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DISSERTATION

LIFETIME ANALYSIS FOR FUZZY DATA

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To My Family

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DECLARATION OF AUTHORSHIP

I hereby declare that I have written this Doctoral Thesis independently, that I have completely specified the utilized sources and resources and that I have definitely marked all parts of the work - including tables, maps and figures - which belong to other works or to the internet, literally or extracted, by referencing the source as borrowed.

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ABSTRACT

Statistics is the science of learning from data. Data are frequently presented in the form of numbers, vectors, or functions, generally containing measurements of some phenomena. During this process of data collection measurements are recorded as precise numbers, and countless techniques are available to model or to draw inference from these measurements.

But in practical situations especially dealing with continuous variables there are two types of uncertainty in daily life data, one is variation among the observations and another is imprecision of single observations. Keep in mind that variation among observations is different from imprecision, which is also called fuzziness.

Classical statistical tools are based on precise observations and do nothing with fuzziness. By ignoring fuzziness of the observations we may lose information and get misleading results.

Therefore, fuzziness of the single observations should be considered and modeled by fuzzy numbers.

To consider fuzziness of the single observations in drawing inference the idea of fuzzy sets was first introduced by Zadeh in 1965. According to him in the physical world many quantities do not have precise values but are more or less fuzzy.

Also if we consider some examples in linguistic description like, a class of good or bad teachers, class of good-looking women, high or low temperature, in all these situations one cannot characterize it in classical mathematical set notation. In the same way there are a lot of situations for which we cannot define precise criteria for the membership to a set.

The analysis techniques of life time data can be traced back centuries but the prompt development started about few decades ago, and since then a significant number of books and research papers has been published. Most of these publications are based on precise life time observations.

It has already been shown that life time observations are not precise numbers but more or less fuzzy. Therefore, the analysis techniques need to be generalized in such a way that in addition to the stochastic variation, fuzziness of the observations are also integrated.

Some work has been done dealing with fuzzy life time data, but still in most of the situations it is ignored.

In this study some popular approaches of survival analysis are generalized in such a way that fuzziness of the observations is also considered for the analysis to obtain appropriate results.

The proposed estimators are based on fuzzy life time data.

In addition to these techniques some parametric and non-parametric techniques from Accelerated Life Testing are also generalized for fuzzy life time data.

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INTRODUCTION

1.1 FUZZY INFORMATION

Statistics is the science of learning from data. Data are usually presented in the form of numbers, vectors, or functions, generally containing measurements of some phenomena. During the data collection process measurements are recorded as precise numbers, and countless techniques (Stochastic models) are available to model or to draw inference from measurements. Stochastic models are used to model variation among the precise observations. But in practical situations especially dealing with continuous variables there are two types of uncertainty in daily life data, one is variation among the observations and another is imprecision of single observations. Keep in mind that variation among observations is different from imprecision, which is also called *fuzziness*.

The common variables which are used in daily life are life time of an object, amount of rain, amount of carbon emission, color intensity of light, height of a tree etc. All these variables are of continuous nature and theoretically measured as precise numbers, but it is worth mentioning that an exact measurement of a real continuous variable is impossible, because they are more or less fuzzy.

Classical statistics (Stochastic models) considers only variation among the precise observations and ignores the fuzziness. By doing so we may lose information and get misleading results. Therefore fuzziness of the single observations should be considered and modeled by so-called fuzzy numbers (Viertl, 2011).

To consider the imprecision of single observations the idea of fuzzy sets was first introduced by Zadeh in 1965. According to him in the physical world you may classify some animal or plants like cow, horse, birds, etc. but some have unclear status like bacteria, starfish etc. Also if we consider some examples in linguistic description, like a class of good students, class of good-looking women, high blood pressure, in all these situations one cannot characterize it in classical mathematical set notation. In the same way there are a lot of situations for which we cannot define precise criteria for the membership to a set (Zadeh, 1965).

In classical set theory a two valued characteristic function, called *indicator function*, is used to represent whether an element t is in a subset A of a universal set M or not, as mentioned in equation (1.1), by the indicator function $\mathbb{1}_A(\cdot)$ which is defined by

$$\mathbb{1}_A(t) = \left\{ \begin{matrix} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{matrix} \right\} \qquad \forall t \in M. \tag{1.1}$$

Since fuzzy set theory is an extension of classical set theory, in which the two-valued logic is extended to a multi-valued logic, therefore the indicator function mentioned in equation (1.1) is generalized to the so-called *membership function* μ_{A^*} of a fuzzy subset A^* of M, i.e. equation (1.2):

$$\mu_{A^*}(t) = \left\{ \begin{array}{ll} 1 & \text{if } t \in \text{core } A^* \\ \delta \in (0\,,1) & \text{if } t \text{ belongs to } A^* \text{ to some degree } \delta \\ 0 & \text{if } t \notin A^* \end{array} \right\} \qquad \forall t \in M \tag{1.2}$$

Where core of a fuzzy subset A^* is the set of all points t in M such that $\mu_{A^*}(t) = 1$.

The membership function maps the elements from a universal set M to the interval $[0\,,1]$ (Szeliga, 2004).

1.2 SUPPORT OF FUNCTIONS

The support $supp(f)$ of a function $f\colon M \to \mathbb{R}$ is the set of points of M for which the function is not equal to zero, i.e. $f(t) \neq 0 \quad \forall\, t \in supp(f)$.

Some work has been done to deal with the imprecision of observations e.g., (Pak et al., 2013), (Nakama, 2013), (Wu, 2009), (Huang et al., 2006), (Buckley, 2006), (Nguyen and Wu, 2006), (Lee, 2006), (Hung and Liu, 2004), (D'Urso, 2003), (Zimmermann, 2001), (Viertl, 1997), (Klir and Yuan, 1995), (Tzafestas and Venetsanopoulos, 1994), (Frühwirth-Schnatter, 1993).

1.3 FUZZY NUMBERS

So-called fuzzy numbers t^* are special fuzzy subsets of $\mathbb{R}$ determined by their so-called *characterizing function* $\xi(\cdot)$ which is a real function of one real variable satisfying the following conditions 1-3:

1. $\xi : \mathbb{R} \to [0\,,1]$.

2. For all $\delta \in (0\,,\,1]$ the so-called δ-cut $C_\delta(t^*) := \{t \in \mathbb{R} : \xi\,(t) \geq \delta\}$ is a finite union of compact intervals $[a_{j,\delta}\,,\,b_{j,\delta}]$, i.e.

$$C_\delta(t^*) \;=\; \bigcup_{j=1}^{k_\delta} [a_{j,\delta}\,,\,b_{j,\delta}] \;\neq\; \emptyset.$$

3. The support of $\xi(\cdot)$ is bounded, i.e.

$$supp[\xi(\cdot)] \;:=\; \{t \,\in\, \mathbb{R} : \xi\,(t) \,>\, 0\,\} \,\subseteq\, [a\,,\,b]$$

with $-\infty \,<\, a \,<\, b \,<\, \infty$.

For details see (Viertl, 2011).

The set of all fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

If all δ-cuts of a fuzzy number are non-empty closed bounded intervals, the corresponding fuzzy number is called *fuzzy interval*.

1.4 Remark

The family $(C_\delta(t^*); \quad \delta \in (0\,,\,1])$ is nested, i.e. for $\delta_1 \,<\, \delta_2$ we have $C_{\delta_1}(t^*) \supseteq C_{\delta_2}(t^*)$.

1.5 Characterizing functions

How to obtain the characterizing function of a fuzzy number is an important issue. For one-dimensional fuzzy quantities a simple way to obtain it is the following:

Consider an experiment to measure the light intensity of a picture on the screen. Let t_1, t_2, ..., t_N be the discrete values of the variable and $h(t_i)$ be the color intensities at position t_i, and Δt the constant distance between the points t_i . Then the discrete analog of the derivative $h'(\cdot)$ is given by the step function $\eta(\cdot)$ which is constant in the intervals $\left(t_i - t_{i+1}\right)$:

$$\eta(t) := |h(t_{i+1}) - h(t_i)| \text{ for } t \in (t_i,\ t_{i+1})$$

and

$$\eta(t_j) := \max\left\{|h(t_{i+1}) - h(t_i)|\ ,\ |h(t_i) - h(t_{i-1})|\right\} \qquad \text{for} \quad i = 2(1)n-1.$$

The corresponding characterizing function $\xi(\cdot)$ is obtained in the following way:

$$\xi(t) := \frac{\eta(t)}{\max\left\{\eta(t)\ :\ t \in \mathbb{R}\right\}} \qquad t \in \mathbb{R}$$

For details see (Viertl, 2011).

For continuous variables the characterizing function of a fuzzy observation t^*, given by a color intensity transition $g(\cdot)$ can be obtained through

$$\xi(t) := \frac{|g'(t)|}{\max\left\{|g'(x)|\ :\ x \in \mathbb{R}\right\}} \qquad \forall t \in \mathbb{R},$$

where $g(\cdot)$ is a function of the continuous variable t, and $g'(\cdot)$ is the derivative of the function $g(\cdot)$.

For example to obtain the characterizing function for the non-precise water level of a river, one can observe the wetness of the survey rod. The intensity of wetness $W(h)$ is a function of height h. Taking the derivative of $W(\cdot)$, resulting in $W'(\cdot)$, the corresponding characterizing function of the fuzzy water level will be given by its values

$$\xi(h) := \frac{|W'(h)|}{\max\{|W'(x)| : x \in \mathbb{R}\}} \qquad \forall h \in \mathbb{R}$$

For the proof see (Viertl, 1997).

1.6 FUZZY VECTORS

For a n-dimensional fuzzy vector $\underline{t}^*$ the so-called *vector-characterizing function* $\zeta(., ..., .)$ is a function of n real variables t_1, t_2, ..., t_n satisfying the following three conditions:

1. $\zeta : \mathbb{R}^n \to [0, 1]$.

2. For all $\delta \in (0, 1]$ the so-called δ-cut $C_\delta(\underline{t}^*) := \{\underline{t} \in \mathbb{R}^n : \zeta(\underline{t}) \geq \delta\}$ is non-empty, and a finite union of simply connected and closed bounded sets.

3. The support of $\zeta(., ..., .)$, defined by $supp[\zeta(., ..., .)] := \{\underline{t} \in \mathbb{R}^n : \zeta(\underline{t}) > 0\}$ is a bounded set.

See (Viertl, 2011).

The set of all n-dimensional fuzzy vectors is denoted by $\mathcal{F}(\mathbb{R}^n)$.

If all δ-cuts of a n-dimensional fuzzy vector are simply connected compact sets, then the corresponding n-dimensional fuzzy vector is called n-dimensional fuzzy interval.

1.7 TRIANGULAR NORMS

A vector $(t_1^*, t_2^*, ..., t_n^*)$ of fuzzy numbers t_i^*, $i = 1(1)n$ is not a fuzzy vector. For the generalization of statistical inference, functions defined on sample spaces are essential. Therefore it is basic to form fuzzy elements in the sample space $M \times M \times ... \times M$ where M denotes the observation space of a random quantity. It is necessary to form fuzzy vectors from fuzzy samples. This is done by so-called *triangular norms*, also called *t-norms*.

Definition: A function T: $[0, 1] \times [0, 1] \to [0, 1]$ is called t-norm, if $\forall\, x, y, z, \in [0, 1]$ the following conditions are fulfilled:

1. $T(x, y) = T(y, x)$
2. $T(T(x, y), z) = T(x, T(y, z))$
3. $T(x, 1) = x$
4. $x \leq y \Rightarrow T(x, z) \leq T(y, z)$

Examples of t-norms are:

1. Minimum t-norm $T(x, y) = \min\{x, y\} \qquad \forall\,(x, y) \in [0, 1]^2$
2. Product t-norm $T(x, y) = x \cdot y \qquad \forall\,(x, y) \in [0, 1]^2$
3. Limited sum t-norm $T(x, y) = \max\{x + y - 1, 0\}$
   $\forall\,(x, y) \in [0, 1]^2$

see (Viertl, 2011).

1.8 Extension principle

This is the generalization of an arbitrary function $g\colon M \to N$ for fuzzy argument value a^* in M. Let a^* be a fuzzy element of M with membership function $\mu : M \to [0, 1]$, then the fuzzy value $y^* = g(a^*)$ is the fuzzy element y^* in N whose membership function $\nu(\cdot)$ is defined by

$$\nu(y) := \begin{Bmatrix} sup\,\{\mu(a) : a \in M,\ g(a) = y\} & \text{if } \exists a : g(a) = y \\ 0 & \text{if } \nexists a : g(a) = y \end{Bmatrix} \quad \forall y \in N.$$

See (Klir and Yuan, 1995).

1.9 THEOREM 1

For a continuous function $g\colon \mathbb{R} \to \mathbb{R}$, and fuzzy interval t^* the following holds true:

$$C_\delta\left[g(t^*)\right] = \left[\min_{t\in C_\delta(t^*)} g(t)\ ,\ \max_{t\in C_\delta(t^*)} g(t)\right] \qquad \forall \delta \in (0, 1]$$

For the proof see (Viertl, 2011).

1.10 LEMMA 1

Denoting by $\mathbb{1}_A(\cdot)$ the indicator function of the set $A \subseteq \mathbb{R}$, for any characterizing function of a fuzzy number the following is valid:

$$\xi(t) = \max\left\{\delta\cdot\mathbb{1}_{C_\delta(t^*)}(t) :\ \delta \in [0, 1]\right\} \qquad \forall t \in \mathbb{R}$$

For the proof see (Viertl, 2011).

1.11 THEOREM 2

For a continuous function g: $\mathbb{R}^n \to \mathbb{R}$, and fuzzy n-dimensional interval $\underline{t}^*$ the following holds true:

$$C_\delta\left[g(\underline{t}^*)\right] = \left[\min_{\underline{t}\in C_\delta(\underline{t}^*)} g(\underline{t})\,,\ \max_{\underline{t}\in C_\delta(\underline{t}^*)} g(\underline{t})\right] \qquad \forall \delta \in (0\,,\,1]$$

For the proof see (Viertl, 2011).

1.12 LEMMA 2

For any fuzzy vector $\underline{t}^*$ with vector-characterizing function $\zeta(.,...,.)$ the following is valid:

$$\zeta\,(\underline{t}) = \max\left\{\delta\cdot\mathbb{1}_{C_\delta(\underline{t}^*)}(\underline{t})\,:\ \delta\in[0\,,\,1]\right\} \qquad \forall\,\underline{t}\in\mathbb{R}^n$$

For the proof see (Viertl, 2011).

1.13 REMARK

It should be noted that not all families $(A_\delta\,;\ \delta\ \in\ (0\,,\,1])$ of nested finite unions of compact intervals are the δ-cuts of a fuzzy number. But the following construction lemma holds:

1.14 CONSTRUCTION LEMMA

Let $(A_\delta\,;\ \delta\ \in (0\,,\,1])$ with $A_\delta = \bigcup_{j=1}^{k_\delta}\left[a_{j,\delta}\,,\ b_{j,\delta}\right]$ be a nested family of non-empty subsets of $\mathbb{R}$. Then the characterizing function of the generated fuzzy number is given by

$$\xi\,(t) = \sup\left\{\delta\cdot\mathbb{1}_{A_\delta}(t)\ :\ \delta\in[0\,,\,1]\right\} \qquad \forall\,t\in\mathbb{R}.$$

For the proof compare (Viertl and Hareter, 2006).

1.15 COMBINATION OF FUZZY NUMBERS

Consider a sample of size n from a stochastic quantity T, i.e. $t_1, t_2, ..., t_n$ then each t_i is an element of the observation space $M_T \subseteq \mathbb{R}$. In case of life times $M_T \subseteq [0, \infty]$, and $(t_1, t_2, ..., t_n)$ is an element of the Cartesian product $M_T \times M_T \times ... \times M_T$ of n copies of M_T, called *sample space*, denoted by M_T^n.

On the other side, if we have fuzzy observations t_i^*, $i = 1(1)n$ with characterizing functions $\xi_i(\cdot)$ as fuzzy elements of M_T, then $(t_1^*, t_2^*, ..., t_n^*)$ is not a fuzzy element of M_T^n. Therefore to obtain a fuzzy element of M_T^n usually the minimum t-norm is used. For the vector-characterizing function of the so-called *combined fuzzy sample* $\underline{t}^*$ applying the minimum t-norm, i.e.

$$\zeta\,(t_1, t_2, ..., t_n) = \min\{\,\xi(t_1), \xi(t_2), ..., \xi(t_n)\} \quad \forall\ (t_1, t_2, ..., t_n)\ \in \mathbb{R}^n,$$

a fuzzy element of $M_T^n \subseteq \mathbb{R}^n$ is obtained.

For the δ-cuts of the combined fuzzy sample the following holds:

$$C_\delta\,[\zeta\,(., ..., .)] = \times_{i=1}^{n} C_\delta\,[\xi_i(\cdot)] \qquad \forall\delta \in (0\,, 1]$$

see (Viertl, 2011)

1.16 FUZZY VALUED EMPIRICAL DISTRIBUTION FUNCTIONS

Let $t_1^*, t_2^*, ..., t_n^*$ be fuzzy intervals with corresponding δ-cuts $C_\delta\left(t_i^*\right) = \left[\underline{t}_{i,\delta}\,, \bar{t}_{i,\delta}\right] \qquad \forall\delta \in (0\,, 1]$ and $i = 1(1)n$.

The empirical cumulative distribution function for the given fuzzy sample is denoted as $F^*(\cdot)$, where $\underline{F}_\delta(\cdot)$ and $\overline{F}_\delta(\cdot)$ represent the corresponding lower and upper δ-level functions of the so-called fuzzy empirical distribution function $F^*(\cdot)$.

The lower and upper δ-level functions are defined through the following equations:

$$\overline{F}_\delta(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,\,t]}(\underline{t}_{i,\delta}) \qquad \forall t \in \mathbb{R}$$

and

$$\underline{F}_\delta(t) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{(-\infty,\,t]}(\bar{t}_{i,\delta}) \qquad \forall t \in \mathbb{R}.$$

See (Viertl, 2011).

2

MATHEMATICAL OPERATIONS FOR FUZZY NUMBERS

Fuzzy set theory is the generalization of classical set theory, therefore, the mathematical operations addition, subtraction, multiplication, and division can be generalized for fuzzy quantities (Viertl, 2011).

2.1 ADDITION OF FUZZY NUMBERS

For two fuzzy numbers t_1^* and t_2^* with characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ respectively, the generalized addition $t_1^* \oplus t_2^*$ has to obey conditions, i.e. to generalize the addition of real numbers and to generalize interval arithmetic.

The characterizing function $\psi(\cdot)$ of the generalized addition given as:

$$\psi(z) := \sup_{t_1,t_2} \{\min\{\xi_1(t_1), \xi_2(t_2)\} : t_1 + t_2 = z\} \qquad \forall z \in \mathbb{R}$$

For fuzzy intervals t_1^* and t_2^* the generalized addition can be defined using δ-cuts.

Let $C_\delta(t_1^*) = [a_{1,\delta}\,, b_{1,\delta}]$ and $C_\delta(t_2^*) = [a_{2,\delta}\,, b_{2,\delta}] \quad \forall\ \delta\ \in\ (0\,,1]$ be the δ-cuts of t_1^* and t_2^*, then the δ-cut of $t_1^* \oplus t_2^*$ is given by

$$C_\delta(t_1^* \oplus t_2^*) = [a_{1,\delta} + a_{2,\delta}\,,\; b_{1,\delta} + b_{2,\delta}] \qquad \forall \delta \in (0\,,1].$$

The characterizing function can be obtained by the above mentioned Construction Lemma.

2.2 SCALAR ADDITION

A special case of addition is scalar addition, i.e. adding a constant to the fuzzy number. Let c be constant $c \in \mathbb{R}$ which is added to the fuzzy number t^* whose characterizing function is $\xi(\cdot)$. The characterizing function of the new fuzzy number, i.e. $t^* \oplus c$ is denoted by $\eta(\cdot)$ and is defined as

$$\eta(t) := \xi(t - c) \qquad \forall t \in \mathbb{R}.$$

2.3 SCALAR MULTIPLICATION

For scalar multiplication a fuzzy number t^* with characterizing function $\xi(\cdot)$ is multiplied by a constant $c \in \mathbb{R}\,(\neq 0)$ in the following way: The characterizing function of the resulting fuzzy number $c \odot t^*$ is denoted by $\eta(\cdot)$, and can be defined as

$$\eta(t) := \xi(t/c) \qquad \forall\, t \in \mathbb{R}.$$

2.4 MULTIPLICATION OF FUZZY NUMBERS

For two fuzzy numbers t_1^* and t_2^* with characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ respectively, the characterizing function $\psi(\cdot)$ of the generalized multiplication $t_1^* \odot t_2^*$ is given by its values

$$\psi(z) := \sup\left\{\min\left\{\xi_1(t_1), \xi_2(t_2)\right\} : t_1 \cdot t_2 = z\right\} \qquad \forall z \in \mathbb{R}.$$

For fuzzy intervals t_1^* and t_2^* the lower and upper ends of the δ-cuts of the obtained fuzzy interval can be obtained using δ-cuts:

$$C_\delta\left(t_1^* \odot t_2^*\right) = \left[\min_{(t_1,t_2)\in\times_{i=1}^2 C_\delta(t_i^*)} t_1 \cdot t_2\,,\ \max_{(t_1,t_2)\in\times_{i=1}^2 C_\delta(t_i^*)} t_1 \cdot t_2\right] \qquad \forall \delta \in (0\,,1]$$

2.5 SUBTRACTION OF FUZZY NUMBERS

The difference of two fuzzy numbers t_1^*, t_2^* is defined as $t_1^* \ominus t_2^* := t_1^* \oplus (-t_2^*)$, where $-t_2^*$ is defined by $(-1) \odot t_2^*$ from section 2.3.

2.6 INVERSE OF FUZZY NUMBERS

For fuzzy interval t^* with characterizing function $\xi(\cdot)$ obeying $supp\,[\xi(\cdot)] \not\ni 0$, the *fuzzy inverse* can be defined as $(t^*)^{-1} = \frac{1}{t^*}$ whose characterizing function is given by the following definition:

$$\psi(t) := \left\{\begin{array}{ll} \xi\left(\frac{1}{t}\right) & \text{for } \frac{1}{t} \in supp\,[\xi(\cdot)] \\ 0 & otherwise \end{array}\right\} \qquad \forall t \in \mathbb{R}$$

2.7 DIVISION OF FUZZY NUMBERS

For two fuzzy numbers t_1^* and t_2^* given that $0 \notin supp[t_2^*]$ having characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$, the fuzzy quotient

$$t_1^* \oslash t_2^* := t_1 \odot [(t_2^*)^{-1}]$$

has characterizing function $\psi(\cdot)$ which can be obtained by

$$\psi(z) := \sup_{t_1, t_2} \{\min\{\xi_1(t_1), \xi_2(t_2)\} : t_1/t_2 = z\} \quad \forall z \in \mathbb{R}.$$

For two fuzzy intervals t_1^* and t_2^* obeying $0 \notin supp[t_2^*]$ having characterizing functions $\xi_1(\cdot)$ and $\xi_2(\cdot)$ with δ-cuts $C_\delta(t_1^*) = [a_{1,\delta}\,,\, b_{1,\delta}]$ and $C_\delta(t_2^*) = [a_{2,\delta}\,,\, b_{2,\delta}] \quad \forall \delta \in (0\,,1]$ respectively, the δ-cuts of the new fuzzy number obtained from the generalized division of fuzzy intervals is defined by

$$C_\delta\,(t_1^* \oslash t_2^*) = \left[\min_{(t_1,t_2)\in \times_{i=1}^{2} C_\delta(t_i^*)} t_1/t_2\,,\ \max_{(t_1,t_2)\in \times_{i=1}^{2} C_\delta(t_i^*)} t_1/t_2\right] \qquad \forall \delta \in (0\,,1].$$

2.8 MINIMUM AND MAXIMUM OF FUZZY NUMBERS

If there are n fuzzy intervals, i.e. $t_1^*, t_2^*, ..., t_n^*$ with corresponding characterizing functions $\xi_1(\cdot), \xi_2(\cdot), ..., \xi_n(\cdot)$ respectively, then its δ-cuts are denoted as $C_\delta(t_i^*) = \left[\underline{t}_{i,\delta}\,,\, \overline{t}_{i,\delta}\right] \quad \forall\,\delta \in (0\,,1]$ and $i = 1(1)n$. Then the minimum $t^*_{\min}$ and the maximum $t^*_{\max}$ of the fuzzy numbers are fuzzy intervals respectively, with δ-cuts $C_\delta(t^*_{\min})$ and $C_\delta(t^*_{\max})$. These are defined by

$$C_\delta(t^*_{\min}) := \left[\min\left\{\underline{t}_{i,\delta} : \ i = 1(1)n\right\}\ ,\ \min\left\{\overline{t}_{i,\delta} : \ i = 1(1)n\right\}\right] \qquad \forall \delta \in (0\,,\,1],$$

and

$$C_\delta(t^*_{\max}) := \left[\max\left\{\underline{t}_{i,\delta} : \ i = 1(1)n\right\}\ ,\ \max\left\{\overline{t}_{i,\delta} : \ i = 1(1)n\right\}\right] \qquad \forall \delta \in (0\,,\,1].$$

Remark: Similarly the classical set operations Equality, Union, Intersection, Complement, Product can be generalized to fuzzy sets (Lee, 2006).

2.9 EQUALITY OF FUZZY SETS

Let A^* and B^* be two fuzzy sets in M then both are equal, i.e. $A^* = B^*$, iff their membership functions are identical.

2.10 COMPLEMENT OF FUZZY SETS

For a fuzzy set A^* of M with membership function $\mu_{A^*}(\cdot)$, the complement of A^* is denoted by $\overline{A}^*$ with membership function

$$\mu_{\overline{A}^*}(t) = 1 - \mu_{A^*}(t) \qquad \forall t \in M.$$

2.11 Union of fuzzy sets

For two fuzzy sets A^* and B^* with corresponding membership functions $\mu_{A^*}(\cdot)$ and $\mu_{B^*}(\cdot)$ respectively, the union of the fuzzy sets can be defined by

$$A^* \cup B^* \mathrel{\hat{=}} \mu_{A^* \cup B^*}(t) \mathrel{\hat{=}} \max\{\mu_{A^*}(t), \mu_{B^*}(t)\} \quad \forall t \in M.$$

For arbitrary families of fuzzy sets $A_i^* \mathrel{\hat{=}} \mu_i(\cdot)$ with index set I the union of fuzzy sets is obtained in the following way

$$\bigcup_{i \in I} A_i^* \mathrel{\hat{=}} \mu(t) := \sup\{\mu_i(t) : i \in I\} \qquad \forall t \in M.$$

2.12 Intersection of fuzzy sets

For two fuzzy sets A^* and B^* with corresponding membership functions $\mu_{A^*}(\cdot)$ and $\mu_{B^*}(\cdot)$ respectively, the intersection is defined as

$$A^* \cap B^* \mathrel{\hat{=}} \mu_{A^* \cap B^*}(t) = \min\{\mu_{A^*}(t), \mu_{B^*}(t)\} \qquad \forall t \in M.$$

For arbitrary families of fuzzy sets $A_i^* \mathrel{\hat{=}} \mu_i(\cdot)$ with index set I, the intersection of fuzzy sets is obtained as

$$\bigcap_{i \in I} A_i^* \mathrel{\hat{=}} \mu(t) := \inf\{\mu_i(t) : i \in I\} \qquad \forall t \in M.$$

3

LIFE TIME ANALYSIS

Survival analysis can generally be defined as the collection of techniques for analyzing so-called life time data. In broad sense one can say life time is "the time to the occurrence of a specified event". Life time is also called survival time, failure time, or event time, and is usually measured in hours, days, weeks, months, or years.

