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Subdirectly irreducible commutative multiplicatively idempotent semirings

Ivan Chajda and Helmut Länger

Abstract. Commutative multiplicatively idempotent semirings were studied by the authors and F. Švrček, where the connections to distributive lattices and unitary Boolean rings were established. The variety of these semirings has nice algebraic properties and hence there arose the question to describe this variety, possibly by its subdirectly irreducible members. For the subvariety of so-called Boolean semirings, the subdirectly irreducible members were described by F. Guzmán. He showed that there were just two subdirectly irreducible members, which are the 2-element distributive lattice and the 2-element Boolean ring. We are going to show that although commutative multiplicatively idempotent semirings are at first glance a slight modification of Boolean semirings, for each cardinal $n > 1$, there exist at least two subdirectly irreducible members of cardinality n and at least 2^n such members if n is infinite. For $n \in \{2, 3, 4\}$ the number of subdirectly irreducible members of cardinality n is exactly 2.

1. Introduction

Semirings form a useful tool in investigations both in algebra and computer science, see e.g., [4] for details. Among them, an important role is played by the variety $\mathcal{I}$ of multiplicatively idempotent semirings, since these are close to Boolean rings, which form a base for the classical propositional calculus and are used also in computer science, see e.g., [4]. However, in some considerations, it is appropriate to study more general structures than rings and thus the concept of commutative multiplicatively idempotent semirings is a natural generalization. Especially, the concept of a so-called Boolean semiring was introduced by F. Guzmán in [3]. However, we believe that also a more general case is worth studying. Our paper is devoted to the problem of determining the subdirectly irreducible members of the variety of commutative members of $\mathcal{I}$. Contrary to the case studied by F. Guzmán, we will show that there exists an infinite number of subdirectly irreducible members that are, moreover, linearly ordered, and an infinite number of such members that are not linearly ordered.

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In fact, we prove that such algebras form a proper class, i.e., the variety of commutative multiplicatively idempotent semirings is not residually small.

2. Basic notions

Since there exist various definitions of semirings in literature, we will use the following one taken from the monograph [2] by J. S. Golan.

Definition 2.1. A *semiring* is an algebra $\mathbf{S} = (S, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying (i)–(iv).

(i) $(S, +, 0)$ is a commutative monoid.
(ii) $(S, \cdot, 1)$ is a monoid.
(iii) The operation $\cdot$ is distributive with respect to $+$.
(iv) $x0 = 0x = 0$ for all $x \in S$.

Definition 2.2. A semiring is called

- *commutative* if it satisfies the identity $xy = yx$;
- *multiplicatively idempotent* if it satisfies the identity $xx = x$;
- *additively idempotent* if it satisfies the identity $x + x = x$.

In the following, $\mathcal{C}$ denotes the variety of commutative multiplicatively idempotent semirings and $\mathcal{B}$ the subvariety of $\mathcal{C}$ defined by the identity $1+x+x = 1$. The members of $\mathcal{B}$ are called *Boolean semirings* (cf. [3]).

That $\mathcal{B} \neq \mathcal{C}$, follows from Example 2.4 below.

Example 2.3. Every Boolean ring and every bounded distributive lattice belongs to $\mathcal{B}$.

It is easy to see that up to isomorphism there exist exactly two two-element semirings, namely the two-element Boolean ring and the two-element lattice.

As proved by F. Guzmán, these are the only subdirectly irreducible members of $\mathcal{B}$. The following example exhibits a three-element member of $\mathcal{C}$ that does not belong to $\mathcal{B}$.

Example 2.4. As we will see later, the semiring $\mathbf{A}$ with universe $\{0, a, 1\}$ and operations defined by

$+$	0	a	1
0	0	a	1
a	a	a	a
1	1	a	1

and

$\cdot$	0	a	1
0	0	0	0
a	0	a	a
1	0	a	1

is a subdirectly irreducible member of $\mathcal{C}$ that does not belong to $\mathcal{B}$ since $1+a+a = a + a = a \neq 1$. Therefore, $\mathbf{A}$ is neither a Boolean ring nor a distributive lattice nor isomorphic to a nontrivial subdirect product of these.

