Val}^{\infty}, \quad (\mu,\nu) \mapsto \mu \cdot \nu,$$

which is uniquely determined by the following two properties:

(i) The product is continuous in the Gårding topology.

(ii) If $L_1, L_2 \in \mathcal{K}^n$ are strictly convex and smooth, then

$$(\mu_{L_1} \cdot \mu_{L_2})(K) = V_{2n}(\iota(K) + L_1 \times L_2),$$

where $\iota : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is defined by $\iota(x) = (x,x)$.

Endowed with this multiplicative structure, $\mathbf{Val}^\infty$ becomes an associative and commutative algebra which is graded by the degree of homogeneity and with unit given by the Euler characteristic.

The next example was computed in [Ale04b].

Example:

Let $L_1, \dots, L_{n-i} \in \mathcal{K}^n$ and $M_1, \dots, M_i \in \mathcal{K}^n$ be strictly convex and smooth. If $\mu \in \mathbf{Val}_i^\infty$ and $\nu \in \mathbf{Val}_{n-i}^\infty$ are defined by

$$\mu(K) = V(K[i],L_1,\dots,L_{n-i}) \quad \text{and} \quad \nu(K) = V(K[n-i],M_1,\dots,M_i),$$

then

$$(\mu \cdot \nu)(K) = \binom{n}{i}^{-1} V(-L_1,\dots,-L_{n-i},M_1,\dots,M_i)\, V_n(K). \tag{2.3}$$

The above example is just a special case of the more general fact that the Alesker product gives rise to a non-degenerate bilinear pairing between smooth valuations of complementary degree.

Theorem 2.4 [Ale04b] *For every $0 \leq i \leq n$, the continuous bilinear pairing*

$$<\cdot,\cdot> : \mathbf{Val}_i^\infty \times \mathbf{Val}_{n-i}^\infty \to \mathbf{Val}_n, \qquad (\mu,\nu) \mapsto \mu \cdot \nu,$$

is non-degenerate. In particular, the induced Poincaré duality map

$$\mathbf{Val}_i^\infty \to \left(\mathbf{Val}_{n-i}^\infty\right)^* \otimes \mathbf{Val}_n, \qquad \mu \mapsto <\mu,\cdot>,$$

is continuous, injective and has dense image with respect to the weak topology.

Here and in the following, for a Fréchet space X, we denote by X^* its topological dual endowed with the weak topology.

Motivated by Theorem 2.4, the notion of generalized valuations was introduced recently in [Ale14]. Before we state the definition, recall that by a classical theorem of Hadwiger [Had57, p. 79] the space $\mathbf{Val}_n$ is spanned by the ordinary volume V_n. In other words, if we do not refer to any Euclidean structure, then $\mathbf{Val}_n \cong \mathscr{D}(V)$, where $\mathscr{D}(V)$ denotes the vector space of all densities on an n-dimensional vector space V. We refer to the appendix of [Berb] for details on these notions.

Definition 2.5. *The space of* generalized valuations *is defined by*

$$\mathbf{Val}^{-\infty} = (\mathbf{Val}^\infty)^* \otimes \mathscr{D}(V)$$

and we define the space of generalized valuations of degree $i \in \{0, \dots, n\}$ by

$$\mathbf{Val}_i^{-\infty} = \left(\mathbf{Val}_{n-i}^{\infty}\right)^* \otimes \mathscr{D}(V).$$

By Theorem 2.4, we have a canonical embedding with dense image

$$\mathbf{Val}^{\infty} \hookrightarrow \mathbf{Val}^{-\infty}.$$

Thus, $\mathbf{Val}^{-\infty}$ can be seen as a completion of $\mathbf{Val}^{\infty}$ in the weak topology.

In order to establish Theorem 3.3, we need the following new classification of generalized valuations of degree 1. A proof of this theorem was given by Semyon Alesker and is included in the appendix of [Berb].

Theorem 2.6. *The map*

$$C_{\mathrm{o}}^{\infty}(S^{n-1}) \to \mathbf{Val}_1^{\infty}, \qquad f \mapsto \left(K \mapsto \int_{S^{n-1}} f(u)\, h(K,u)\, du\right),$$

is an isomorphism of Fréchet spaces which extends uniquely by continuity in the weak topologies to an isomorphism

$$C_{\mathrm{o}}^{-\infty}(S^{n-1}) \to \mathbf{Val}_1^{-\infty}.$$

Note that, by Theorem 2.6, if $\gamma \in \mathbf{Val}_1^{-\infty}$ and $T_\gamma \in C_{\mathrm{o}}^{-\infty}(S^{n-1})$ is the corresponding distribution, then we can evaluate γ on convex bodies $K \in \mathcal{K}^n$ with smooth support function by

$$\gamma(K) := \langle h(K,\cdot), T_\gamma \rangle.$$

Next we briefly recall the Hard Lefschetz operators on smooth translation-invariant scalar-valued valuations. It is well known that McMullen's decomposition (2.1) of the space $\mathbf{Val}$ implies a general Steiner type formula for continuous translation-invariant valuations which, in turn, gives rise to a derivation operator $\Lambda : \mathbf{Val} \to \mathbf{Val}$ defined by

$$(\Lambda\mu)(K) = \left.\frac{d}{dt}\right|_{t=0} \mu(K + tB).$$

Note that Λ commutes with the action of $\mathrm{O}(n)$ and that it preserves parity. Moreover, if $\mu \in \mathbf{Val}_i$, then $\Lambda\mu \in \mathbf{Val}_{i-1}$.

The importance of the operator Λ became evident from a Hard Lefschetz type theorem established by Alesker [Ale03] for even valuations and by Bernig and Bröcker [Ber07b] for general valuations. More recently, a dual version of this fundamental result was established in [Ale04a; Ale11a]. There, the derivation operator Λ is replaced by an integration operator $\mathfrak{L} : \mathbf{Val} \to \mathbf{Val}$ defined by

$$(\mathfrak{L}\mu)(K) = (V_1 \cdot \mu)(K) = \int_{A(n,n-1)} \mu(K \cap E)\, dE, \tag{2.4}$$

where here and in the following, $A(n,k)$ denotes the affine Grassmannian of k-planes in $\mathbb{R}^n$ and integration is with respect to a (suitably normalized) invariant measure. The original definition of $\mathfrak{L}$ corresponds to the first equality in (2.4) and it was proved by Bernig [Ber07a] that the second equality holds. We also note that $\mathfrak{L}$ commutes with the action of $\mathrm{O}(n)$ and that it preserves parity. Moreover, if $\mu \in \mathbf{Val}_i$, then $\mathfrak{L}\mu \in \mathbf{Val}_{i+1}$.

2.2 Minkowski valuations

A map $\Phi\colon \mathcal{K}^n \to \mathcal{K}^n$ is called a *Minkowski valuation* if

$$\Phi K + \Phi L = \Phi(K \cup L) + \Phi(K \cap L)$$

whenever $K \cup L \in \mathcal{K}^n$ and addition on $\mathcal{K}^n$ is Minkowski addition. Systematic investigations of Minkowski valuations have only been initiated about a decade ago by Ludwig [Lud02; Lud03; Lud05]. These valuations arise naturally from data about projections and sections of convex bodies and form an integral part of geometric tomography. As first examples we mention here the projection body maps $\Pi_i : \mathcal{K}^n \to \mathcal{K}^n$ of order $i \in \{1, \dots, n-1\}$, defined by

$$h(\Pi_i K,u) = V_i(K|u^{\perp}), \qquad u \in S^{n-1}.$$

We refer to Section 2.3 for further details on projection bodies.

While the entire family Π_i is translation-invariant and $\mathrm{SO}(n)$-equivariant, the classic projection body map Π_{n-1} is the only one among them which intertwines *linear* transformations (see [Lud02]). In fact, there is only a small number of Minkowski valuations which are compatible with affine transformations (see [Aba12; Aba11; Hab12; Lud05; Sch12a; Wan11] for their classification).

The trivial Minkowski valuation maps every convex body to the set containing only the origin. For $0 \le j \le n$, we denote by $\mathbf{MVal}_j$ the set of all continuous, translation-invariant and $\mathrm{SO}(n)$-equivariant Minkowski valuations of degree j. In the next lemma, we state basic properties of such Minkowski valuations which are well known (cf. [Ale11b; Par12; Sch10]) and are needed in what follows.

Lemma 2.7. *If $\Phi_j \in \mathbf{MVal}_j$, $0 \le j \le n$, then the following statements hold:*

(a) The Steiner point of $\Phi_j K$ is at the origin, that is, $s(\Phi_j K) = o$ for every $K \in \mathcal{K}^n$.

(b) There exists $r_{\Phi_j} \ge 0$ such that

$$W_{n-1}(\Phi_j K) = r_{\Phi_j} W_{n-j}(K)$$

for every $K \in \mathcal{K}^n$. If Φ_j is non-trivial, then $r_{\Phi_j} > 0$.

(c) The $\mathrm{SO}(n-1)$-invariant valuation $\nu_j \in \mathbf{Val}_j$ defined by

$$\nu_j(K) = h(\Phi_j K,\bar{e})$$

uniquely determines Φ_j and is called the associated real-valued valuation *of the Minkowski valuation $\Phi_j \in \mathbf{MVal}_j$.*

Lemma 2.7 (c) motivated the following definition which first appeared in [Sch10].

Definition 2.8. *A Minkowski valuation $\Phi_j \in \mathbf{MVal}_j$, $0 \leq j \leq n$, is called* smooth *if its associated real-valued valuation ν_j is smooth.*

By $\mathbf{MVal}_j^\infty$ we will denote the set of translation-invariant and $\mathrm{SO}(n)$-equivariant *smooth* Minkowski valuations. Recall that smooth translation-invariant scalar-valued valuations are dense in all continuous translation-invariant scalar-valued valuations. However, this does not *directly* imply the same for Minkowski valuations; instead, additional arguments were needed for the proof which was given in [Sch10] for even and in [Sch] for general Minkowski valuations.

We are now in a position to state a recent Hadwiger type theorem for smooth Minkowski valuations which is the key to the proof of Theorem 3.3.

Theorem 2.9 [Sch] *If $\Phi_j \in \mathbf{MVal}_j^\infty$, $j \in \{1, \dots, n-1\}$, then there exists a unique $f \in C_{\mathrm{o}}^\infty(S^{n-1}, \bar{e})$, called the* generating function *of Φ_j, such that for every $K \in \mathcal{K}^n$,*

$$h(\Phi_j K, \cdot) = S_j(K, \cdot) * f. \tag{2.5}$$

In fact, a more general version of Theorem 2.9 for merely continuous Minkowski valuations in $\mathbf{MVal}_j$ was also established in [Sch], which are generated by $SO(n-1)$-invariant measures. At the time of the preparation of the article [Berb], this result was not yet available. Therefore Theorem 2.9 is used in [Berb] and in this thesis.

Examples:

(a) Kiderlen [Kid06] proved (in a slightly different form) that if $\Phi_1 \in \mathbf{MVal}_1^\infty$, then there exists a unique $g \in C_{\mathrm{o}}^\infty(S^{n-1}, \bar{e})$ such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_1 K, \cdot) = h(K, \cdot) * g. \tag{2.6}$$

In order to see how (2.6) is related to Theorem 2.9, we use (1.26) and the fact that $\Box_n : C_{\mathrm{o}}^\infty(S^{n-1}) \to C_{\mathrm{o}}^\infty(S^{n-1})$ is a bijective multiplier transformation to obtain a function $f \in C^\infty(S^{n-1}, \bar{e})$ with $\Box_n f = g$ and conclude that

$$h(\Phi_1 K, \cdot) = h(K, \cdot) * g = h(K, \cdot) * \Box_n f = \Box_n h(K, \cdot) * f = S_1(K, \cdot) * f.$$

(b) The case $j = n-1$ of Theorem 2.9 was first proved (in a more general form) in [Sch07]. Moreover, it was also shown there that $\Phi_{n-1} \in \mathbf{MVal}_{n-1}^\infty$ is *even* if and only if there exists an o-symmetric body of revolution $L \in \mathcal{K}^n$ with smooth support function such that for every $K \in \mathcal{K}^n$,

$$h(\Phi_{n-1} K, \cdot) = S_{n-1}(K, \cdot) * h(L, \cdot).$$

(c) For $i \in \{1, \dots, n-1\}$, the support function of the projection body map of order i (cf. also Section 2.3), $\Pi_i \in \mathbf{MVal}_i$, is given by

$$h(\mathit{\Pi}_i K,u) = V_i(K|u^\perp) = \frac{1}{2}\int_{S^{n-1}} |u\cdot v|\, dS_i(K,v), \qquad u \in S^{n-1}.$$

Note that $\mathit{\Pi}_i$ is continuous but *not* smooth. Its (merely) continuous generating function is given by $f(u) = \frac{1}{2}|u\cdot\bar{e}|$, $u \in S^{n-1}$.

(d) For $i \in \{2,\dots,n\}$, the (normalized) mean section operator of order i, denoted by $\mathrm{M}_i \in \mathbf{MVal}_{n+1-i}$, was first defined in [Goo92] by

$$h(\mathrm{M}_i K,\cdot) = \int_{A(n,i)} h(\mathrm{J}(K\cap E),\cdot)\, dE.$$

Here, $\mathrm{J} \in \mathbf{MVal}_1$ is defined by $\mathrm{J}K = K - s(K)$, where $s : \mathcal{K}^n \to \mathbb{R}^n$ is the Steiner point map. Recently, Goodey and Weil [Goo14] proved that the generating functions of the mean section operators are (up to normalization) the zonal functions $\breve{g}_i \in L_1(S^{n-1},\bar{e})$ determined by C. Berg's functions g_i on $[-1,1]$. More precisely,

$$h(\mathrm{M}_i K,\cdot) = p_{n,i}\, S_{n+1-i}(K,\cdot) * \breve{g}_i, \tag{2.7}$$

with constants $p_{n,i}$ which were explicitly determined in [Goo14].

The integration operator $\mathfrak{L}$ on translation-invariant scalar-valued valuations can be extended to Minkowski valuations by extending the identity (2.4). For $\mathit{\Phi} \in \mathbf{MVal}_j$, there exists $\mathfrak{L}\mathit{\Phi} \in \mathbf{MVal}_{j+1}$ such that

$$h((\mathfrak{L}\mathit{\Phi})(K),\cdot) = \int_{A(n,n-1)} h(\mathit{\Phi}(K\cap E),\cdot)\, dE. \tag{2.8}$$

where $A(n,n-1)$ denotes the affine Grassmannian of $n-1$ planes in $\mathbb{R}^n$ and where we integrate with respect to the suitably normalized invariant measure on $A(n,n-1)$. For *scalar-valued* valuations the operator $\mathfrak{L}$ was first defined in [Ale04a] and used to deduce results for valuations of degree i from those for valuations of some degree $j < i$. As an operator on Minkowski valuations, $\mathfrak{L}$ was first considered in [Sch15].