The specified event depends on the field under study, it may be death or recovery of a patient from a disease in biomedical sciences, failure of a mechanical equipment in engineering sciences, divorce in sociology, change of residence in demography etc. (Lee and Wang, 2013).

The analysis techniques of life time data can be traced back centuries but the rapid development started about few decades ago, especially World War II stimulated interest in the reliability of military equipments (Miller, 2011).

The prominence of survival analysis is to predict the probability of response, mean survival time, identifying the prognostic factors related to the life time of units, and to compare the survival distributions. Models used for survival times are usually termed as time-to event models (Lee and Wang, 2013).

A significant number of books and research papers have already been written for the analysis of life time data, including reliability analysis, e.g. (Hosmer and Lemeshow, 1999), (Meeker and Escobar, 1998), (Ibrahim et al., 2001), (Deshpande and Purohit, 2005), (Kleinbaum and Klein, 2005), (Nelson, 2005), (Hamada et al., 2008), (Couallier et al., 2013).

3.1 CENSORING

In the time-to-event data generally it is very challenging to wait till the death or failure of all units. One has to terminate the experiment due to time or other constraints. The observations which still stay at the termination of the experiment, which have not been observed completely are called *censored observations* (Kalbfleisch and Prentice, 2011).

3.1.1 *Type I censoring*

When the termination time of an experiment is fixed in advance then the survived items at the time of termination are right censored, which is called *type* I *censoring*.

3.1.2 *Type II censoring*

When it is decided that the experiment will be terminated when the r^{th} failure/death occurs, and considering the rest of the items censored are called *type* II *censoring*.

According to (Lee and Wang, 2013) probability density function, survival function and hazard rate are explained below.

3.2 Probability density function

For a non-negative random variable T the probability density function is defined as

$$f(t) = \lim_{\Delta t \to 0} \frac{P\{\text{an unit dying in the interval}(t,\, t + \Delta t)\}}{\Delta t},$$

it is the limit of the probability that a unit will fail in a short time interval t to Δt.

The probability density can be estimated by

$$\hat{f}(t) = \frac{\text{Number of units dying in the interval}\,(t,\, t + \Delta t)}{(\text{Total number of units}) \cdot (\text{Interval width})}$$

3.3 Survival function

Let T be a non-negative random variable denoting the waiting time until a specified event occurred. For instance consider the event is death and the waiting time until the death of a unit, is survival time. If time t is some specified time then the survival function is conventionally denoted by S, which is defined as

$$S(t) = Pr(\text{Unit will survive time } t)$$

$$S(t) = Pr(T{>}t) \quad \forall\, t \geq 0.$$

From the cumulative distribution function $F(\cdot)$ of T it can also be defined as

$$S(t) = 1 - Pr(\text{Unit will fail not later than } t)$$

$$S(t) = 1 - F(t) \qquad \forall\, t \geq 0.$$

With the properties

$$S(0) \;=\; 1 \quad \text{and} \quad \lim_{t\to\infty} S(t) \;=\; 0.$$

The survival function can be estimated by

$$\hat{S}(t) = \frac{\text{Number of units surviving longer than } t}{\text{Total number of units}}.$$

3.4 HAZARD RATE

The hazard rate $h(\cdot)$ of survival time T gives the conditional failure rate, it can be defined as the limit of the probability that an unit will fail in a short time interval $(t,\, t + \Delta t)$, given that it survived to time t:

$$h(t) := \lim_{\Delta t\to 0} \frac{P\left\{\text{an unit at time } t \text{ fails in time interval } (t,\, t+\Delta t) | T > t\right\}}{\Delta t}$$

$$\hat{h}(t) = \frac{\text{Number of units dying in the interval beginning at time } t}{(\text{Number of units surviving at } t) \cdot (\text{Interval width})}$$

It can be written as

$$h(t) = \frac{f(t)}{S(t)} \quad \forall t \geq 0,$$

and can be estimated by

$$\hat{h}(t) = \frac{\hat{f}(t)}{\hat{S}(t)} \quad \forall t \geq 0.$$

Many parametric and non-parametric approaches are available to model life time data, which are explained below.

3.5 Kaplan-Meier estimator

Let $0 \leq t_1 \leq t_2 \leq t_3 \leq ... \leq t_n$ be n life times from a given population, where n_i, and d_i denote the number of observations at risk, and number of deaths at time t_i respectively.

If d_i denotes the number of deaths at time t_i, frequently it is either 0 or 1, but tied survival times are possible. In that case d_i may be greater than 1.

The Kaplan-Meier estimator can be expressed as

$$S(t) = \prod_{t_i \leq t} (1 - \frac{d_i}{n_i}) \qquad \forall\ t\ \geq 0.$$

(Kaplan and Meier, 1958).

3.6 Exponential distribution

For survival time T with exponential distribution we have the following:

$$f(t|\lambda) = \lambda e^{-\lambda t} \qquad \forall\ t \geq 0 \text{ with } \lambda > 0$$
$$h(t) = \lambda \qquad \forall\ t\ \geq 0$$
$$S(t) = e^{-\lambda t} \qquad \forall\ t\ \geq 0$$

Based on a complete sample t_1, t_2, ..., t_n of T the Maximum Likelihood Estimator for the parameter of the exponential distribution is

$\hat{\lambda} = \frac{1}{\frac{\sum_{i=1}^{n} t_i}{n}}$. An estimator for the hazarad rate is $\hat{h}(t) = \hat{\lambda}$, and for the survival function is $\hat{S}(t) = e^{-\hat{\lambda}t} \qquad \forall\ t\ \geq 0.$

3.7 TWO PARAMETERS WEIBULL DISTRIBUTION

The two parameters Weibull distribution is one of the most popular distributions to model time to event data. Its density is defined by

$$f(t|\,\tau, \beta) = \frac{\beta}{\tau}\left(\frac{t}{\tau}\right)^{\beta-1} exp\left\{-\left(\frac{t}{\tau}\right)^{\beta}\right\} \qquad \forall\ t > 0,\ \text{with}\ \tau > 0,\ \ \beta > 0.$$

$\tau :\ Scale\ Parameter;\ also\ called\ characteristic\ life\ time$

$\beta :\ Shape\ Parameter$

The Maximum Likelihood Estimators of the corresponding parameters, based on a complete sample t_1, t_2, ..., t_n are given by

$$\hat{\tau} = \left\{\frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\beta}}\right\}^{\frac{1}{\hat{\beta}}} \tag{3.1}$$

and

$$\frac{1}{\hat{\beta}} = \left\{\frac{\sum_{i=1}^{n} t_i^{\hat{\beta}} \ln(t_i)}{\sum_{i=1}^{n} t_i^{\hat{\beta}}} - \frac{1}{n}\sum_{i=1}^{n} \ln(t_i)\right\} \tag{3.2}$$

$$\hat{h}(t) = \frac{\hat{\beta} t^{\hat{\beta}-1}}{\hat{\tau}^{\hat{\beta}}} \quad \forall\ t\ \geq 0$$

$$\hat{S}(t) = e^{-(\frac{t}{\hat{\tau}})^{\hat{\beta}}} \quad \forall\ t\ \geq 0.$$

The right hand side of equation (3.2) is a monotone increasing function in $\hat{\beta}$, and the left hand side is a decreasing function. It follows that where they intersect will be the value of $\hat{\beta}$. Then putting that value in equation (3.1) the estimate $\hat{\tau}$ is obtained.

The Maximum Likelihood Estimators of the parameters τ and β for type II censored data are

$$\hat{\tau} = \left[\frac{1}{r} \left\{ \sum_{i=1}^{r} t_{(i)}^{\hat{\beta}} + (n-r) t_{(r)}^{\hat{\beta}} \right\} \right]^{\frac{1}{\hat{\beta}}} \tag{3.3}$$

and

$$\frac{1}{\hat{\beta}} = \left[\frac{\sum_{i=1}^{r} t_{(i)}^{\hat{\beta}} \ln(t_{(i)}) + (n-r) t_{(r)}^{\hat{\beta}} \ln(t_{(r)})}{\sum_{i=1}^{r} t_{(i)}^{\hat{\beta}} + (n-r) t_{(r)}^{\hat{\beta}}} - \frac{1}{r} \sum_{i=1}^{r} \ln(t_{(i)}) \right] \tag{3.4}$$

respectively.

The right hand side of equation (3.4) is a monotone increasing function in β and the left hand side is a decreasing function. It follows that where they intersect will be the value of $\hat{\beta}$. Then putting that value in equation (3.3) an estimate of $\hat{\tau}$ is obtained.

For type I censoring $t_{(r)}$ should be replaced by the pre-specified censoring time (Balakrishnan and Kateri, 2008).

3.8 THREE PARAMETERS WEIBULL DISTRIBUTION

The density and reliability function of the three parameters Weibull distribution are given by

$$f(t|\alpha,\beta,\gamma) = \frac{\gamma}{\beta}\left(\frac{t-\alpha}{\beta}\right)^{\gamma-1} exp\left\{-\left(\frac{t-\alpha}{\beta}\right)^{\gamma}\right\} \quad \forall\, t\ \geq \alpha.$$

α : *Location Parameter*

β : *Scale Parameter*

γ : *Shape Parameter*

The parameters of the three parameters Weibull distribution can be estimated by (Cran, 1988):

$$\hat{\alpha} = \frac{\overline{m}_1 \cdot \overline{m}_4 \ - \ \overline{m}_2^2}{\overline{m}_1 + \overline{m}_4 - 2\overline{m}_2}$$

$$\hat{\gamma} = \frac{\ln 2}{\ln(\overline{m_1} - \overline{m}_2) - \ln(\overline{m}_2 - \overline{m}_4)}$$

$$\hat{\beta} = \frac{\overline{m}_1 - \hat{\alpha}}{\Gamma\left(1 + \frac{1}{\hat{\gamma}}\right)}$$

$$\overline{m}_k = \sum_{r=0}^{n-1}\left(1-\frac{r}{n}\right)^k \left(t_{(r+1)} - t_{(r)}\right), \quad t_{(0)} = 0$$

and the three parameters Weibull reliability function is given by

$$R(t) \ = exp\left\{-\left(\frac{t-\alpha}{\beta}\right)^{\gamma}\right\} \qquad \forall\, t \geq \alpha.$$

3.9 GAMMA DISTRIBUTION

For the random variable T the Gamma distribution is defined by the density

$$f(t|\,\lambda,\gamma) = \frac{\lambda}{\Gamma(\gamma)}(\lambda t)^{\gamma-1}e^{-\lambda t} \quad \text{with } \lambda > 0,\ \gamma > 0.$$

The cumulative distribution function is given by

$$F(t) = \int_0^t \frac{\lambda}{\Gamma(\gamma)}(\lambda t)^{\gamma-1}e^{-\lambda t}\,dt \quad \forall\, t\ \geq 0.$$

The survival function $S(t)$ can be simply written as

$$S(t) = 1 - F(t) \quad \forall\, t\ \geq 0,$$

and the hazard rate is

$$h(t) = f(t)/S(t).$$

According to (Lee and Wang, 2013) the moment estimator and a corrected moment estimator for λ and γ are

$$\hat{\lambda}_b = \frac{\sum_{i=1}^n t_i}{\sum_{i=1}^n (t_i - \bar{t})^2}$$

and

$$\hat{\gamma}_b = \frac{\left(\sum_{i=1}^n t_i\right)^2}{\sum_{i=1}^n (t_i - \bar{t})^2}$$

where $\bar{t}$ represents the mean of the life time observations.

These estimators are biased, the bias corrected estimators are,

$$\hat{\gamma} = \frac{\hat{\gamma}_b}{(1 + \frac{2}{n})} - \frac{3}{n}$$

$$\hat{\lambda} = (\frac{\hat{\gamma}}{\bar{t}})(1 - \frac{1}{n\hat{\gamma}})$$

$$\hat{F}(t) = \int_0^t \frac{\hat{\lambda}}{\Gamma(\hat{\gamma})}(\hat{\lambda}t)^{\hat{\gamma}-1}e^{-\hat{\lambda}t}\, dt.$$

$$\hat{S}(t) = 1 - \hat{F}(t)$$

$$\hat{h}(t) = \hat{f}(t)/\hat{S}(t)$$

respectively.

3.10 LOGNORMAL DISTRIBUTION

For the survival time T such that $\ln T$ is normally distributed with mean μ and variance σ^2, the probability density of the lognormal distribution is given by its values for $t > 0$:

$$f(t|\mu, \sigma) = \frac{1}{t\sigma\sqrt{2\pi}} exp\left[-\frac{1}{2\sigma^2}(\ln t - \mu)^2\right] \quad \text{with} \quad \mu > 0, \sigma > 0$$

The cumulative distribution function can be written as

$$F(t) = \frac{1}{t\sigma\sqrt{2\pi}} \int_0^t exp\left[-\frac{1}{2\sigma^2}(\ln t - \mu)^2\right]\, dt.$$

The survival function $S(\cdot)$ of the lognormal distribution can simply be written as

$$S(t) = 1 - F(t),$$

and the hazard rate is

$$h(t) = f(t)/S(t) \quad \forall\ t\ \geq 0.$$

The Maximum Likelihood Estimators of the parameters of the lognormal distribution given in (Lee and Wang, 2013) are

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \ln t_i}{n}$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\left[\sum_{i=1}^{n}(\ln t_i)^2 - \frac{\left(\sum_{i=1}^{n} \ln t_i\right)^2}{n}\right].$$

$\hat{\mu}$ is an unbiased estimate, but $\hat{\sigma}^2$ is not. An unbiased estimator for σ^2 can be obtained as $s^2 = \hat{\sigma}^2[n/(n-1)]$.
An estimate for $F(t)$ is given by

$$\hat{F}(t) = \frac{1}{ts\sqrt{2\pi}} \int_0^t exp\left[-\frac{1}{2s^2}(\ln t - \hat{\mu})^2\right] dt.$$

Estimates for $S(\cdot)$ and $h(\cdot)$ are

$$\hat{S}(t) = 1 - \hat{F}(t)$$

and

$$\hat{h}(t) = \hat{f}(t)/\hat{S}(t)$$

respectively.

3.11 A NEW MODEL FOR LIFETIME DISTRIBUTION WITH BATHTUB SHAPED FAILURE RATE

A new model for lifetime distribution with bathtub shaped failure rate was proposed by (Haupt and Schäbe, 1992) with probability density function

$$f(t) = \frac{1+2\beta}{2t_0\sqrt{\beta^2 + (1+2\beta)t/t_0}} \quad \text{for} \quad 0 \leq t \leq t_0 \quad \text{with} \ -1/2 < \beta < \infty$$

and failure rate

$$h(t) = \frac{1+2\beta}{(2t_0\sqrt{\beta^2 + (1+2\beta)t/t_0})(1+\beta - \sqrt{\beta^2 + (1+2\beta)t/t_0})} \quad \text{for} \quad 0 \leq t \leq t_0.$$

The failure rate of the given distribution is bathtub shaped for $-1/3 < \beta < 1$. For precise life time observations t_1, t_2, ..., t_n and assuming that $t_1 < t_2 < ... < t_n$, then the maximum likelihood estimators of the parameters are

$$\hat{t}_0 = t_{(n)}$$

$$\frac{2}{1+2\hat{\beta}} = \frac{1}{n}\sum_{i=1}^{n} \frac{\hat{\beta}_i + t_{(i)}/\hat{t}_0}{\beta^2 + (1+2\hat{\beta})t_{(i)}/\hat{t}_0}.$$

For the proof see (Haupt and Schäbe, 1992).

3.12 A new two-parameter lifetime distribution with bathtub shaped failure rate

Another new two-parameter lifetime distribution was proposed by (Chen, 2000) with probability density function

$$f(t|\lambda, \beta) = \lambda\beta e^{\lambda(1-e^{t^\beta})} e^{t^\beta} t^{\beta-1} \qquad \forall t > 0 \quad \text{with} \quad \lambda > 0, \beta > 0,$$

with failure rate

$$h(t) = \lambda\beta e^{t^\beta} t^{\beta-1} \quad \forall t > 0.$$

For precise life time observations $t_1, t_2, ..., t_n$ and assuming that $t_{(1)} < t_{(2)} < ... < t_{(k)}$ are the first k order statistics, where $k \leq n$, then the maximum likelihood estimator $\hat{\beta}$ of the parameter β is the solution of the following equation:

$$\tfrac{k}{\beta} + \textstyle\sum_{i=1}^{k} \ln t_{(i)} + \sum_{i=1}^{k} \left(t_{(i)}^{\beta} \ln t_{(i)} \right) - \frac{k \left[\sum_{i=1}^{k} \left(e^{t_{(i)}^{\beta}} t_{(i)}^{\beta} \ln t_{(i)} \right) + (n-k) \left(e^{t_{(k)}^{\beta}} t_{(k)}^{\beta} \ln t_{(k)} \right) \right]}{\sum_{i=1}^{k} e^{t_{(i)}^{\beta}} - n - (n-k) e^{t_{(k)}^{\beta}}} = 0$$

The Maximum Likelihood Estimator for λ is given by

$$\hat{\lambda} = \frac{k}{\sum_{i=1}^{k} e^{t_{(i)}^{\hat{\beta}}} - n - (n-k) e^{t_{(k)}^{\hat{\beta}}}}.$$

For the proof see (Chen, 2000).

4

ACCELERATED LIFE TESTING

To increase life time of mechanical components always remained a prime interest for the industry. For estimation of the reliability of a component various life tests were made in available environmental conditions. The reliability tests require large data, and usually many manufactured components have long life time, therefore it was very time consuming to get information about the life time of components. Similarly in practical applications it was not possible to have the same environmental conditions for the components. One of the initially papers to test units under different conditions is "Accelerated life testing of capacitors" by (Levenbach, 1957) which was a good reason to test an electronic equipment under different environmental conditions.

Accelerated Life Testing (ALT) contain popular standard statistical tools to get information on life time and make inference about various mechanical products/systems, without waiting longer as the mean life time of that product.

For reliability tests of mechanical components usually the tests are performed under different stress levels which are more severe than usual stress level.

If we describe life time by a stochastic model $f(t|\theta)$ with parameter θ then the stress dependence of the life time distribution can be written in the form

$$\theta(S) = \psi(S; A, B, ...),$$

where $\psi(\cdot)$ is expected to be some identified function with stress level S and unknown constants $A, B, ...$

Some important accelerated life testing techniques are explained in the following sections.

4.1 CONSTANT STRESS LEVELS

In constant stress level accelerated life testing the units are tested under constant stress levels S_1, S_2, ..., S_k and life times of the failed units are recorded.

Consider an exponential distribution which is one of the most popular distributions to deal with life time data. Its density is $f(t|\theta) = \frac{1}{\theta}\exp\left[-(\frac{t}{\theta})\right]$ for $t \geq 0$ and $\theta > 0$ with failure rate $\lambda = \frac{1}{\theta}$. In order to estimate the parameter under different stress levels, i.e. S_1, S_2, ..., S_k, we get the data set $\{S_i, n_i, r_i\}, i = 1(1)k$, where n_i is the number of units under stress S_i, and r_i is the number of failures under i^{th} stress level. Maximum Likelihood Estimator for the parameter θ_i is

$$\hat{\theta}_i = \left[\frac{\sum_{j=1}^{r_i} t_{i,(j)} + (n_i - r_i)t_{i,(r_i)}}{r_i}\right],$$

where $t_{i,(j)}$, $j = 1(1)r_i$ denote the observed r_i ordered times of failure under the stress level S_i (Viertl, 1988).

4.2 POWER RULE MODEL

The power rule model for precise data in (Viertl, 1988) is defined as $\theta(S) = \frac{C}{S^A}$ where S is stress level, A and C are unknown constants.

For estimation of A and C the modified power rule model is given as

$$\hat{\theta}_i = \hat{\theta}(S_i) = \frac{\hat{C}}{\left(\frac{S_i}{\overline{S}}\right)^{\hat{A}}}$$

where $\overline{S} = \prod_{i=1}^{k} S^{r_i / \sum_{i=1}^{k} r_i}$.

In the given equation r_i denotes the number of failures under stress level S_i for $i = 1(1)k$.

If the samples are considered independent of each other for different stress levels then the Maximum Likelihood Estimators for A and C can be obtained by the following equations:

$$\sum_{i=1}^{k} r_i \cdot \hat{\theta}_i \cdot (\frac{S_i}{\overline{S}})^{\hat{A}} \cdot \ln(\frac{S_i}{\overline{S}}) \ = 0$$

$$\hat{C} = \frac{\sum_{i=1}^{k} r_i \cdot \hat{\theta}_i \cdot (\frac{S_i}{\overline{S}})^{\hat{A}}}{\sum_{i=1}^{k} r_i}$$

Solving the first equation for $\hat{A}$, and afterward the second one for $\hat{C}$ estimates are obtained.

4.3 STEP-STRESS MODEL

Step-stress life test is a particular type of accelerated life testing approach. In this approach the units are observed under stress for some pre-specified time and recording the number of failures, and then the stress level changes to another level, and recording the number of failures and so on.

Consider a simple two step-stress test in which the life times follow exponential distributions, i.e. Ex(θ_1) and Ex(θ_2) at stress level S_1 for a specified time τ and then change to S_2.

The general form of the cumulative distribution function of time to event data can be written in the form below:

$$G(t) = \begin{cases} G_1(t) = F_1(t\,;\theta_1) & \text{for } 0 < t < \tau \\ G_2(t) = F_2(t-\tau+u\,;\theta_2) & \text{for } \tau \leq t < \infty \end{cases}$$

where

$$F_k(t;\,\theta_k) = 1 - exp\left\{-\frac{t}{\theta_k}\right\} \quad \text{for } t \geq 0\;;\; \theta_k \geq 0$$

with

$$u = \frac{\theta_2}{\theta_1}\tau.$$

For exponential distributions the cumulative distribution function under step-stress will be of the form:

$$G(t) = \begin{cases} G_1(t) = 1 - exp\left\{-\frac{t}{\theta_1}\right\} & \text{for } 0 < t < \tau \\ G_2(t) = 1 - exp\left\{-\frac{\tau}{\theta_1} + \frac{\tau - t}{\theta_2}\right\} & \text{for } \tau \leq t < \infty, \end{cases}$$

with corresponding density

$$g(t) = \begin{cases} g_1(t) = \frac{1}{\theta_1} exp\left\{-\frac{t}{\theta_1}\right\} & \text{for} \quad 0 < t < \tau \\ g_2(t) = \frac{1}{\theta_2} exp\left\{-\frac{t-\tau}{\theta_2} - \frac{\tau}{\theta_1}\right\} & \text{for } \tau \le t < \infty. \end{cases}$$

Let n units be placed under stress level S_1 for a specified time τ and then stress is changed to S_2 until r failures in total occur with $r < n$.

The experiment is terminated on the occurrence of r^{th} failure/death. This type of censoring is called *type* II *censoring*.

Let n_1 be the number of failed units till time τ and total failures are fixed to r showing type II censored data, then under condition $1 \le n_1 \le r - 1$ the likelihood function is denoted as

$$L\left(\theta_1, \theta_2; t_{1,(1)}, t_{1,(2)}, ..., t_{1,(n_1)}, t_{2,(n_1+1)}, ..., t_{2,(r)}\right),$$

and the corresponding maximum likelihood estimators are

$$\hat{\theta}_1 = \left[\frac{\sum_{j=1}^{n_1} t_{1,(j)} + (n - n_1)\tau}{n_1}\right]$$

$$\hat{\theta}_2 = \left[\frac{\sum_{j=n_1+1}^{r} (t_{2,(j)} - \tau) + (n - r)(t_{2,(r)} - \tau)}{r - n_1}\right].$$

(Balakrishnan et al., 2007)

4.4 NON-PARAMETRIC ESTIMATION

According to (Shaked et al., 1979), let S_1, S_2, ..., S_k be denoting accelerated stress levels and $t_{i,l}$, $i = 1(1)k$, $l = 1(1)n_i$ are the corresponding life times under accelerated stress levels. Let F_1, F_2, ..., F_k are the corresponding Cdf's of the life times.

Let n_i be the number of units tested under stress level S_i where $i = 1(1)k$, and $N = \sum_{i=1}^{k} n_i$ is the total number of units tested.
The data will be transformed by the form $\overline{T}_i = \frac{1}{n_i}\sum_{l=1}^{n_i} t_{il}$ where $i = 1(1)k$.

The scale factor between F_i and F_j is denoted by α_{ij}, where

$$\alpha_{ij} = \left(S_i/S_j\right)^{\gamma}, i \neq j$$

$$\gamma = \ln(\alpha_{ij})/\left(\ln S_i - \ln S_j\right).$$

The estimator of α_{ij} is given by

$$\hat{\alpha}_{ij} = \overline{T}_i/\overline{T}_j \quad \text{for} \quad i \neq j,$$

and the estimate of γ can be obtained as

$$\hat{\gamma}_{ij} = \ln(\hat{\alpha}_{ij})/\left(\ln S_i - \ln S_j\right) \quad \text{for} \quad i \neq j$$

which can be written as

$$\hat{\gamma}_{ij} = \frac{\ln\left(\frac{\overline{T}_i}{\overline{T}_j}\right)}{\ln\left(\frac{S_j}{S_i}\right)} \quad \text{for} \quad i \neq j.$$

An overall estimator of γ can be obtained as weighted average of the $\hat{\gamma}_{ij}$'s:

$$\hat{\gamma} = \frac{\sum_{i=1}^{k} \sum_{j=i+1}^{k} \ln(\frac{S_j}{S_i}) \ln(\frac{\overline{T}_i}{\overline{T}_j})}{\sum_{i=1}^{k} \sum_{j=i+1}^{k} (\ln(\frac{S_j}{S_i}))^2}.$$

4.5 ACCELERATION FUNCTIONS

In non-parametric accelerated life testing, acceleration functions are considered as important modeling technique. Main aim of this analysis is to estimate the relationship between the cumulative distribution functions of life times that are tested under stress level S_1 and S_2 respectively. Linear acceleration functions and power type acceleration functions are commonly used in accelerated life testing (Viertl, 1988).

4.6 LINEAR ACCELERATION FUNCTIONS

For life time observations which are tested under different stress levels a linear acceleration function can be simply defined as

$$a(t) = \alpha \cdot t \qquad \forall\, t \geq 0.$$

This means for the relationship between the corresponding cdf's

$$F(t|S_2) = F(\alpha_{1,2} \cdot t|S_1) \qquad \forall\, t \geq 0.$$

For ordered life time observations of two samples, i.e. $t_{1,(1)}, t_{1,(2)}, ..., t_{1,(n)}$ and $t_{2,(1)}, t_{2,(2)}, ..., t_{2,(n)}$ obtained under two stress levels $S_1 < S_2$ respectively, the relative acceleration coefficient $\alpha_{1,2}$ can be estimated by

$$\widehat{\ln\alpha_{1,2}} = \frac{1}{m}\sum_{i=1}^{m} \ln\frac{t_{1,(i)}}{t_{2,(i)}}$$

where $m \leq n$.

4.7 POWER TYPE ACCELERATION FUNCTIONS

The relationship between the cumulative distribution functions of life times under different stress levels through a power type acceleration function can be written as

$$F(t|S_2) = F(\alpha_{1,2} \,.\, t^{\beta_{1,2}}|S_1) \qquad \forall\, t \geq 0, \tag{4.1}$$

where T_i denotes the random variable describing life time under stress level S_i. The mentioned relationship can be estimated based on the above stochastic equation (4.1).

Let $t_{1,(1)}, ..., t_{1,(n)}$ and $t_{2,(1)}\,, ..., t_{2,(n)}$ be independent ordered samples under stress S_1 and S_2 respectively.

For the parameters $\alpha_{1,2}$ and $\beta_{1,2}$ the corresponding estimators given in (Viertl, 1988) are

$$\hat{\beta}_{1,2} = \frac{1}{m}\sum_{k=1}^{m} \frac{\ln t_{1,(k+1)} - \ln t_{1,(k)}}{\ln t_{2,(k+1)} - \ln t_{2,(k)}}$$

and

$$\widehat{\ln \alpha_{1,2}} = \frac{1}{m} \sum_{k=1}^{m} \left[\ln t_{1,(k)} - \hat{\beta}_{1,2} \ln t_{2,(k)} \right]$$

where $m \leq n$.