If $\mathbf{S} \in \mathcal{C}$ is a semiring with universe S, then $(S, \cdot, 0, 1)$ is a bounded meet-semilattice. In the following, let $\leq$ denote the corresponding partial order relation. If $\mathbf{S}$ is an additively idempotent semiring, then $(S, +, 0)$ is a join-semilattice with smallest element 0.

3. Structure of subdirectly irreducible semirings

In order to study subdirectly irreducible semirings, we need to describe their congruences. It is useful to introduce congruences of the following shape.

Definition 3.1. For every commutative semiring $\mathbf{S}$ with universe S and every $a \in S$, put $\Theta_a := \{(x, y) \in S^2 \mid xa = ya\}$.

Lemma 3.2. *If* $\mathbf{S}$ *is a commutative semiring with universe* S *and* $a \in S$*, then* $\Theta_a \in \operatorname{Con} \mathcal{S}$.

Proof. Obviously, Θ_a is an equivalence relation on S. Moreover, $(b, c) \in \Theta_a$ and $d \in S$ imply $(b + d, c + d), (bd, cd) \in \Theta_a$. □

Lemma 3.3. *Let* $\mathbf{S} \in \mathcal{C}$ *be a semiring with universe* S*,* $a, b, c \in S$*, and* $\Theta \in \operatorname{Con} \mathcal{S}$*. Then* (i)–(iii) *hold.*

(i) *If* $a \leq b \leq c$ *and* $(a, c) \in \Theta$*, then* $(a, b) \in \Theta$.
(ii) $\Theta_a = \Delta$ $(:= \{(x, x) \mid x \in S\})$ *if and only if* $a = 1$.
(iii) *If* $a, b \leq c$*, then* $a + b \leq c$.

Proof. (i): We have $(a, b) = (ab, cb) \in \Theta$.

(ii): Obviously, $\Theta_1 = \Delta$. If, conversely, $\Theta_a = \Delta$, then $(a, 1) \in \Theta_a = \Delta$, and hence $a = 1$.

(iii) We have $(a + b)c = ac + bc = a + b$. □

Remark 3.4. Lemma 3.3(i) says that the classes of congruences on a member of $\mathcal{C}$ are convex with respect to $\leq$.

In the following proposition, we apply congruences of the form Θ_a in order to describe the monolith of subdirectly irreducible members of $\mathcal{C}$.

Proposition 3.5. *If* $\mathbf{S}$ *is a subdirectly irreducible member of* $\mathcal{C}$*, then* $S \setminus \{1\}$ *has a greatest element* a *and* $\Delta \cup \{a, 1\}^2$ *is the monolith* μ *of* $\mathbf{S}$.

Proof. Let $b, c, d \in S$ and assume $(b, c) \in \mu$ and $b \neq c$. Then according to Lemma 3.3, $1 \notin \{b, c\}$ would imply $\Theta_b, \Theta_c \neq \Delta$, and hence $\mu \subseteq \Theta_b \cap \Theta_c$, whence $b = bb = cb = bc = cc = c$, a contradiction. Hence, $1 \in \{b, c\}$. This shows that there exists an element a of $S \setminus \{1\}$ with $\mu = \Delta \cup \{a, 1\}^2$. If $d \neq 1$, then $\Delta_d \neq \Delta$, and hence $\mu \subseteq \Delta_d$, whence $da = ad = 1d = d$, i.e., $d \leq a$. This shows that a is the greatest element of $(S \setminus \{1\}, \leq)$. □

Lemma 3.6. *If* $\mathbf{S}$ *is a subdirectly irreducible member of* $\mathcal{C}$ *with universe* S*,* $|S| > 2$*, and monolith* $\mu = \Delta \cup \{a, 1\}^2$*, then there exists an element* b *of* $S \setminus \{0, 1\}$ *with* $b + 1 \neq 1$.