It was proved in [Par12] that the derivation operator $\mathit{\Lambda}$ can be extended to continuous translation-invariant Minkowski valuations, as well:

$$h((\mathit{\Lambda}\mathit{\Phi})(K),\cdot) = \left.\frac{d}{dt}\right|_{t=0} h(\mathit{\Phi}(K+tB),\cdot). \tag{2.9}$$

Note that in this case it is not trivial that the right-hand side actually defines the support function of a convex body; this was proved in [Par12].

If $\mathit{\Phi}_j \in \mathbf{MVal}_j^\infty$, $1 \le j \le n-1$, with associated real-valued valuation $\nu_j \in \mathbf{Val}_j^\infty$, then the associated real-valued valuations of $\mathfrak{L}\mathit{\Phi}_j$ and $\mathit{\Lambda}\mathit{\Phi}_j$ are given by $\mathfrak{L}\nu_j \in \mathbf{Val}_{j+1}^\infty$ and $\mathit{\Lambda}\nu_j \in \mathbf{Val}_{j+1}^\infty$, respectively. In particular, we have $\mathfrak{L}\mathit{\Phi}_j \in \mathbf{MVal}_{j+1}^\infty$ and $\mathit{\Lambda}\mathit{\Phi}_j \in \mathbf{MVal}_{j-1}^\infty$.

In view of Theorem 2.9, it is a natural problem to determine the induced action of the $\mathrm{SO}(n)$-equivariant operators $\mathit{\Lambda}$ and $\mathfrak{L}$ on the generating functions of smooth Minkowski valuations. This was done in [Sch15] and is the content of the following theorem.

Theorem 2.10 [Sch15] *Suppose that* $\Phi_j \in \mathbf{MVal}_j^\infty$ *and let* $f \in C_o^\infty(S^{n-1},\bar{e})$ *be the generating function of* Φ_j*.*

(a) If $2 \leq j \leq n-1$*, then the generating function of* $\Lambda\Phi_j$ *is given by* jf*.*

(b) If $1 \leq j \leq n-2$*, then there exists a constant* $c_{n,j} > 0$ *such that the generating function of* $\mathfrak{L}\Phi_j$ *is given by* $c_{n,j}\,\square_{n-j+1} f * \breve{g}_{n-j}$*.*

In particular, the map $\Lambda : \mathbf{MVal}_j^\infty \to \mathbf{MVal}_{j-1}^\infty$ *is injective for all* $2 \leq j \leq n-1$ *and* $\mathfrak{L} : \mathbf{MVal}_j^\infty \to \mathbf{MVal}_{j+1}^\infty$ *is injective for all* $1 \leq j \leq n-2$*.*

The constants $c_{n,j}$ from Theorem 2.10 (b) were explicitly determined in [Sch15]. We will now give the definition of an important subclass of Minkowski valuations, that inhabits log-concavity properties (cf. Section 3.2).

Definition 2.11. *For* $1 \leq i,j \leq n-1$*, let* $\mathbf{MVal}_{j,i}^\infty \subseteq \mathbf{MVal}_j^\infty$ *be defined by*

$$\mathbf{MVal}_{j,i}^\infty = \begin{cases} \Lambda^{i-j}(\mathbf{MVal}_i^\infty) & \text{if } i > j, \\ \mathbf{MVal}_j^\infty & \text{if } i \leq j. \end{cases}$$

We write $\mathbf{MVal}_{j,i}$ *for the closure of* $\mathbf{MVal}_{j,i}^\infty$ *in the topology of uniform convergence on compact subsets.*

By Theorem 2.10 the map $\Lambda : \mathbf{MVal}_j^\infty \to \mathbf{MVal}_{j-1}^\infty$ is injective for $2 \leq j \leq n$. Thus, for $i > j$, the inverse map $(\Lambda^{i-j})^{-1} : \mathbf{MVal}_{j,i}^\infty \to \mathbf{MVal}_i^\infty$ is well-defined and will be denoted by Λ^{j-i}. From Theorem 2.10 (a) and Examples (a) and (b) above, we can also deduce more information about the classes $\mathbf{MVal}_{j,i}^\infty$.

Corollary 2.12.

(a) Suppose that $1 \leq i,j \leq n-1$*,* $\Phi_j \in \mathbf{MVal}_j^\infty$*, and let* $f \in C_o^\infty(S^{n-1},\bar{e})$ *be the generating function of* Φ_j*. Then* $\Phi_j \in \mathbf{MVal}_{j,i}^\infty$ *if and only if* $S_i(K,\cdot) * f$ *is a support function for every* $K \in \mathcal{K}^n$*.*

(b) $\mathbf{MVal}_{1,n-1}^\infty \subsetneq \mathbf{MVal}_1^\infty$*.*

Proof. Statement (a) is a direct consequence of the definition of $\mathbf{MVal}_{j,i}^\infty$ and Theorem 2.10 (a).

In order to prove (b), let $\Phi_1 \in \mathbf{MVal}_1^\infty$ be even and let $f \in C_o^\infty(S^{n-1},\bar{e})$ be the generating function of Φ_1. Then, by (a) and Example (b) from above, $\Phi_1 \in \mathbf{MVal}_{1,n-1}^\infty$ if and only if $f = h(L,\cdot)$ for some o-symmetric body of revolution $L \in \mathcal{K}^n$. In this case, we have

$$h(\Phi_1 K,\cdot) = S_1(K,\cdot) * h(L,\cdot) = \square_n h(K,\cdot) * h(L,\cdot) = h(K,\cdot) * s_1(L,\cdot),$$

where $s_1(L,\cdot) = \square_n h(L,\cdot)$ is the smooth density of $S_1(L,\cdot)$. It was proved by Kiderlen [Kid06] that for any (even) non-negative $g \in C_o^\infty(S^{n-1},\bar{e})$, (2.6) defines an (even) Minkowski valuation in $\mathbf{MVal}_1^\infty$. Since the set of area measures of order 1 is nowhere dense in $\mathcal{M}_o$, this proves the claim. $\square$

Note that, by Corollary 2.12 (b), in general $\mathbf{MVal}_{j,i}^{\infty} \subsetneq \mathbf{MVal}_j^{\infty}$ for $i > j$. Explicit examples of Minkowski valuations in $\mathbf{MVal}_j$ with generating functions which do not generate a Minkowski valuation in $\mathbf{MVal}_{n-1}$ are provided by the mean section operators. This follows from (2.7) and the case $i = n-1$ of Theorem 2.9 for continuous Minkowski valuations established in [Sch07], where it was proved that $\Phi_{n-1} \in \mathbf{MVal}_{n-1}$ is generated by a *continuous* function $f \in C_o(S^{n-1},\bar{e})$. However, C. Berg's functions g_i are not continuous on $[-1,1]$ for $i \geq 5$.

We end this section with another remark concerning Corollary 2.12 (a): Generating functions or earlier versions of Theorem 2.9, respectively, were the critical tool used in the proofs of the first Brunn–Minkowski type inequalities for Minkowski valuations. We will see in Section 3.2 that the Hard Lefschetz operators on Minkowski valuations (which were introduced only recently) and Theorem 2.9 both naturally lead to the same classes $\mathbf{MVal}_{j,i}$ for which we can establish such inequalities.

2.3 Generalized projection and centroid bodies

In this section we recall the definition of the projection body maps of order i and of the centroid body and generalize these notions to Minkowski valuations generated by zonoids of revolution, where we follow the approach of [Hab].

The projection body maps of order $i \in 1, \ldots, n-1$ are the Minkowski valuations given by

$$h(\Pi_i K, \cdot) = V_i(K|u^{\perp}) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| dS_i(K,v), \quad u \in S^{n-1}.$$

These maps $\Pi_i \colon \mathcal{K}^n \to \mathcal{K}^n$ are translation-invariant, i-homogeneous and $SO(n)$-equivariant. The map $\Pi = \Pi_{n-1}$ is $SL(n)$-contravariant, that is, $\Pi \Phi K = \Phi^{-\top} \Pi K$ for $\Phi \in SL(n)$. The projection body operator has been investigated extensively (for example in [Aba11; Lut00b; Lut10a]) and generalizations in various settings have also been defined.

It is possible (cf. [Goo92]) to write the support function of $\Pi_i K$ in a way that will prove useful later on as

$$h(\Pi_i K,\cdot) = \frac{\kappa_{n-1}}{\kappa_n} R_{n-i,1} \mathrm{vol}_i(K|\cdot^{\perp}). \tag{2.10}$$

Considering identity (1.3), there is also another way (cf. [Lut86b]) to write the i-th projection body as

$$\Pi\,[K]_i = \Pi_i K.$$

The projection body map $\Pi = \Pi_{n-1}$ has been generalized by Haberl and Schuster [Hab] to Minkowski valuations generated by zonoids of revolution. We will in the following recall their results, starting with a well-known characterization of zonoids.

Proposition 2.13 (see, e.g., [Sch14, Theorem 3.5.3])**.** *A convex body $K \in \mathcal{K}^n$ is a zonoid with center at o if and only if its support function can be represented in the form*

$$h(K,u) = \int_{S^{n-1}} |u \cdot v| d\mu(v), \quad u \in S^{n-1}$$

with some uniquely determined even non-negative measure μ on S^{n-1}.

For a non-trivial, o-symmetric zonoid of revolution $Z(\bar{e})$ with axis of revolution $\bar{e}$ (a fixed pole on the sphere), we thus have

$$h(Z(\bar{e}),u) = \int_{S^{n-1}} |u \cdot w| d\mu(w) = \int_{SO(n)} |u \cdot \phi\bar{e}| d\breve{\mu}(\phi), \tag{2.11}$$

where $\breve{\mu}$ is the measure introduced in Section 1.3. If $Z(\bar{e})$ is rotated such that $v \in S^{n-1}$ becomes the new axis, we abbreviate $Z(v) := \theta_v Z(\bar{e})$, where $v = \theta_v \bar{e}$ with $\theta_v \in SO(n)$. The support function of this new zonoid $Z(v)$ is then given by

$$h(Z(v),u) = \int_{SO(n)} |u \cdot \phi v| d\breve{\mu}_v(\phi), \tag{2.12}$$

since we can compute

$$h(Z(v),u) = h(Z(e_n), \vartheta_v^{-1} v) = \int_{SO(n)} |u \cdot \vartheta_v \phi \vartheta_v^{-1} v| d\breve{\mu}(\phi) = \int_{SO(n)} |u \cdot \phi v| d\breve{\mu}_v(\phi),$$

where $\breve{\mu}_v = t_v \# \breve{\mu}$ with $t_v(\phi) = \vartheta_v \phi \vartheta_v^{-1}$. In particular, we note that

$$h(Z(v),u) = h(Z(u),v), \quad u,v \in S^{n-1}. \tag{2.13}$$

In [Hab], Haberl and Schuster investigated the Minkowski valuations $\Phi_Z : \mathcal{K}^n \to \mathcal{K}^n$ generated by an o-symmetric zonoid of revolution Z with rotation axis $\bar{e}$, by

$$h(\Phi_Z K, u) = \frac{1}{2} \int_{S^{n-1}} h(Z(v),u) dS_{n-1}(K,v), \tag{2.14}$$

which for $Z = [-\bar{e},\bar{e}]$ is just the projection body. With the help of the spherical convolution, this can also be written in the form

$$h(\Phi_Z K, \cdot) = \frac{1}{2} S_{n-1}(K, \cdot) * h(Z(\bar{e}), \cdot). \tag{2.15}$$

We are going to extend this definition to include the projection bodies of order i. For Φ_Z as defined above, by the definition of the mixed area measure, there exists a continuous operator

$$\Phi_Z \colon \overbrace{\mathcal{K}^n \times \cdots \times \mathcal{K}^n}^{n-1} \to \mathcal{K}^n,$$

symmetric in its arguments such that for $K_1, \ldots, K_m \in \mathcal{K}^n$ and $\lambda_1, \ldots \lambda_m \geq 0$

$$\Phi_Z(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1,\ldots,i_{n-1}} \lambda_{i_1} \cdots \lambda_{i_{n-1}} \Phi_Z(K_{i_1}, \ldots, K_{i_{n-1}}),$$

where the coefficients are explicitely given by the formula

$$h(\Phi_Z(K_1, \ldots, K_{n-1}), u) = \frac{1}{2} \int_{S^{n-1}} h(Z(v),u) dS_{n-1}(K_1, \ldots, K_{n-1},v).$$

We will also use the abbreviation

$$h(\Phi_{Z,i}K,u) = \frac{1}{2}\int_{S^{n-1}} h(Z(v),u)dS_i(K,v), \tag{2.16}$$

where again $Z = [-\bar{e},\bar{e}]$ gives just the projection body of order i. We recall the integral representation of Φ_Z by Haberl and Schuster, which is essential to their proof of Theorem 4.1.

Lemma 2.14 [Hab] *For $K \in \mathcal{K}^n$ and Z as above, the support function of $\Phi_Z K$ has the representation*

$$h(\Phi_Z K,u) = \int_{SO(n)} h(\Pi K,\phi u)d\breve{\mu}_u(\phi), \quad u \in S^{n-1}. \tag{2.17}$$

We are now able to prove an analogous representation for $\Phi_{Z,i}K$.

Lemma 2.15. *For $K \in \mathcal{K}^n$, Z as above and for $0 < i < n-1$, the support function of $\Phi_{Z,i}K$ has the representation*

$$h(\Phi_{Z,i}K,u) = \int_{SO(n)} h(\Pi_i K,\phi u)d\breve{\mu}_u(\phi). \tag{2.18}$$

Proof. By (2.16), (2.12), and by using Fubini's theorem, we have

$$\begin{aligned} h(\Phi_{Z,i}K,u) &= \frac{1}{2}\int_{S^{n-1}} h(Z(v),u)dS_i(K,v) \\ &= \frac{1}{2}\int_{S^{n-1}}\int_{SO(n)} |v\cdot\phi u|d\breve{\mu}_u(\phi)dS_i(K,v) \\ &= \int_{SO(n)} h(\Pi_i K,\phi u)d\breve{\mu}_u(\phi). \end{aligned}$$

□

In the same way projection bodies can be generalized to Minkowski valuations generated by Z, we will now generalize the centroid body map. For $K \in \mathcal{K}^n_{(o)}$ the *centroid body* is given by

$$h(\Gamma K,u) = \frac{1}{V(K)}\int_K |u\cdot x|dx.$$

This can be rewritten by using polar coordinates to

$$h(\Gamma K,u) = \frac{1}{(n+1)V(K)}\int_{S^{n-1}} |u\cdot v|\rho(K,v)^{n+1}dS(v).$$

The famous Busemann–Petty centroid inequality then compares the volume of the centroid body to the volume of a convex body $K \in \mathcal{K}^n_{(o)}$ by

$$V(K)V(\Gamma K)^{-1} \leq V(B)V(\Gamma B)^{-1}. \tag{2.19}$$

This inequality has been proved by Petty [Pet61], and was extended to the L_p setting by Lutwak, Yang, and Zhang [Lut00b], Campi and Gronchi [Cam02] and Haberl and Schuster [Hab09b].

