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Ranking Sets of Objects

How to Deal with Impossibility Results

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Vienna, 18th August, 2020

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Declaration of Authorship

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I hereby declare that I have written this Doctoral Thesis independently, that I have completely specified the utilized sources and resources and that I have definitely marked all parts of the work - including tables, maps and figures - which belong to other works or to the internet, literally or extracted, by referencing the source as borrowed.

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Kurzfassung

Präferenzen zwischen verschiedenen Alternativen zu modellieren ist eine wichtige Herausforderung für viele Bereiche der Künstlichen Intelligenz Forschung. Besonders zentral ist die Modellierung von Präferenzen zum Beispiel für *wissensbasierte Systeme* und insbesondere für *Computational Social Choice*. Wenn die Zahl der Alternativen klein genug ist, werden Präferenzen meist als totale Ordnungen modelliert. Allerdings sind die Alternativen in vielen Anwendungen „kombinatorischer“ Natur, das heißt die Alternativen sind Mengen von Objekten. Diese Mengen können zum Beispiel Sammlungen von Gütern sein, wenn es um Verteilungsprobleme geht, oder Komitees, wenn es um Wahlen geht. In solchen Situationen steigt die Anzahl an Alternativen exponentiell mit der Anzahl an Objekten, weshalb es nicht praktikabel ist Akteure nach ihrer vollen Präferenzordnung zu fragen.

Ein weit verbreiteter Ansatz, um dieses Problem zu umgehen, ist es die Präferenzordnung auf den Mengen von Objekten aus einer Präferenzordnung auf den Objekten abzuleiten. Dies wird auch eine Ordnung von Objekten auf Mengen von Objekten heben oder *liften* genannt. Dieser Lifting-Prozess wird häufig von Axiomen geleitet, die die geliftete Ordnung erfüllen soll. Unglücklicherweise sagen uns bekannte Unmöglichkeitsresultate, die von Kannai und Peleg sowie von Barberà und Pattanaik bewiesen wurden, dass einige wünschenswerte Axiome, nämlich *Dominance* und *Independence* sowie *Dominance* und *Strict Independence*, nicht gleichzeitig erfüllt werden können, falls alle möglichen nicht-leeren Mengen von Objekten geordnet werden sollen. Allerdings ist es manchmal möglich die Axiome gleichzeitig zu erfüllen, falls *nicht alle* nicht-leeren Mengen geordnet werden müssen. Es ist jedoch nicht bekannt für welche Familien von Mengen Dominance und (Strict) Independence gleichzeitig erfüllt werden können. Dies wirft zwei Fragen auf: (1) Ist es effizient möglich mit Hilfe eines Computers zu entscheiden, ob es für eine gegebene Familie von Mengen möglich ist Dominance und (Strict) Independence gleichzeitig zu erfüllen? (2) Ist es möglich, diese Familien zu klassifizieren?

Eine wichtige Beobachtung ist, dass Dominance und (Strict) Independence für eine spezifische Familie von Mengen zu unterschiedlichen Graden gleichzeitig erfüllbar sein können. Sie können für eine spezielle Präferenzordnung $\leq$ auf den Objekten gleichzeitig erfüllbar sein, für alle Präferenzordnungen oder wenigstens für eine Präferenzordnung. Im ersten Fall sagen wir, die Familie ist $\leq$-*orderable*, im zweiten Fall nennen wir die Familie *strongly orderable* und im letzten Fall *weakly orderable*.

Die erste der oben gestellten Fragen wird in dieser Doktorarbeit beantwortet, indem ein fast vollständiges Bild über die Komplexität der Frage, ob eine Familie $\leq$-orderable, strongly orderable oder weakly orderable im Bezug auf Dominance und Independence bzw. Dominance und Strict Independence ist, gegeben wird. Im besonderen wird gezeigt, dass es in fast allen Fällen nicht effizient möglich ist zu entscheiden, ob Dominance und (Strict) Independence gleichzeitig erfüllt werden können, falls die Präferenzordnung auf der Familie von Mengen total sein muss. Falls die geliftete Ordnung nur partiell sein muss, werden die meisten Probleme effizient entscheidbar, mit der Ausnahme der Frage, ob eine Familie strongly oder weakly orderable im Bezug auf Dominance und Strict Independence ist.