Proof. Suppose $x+1=1$ for all $x \in S \setminus \{0,1\}$. Then $x+1=1$ for all $x \in S \setminus \{1\}$, and hence according to Lemma 3.3(iii), the relation $\Delta \cup [0,a]^2$ (where $[0,a] := \{x \in S \mid x \leq a\}$) would be a non-trivial congruence on $\mathbf{S}$ not including μ, contradicting the subdirect irreducibility of $\mathbf{S}$. This can be seen as follows: if $b,c,d \in [0,a]$, then $b+d, c+d \in [0,a]$ according to Lemma 3.3(iii). Moreover, $b+1=1=c+1$. Hence, there exists an element e of $S \setminus \{0,1\}$ with $e+1 \neq 1$. □

4. Constructing subdirectly irreducible semirings

Definition 4.1. For every integer $n>1$ let $\mathbf{S}_n$ be the semiring with universe $S_n = \{1,\dots,n\}$ and operations defined for $x,y \in S_n$, as follows:

$$x+y := \begin{cases} \max(x,y), & \text{for } x \text{ and } y \text{ odd,} \\ y, & \text{for } x \text{ odd and } y \text{ even,} \\ x, & \text{for } x \text{ even and } y \text{ odd,} \\ \min(x,y), & \text{for } x \text{ and } y \text{ even;} \end{cases}$$

$$xy := \min(x,y)$$

Here the operation symbols 0 and 1 are interpreted as 1 and n, respectively. Moreover, let $\mathbf{T}_n$ be the semiring that coincides with $\mathbf{S}_n$ with the only exception that $n+n := n-1$ instead of $n+n := n$.

Observe that in $\mathbf{S}_n$, we have $x+y \in \{x,y\}$ for all $x,y \in S_n$. This is not true for $\mathbf{T}_n$.

Now we are ready to prove that for any integer $n>1$, there exist subdirectly irreducible members of $\mathcal{C}$ of cardinality n that are, moreover, linearly ordered with respect to the induced semilattice order.

Theorem 4.2. *For every integer $n>1$, the semirings $\mathbf{S}_n$ and $\mathbf{T}_n$ are non-isomorphic subdirectly irreducible members of the variety $\mathcal{C}$. Moreover, we have $\mathbf{S}_2, \mathbf{T}_2 \in \mathcal{B}$ and $\mathbf{S}_n, \mathbf{T}_n \notin \mathcal{B}$ if $n>2$. Finally, the congruence lattices of both $\mathbf{S}_n$ and $\mathbf{T}_n$ are n-element chains.*

Proof. First we consider $\mathbf{S}_n$. Define a linear order relation $\leq_1$ on S_n by

$$1 \leq_1 3 \leq_1 \cdots \leq_1 n-1 \leq_1 n \leq_1 \cdots \leq_1 4 \leq_1 2 \quad \text{if } n \text{ is even;}$$
$$1 \leq_1 3 \leq_1 \cdots \leq_1 n \leq_1 n-1 \leq_1 \cdots \leq_1 4 \leq_1 2 \quad \text{if } n \text{ is odd.}$$

It can be easily checked that $+$ is the maximum operation with respect to $\leq_1$ and $\cdot$ the minimum operation with respect to $\leq$. From this, it follows that $(S_n,+,1)$ and $(S_n,\cdot,n)$ are commutative monoids. It can be easily checked that $\cdot$ is distributive with respect to $+$. Moreover, $x1 = 1x = 1$ for all $x \in S_n$ and, moreover, $\cdot$ is commutative and idempotent. Hence, $\mathbf{S}_n \in \mathcal{C}$. Now let $\Theta \in (\operatorname{Con} \mathbf{S}_n) \setminus \{\Delta\}$ and let a denote the smallest element of S_n belonging to a non-singleton class of Θ. According to Remark 3.4, $(a, a+1) \in \Theta$. Here and in the next sentence, $a+1$ denotes the successor of the integer a. Now,

$$(a+n, (a+1)+n) = \begin{cases} (n, a+1), & \text{if } a \text{ is odd,} \\ (a, n), & \text{if } a \text{ is even.} \end{cases}$$