In [Sch06], a generalized centroid body depending on Z as above analogous to the generalization of the projection body was introduced.

Definition 2.16. *For $K \in \mathcal{K}^n_{(o)}$, the centroid body $\varGamma_Z K$ generated by Z is defined by*

$$h(\varGamma_Z K, u) = \frac{1}{V(K)} \int_K h(Z(x), u) dx, \tag{2.20}$$

where we write $Z(x) = \|x\| Z(u)$ for $x = \|x\| u$, $u \in S^{n-1}$.

This can again be rewritten to

$$h(\varGamma_Z K, v) = \frac{1}{(n+1)V(K)} \int_{S^{n-1}} h(Z(v), u) \rho(K, u)^{n+1} dS(u), \quad v \in S^{n-1}. \tag{2.21}$$

Using this identity, an alternative integral representation of the support function of the generalized centroid body can be computed.

Lemma 2.17. *If $K \in \mathcal{K}^n_{(o)}$ and Z defined as above, then*

$$h(\varGamma_Z K, v) = \int_{SO(n)} h(\varGamma K, \phi v) d\breve{\mu}_v(\phi). \tag{2.22}$$

Proof. By (2.21),(2.12), and Fubini's theorem, we have

$$\begin{aligned} h(\varGamma_Z K, v) &= \frac{1}{(n+1)V(K)} \int_{S^{n-1}} h(Z(v), u) \rho(K, u)^{n+1} dS(u) \\ &= \frac{1}{(n+1)V(K)} \int_{S^{n-1}} \int_{SO(n)} |u \cdot \phi v| d\breve{\mu}_v(\phi) \rho(K, u)^{n+1} dS(u) \\ &= \frac{1}{(n+1)V(K)} \int_{SO(n)} \int_{S^{n-1}} |u \cdot \phi v| \rho(K, u)^{n+1} dS(u) d\breve{\mu}_v(\phi) \\ &= \int_{SO(n)} h(\varGamma K, \phi v) d\breve{\mu}_v(\phi). \end{aligned}$$

□

2.4 L_p Minkowski valuations

For $p > 1$, an operator $\varPhi \colon \mathcal{K}^n_o \to \mathcal{K}^n_o$ is called an L_p *Minkowski valuation* if

$$\varPhi(K \cup L) +_p \varPhi(K \cap L) = \varPhi(K) +_p \varPhi(L)$$

whenever $K, L, K \cup L \in \mathcal{K}^n_o$. L_p Minkowski valuations were first investigated by Ludwig [Lud05].

For $K \in \mathcal{K}^n_{(o)}$ and $p \geq 1$, Lutwak, Yang, and Zhang introduced the L_p *projection body* in [Lut00b]. With a slightly different normalization, we will write

$$h(\Pi_{\mathbf{p}} K,u)^p = \frac{1}{2} \int_{S^{n-1}} |u \cdot v|^p dS_{\mathbf{p}}(K,v).$$

We note that with this normalization, $\Pi_{\mathbf{1}} K$ is just the classical projection body $\Pi_{n-1} K$ and therefore $\Pi_{\mathbf{1}} B = \kappa_{n-1} B$.

The L_p projection body was generalized by Haberl and Schuster to $SO(n)$-equivariant L_p Minkowski valuations. For $1 \leq p < \infty$, a convex body $K \in \mathcal{K}^n_n$ is called an L_p zonoid, if its support function can be represented in the form

$$h(K,u)^p = \int_{S^{n-1}} |u \cdot v|^p d\mu(v), \quad u \in S^{n-1},$$

with some even finite Borel measure μ on S^{n-1} which is uniquely determined if p is not an even integer. In the following, we will denote by $Z(\bar{e})$ an o-symmetric L_p zonoid of revolution with revolution axis $\bar{e}$. For $K \in \mathcal{K}^n_{(o)}$, Haberl and Schuster defined the L_p Minkowski valuations generated by an L_p zonoid Z by

$$h(\Phi_{Z,\mathbf{p}} K,u)^p = \frac{1}{2} \int_{S^{n-1}} h(Z(u),v)^p dS_{\mathbf{p}}(K,v).$$

We refer to Section 4.1 for important volume inequalities by Lutwak, Yang, and Zhang for the operator $\Pi_{\mathbf{p}}$ and generalizations of these inequalities by Haberl and Schuster for $\Phi_{Z,\mathbf{p}}$.

The L_p centroid body of a star body K is defined by

$$h(\Gamma_{\mathbf{p}} K,u)^p = \frac{1}{V(K)} \int_K |u \cdot x|^p dx$$

and can also be generalized to the L_p centroid body generated by Z, given by

$$h(\Gamma_{Z,\mathbf{p}} K,u)^p = \frac{1}{V(K)} \int_K h(Z(u),x)^p dx. \tag{2.23}$$

We remark that this new definition again includes the generalized centroid body $\Gamma_Z K$ as a special case, since $h(\Gamma_{Z,\mathbf{1}} K,u) = h(\Gamma_Z K,u)$ is the classical centroid body. By using polar coordinates, this can be written as

$$h(\Gamma_{Z,\mathbf{p}} K,u)^p = \frac{1}{(n+p)V(K)} \int_{S^{n-1}} h(Z(u),v)^p \rho(K,v)^{n+p} dS(v). \tag{2.24}$$

For further results on these operators, we refer the reader to Section 4.1.

2.5 Generalized intersection bodies

In this section we recall the definition of intersection bodies of order i for $0 < i \le n-1$ and generalize this notion to a subclass of radial Minkowski valuations.

A *radial Minkowski valuation* is a map $\Psi\colon \mathcal{S}_o^n \to \mathcal{S}_o^n$ satisfying

$$\Psi K \tilde{+}_1 \Psi L = \Psi(K \cup L) \tilde{+}_1 \Psi(K \cap L).$$

These maps have been investigated in [Hab09a; Lud06; Sch06]. An important example of a radial Minkowski valuation is the intersection body. For $K \in \mathcal{S}_o^n$, the intersection body is the unique star body $\mathrm{I}K$ defined by

$$\rho(\mathrm{I}K, u) = V_{n-1}(K \cap u^\perp), \quad u \in S^{n-1}.$$

It was first explicitly defined by Lutwak in [Lut88b] but has appeared earlier in [Bus49]. Lutwak also showed in [Lut91] that the intersection body operator satisfies $\mathrm{I}\phi K = \phi^{-\top}\mathrm{I}K$ for $\phi \in SL(n)$. Dual to the projection body, Ludwig [Lud06] characterized the intersection body as the only $SL(n)$ compatible radial Minkowski valuation. By equation (1.5), the radial function of the intersection body can also be written for $u \in S^{n-1}$ as

$$\rho(\mathrm{I}K, u) = (R\rho(K,\cdot)^{n-1})(u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho(K,v)^{n-1} d\lambda_{S^{n-1}\cap u^\perp}(v), \tag{2.25}$$

where $\lambda_{S^{n-1}\cap \bar{e}^\perp}$ denotes the invariant measure concentrated on $S^{n-1} \cap \bar{e}^\perp$ with total mass κ_{n-1} and $*$ is the convolution on the sphere (confer Section 1.3 for details).

For $K_1, \dots, K_{n-1} \in \mathcal{S}_o^n$, *mixed intersection bodies* were introduced in [Lei98] by

$$\rho(\mathrm{I}(K_1, \dots, K_{n-1}), u) = \tilde{v}(K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp),$$

where $\tilde{v}(K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp)$ denotes the $(n-1)$-dimensional mixed volume of the star bodies $K_1 \cap u^\perp, \dots, K_{n-1} \cap u^\perp$ in $u^\perp$. This can be rewritten in terms of the integral representation

$$\rho(\mathrm{I}(K_1, \dots, K_{n-1}), u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^\perp} \rho(K_1,v)\cdots\rho(K_{n-1},v) d\lambda_{S^{n-1}\cap u^\perp}(v).$$

If $K_1 = \cdots = K_i = K$ and $K_{i+1} = \cdots = K_{n-1} = B$, then we introduce the intersection body maps of order i by

$$\rho(\mathrm{I}_i K, u) := \rho(\mathrm{I}(K[i], B[n-i-1]), u) = \tilde{v}(K \cap u^\perp[i], B \cap u^\perp[n-1-i]).$$

They were first defined by Zhang in [Zha94], where he also showed that

$$\rho(\mathrm{I}_i K, u) = \frac{\kappa_{n-1}}{\kappa_i}\left[R_{n-i,1} vol_i^\perp(K \cap \cdot)\right](\bar{u}).$$

It is easy to see from (1.13) that the operators I_i are $SO(n)$-equivariant. For $K \in \mathcal{S}_o^n$ and

$0 < i < n-1$, the i-th intersection bodies also satisfy a Steiner type formula

$$\mathrm{I}(K\tilde{+}_1 rB) = \sum_{i=0}^{n-1} \binom{n-1}{i} r^{n-1-i} \mathrm{I}_i K.$$

In analogy to Proposition 2.13, the class of intersection bodies is given by those $L \in \mathcal{S}_o^n$ such that there is a finite non-negative Borel measure ν on S^{n-1} with $\rho(L,\cdot) = R\nu$, in the sense that

$$\int_{S^{n-1}} \rho(L,u) f(u) du = \int_{S^{n-1}} Rf(u) d\nu(u)$$

for every $f \in C(S^{n-1})$. By (2.25), the intersection body of a star body is an example of a body from the class of intersection bodies. We define the operators Ψ_ν for $\nu \in \mathcal{M}_+(S^{n-1}, \bar{e})$ by

$$\rho(\Psi_\nu K, \cdot) := \rho^{n-1}(K,\cdot) * R\nu = \rho^{n-1}(K,\cdot) * (\nu * \lambda_{S^{n-1}\cap \bar{e}^\perp}). \tag{2.26}$$

In analogy to definition (2.15), for an intersection body L with $\rho(L,\cdot) = R\nu$ this just becomes

$$\rho(\Psi_L K, \cdot) := \rho^{n-1}(K,\cdot) * \rho(L,\cdot).$$

These radial valuations satisfy a Steiner type formula by (1.9). We introduce the corresponding generalizations for the i-th intersection bodies by

$$\Psi_\nu(K\tilde{+} rB) = \sum_{i=0}^{n-1} \binom{n-1}{i} r^{n-1-i} \Psi_{\nu,i} K.$$

They can be written in the form

$$\rho(\Psi_{\nu,i} K, \cdot) := R\rho^i(K,\cdot). \tag{2.27}$$

By identity (1.13), the operators $\Psi_{\nu,i}$ are $\mathrm{SO}(n)$-equivariant. In analogy to Lemma 2.15, it is now possible to get a corresponding integral representation also for the $\Psi_{\nu,i}$.

Lemma 2.18. *For $K \in \mathcal{S}_o^n$ and $\nu \in \mathcal{M}_+(S^{n-1}, \bar{e})$, the radial function of $\Psi_{\nu,i} K$, $1 \le i \le n-1$ has the representation*

$$\rho(\Psi_{\nu,i} K, \eta\bar{e}) = (\rho(\mathrm{I}_i K, \cdot) * \nu)(\eta\bar{e}) = \int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, \phi\eta) d\mu_\eta(\phi),$$

where $\mu_\eta := h_\eta \# \check{\nu}$.

Proof. The first equation follows from the commutativity of the spherical convolution of zonal measures and the associativity of convolution (1.15), since

$$\begin{aligned} \rho(\Psi_{\nu,i} K, \cdot) &= \rho^i(K,\cdot) * (\nu * \lambda_{S^{n-1}\cap \bar{e}^\perp}) \\ &= (\rho^i(K,\cdot) * \lambda_{S^{n-1}\cap \bar{e}^\perp}) * \nu \\ &= \rho(\mathrm{I}_i K, \cdot) * \nu. \end{aligned}$$

The normalization $\nu(S^{n-1}) = n\kappa_n$ corresponds to $\rho(\Psi_\nu B, \cdot) = 1$ or $\check{\nu}(O(n)) = 1$. If we rewrite this with the corresponding measure on $O(n)$, we obtain

$$\rho(\Psi_{\nu,i} K, \eta\bar{e}) = \int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, \eta\varphi^{-1}) d\check{\nu}(\varphi).$$

Setting $h_\eta(\varphi) := \eta\varphi^{-1}\eta^{-1}$ and using the change of variables formula yields

$$\int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, \eta\varphi^{-1}) d\check{\nu}(\varphi) = \int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, h_\eta(\varphi)\eta) d\check{\nu}(\varphi) = \int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, \phi\eta) d(h_\eta \# \check{\nu})(\phi).$$

With $\mu_\eta := h_\eta \# \check{\nu}$ we therefore obtain the desired representation

$$\rho(\Psi_{\nu,i} K, \cdot) = \int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, \phi\eta) d\mu_\eta(\phi).$$

□

CHAPTER 3

Log-concavity properties of Minkowski valuations

3.1 Log-concavity properties of classical functionals

The fundamental log-concavity property of the volume functional is expressed by the multiplicative form of the Brunn–Minkowski inequality:

$$V_n((1-\lambda)K+\lambda L) \geq V_n(K)^{1-\lambda}V_n(L)^{\lambda}, \tag{3.1}$$

where K and L are convex bodies (non-empty compact convex sets) in $\mathbb{R}^n$ with non-empty interiors, $0<\lambda<1$, and $+$ denotes Minkowski addition. Equality holds in (3.1) if and only if K and L are translates of each other. The excellent survey of Gardner [Gar02] gives a comprehensive overview of different aspects and consequences of the Brunn–Minkowski inequality.

A consequence of the Minkowski inequality (1.8) and the homogeneity of quermassintegrals is the (multiplicative) Brunn–Minkowski inequality for quermassintegrals: If $2 \leq i \leq n$ and $K, L \in \mathcal{K}^n$ have dimension at least i, then for all $\lambda \in (0,1)$,

$$W_{n-i}((1-\lambda)K+\lambda L) \geq W_{n-i}(K)^{1-\lambda}W_{n-i}(L)^{\lambda}, \tag{3.2}$$

with equality if and only if K and L are translates of each other.