Zur Beantwortung der zweiten Frage werden nur spezielle Familien von Mengen betrachtet, nämlich solche, die zusammenhängende Teilgraphen in einem Graphen induzieren. Für diese Familien werden alle drei Grade der gleichzeitigen Erfüllbarkeit für Dominance und Strict Independence klassifiziert. Außerdem werden starke notwendige und hinreichende Bedingungen für strong orderability im Bezug auf Dominance und Independence bewiesen.

Abstract

Modeling preferences over alternatives is a major challenge in many areas of AI, for example in knowledge representation and, especially, in computational social choice. If the number of alternatives is small enough, preferences are most often modeled as a total order. However, in many applications the alternatives are 'combinatorial', i.e., sets of objects. These sets can be, for example, bundles of goods in packing or allocation problems or committees in voting. In these situations, the number of alternatives grows exponentially with the number of objects which makes it unfeasible for agents to specify a full preference relation over all alternatives.

A widely used approach to circumvent this problem is to derive a preference order on sets of objects from a preference order on the objects. We call this lifting an order from objects to sets of objects. This process is often guided by axioms postulating properties the lifted order on sets should have. However, well-known impossibility results by Kannai and Peleg and by Barberà and Pattanaik tell us that some desirable axioms – namely dominance together with independence or strict independence – are not jointly satisfiable if *all* non-empty sets of objects are to be ordered. On the other hand, if *not all* non-empty sets of objects are to be ordered, the axioms are jointly satisfiable for some families of sets. However, it is not known on which families dominance and (strict) independence can be jointly satisfied and on which they are incompatible. This raises two questions: (1) Is it computationally difficult to decide for a given collection of sets whether dominance and (strict) independence are jointly satisfiable? (2) Is it possible to characterize collections of sets for which dominance and (strict) independence are jointly satisfiable?

Observe that the axioms can be compatible to three different degrees. Dominance and (strict) independence can be jointly satisfiable on a family of sets for *a specific linear order* $\leq$ on the objects or, alternatively, for *every* or *at least one linear order* on the objects. In the first case, we say that the family is $\leq$-*orderable*, and in the latter cases we say that the family is *strongly* resp. *weakly* orderable.

The first question is answered in this thesis by giving a nearly complete picture of the complexity of deciding whether a family is strongly, weakly or $\leq$-orderable with respect to dominance and independence or dominance and strict independence. In particular, it is shown that it is almost always intractable to decide orderability when the lifted order needs to be total. If the lifted order only needs to be partial, then most problems become tractable with the exception of strong and weak orderability with respect to dominance and strict independence, which remain intractable.

For the second question, we focus on families of sets that induce connected subgraphs in graphs. For such families we characterize strong, weak and $\leq$-orderability with respect to dominance and strict independence, and obtain a tight bound on the class of families that are strongly orderable with respect to dominance and independence.

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CHAPTER 1

Introduction

When agents, individually or as a group, make a decision to select one of several options, they refer to their ranking of (or preference order on) the available choices. In a single-agent setting, the agent simply selects an option that she prefers the most. In a group setting, agents communicate their preferences in order to form a collective decision. This can either be an informal process or a formal election where participants submit their preferences in a predefined way and voting rules are used to determine the option to select.

If the options are atomic objects, then preferences can be intrinsic, i.e., they do not depend on my conscious beliefs and decisions (Hansson & Grüne-Yanoff, 2018). As Schopenhauer puts it:

> Du kannst thun was du willst: aber du kannst, in jedem gegebenen Augenblick deines Lebens, nur Ein Bestimmtes wollen und schlechterdings nichts Anderes, als dieses Eine.[1] (Schopenhauer, 1839, p. 62f)

In this case, our ability to model and reason about the process by which preferences are formed is very limited and we have to treat preferences as atomic facts.

However, in many situations, having a preference order on individual objects is not enough for decision making and the ability to compare *sets* of alternatives is needed. In such a situation, the preferences over sets will often be derived from the preferences over the individual objects. When offered the choice between two cups of ice cream, one containing vanilla and chocolate ice cream and the other one strawberry and lemon ice cream, my final choice depends on my preferences on the individual flavors. As we are now dealing with derived preferences, we can try to model and study the process by which these preferences are derived. On a high level, we can say that this thesis studies a specific model of this process. Studying such models is of theoretical interest

[1] *Translation by the author:* You can do what you want, but in every moment of your life you can only want one thing and simply nothing else than this one thing.