5

LIFE TIME ANALYSIS AND FUZZY DATA

Measurement actions are performed for well-defined purposes. In general people measure to know. This requires special attention to record the information given by measurements, and it has to be recorded in intervals to avoid misunderstanding.

In the technologic use of measurement instruments it is evident how the concept like "exactness" or "equality" needs to be banned. Even on the measurement obtained from a high quality instrument one can only believe that it is exact, but the characteristic exact cannot be obtained in reality. This can be confirmed practically by repeating measurement of some continuous phenomenon (Barbato et al., 2013).

These arguments also support the idea explained in (Viertl, 2006), that in fact there are two types of uncertainty in the measurements, one is variation among the observations and another is fuzziness.

From the centuries classical statistical tools are developed to model variation among the observations, without considering the imprecision of single observations.

Classical statistics (Stochastic models) consider only variation among the precise observations and ignores the fuzziness. By doing so we may lose information and get misleading results (Viertl, 2011).

From the centuries life time analyses are used to model life time data in efficient way. In (Viertl, 2009), it has been shown that life time observations are not precise numbers but fuzzy, therefore life time analysis techniques should be generalized to deal with fuzziness of the observations.

5.1 GENERALIZED KAPLAN-MEIER ESTIMATOR

The Kaplan-Meier estimator is one of the popular methods for precise survival times. It is natural that life time is of continuous nature, therefore it is unrealistic to deal life time observations as precise numbers. Consequently, fuzzy numbers are more suitable and realistic to describe real survival times.

For fuzzy life time observations t_1^*, t_2^*, ..., t_n^* having δ-cuts $C_\delta(t_i^*) = [\underline{t}_{i,\delta}\,,\, \overline{t}_{i,\delta}] \qquad \forall \delta \in (0\,,\,1]$ and $i = 1(1)n$, the generalized Kaplan-Meier estimator is denoted by $S^*(t)$ and its upper and lower δ-level curves are obtained in the following way:

$$C_\delta(S^*(t)) = \left[\min_{\underline{t} \in \times_{i=1}^n C_\delta(t_i^*)} S(\underline{t})\,,\, \max_{\underline{t} \in \times_{i=1}^n C_\delta(t_i^*)} S(\underline{t}) \right] \quad \forall\, \delta \in (0\,,\,1]$$

with $\underline{t} = (t_1, t_2, ..., t_n) \in [0\,,\,\infty)^n \quad \forall\, \delta \in (0\,,\,1]$.

Where $\underline{S}(t) = \min_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} S(\underline{t})$ is the lower end of the δ-cut which defines the lower δ-level curve and $\overline{S}(t) = \max_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} S(\underline{t})$ is the upper end of the δ-cut which defines the upper δ-level curve.

The above mathematical calculations are realized through the following algorithm:

1. The values for δ are taken from 0 to 1 with an increment $\Delta \in (0, 1)$.

2. For a given value of δ calculate the δ-cuts of the fuzzy combined sample $\underline{t}^*$.

3. Taking minimum and maximum from the δ-cuts to generate hypothetical classical samples.

4. Kaplan-Meier survival probabilities are calculated and the Kaplan-Meier survival curves are drawn for fixed δ-level.

5. Steps 2-4 are performed for each $\delta = 0(\Delta)1$.

Figure 1: Characterizing functions of a fuzzy life times sample

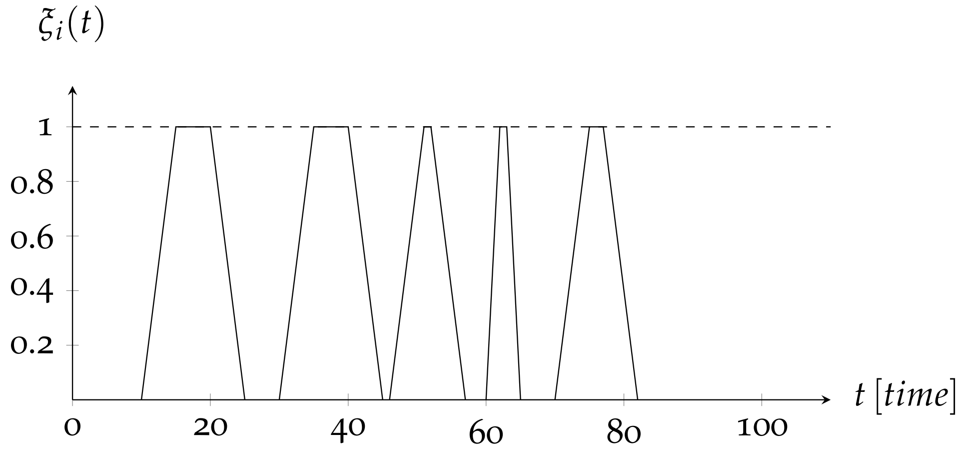
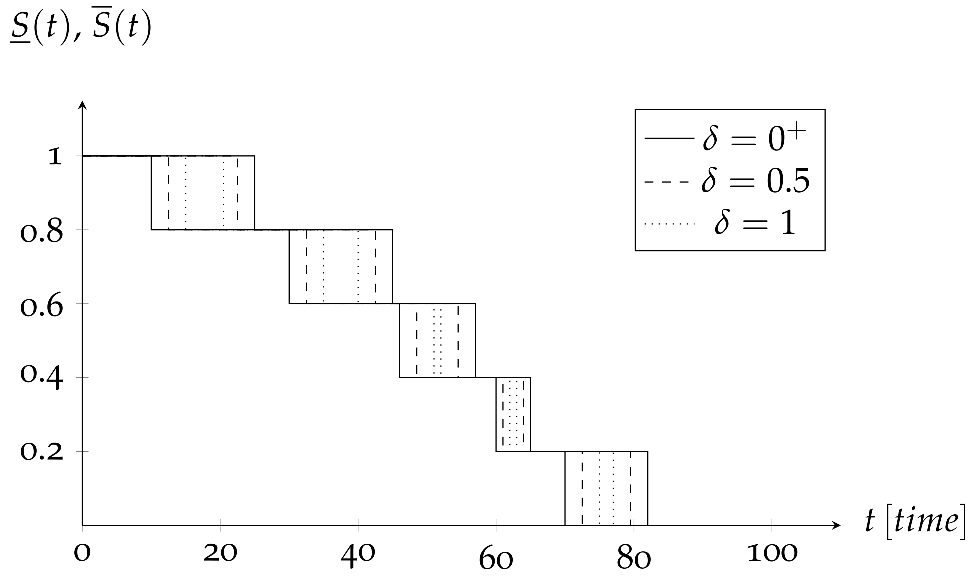


Figure 2: Some lower and upper δ-level curves of the Generalized Kaplan-Meier Estimator



The curves for δ show the boundaries of the supports of the corresponding characterizing functions.

The generalized Kaplan-Meier estimator incorporates fuzziness of the observations, therefore the results based on the generalized Kaplan-Meier estimator are more suitable and realistic.

5.2 Fuzzy maximum likelihood estimators of the two parameter Weibull distribution for fuzzy censored data

The two parameter Weibull distribution is considered as the most important distribution for modeling life time data. Therefore, maximum likelihood estimators need to be generalized for fuzzy observations $t_1^*, t_2^*, ..., t_n^*$. Fuzzy maximum likelihood estimators are denoted by $\hat{\tau}^*$ and $\hat{\beta}^*$.

Where $\hat{\tau}$ and $\hat{\beta}$ are the solutions of the equations mentioned in section 3.1 and 3.2 respectively.

The generating family of intervals $(B_\delta; \quad \delta \in (0, 1])$ for the fuzzy estimator $\hat{\beta}^*$ is obtained by

$$B_\delta\left(\hat{\beta^*}\right) = \left[\min_{\underline{t} \in \times_{i=1}^n C_\delta(t_i^*)} \hat{\beta}\, , \max_{\underline{t} \in \times_{i=1}^n C_\delta(t_i^*)} \hat{\beta}\right].$$

In a similar way the generating family of intervals $(A_\delta;\ \delta \in (0, 1])$ for the fuzzy estimate $\hat{\tau}^*$ can be obtained:

$$A_\delta\left(\hat{\tau^*}\right) = \left[\min_{\underline{t} \in \times_{i=1}^n C_\delta(t_i^*)} \hat{\tau}\, , \max_{\underline{t} \in \times_{i=1}^n C_\delta(t_i^*)} \hat{\tau}\right].$$

with $\left(\underline{t} = t_{(1)}, t_{(2)}, ..., t_{(r)}\right)$.

The characterizing functions $\phi(\cdot)$ of $\hat{\tau}^*$ and $\psi(\cdot)$ of $\hat{\beta}^*$ are given by the construction lemma in the following way:

$$\phi(\tau) = sup\left\{\delta \cdot \mathbb{1}_{A_\delta}(\tau) \,:\, \delta \in [0, 1]\right\} \qquad \forall\, \tau \in [0, \infty)$$

$$\psi(\beta) = sup\left\{\delta\cdot\mathbb{1}_{B_\delta}(\beta)\ :\ \delta \in [0\,,1]\right\} \qquad \forall\,\beta \in [0\,,\infty)$$

The above mathematical calculations can be done approximately by the following algorithm:

1. The values for δ are taken from 0 to 1 with an increment $\triangle \in (0\,,1)$.

2. For a given value of δ all δ-cuts of the fuzzy observations are determined.

3. Taking values from the δ-cuts to get hypothetical classical samples.

4. From these hypothetical classical samples at a given level δ, calculate the classical estimates.

5. In order to construct the generalized (fuzzy) estimators take minimum and maximum values from these estimates and consider it as the end points of the family (A_δ and B_δ ; $\delta \in (0\,,1]$) of generating intervals A_δ and B_δ for the characterizing functions of the fuzzy estimators at a given level of δ.

6. Steps 2-5 are performed for each estimator for $\delta = 0\,(\triangle)\,1$.

7. From all these generating intervals A_δ and B_δ obtained for each δ (i.e. $\delta = 0\,(\triangle)\,1$) through the above mentioned Construction Lemma the characterizing functions of the fuzzy estimates of the parameters are obtained approximately.

Figure 3: Sample of Type I censoring with fuzzy censoring time

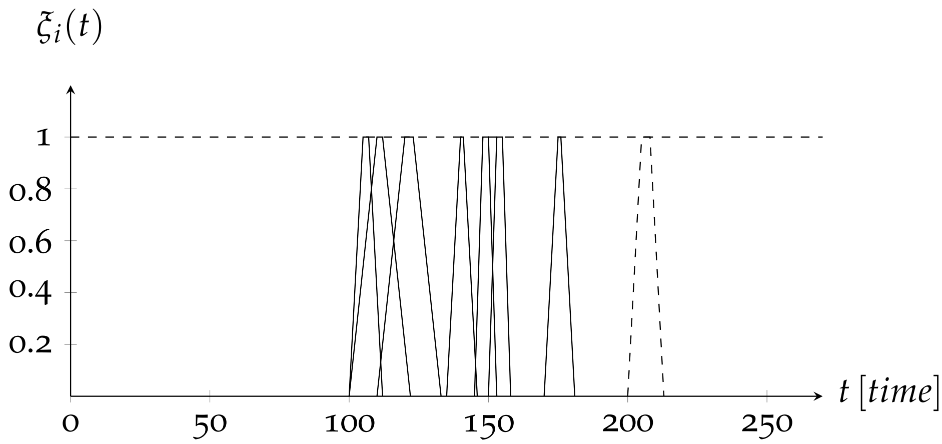


Figure 4: Characterizing function of $\hat{\tau}^*$ for type I censoring data with fuzzy censoring time from figure 3

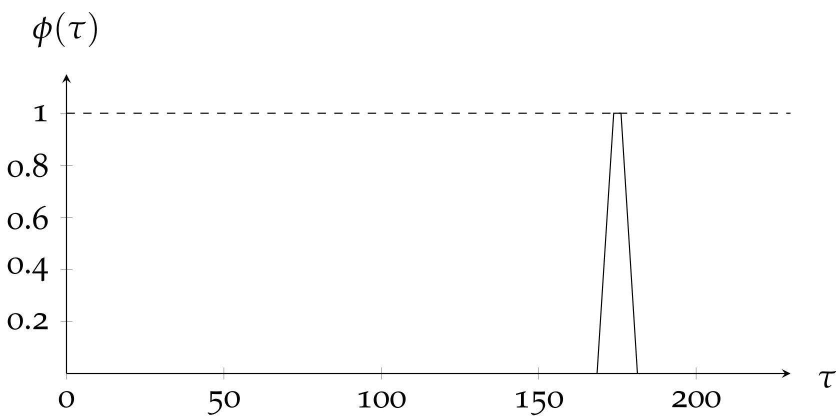
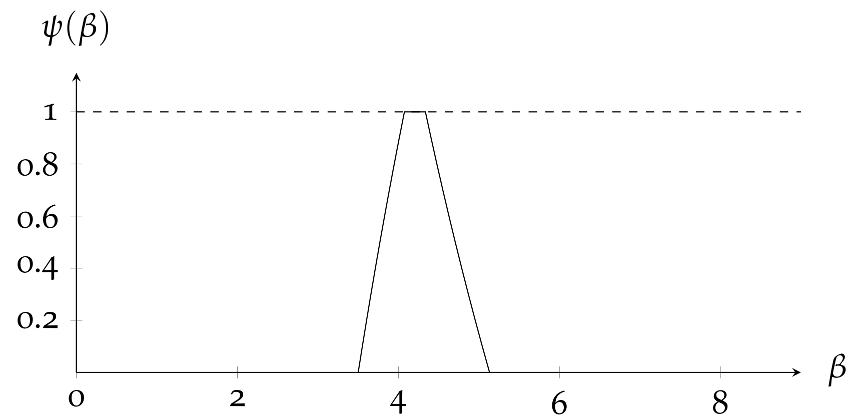


Figure 5: Characterizing function of $\hat{\beta}^*$ for type I censoring with fuzzy censoring time from figure 3



Next the termination time of the experiment is assumed to be a precise number. Given below in figure 6 is a sample of 8 units with 7 failures and 1 censored observation, with precise censoring time. Figure 7 and figure 8 show the characterizing functions of the fuzzy estimates of the Weibull parameters based on fuzzy failure times and precise censoring time from figure 6.

Figure 6: Sample of Type I censoring with precise censoring time

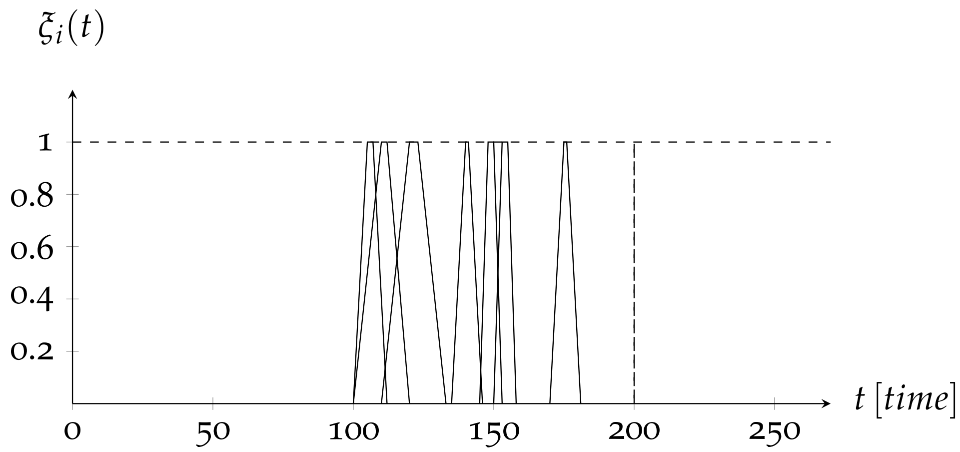


Figure 7: Characterizing function of $\hat{\tau}^*$ for type I censoring data from figure 6 with precise censoring time

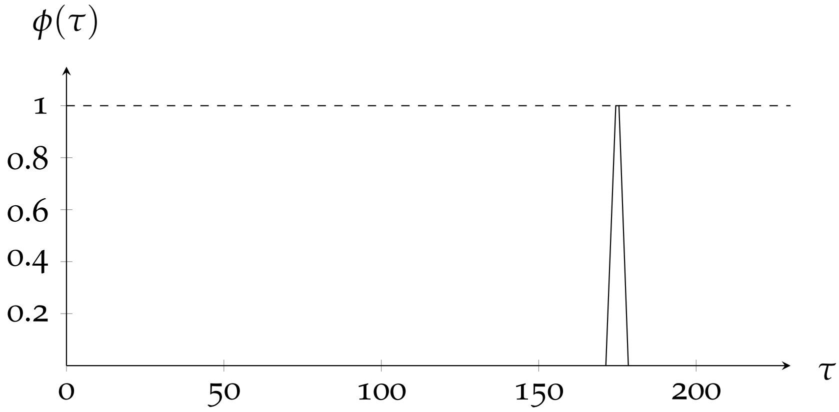
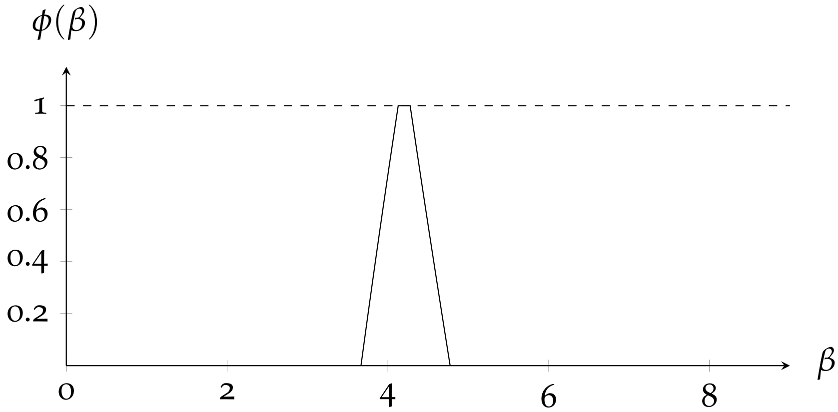


Figure 8: Characterizing function of $\hat{\beta}^*$ for type I censoring with precise censoring time from figure 6



For type II censoring considering 8 fuzzy failure times and 2 censored observations are given in figure 9. The characterizing functions of the corresponding fuzzy estimates are given in figure 10 and figure 11.

Figure 9: Sample of Type II censoring with fuzzy failure times

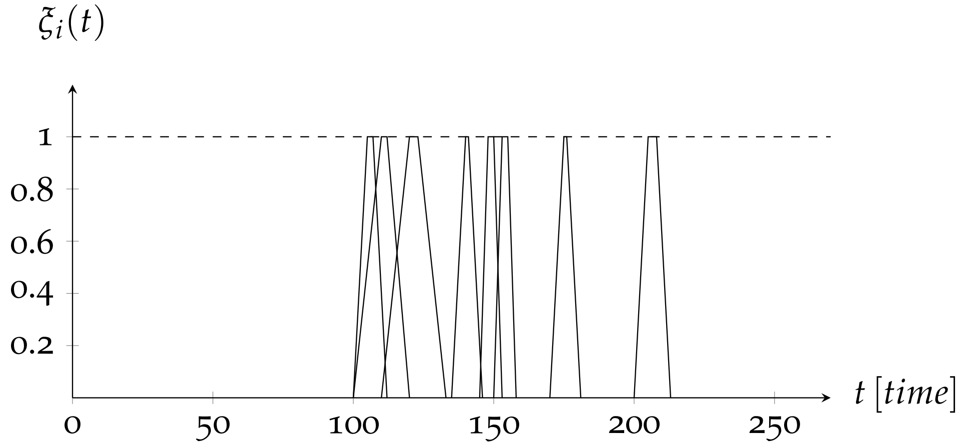


Figure 10: Characterizing function of $\hat{\tau}^*$ for type II censoring with fuzzy failure times from figure 9

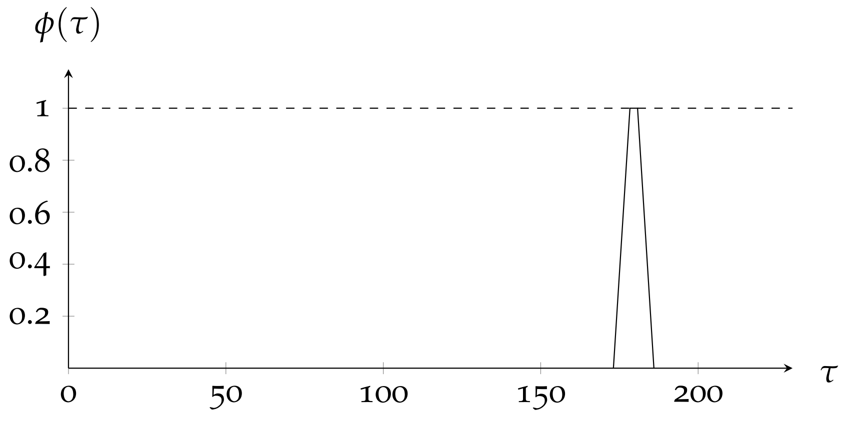
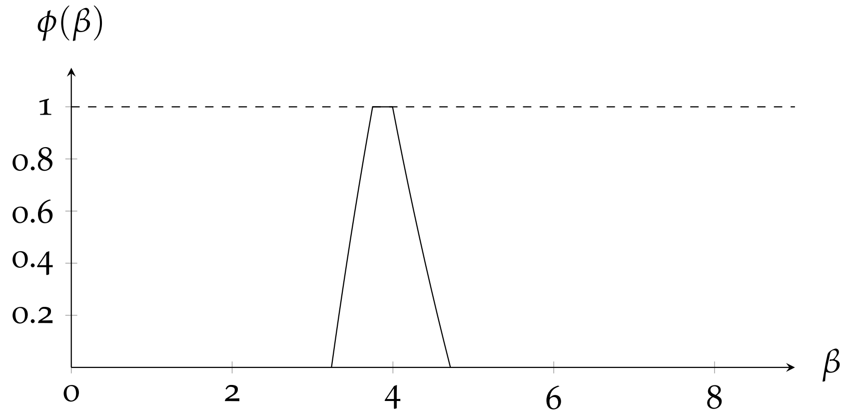


Figure 11: Characterizing function of $\hat{\beta}^*$ for type II censoring with fuzzy failure times from figure 9



5.3 GENERALIZED ESTIMATION FOR EXPONENTIAL DISTRIBUTION

Based on fuzzy life time observations t_1^*, t_2^*, ..., t_n^* the generalized (fuzzy) estimates for the parameter, survival function, and hazard rate of the exponential distribution are denoted by $\hat{\lambda}^*$, $\hat{S}^*(t)$, and $\hat{h}^*(t)$ respectively. The δ-cuts of the parameter estimate can be written as

$$C_\delta[\hat{\lambda}^*] = [\underline{\lambda}_\delta \, , \, \overline{\lambda}_\delta] \qquad \forall\, \delta \in (0\, , 1],$$

and the δ-level curves for the survival function, and hazard rate are defined as

$$C_\delta[\hat{S}^*(t)] = \left[\underline{S}_\delta(t)\, , \, \overline{S}_\delta(t)\right] \qquad \forall\, \delta \in (0\, , 1],$$

and

$$C_\delta[\hat{h}^*(t)] = \left[\underline{h}_\delta(t)\, , \, \overline{h}_\delta(t)\right] \qquad \forall\, \delta \in (0\, , 1],$$

respectively.

These δ-cuts and δ-level curves for the exponential distribution can be formulated for fuzzy observations $(t_1^*, t_2^*, ..., t_n^*)$ having δ-cuts $C_\delta[t_i^*] = \left[\underline{t}_{i,\delta}\,,\ \overline{t}_{i,\delta}\right] \quad \forall\delta \in (0\,,1]$ as:

$$C_\delta[\hat{\lambda}^*] = \left[\frac{1}{\frac{\sum_{i=1}^n \overline{t}_{i,\delta}}{n}}\,,\ \frac{1}{\frac{\sum_{i=1}^n \underline{t}_{i,\delta}}{n}}\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta[\hat{S}^*(t)] = \left[e^{-\overline{\lambda}_\delta t}\,,\ e^{-\underline{\lambda}_\delta t}\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta[\hat{h}^*(t)] = \left[\underline{\lambda}_\delta\,,\ \overline{\lambda}_\delta\right] \qquad \forall\,\delta \in (0\,,1]$$

The characterizing functions can be constructed by the mentioned Construction lemma.

Characterizing functions of fuzzy life time observations and the fuzzy parameter estimator are given in figure 12 and figure 13 respectively.

Some δ-level curves of the fuzzy estimated survival function and fuzzy estimate of the hazard rate are given in figure 14 and figure 15.