A further generalization of inequality (3.2) (where the equality conditions are not yet known) is the following (see [Sch14, p. 406]): If $0 \leq i \leq n-2$, $K, L, K_1, \ldots, K_i \in \mathcal{K}^n$ and $\mathbf{C} = (K_1,\ldots,K_i)$, then for all $\lambda \in (0,1)$,

$$V_i((1-\lambda)K+\lambda L,\mathbf{C}) \geq V_i(K,\mathbf{C})^{1-\lambda}V_i(L,\mathbf{C})^{\lambda}. \tag{3.3}$$

The problem of finding sharp bounds for the volume of projection bodies, given the volume of the original body, has sparked the investigation of log-concavity properties of another class of geometric functionals associated with a convex body. In 1993, Lutwak [Lut93a] established not only Brunn–Minkowski type inequalities for the volume of projection bodies, but for all the *intrinsic volumes* of projection bodies of arbitrary order. In an equivalent multiplicative form, his result states the following: If $K, L \in \mathcal{K}^n$ have non-empty interiors,

$1 \leq i \leq n$, and $2 \leq j \leq n-1$, then for all $\lambda \in (0,1)$,

$$V_i(\mathit{\Pi}_j((1-\lambda)K + \lambda L)) \geq V_i(\mathit{\Pi}_j K)^{1-\lambda} V_i(\mathit{\Pi}_j L)^{\lambda}, \tag{3.4}$$

with equality if and only if K and L are translates of each other. In the next section, we will discuss various generalizations of these results.

3.2 Extensions of the classical results

Inequalities (3.4) have been generalized in different directions; Abardia and Bernig [Aba11], for example, extended (3.4) to the entire class of *complex* projection bodies.

In this thesis we investigate a common generalization of Lutwak's Brunn–Minkowski inequalities for projection bodies and inequality (3.1), more specifically, its version for all the intrinsic volumes. To be more precise, we establish new log-concavity properties of intrinsic volumes of convex-body-valued *valuations* which intertwine rigid motions.

In recent years, it has become apparent that several geometric inequalities for projection bodies and, more general, valuations intertwining the group of affine transformations, in fact, hold for much larger classes of valuations intertwining merely rigid motions. First such results were obtained in [Sch06], where the Brunn–Minkowski inequalities for projection bodies of Lutwak were generalized to translation-invariant and $\mathrm{SO}(n)$-*equivariant* Minkowski valuations of degree $n-1$ and then in [Sch10] for *even* valuations in $\mathbf{MVal}_j$ in the case $i = j+1$. The assumption on the parity could later be omitted in [Ale11b]. Although considerable efforts have been invested ever since to show that these log-concavity properties extend to Minkowski valuations of arbitrary degree (see [Ale11b; Par12; Sch10]), the conjectured complete family of inequalities has only partially been obtained.

In the Euclidean setting, the most general result to date can be stated (in multiplicative form) as follows:

Theorem 3.1 [Par12] *Let $\mathit{\Phi}_j \in \mathbf{MVal}_j$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ and $1 \leq i \leq j+1$, then for all $\lambda \in (0,1)$,*

$$V_i(\mathit{\Phi}_j((1-\lambda)K + \lambda L)) \geq V_i(\mathit{\Phi}_j K)^{1-\lambda} V_i(\mathit{\Phi}_j L)^{\lambda}. \tag{3.5}$$

If K and L are of class C^2_+, then equality holds if and only if K and L are translates of each other.

Note that Theorem 3.1 establishes (3.5) only for $1 \leq i \leq j+1$, while in Lutwak's family of inequalities (3.4) the range of i does not depend on j. For the inequalities established so far, two different approaches were used. While in [Sch06] and [Sch10] integral representations of (even) Minkowski valuations which are translation-invariant and $\mathrm{SO}(n)$-equivariant were crucial, in [Par12] the Hard Lefschetz *derivation* operator on Minkowski valuations from (2.9), together with a symmetry property of bivaluations [Ale11b], were crucial. The key to the proof of Theorem 3.1 was the following generalization of a symmetry property of bivaluations obtained in [Ale11b].

Theorem 3.2 ([Par12]). *Let $\Phi_j \in \mathbf{MVal}_j$, $2 \leq j \leq n-1$. If $1 \leq i \leq j+1$, then*

$$W_{n-i}(K,\Phi_j L) = \frac{(i-1)!}{j!}\, W_{n-j-1}(L,(\Lambda^{j+1-i}\Phi_j)(K)) \tag{3.6}$$

for every $K, L \in \mathcal{K}^n$.

We will see that Theorem 3.2 follows directly from a recently obtained integral representation of Minkowski valuations intertwining rigid motions.

An obvious idea for a proof of (3.5) for the remaining cases $j+2 \leq i \leq n$ is to establish a counterpart of Theorem 3.2 for the Hard Lefschetz integration operator given by (2.8). However, the situation is more delicate in this case and we will see that a full analog of (3.6) only holds for a subclass of Minkowski valuations, namely the class given by Definition 2.11. Our counterpart of Theorem 3.2 can be stated as follows:

Theorem 3.3. *Let $\Phi_j \in \mathbf{MVal}_j^\infty$, $2 \leq j \leq n-1$. For $j+2 \leq i \leq n$ and every convex body $L \in \mathcal{K}^n$, there exists a* generalized valuation *$\gamma_{i,j}(L,\cdot) \in \mathbf{Val}_1^{-\infty}$ such that*

$$W_{n-i}(K,\Phi_j L) = \gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K))$$

for every $K \in \mathcal{K}^n$. Moreover, if $\Phi_j \in \mathbf{MVal}_{j,i-1}^\infty$, then

$$\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \frac{(i-1)!}{j!} W_{n-1-j}(L,(\Lambda^{j+1-i}\Phi_j)(K)).$$

Proof. First define an isomorphism $\Theta_j : C_o^\infty(S^{n-1}) \to C_o^\infty(S^{n-1})$ by

$$\Theta_j \zeta = c_{n,j}\, \Box_{n-j+1}\zeta * \breve{g}_{n-j} = c_{n,j}\, \zeta * \Box_{n-j+1}\breve{g}_{n-j},$$

where the constant $c_{n,j} > 0$ is as in Theorem 2.10 (b). Here, the second equality follows from the fact that multiplier transformations commute and $\Box_{n-j+1}\breve{g}_{n-j}$ is to be understood in the sense of distributions, where we use the canonical extension of the self-adjoint operator $\Box_{n-j+1}$ to $C_o^{-\infty}(S^{n-1})$.

Let $\tau_{\bar{e}} = \delta_{\bar{e}} - \pi_1\delta_{\bar{e}} \in \mathcal{M}_o(S^{n-1})$, where $\delta_{\bar{e}}$ is the Dirac measure supported in $\bar{e} \in S^{n-1}$. Then, by (1.14), $\zeta * \tau_{\bar{e}} = \zeta$ for every $\zeta \in C_o^\infty(S^{n-1})$. Now, since $\Box_k \breve{g}_k = \tau_{\bar{e}}$, it follows from Theorem 2.10 (b) that if $f \in C_o^\infty(S^{n-1},\bar{e})$ is the generating function of Φ_j, then $\mathfrak{L}^{i-j-1}\Phi_j \in \mathbf{MVal}_{i-1}^\infty$ is generated by

$$\Theta_{i-2}\Theta_{i-1}\cdots\Theta_{j+1}\Theta_j f = q_{n,i,j}\, \Box_{n-j+1} f * \breve{g}_{n-i+2}, \tag{3.7}$$

where $q_{n,i,j} = \prod_{m=j}^{i-2} c_{n,m} > 0$. Note that the inverse of the isomorphism (3.7) is, for $\zeta \in C_o^\infty(S^{n-1})$, given by

$$q_{n,i,j}^{-1}\, \Box_{n-i+2}\zeta * \breve{g}_{n-j+1}.$$

For every $L \in \mathcal{K}^n$ we define a distribution $T_{i,j}(L) \in C_o^{-\infty}(S^{n-1})$ by

$$\langle \zeta, T_{i,j}(L)\rangle = q_{n,i,j}^{-1} \int_{S^{n-1}} (\Box_{n-i+2}\zeta * \breve{g}_{n-j+1})(u)\, dS_j(L,u)$$

for $\zeta \in C_{\mathrm{o}}^{\infty}(S^{n-1})$. Let $\gamma_{i,j}(L,\cdot) \in \mathbf{Val}_1^{-\infty}$ be the generalized valuation corresponding to $T_{i,j}(L)$ determined by Theorem 2.6.

Since $\mathfrak{L}^{i-j-1}\Phi_j$ is smooth, it follows that $h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot)$ is smooth for every $K \in \mathcal{K}^n$. Hence, we can evaluate $\gamma_{i,j}(L,\cdot)$ on $(\mathfrak{L}^{i-j-1}\Phi_j)(K)$. Using that

$$h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot) = q_{n,i,j}\, S_{i-1}(K,\cdot) * (\Box_{n-j+1} f * \breve{g}_{n-i+2}),$$

we obtain

$$\begin{aligned}\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) &= \langle h((\mathfrak{L}^{i-j-1}\Phi_j)(K),\cdot), T_{i,j}(L)\rangle \\ &= \int_{S^{n-1}} (S_{i-1}(K,\cdot) * f)(u)\, dS_j(L,u).\end{aligned}$$

Now on the one hand it follows from (1.16), that

$$\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \int_{S^{n-1}} (S_j(L,\cdot) * f)(u)\, dS_{i-1}(K,u) = W_{n-i}(K,\Phi_j L).$$

On the other hand, if $\Phi_j \in \mathbf{MVal}_{j,i-1}^{\infty}$, then, by Theorem 2.10 (a),

$$S_{i-1}(K,\cdot) * f = \frac{(i-1)!}{j!} h((\Lambda^{j+1-i}\Phi_j)(K),\cdot)$$

and, thus,

$$\gamma_{i,j}(L,(\mathfrak{L}^{i-j-1}\Phi_j)(K)) = \frac{(i-1)!}{j!} W_{n-1-j}(L,(\Lambda^{j+1-i}\Phi_j)(K))$$

which completes the proof. $\square$

Note that, by Theorem 1.6 and Lemma 1.4, $\Box_{n-i+2}\breve{g}_{n-j+1} \in C_{\mathrm{o}}^{-\infty}(S^{n-1})$ and that if $f \in C_{\mathrm{o}}(S^{n-1})$, then also

$$f * \Box_{n-i+2}\breve{g}_{n-j+1} = \Box_{n-i+2} f * \breve{g}_{n-j+1} \in C_{\mathrm{o}}^{-\infty}(S^{n-1}). \tag{3.8}$$

However, in general (3.8) does *not* define a continuous function on S^{n-1} if f is merely continuous.

Next, we note that using Theorem 2.9 we can also give a new and short proof of Theorem 3.2: If $\Phi_j \in \mathbf{MVal}_j^{\infty}$, $2 \leq j \leq n-1$, has generating function $f \in C_{\mathrm{o}}^{\infty}(S^{n-1})$ and $1 \leq i \leq j+1$, then, by (1.16) and Theorem 2.10 (a),

$$\begin{aligned}W_{n-i}(K,\Phi_j L) &= \int_{S^{n-1}} (S_{i-1}(K,\cdot) * f)(u)\, dS_j(L,u) \\ &= \frac{(i-1)!}{j!}\, W_{n-j-1}(L,(\Lambda^{j+1-i}\Phi_j)(K))\end{aligned}$$

for every $K, L \in \mathcal{K}^n$.

Putting together Theorem 3.2 and Theorem 3.3, we obtain the following.

Corollary 3.4. *For $1 \le i \le n$, $2 \le j \le n-1$, and $\Phi_j \in \mathbf{MVal}^{\infty}_{j,i-1}$, we have*

$$W_{n-i}(K,\Phi_j L) = \frac{(i-1)!}{j!}\, W_{n-1-j}(L,(\Lambda^{j+1-i}\Phi_j)(K)) \tag{3.9}$$

for every $K, L \in \mathcal{K}^n$.

Using Theorem 3.3, we establish the following result.

Theorem 3.5. *Let $1 \le i \le n$ and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \le j \le n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ have non-empty interiors, then for all $\lambda \in (0,1)$,*

$$W_{n-i}(\Phi_j((1-\lambda)K + \lambda L)) \ge W_{n-i}(\Phi_j K)^{1-\lambda} W_{n-i}(\Phi_j L)^{\lambda}, \tag{3.10}$$

with equality if and only if K and L are translates of each other.

Since $\mathbf{MVal}_{j,i-1} = \mathbf{MVal}_j$ for $i \le j+1$, Theorem 3.5 includes both Lutwak's inequalities (3.4) and Theorem 3.1 as special cases. Also note that the smoothness assumption for the bodies K and L in the equality conditions of (3.5) is no longer required. This follows from new monotonicity properties of the Minkowski valuations in $\mathbf{MVal}_{j,i-1}$ (confer Lemma 3.7).

We also show that our proof of Theorem 3.5 can be modified to yield an even stronger result. More precisely, we show that (3.10) not only holds for the usual Minkowski addition but, in fact, for all commutative Orlicz Minkowski additions introduced by Gardner, Hug, and Weil [Gar14]. In particular, this includes all the L_p Minkowski additions. We first state the result.

Theorem 3.6. *Let $\varphi \in \Theta_1$, $1 \le i \le n$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \le j \le n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ contain the origin, then for all $\lambda \in (0,1)$,*

$$W_{n-i}(\Phi_j(K +_{\varphi,\lambda} L)) \ge W_{n-i}(\Phi_j K)^{1-\lambda} W_{n-i}(\Phi_j L)^{\lambda}. \tag{3.11}$$

When φ is strictly convex and K and L have non-empty interiors, equality holds if and only if $K = L$.

For the proof of Theorem 3.6 and in order to establish the equality cases in Theorem 3.5, we need the following monotonicity property of Minkowski valuations:

Lemma 3.7. *Suppose that $1 \le i \le n$, $2 \le j \le n-1$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$ be non-trivial. If $K,L \in \mathcal{K}^n$ have non-empty interiors, then $K \subseteq L$ implies that*

$$W_{n-i}(\Phi_j K) \le W_{n-i}(\Phi_j L), \tag{3.12}$$

with equality if and only if $K = L$. In particular, $W_{n-i}(\Phi_j K) > 0$ for every $K \in \mathcal{K}^n$ with non-empty interior.

Proof. We first assume that $i \ge 2$, that Φ_j is smooth and that K and L are of class C^2_+. In this case, it was proved in [Par12, p. 992] that $\Phi_j K$ and $\Phi_j L$ also have non-empty interiors.