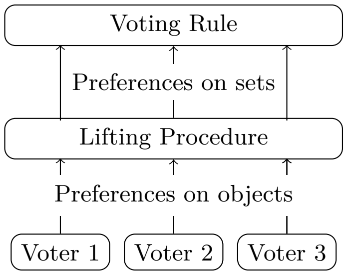

Figure 1.1: The use of lifted orders in voting

but also of practical importance as the number of possible combinations of objects grows exponentially with the number of objects which often makes it unfeasible for agents to specify a full preference relation over sets of objects. To circumvent this problem, the agents' true preference order on sets could be approximated by an order that can be derived from their preference order on individual objects.

We observe that in applications, the set of objects usually either represents a collection of objects that the agent receives as a whole or a set of possible outcomes from which one will be selected. The first case is often called the conjunctive interpretation and the second case the disjunctive interpretation. Both approaches occur commonly in applications in computer science. Important examples for the conjunctive interpretation are the following:

- When an agent is to select a set of objects subject to some constraints, as in the *knapsack* problem, the agent's preferences on the family of feasible sets is often derived from the utility of individual objects for the agent.
- A similar approach is often used in the problem of fair allocation of indivisible goods (Bouveret, Chevaleyre & Maudet, 2016), where knowledge of the agents' rankings of sets of goods is necessary to ensure that the goods are distributed fairly.
- Similarly, knowledge of agents' preferences on collections of objects is required for determining optimal matchings and assignments (Roth & Sotomayor, 1990).
- Finally, derived preferences could be used in voting if a group of agents has to jointly select a set of objects or a committee (Maly, 2020). Instead of using multiwinner voting rules, voters could specify their preference orders on objects or candidates which are lifted to orders on sets or committees. Then, single winner voting rules could be applied to these lifted orders. Figure 2.1 illustrates this approach to voting.

Some examples for the disjunctive interpretation are:

- Studying strategic behaviors in voting with a tie-breaking mechanism (Barberà, 1977; Fishburn, 1972; Bossert, 1989; Brandt & Brill, 2011; Brandt, Saile & Stricker,

2018) requires a ranking of agents' preferences on sets of tied candidates while only rankings over candidates are known.

- Moreover, rankings on sets of possible outcomes are needed in decision making, when there is uncertainty about the consequences of an action (Larbi, Konieczny & Marquis, 2010).

If the preferences over objects are given by a utility function, the utility function (preference) on sets of objects can often be derived by assuming some form of additivity. In the most straightforward setting, the utility of a set is just the sum of the utilities of its objects. This is a common setting, used for instance in the knapsack problem and fair division, though, for example, synergies between objects make more complicated models necessary (Bouveret et al., 2016).

Unfortunately, utility functions are often neither a realistic nor a practical model of preferences (Suzumura, 2002). In this case an alternative and more abstract framework, known as the *ordinal* setting, is often used. This setting will be the focus of this thesis. It assumes that preferences on objects in a set X are represented by an order relation on X. The objective is to *lift* this order to an order on a family of non-empty subsets of X. We call this the *order lifting problem.* The problem of lifting an order relation on X to an order on the family of *all* non-empty subsets of X has been extensively studied. The paper by Barberà, Bossert & Pattanaik (2004) provides an excellent and extensive overview of this research area. The results can roughly be divided into two groups, those concerned with studying specific ways to lift an order on objects to an order on sets of objects (Fishburn, 1972; Gärdenfors, 1976; Kelly, 1977), and those following the "axiomatic" approach, where one postulates desirable properties a lifted order should have and seeks conditions that would guarantee the existence of such a lifting (Barberà, 1977; Barberà, Barret & Pattanaik, 1984; Moretti & Tsoukiàs, 2012). Among these properties *dominance* and *independence* are the most studied ones. Informally speaking, dominance ensures that adding an element which is better (worse) than all elements in a set, makes the augmented set better (worse) than the original one. Independence, on the other hand, states that adding a new element a to sets A and B where A is already known to be preferred over B, must not make $B \cup \{a\}$ be preferred over $A \cup \{a\}$ or, in the strict variant, $A \cup \{a\}$ should remain preferred over $B \cup \{a\}$. We refer to the strict variant of independence by *strict independence.* In what follows, we write "(strict) independence" whenever a statement holds for independence as well as its strict counterpart. A further basic property, called *extension* rule, states that the singletons $\{a\}$ and $\{b\}$ have to be ordered the same way as elements a and b are ordered in the underlying order. We shall call properties like dominance, (strict) independence, and extensions simply *axioms.*