Figure 12: Characterizing functions of the fuzzy life times sample

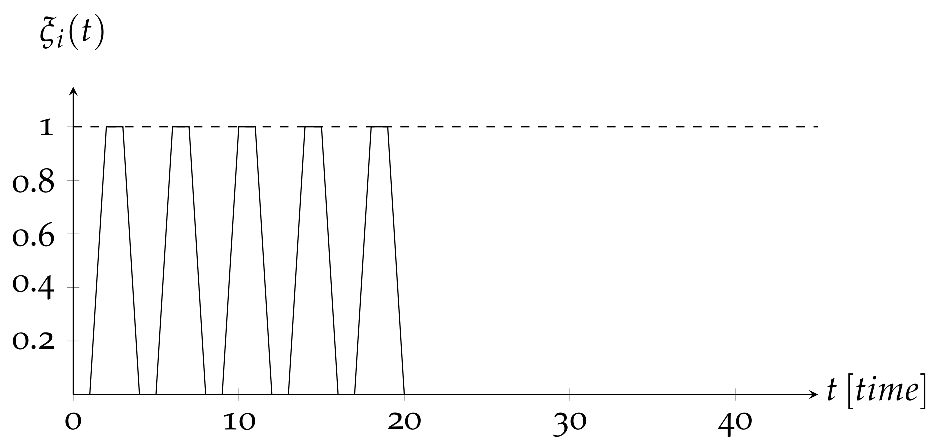


Figure 13: Characterizing function of the fuzzy estimator $\hat{\lambda}^*$

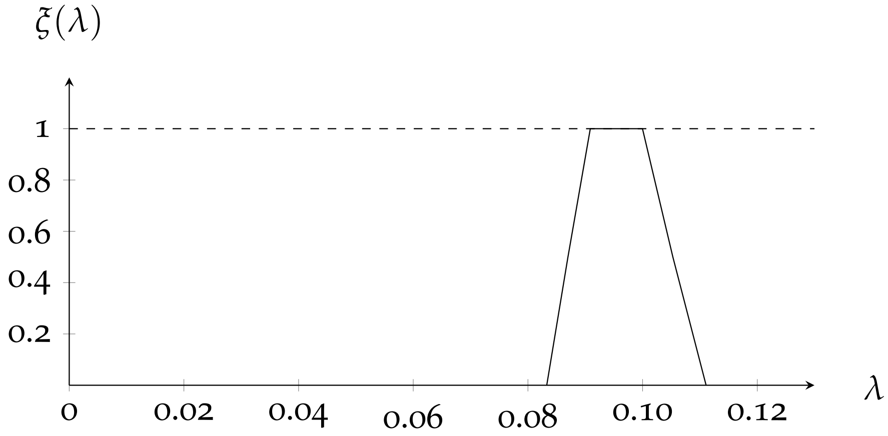
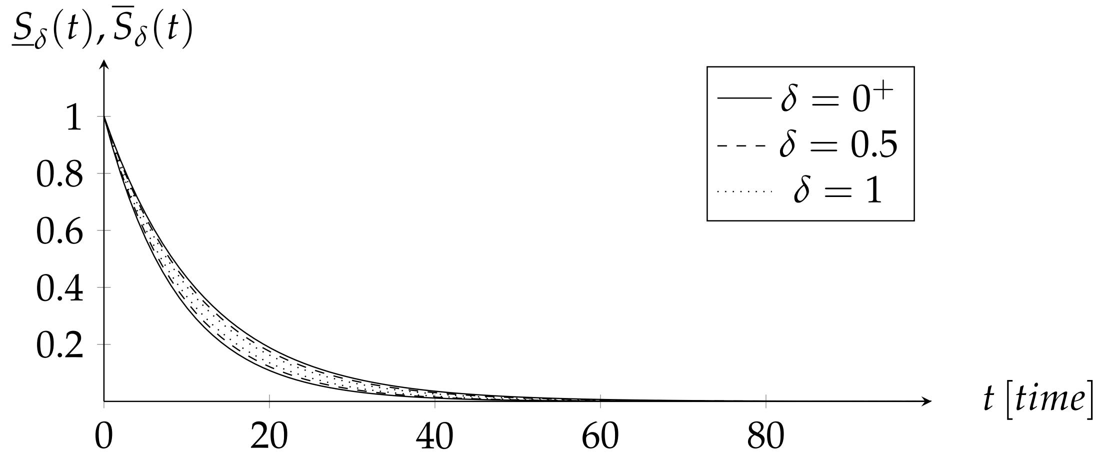
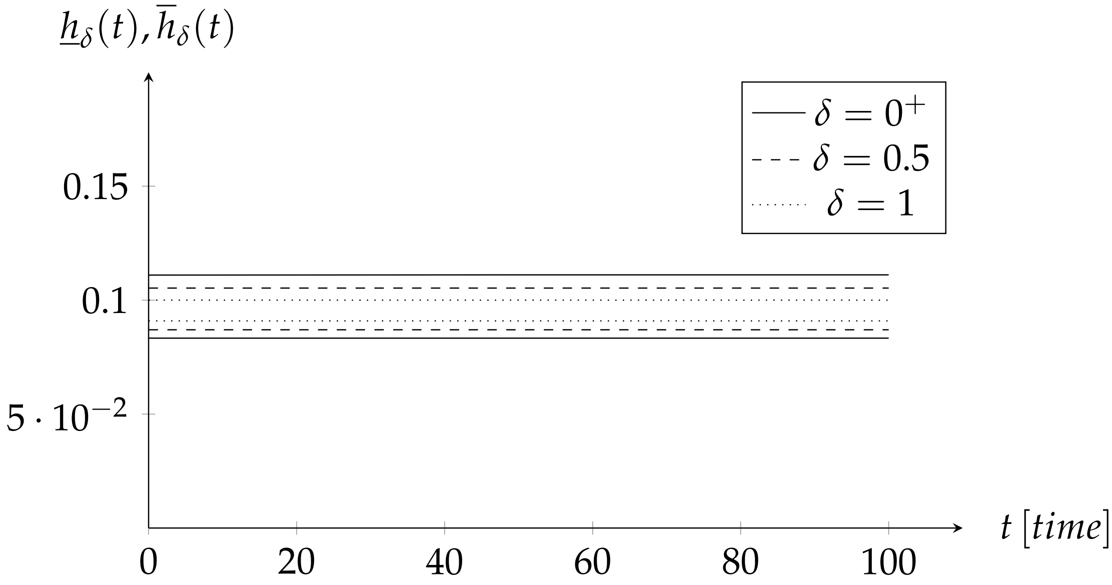


Figure 14: Some Upper and Lower δ-level curves of the fuzzy estimate of the survival function



The curves for $\delta = 0^+$ show the boundaries of the supports of the corresponding characterizing functions.

Figure 15: Some Upper and Lower δ-level curves of the fuzzy estimate of the hazard rate



5.4 GENERALIZED ESTIMATION FOR WEIBULL DISTRIBUTIONS

For the parameter estimation of Weibull distributions the generalized (fuzzy) estimators for the parameters, survival function, and the hazard rate based on fuzzy life times t_1^*, t_2^*, ..., t_n^* are denoted as $\hat{\tau}^*$, $\hat{\beta}^*$, $\hat{S}^*(t)$, and $\hat{h}^*(t)$ respectively.

The δ-cuts of $\hat{\tau}^*$ and $\hat{\beta}^*$, and δ-level curves of $\hat{S}^*(t)$ and $\hat{h}^*(t)$ are the following:

$$C_\delta\left[\hat{\tau}^*\right] = \left[\underline{\tau}_\delta\,,\ \overline{\tau}_\delta\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta\left[\hat{\beta}^*\right] = \left[\underline{\beta}_\delta\,,\ \overline{\beta}_\delta\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta\left[\hat{S}^*(t)\right] = \left[\underline{S}_\delta(t)\,,\ \overline{S}_\delta(t)\right] \qquad \forall\,\delta \in (0\,,1]$$

and

$$C_\delta\left[\hat{h}^*(t)\right] = \left[\underline{h}_\delta(t)\,,\ \overline{h}_\delta(t)\right] \qquad \forall\,\delta \in (0\,,1]$$

respectively.

These can be formulated by theorem 2 from section 1.11 in the following way:

$$C_\delta\left[\hat{\beta}^*\right] = \left[\min_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\beta}\,,\ \max_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\beta}\right] \qquad \forall\,\delta \in (0\,,\,1]$$

$$C_\delta\left[\hat{\tau}^*\right] = \left[\left\{\frac{1}{n}\sum_{i=1}^{n} \underline{t}_{i,\delta}^{\underline{\beta}_\delta}\right\}^{\frac{1}{\overline{\beta}_\delta}}\,,\ \left\{\frac{1}{n}\sum_{i=1}^{n} \overline{t}_{i,\delta}^{\overline{\beta}_\delta}\right\}^{\frac{1}{\underline{\beta}_\delta}}\right] \qquad \forall\,\delta \in (0\,,\,1]$$

$$C_\delta\left[\hat{S}^*(t)\right] = \left[e^{-(\frac{t}{\underline{\tau}_\delta})^{\overline{\beta}_\delta}}\,,\ e^{-(\frac{t}{\overline{\tau}_\delta})^{\underline{\beta}_\delta}}\right] \qquad \forall\,\delta \in (0\,,\,1]$$

$$C_\delta\left[\hat{h}^*(t)\right] = \left[\frac{\underline{\beta}_\delta t^{\underline{\beta}_\delta - 1}}{\overline{\tau}_\delta^{\overline{\beta}_\delta}}\,,\ \frac{\overline{\beta}_\delta t^{\overline{\beta}_\delta - 1}}{\underline{\tau}_\delta^{\underline{\beta}_\delta}}\right] \qquad \forall\,\delta \in (0\,,\,1]$$

The characterizing functions can be constructed from the above mentioned Construction lemma.

Figure 16: Characterizing function of the fuzzy estimator $\hat{\tau}^*$ from the sample in figure 12

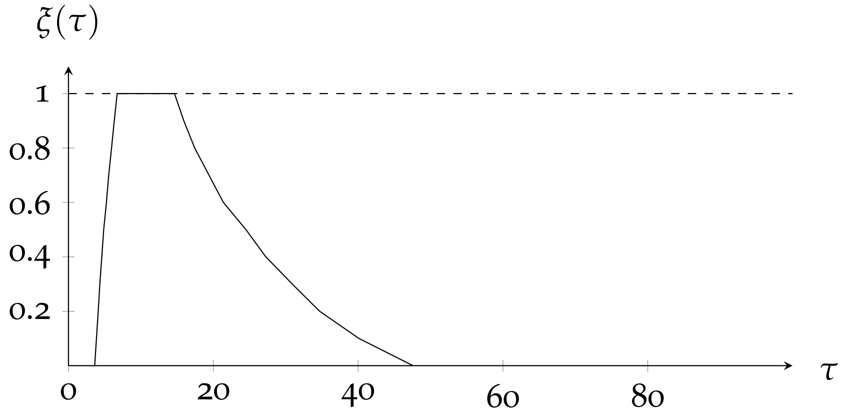


Figure 17: Characterizing function of the fuzzy estimator $\hat{\beta}^*$ from the sample in figure 12

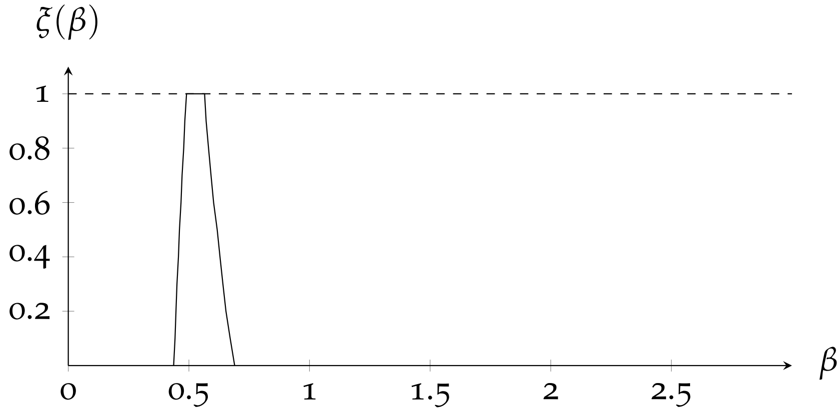


Figure 18: Some Upper and Lower δ-level curves of the fuzzy estimate of the survival function

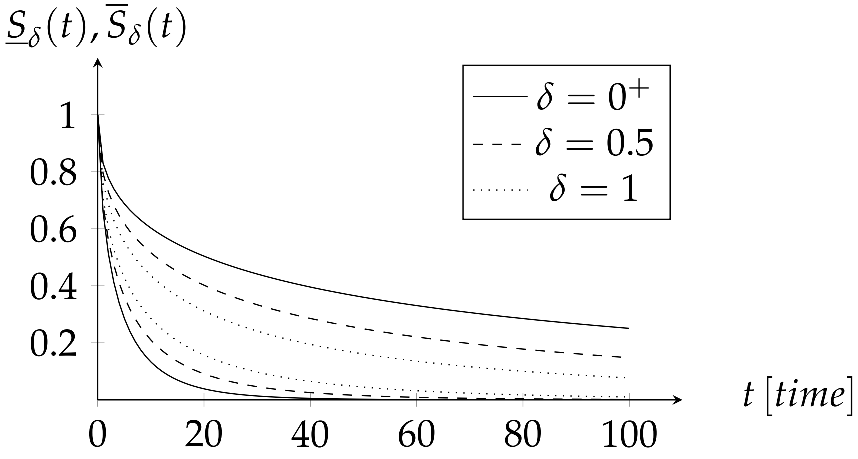
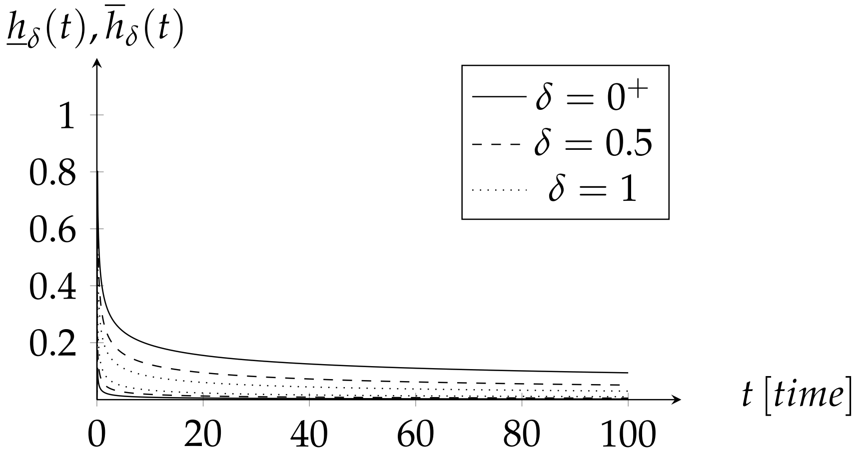


Figure 19: Upper and Lower δ-level curves of the fuzzy estimate for the hazard rate



5.5 GENERALIZED ESTIMATION FOR GAMMA DISTRIBUTIONS

Gamma distribution explained in section 3.9 is based on precise life time observations. Therefore the estimators $\hat{\lambda}$ and $\hat{\gamma}$ for parameters, survival function and hazard rate mentioned in section 3.9 need to be generalized for fuzzy life times.

The generalized (fuzzy) estimators for parameters, survival function, and hazard rate of the gamma distribution based on fuzzy life times can be written as $\hat{\lambda}^*$, $\hat{\gamma}^*$, $\hat{S}^*(t)$, and $\hat{h}^*(t)$ respectively.

The δ-cuts of the estimators and δ-level curves of the fuzzy estimates of the survival function and hazard rate are denoted by

$$C_\delta\left[\hat{\lambda}^*\right] = \left[\underline{\lambda}_\delta\,,\,\overline{\lambda}_\delta\right] \qquad \forall\,\delta \in (0\,,1],$$

$$C_\delta\left[\hat{\gamma}^*\right] = \left[\underline{\gamma}_\delta\,,\,\overline{\gamma}_\delta\right] \qquad \forall\,\delta \in (0\,,1],$$

$$C_\delta\left[\hat{S}^*(t)\right] = \left[\underline{S}_\delta(t)\,,\,\overline{S}_\delta(t)\right] \qquad \forall\,\delta \in (0\,,1],$$

and

$$C_\delta\left[\hat{h}^*(t)\right] = \left[\underline{h}_\delta(t)\,,\,\overline{h}_\delta(t)\right] \qquad \forall\,\delta \in (0\,,1]$$

respectively.

Using the notation $\underline{t} = (t_1, t_2, \ldots, t_n)$ the above δ-cuts of the fuzzy estimates are formulated as

$$C_\delta\left[\hat{\lambda^*}\right] = \left[\min_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\lambda}\,,\ \max_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\lambda}\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta\left[\hat{\gamma}^*\right] = \left[\min_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\gamma}\ ,\ \max_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\gamma}\right] \qquad \forall\,\delta \in (0\,,\,1]$$

$$C_\delta\left[\hat{f}^*(t)\right] = \left[f(t|\overline{\lambda}_\delta,\,\overline{\gamma}_\delta)\ ,\ f(t|\underline{\lambda}_\delta,\,\underline{\gamma}_\delta)\right] \qquad \forall\,\delta \in (0\,,\,1]$$

$$C_\delta\left[\hat{F}^*(t)\right] = \left[F(t|\overline{\lambda}_\delta,\,\overline{\gamma}_\delta)\ ,\ F(t|\underline{\lambda}_\delta,\,\underline{\gamma}_\delta)\right] \qquad \forall\,\delta \in (0\,,\,1].$$

The lower and upper δ-level curves of the fuzzy estimates of the survival function can be defined as

$$\underline{S}_\delta(t) = 1 - \left[F(t|\underline{\lambda}_\delta,\,\underline{\gamma}_\delta)\right] \qquad \forall\,\delta \in (0\,,\,1]$$

$$\overline{S}_\delta(t) = 1 - \left[F(t|\overline{\lambda}_\delta,\,\overline{\gamma}_\delta)\right] \qquad \forall\,\delta \in (0\,,\,1].$$

The lower and upper δ-level curves of the fuzzy estimates of the hazard rate are defined as

$$\underline{h}_\delta(t) = \underline{f}_\delta(t)/\overline{S}_\delta(t) \qquad \forall\,\delta \in (0\,,\,1]$$

$$\overline{h}_\delta(t) = \overline{f}_\delta(t)/\underline{S}_\delta(t) \qquad \forall\,\delta \in (0\,,\,1].$$

Characterizing functions of fuzzy life time observations and fuzzy estimates of the Gamma distribution are depicted in figures 20-24.

Figure 20: Characterizing functions of the fuzzy life times sample

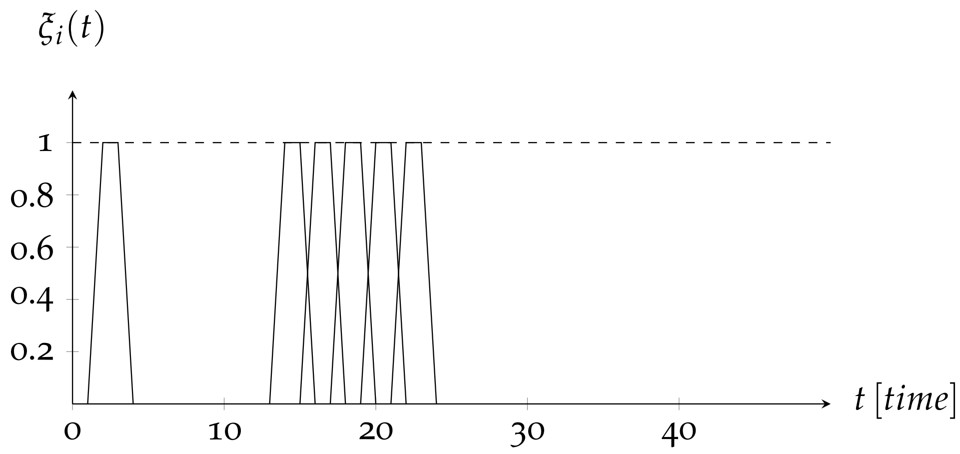


Figure 21: Characterizing function of the fuzzy estimator $\hat{\lambda}^*$ based on the fuzzy sample shown in figure 9

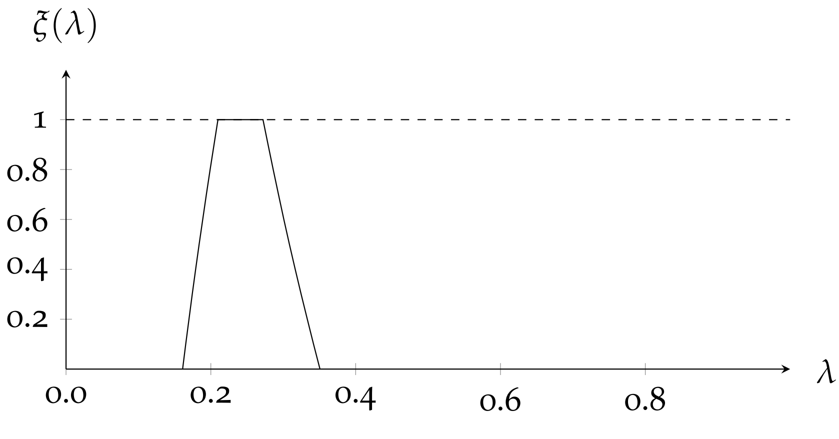


Figure 22: Characterizing function of the fuzzy estimator $\hat{\gamma}^*$ based on the fuzzy sample shown in figure 9

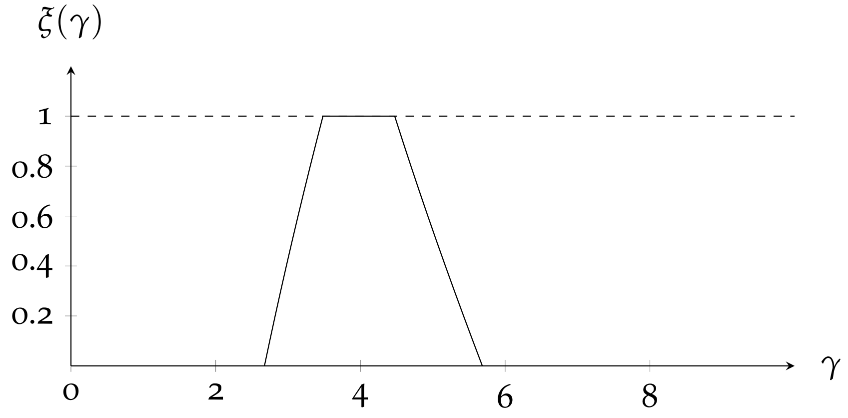


Figure 23: Upper and Lower δ-level curves of the fuzzy estimate of the survival function

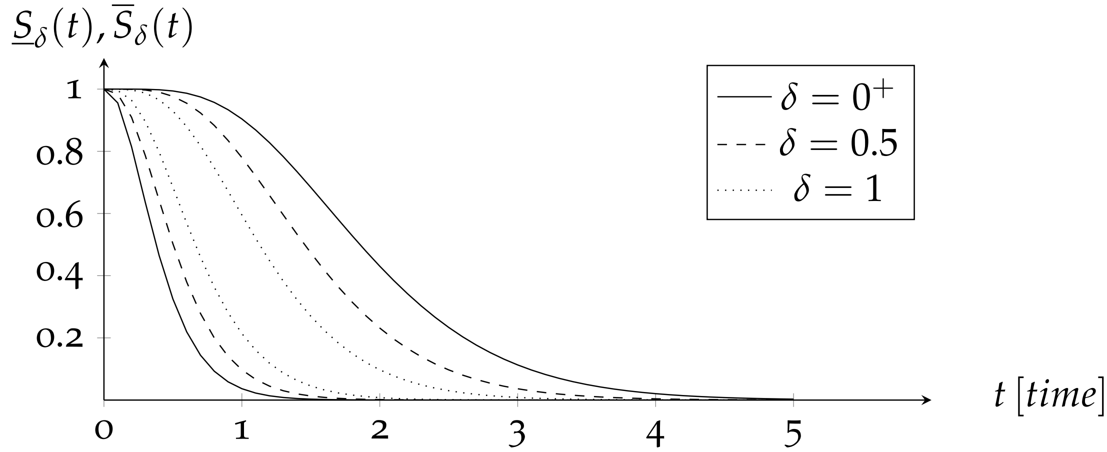
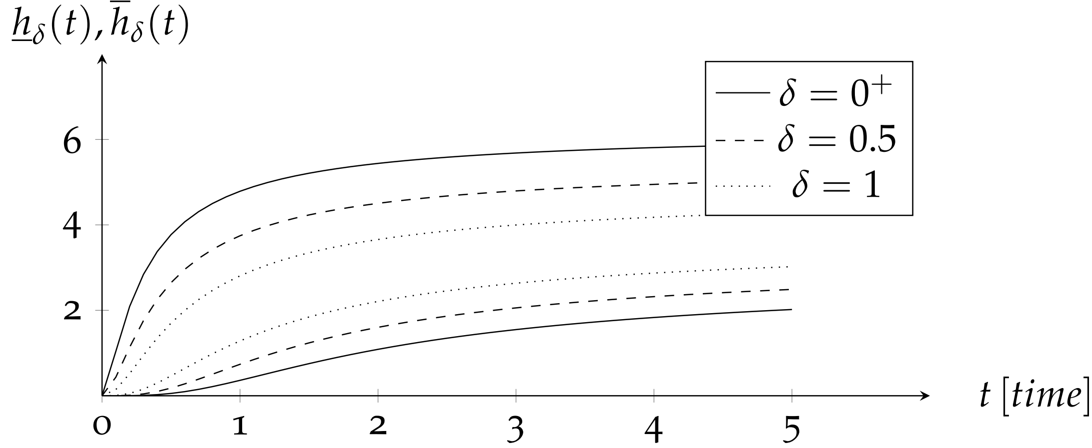


Figure 24: Upper and Lower δ-level curves of the fuzzy estimate of the hazard rate



5.6 GENERALIZED ESTIMATION FOR LOGNORMAL DISTRIBUTIONS

Lognormal distribution, parameter estimators, estimators of survival function and hazard rate explained in section 3.10 are based on precise life time observations. The explained estimates need to be generalized for fuzzy life times.

The generalized estimators for the parameters, survival function, and hazard rate of the lognormal distribution based on fuzzy life times are denoted as $\hat{\mu}^*$, $\hat{s}^*$, $\hat{S}^*(t)$ and $\hat{h}^*(t)$ respectively.

The δ-cuts of the estimators and δ-level curves of the fuzzy estimates of the survival function and hazard rate can be written as

$$C_\delta \left[\hat{\mu}^*\right] = \left[\underline{\mu}_\delta \, , \, \overline{\mu}_\delta\right] \qquad \forall \, \delta \in (0\, , 1]$$

$$C_\delta \left[\hat{s}^*\right] = \left[\underline{s}_\delta \, , \, \overline{s}_\delta\right] \qquad \forall \, \delta \in (0\, , 1]$$

$$C_\delta \left[\hat{S}^*(t)\right] = \left[\underline{S}_\delta(t) \, , \, \overline{S}_\delta(t)\right] \qquad \forall \, \delta \in (0\, , 1]$$

and

$$C_\delta\left[\hat{h}^*(t)\right] = \left[\underline{h}_\delta(t)\,,\,\overline{h}_\delta(t)\right] \qquad \forall\,\delta \in (0\,,1]$$

respectively. These can be formulated by using theorem 2, section 1.11:

$$C_\delta\left[\hat{\mu}^*\right] = \left[\min_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\mu}\,,\ \max_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{\mu}\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta\left[\hat{s}^*\right] = \left[\min_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{s}\,,\ \max_{\underline{t}\,\in\,\times_{i=1}^{n} C_\delta(t_i^*)} \hat{s}\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta\left[\hat{f}^*(t)\right] = \left[f(t|\underline{\mu}_\delta, \underline{s}_\delta)\,,\ f(t|\overline{\mu}_\delta, \overline{s}_\delta)\right] \qquad \forall\,\delta \in (0\,,1]$$

$$C_\delta\left[\hat{F}^*(t)\right] = \left[F(t|\underline{\mu}_\delta, \underline{s}_\delta)\,,\ F(t|\overline{\mu}_\delta, \overline{s}_\delta)\right] \qquad \forall\,\delta \in (0\,,1]$$

The lower and upper δ-level curves of the fuzzy estimates of the survival function are defined as

$$\underline{S}_\delta(t) = 1 - \left[F(t|\overline{\mu}_\delta, \overline{s}_\delta)\right] \qquad \forall\,\delta \in (0\,,1]$$

$$\overline{S}_\delta(t) = 1 - \left[F(t|\underline{\mu}_\delta, \underline{s}_\delta)\right] \qquad \forall\,\delta \in (0\,,1].$$

The lower and upper δ-level curves of the fuzzy estimate of the hazard rate are defined by

$$\underline{h}_\delta(t) = \underline{f}_\delta(t)/\overline{S}_\delta(t) \qquad \forall\,\delta \in (0\,,1]$$

$$\overline{h}_\delta(t) = \overline{f}_\delta(t)/\underline{S}_\delta(t) \qquad \forall\,\delta \in (0\,,1].$$

Figure 25: Characterizing function of the fuzzy estimator $\hat{\mu}^*$ from the fuzzy sample shown in figure 20

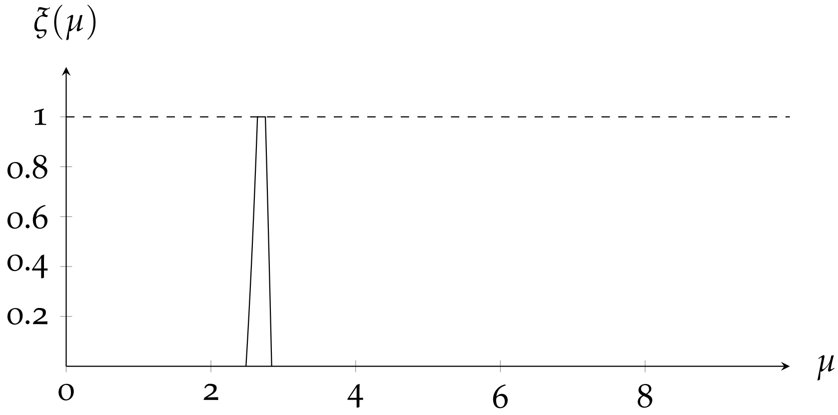


Figure 26: Characterizing function of the fuzzy estimator $\hat{s}^*$ based on the fuzzy sample shown in figure 20