Hence, $(a, n) \in \Theta$, which shows $\Theta = \Delta \cup [a, n]^2$. By considering all possible cases for $b, c, d \in S_n$ with $a \leq b < c$, one sees that $(b+d, c+d) \in \Theta$. Since all classes of Θ are convex, Θ is compatible with $\cdot$. This shows $\Theta \in \operatorname{Con} \mathbf{S}_n$. Therefore, $\operatorname{Con} \mathbf{S}_n = \{\Delta \cup [x, 1]^2 \mid x \in S_n\}$, and hence $\mathbf{S}_n$ is subdirectly irreducible with monolith $\Delta \cup \{n-1, n\}^2$ and the congruence lattice of $\mathbf{S}_n$ forms an n-element chain. Now consider $\mathbf{T}_n$. Since for $x \in S_n \setminus \{n\}$, we have

$$(n+n)+x = (n-1)+x = n-1 = n+n = n+(n+x) \quad \text{if } x \text{ is odd, and}$$
$$(n+n)+x = (n-1)+x = x = n+x = n+(n+x) \quad \text{if } x \text{ is even;}$$

so $(S_n, +, 1)$ is a commutative monoid. Since

$$(n+n)x = (n-1)x = x = x+x = nx+nx \quad \text{for } x \in S_n \setminus \{n\}, \text{ and}$$
$$(n+n)n = (n-1)n = n-1 = n+n = nn+nn,$$

the multiplication $\cdot$ is distributive with respect to $+$. Hence, $\mathbf{T}_n \in \mathcal{C}$. Similarly as for $\mathbf{S}_n$, it follows that $\mathbf{T}_n$ is subdirectly irreducible with monolith $\Delta \cup \{n-1, n\}^2$ and $\operatorname{Con} \mathbf{T}_n = \operatorname{Con} \mathbf{S}_n$. Since $\mathbf{S}_n$ is additively idempotent and $\mathbf{T}_n$ is not, we have $\mathbf{S}_n \not\cong \mathbf{T}_n$. It is easy to see that $\mathbf{S}_2, \mathbf{T}_2 \in \mathcal{B}$. If $n > 2$, then $n+2+2 = n+2 = 2 \neq n$, and hence $\mathbf{S}_n, \mathbf{T}_n \notin \mathcal{B}$. □

In order to obtain the main result of the paper, we do not need the Axiom of Choice (AC) (which is usually assumed in mathematical papers); for our purposes, the weaker assumption of the Boolean Prime Ideal Property (BPI) is sufficient. This property is strictly weaker than AC. It says that every proper ideal of a Boolean algebra is contained in a prime ideal. It is equivalent to the property that every proper filter of a Boolean algebra is contained in an ultrafilter. Exactly this condition is used in order to prove the Representation Theorem for Boolean algebras. It follows from the BPI that every set can be linearly ordered. It is well known that AC is equivalent to the fact that every set can be well-ordered.

Definition 4.3. For every infinite cardinal k, let $\mathbf{C}_k = (C_k, \leq_2, 0, 1)$ be a bounded chain of cardinality k and $\mathbf{U}_k$ the semiring with universe $U_k := C_k \times \{1, 2\}$ and operations defined as follows:

$$(x, i) + (y, j) := \begin{cases} (\max_{\leq_2}(x, y), 1) & \text{for } (i, j) = (1, 1), \\ (y, 2) & \text{for } (i, j) = (1, 2), \\ (x, 2) & \text{for } (i, j) = (2, 1), \\ (\min_{\leq_2}(x, y), 2) & \text{for } (i, j) = (2, 2), \end{cases}$$

and

$$(x,i)(y,j) := \begin{cases} (x,i) & \text{for } x < y, \\ (x,\min(i,j)) & \text{for } x = y, \\ (y,j) & \text{for } x > y, \end{cases}$$

for $(x,i),(y,j) \in U_k$. Here the operation symbols 0 and 1 are interpreted as $(0,1)$ and $(1,2)$, respectively. Moreover, let $\mathbf{V}_k$ denote the semiring that coincides with $\mathbf{U}_k$ with the only exception that $(1,2)+(1,2) := (1,1)$ instead of $(1,2)+(1,2) := (1,2)$.