The most striking results in the group of axiomatic approaches are known as *impossibility* theorems. They say that some natural desiderata are inherently incompatible and cannot be achieved together (Kannai & Peleg, 1984; Barberà et al., 2004; Geist & Endriss, 2011). For instance, given an ordered set X with $|X| \geq 6$, orders on $\mathcal{P}(X) \setminus \{\emptyset\}$ satisfying dominance and independence are not possible (Kannai & Peleg, 1984). Moreover, satisfying dominance and strict independence is already impossible if X has at least

3 elements (Barberà & Pattanaik, 1984). Since these impossibility results usually seek liftings to the family of *all* non-empty subsets of a set, they put very strong constraints on the lifted order, constraints that cannot be satisfied together. However, one is often only interested in comparing sets from much smaller families of sets. In many allocation problems, there are constraints on the possible bundles of objects that an agent can receive. For example, it is often demanded that bundles of objects must be connected according to some topology (Stromquist, 1980; Bouveret, Cechlárová, Elkind, Igarashi & Peters, 2017). Similarly, if we need to rank sets of possible outcomes in decision making under uncertainty, it is rarely the case that every conceivable set of outcomes describes the possible outcomes of an available action. Furthermore, in multiwinner or combinatorial voting it is common to have restrictions on the set of possible committees (Lang & Xia, 2016; Kilgour, 2016). Now, it turns out that it is possible to construct arbitrarily large families of sets – for example families of disjoint sets – that can be ordered with an order satisfying dominance and (strict) independence.

Motivated by this observation, we ask for which families of sets do the aforementioned impossibility result hold and for which families can they be avoided. In particular, this thesis explores the following two questions:

Main Questions of This Thesis

1. Is it computationally difficult to decide for a given collection of sets whether dominance and (strict) independence are jointly satisfiable?
2. Is it possible to characterize collections of sets for which dominance and (strict) independence are jointly satisfiable?

First of all, the first question is important from a practical standpoint. Many possible applications of the axiomatic approach to the order lifting problem depend on the tractability of deciding whether lifting to an order satisfying some axioms is possible. Consider for example an application in knowledge representation, where we may want to offer the user the ability to rank the results of a query based on an importance ranking on the atoms. Then, we need to decide whether we can offer a ranking of the results that satisfies dominance and (strict) independence in a timely fashion.

Moreover, the first question is also of theoretical interest as it tells us if the compatibility of dominance and (strict) independence coincides with some simple (to check) property of the family of sets and if there is a simple order that satisfies dominance and (strict) independence whenever this is possible. It turns out that, in general, the answer to both questions is no!

This motivates us to restrict our focus to "well-behaved" families of sets when trying to answer the second question. Namely we focus on families of sets that are defined by the condition of connectivity in a given graph. More precisely, we seek characterizations of graphs (topologies) for which the impossibility results still hold and for which lifting to orders satisfying prescribed postulates is possible. Because these families are so "structured", we are able to formally capture intuitive ideas about what makes dominance and (strict) independence incompatible in more general families of sets. For example, intuitively, dominance and strict independence are only incompatible when there is some form of cyclicity in the family of sets. This intuition is captured in a result that states that dominance and strict independence are always compatible on a family characterized by a graph if and only if the graph is acyclic.