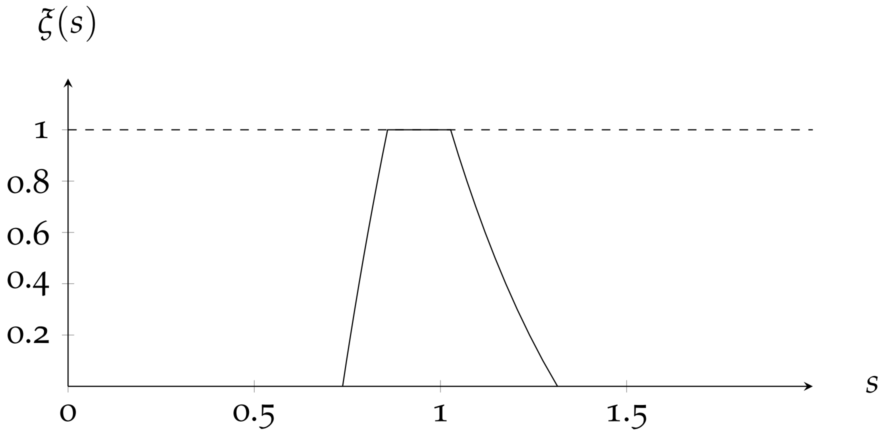


Figure 27: Upper and Lower δ-level curves of the fuzzy estimate of the survival function

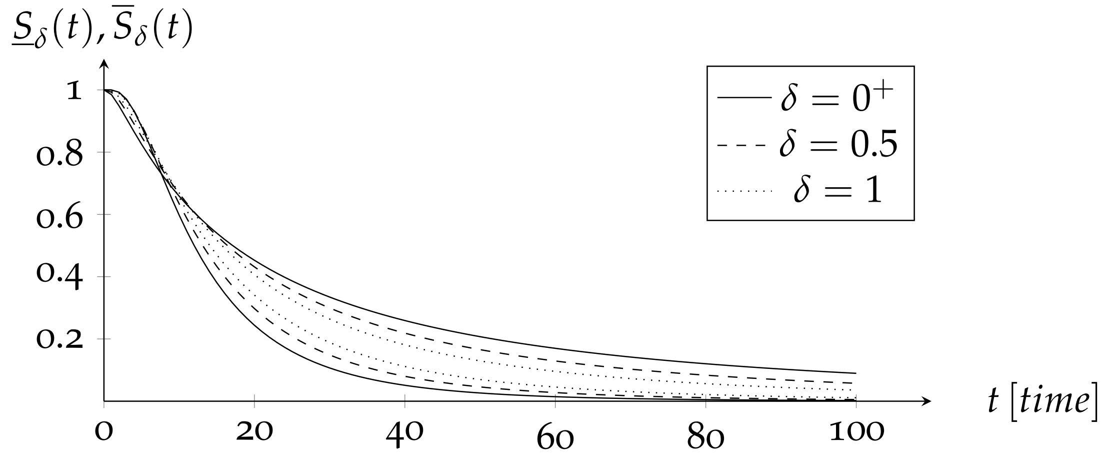
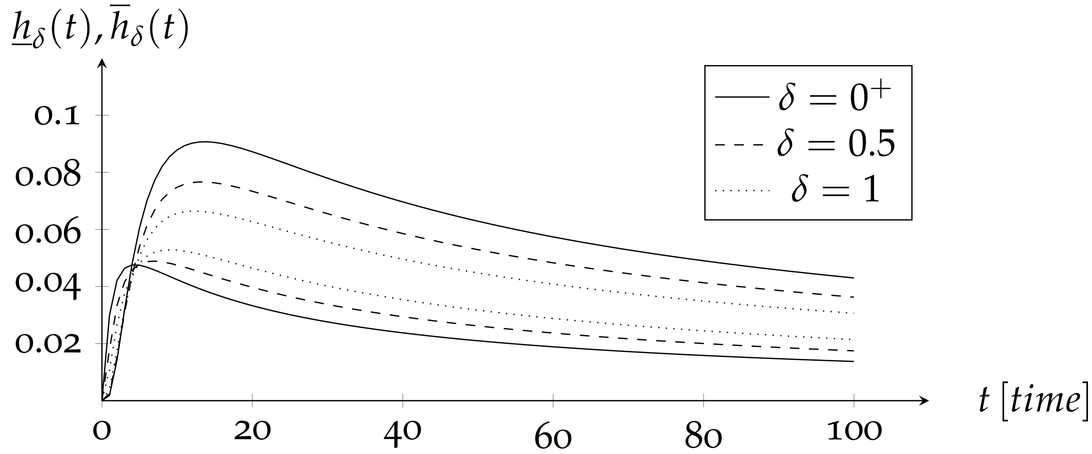


Figure 28: Upper and Lower δ-level curves of the fuzzy estimate of the hazard rate



5.7 GENERALIZED ESTIMATION FOR THE THREE PARAMETER WEIBULL DISTRIBUTION

Estimators $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ for three parameter Weibull distribution explained in section section 3.8 are based on precise life time observations. The generalized estimates based on fuzzy life time observations are explained below.

For fuzzy life time observations $t_1^*, t_2^*, \dots, t_n^*$, if $\hat{\alpha}^*$ is a fuzzy parameter estimator for the location parameter of the three parameter Weibull distribution based on fuzzy data, then its δ-cuts are denoted by

$$C_\delta(\hat{\alpha}^*) = [\underline{\alpha}_\delta\,, \overline{\alpha}_\delta] \qquad \forall \delta \in (0\,, 1],$$

where

$$\underline{\alpha}_\delta = \min_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \frac{\overline{m}_1 \cdot \overline{m}_4 \; - \; \overline{m}_2^2}{\overline{m}_1 + \overline{m}_4 - 2\overline{m}_2}$$

and

$$\overline{\alpha}_\delta = \max_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \frac{\overline{m}_1 \cdot \overline{m}_4 \; - \; \overline{m}_2^2}{\overline{m}_1 + \overline{m}_4 - 2\overline{m}_2}.$$

In a similar way $\hat{\beta}^*$, $\hat{\gamma}^*$ are fuzzy estimators of scale and shape parameter of the three parameter Weibull distribution respectively. Its δ-cuts are defined by

$$C_\delta\big(\hat{\beta}^*\big) = \Big[\underline{\beta}_\delta\,, \overline{\beta}_\delta\Big] \qquad \forall \delta \in (0\,, 1],$$

$$\underline{\beta}_\delta = \min_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \frac{\overline{m}_1 - \hat{\alpha}}{\Gamma\left(1 + \frac{1}{\hat{\gamma}}\right)}$$

and

$$\overline{\beta}_\delta = \max_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \frac{\overline{m}_1 - \hat{\alpha}}{\Gamma\left(1 + \frac{1}{\hat{\gamma}}\right)}$$

where

$$C_\delta(\hat{\gamma}^*) = \left[\underline{\gamma}_\delta\,,\,\overline{\gamma}_\delta\right] \qquad \forall \delta \in (0\,,1],$$

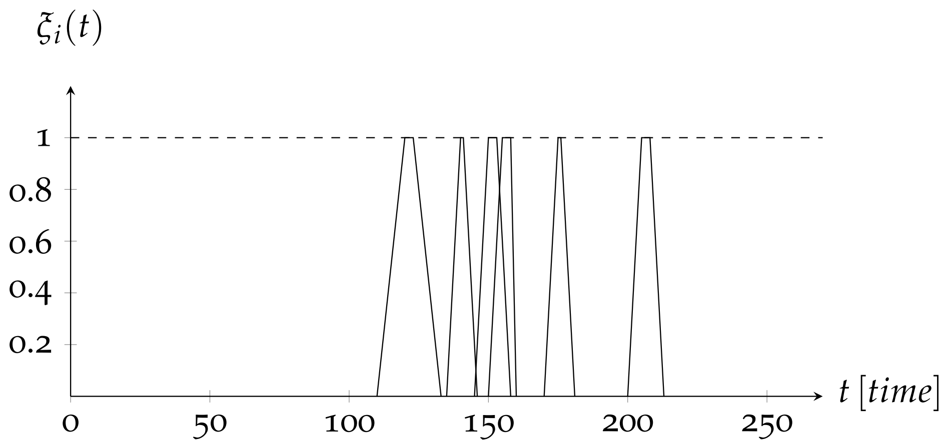
$$\underline{\gamma}_\delta = \min_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \frac{\ln 2}{\ln(\overline{m}_1 - \overline{m}_2) - \ln(\overline{m}_2 - \overline{m}_4)}$$

and

$$\overline{\gamma}_\delta = \max_{\underline{t} \in \times_{i=1}^{n} C_\delta(t_i^*)} \frac{\ln 2}{\ln(\overline{m}_1 - \overline{m}_2) - \ln(\overline{m}_2 - \overline{m}_4)}.$$

As an example take fuzzy life time observations with trapezoidal characterizing functions as given in figure 29.

Figure 29: Trapezoidal fuzzy life times



For the construction of the characterizing function $\psi(\cdot)$ of the fuzzy estimates of the Weibull parameters the following steps are applied:

1. The values for δ are taken from 0 to 1 with an increment 0.1.

2. For a given value of δ all δ-cuts of the fuzzy observations are determined.

3. Taking 10 values from all δ-cuts (for n observations we obtain 10^n values) we obtain hypothetical classical samples.

4. From these hypothetical classical samples at a given level δ, the standard classical estimates of the parameters are calculated.

5. In order to construct the characterizing function of a generalized (fuzzy) estimator $\hat{\theta}^*$ the minimum and maximum values from these estimates are taken and are considered as the end points of the family $(A_\delta;\ \delta \in (0\,,1])$ of generating intervals A_δ of the characterizing function of the fuzzy estimators at a given level of δ.

6. Steps 3-5 are performed for each for $\delta = 0(0.1)1$.

7. From all these generating intervals A_δ obtained for each δ (i.e. $\delta = 0^+$, 0.1, 0.2, ..., 1) through the above mentioned Construction lemma an approximation of the characterizing functions of the fuzzy estimates of the parameters are obtained.

In figures 30 - 32 the characterizing functions of the fuzzy estimates of the Weibull parameters α, β, and γ based on the fuzzy sample from figure 29 are depicted:

Figure 30: Characterizing function of the fuzzy estimate $\hat{\alpha}^*$

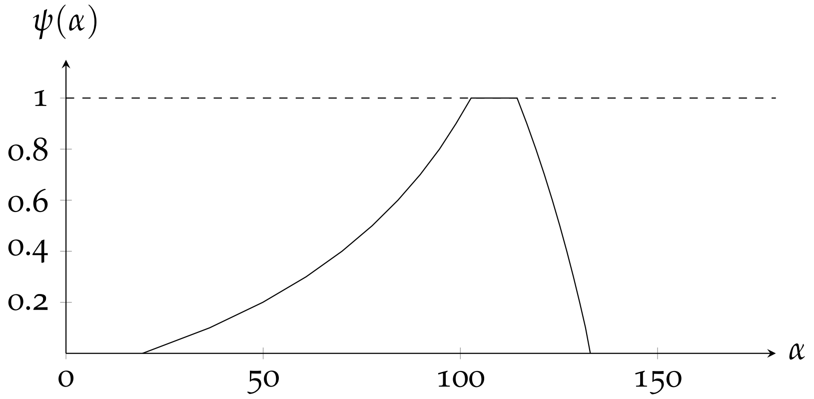


Figure 31: Characterizing function of the fuzzy estimate $\hat{\beta}^*$

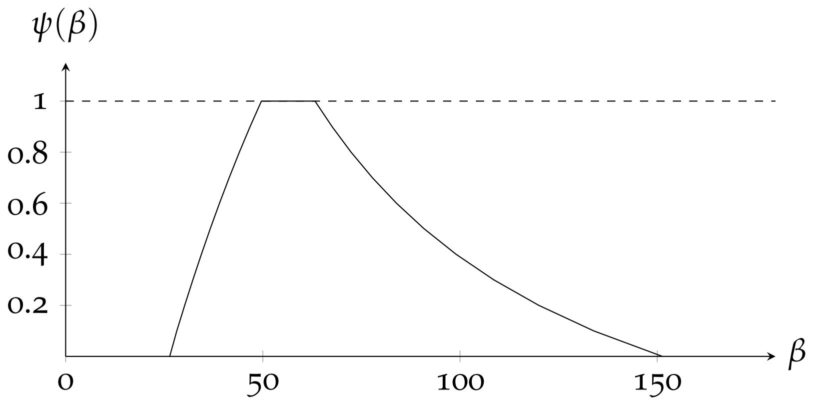
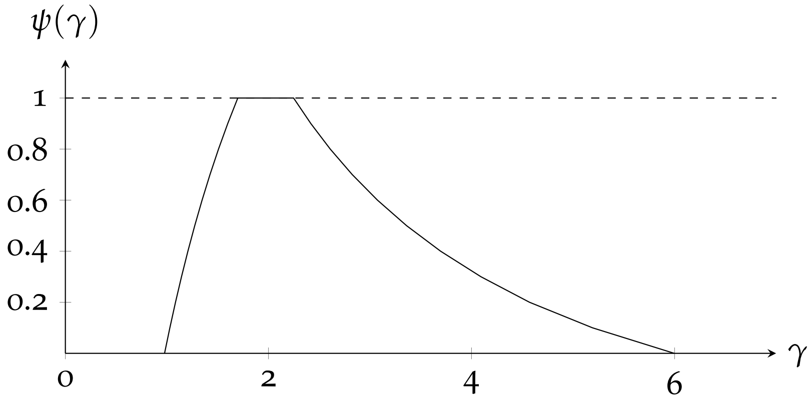


Figure 32: Characterizing function of the fuzzy estimate $\hat{\gamma}^*$



5.8 ESTIMATION OF THE RELIABILITY FUNCTION

The lower and upper δ-level curves of a fuzzy estimate $R^*(\cdot)$ of the reliability function $R(\cdot)$ of a three parameter Weibull distribution are obtained from the corresponding generating family $(A_\delta;\ \delta \in (0\,,1])$:

$$A_\delta\Big(R^*(x)\Big) = \Bigg[\inf_{\substack{\underline{x} \in \times_{i=1}^n C_\delta(x_i^*) \\ \hat{\alpha}_\delta \in [\underline{\alpha}_\delta\,,\overline{\alpha}_\delta] \\ \hat{\beta}_\delta \in [\underline{\beta}_\delta\,,\overline{\beta}_\delta] \\ \hat{\gamma}_\delta \in [\underline{\gamma}_\delta\,,\overline{\gamma}_\delta]}} \left\{ exp \left\{ - \left(\frac{x - \hat{\alpha}_\delta}{\hat{\beta}_\delta} \right)^{\hat{\gamma}_\delta} \right\} \right\},$$

$$\sup_{\substack{\underline{x} \in \times_{i=1}^n C_\delta(x_i^*) \\ \hat{\alpha}_\delta \in [\underline{\alpha}_\delta\,,\overline{\alpha}_\delta] \\ \hat{\beta}_\delta \in [\underline{\beta}_\delta\,,\overline{\beta}_\delta] \\ \hat{\gamma}_\delta \in [\underline{\gamma}_\delta\,,\overline{\gamma}_\delta]}} \left\{ exp \left\{ - \left(\frac{x - \hat{\alpha}_\delta}{\hat{\beta}_\delta} \right)^{\hat{\gamma}_\delta} \right\} \right\} \Bigg] \qquad \forall \delta \in (0\,,1], \quad t \geq \alpha, \quad (A)$$

The fuzzy estimate of the reliability function of the three parameter Weibull distribution is obtained through the following algorithm:

1. The values taken for δ are 0, 0.5, 1.

2. For a given value of δ all δ-cuts of the fuzzy observations are determined.

3. Taking 10 values from all δ-cuts (for n observations we obtain 10^n values) hypothetical classical samples are obtained.

4. From these hypothetical classical samples at a given level δ, the standard classical estimates of the parameters are calculated.

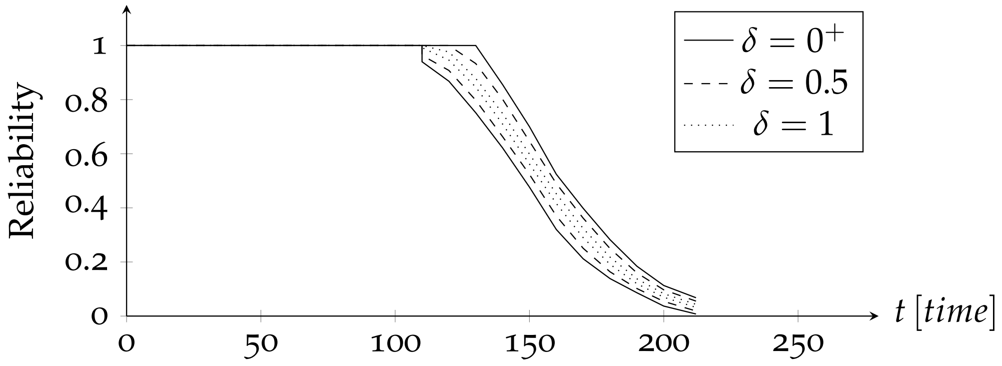
5. Based on these estimates the values of the reliability function given in equation (A) are calculated for times 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 212.

6. The lower and upper δ-level curves of the fuzzy estimate of the reliability function are constructed by taking minimum and maximum values of the classical estimated values of the reliability function at time points 110, 120, 130, 140, 150, 160, 170, 180, 190, 200, 212 at a given level of δ.

7. The minimum values make the lower δ-level curve, and the maximum values make the upper δ-level curve.

8. Steps 2-7 are performed for $\delta = 0, 0.5$, and 1.

Figure 33: Lower and Upper δ-level curves of the Fuzzy Reliability Function estimate based on the fuzzy sample given in figure 29.



5.9 A NEW MODEL FOR LIFETIME DISTRIBUTION WITH BATHTUB SHAPED FAILURE RATE

Let $t_1^*, t_2^*, \dots, t_n^*$ be denoting fuzzy life time observations with corresponding δ-cuts $C_\delta(t_i^*) = \left[\underline{t}_{i,\delta}\,,\,\overline{t}_{i,\delta}\right] \quad \forall \delta \in (0\,,1]$.

Based on fuzzy life time data the generalized (fuzzy) estimators for the parameters explained in 3.11 are denoted by $\hat{t}_0^*$ and $\hat{\alpha}^*$.

The generating families of intervals for the fuzzy estimators $\hat{t}_0^*$ and $\hat{\alpha}^*$ are denoted as $(A_\delta(\hat{t}_0^*);\ \forall\delta \in (0\,,1])$ and $(B_\delta(\hat{\alpha}^*);\ \forall\delta \in (0\,,1])$, respectively and are obtained by the following equations:

$$A_\delta(\hat{t}_0^*) = \left[\underline{t}_{(n),\delta}\,,\,\overline{t}_{(n),\delta}\right] \quad \forall\delta \in (0\,,1]$$

where $\underline{t}_{0,\delta} = \underline{t}_{(n),\delta}$ and $\overline{t}_{0,\delta} = \overline{t}_{(n),\delta}$ denote lower and upper ends of the δ-cut of the largest order statistic respectively.

For the lower and upper ends of the generating family of intervals for the fuzzy estimator $\hat{\alpha}^*$, the estimator explained in section 3.11 is generalized in the following way:

Solving the following equations

$$\text{for } \alpha_{1,\delta}: \quad \frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha + \underline{t}_{(i),\delta}/\underline{t}_{0,\delta}}{\alpha^2 + (1+2\alpha)\overline{t}_{(i),\delta}/\underline{t}_{0,\delta}} = 0$$

$$\text{for } \alpha_{2,\delta}: \quad \frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha + \overline{t}_{(i),\delta}/\overline{t}_{0,\delta}}{\alpha^2 + (1+2\alpha)\underline{t}_{(i),\delta}/\overline{t}_{0,\delta}} = 0$$

$$\text{for } \alpha_{3,\delta}: \quad \frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha + \underline{t}_{(i),\delta}/\overline{t}_{0,\delta}}{\alpha^2 + (1+2\alpha)\underline{t}_{(i),\delta}/\overline{t}_{0,\delta}} = 0$$

$$\text{for } \alpha_{4,\delta}: \quad \frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha + \overline{t}_{(i),\delta}/\underline{t}_{0,\delta}}{\alpha^2 + (1+2\alpha)\overline{t}_{(i),\delta}/\underline{t}_{0,\delta}} = 0$$

and defining

$$\underline{\alpha}_\delta = min\left[\alpha_{1,\delta},\, \alpha_{2,\delta},\, \alpha_{3,\delta},\, \alpha_{4,\delta}\right]$$

$$\overline{\alpha}_\delta = max\left[\alpha_{1,\delta},\, \alpha_{2,\delta},\, \alpha_{3,\delta},\, \alpha_{4,\delta}\right],$$

the generating family $\Big(B_\delta\left(\hat{\alpha}^*\right) = \left[\underline{\alpha}_\delta\,,\, \overline{\alpha}_\delta\right] \quad \forall \delta \in (0\,,\,1]\Big)$ is obtained.

The characterizing functions for the fuzzy parameter estimates are obtained by the mentioned Construction lemma.

Figure 34: Characterizing functions of a fuzzy sample

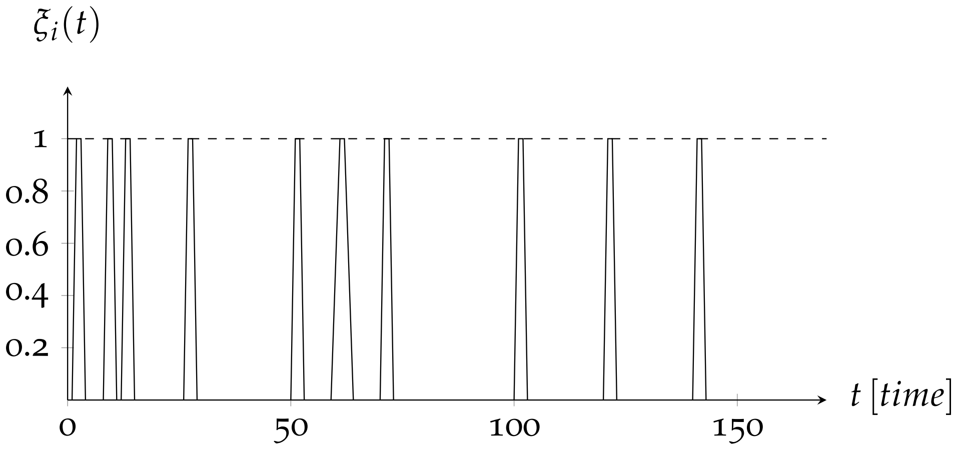


Figure 35: Characterizing function of the fuzzy estimator $\hat{t}_0^*$

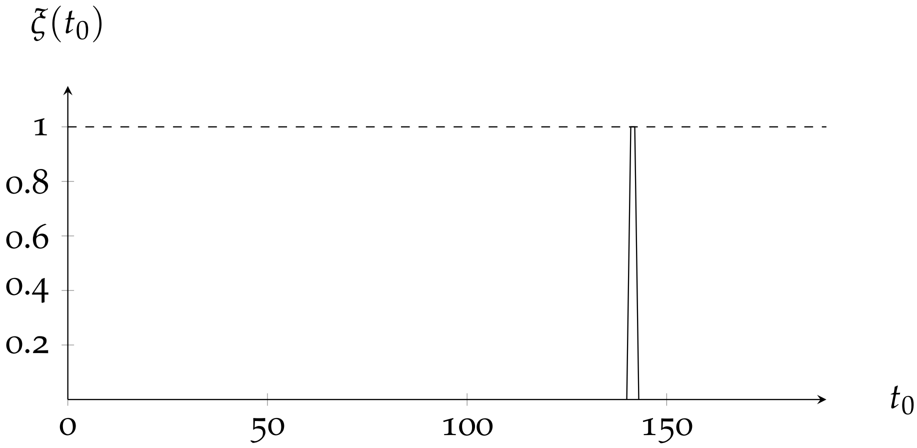
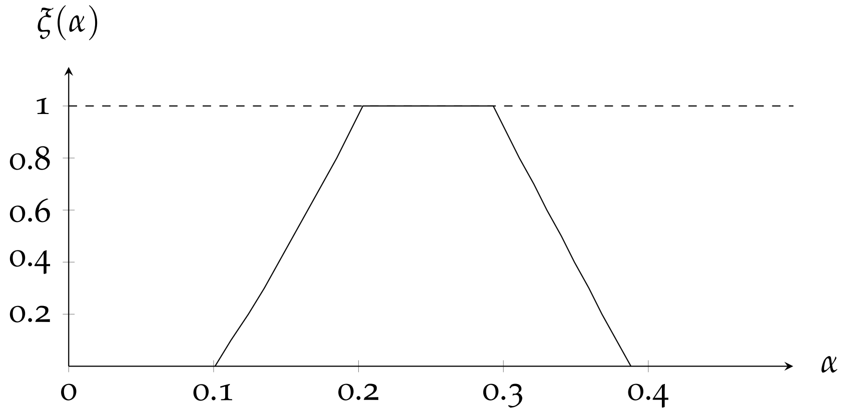


Figure 36: Characterizing function of the fuzzy estimator $\hat{\alpha}^*$



Based on fuzzy estimates of t_0 and α the δ-level curves of the fuzzy estimate of hazard rate are given by $C_\delta(h^*(t)) = [\underline{h}_\delta(t)\,,\, \overline{h}_\delta(t)] \quad \forall \delta \in (0\,,\,1]$.