Observe that in $\mathbf{U}_k$, we have $x+y \in \{x,y\}$ for all $x,y \in U_k$. This is not true for $\mathbf{V}_k$.

Observe also that if k is a positive integer, then $\mathbf{U}_k \cong \mathbf{S}_{2k}$ and $\mathbf{V}_k \cong \mathbf{T}_{2k}$. Indeed, these isomorphisms are established by the following bijections: if $\mathbf{C}_k$ denotes the chain $a_1 <_2 \cdots <_2 a_k$, then (a_i, j) corresponds to $2i-2+j$ for $i = 1,\ldots,k$ and $j = 1,2$.

The next theorem proves the existence of subdirectly irreducible linearly ordered semirings in $\mathcal{C}$ of arbitrary infinite cardinality under the condition of BPI.

Theorem 4.4. *For every infinite cardinal k, the semirings $\mathbf{U}_k$ and $\mathbf{V}_k$ are non-isomorphic subdirectly irreducible members of $\mathcal{C} \setminus \mathcal{B}$ of cardinality k.*

Proof. First we consider $\mathbf{U}_k$. Define two binary relations $\leq_1$ and $\leq$ on U_k, for all $(x,i),(y,j) \in U_k$, by

$(x,i) \leq_1 (y,j)$ if and only if $((i,j) = (1,1)$ and $x \leq_2 y)$
or $(i,j) = (1,2)$ or $((i,j) = (2,2)$ and $x \geq_2 y)$;

$(x,i) \leq (y,j)$ if and only if $x <_2 y$ or $(x = y$ and $i \leq_2 j)$.

It can be easily checked that $(U_k, \leq_1)$ and $(U_k, \leq)$ are chains and that $+$ is the maximum operation with respect to $\leq_1$ and $\cdot$ the minimum operation with respect to $\leq$. From this, it follows that $(U_k, +, (0,1))$ and $(U_k, \cdot, (1,2))$ are commutative monoids. It can be easily checked that $\cdot$ is distributive with respect to $+$. Moreover, $(x,i)(0,1) = (0,1)(x,i) = (0,1)$ for all $(x,i) \in U_k$, and $\cdot$ is commutative and idempotent. Hence, $\mathbf{U}_k \in \mathcal{C}$. Because for all $x \in C_k$,

$$((1,1)+(x,1),(1,2)+(x,1)) = ((1,1),(1,2)) \text{ and}$$
$$((1,1)+(x,2),(1,2)+(x,2)) = ((x,2),(x,2)),$$

we have $\Delta \cup \{(1,1),(1,2)\}^2 \in \operatorname{Con} \mathbf{U}_k$. Now let $\Theta \in (\operatorname{Con} \mathbf{U}_k) \setminus \{\Delta\}$. Then there exist $(a,i),(b,j) \in U_k$ with $(a,i) < (b,j)$ and $((a,i),(b,j)) \in \Theta \setminus \Delta$. If $i = j = 1$, then $((a,1),(a,2)) \in \Theta$ according to Remark 3.4. If $i = j = 2$, then $((b,1),(b,2)) \in \Theta$ according to Remark 3.4. Hence, there exist $c,d \in C_k$ with $((c,1),(d,2)) \in \Theta$. Now $((c,1)+(1,2),(d,2)+(1,2)) = ((1,2),(d,2))$, and hence $\Theta \supseteq \Delta \cup \{(1,1),(1,2)\}^2$. Therefore, $\Delta \cup \{(1,1),(1,2)\}^2$ is the monolith of $\mathbf{U}_k$, and hence $\mathbf{U}_k$ is subdirectly irreducible.