Furthermore, such families appear in interesting applications. Indeed, if in fair allocation the set of goods are offices and labs in a new research building and agents are research groups, it is natural to only consider allocations that form topologically contiguous areas. For instance, if the building consists of a single long hall of rooms, legal allocations are only those that split this hall into segments. In such situations, only preferences that research groups may have on contiguous segments of rooms need to be taken into account (Bouveret et al., 2017). For another example, we might consider a problem of farmland fragmentation, where individual farms consist of many small non-contiguous plots of land as the result of divisions of farms among heirs, and acquiring ownership through marriage (King & Burton, 1982). Land consolidation was proposed as a method to improve economic performance. The objective of land consolidation is to reallocate the plots so that they form large contiguous land areas. In both cases, the topology of the set of goods can be modeled by a graph and valid sets of goods are those that induce in this graph a connected subgraph.

Now, we observe that there are three ways to formulate the question whether dominance and (strict) independence are jointly satisfiable on a given family of sets: We can ask whether dominance and (strict) independence are compatible for all possible rankings of the objects or, conversely, whether dominance and (strict) independence are compatible for no ranking of the objects. Alternatively, we can ask if dominance and (strict) independence are compatible for a specific ranking of the objects. This motivates the definition of three types of "orderability" that need to be distinguished:

1. We call families of sets for which *any possible* order on the elements can be lifted to an order satisfying dominance and (strict) independence *strongly orderable* with respect to dominance and (strict) independence.

2. Similarly, we call families of sets for which *at least one* order on the elements can be lifted to an order satisfying dominance and (strict) independence *weakly orderable* with respect to dominance and (strict) independence.

3. Finally, if for a family of sets a specific linear order $\leq$ can be lifted to an order satisfying dominance and (strict) independence, we call that family $\leq$-*orderable* with respect to dominance and (strict) independence.

From a theoretical point of view all three orderability properties are of equal interest. Which of them is the most important for applications depends on the envisioned use case. When ordering the results of a query or actions based on possible outcomes, it often suffices to know if a family is orderable after the order on the atoms or outcomes has been elicited and we don't care if the family is orderable with respect to other orders. Therefore, in these contexts $\leq$-orderability is the most important property. On the other hand, for applications in voting or other social choice problems, it is necessary to fix a voting method before the ballots are collected. Therefore, it is critically important to know for a given family of sets if dominance and (strict) independence are compatible for any preference order the agents may report. Hence, strong orderability is the most crucial property in these applications. Finally, weak orderability is arguably the least important property from a practical standpoint. However, in some applications we may want to use lifted rankings when possible and only require more information from the users if they submitted a ranking that can not be lifted. In this case, it is valuable to know if there is no chance that a submitted ranking can be lifted such that we can ask for more information immediately. In other words, it is valuable to know whether a family of is not weakly orderable. For this reason, we also study weak orderability but focus more on strong and $\leq$-orderability.

Another degree of freedom in our main questions is whether the lifted order needs to be total or if a partial order suffices. For example, it is known that dominance and independence can be jointly satisfied by a partial order but not by a total order. Dominance and strict independence, on the other hand, are not jointly satisfiable even for partial orders. Now, for example in fair allocation it is usually necessary to produce a total order on bundles for an allocation procedure to work. However, in social choice some authors argue that it is more sensible to only require incomplete preferences when dealing with combinatorial domains (Boutilier & Rosenschein, 2016). Voting rules that facilitate the aggregation of partial orders or even weaker preference models exist (Xia & Conitzer, 2011; Terzopoulou & Endriss, 2019), therefore, it is possible to just lift to partial orders when applying the order lifting approach in voting.

This means the two main questions should be studied for all three aforementioned degrees of orderability and for partial and total orders. In the following section, we discuss the progress this thesis makes towards this goal.

Contributions

On a high level, the main contribution of this thesis is being the first comprehensive study of the effect of dropping the requirement that the whole power set needs to be ordered when determining the (im)possibility of ranking sets of objects in the axiomatic approach. This includes a (straightforward) adaption of the axioms to this setting. Furthermore, the work presented in this thesis is, to our knowledge, the first study of complexity questions that arise in the axiomatic approach. Let us now give a more detailed overview over the main contributions, which can be split into two parts:

The first part is a nearly complete picture of the complexity of deciding whether a family of sets is weakly, strongly or $\leq$-orderable with respect to dominance and (strict) independence and with respect to dominance, (strict) independence and the extension rule. In detail, the following results are included in this thesis:

1. If we expect the lifted order to be total, deciding if a family of sets is strongly or $\leq$-orderable with respect to dominance and independence or dominance and strict

independence is intractable. This also holds if the extension rule is additionally required. For both combinations of axioms, deciding if a family is $\leq$-orderable is NP-complete and deciding if a family is strongly orderable is Π_2^p-complete. The latter result also implies that it is not possible to find an order satisfying dominance and (strict) independence in polynomial time even if one already knows that a given family is strongly orderable.