Moreover, by (3.9) and the monotonicity of mixed volumes, we have for every $Q \in \mathcal{K}^n$,

$$\begin{aligned} W_{n-i}(Q,\Phi_j L) &= \frac{(i-1)!}{j!} W_{n-1-j}(L,(\Lambda^{j+1-i}\Phi_j)(Q)) \\ &\geq \frac{(i-1)!}{j!} W_{n-1-j}(K,(\Lambda^{j+1-i}\Phi_j)(Q)) = W_{n-i}(Q,\Phi_j K). \end{aligned}$$

Thus, taking $Q = \Phi_j L$ and using inequality (1.8), yields

$$W_{n-i}(\Phi_j L)^i \geq W_{n-i}(\Phi_j L,\Phi_j K)^i \geq W_{n-i}(\Phi_j L)^{i-1} W_{n-i}(\Phi_j K)$$

which implies (3.12) since $W_{n-i}(\Phi_j L) > 0$. If $\Phi_j \in \mathbf{MVal}_{j,i-1}$ is not smooth and K and L are arbitrary, (3.12) follows by approximation.

In order to establish the equality conditions first note that, by the $\mathrm{SO}(n)$-equivariance of Φ_j, the convex body $\Phi_j B$ must be an o-symmetric ball. Moreover, from Lemma 2.7 (b), it follows that $\Phi_j B = r_{\Phi_j} B$, where $r_{\Phi_j} > 0$. Thus, since K and L have non-empty interiors, we conclude from (3.12) that $W_{n-i}(\Phi_j K), W_{n-i}(\Phi_j L) > 0$ or, equivalently, that $\Phi_j K$ and $\Phi_j L$ have dimension at least i holds for all $K, L \in \mathcal{K}^n$ with non-empty interiors.

Assume now that equality holds in (3.12). Then, by the equality conditions of (1.8) and Lemma 2.7 (a), there exists an $\alpha > 0$ such that $\Phi_j K = \alpha\,\Phi_j L$. It follows from equality in (3.12) that $\alpha = 1$. Thus, by Lemma 2.7 (b), we have

$$W_{n-j}(K) = r_{\Phi_j}^{-1} W_{n-1}(\Phi_j K) = r_{\Phi_j}^{-1} W_{n-1}(\Phi_j L) = W_{n-j}(L). \tag{3.13}$$

Using again the monotonicity of mixed volumes and (1.8), we obtain

$$W_{n-j}(L)^j = W_{n-j}(L,L)^j \geq W_{n-j}(L,K)^j \geq W_{n-j}(L)^{j-1} W_{n-j}(K).$$

From (3.13) and the equality conditions of inequality (1.8), we conclude that K is a translate of L. But since $K \subseteq L$, we must have $K = L$.

Inequality (3.12) for $i = 1$ follows directly from Lemma 2.7 (b) and the monotonicity of quermassintegrals. If equality holds in (3.12) for $i = 1$, then we have (3.13) and therefore, as before, obtain that $K = L$. □

In contrast to Lemma 3.7, we note that not every Minkowski valuation $\Phi_j \in \mathbf{MVal}_{j,i-1}$ is monotone with respect to set inclusion (cf. [Kid06]). However, all known examples of Minkowski valuations $\Phi_j \in \mathbf{MVal}_j$, $1 \leq j \leq n-1$ are *weakly monotone* , that is, for every pair of convex bodies $K, L \in \mathcal{K}^n$ such that $K \subseteq L$, there exists a vector $x(K,L) \in \mathbb{R}^n$ such that

$$\Phi_j K \subseteq \Phi_j L + x(K,L).$$

It is an open problem whether all translation-invariant and $\mathrm{SO}(n)$-equivariant Minkowski valuations are weakly monotone. Using arguments as in the proof of Lemma 3.7, we can show the following.

Proposition 3.8. *Suppose that $2 \leq j \leq n-1$. If $\Phi_j \in \mathbf{MVal}_{j,n-1}$, then Φ_j is weakly monotone.*

Proof. Without loss of generality we may assume that Φ_j is smooth. If $K, L \in \mathcal{K}^n$ such that $K \subseteq L$, then, as in Lemma 3.7, it follows from (3.9) and the monotonicity of mixed volumes that for every $Q \in \mathcal{K}^n$,

$$\begin{aligned} W_0(Q,\Phi_j L) &= \frac{(n-1)!}{j!} W_{n-1-j}(L,(\Lambda^{j+1-n}\Phi_j)(Q)) \\ &\geq \frac{(n-1)!}{j!} W_{n-1-j}(K,(\Lambda^{j+1-n}\Phi_j)(Q)) = W_0(Q,\Phi_j K). \end{aligned}$$

But it is well known (cf. [Sch07, Corollary 4.3]) that $W_0(Q,\Phi_j K) \leq W_0(Q,\Phi_j L)$ for every $Q \in \mathcal{K}^n$ implies $\Phi_j K \subseteq \Phi_j L + x$ for some $x \in \mathbb{R}^n$. $\square$

We return now to the proof of Theorem 3.5.

Proof of Theorem 3.5. First we assume that $i \geq 2$ and that Φ_j is smooth. We also use the abbreviations $K_\lambda = (1-\lambda)K + \lambda L$ and $Q = \Phi_j K_\lambda$. Then, by (3.9),

$$W_{n-i}(\Phi_j K_\lambda) = W_{n-i}(Q,\Phi_j K_\lambda) = \frac{(i-1)!}{j!} W_{n-1-j}(K_\lambda,(\Lambda^{j+1-i}\Phi_j)(Q)).$$

From an application of inequality (3.3), we therefore obtain

$$W_{n-i}(\Phi_j K_\lambda) \geq \frac{(i-1)!}{j!} W_{n\text{-}1\text{-}j}(K,(\Lambda^{j+1-i}\Phi_j)(Q))^{1-\lambda} W_{n\text{-}1\text{-}j}(L,(\Lambda^{j+1-i}\Phi_j)(Q))^{\lambda}.$$

Thus, using (3.9) again, we obtain

$$W_{n-i}(\Phi_j K_\lambda)^i \geq W_{n-i}(Q,\Phi_j K)^{i(1-\lambda)} W_{n-i}(Q,\Phi_j L)^{i\lambda}.$$

Now, if $\Phi_j \in \mathbf{MVal}_{j,i-1}$ is not smooth, then this inequality still follows by approximation. Hence, using (1.8) and the fact that, by Lemma 3.7, $W_{n-i}(Q) > 0$, we arrive at

$$W_{n-i}(\Phi_j K_\lambda)^i \geq W_{n-i}(Q)^{i-1} W_{n-i}(\Phi_j K)^{1-\lambda} W_{n-i}(\Phi_j L)^{\lambda},$$

which, by the definitions of $\Phi_j K_\lambda$ and Q, is the desired inequality (3.10).

In order to establish the equality conditions, first note that by Lemma 3.7, $\Phi_j K$, $\Phi_j L$ and $\Phi_j K_\lambda$ all have dimension at least i. Therefore, the equality conditions of inequality (1.8) imply that $\Phi_j K$ is homothetic to $\Phi_j K_\lambda$, which is in turn homothetic to $\Phi_j L$. In fact, by Lemma 2.7 (a), they have to be dilates of one another, that is, there exist $t_1, t_2 > 0$ such that

$$t_1 \Phi_j K = \Phi_j K_\lambda = t_2 \Phi_j L,$$

where $1 = t_1^{1-\lambda} t_2^{\lambda}$, by the equality in (3.10). Moreover, an application of Lemma 2.7 (b) yields $t_1 W_{n-j}(K) = W_{n-j}(K_\lambda) = t_2 W_{n-j}(L)$. Consequently, we have

$$W_{n-j}(K_\lambda) = W_{n-j}(K)^{1-\lambda} W_{n-j}(L)^{\lambda}.$$

By the equality conditions of inequality (3.2), this is possible only if K and L are translates.

This completes the proof for $i \geq 2$. If $i = 1$, then the statement is an immediate consequence of Lemma 2.7 (b) and (3.2). $\square$

It remains to complete the proof of Theorem 3.6.

Proof of Theorem 3.6. First note that inequality (3.11) follows from Lemma 1.9, Lemma 3.7, and Theorem 3.5. In order to establish the equality conditions, let φ be strictly convex and let K and L have non-empty interiors. It follows from the equality conditions of Lemma 3.7 that

$$K +_{\varphi,\lambda} L = (1-\lambda)K + \lambda L. \tag{3.14}$$

We want to show that this is possible only if $K = L$ or, equivalently, if $h(K,u) = h(L,u)$ for all $u \in S^{n-1}$. If $h(K,u) = h(L,u) = 0$, then there is nothing to prove. Therefore, we may assume that $h(K +_{\varphi,\lambda} L,u) > 0$. Now from the definition of the Orlicz convex combination and (3.14), together with the convexity of φ and our assumption that $\varphi(1) = 1$, we obtain

$$\varphi\left(\frac{(1-\lambda)h(K,u) + \lambda h(L,u)}{h(K +_{\varphi,\lambda} L,u)}\right) = 1.$$

But since we have assumed that φ is strictly convex, this implies that $h(K,u) = h(L,u)$. $\square$

Like the classical inequality (3.2), Theorem 3.5 as well as Theorem 3.6 in case of a homogeneous addition are equivalent to corresponding additive versions. Here we state one such additive version for L_p Minkowski addition.

Corollary 3.9. *Let $p > 1$, $1 \leq i \leq n$, and let $\Phi_j \in \mathbf{MVal}_{j,i-1}$, $2 \leq j \leq n-1$, be non-trivial. If $K, L \in \mathcal{K}^n$ contain the origin in their interiors, then*

$$V_i(\Phi_j((1-\lambda)\cdot K +_p \lambda \cdot L))^{\frac{p}{ij}} \geq (1-\lambda)V_i(\Phi_j K)^{\frac{p}{ij}} + \lambda V_i(\Phi_j L)^{\frac{p}{ij}},$$

with equality if and only if K and L are dilates of each other.

We finally remark that the special case $j = n-1$ of Corollary 3.9 was recently obtained by Wang [Wan13].

CHAPTER 4

Lutwak–Petty projection inequalities for Minkowski valuations and their duals

4.1 Petty projection inequality and generalizations

The *polar Petty projection inequality* [Pet71] states that for the operator $\mathit{\Pi} = \mathit{\Pi}_{n-1}$, the inequality

$$V(K)^{n-1}V(\mathit{\Pi}^*K) \leq V(B)^{n-1}V(\mathit{\Pi}^*B)$$

holds for all $K \in \mathcal{K}_n^n$, with equality if and only if K is an ellipsoid. This now classical result is considerably stronger than the Euclidean isoperimetric inequality and still has significant impact on current research. Although the projection body map was already introduced by Minkowski at the turn of the previous century, its fundamental role in convex geometry only became apparent through the work of Petty [Pet61; Pet67], Schneider [Sch67] and Bolker [Bol69]. Various generalizations of the polar Petty projection inequality have been obtained in recent years (see [Bör13; Hab09b; Lut00b; Lut10a] for extensions to the L_p and the Orlicz–Brunn–Minkowski theories and [Wan12] for the extension to sets of finite perimeter). In [Lut85], Lutwak established a version of this inequality for projection bodies of all orders, the Lutwak–Petty projection inequalities. His result states that for $K \in \mathcal{K}_n^n$ and $0 < i < n-1$,

$$V(K)^iV(\mathit{\Pi}_i^*K) \leq V(B)^iV(\mathit{\Pi}_i^*B), \tag{4.1}$$

with equality if and only if K is a ball. The inequalities (4.1) strengthen the classical inequalities between volume and quermassintegrals, since

$$\kappa_n^{n-i}V(K)^i \leq \kappa_{n-1}^{-n}\kappa_n^{n+1}V^{-1}(\mathit{\Pi}_i^*K) \leq W_{n-i}^n(K).$$

A generalization of the polar Petty projection inequality to Minkowski valuations has recently been established by Haberl and Schuster [Hab] for the operators $\mathit{\Phi}_Z$, whose definition we recalled in Section 2.3. These Minkowski valuations are homogeneous of degree $n-1$ and form an infinite-dimensional cone containing the projection body, which is obtained by just taking $Z = [-\bar{e},\bar{e}]$. Using a new integral representation of these valuations, Haberl and Schuster established a family of inequalities each of which refines again the relation between volume and surface area expressed by the Euclidean isoperimetric inequality. Their generalization can be stated as follows:

Theorem 4.1 [Hab] *Let $K \in \mathcal{K}^n_n$. Then*

$$V(K)^{n-1}V(\Phi_Z^* K) \leq V(B)^{n-1}V(\Phi_Z^* B), \tag{4.2}$$

with equality

- *for $Z = [-\bar{e},\bar{e}]$ if and only if K is an ellipsoid and*
- *for $Z \neq [-\bar{e},\bar{e}]$ if and only if K is a ball.*

Since the article [Hab] is still in preparation, in order for this thesis to be self-contained, we will state the proof of Theorem 4.1 in dimensions $n \geq 3$, taken from an early version of [Hab].

Proof of Theorem 4.1. In order to prove the inequality, we use (2.17) to write

$$\begin{aligned} V(\Phi_Z^* K) &= \frac{1}{n}\int_{S^{n-1}} h(\Phi_Z K, u)^{-n} du \\ &= \frac{1}{n}\int_{S^{n-1}} \left(\int_{\mathrm{SO}(n)} h(\Pi K, \phi u) d\breve{\mu}_u(\phi)\right)^{-n} du. \end{aligned}$$

Since $\mu_u(\mathrm{SO}(n)) = 1$, we can use Jensen's inequality to get

$$V(\Phi_Z^* K) \leq \frac{1}{n}\int_{S^{n-1}}\int_{\mathrm{SO}(n)} h(\Pi K, \phi u)^{-n} d\breve{\mu}_u(\phi) du. \tag{4.3}$$

We denote by σ the right-invariant Haar measure on $\mathrm{SO}(n)$. Since Φ_Z is $\mathrm{SO}(n)$-equivariant, we obtain

$$\begin{aligned} V(\Phi_Z^* K) &= \int_{\mathrm{SO}(n)} V(\Phi_Z^*(\vartheta^{-1}K)) d\sigma(\vartheta) \\ &\leq \frac{1}{n}\int_{\mathrm{SO}(n)}\int_{S^{n-1}}\int_{\mathrm{SO}(n)} h(\Pi K, \vartheta\phi u)^{-n} d\breve{\mu}_u(\phi) du d\sigma(\vartheta) \\ &= \frac{1}{n}\int_{S^{n-1}}\int_{\mathrm{SO}(n)}\int_{\mathrm{SO}(n)} h(\Pi K, \vartheta\phi u)^{-n} d\sigma(\vartheta) d\breve{\mu}_u(\phi) du. \end{aligned}$$

By the right-invariance of σ and again using $\mu_u(\mathrm{SO}(n)) = 1$, it follows that

$$\begin{aligned} &\frac{1}{n}\int_{S^{n-1}}\int_{\mathrm{SO}(n)}\int_{\mathrm{SO}(n)} h(\Pi K, \vartheta\phi u)^{-n} d\sigma(\vartheta) d\breve{\mu}_u(\phi) du \\ &\quad = \frac{1}{n}\int_{S^{n-1}}\int_{\mathrm{SO}(n)}\int_{\mathrm{SO}(n)} h(\Pi K, \vartheta u)^{-n} d\sigma(\vartheta) d\breve{\mu}_u(\phi) du \\ &\quad = \frac{1}{n}\int_{S^{n-1}}\int_{\mathrm{SO}(n)} h(\Pi K, \vartheta u)^{-n} d\sigma(\vartheta) du. \end{aligned}$$

Rewriting as before then immediately gives

$$\begin{aligned}V(\Phi_Z^* K) &\leq \frac{1}{n} \int_{\mathrm{SO}(n)} \int_{S^{n-1}} h(\Pi \vartheta^{-1} K, u)^{-n} du d\sigma(\vartheta) \\ &= \int_{\mathrm{SO}(n)} V(\Pi^* \vartheta^{-1} K) d\sigma(\vartheta) \\ &= V(\Pi^* K).\end{aligned}$$

Combining this with the Petty projection inequality, we arrive at (4.2).