Now we consider $\mathbf{V}_k$. Since

$$\begin{aligned}((1,2)+(1,2))+(x,1) &= (1,1)+(x,1) = (1,1) = (1,2)+(1,2)\\ &= (1,2)+((1,2)+(x,1)) \quad \text{for all } x \in C_k,\\ ((1,2)+(1,2))+(x,2) &= (1,1)+(x,2) = (x,2) = (1,2)+(x,2)\\ &= (1,2)+((1,2)+(x,2)) \quad \text{for all } x \in C_k \setminus \{1\},\end{aligned}$$

$(U_k, +, (0,1))$ is a commutative monoid. Since

$$\begin{aligned}((1,2)+(1,2))(x,i) &= (1,1)(x,i) = (x,i) = (x,i)+(x,i)\\ &= (1,2)(x,i)+(1,2)(x,i) \quad \text{for all } (x,i) \in U_k \setminus \{(1,2)\},\\ ((1,2)+(1,2))(1,2) &= (1,1)(1,2) = (1,1) = (1,2)+(1,2)\\ &= (1,2)(1,2)+(1,2)(1,2),\end{aligned}$$

the multiplication $\cdot$ is distributive with respect to $+$. Hence, $\mathbf{V}_k \in \mathcal{C}$. Since $((1,1)+(1,2),(1,2)+(1,2)) = ((1,2),(1,1))$, so $\Delta \cup \{(1,1),(1,2)\}^2 \in \operatorname{Con} \mathbf{V}_k$. Similarly, one can see that $\mathbf{V}_k$ is subdirectly irreducible. Since $\mathbf{U}_k$ is additively idempotent and $\mathbf{V}_k$ is not, $\mathbf{U}_k \not\cong \mathbf{V}_k$. We have $\mathbf{U}_k, \mathbf{V}_k \notin \mathcal{B}$ because

$$(1,2)+(0,2)+(0,2) = (1,2)+(0,2) = (0,2) \neq (1,2). \qquad \square$$

Contrary to the variety of Boolean semirings, we have the following.

Corollary 4.5. *In $\mathcal{C}$, there exist subdirectly irreducible members of arbitrary cardinality, and hence $\mathcal{C}$ is residually large.*

It is natural to ask if the constructed semirings $\mathbf{S}_n$ and $\mathbf{T}_n$ are the only subdirectly irreducible members of $\mathcal{C}$ with cardinality n. In what follows, we prove that for $n = 2,3,4$, this is true. On the other hand, there exists a five-element subdirectly irreducible semiring in $\mathcal{C}$ that is not linearly ordered, as is shown below.

Theorem 4.6. *For $n = 2,3,4$, the semirings $\mathbf{S}_n$ and $\mathbf{T}_n$ are the only n-element subdirectly irreducible members of $\mathcal{C}$.*

Proof. The case $n = 2$ is clear. Now consider the case $n = 3$. Let $\mathbf{S}$ be a three-element subdirectly irreducible member of $\mathcal{C}$. According to Proposition 3.5, S is of the form $\{0, a, 1\}$ with $0 < a < 1$. We have

$$(a+1)a = a+a \quad \text{and} \quad (1+1)a = a+a.$$

Hence, since the semigroup $(S, \cdot)$ has no zero divisors,

$$a+1 = 0 \text{ if and only if } a+a = 0, \text{ and } 1+1 = 0 \text{ if and only if } a+a = 0.$$

Therefore, $0 \in \{a+a, a+1, 1+1\}$ if and only if $a+a = a+1 = 1+1 = 0$. But this case is not possible since $(a+a)+1 = 0+1 = 1 \neq a = a+0 = a+(a+1)$.

Hence, $0 \notin \{a+a, a+1, 1+1\}$, and therefore $a+a = a$ according to Lemma 3.3. The following four cases remain:

	1	2	3	4
$a+1$	a	a	1	1
$1+1$	a	1	a	1 .