2. If we expect the lifted order to be total, deciding if a family of sets is weakly orderable with respect to dominance and strict independence is NP-complete and hence also intractable. This also holds if the extension rule is additionally required.

3. If we drop the requirement that the lifted order needs to be total, it was already known that dominance and independence are always jointly satisfiable. Additionally, we show that it can be decided in polynomial time if a family is $\leq$-orderable with respect to dominance and strict independence. However, weak and strong orderability with respect to dominance and strict independence are still intractable. The former is NP-complete whereas the later is coNP-complete. Again, all these results still hold if the extension rule is additionally required.

4. Finally, we observe that these results assume that the family of sets is given explicitly. As this assumption is not satisfied in many interesting applications, we show – for a specific succinct representation that is well studied in the literature – that succinct representation can lead to an exponential blow up in complexity.

The second part of our contributions concern families that are represented by graphs, for which we present a complete characterization of strong, weak and $\leq$-orderability with respect to dominance and strict independence. For strong and weak orderability, these characterizations also hold if the extension rule is additionally required. Moreover, in the same setting, a tight bound on the class of families that are strongly orderable under dominance and independence and the class of families that are strongly orderable under dominance, independence and extension are achieved. In detail, these results look as follows:

1. We show that the disjoint union of orderable sets yields an orderable set as well. This enables us to fully describe strong and weak orderability by characterizing the two concepts for connected graphs.

2. We fully characterize $\leq$-orderable connected graphs with respect to the axioms of dominance and strict independence. Furthermore, for these two axioms, we show that the class of strongly orderable graphs is that of trees and the class of weakly orderable graphs is that of connected bipartite graphs. The latter two results also hold if, in addition, the extension rule is required.

3. We give a full characterization of strong orderability with respect to dominance and independence for *two-connected* graphs. Here we observe that, except for some

smaller special cases, two-connected graphs are strongly orderable with respect to dominance and independence if and only if they are cycles or if they do not contain a cycle of length five or more. This result holds also if we additionally require the extension axiom. Finally, we give a nearly complete picture for strong orderability with respect to dominance and independence and with respect to dominance, independence and extension for arbitrary graphs.

Outline of the Thesis

This thesis is structured as follows:

- In Chapter 2, we review the most relevant scientific background necessary to understand this thesis. First, we discuss how preferences are modeled in the ordinal approach (Section 2.1). Then, we discuss some basic properties of graphs and hypergraphs, mainly in order to fix notation (Section 2.2). Finally, we give a very brief introduction to the most important concepts from complexity theory, as far as they are relevant for this thesis (Section 2.3).
- In Chapter 3, we introduce the axiomatic approach to the order lifting problem. Here, we focus mainly on our four main axioms, dominance, independence, strict independence and the extension rule as well as the related, known impossibility results. First, we give a very short history of the order lifting problem and define it formally (Section 3.1 and 3.2). Then, we introduce our main axioms in Section 3.3 and illustrate them with several examples. Afterwards, we discuss famous impossibility results regarding dominance and independence or dominance and strict independence by Kannai and Peleg resp. Barberà and Pattanaik (Section 3.4). In Section 3.5, we formally introduce strong, weak and $\leq$-orderability. To conclude the chapter, we discuss some variations of our main axioms and prove some helpful lemmas in Section 3.6.
- Then, we move to our main results which are presented in Chapter 4 and 5. In Chapter 4, we study the complexity of deciding whether a family of sets is strongly, weakly or $\leq$-orderable. First, we focus on strong and $\leq$-orderability with respect to total orders (Section 4.1). Then, we study the effect of dropping the requirement that the lifted order needs to be total on the complexity of deciding whether a family is strongly or $\leq$-orderable (Section 4.2). In Section 4.3, we study the effect of a succinct representation of the family of sets on the results obtained in the two previous sections. We then turn our attention to weak orderability with respect to total and partial orders in Section 4.4. Finally, we consider the effect of strengthening dominance on the complexity of $\leq$-orderability (Section 4.5).
- In Chapter 5, we present our characterization results for families that are represented by graphs. First, we formally introduce families that are represented by graphs in Section 5.1. Then, we present our results, first for strict independence (Section 5.2) and then for "regular" independence (Section 5.3).