To get the equality conditions, we assume that $V(\Phi_Z^* K) = V(\Pi^* K)$, with $\Phi_Z \neq \Pi$. We first note that by the definition of $\breve{\mu}_u$,

$$\int_{\mathrm{SO}(n)} h(\Pi K, \phi u) d\breve{\mu}_u(\phi) = \int_{\mathrm{SO}(n)} h(\Pi K, \vartheta_u \phi \vartheta_u^{-1} u) d\breve{\mu}(\phi) = \int_{S^{n-1}} h(\Pi K, \vartheta_u v) d\mu(v).$$

Equality in (4.3) therefore holds if and only if

$$\forall u \in S^{n-1}\ \forall \vartheta \in \mathrm{SO}(n) \text{ with } \vartheta \bar{e} = u \colon h(\Pi K, \vartheta_u v) = c_{\vartheta_u}$$

for μ-almost every $v \in S^{n-1}$. This means that equality in (4.3) holds if and only if

$$\forall \vartheta \in \mathrm{SO}(n) \colon h(\Pi K, \vartheta v) = c_\vartheta \tag{4.4}$$

for μ-almost every $v \in S^{n-1}$. We want to show that this holds exactly in the case when $h(\Pi K, \cdot)$ is constant everywhere on S^{n-1}. We first give an overview of the main steps of the proof. We are going to show that there is a circle on S^{n-1} such that this condition holds on the whole circle, i.e.,

$$\exists t \in (-1,1)\ \forall \vartheta \in \mathrm{SO}(n) \colon h(\Pi K, \vartheta v) = c_\vartheta \quad \forall v \in S^{n-1} \cap H_{\bar{e},t}. \tag{4.5}$$

Assume that this has already been proven, then in particular, $h(\Pi K, v) = c_{id}$ for $v \in H_{\bar{e},t} \cap S^{n-1}$. For a rotation $\vartheta' \in S^{n-1}$, on the rotated circle $\vartheta'(H_{\bar{e},t} \cap S^{n-1}) = H_{\vartheta' \bar{e},t} \cap S^{n-1}$, the support function satisfies

$$h(\Pi K, u) = c_{\vartheta'}, \quad u \in H_{\vartheta' \bar{e},t} \cap S^{n-1}.$$

Thus, if we choose the rotation $\vartheta' \in \mathrm{SO}(n)$ such that these circles intersect, we therefore get $c_{id} = c_{\vartheta'}$. It is possible to reach any point on S^{n-1} with a finite number of rotations, thus $h(\Pi K, \cdot)$ is constant everywhere on S^{n-1}.

Since $V(\Phi_Z^* K) = V(\Pi^* K)$, equality in Theorem 4.1 can only hold if equality in the Petty projection inequality holds and thus K is an ellipsoid. Since the only ellipsoids where $h(\Pi K, \cdot)$ is constant are balls, we conclude that in the case $\Phi_Z \neq \Pi$, K has to be a ball.

It remains to show (4.5). We first show that

$$\exists t \in (-1,1)\ \forall \varepsilon > 0\ \forall x \in H_{\bar{e},t} \cap S^{n-1} \colon \bar{\mu}(B_\varepsilon(x) \cap S^{n-1}) > 0. \tag{4.6}$$

Assume that (4.6) does not hold, i.e., that for all $t \in (-1,1)$ there is an $\varepsilon_t > 0$ and $x_t \in H_{\bar{e},t} \cap S^{n-1}$ such that $\bar{\mu}(B_\varepsilon(x_t) \cap S^{n-1}) = 0$. Since $\bar{\mu}$ is $\mathrm{SO}(n-1)$-invariant, it follows that $\bar{\mu}(B_\varepsilon(x) \cap S^{n-1}) = 0$ for all $x \in H_{\bar{e},t} \cap S^{n-1}$. From this, we conclude that for each $t \in (-1,1)$ there is an $\alpha(t) > 0$ such that for the whole closed strip $S_{\alpha(t),t} = H^+_{\bar{e},t-a} \cap H^-_{\bar{e},t+a}$ we get that $\bar{\mu}(S_{\alpha(t),t} \cap S^{n-1}) = 0$ and therefore that $\bar{\mu}(\mathrm{int}(S_{\alpha(t),t}) \cap S^{n-1}) = 0$. For $\beta \in (-1,1)$, we conclude that $\bar{\mu}(S_{\beta,0} \cap S^{n-1}) = 0$ (since $S_{\beta,0}$ is contained in a countable union of open strips where, by compactness, a finite subcovering can be chosen). By writing

$$S^{n-1} \setminus \{\pm\bar{e}\} = \bigcup_{k \in \mathbb{N}} S_{1-1/k,0} \cap S^{n-1}$$

and since $S_{1-1/k,0} \uparrow S^{n-1} \setminus \{\pm\bar{e}\}$ and $\bar{\mu}$ is continuous from below, we conclude that

$$\bar{\mu}(S^{n-1} \setminus \{\pm\bar{e}\}) = 0.$$

This would imply

$$\bar{\mu} = c'(\delta_{\bar{e}} + \delta_{-\bar{e}}) \tag{4.7}$$

for a $c' > 0$, which is a contradiction to $\Phi_Z \neq \Pi$. By (4.4), we know that there is a set of measure zero $A_\vartheta \subseteq S^{n-1}$ such that

$$h(\Pi K, \vartheta v) = c_\vartheta, \quad \forall v \in A^c_\vartheta.$$

For $\vartheta \in \mathrm{SO}(n)$ and $x \in S^{n-1} \cap H_{\bar{e},t}$, this implies that for every $\varepsilon > 0$, $B_\varepsilon(x) \cap A^c_\vartheta \neq \emptyset$ (since otherwise there would be an $\varepsilon > 0$ such that $\bar{\mu}(B_\varepsilon(x_t) \cap S^{n-1}) = 0$). Thus, there is a sequence (x_k), $x_k \in A^c_\vartheta$ with $x_k \to x$ and therefore $h(\Pi K, \vartheta x) = c_\vartheta$ for $x \in H_{\bar{e},t} \cap S^{n-1}$. This finishes the proof of (4.5) and therefore the proof of the equality conditions of Theorem 4.1. □

In 1985, Lutwak [Lut85] showed that the polar Petty projection inequality can be used to obtain similar volume inequalities for polar projection bodies of all orders which strengthen the classical inequalities comparing the volume and the intrinsic volumes of a convex body. Even more general, he proved that an analog of the polar Petty projection inequality holds for mixed projection bodies. These operators stem from a polarization of the projection body map of order $n-1$ and were first introduced by Süss [Süs29] and later thoroughly studied by Lutwak [Lut85; Lut86b; Lut90; Lut93a]. Although such polarizations do not exist in general for Minkowski valuations, as was shown in [Par13], Schuster [Sch06] proved the existence for $(n-1)$-homogeneous $\mathrm{SO}(n)$-equivariant translation-invariant Minkowski valuations. In particular, for Φ_Z this polarization exists (confer Section 2.3 for the definition). For $Z = [-\bar{e},\bar{e}]$, this reduces to the classical mixed projection body. Our first result is a generalization of Lutwak's polar projection inequality to the new mixed operators Φ_Z.

Theorem 4.2 (Generalized polar projection inequality). *If $K_1, \dots, K_{n-1}$ are convex bodies in $\mathcal{K}^n_n$, then*

$$V(K_1)\cdots V(K_{n-1})V(\Phi^*_Z(K_1,\dots,K_{n-1})) \le V(\Phi^*_Z B)V(B)^{n-1}, \tag{4.8}$$

with equality if and only if the K_i are

- *homothetic ellipsoids for $Z = [-\bar{e},\bar{e}]$,*
- *homothetic balls for $Z \neq [-\bar{e},\bar{e}]$.*

When $\Phi_Z = \Pi$, Theorem 4.2 reduces to Lutwak's polar projection inequality [Lut85]. The proof of Theorem 4.2 relies on a connection between Theorem 4.1 and a generalization of the Busemann–Petty centroid inequality (discovered for Π by Lutwak [Lut86a]).

The Busemann–Petty centroid inequality was first conjectured by Blaschke and later proven by Petty [Pet61], who deduced it from a reformulation of the Busemann random simplex inequality [Bus53]. In [Lut85], Lutwak showed that the Busemann–Petty centroid inequality can be used to derive a version of the Petty projection inequality for mixed projection bodies. The approach for proving our new results makes use of Lutwak's techniques for generalized centroid bodies Γ_Z (see Section 2.3 for the definition), which were introduced in [Sch06]. We establish an analog of inequality (2.19) for these operators:

Theorem 4.3 (Generalized Busemann-Petty centroid inequality). *Let $K \in \mathcal{K}^n_{(o)}$. Then*

$$V(K)V(\Gamma_Z K)^{-1} \le V(B)V(\Gamma_Z B)^{-1}, \tag{4.9}$$

with equality

- *for $Z = [-\bar{e},\bar{e}]$ if and only if K is an ellipsoid centered at the origin and*
- *for $Z \neq [-\bar{e},\bar{e}]$ if and only if K is a ball centered at the origin.*

We also show that the Haberl–Schuster inequality from Theorem 4.1 is related to this new inequality in the same way that the Petty projection inequality relates to the Busemann–Petty centroid inequality [Lut85] and use Theorem 4.3 to prove the generalized polar projection inequality from Theorem 4.8.

We begin with the proof of Theorem 4.3. To this end, we use an observation of Lutwak, who remarked in [Lut85] that a corollary to the polar centroid inequality combined with the polar Petty projection inequality can be used to obtain the Busemann–Petty centroid inequality. Our first aim therefore is to establish a generalization of the polar centroid inequality for the generalized centroid body. In the following, $Z = Z(v)$ will always be an o-symmetric zonoid of revolution with axis of revolution v.

Theorem 4.4 (Generalized polar centroid inequality with equality cases). *For $K_n \in \mathcal{K}^n_{(o)}$ and $K_1, \dots, K_{n-1} \in \mathcal{K}^n$,*

$$V(K_n) \le \left(\frac{n+1}{2}\right)^n V^n(K_1,\dots,K_{n-1},\Gamma_Z K_n)V(\Phi^*_Z(K_1,\dots,K_{n-1})), \tag{4.10}$$

with equality if and only if K_n is a dilation of $\Phi^*_Z(K_1,\dots,K_{n-1})$.

Proof. We will need the representation

$$V(K_1,\ldots,K_{n-1},\mathit{\Gamma}_Z K_n)=\frac{1}{n}\int_{S^{n-1}} h(\mathit{\Gamma}_Z K_n,v)dS(K_1,\ldots,K_{n-1},v).$$

Using this, we get by Fubini's theorem

$$\begin{aligned}(n+1)&V(K_n)V(K_1,\ldots,K_{n-1},\mathit{\Gamma}_Z K_n)\\ &=\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}} h(Z(v),u)\rho(K_n,u)^{n+1}dS(u)dS(K_1,\ldots,K_{n-1},v)\\ &=\frac{1}{n}\int_{S^{n-1}}\int_{S^{n-1}} h(Z(v),u)dS(K_1,\ldots,K_{n-1},v)\rho(K_n,u)^{n+1}dS(u)\\ &=\frac{2}{n}\int_{S^{n-1}} h(\mathit{\Phi}_k(K_1,\ldots,K_{n-1}),u)\rho(K_n,u)^{n+1}dS(u).\end{aligned}$$

Applying Hölder's inequality with $p=(n+1)/n$, $q=n+1$ and

$$\begin{aligned}f(u)&=\rho(K_n,u)^n h(\mathit{\Phi}_Z(K_1,\ldots,K_{n-1}),u)^{\frac{n}{n+1}},\\ g(u)&=h(\mathit{\Phi}_Z(K_1,\ldots,K_{n-1}),u)^{-\frac{n}{n+1}}\end{aligned}$$

gives

$$\begin{aligned}\left(\int_{S^{n-1}}\rho(K_n,u)^n dS(u)\right)^{n+1}\leq\left(\int_{S^{n-1}}\rho(K_n,u)^{n+1}h(\mathit{\Phi}_Z(K_1,\ldots,K_{n-1}),u)dS(u)\right)^n\times\\ \int_{S^{n-1}} h(\mathit{\Phi}_Z(K_1,\ldots,K_{n-1}),u)^{-n}dS(u).\end{aligned}$$

Simplifying, we arrive at

$$\left(nV(K_n)\right)^{n+1}\leq\left(n\frac{(n+1)}{2}V(K_n)V(K_1,\ldots,K_{n-1},\mathit{\Gamma}_Z K_n)\right)^n nV(\mathit{\Phi}_Z^*(K_1,\ldots,K_{n-1}))$$

and thus at

$$V(K_n)\leq\left(\frac{n+1}{2}\right)^n V^n(K_1,\ldots,K_{n-1},\mathit{\Gamma}_Z K_n)V(\mathit{\Phi}_Z^*(K_1,\ldots,K_{n-1})).$$

For the equality conditions, note that equality in Hölder's inequality holds if and only if

$$f(u)=c\frac{g^q(u)}{g(u)},$$

which for the functions defined above is just

$$\rho(K_n,u)^n h(\mathit{\Phi}_Z(K_1,\ldots,K_{n-1}),u)^n=c.$$

This holds if and only if K_n is a dilation of $\mathit{\Phi}_Z^*(K_1,\ldots,K_{n-1})$. $\square$

An easy corollary to the equality cases in Theorem 4.4 which will be needed later is the following:

Corollary 4.5. *If $K_1,\ldots,K_{n-1}$ are convex bodies in $\mathcal{K}_n^n$, then*

$$V(K_1,\ldots,K_{n-1},\Gamma_Z\Phi_Z^*(K_1,\ldots,K_{n-1})) = \frac{2}{(n+1)}. \tag{4.11}$$

Proof. Set $K_n = \Phi_Z^*(K_1,\ldots,K_{n-1})$ in inequality (4.4). □

In the following, we will be mainly interested in the special case of Theorem 4.4 for $K_1,\ldots,K_{n-1} = \Gamma_Z K$ and $K_n = K$, where K contains the origin in the interior:

$$V(K) \leq \left(\frac{n+1}{2}\right)^n V^n(\Gamma_Z K)V(\Phi_Z^*\Gamma_Z K). \tag{4.12}$$

We are now able to prove the Busemann–Petty centroid inequality for the operator Γ_Z.