Cases 3 and 4 are not possible since $\Delta \cup \{0, a\}^2 \in \operatorname{Con} \mathbf{S}$, contradicting the subdirect irreducibility of $\mathbf{S}$. In cases 1 and 2, $\mathbf{S} \cong \mathbf{T}_3$ and $\mathbf{S} \cong \mathbf{S}_3$, respectively.

Finally, consider the case $n = 4$. Let $\mathbf{T}$ be a four-element subdirectly irreducible member of $\mathcal{C}$. According to Proposition 3.5, T is of the form $\{0, b, a, 1\}$ with $0 < b < a < 1$. We have

$$(b+1)b = b+b, \qquad (b+1)a = b+a,$$
$$(a+1)b = b+b, \qquad (a+1)a = a+a,$$
$$(1+1)b = b+b.$$

Hence, since the semigroup $(T, \cdot)$ has no zero divisors, we have

$b+1 = 0$ if and only if $b+b = 0$, $\quad b+1 = 0$ if and only if $b+a = 0$,

$a+1 = 0$ if and only if $b+b = 0$, $\quad a+1 = 0$ if and only if $a+a = 0$,

$1+1 = 0$ if and only if $b+b = 0$.

Therefore, $0 \in \{b+b, b+a, b+1, a+a, a+1, 1+1\}$ if and only if $b+b = b+a = b+1 = a+a = a+1 = 1+1 = 0$. In this case, $\Delta \cup \{b, a\}^2 \in \operatorname{Con} \mathbf{T}$, contradicting the subdirect irreducibility of $\mathbf{T}$. Hence, $0 \notin \{b+b, b+a, b+1, a+a, a+1, 1+1\}$. If $1 \notin \{b+1, a+1\}$ or $b+1 = a+1 = 1$, then $\Delta \cup \{b, a\}^2 \in \operatorname{Con} \mathbf{T}$, contradicting the subdirect irreducibility of $\mathbf{T}$. Hence, $\{b+1, a+1\} \in \{\{b, 1\}, \{a, 1\}\}$. Now we have the following possibilities:

	1	2	3	4	5	6	7	8	9	10	11	12
$b+1$	b	b	b	a	a	a	1	1	1	1	1	1
$a+1$	1	1	1	1	1	1	b	b	b	a	a	a
$1+1$	b	a	1	b	a	1	b	a	1	b	a	1 .

We also have $b+a = (b+1)a$, $b+b = (1+1)b$ and $a+a = (1+1)a$. Because of $(a+1)a = a+a = (1+1)a$, the cases 1, 4, 8, 9, and 10 are not possible. Cases 5 and 6 are not possible since $(b+a)+1 = a+1 = 1 \neq a = b+1 = b+(a+1)$. Cases 7, 11, and 12 are not possible since $\Delta \cup \{0, b\}^2 \in \operatorname{Con} \mathbf{T}$, contradicting the subdirect irreducibility of $\mathbf{T}$. In cases 2 and 3, $\mathbf{T} \cong \mathbf{T}_4$ and $\mathbf{T} \cong \mathbf{S}_4$, respectively. □

Finally, we show that there are further subdirectly irreducible members of $\mathcal{C}$ that are not linearly ordered.

In the following, let $\mathbf{B} = (B, \vee, \wedge, ', 0, a)$ be a non-trivial Boolean algebra.

Theorem 4.7. *The semiring* $\mathbf{B}^1$ *with universe* $S = B \cup \{1\}$ (*where* $1 \notin B$) *and operations defined for* $x, y \in S$ *by*

$$x + y := \begin{cases} x \vee y & \text{if } x, y \leq a, \\ 1 & \text{if } (x, y) \in \{(0,1), (1,0)\}, \\ a & \text{otherwise,} \end{cases}$$

$$xy := \begin{cases} x \wedge y & \text{if } x, y \leq a, \\ y & \text{if } x = 1, \\ x & \text{if } y = 1, \end{cases}$$

is a subdirectly irreducible member of $\mathcal{C}$.