- Finally, in Chapter 6, we summarize and discuss the results presented in the previous two chapters. Moreover, we consider directions for future research that are opened by this work.

Related Work

The axiomatic approach to the order lifting problem has now been seriously studied for nearly forty years. Hence, for a general overview over earlier research in this area we refer the reader to the excellent survey by Barberà et al. (2004). More recent developments include characterization results for decision making under complete uncertainty (Larbi et al., 2010; Bossert & Suzumura, 2012) as well as novel approaches for ranking sets of interacting elements (Moretti & Tsoukiàs, 2012; Lucchetti, Moretti & Patrone, 2015). Furthermore, Geist & Endriss (2011) managed to show several new impossibility results with computer generated proofs using SAT-solving methods. Finally, the social ranking problem of ranking elements based on a ranking of subsets, which could be seen as the dual of the order lifting problem, has received a lot of attention lately. The problem was first introduced by Moretti & Öztürk (2017) and significant progress has already been made for example by Haret, Khani, Oztürk & Meltem (2018) as well as Khani, Moretti & Oztürk (2019) and Bernardi, Lucchetti & Moretti (2019). However, we have to mention that, to our knowledge, all of the previous works on the axiomatic approach to the order lifting problem required a ranking of all possible subsets, with the notable exception of Bossert (1995) who studied rankings of subsets with fixed cardinality. In this sense, the work presented here is clearly different from the existing literature on the order lifting problem.

More generally, the work presented in this thesis can also be seen as a contribution to an area of research in AI concerned with models of preferences for *combinatorial domains*. One successful approach to dealing with combinatorial domains that has been extensively studied is the use of concise *implicit* preference models (Domshlak, Hüllermeier, Kaci & Prade, 2011; Kaci, 2011). Our research can be seen as belonging to this line of research. We study the question whether a preference order on a collection of subsets of a set could be obtained by lifting a strict preference order on elements of that set in a way to satisfy some natural postulates *lifted* preference orders on collections of sets should satisfy. Whenever it is the case, the original order on elements of a set can serve as a concise representation of the one *lifted* to a much larger domain of subsets of that set. As we note above, in the present thesis, we concentrate purely on the question whether lifting is possible at all. The question how to reason about lifted orders based on the "ground" information about preferences on elements is left for future research. In contrast to our approach, most implicit preference models in the literature either build on logical languages (Dubois & Prade, 1991; Brewka, Benferhat & Berre, 2004; Brewka, Niemelä & Truszczyński, 2003) or employ intuitive graphical representations such as lexicographic trees (Booth et al., 2010; Bräuning & Hüllermeier, 2012; Liu & Truszczynski, Liu & Truszczynski), CP-nets (Boutilier, Brafman, Domshlak, Hoos & Poole, 2004) or CI-nets (Bouveret, Endriss & Lang, 2009). To this date, it is unclear whether the rankings obtained by such formalisms satisfy desirable properties that are formalized in the axiomatic approach. Therefore, our work can also be seen as a starting point for more general investigations on (im)possibility results in these formalisms.

Similarly, many different approaches to dealing with combinatorial domains are studied in the context of voting. Lang & Xia (2016) give a thorough overview of the most relevant ideas. The easiest solution is often to vote on each candidate separately. However, this approach only works well if the voters have separable preferences, i.e., if the preference on having a candidate in the committee is independent on who else is in the committee (Lacy & Niou, 2000). Another option is eliciting the top ranked committee (Brams, Kilgour & Sanver, 2007). Then one can, for example, infer a preference order on the committees via a distance measure like the Hamming distance (Laffond & Lainé, 2009; Çuhadaroğlu & Lainé, 2012). This approach minimizes the communication cost but only takes very little of the agent's full preferences into account. Alternatively, one can ask the agents to specify their preferences using a CP-net or another implicit preference representation as discussed above. This can be very effective but requires the agents to learn a non-trivial preference representation. In many cases this is an unacceptable requirement. Finally, there is a large number of well-studied voting rules that select a winning committee directly from a preferences over candidates (Faliszewski, Skowron, Slinko & Talmon, 2017). This approach however has mainly been studied for committees of fixed size, though exceptions exist especially in the context of approval voting, see for example Kilgour (2016) and Faliszewski, Slinko & Talmon (2020). In contrast, the order lifting approach discussed in this thesis works for arbitrary sets of admissible committees.