Proof of Theorem 4.3. We combine inequality (4.12) and Theorem 4.1 applied to Γ_Z to obtain

$$V(K) \leq \left(\frac{n+1}{2}\right)^n V(B)^{n-1}V(\Phi_Z^*B)V(\Gamma_Z K). \tag{4.13}$$

Since $\Phi_Z B$ is a ball, one easily sees that the constants in (4.9) and (4.13) are the same.

The equality conditions are derived by considering that equality in inequality (4.2) applied to the convex body Γ_Z holds for $Z \neq [-\bar{e},\bar{e}]$ if and only if $\Gamma_Z K$ is a ball. Since by the equality conditions of (4.10) $K = c\Phi_Z^*(\Gamma_Z K)$, this is only possible if K is a ball. □

As an easy corollary of the generalized Busemann–Petty centroid inequality, we note a version of the general centroid inequality for the generalized centroid body. It can directly be obtained by using (1.6) with equality conditions.

Corollary 4.6 (Generalized general centroid inequality)**.** *Let $K_1,\ldots,K_{n-1} \in \mathcal{K}_n^n$ and $K_n \in \mathcal{K}_{(o)}^n$. Then*

$$V(K_1)\cdots V(K_n) \leq \left(\frac{n+1}{2}\right)^n V(\Phi_Z^*B)V(B)^{n-1}V^n(K_1,\ldots,K_{n-1},\Gamma_Z K_n), \tag{4.14}$$

with equality if and only if each K_i is homothetic to K_n and additionally

- *for $Z = [-\bar{e},\bar{e}]$, K_n is an ellipsoid centered at the origin and*
- *for $Z \neq [-\bar{e},\bar{e}]$, K_n is a ball centered at the origin.*

The proof of Theorem 4.2 now follows easily:

Proof of 4.2. Combine (4.14) and (4.11) and consider the equality conditions of the polar projection inequalities for Π^* (confer [Lut85]). □

The special case of Theorem 4.2 for $K_1, \dots, K_{n-1} = K$ is of course the result of Haberl and Schuster. The special case for $K_1, \dots, K_i = K$ and $K_{i+1}, \dots, K_{n-1} = B$ can be written as

$$V(K)^i V(\Phi^*_{Z,i}K) \leq V(B)^i V(\Phi^*_{Z,i}B) \tag{4.15}$$

and is a direct extension of the Lutwak–Petty projection inequalities. If we normalize Z such that $\Phi_Z B = \Pi B$, then, by Theorem 4.4, this implies the following inequality between volume and W_{n-i}:

$$\kappa_n^{n-i} V(K)^i \leq \kappa_{n-1}^{-n} \kappa_n^{n+1} V^{-1}(\Phi^*_{Z,i}K) \leq W^n_{n-i}(K).$$

For $Z = B$, this is the classical inequality between volume and quermassintegral, and for $Z = [-\bar{e},\bar{e}]$, this is the Lutwak-Petty projection inequality for the projection body of order i. We remark that the special case (4.15) could also be obtained quite easily by using the Haberl–Schuster inequality and mixed bodies, since

$$\Phi_{Z,i}K = \Phi_Z \left[K\right]_i .$$

We use Corollary 1.1 to obtain

$$\begin{aligned} V(K)^i V(\Phi^*_{Z,i}K) &\leq V([K]_i)^{n-1} V(\Phi^*_Z \left[K\right]_i) \frac{1}{V(B)^{n-i-1}} \\ &\leq V(B)^i V(\Phi^*_{Z,i}B). \end{aligned} \tag{4.16}$$

There is more to be gained by reviewing inequalities (4.15) in yet another light. Haberl and Schuster proved in [Hab] that the Petty projection inequality is the strongest among their family of inequalities. More precisely, if $\Phi_Z B = \Pi B$, then

$$V(K)^{n-1} V(\Phi^*_Z K) \leq V(K)^{n-1} V(\Pi^* K) \leq V(B)^{n-1} V(\Pi^* B). \tag{4.17}$$

The significance of this observation lies in the fact that the whole family of Euclidean inequalities from Theorem 4.1 is dominated by the only one which is affine in nature. This follows from a result of Ludwig [Lud05], who first characterized the projection body as the only continuous translation-invariant $SL(n)$-contravariant Minkowski valuation. We will give another proof of the inequalities (4.16) using techniques of Haberl and Schuster to identify Lutwak's results as the strongest members of our family. Although these are not affine, we will show that the volume of the polar projection body of order i is dominated by a corresponding affine quermassintegral, which is an affine invariant. To this end we prove the following theorem:

Theorem 4.7. *Let $K \in \mathcal{K}^n_n$ and $Z(v)$ as above, normalized such that $\Phi_Z B = \Pi B$. Then, for $1 \leq i \leq n-1$,*

$$V(\Phi^*_{Z,i}K) \leq V(\Pi^*_i K) \leq \frac{\kappa_n^{n+1}}{\kappa_{n-1}^n} A_{n-i}(K)^{-n}. \tag{4.18}$$

Proof. In order to prove the inequality on the left-hand side, we use (2.18) to write

$$\begin{aligned} V(\Phi_{Z,i}^* K) &= \frac{1}{n} \int_{S^{n-1}} h(\Phi_{Z,i} K, u)^{-n} du \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\int_{\mathrm{SO}(n)} h(\Pi_i K, \phi u) d\check{\mu}_u(\phi) \right)^{-n} du. \end{aligned}$$

Since $\mu_u(\mathrm{SO}(n)) = 1$, we can use Jensen's inequality to get

$$V(\Phi_{Z,i}^* K) \leq \frac{1}{n} \int_{S^{n-1}} \int_{\mathrm{SO}(n)} h(\Pi_i K, \phi u)^{-n} d\check{\mu}_u(\phi) du.$$

We denote by σ the right-invariant Haar measure on $\mathrm{SO}(n)$. Since $\Phi_{Z,i}$ is $\mathrm{SO}(n)$-equivariant, we obtain

$$\begin{aligned} V(\Phi_{Z,i}^* K) &= \int_{\mathrm{SO}(n)} V(\Phi_{Z,i}^* (\vartheta^{-1} K)) d\sigma(\vartheta) \\ &\leq \frac{1}{n} \int_{\mathrm{SO}(n)} \int_{S^{n-1}} \int_{\mathrm{SO}(n)} h(\Pi_i K, \vartheta \phi u)^{-n} d\check{\mu}_u(\phi) du d\sigma(\vartheta) \\ &= \frac{1}{n} \int_{S^{n-1}} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} h(\Pi_i K, \vartheta \phi u)^{-n} d\sigma(\vartheta) d\check{\mu}_u(\phi) du. \end{aligned}$$

By the right-invariance of σ and again using $\mu_u(\mathrm{SO}(n)) = 1$, it follows that

$$\begin{aligned} &\frac{1}{n} \int_{S^{n-1}} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} h(\Pi_i K, \vartheta \phi u)^{-n} d\sigma(\vartheta) d\check{\mu}_u(\phi) du \\ &\qquad = \frac{1}{n} \int_{S^{n-1}} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} h(\Pi_i K, \vartheta u)^{-n} d\sigma(\vartheta) d\check{\mu}_u(\phi) du \\ &\qquad = \frac{1}{n} \int_{S^{n-1}} \int_{\mathrm{SO}(n)} h(\Pi_i K, \vartheta u)^{-n} d\sigma(\vartheta) du. \end{aligned}$$

Rewriting as before then immediately gives

$$\begin{aligned} V(\Phi_{Z,i}^* K) &\leq \frac{1}{n} \int_{\mathrm{SO}(n)} \int_{S^{n-1}} h(\Pi_i \vartheta^{-1} K, u)^{-n} du d\sigma(\vartheta) \\ &= \int_{\mathrm{SO}(n)} V(\Pi_i^* \vartheta^{-1} K) d\sigma(\vartheta) \\ &= V(\Pi_i^* K). \end{aligned}$$

For the right-hand side of (4.18), we first recall the fact that

$$\int_{S^{n-1}} f(u^\perp) du = n\kappa_n \int_{G(n,n-1)} f(L) dL = n\kappa_n \int_{G(n,1)} f(\bar{u}^\perp) d\bar{u},$$

for $f \in C(G(n,n-1))$. We use (2.10) and identity (1.12) to get

$$\begin{aligned} V(\Pi_i^* K) &= \frac{1}{n}\int_{S^{n-1}} h(\Pi_i K, u)^{-n} du \\ &= \frac{1}{n}\frac{\kappa_i^n}{\kappa_{n-1}^n} n\kappa_n \int_{G(n,1)} \left[R_{n-i,1} vol_i(K|\cdot^{\perp})\right]^{-n}(\bar{u}) d\nu_1(\bar{u}) \\ &= \frac{1}{n}\frac{\kappa_i^n}{\kappa_{n-1}^n} n\kappa_n \int_{G(n,1)} \left[(R_{i,n-1} vol_i(K|\cdot))^{-n}\right]^{\perp}(\bar{u}) d\nu_1(\bar{u}). \end{aligned}$$

By Jensen's inequality and the normalization of the measure on $G(n,i)^L$, it follows that

$$(R_{i,n-1} vol_i(K|\cdot))^{-n}(L) \le (R_{i,n-1} vol_i^{-n}(K|\cdot))(L).$$

Thus, we obtain

$$\begin{aligned} V(\Pi_i^* K) &\le \frac{1}{n}\frac{\kappa_i^n}{\kappa_{n-1}^n} n\kappa_n \int_{G(n,1)} \left[R_{i,n-1} vol_i^{-n}(K|\cdot)\right]^{\perp}(\bar{u}) d\nu_1(\bar{u}) \\ &= \frac{1}{n}\frac{\kappa_i^n}{\kappa_{n-1}^n} n\kappa_n \int_{G(n,1)} \int_{G(n,i)^{u^\perp}} vol_i^{-n}(K|E) d\nu_i^{u^\perp}(E) d\nu_1(\bar{u}) \\ &= \frac{1}{n}\frac{\kappa_i^n}{\kappa_{n-1}^n} n\kappa_n \int_{G(n,n-1)} \int_{G(n,i)^L} vol_i^{-n}(K|E) d\nu_i^L(E) d\nu_{n-1}(L). \end{aligned}$$

Since, by (1.11), we have

$$\int_{G(n,n-1)} \left[R_{i,n-1} vol_i^{-n}(K|\cdot)\right](L) d\nu_{n-1}(L) = \int_{G(n,i)} vol_i^{-n}(K|E) R_{n-1,i}(1) d\nu_i(E),$$

we finally conclude that

$$\begin{aligned} V(\Pi_i^* K) &\le \frac{1}{n}\frac{\kappa_i^n}{\kappa_{n-1}^n} n\kappa_n \int_{G(n,i)} vol_i^{-n}(K|E) R_{n-1,i}(1) d\nu_i(E) \\ &= \frac{\kappa_n^{n+1}}{\kappa_{n-1}^n} A_{n-i}(K)^{-n}. \end{aligned}$$

□

Using Theorem 4.7 and combining it with Conjecture 1.2, we would get

$$V(K)^i V(\Phi_{Z,i}^* K) \le V(K)^i V(\Pi_i^* K) \le \frac{\kappa_n^{n+1}}{\kappa_{n-1}^n} V(K)^i A_{n-i}(K)^{-n} \le \frac{\kappa_n^{i+1}}{\kappa_{n-1}^n}.$$

Theorem 4.7 therefore allows us to recover inequality (4.16) from inequality (4.1) in the form

$$V(K)^i V(\Phi_{Z,i}^* K) \le V(K)^i V(\Pi_i^* K) \le \frac{\kappa_n^{i+1}}{\kappa_{n-1}^n},$$

and to relate our results to an important conjecture by Lutwak on the relation between affine quermassintegrals and volume [Lut88a] (cf. Section 1.1).

Applying the techniques used in the proof of Theorem 4.7 to the operator Γ_Z^*, we can also relate the volume of the polar generalized centroid body to the volume of the convex body. This inequality is again strengthened by its classical counterpart.

Theorem 4.8. *Let $K \in \mathcal{K}^n_{(o)}$ and $Z \neq [-\bar{e},\bar{e}]$ as above, normalized such that $\Phi_Z B = \Pi B$. Then*

$$V(\Gamma_Z^* K)V(K) \leq V(\Gamma^* K)V(K) \leq V(\Gamma^* B)V(B), \tag{4.19}$$

with equality if and only if K is a ball centered at the origin.

Proof. The equation on the right-hand side is a well known fact, derived by combining inequality (1.4) with inequality (2.19) and rewriting the constants afterwards. To prove the inequality on the left-hand side, we use (2.22) to write

$$\begin{aligned} V(\Gamma_Z K) &= \frac{1}{n}\int_{S^{n-1}} h(\Gamma_Z K, u)^{-n} du \\ &= \frac{1}{n}\int_{S^{n-1}} \left(\int_{\mathrm{SO}(n)} h(\Gamma K, \phi u) d\check{\mu}_u(\phi) \right)^{-n} du. \end{aligned}$$

Since $\mu_u(\mathrm{SO}(n)) = 1$, we can use Jensen's inequality to get

$$V(\Phi_{Z,i}^* K) \leq \frac{1}{n}\int_{S^{n-1}} \int_{\mathrm{SO}(n)} h(\Gamma K, \phi u)^{-n} d\check{\mu}_u(\phi) du.$$

Since Γ_Z is $\mathrm{SO}(n)$-equivariant, by the same methods as in the proof of Theorem 4.7 we obtain

$$V(\Gamma_Z K) \leq \frac{1}{n}\int_{S^{n-1}} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} h(\Gamma K, \vartheta\phi u)^{-n} d\sigma(\vartheta) d\check{\mu}_u(\phi) du.$$

By the right-invariance of σ and again using $\mu_u(\mathrm{SO}(n)) = 1$, it follows that

$$\begin{aligned} &\frac{1}{n}\int_{S^{n-1}} \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} h(\Gamma K, \vartheta\phi u)^{-n} d\sigma(\vartheta) d\check{\mu}_u(\phi) du \\ &\qquad = \frac{1}{n}\int_{S^{n-1}} \int_{\mathrm{SO}(n)} h(\Gamma K, \vartheta u)^{-n} d\sigma(\vartheta) du. \end{aligned}$$

Rewriting as before then immediately gives

$$\begin{aligned} V(\Gamma_Z K) &\leq \frac{1}{n}\int_{\mathrm{SO}(n)} \int_{S^{n-1}} h(\Gamma\vartheta^{-1} K, u)^{-n} du d\sigma(\vartheta) \\ &= \int_{\mathrm{SO}(n)} V(\Gamma\vartheta^{-1} K) d\sigma(\vartheta) \\ &= V(\Gamma K). \end{aligned}$$

By the equality conditions of Theorem 4.3, K has to be a ball centered at the origin. $\square$

We remark that the inequality

$$V(\Gamma_Z^* K)V(K) \le V(\Gamma^* B)V(B)$$

does hold also without the assumption on the normalization of Z. This can be shown as before using Blaschke–Santaló and the generalized centroid inequality. The case $Z = B$ then gives the inequality

$$V(K)^{-\frac{(2n+1)}{n}} I_1(K) \ge V(B)^{-\frac{(2n+1)}{n}} I_1(B),$$

where $I_1(K) = \int_K ||x||dx$ is the first moment of the convex body K.