Proof. First observe that $(S, +, 0)$ is a commutative groupoid with neutral element, $(S, \cdot, 1)$ is a semilattice, and $x0 = 0x = 0$ for all $x \in S$. The laws $(x+y)+z = x+(y+z)$ and $(x+y)z = xz+yz$ can be checked by considering the following cases:

(i) $0 \in \{x, y, z\}$; (ii) $x, y, z \leq a$; (iii) $x, y, z > 0$ and $1 \in \{x, y, z\}$.

Hence, $\mathbf{B}^1 \in \mathcal{C}$. Now put $\Theta_0 := \Delta \cup \{a, 1\}^2$ and let $b \in S$. If $b = 0$, then $(a + b, 1 + b) = (a, 1) \in \Theta_0$. If $b \neq 0$, then $(a + b, 1 + b) = (a, a) \in \Theta_0$. If $b = 1$, then $(ab, 1b) = (a, 1) \in \Theta_0$. If $b \neq 1$, then $(ab, 1b) = (b, b) \in \Theta_0$. Hence, $\Theta_0 \in \operatorname{Con} \mathbf{B}^1$.

Now assume $\Theta \in (\operatorname{Con} \mathbf{B}^1) \setminus \{\Delta\}$. Then there exists a $(c, d) \in \Theta$ with $c \neq d$. We prove $(a, 1) \in \Theta$, which implies $\Theta \supseteq \Theta_0$. If $c = 1$, then $(1, d) \in \Theta$ and $(a, d) = (ca, da) \in \Theta$, whence $(a, 1) \in \Theta$. If $d = 1$, then $(c, 1) \in \Theta$ and $(c, a) = (ca, da) \in \Theta$, whence $(a, 1) \in \Theta$. If $c, d \neq 1$ and $cd \neq c$, then $cd < c$, which together with $c = cd + c(cd)'$, shows $c(cd)' \neq 0$, and hence

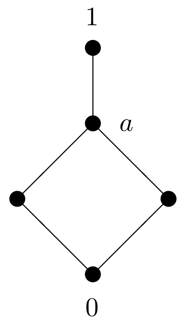
$$(a, 1) = (cc(cd)' + 1, dc(cd)' + 1) \in \Theta.$$

If $c, d \neq 1$ and $cd \neq d$, then $cd < d$, which together with $d = cd + d(cd)'$ shows $d(cd)' \neq 0$, and hence

$$(1, a) = (cd(cd)' + 1, dd(cd)' + 1) \in \Theta.$$

This shows that $\mathbf{B}^1$ is subdirectly irreducible, which completes the proof. □

If $|B| = 4$, then the Hasse diagram of the meet-semilattice $(S, \cdot)$ is shown below. It is easy to see that $\mathbf{B}^1 \cong \mathbf{T}_3$ if $|B| = 2$. Moreover, it is well known that for an infinite cardinal n, there exist 2^n mutually non-isomorphic Boolean algebras of cardinality n (cf. [5]).



5. Summary

Theorem 5.1.

(i) *For every integer $n > 1$, there exist at least two non-isomorphic subdirectly irreducible members of $\mathcal{C}$ of cardinality n.*

(ii) *For every integer $m > 1$, there exist at least three non-isomorphic subdirectly irreducible members of $\mathcal{C}$ of cardinality $2^m + 1$.*

(iii) *For every infinite cardinal n, there exist at least 2^n non-isomorphic subdirectly irreducible members of $\mathcal{C}$ of cardinality n.*

Our next goal is to study varieties of semirings between $\mathcal{B}$ and $\mathcal{C}$. For this, we first have to investigate which subdirectly irreducible members distinct from $\mathbf{T}_3$ are in the variety generated by $\mathbf{T}_3$. Moreover, we have to determine identities holding in these varieties and we have to check if these varieties are residually small or large. This should be our next topic of research.

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Ivan Chajda
Department of Algebra and Geometry, Palacký University Olomouc, 17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: ivan.chajda@upol.cz

Helmut Länger
Institute of Discrete Mathematics and Geometry, TU Wien, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria
e-mail: helmut.laenger@tuwien.ac.at