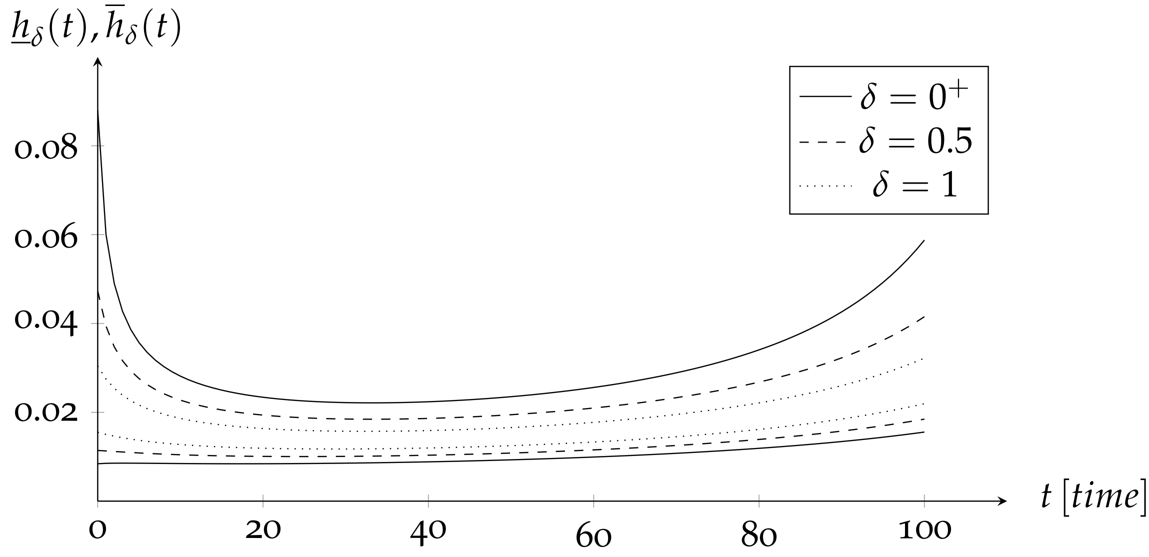
The corresponding δ-level curves are obtained as

$$\underline{h}_\delta(t) = \frac{1+2\underline{\alpha}_\delta}{(2\overline{t}_{0,\delta}\sqrt{\overline{\alpha}_\delta^2+(1+2\overline{\alpha}_\delta)t/\underline{t}_{0,\delta}})(1+\overline{\alpha}_\delta-\sqrt{\underline{\alpha}_\delta^2+(1+2\underline{\alpha}_\delta)t/\overline{t}_{0,\delta}})} \quad \forall \delta \in (0\,,\,1]$$

and

$$\overline{h}_\delta(t) = \frac{1+2\overline{\alpha}_\delta}{(2\underline{t}_{0,\delta}\sqrt{\underline{\alpha}_\delta^2+(1+2\underline{\alpha}_\delta)t/\overline{t}_{0,\delta}})(1+\underline{\alpha}_\delta-\sqrt{\overline{\alpha}_\delta^2+(1+2\overline{\alpha}_\delta)t/\underline{t}_{0,\delta}})} \quad \forall \delta \in (0\,,\,1].$$

Figure 37: Some lower and upper δ-level curves of the fuzzy estimate of the hazard rate



5.10 NEW TWO-PARAMETER LIFETIME DISTRIBUTION WITH BATH-TUB SHAPED FAILURE RATE

For fuzzy life times fuzzy parameter estimators and a fuzzy estimator for the hazard rate for the new two-parameter distribution defined by (Chen, 2000) can be written as:

$$C_\delta(\hat{\beta}^*) = [\underline{\beta}_\delta \,,\, \overline{\beta}_\delta] \quad \forall \delta \in (0\,,\,1]$$

$$C_\delta(\hat{\lambda}^*) = [\underline{\lambda}_\delta \,,\, \overline{\lambda}_\delta] \quad \forall \delta \in (0\,,\,1]$$

$$C_\delta(h^*(t)) = [\underline{h}_\delta(t)\,,\, \overline{h}_\delta(t)] \quad \forall \delta \in (0\,,\,1]$$

Lower and upper limits of the generalized families of intervals for the fuzzy estimator $\hat{\beta}^*$ are approximated through the following equations:

$$\beta_{1,\delta} = \tfrac{k}{\beta} + \textstyle\sum_{i=1}^{k} \ln \underline{t}_{(i),\delta} + \sum_{i=1}^{k} \left(\underline{t}_{(i),\delta}^{\beta} \ln \underline{t}_{(i),\delta} \right) -$$

$$\frac{k \left[\sum_{i=1}^{k} \left(e^{\underline{t}_{(i),\delta}^{\beta}} \underline{t}_{(i),\delta}^{\beta} \ln \underline{t}_{(i),\delta} \right) + (n-k) \left(e^{\underline{t}_{(k),\delta}^{\beta}} \underline{t}_{(k),\delta}^{\beta} \ln \underline{t}_{(k),\delta} \right) \right]}{\sum_{i=1}^{k} e^{\underline{t}_{(i),\delta}^{\beta}} - n - (n-k) e^{\underline{t}_{(k),\delta}^{\beta}}} = 0$$

and

$$\beta_{2,\delta} = \tfrac{k}{\beta} + \textstyle\sum_{i=1}^{k} \ln \bar{t}_{(i),\delta} + \sum_{i=1}^{k} \left(\bar{t}_{(i),\delta}^{\beta} \ln \bar{t}_{(i),\delta} \right) -$$

$$\frac{k \left[\sum_{i=1}^{r} \left(e^{\bar{t}_{(i),\delta}^{\beta}} \bar{t}_{(i),\delta}^{\beta} \ln \bar{t}_{(i),\delta} \right) + (n-k) \left(e^{\bar{t}_{(k),\delta}^{\beta}} \bar{t}_{(k),\delta}^{\beta} \ln \bar{t}_{(k),\delta} \right) \right]}{\sum_{i=1}^{k} e^{\bar{t}_{(i),\delta}^{\beta}} - n - (n-k) e^{\bar{t}_{(r),\delta}^{\beta}}} = 0$$

where

$$A_{\delta}(\hat{\beta}^{*}) = [\{min(\beta_{1,\delta},\ \beta_{2,\delta})\}\ ,\ \{max(\beta_{1,\delta}\ ,\ \beta_{2,\delta})\}]\ ; \quad \forall \delta \in (0\ ,\ 1].$$

where $\underline{\beta}_{\delta} = min[\beta_{1,\delta}\ ,\ \beta_{2,\delta}]$ and $\overline{\beta}_{\delta} = max[\beta_{1,\delta}\ ,\ \beta_{2,\delta}]$.

The characterizing function of $\hat{\beta}^{*}$ can be obtained by the mentioned Construction lemma.

Lower and upper limits of the generating family of intervals for the fuzzy estimator $\hat{\lambda}^{*}$ are approximated through the following equations:

$$A_{\delta}(\hat{\lambda}^{*}) = \left[\frac{k}{\sum_{i=1}^{k} e^{\bar{t}_{(i),\delta}^{\overline{\beta}_{\delta}}} - n - (n-k) e^{\underline{t}_{(k),\delta}^{\underline{\beta}_{\delta}}}}\ ,\ \frac{k}{\sum_{i=1}^{r} e^{\underline{t}_{(i),\delta}^{\underline{\beta}_{\delta}}} - n - (n-k) e^{\bar{t}_{(k),\delta}^{\overline{\beta}_{\delta}}}} \right]$$

$$\forall \delta \in (0\ ,\ 1]$$

where $\underline{\lambda}_\delta$ and $\overline{\lambda}_\delta$ represent the lower and upper limits of the interval at level δ. The characterizing function of $\hat{\lambda}^*$ can be obtained by the Construction lemma.

Given below in figures 38-40 are characterizing functions of fuzzy sample, $\hat{\beta}^*$, and $\hat{\lambda}^*$ respectively.

Figure 38: Characterizing functions of a fuzzy sample

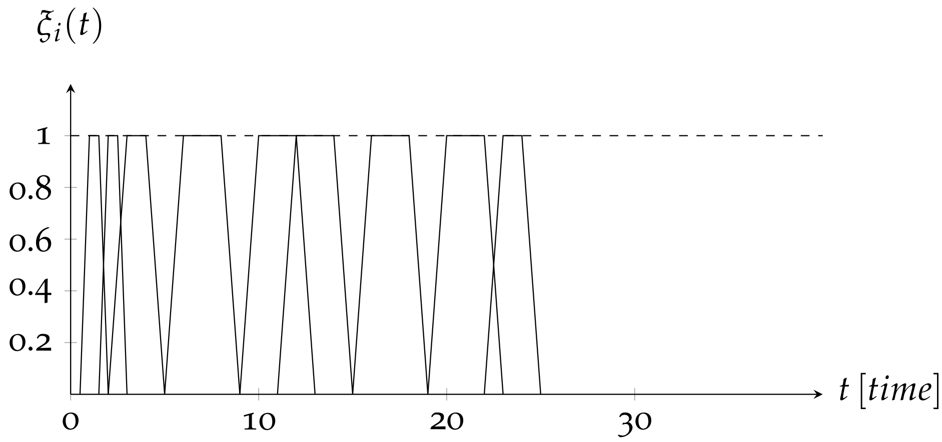


Figure 39: Characterizing function of the fuzzy estimator $\hat{\beta}^*$

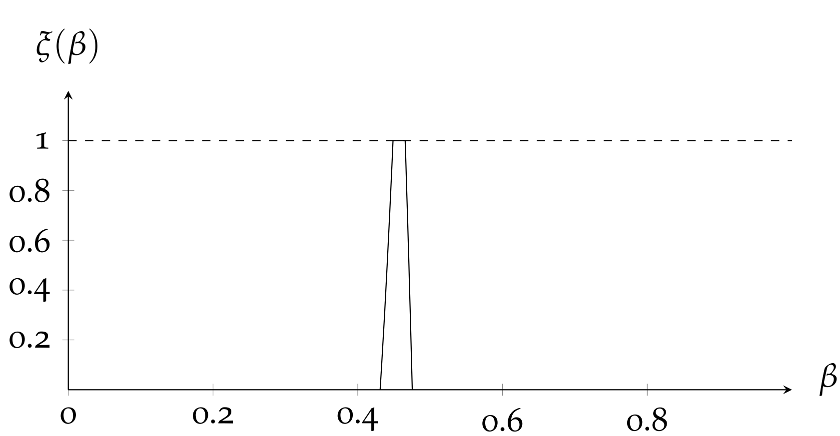
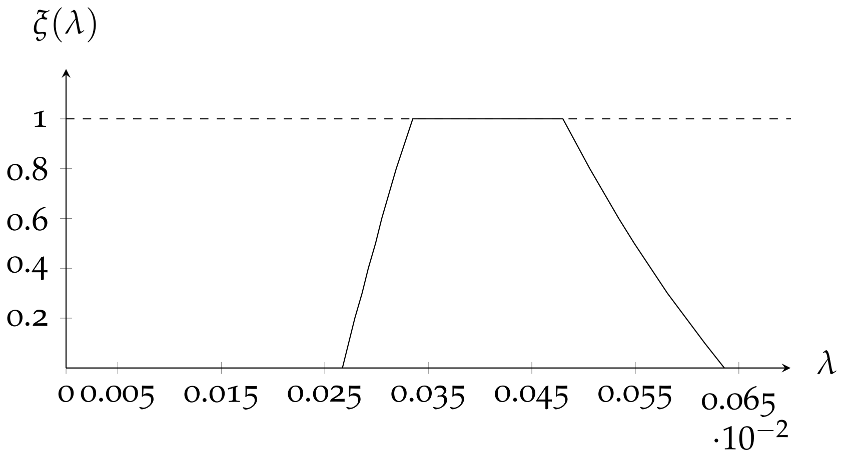


Figure 40: Characterizing function of the fuzzy estimator $\hat{\lambda}^*$

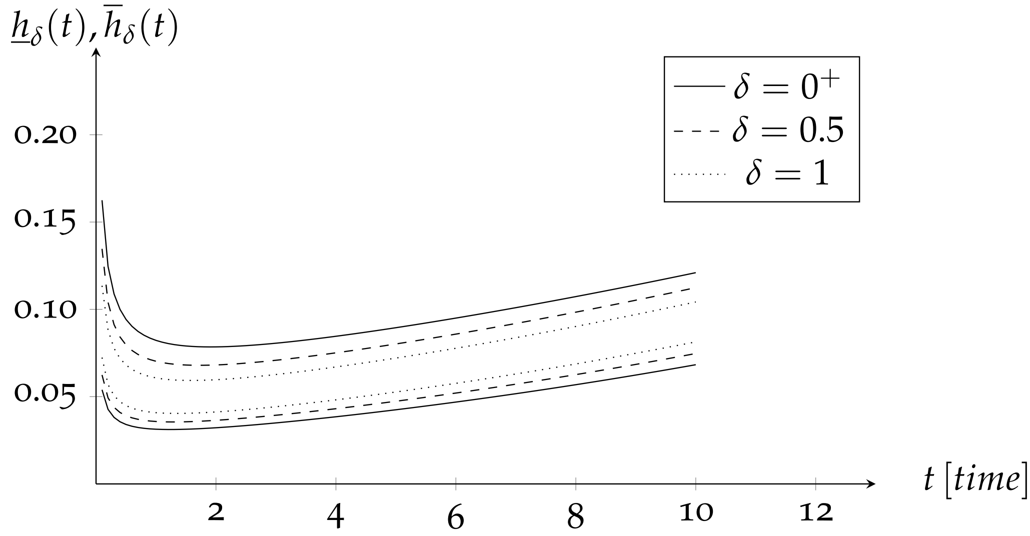


Lower and upper δ-level curves of the fuzzy hazard rate are given by:

$$\underline{h}_\delta(t) = \underline{\lambda}_\delta \underline{\beta}_\delta e^{t^{\overline{\beta}_\delta}} t^{\overline{\beta}_\delta - 1} \qquad \forall \delta \in (0, 1], \qquad \forall t \in [0, \infty)$$

$$\overline{h}_\delta(t) = \overline{\lambda}_\delta \overline{\beta}_\delta e^{t^{\underline{\beta}_\delta}} t^{\underline{\beta}_\delta - 1} \qquad \forall \delta \in (0, 1], \qquad \forall t \in [0, \infty)$$

Figure 41: Some lower and upper δ-level curves of the fuzzy estimate of the hazard rate



5.11 ACCELERATED LIFE TESTING AND FUZZY INFORMATION

In order to obtain generalized (fuzzy) estimators based on fuzzy life time observations for accelerated life testing, the procedures to obtain generalized estimators are given in the following sections.

5.11.1 *Constant Stress Levels and Fuzzy Life Times*

As discussed earlier life time observations are more or less fuzzy and parameter estimators are functions of life time observations. Therefore the estimators explained in section 4.1 are needed to be generalized for fuzzy life time observations, i.e. $\hat{\theta}_i^* = f(t_{i,(j)}^*,\ j = 1(1)r_i)$, where $t_{i,(j)}^*$ denote the fuzzy life times of r_i failed units under the stress level S_i.

Since the parameter estimator $\hat{\theta}_i^*$ is becoming a fuzzy value, therefore its δ-cuts are denoted as $C_\delta\left(\hat{\theta}_i^*\right) = \left[\underline{\theta}_{i,\delta}\ ,\ \overline{\theta}_{i,\delta}\right] \quad \forall\ \delta\ \in (0\,,1]$.

To obtain $\underline{\theta}_{i,\delta}$ and $\overline{\theta}_{i,\delta}$ the proposed estimators are explained below:

Under constant stress levels $S_i, i = 1(1)k$ the generating family of the fuzzy estimator $\hat{\theta}_i^*$ from section 4.1 for the parameters are defined by

$$A_\delta(\hat{\theta}_i^*) = \left[\frac{\sum_{j=1}^{r_i} \underline{t}_{i,(j),\delta} + (n_i - r_i)\underline{t}_{i,(r_i),\delta}}{r_i}\ ,\ \frac{\sum_{j=1}^{r_i} \overline{t}_{i,(j),\delta} + (n_i - r_i)\overline{t}_{i,(r_i),\delta}}{r_i}\right]$$

$$\forall\ \delta\ \in\ (0\,,1]$$

where $\left[\underline{t}_{i,(j),\delta}\ ,\ \overline{t}_{i,(j),\delta}\right] \quad \forall\ \delta\ \in\ (0\,,1]$ are the δ-cuts of the fuzzy life times $t_{i,(j)}^*$.

The characterizing function of $\hat{\theta}_i^*$ is obtained by the Construction lemma for characterizing functions.

For the construction of the characterizing functions of the fuzzy estimates as an example we have $S_1 = 1,\ n_1 = 5,\ r_1 = 3,$ and
$S_2 = 2.5,\ n_2 = 5,\ r_2 = 4.$

The characterizing functions of the observed fuzzy life times and fuzzy estimates of the parameters are given in figure 42, figure 43, and figure 44 respectively.

Figure 42: Fuzzy sample under stress S_1

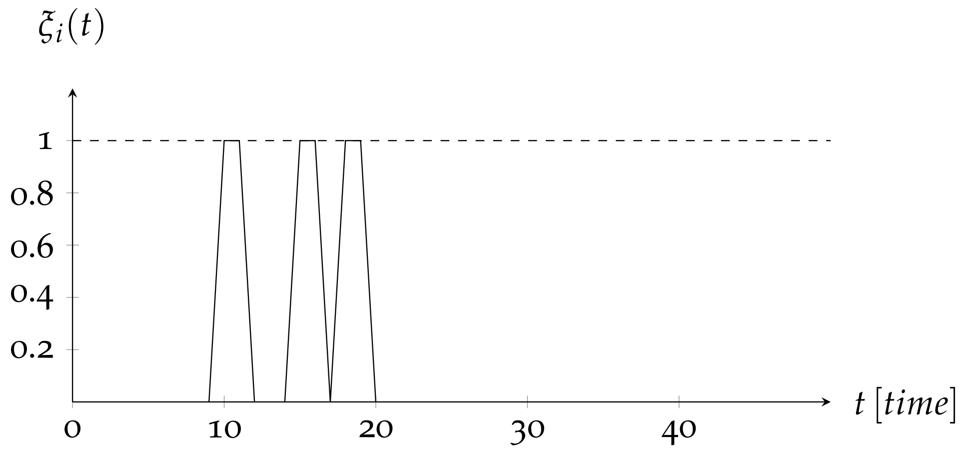


Figure 43: Fuzzy sample under stress S_2

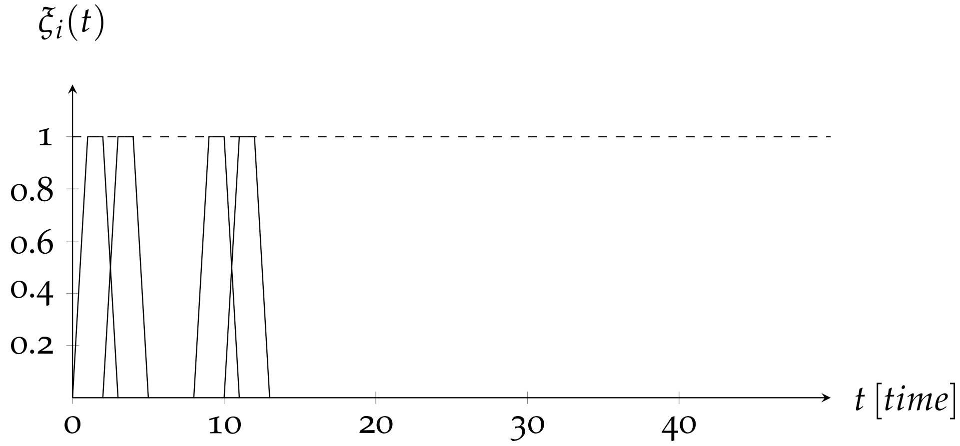
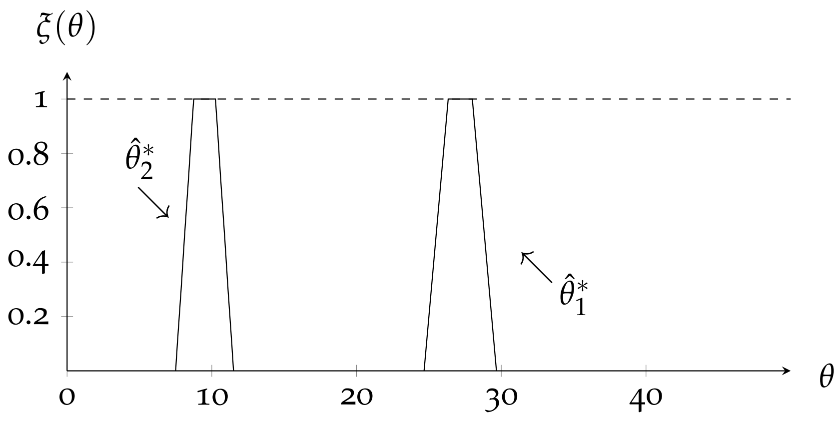


Figure 44: Characterizing functions of $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$



5.11.2 *Power Rule Model and Fuzzy Life Times*

But as discussed earlier life time observations are not precise numbers but fuzzy. Therefore the generalized estimators for θ_i explained in section 4.2 are written as $\hat{\theta}_i^*$, $i = 1(1)k$ for fuzzy life time data.

The estimates of A and C will also be fuzzy because they are functions of $\hat{\theta}_i^*$, defined by

$\hat{C}^* = f_1(\hat{\theta}_i^*)$, $\hat{A}^* = f_2(\hat{\theta}_i^*)$ and also $\hat{\theta}^*(S_i) = \frac{\hat{C}^*}{\left(\frac{S_i}{S}\right)^{\hat{A}^*}}$.

Therefore its δ-cuts are

$$C_\delta(\hat{A}^*) = \left[\underline{A}_\delta \,,\, \overline{A}_\delta\right] \quad \forall\, \delta \,\in\, (0\,,1]$$

$$C_\delta(\hat{C}^*) = \left[\underline{C}_\delta \,,\, \overline{C}_\delta\right] \quad \forall\, \delta \,\in\, (0\,,1]$$

$$C_\delta(\hat{\theta}^*(S_i)) = \left[\underline{\theta}_\delta(S_i) \,,\, \overline{\theta}_\delta(S_i)\right] \quad \forall\, \delta \,\in\, (0\,,1].$$

For the estimation of lower and upper ends of the generating families of intervals of A^*, C^*, and $\hat{\theta}^*(S_i)$ respectively the proposed estimators are explained below:

The generating families of intervals of the fuzzy estimators $\hat{A}^*$ and $\hat{C}^*$ are denoted as $\left[\underline{A}_\delta\,,\overline{A}_\delta\right]$ and $\left[\underline{C}_\delta\,,\overline{C}_\delta\right]$, respectively. In order to obtain them we have the following equations:

$$\underline{A}_\delta = \min\left\{\sum_{i=1}^{k} r_i \cdot \underline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^A \cdot \ln(\frac{S_i}{\overline{S}}) \;=0\,, \quad \sum_{i=1}^{k} r_i \cdot \overline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^A \cdot \ln(\frac{S_i}{\overline{S}}) \;=\; 0\right\}$$

$$\forall\, \delta \,\in\, (0\,,1]$$

$$\overline{A}_\delta = \max\left\{ \sum_{i=1}^{k} r_i \cdot \underline{\theta}_{i,\delta} \cdot (\tfrac{S_i}{\overline{S}})^{A} \cdot \ln(\tfrac{S_i}{\overline{S}}) \ = 0\,, \quad \sum_{i=1}^{k} r_i \cdot \overline{\theta}_{i,\delta} \cdot (\tfrac{S_i}{\overline{S}})^{A} \cdot \ln(\tfrac{S_i}{\overline{S}}) \ = \ 0 \right\}$$

$$\forall\, \delta \ \in \ (0\,, 1]$$

and

$$\underline{C}_\delta = \min\left\{ \frac{\sum_{i=1}^{k} r_i \cdot \underline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\underline{A}_\delta}}{\sum_{i=1}^{k} r_i}\,, \ \frac{\sum_{i=1}^{k} r_i \cdot \underline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\overline{A}_\delta}}{\sum_{i=1}^{k} r_i}\,, \ \frac{\sum_{i=1}^{k} r_i \cdot \overline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\underline{A}_\delta}}{\sum_{i=1}^{k} r_i}\,, \ \frac{\sum_{i=1}^{k} r_i \cdot \overline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\overline{A}_\delta}}{\sum_{i=1}^{k} r_i} \right\}$$

$$\forall\, \delta \ \in \ (0\,, 1]$$

$$\overline{C}_\delta = \max\left\{ \frac{\sum_{i=1}^{k} r_i \cdot \underline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\underline{A}_\delta}}{\sum_{i=1}^{k} r_i}\,, \ \frac{\sum_{i=1}^{k} r_i \cdot \underline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\overline{A}_\delta}}{\sum_{i=1}^{k} r_i}\,, \ \frac{\sum_{i=1}^{k} r_i \cdot \overline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\underline{A}_\delta}}{\sum_{i=1}^{k} r_i}\,, \ \frac{\sum_{i=1}^{k} r_i \cdot \overline{\theta}_{i,\delta} \cdot (\frac{S_i}{\overline{S}})^{\overline{A}_\delta}}{\sum_{i=1}^{k} r_i} \right\}$$

$$\forall\, \delta \ \in \ (0\,, 1]$$

The characterizing functions for $\hat{C}^*$ and $\hat{A}^*$ are obtained by the above mentioned Construction lemma.

The fuzzy estimate for the parameters of $\theta(S_i)$ are obtained by the following generating families:

$$A_\delta(\hat{\theta}^*(S_1)) = \left[\frac{\underline{C}_\delta}{(S_1/\overline{S})^{\underline{A}_\delta}}\,, \ \frac{\overline{C}_\delta}{(S_1/\overline{S})^{\overline{A}_\delta}} \right] \quad \forall\, \delta \ \in \ (0\,, 1]$$

$$A_\delta(\hat{\theta}^*(S_2)) = \left[\frac{\underline{C}_\delta}{(S_2/\overline{S})^{\overline{A}_\delta}}\,, \ \frac{\overline{C}_\delta}{(S_2/\overline{S})^{\underline{A}_\delta}} \right] \quad \forall\, \delta \ \in \ (0\,, 1]$$

From these generating families of intervals the characterizing functions are obtained by the Construction lemma.

The characterizing functions of $\hat{A}^*$, $\hat{C}^*$, and $\hat{\theta}^*(S_i)$ are depicted in figures 45-47 and, fuzzy life times are considered from figures 42-43.

Figure 45: Characterizing function of $\hat{A}^*$

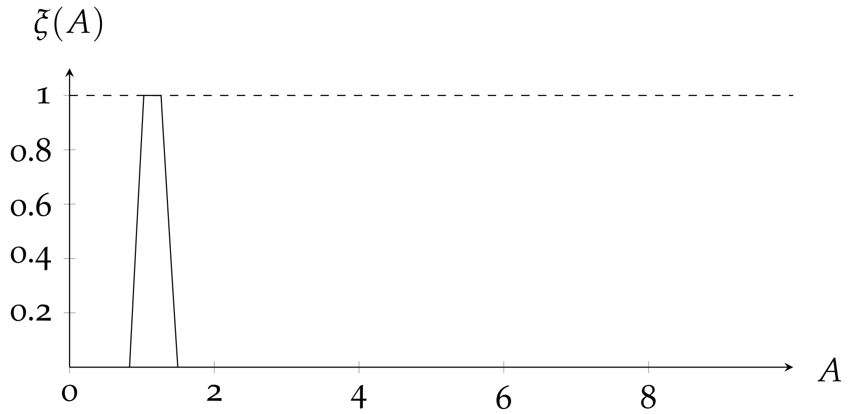


Figure 46: Characterizing function of $\hat{C}^*$

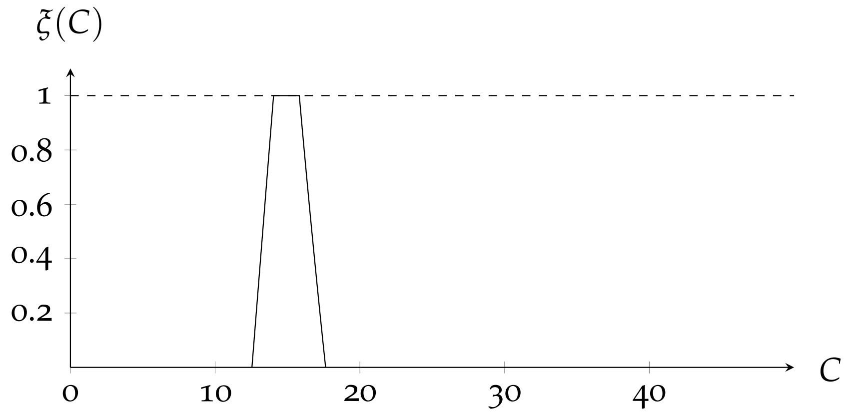
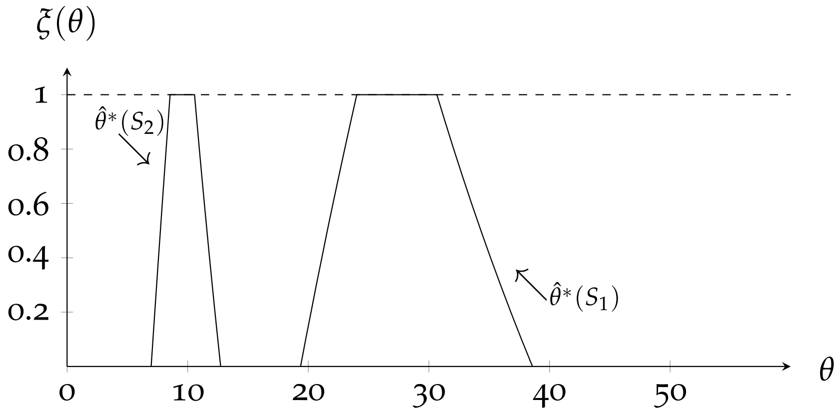


Figure 47: Characterizing functions for $\hat{\theta}^*(S_i)$



5.11.3 *Step-Stress Model and Fuzzy Life Times*

As discussed earlier life time observations are not precise numbers but fuzzy. Therefore the likelihood function and parameter estimators for the step-stress model are required to be generalized for fuzzy life time observations. The likelihood function for fuzzy data can be generalized by the extension principle and is denoted as

$$L^*(\theta_1,\, \theta_2;\, t^*_{1,(1)}, t^*_{1,(2)}, ..., t^*_{1,(n_1)}, t^*_{2,(n_1+1)}, ..., t^*_{2,(r)}).$$

For fuzzy life time observations the generalized parameter estimators are denoted as $\hat{\theta}^*_1$ and $\hat{\theta}^*_2$. Since these estimators will be functions of fuzzy life time observations, therefore its δ-cuts are denoted by

$$C_\delta(\hat{\theta}^*_1) = \left[\underline{\theta}_{1,\delta}\ ,\ \overline{\theta}_{1,\delta}\right] \qquad \forall\, \delta\, \in (0\, ,\, 1]$$

and

$$C_\delta(\hat{\theta}^*_2) = \left[\underline{\theta}_{2,\delta}\ ,\ \overline{\theta}_{2,\delta}\right] \qquad \forall\, \delta\, \in (0\, ,\, 1].$$

For the step-stress model explained in section 4.3, fuzzy estimates of the corresponding parameters can be obtained through the following equations:

$$A_\delta(\hat{\theta}_1^*) = \left[\frac{\sum_{j=1}^{n_1} \underline{t}_{1,(j),\delta} + (n-n_1)\tau}{n_1} \, , \, \frac{\sum_{j=1}^{n_1} \bar{t}_{1,(j),\delta} + (n-n_1)\tau}{n_1} \right] \quad \forall\, \delta \in (0\,,1]$$

$$A_\delta(\hat{\theta}_2^*) = \left[\frac{\sum_{j=n_1+1}^{r} (\underline{t}_{2,(j),\delta} - \tau) + (n-r)(\underline{t}_{2,(r),\delta} - \tau)}{r-n_1} \, , \, \frac{\sum_{j=n_1+1}^{r} (\bar{t}_{2,(j),\delta} - \tau) + (n-r)(\bar{t}_{2,(r),\delta} - \tau)}{r-n_1} \right]$$

$$\forall\, \delta \in (0\,,1]$$

From these generating families of intervals using the mentioned Construction lemma the characterizing functions can be obtained.