Next, we are going to show that the volume of $\Phi_Z^* K$ can be compared to the affine surface area. Let $K \in \mathcal{K}_n^n$ have a continuous positive curvature function $f(K,\cdot)\colon S^{n-1} \to (0,\infty)$ and let M be the star body defined by $\rho(M,u) = f(K,u)^{1/(n+1)}$. Then

$$\begin{aligned} h(\Gamma_Z M, v) &= \frac{1}{(n+1)V(M)} \int_{S^{n-1}} h(Z(v),u)\rho(M,u)^{n+1} dS(u) \\ &= \frac{2}{(n+1)V(M)} h(\Phi_Z K, v). \end{aligned}$$

Therefore,

$$(n+1)V(M)\Gamma_Z M = 2\Phi_Z K$$

and thus, by Theorem 4.3, we can compare the volume of $\Phi_Z K$ to the affine surface area:

$$n^{n+1} V(B)^{n-1} V(\Phi_Z^* B) V(\Phi_Z K) \ge \Omega(K)^{n+1}. \tag{4.20}$$

We briefly remark that combining inequality (4.20) with the conjectured Petty projection inequality [Pet71] for $\Phi_Z K$ would allow a connection to the affine isoperimetric inequality

$$n^{n+1} V(B)^2 V(K)^{n-1} \ge \Omega(K)^{n+1}.$$

In the L_p setting, an analog of the Petty projection inequality was proven by Lutwak, Yang, and Zhang. Their result can be stated as follows:

Theorem 4.9 [Lut00b] *For $K \in \mathcal{K}_{(o)}^n$ and $1 < p < \infty$,*

$$V(K)^{n-p} V(\Pi_{\mathbf{p}}^* K)^p \le V(B)^{n-p} V(\Pi_{\mathbf{p}}^* B)^p,$$

with equality if and only if K is an ellipsoid centered at the origin.

There is connection between this inequality and the L_p centroid body, analog to the Minkowski setting. In [Lut00b], Lutwak, Yang, and Zhang proved an L_p version of the Busemann–Petty centroid where they showed that the L_p centroid body $\Gamma_p K$ of a star body K satisfies for $1 \leq p < \infty$

$$V(B)V(\mathit{\Gamma}_{\mathbf{p}}B)^{-1} \geq V(K)V(\mathit{\Gamma}_{\mathbf{p}}K)^{-1}, \tag{4.21}$$

with equality if and only if K is an ellipsoid centered at the origin.

In [Hab], Haberl and Schuster also investigated the generalization of Theorem 4.1 to the L_p setting, generalizing therefore the L_p Petty projection inequality of Lutwak, Yang, and Zhang. They showed that the operators $\mathit{\Phi}_{Z,\mathbf{p}}$ satisfy again L_p versions of the Petty projection inequality:

Theorem 4.10 [Hab] *For $K \in \mathcal{K}^n_{(o)}$ and $1 < p < \infty$,*

$$V(K)^{n-p}V(\mathit{\Phi}^*_{Z,\mathbf{p}}K)^p \leq V(B)^{n-p}V(\mathit{\Phi}^*_{Z,\mathbf{p}}B)^p,$$

with equality

- *for $Z = [-\bar{e},\bar{e}]$ if and only if K is an ellipsoid centered at the origin,*
- *for $Z \neq [-\bar{e},\bar{e}]$ if and only if K is a ball centered at the origin.*

One aim of this thesis is to establish a connection between Theorem 4.10 and the newly defined generalized L_p centroid body maps introduced in Section 2.4 and to proof an analog of inequality (4.21) for the operators $\mathit{\Phi}_{Z,\mathbf{p}}$, where we will follow the approach of [Lut00b]. In the following, we will always assume $1 < p < \infty$. We start with a lemma.

Lemma 4.11. *For $K,L \in \mathcal{K}^n_{(o)}$, we have*

$$V_p(L, \mathit{\Gamma}_{Z,\mathbf{p}}K) = \frac{2}{(n+p)V(K)}\tilde{V}_{-p}(K, \mathit{\Phi}^*_{Z,\mathbf{p}}L). \tag{4.22}$$

Proof. By (1.39), (2.23) and (2.24), we get

$$\begin{aligned} V_p(L;\mathit{\Gamma}_{Z,\mathbf{p}}K) &= \frac{1}{n}\int_{S^{n-1}} h(\mathit{\Gamma}_{Z,\mathbf{p}}K,u)^p dS_{\mathbf{p}}(L,u) \\ &= \frac{1}{nV(K)}\int_{S^{n-1}}\int_K h(Z(u),x)^p dx \, dS_{\mathbf{p}}(L,u) \\ &= \frac{1}{n(n+p)V(K)}\int_{S^{n-1}}\int_{S^{n-1}} h(Z(u),v)^p \rho(K,v)^{n+p} dS(v) dS_{\mathbf{p}}(L,u). \end{aligned}$$

Applying Fubini's theorem and the definition of $\mathit{\Phi}_{Z,\mathbf{p}}$ gives

$$V_p(L;\mathit{\Gamma}_{Z,\mathbf{p}}K) = \frac{2}{n(n+p)V(K)}\int_{S^{n-1}} h(\mathit{\Phi}_{Z,\mathbf{p}}L,v)^p \rho(K,v)^{n+p} dS(v).$$

Finally, by (1.40), we obtain (4.22). □

If we set $L = \mathit{\Gamma}_{Z,\mathbf{p}}K$ or $K = \mathit{\Phi}^*_{Z,\mathbf{p}}L$ in (4.22), respectively, we immediately obtain:

Lemma 4.12. *Let $K \in \mathcal{K}^n_{(o)}$. Then*

$$V(\mathit{\Gamma}_{Z,\mathbf{p}}K) = \frac{2}{(n+p)V(K)}\tilde{V}_{-p}(K, \mathit{\Phi}^*_{Z,\mathbf{p}}\mathit{\Gamma}_{Z,\mathbf{p}}K). \tag{4.23}$$

Let $L \in \mathcal{K}^n_{(o)}$. Then

$$V_p(L, \mathit{\Gamma}_{Z,\mathbf{p}}\mathit{\Phi}^*_{Z,\mathbf{p}}L) = \frac{2}{(n+p)}. \tag{4.24}$$

We are now finally able to prove the L_p version of the generalized centroid inequality.

Theorem 4.13. *Let $K \in \mathcal{K}^n_{(o)}$. Then*

$$V(B)V(\mathit{\Gamma}_{Z,\mathbf{p}}B)^{-1} \geq V(K)V(\mathit{\Gamma}_{Z,\mathbf{p}}K)^{-1},$$

with equality

- *for $Z = [-\bar{e},\bar{e}]$ if and only if K is an ellipsoid centered at the origin,*
- *for $Z \neq [-\bar{e},\bar{e}]$ if and only if K is a ball centered at the origin.*

Proof. We use (4.23) and (1.41) to obtain

$$V(\mathit{\Gamma}_{Z,\mathbf{p}}K) \geq \frac{2}{(n+p)}V(K)^{\frac{p}{n}}\ V(\mathit{\Phi}^*_{Z,\mathbf{p}}\mathit{\Gamma}_{Z,\mathbf{p}}K)^{-\frac{p}{n}}.$$

Applying Theorem 4.10 yields

$$V(\mathit{\Gamma}_{Z,\mathbf{p}}K)V(B)^{\frac{n-p}{p}}V(\mathit{\Phi}^*_{Z,\mathbf{p}}B) \geq V(\mathit{\Gamma}_{Z,\mathbf{p}}K)^{\frac{n}{p}}V(\mathit{\Phi}^*_{Z,\mathbf{p}}\mathit{\Gamma}_{Z,\mathbf{p}}K) \geq \left(\frac{2}{n+p}\right)^{\frac{n}{p}} V(K).$$

The constants can easily be rewritten to get the formulation of the theorem.
For the equality cases in the case $Z \neq [-\bar{e},\bar{e}]$, we combine the equality cases of (1.41) and Theorem 4.10. The case $Z = [-\bar{e},\bar{e}]$ is inequality (4.21). $\square$

We remark that it is easy to recover Theorem 4.10 by using (4.24) and Theorem 4.13.

4.2 Busemann intersection inequality and generalizations

One of the most important inequalities of the dual Brunn–Minkowski theory concerns the relation between the volume of a star body and its intersection body. The Busemann intersection inequality states that for $K \in \mathcal{S}^n_o$,

$$V(IK) \leq \frac{\kappa^n_{n-1}}{\kappa^{n-1}_n}V(K)^{n-1}, \tag{4.25}$$

with equality if and only if K is an ellipsoid centered at the origin. This inequality was obtained by Busemann for convex bodies [Bus53], who derived it by using the Busemann random simplex inequality. Later, Petty [Pet61] observed that it also holds for star bodies.

A quite recent result on intersection bodies is a generalization of inequality (4.25) by Leng and Lu [Lu08] to i-intersection bodies. For these operators, it follows from their results (combine Lemma 3.2, Lemma 3.3 from [Lu08] and inequality (3.9)) that for $K \in \mathcal{S}_o^n$ and $0 < i < n-1$,

$$V(\mathrm{I}_i K) \leq \frac{\kappa_{n-1}^n}{\kappa_n^{i-1}} V(K)^i, \tag{4.26}$$

with equality if and only if K is a ball centered at the origin.

We are now going to prove that there is also an extension of (4.25) and (4.26) for radial Minkowski valuations parametrized by positive zonal measures on S^{n-1}, similar to the generalization of the projection inequalities. Dual to Φ_Z, we introduced the operators Ψ_ν in Section 2.5. These operators are $(n-1)$-homogeneous, $\mathrm{SO}(n)$-equivariant and satisfy the valuation property with respect to radial addition. Our analogue of Theorem 4.7 for these radial valuations Ψ_ν can be stated as follows:

Theorem 4.14. *Let $K \in \mathcal{S}_o^n$ and $0 < i \leq n-1$. Using the normalization $\nu(S^{n-1}) = n\kappa_n$, the following chain of inequalities holds:*

$$V(\Psi_{\nu,i}K) \leq V(I_i K) \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-1}} \tilde{A}_{n-i}(K)^n \leq \frac{\kappa_{n-1}^n}{\kappa_n^{i-1}} V(K)^i. \tag{4.27}$$

Proof. Using the representation from Lemma 2.18, we get

$$\begin{aligned} V(\Psi_{\nu,i}K) &= \frac{1}{n} \int_{S^{n-1}} \rho(\Phi_{\nu,i}K, u)^n du \\ &= \kappa_n \int_{\mathrm{SO}(n)} \rho(\Phi_{\nu,i}K, \eta e_n)^n d\eta \\ &= \kappa_n \int_{\mathrm{SO}(n)} \left(\int_{\mathrm{SO}(n)} \rho(\mathrm{I}_i K, \phi\eta e_n) d\mu_\eta(\phi) \right)^n d\eta. \end{aligned}$$

By Jensen's inequality and using $\mu_\eta(\mathrm{SO}(n)) = \nu(\mathrm{SO}(n)) = 1$, we obtain

$$V(\Psi_{\nu,i}K) \leq \kappa_n \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} \rho^n(\mathrm{I}_i K, \phi\eta e_n) d\mu_\eta(\phi) d\eta. \tag{4.28}$$

The $\mathrm{SO}(n)$-equivariance of $\Psi_{\nu,i}$ allows us to write

$$V(\Psi_{\nu,i}K) = \int_{O(n)} V(\Psi_{\nu,i}(\vartheta^{-1}K)) d\sigma(\vartheta).$$

By using (4.28), the right-invariance of σ and $\mu_\eta(\mathrm{SO}(n)) = 1$, we get

$$V(\Psi_{\nu,i}K) \leq \kappa_n \int_{\mathrm{SO}(n)} \int_{\mathrm{SO}(n)} \rho^n(\mathrm{I}_i K, \vartheta\eta e_n) d\sigma(\vartheta) d\eta.$$

Rewriting this gives

$$V(\Psi_{\nu,i}K) \leq V(\mathrm{I}_i K).$$

For the second inequality, we use the same techniques as in the proof of Theorem 4.7 to get

$$
\begin{aligned}
V(\mathrm{I}_i K) &\leq \frac{1}{n} \frac{\kappa_{n-1}^n}{\kappa_i^n} n\kappa_n \int_{Gr(n,i)} vol_i^n(K \cap E) R_{n-1,i}(1) d\nu_i(E) \\
&= \frac{\kappa_{n-1}^n}{\kappa_n^{n-1}} \tilde{A}_{n-i}(K)^n,
\end{aligned}
$$

where we used inequality (1.10) in the last step. This concludes the proof. □

The equality conditions in (4.27) follow immediately from the equality conditions in (4.26). Using Theorem 4.14, we therefore immediately see that

$$
V(\Psi_{\nu,i}K)V(K)^{-i} \leq V(I_i K)V(K)^{-i} \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-1}} \tilde{A}_{n-i}(K)^n V(K)^{-i} \leq \frac{\kappa_{n-1}^n}{\kappa_n^{i-1}}, \tag{4.29}
$$

with equality in the outermost inequality if and only if K is a ball centered at the origin.

Theorem 4.14 now identifies the strongest among this new family of inequalities generalizing Leng and Lu's inequalities, and at the same time compares the results to Grinberg's [Gri91] $SL(n)$-invariant inequality for dual affine quermassintegrals

$$
V(K)^{-i} V(\Psi_{\nu,i}K) \leq V(K)^{-i} V(I_i K) \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-1}} V(K)^{-i} \tilde{A}_{n-i}(K)^n \leq \frac{\kappa_{n-1}^n}{\kappa_n^{i-1}}.
$$

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Index

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