For the construction of characterizing functions of fuzzy parameter estimates, as an example, $n = 20, r = 16, n_1 = 4, \tau = 21, S_1 = 1, S_2 = 2.5$ are considered.

Figure 48: Fuzzy sample

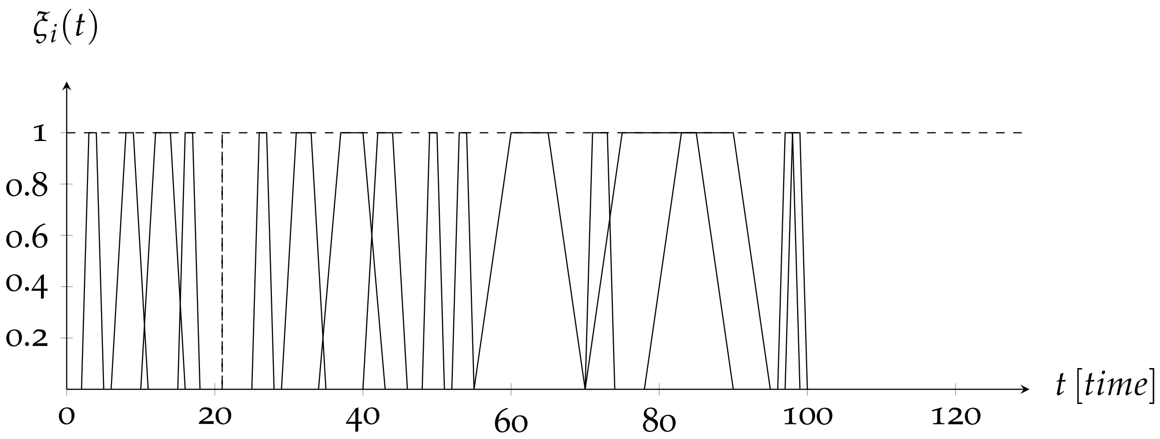


Figure 49: Fuzzy sample before time $\tau = 21$

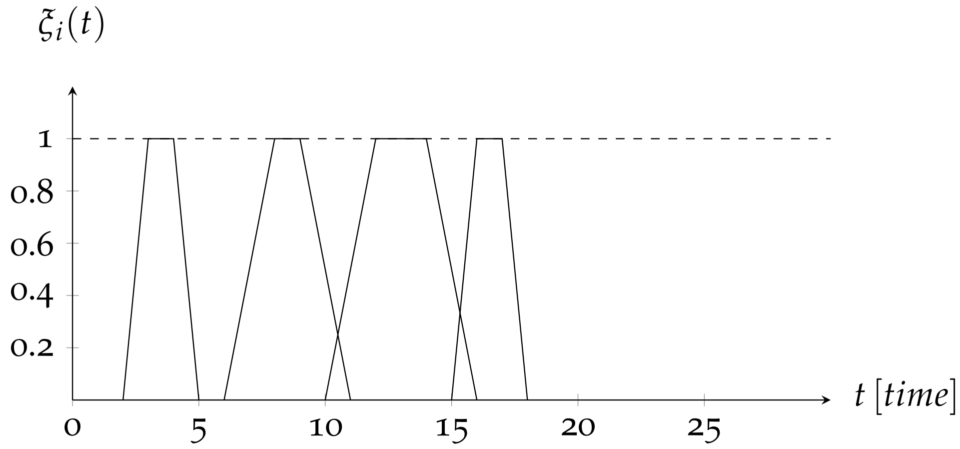


Figure 50: Fuzzy sample after time $\tau = 21$

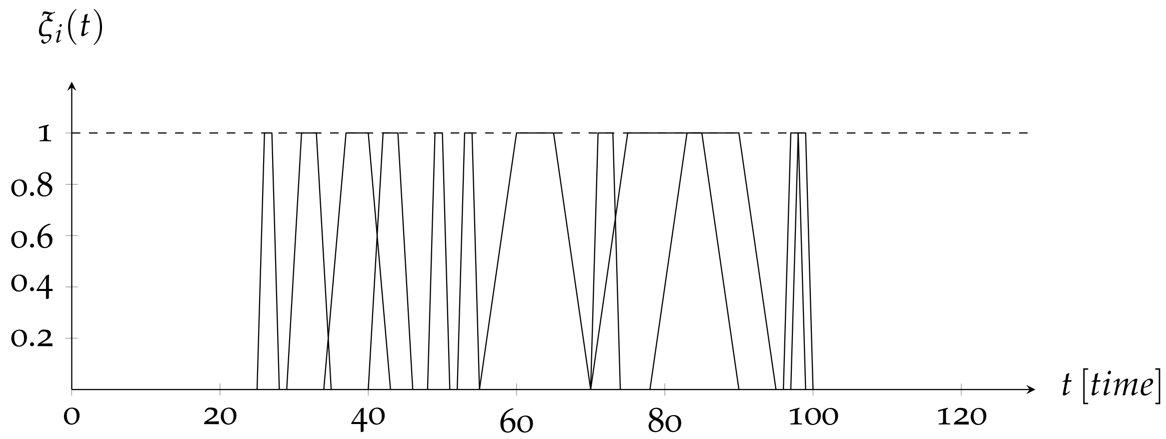
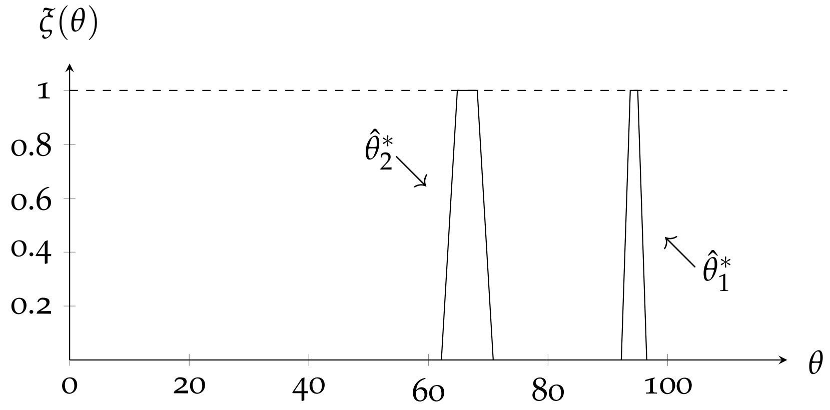


Figure 51: Characterizing functions of $\hat{\theta}_1^*$ and $\hat{\theta}_2^*$ based on the algorithm from section 5.11.3.



5.11.4 *Non-Parametric Estimation for Fuzzy Life Times*

The estimator $\hat{\gamma}$ explained in section 4.4 is based on precise life time observations, and needs to be generalized for fuzzy life time observations. The generalized fuzzy estimator of γ is denoted by $\hat{\gamma}^*$, having δ-cuts

$$C_\delta(\hat{\gamma}^*) = \left[\underline{\gamma}_\delta\,,\,\overline{\gamma}_\delta\right] \quad \forall\,\delta\,\in\,(0\,,1].$$

If we consider precise stress levels S_1, S_2, ..., S_k and fuzzy life time observations $t^*_{i,l}$, $l = 1(1)n_i$ and $i = 1(1)k$ having δ-cuts $C_\delta(t^*_{i,l}) = \left[\underline{t}_{i,l,\delta}\,,\,\overline{t}_{i,l,\delta}\right] \qquad \forall\delta \in (0\,,1]$, then δ-cuts of the mean of these life time observations can be written as

$$C_\delta(\overline{T}^*_i) = \left[\frac{1}{n_i}\sum_{l=1}^{n_i}\underline{t}_{i,l,\delta}\,,\,\frac{1}{n_i}\sum_{l=1}^{n_i}\overline{t}_{i,l,\delta}\right] \quad \forall\,\delta\,\in\,(0\,,1].$$

Defining

$$\underline{\overline{T}}_{i,\delta} = \frac{1}{n_i}\sum_{l=1}^{n_i}\underline{t}_{i,l,\delta} \qquad \text{and} \qquad \overline{\overline{T}}_{i,\delta} = \frac{1}{n_i}\sum_{l=1}^{n_i}\overline{t}_{i,l,\delta}$$

the lower and upper ends of the generating family of intervals $\left([\underline{\gamma}_\delta \,,\, \overline{\gamma}_\delta]; \quad \forall\, \delta \,\in\, (0\,,\,1] \right)$ for the fuzzy estimator of γ can be calculated as:

$$\underline{\gamma}_\delta = \min \left[\frac{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)\left(\ln\left(\frac{\overline{\underline{T}}_{i,\delta}}{\overline{\overline{T}}_{j,\delta}}\right)\right)}{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)^2} \,,\, \frac{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)\left(\ln\left(\frac{\overline{\overline{T}}_{i,\delta}}{\overline{\underline{T}}_{j,\delta}}\right)\right)}{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)^2} \right]$$

$\forall \delta \in (0\,,\,1]$

$$\overline{\gamma}_\delta = \max \left[\frac{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)\left(\ln\left(\frac{\overline{\underline{T}}_{i,\delta}}{\overline{\overline{T}}_{j,\delta}}\right)\right)}{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)^2} \,,\, \frac{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)\left(\ln\left(\frac{\overline{\overline{T}}_{i,\delta}}{\overline{\underline{T}}_{j,\delta}}\right)\right)}{\sum_{i=1}^{k}\sum_{j=i+1}^{k}\left(\ln\left(\frac{S_j}{S_i}\right)\right)^2} \right].$$

$\forall \delta \in (0\,,\,1]$

From this generating family of intervals the characterizing function of the fuzzy estimator $\hat{\gamma}^*$ can be obtained through the above mentioned Construction lemma.

Given below are the characterizing functions of three fuzzy samples, and fuzzy parameter estimates under S_1, S_2, and S_3, respectively.

Figure 52: Fuzzy sample under $S_1 = 36$

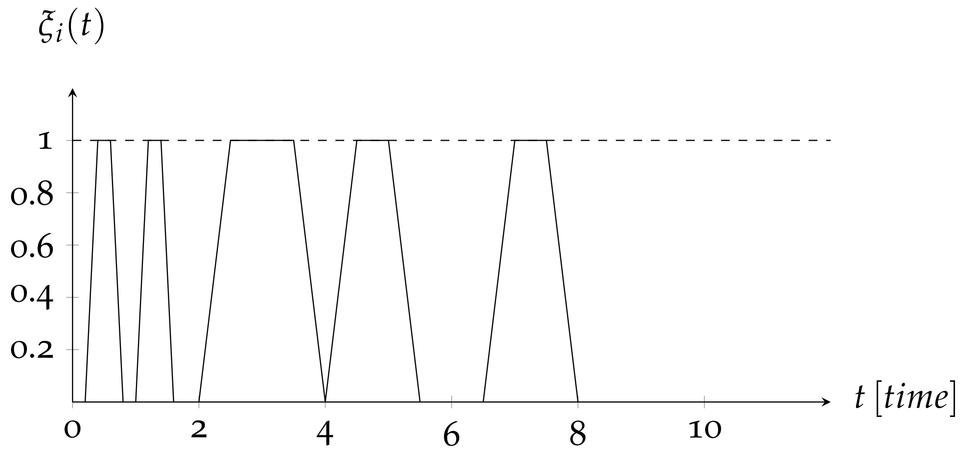


Figure 53: Fuzzy sample under $S_2 = 34$

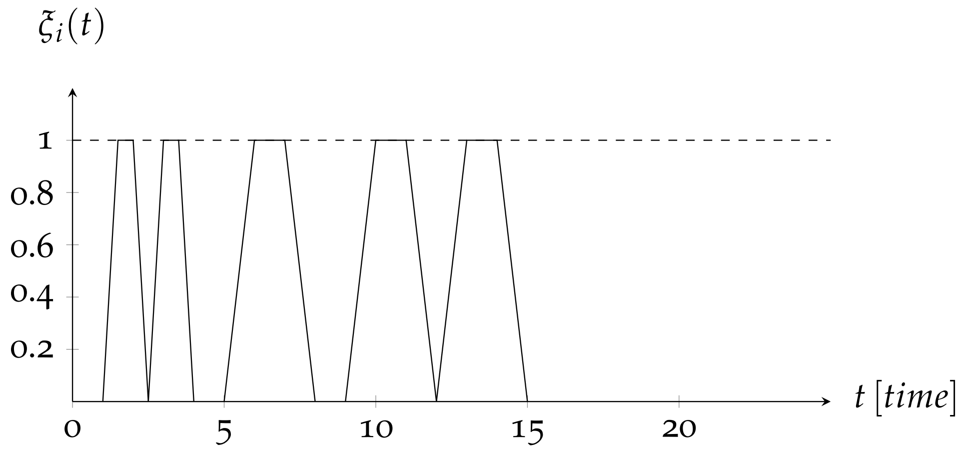


Figure 54: Fuzzy sample under $S_3 = 32$

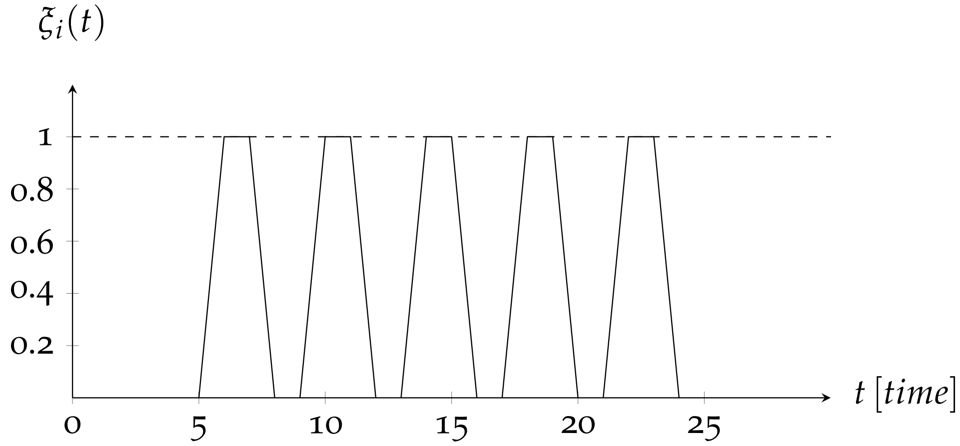
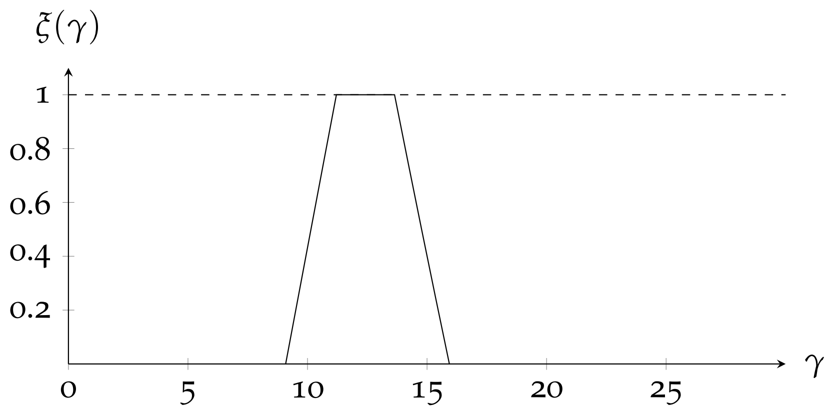


Figure 55: Characterizing function of the fuzzy parameter estimate $\hat{\gamma}^*$



5.11.5 *Linear Acceleration Function and Fuzzy Life Times*

Linear acceleration functions explained in sub-section 4.6 are based on precise life times, therefore this needs to be generalized for fuzzy life times. For fuzzy life time observations $t^*_{1,(1)}, ..., t^*_{1,(n)}$ and $t^*_{2,(1)}, ..., t^*_{2,(n)}$ under stress S_1 and S_2 respectively, the generating family of intervals of the fuzzy estimator $\widehat{\ln\alpha_{1,2}}^*$ is obtained through the following equations:

$$C_\delta(t^*_{i,(j)}) = \left[\underline{t}_{i,(j),\delta}\,,\ \overline{t}_{i,(j),\delta}\right] \quad \forall\,\delta\,\in\,(0\,,\,1], \quad i = 1,2.$$

$$A_\delta(\widehat{\ln\alpha_{1,2}}^*) = \left[\frac{1}{m}\sum_{j=1}^{m}\ln\frac{\underline{t}_{1,(j),\delta}}{\overline{t}_{2,(j),\delta}}\,,\ \frac{1}{m}\sum_{j=1}^{m}\ln\frac{\overline{t}_{1,(j),\delta}}{\underline{t}_{2,(j),\delta}}\right] \quad \forall\,\delta\,\in\,(0\,,\,1]$$

From the generating family of intervals, i.e. $\left(A_\delta(\widehat{\ln\alpha_{1,2}}^*); \quad \forall\delta\in(0\,,\,1]\right)$ the characterizing function of the fuzzy estimator $\widehat{\ln\alpha_{1,2}}^*$ can be obtained by the Construction lemma.

Characterizing functions of fuzzy samples, and fuzzy estimator of the parameter $\ln\alpha_{1,2}$ are depicted in figures 56 to 58 respectively.

Figure 56: Fuzzy sample under $S_1 = 1$

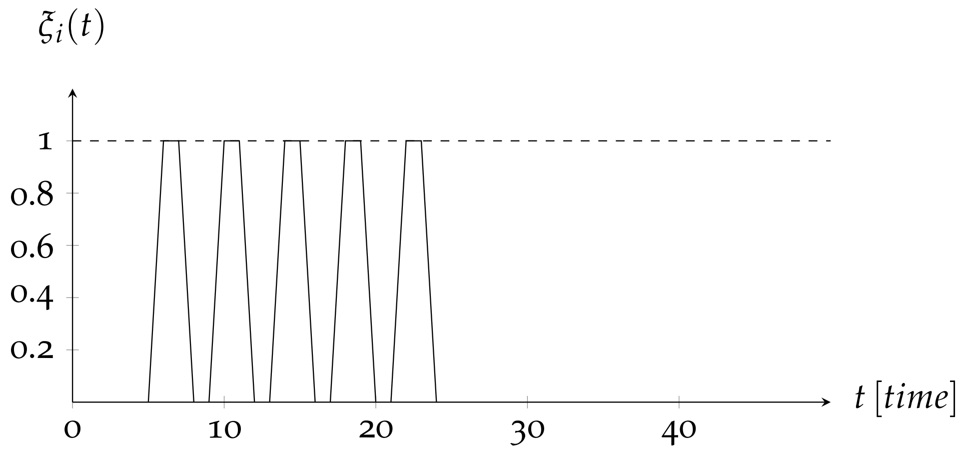


Figure 57: Fuzzy sample under $S_1 = 1.5$

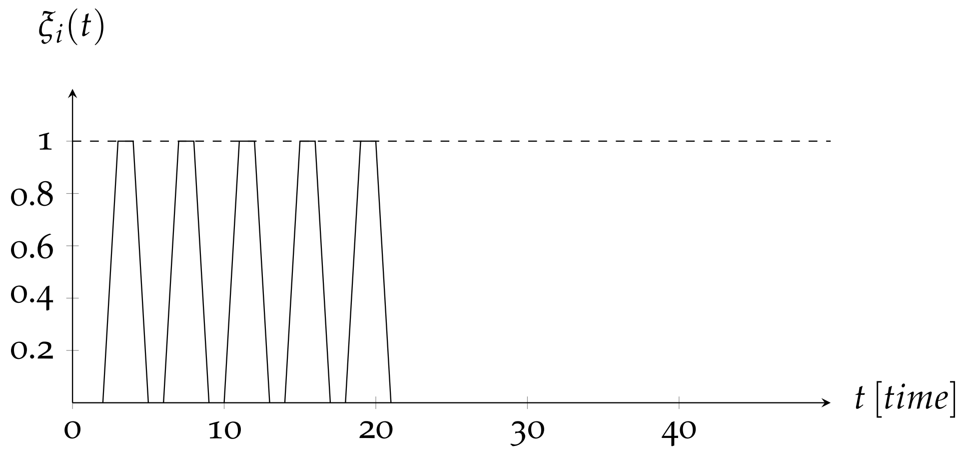
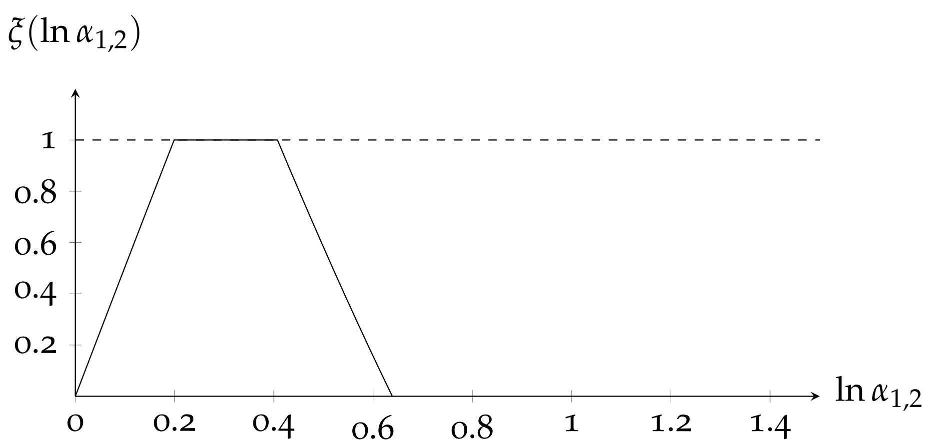


Figure 58: Characterizing function of $\widehat{\ln \alpha_{1,2}}^*$



The empirical cumulative distribution function explained in section 4.6 can be generalized for fuzzy life time data.

The relationship between cumulative distribution functions under different stress levels $S_1 < S_2$ in a non-parametric model is the following:

$$F(t|S_2) = F(\alpha_{1,2} \cdot t|S_1) \qquad \alpha_{1,2} > 1, \quad \forall t > 0 \tag{5.1}$$

Let us consider T_1 to be the life time under stress S_1 and T_2 is the life time under stress S_2.

Then equation (5.1) implies

$$Pr(T_2 \leq t) = Pr(T_1 \leq \alpha_{1,2} \cdot t) = Pr(\frac{T_1}{\alpha_{1,2}} \leq t).$$

If $F(\cdot)$ is invertible it follows

$$T_2 = \frac{T_1}{\alpha_{1,2}}$$

and

$$T_1 = \alpha_{1,2} \cdot T_2. \tag{5.2}$$

Since life time observations are fuzzy, therefore the CDF from equation (5.1) has to be generalized for fuzzy life time data.

For fuzzy life times the generalized CDFs are written as

$$F^*(t|S_2) = F^*(\alpha_{1,2} \cdot t|S_1).$$

Based on precise linear acceleration function and fuzzy life time observations the relationship in equation (5.2) takes the following form:

$$T_1^* = \alpha_{1,2} \cdot T_2^* \tag{5.3}$$

The relationship given in equation (5.3) leads to the conclusion that uncertainty of T_2 implies bigger uncertainty of T_1 by $\alpha_{1,2} > 1$.

Let t_2^* be a fuzzy life time observed under stress S_2.

The extrapolated life time t_1^* given in equation (5.3) is a fuzzy number having δ-cuts

$$C_\delta(t_1^*) = \left[\, \underline{t}_{1,\delta}\, ,\, \overline{t}_{1,\delta} \right] \qquad \forall \delta \in (0\, ,\, 1].$$

Then the lower and upper ends of the generating family of intervals for the transformed life time are defined by

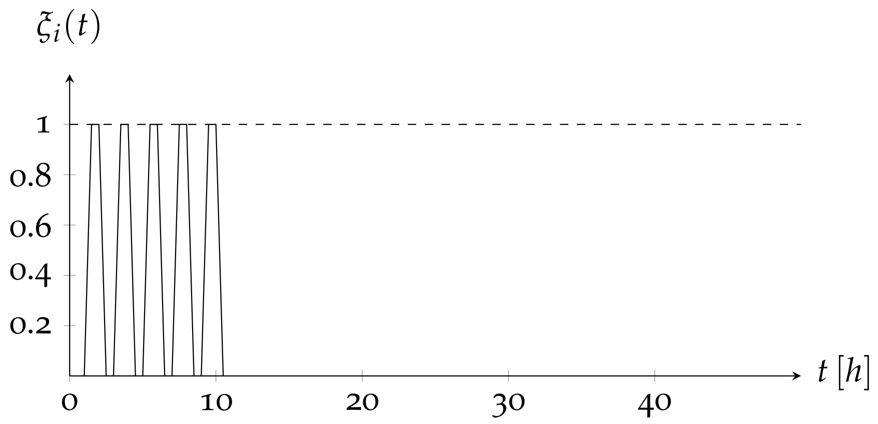
$$\underline{t}_{1,\delta} = \alpha_{1,2} \cdot \underline{t}_{2,\delta} \qquad \forall \delta \in (0\, ,\, 1]$$

$$\overline{t}_{1,\delta} = \alpha_{1,2} \cdot \overline{t}_{2,\delta} \qquad \forall \delta \in (0\, ,\, 1].$$

Example:

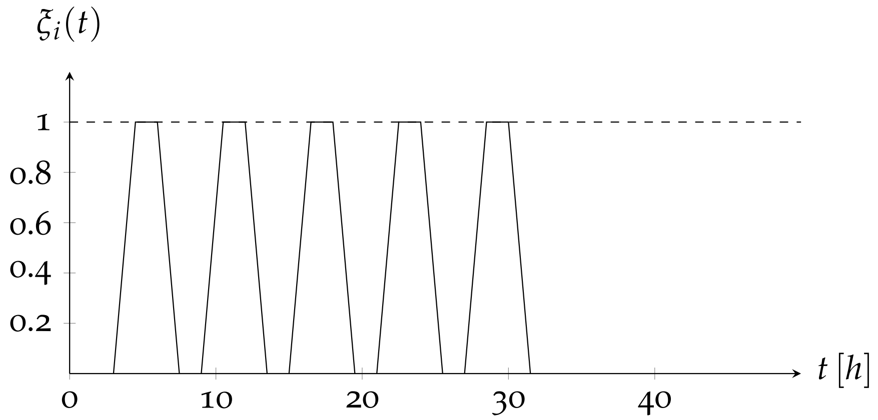
Consider a precise linear acceleration constant $\alpha_{1,2} = 3$ and fuzzy sample $t_{2,1}^*$, $t_{2,2}^*$, ..., $t_{2,5}^*$ under stress level S_2 in figure 59.

Figure 59: Fuzzy sample under $S_2 = 1.5N$



In figure 60 are the characterizing functions of the extrapolated life times under S_1 based on equation (5.3).

Figure 60: Characterizing functions of the transformed fuzzy life times



This means that the δ-cuts of the extrapolated fuzzy life times $t^*_{1,1}$, $t^*_{1,2}$, ..., $t^*_{1,5}$ based on the fuzzy life times $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$ contain more fuzziness.

For the fuzzy empirical CDFs based on fuzzy life times under S_2 and transformed life times data at $\delta = 0^+$ are considered which are given below in figure 61:

$\alpha_{1,2} = 3$

$C_{0^+}(t^*_{2,1}) = [1\,,\,2.5]$

$C_{0^+}(t^*_{2,2}) = [3\,,\,4.5]$

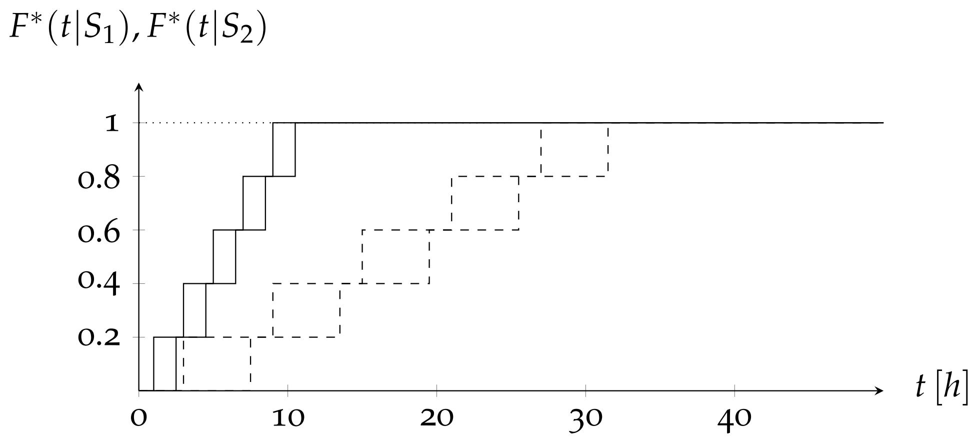
$C_{0^+}(t^*_{2,3}) = [5\,,\,6.5]$

$C_{0^+}(t^*_{2,4}) = [7\,,\,8.5]$

$C_{0^+}(t^*_{2,5}) = [9\,,\,10.5]$

are the lower and upper ends of the support of the fuzzy life times $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$ at $\delta = 0^+$

Figure 61: Lower and upper δ-level curves of the fuzzy empirical CDF based on fuzzy life times and linear acceleration function



The solid lines represent the support functions of the fuzzy empirical CDF based on $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$, and the dashed lines represent the fuzzy empirical CDF for the transformed fuzzy sample obtained by equation (5.3).

5.11.6 *Power Type Acceleration Function and Fuzzy Life Times*

The power type acceleration function explained in section 4.7 is also based on precise life times, therefore this needs to be generalized for fuzzy life times.

Let $t^*_{1,(1)}, \ldots, t^*_{1,(n)}$ and $t^*_{2,(1)}, \ldots, t^*_{2,(n)}$ be the corresponding fuzzy life time observations under stress S_1 and S_2 respectively. Then the δ-cuts of fuzzy life times, and estimators are denoted by

$$C_\delta(t^*_{j,(i)}) = \left[\underline{t}_{i,(j),\delta}\,,\ \overline{t}_{i,(j),\delta}\right] \quad \forall\,\delta \in (0,1],$$

and

$$C_\delta(\hat{\beta}^*_{1,2}) = \left[\underline{\beta}_{1,2,\delta}\,,\ \overline{\beta}_{1,2,\delta}\right] \quad \forall\,\delta \in (0,1],$$

$$C_\delta(\widehat{\ln\alpha_{1,2}}^{\,*}) = \left[\underline{\ln\alpha}_{1,2,\delta}\,,\ \overline{\ln\alpha}_{1,2,\delta}\right] \quad \forall\,\delta \in (0,1].$$

Equations for generating families of intervals for the generalized estimators are given below:

$$A_\delta(\hat{\beta}^*_{1,2}) = \left[\frac{1}{m}\sum_{k=1}^{m}\frac{\ln\underline{t}_{1,(k+1),\delta} - \ln\overline{t}_{1,(k),\delta}}{\ln\overline{t}_{2,(k+1),\delta} - \ln\underline{t}_{2,(k),\delta}}\,,\ \frac{1}{m}\sum_{k=1}^{m}\frac{\ln\overline{t}_{1,(k+1),\delta} - \ln\underline{t}_{1,(k),\delta}}{\ln\underline{t}_{2,(k+1),\delta} - \ln\overline{t}_{2,(k),\delta}}\right]$$

$$\forall\,\delta \in (0,1]$$

$$A_\delta(\widehat{\ln\alpha_{1,2}}^{\,*}) = \left[\frac{1}{m}\sum_{k=1}^{m}[\ln\underline{t}_{1,(k),\delta} - \overline{\hat{\beta}}_{1,2,\delta}\ln\overline{t}_{2,(k),\delta}]\,,\ \frac{1}{m}\sum_{k=1}^{m}[\ln\overline{t}_{1,(k),\delta} - \underline{\hat{\beta}}_{1,2,\delta}\ln\underline{t}_{2,(k),\delta}]\right]$$

$$\forall\,\delta \in (0,1]$$

From these generating families of intervals, i.e. $\left(A_\delta(\widehat{\ln\alpha_{1,2}}^*); \quad \forall\delta \in (0\,,\,1]\right)$ and $\left(A_\delta(\hat{\beta}^*_{1,2}); \quad \forall\delta \in (0\,,\,1]\right)$ the characterizing functions of the fuzzy estimators $\widehat{\ln\alpha_{1,2}}^*$ and $\hat{\beta}^*_{1,2}$ can be obtained through the Construction lemma.

As an example two precise stress levels, i.e. $S_1 = 1, S_2 = 2,$ and $n_1 = n_2 = 5$ are considered.

Characterizing functions of the fuzzy samples, and fuzzy estimates are depicted in figures 59 to 62 respectively.

Figure 62: Fuzzy sample under $S_1 = 1$

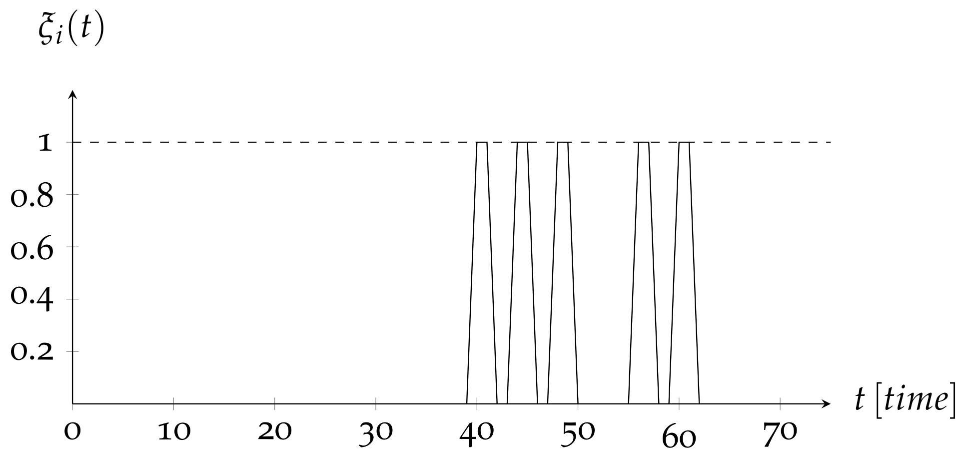


Figure 63: Fuzzy sample under $S_2 = 2$

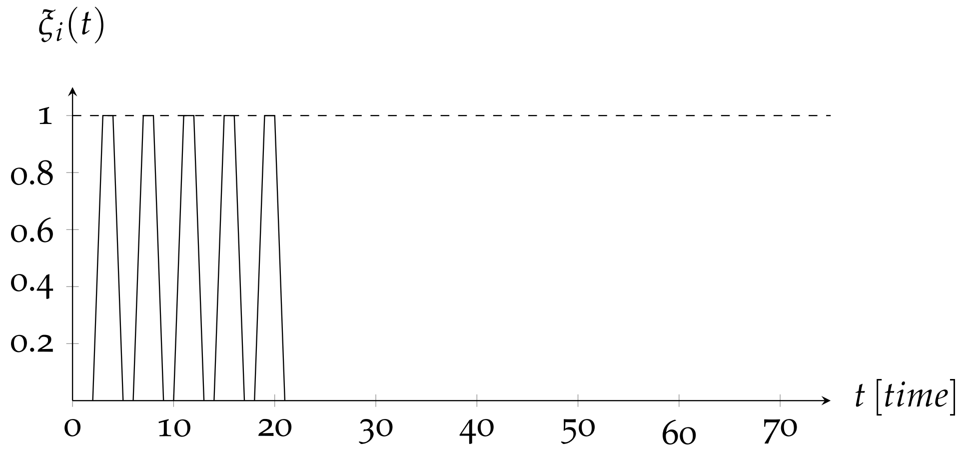


Figure 64: Characterizing function of $\hat{\beta}^*_{1,2}$

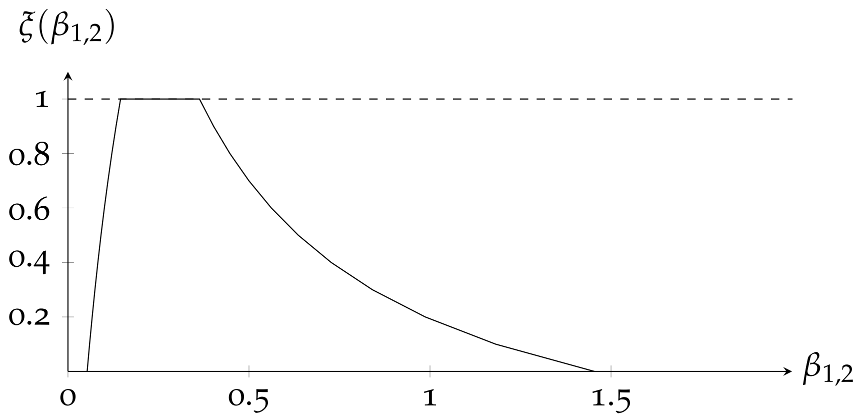
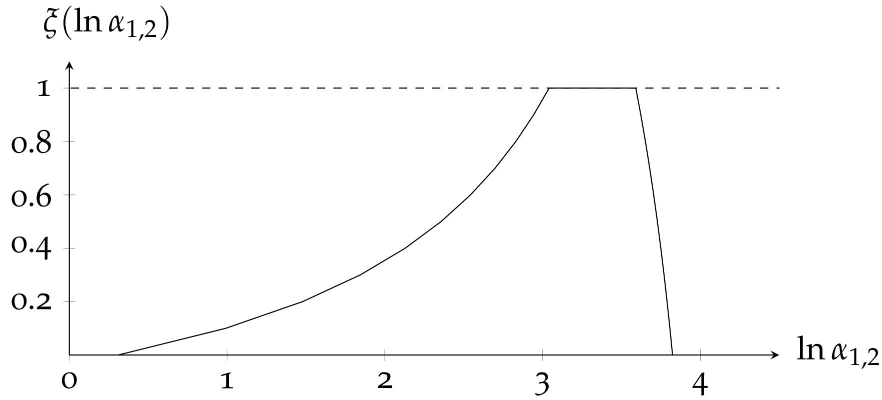


Figure 65: Characterizing functions for $\widehat{\ln \alpha_{1,2}}^*$



Empirical cumulative distribution functions for power type acceleration function explained in section 4.7 can be generalized for fuzzy life time data.

This relationship between cumulative distribution functions under different stress levels $S_1 < S_2$ in a non-parametric model is the following:

$$F(t|S_2) = F(\alpha_{1,2} \cdot t^{\beta_{1,2}}|S_1) \qquad \text{with } \alpha_{1,2} > 1, \text{ and } \beta_{1,2} > 1, \quad \forall t > 0 \tag{5.4}$$

Let us consider T_1 to be the life time under stress S_1 and T_2 is the life time under stress S_2.

Then equation (5.4) implies

$$Pr(T_2 \leq t) = Pr(T_1 \leq \alpha_{1,2} \cdot t^{\beta_{1,2}}) = Pr\left(\left(\frac{T_1}{\alpha_{1,2}}\right)^{1/\beta_{1,2}} \leq t\right).$$

If $F(\cdot)$ is invertible it follows

$$T_2 = \left(\frac{T_1}{\alpha_{1,2}}\right)^{1/\beta_{1,2}}$$

and

$$T_1 = \alpha_{1,2} \cdot T_2^{\beta_{1,2}}. \tag{5.5}$$

Since life time observations are fuzzy, therefore the CDF from equation (5.4) has to be generalized for fuzzy life time data.

For fuzzy life times the generalized CDFs are written as

$$F^*(t|S_2) = F^*(\alpha_{1,2} \cdot t^{\beta_{1,2}}|S_1) \qquad \text{with } \alpha_{1,2}, \text{ and } \beta_{1,2} > 1, \quad \forall t > 0.$$

Based on power type acceleration function and fuzzy life time observations the relationship in equation (5.5) takes the following form:

$$T_1^* = \alpha_{1,2} \cdot \left(T_2^*\right)^{\beta_{1,2}} \tag{5.6}$$

The relationship given in equation (5.6) leads to the conclusion that uncertainty of T_2 implies bigger uncertainty of T_1 by $\alpha_{1,2} > 1$, and $\beta_{1,2} > 1$.

Let t_2^* be a fuzzy life time observed under stress S_2.

The transformed fuzzy life time t_1^* given by equation (5.6) is a fuzzy number having δ-cuts

$$C_\delta(t_1^*) = \left[\underline{t}_{1,\delta}\, , \, \overline{t}_{1,\delta}\right] \qquad \forall \delta \in (0\, , \, 1].$$

Then the lower and upper ends of the generating family of intervals for the transformed life time are defined by

$$\underline{t}_{1,\delta} = \alpha_{1,2} \cdot \underline{t}_{2,\delta}^{\beta_{1,2}} \qquad \forall \delta \in (0\,,\,1]$$

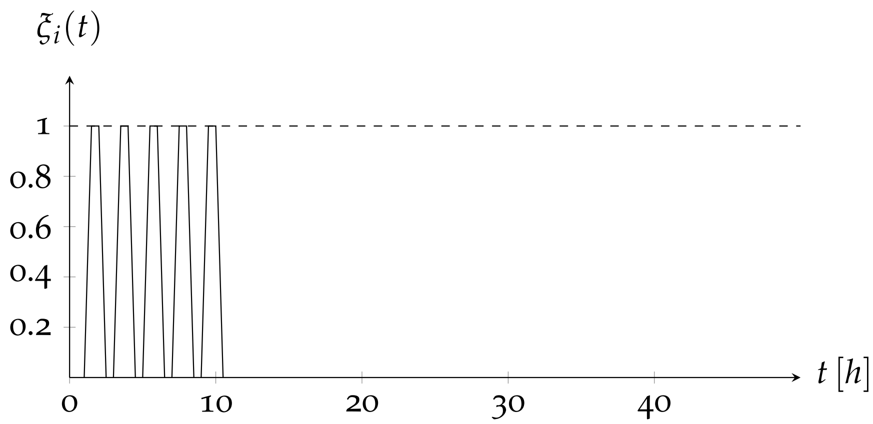
$$\overline{t}_{1,\delta} = \alpha_{1,2} \cdot \overline{t}_{2,\delta}^{\beta_{1,2}} \qquad \forall \delta \in (0\,,\,1].$$

Example:

Consider a power type acceleration function with constants $\alpha_{1,2} = 3$, and $\beta_{1,2} = 1.5$.

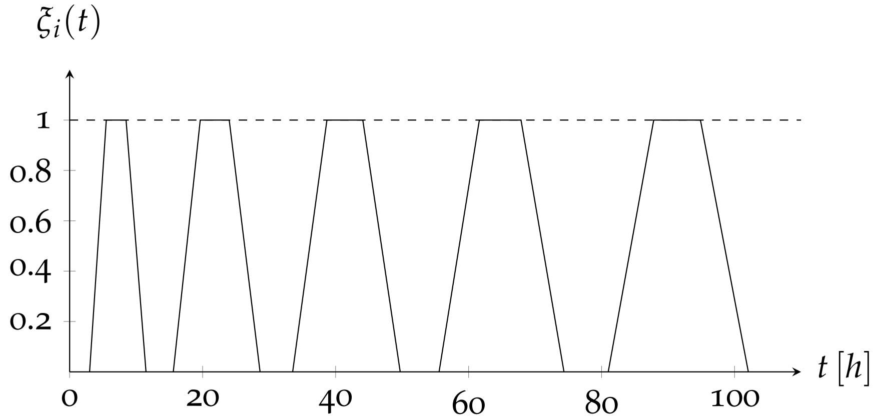
Characterizing functions of a fuzzy sample $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$ under stress level S_2 are given in figure 66.

Figure 66: Fuzzy sample under $S_2 = 2N$



In figure 67 are the characterizing functions of the transformed fuzzy life times under S_1 based on equation (5.6).

Figure 67: Characterizing functions of the transformed fuzzy life times



This means that the δ-cuts of the transformed fuzzy life times $t^*_{1,1}$, $t^*_{1,2}$, ..., $t^*_{1,5}$ based on the fuzzy life times $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$ contain more fuzziness.

For the fuzzy empirical CDF of the fuzzy life times under S_2, and transformed fuzzy life times data at $\delta = 0^+$ the following holds:

$\alpha_{1,2} = 3$

$\beta_{1,2} = 1.5$

$C_{0^+}(t^*_{2,1}) = [1\,,\, 2.5]$

$C_{0^+}(t^*_{2,2}) = [3\,,\, 4.5]$

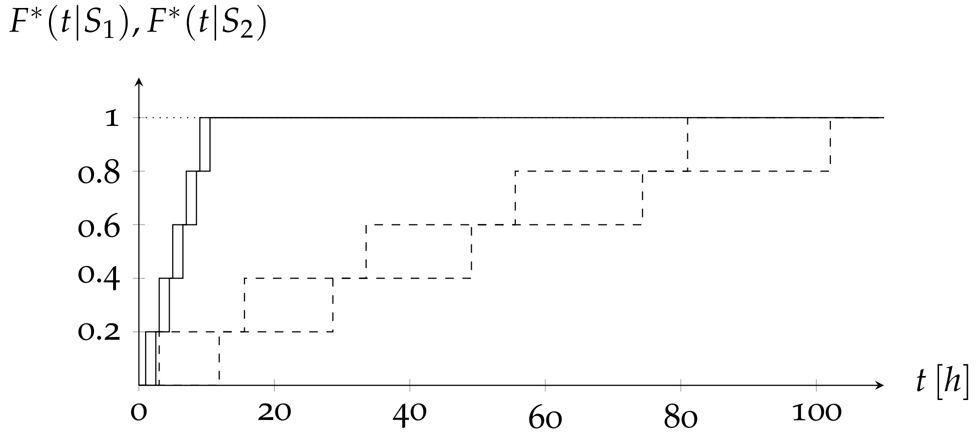
$C_{0^+}(t^*_{2,3}) = [5\,,\, 6.5]$

$C_{0^+}(t^*_{2,4}) = [7\,,\, 8.5]$

$C_{0^+}(t^*_{2,5}) = [9\,,\, 10.5]$

are lower and upper ends of the support of the fuzzy life times $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$ at $\delta = 0^+$.

Figure 68: Lower and upper δ-level curves for $\delta = 0^+$ of the fuzzy empirical CDF based on fuzzy life times for power type acceleration function



The solid lines represent the fuzzy empirical CDF based on $t^*_{2,1}$, $t^*_{2,2}$, ..., $t^*_{2,5}$, and the dashed lines represent the fuzzy empirical CDF for the transformed fuzzy sample obtained by equation (5.6).

This shows the dramatic increase of fuzziness in accelerated life testing.

6

CONCLUSIONS

Statistical analysis techniques of life time data can be traced back centuries, which reflects its importance and resulted into rapid development of various statistical approaches during the last few decades to get refined and relevant results.

The emergence of technological advancement augments the increase in life time of units, which with only few observations draw inference about the aggregate of units. Hence, it is pertinent to utilize all the available information in the best possible manner.

Likewise, the world of real measurements establishes the fact that exact measurement of a continuous quantity is unattainable, and all such quantities are more or less fuzzy. While, most of the classical statistical tools allied with continuous quantities are based on precise observations, and ignore fuzziness.

Furthermore, it has already been mentioned that life time observations are more or less fuzzy; therefore, dealing with life time data fuzziness of individual observations needs to be considered.

For this purpose in addition to stochastic models, the best up to date, fuzzy number approaches are more relevant than precise measurements.

In this study some of the most important techniques pertaining to life time data are generalized for fuzzy life times to draw more suitable and appropriate inference.

Fuzzy parameter estimation, fuzzy survival function estimation and fuzzy hazard rate estimation of the Exponential distribution, Weibull distribution, Gamma distribution, and Lognormal distribution are encompassed in the study.

Furthermore, parameter estimation of bathtub hazard rate distributions and three parameter Weibull distribution were also generalized for fuzzy life time data.

Dealing with life time data, Accelerated Life Testing (ALT) approaches are regarded as one of the most prominent fields of statistics; therefore, these techniques also require generalization for fuzzy life time data.

ALT techniques, for the first time in the present study, are reckoned for fuzzy life time data. Some techniques related to constant stress level, step-stress level, power rule model, non-parametric estimation and acceleration functions are considered and generalized for fuzzy life time data.

Additionally, dramatic increase in the fuzziness of the transformed fuzzy life times by acceleration functions were observed.

The proposed estimators are based on two types of uncertainty: fuzziness of individual observations and stochastic variation among the observations.

Therefore, inference based on the proposed generalized estimators are more suitable and realistic.

7

BIBLIOGRAPHY

Balakrishnan, N. and Kateri, M. (2008). On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Statistics & Probability Letters*, 78(17):2971 – 2975.

Balakrishnan, N., Kundu, D., Ng, H. K. T., and Kannan, N. (2007). Point and interval estimation for a simple step-stress model with type II censoring. *Journal of Quality Technology*, 39(1):35–47.

Barbato, G., Germak, A., Genta, G., and Barbato, A. (2013). *Measurements for Decision Making. Measurements and Basic Statistics*. Esculapio, Milano.

Buckley, J. J. (2006). *Fuzzy Probability and Statistics*. Springer, Berlin Heidelberg.

Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics & Probability Letters*, 49(2):155 – 161.

Couallier, V., Gerville-Réache, L., Huber-Carol, C., Limnios, N., and Mesbah, M. (2013). *Statistical Models and Methods for Reliability and Survival Analysis*. Wiley, London.

Cran, G. W. (1988). Moment estimators for the 3-parameter Weibull distribution. *IEEE Transactions on Reliability*, 37(4):360–363.

Deshpande, J. V. and Purohit, S. G. (2005). *Life Time Data: Statistical Models and Methods*. World Scientific Publishing, Singapore.

D'Urso, P. (2003). Linear regression analysis for fuzzy/crisp input and fuzzy/crisp output data. *Computational Statistics & Data Analysis*, 42(1-2):47–72.

Frühwirth-Schnatter, S. (1993). On fuzzy Bayesian inference. *Fuzzy Sets and Systems*, 60(1):41–58.

Hamada, M., Wilson, A., Reese, C., and Martz, H. (2008). *Bayesian Reliability*. Springer, New York.

Haupt, E. and Schäbe, H. (1992). A new model for a lifetime distribution with bathtub shaped failure rate. *Microelectronics Reliability*, 32(5):633 – 639.

Hosmer, D. and Lemeshow, S. (1999). *Applied Survival Analysis: Regression Modeling of Time to Event Data*. Wiley, New York.

Huang, H.-Z., Zuo, M. J., and Sun, Z.-Q. (2006). Bayesian reliability analysis for fuzzy lifetime data. *Fuzzy Sets and Systems*, 157(12):1674–1686.

Hung, W.-L. and Liu, Y.-C. (2004). Estimation of Weibull parameters using a fuzzy least-squares method. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 12(05):701–711.

Ibrahim, J., Chen, M., and Sinha, D. (2001). *Bayesian Survival Analysis*. Springer, New York.

Kalbfleisch, J. and Prentice, R. (2011). *The Statistical Analysis of Failure Time Data*. Wiley, New Jersey.

Kaplan, E. L. and Meier, P. (1958). Nonparametric estimation from incomplete observations. *Journal of the American Statistical Association*, 53(282):457–481.

Kleinbaum, D. and Klein, M. (2005). *Survival Analysis: A Self-Learning Text*. Springer, New York.

Klir, G. J. and Yuan, B. (1995). *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice-Hall, New Jersey.

Lee, E. T. and Wang, J. W. (2013). *Statistical Methods for Survival Data Analysis*. Wiley, New Jersey.

Lee, K. H. (2006). *First Course on Fuzzy Theory and Applications*. Springer, London.

Levenbach, G. J. (1957). Accelerated life testing of capacitors. *IEEE Transactions on Reliability and Quality Control*, PGRQC(10):9–20.

Meeker, W. and Escobar, L. (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

Miller, R. (2011). *Survival Analysis*. Wiley, New York.

Nakama, T. (2013). *Statistical Procedures for Fuzzy Data in Medical Research*. Springer, Berlin - Heidelberg.

Nelson, W. (2005). *Applied Life Data Analysis*. Wiley, New Jersey.

Nguyen, G. T. and Wu, B. (2006). *Fundamentals of Statistics with Fuzzy Data*. Springer, New York.

Pak, A., Parham, G., and Saraj, M. (2013). Reliability estimation in Rayleigh distribution based on fuzzy lifetime data. *International Journal of System Assurance Engineering and Management*, 5(4):487–494.

Shaked, M., Zimmer, W. J., and Ball, C. A. (1979). A nonparametric approach to accelerated life testing. *Journal of the American Statistical Association*, 74(367):694–699.

Szeliga, E. (2004). Structural reliability - fuzzy sets theory approach. *Journal of Theoretical and Applied Mechanics*, 42(3):651–666.

Tzafestas, S. and Venetsanopoulos, A. (1994). *Fuzzy Reasoning in Information, Decision, and Control Systems*. Kluwer Academic Publishers, Norwell.

Viertl, R. (1988). *Statistical Methods in Accelerated Life Testing*. Vandenhoeck & Ruprecht, Göttingen.

Viertl, R. (1997). On statistical inference for non-precise data. *Environmetrics*, 8(5):541–568.

Viertl, R. (2006). Univariate statistical analysis with fuzzy data. *Computational Statisitcs & Data Anaysis.*, 51(1):133–147.

Viertl, R. (2009). On reliability estimation based on fuzzy lifetime data. *Journal of Statistical Planning and Inference*, 139(5):1750 – 1755.

Viertl, R. (2011). *Statistical Methods for Fuzzy Data*. Wiley, Chichester.

Viertl, R. and Hareter, D. (2006). *Beschreibung und Analyse unscharfer Information: Statistische Methoden für unscharfe Daten*. Springer, Wien.

Wu, H.-C. (2009). Statistical confidence intervals for fuzzy data. *Expert Systems with Applications*, 36(2):2670 – 2676.

Zadeh, L. (1965). Fuzzy sets. *Information and Control*, 8(3):338 – 353.

Zimmermann, H. (2001). *Fuzzy Set Theory - and Its Applications*. Kluwer Academic Publishers, Norwell.