Finally, families defined by a connectivity condition in a given graph as they appear in the characterization results in Chapter 5 have frequently been studied in fair allocation in the last few years. They were first introduced by Suksompong (2017) and Bouveret et al. (2017) independently of each other. Subsequently, the theory was further developed for example by Lonc & Truszczyński (2018), Bilò et al. (2019) and Igarashi & Peters (2019). Moreover, these types of families also appear much earlier in the literature on combinatorial auctions (Conitzer, Derryberry & Sandholm, 2004). In all of these cases, preferences need to be lifted from objects to bundles of objects, though all works assume that the preferences are given as utility functions. To our knowledge, the work presented in this thesis is the first application of this successful paradigm from fair allocation in an ordinal setting.

Publications

The results in this thesis are mainly based on the following five publications:

1. Maly, J. & Woltran, S. (2017a). Ranking specific sets of objects. In *BTW (Workshops)*, volume P-266 of *LNI*, (pp. 193–201). GI.

2. Maly, J. & Woltran, S. (2017b). Ranking specific sets of objects. *Datenbank-Spektrum*, *17*(3), 255–265.

3. Maly, J., Truszczyński, M., & Woltran, S. (2018). Preference orders on families of sets - when can impossibility results be avoided? In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018*, (pp. 433–439). ijcai.org.

4. Maly, J., Truszczynski, M., & Woltran, S. (2019). Preference orders on families of sets - when can impossibility results be avoided? *Journal of Artificial Intelligence Research (JAIR)*, *66*, 1147–1197.

5. Maly, J. (2020). Lifting preferences over alternatives to preferences over sets of alternatives: The complexity of recognizing desirable families of sets. In *Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence*, (pp. 2152–2159).

The second publication is a long version of the first and the fourth is a long version of the third. The complexity results discussed in Chapter 4 mostly are due to Maly & Woltran (2017a,b) and Maly (2020). The results discussed in Chapter 5, on the other hand, are mainly taken from Maly, Truszczyński & Woltran (2018) and Maly, Truszczynski & Woltran (2019).

The author of this thesis has co-authored further publications, which are, to varying degrees, related to the topic of this thesis but do not contain results directly included here. They are listed below:

1. Maly, J. & Woltran, S. (2018). A new logic for jointly representing hard and soft constraints. In Lukasiewicz, T., Peñaloza, R., & Turhan, A. (Eds.), *The Second Workshop on Logics for Reasoning about Preferences, Uncertainty, and Vagueness, PRUV@IJCAR 2018*, volume 2157 of *CEUR Workshop Proceedings*. CEUR-WS.org.

2. Gangl, C., Lackner, M., Maly, J., & Woltran, S. (2019). Aggregating expert opinions in support of medical diagnostic decision-making. In *Knowledge Representation for Health Care/ProHealth, KR4HC 2019*, (pp. 56–62). Position Paper.

3. Maly, J. & Müller, M. (2018). A remark on pseudo proof systems and hard instances of the satisfiability problem. *Mathematical Logic Quarterly*, *64*(6), 418–428.

CHAPTER 2

Background

In this chapter, we will review the scientific background the remainder of this thesis is built upon. First, we will shortly discuss the models of preferences used in this work. Then, we will recall some basic mathematical concepts, mostly to fix a consistent notation. Finally, we give a short introduction to the relevant concepts of complexity theory.

Neither of these sections aspires to be introductions of suitable depth to their respective topics for the uninitiated reader. Instead, references to recommended introductory books are provided, should the reader need them.

2.1 The standard model for preferences: Total orders

In this work, we assume that human preferences can be represented as some kind of mathematical order relation. The standard model for preferences in Social Choice Theory and indeed most fields of economics is a total order (Hansson & Grüne-Yanoff, 2018). This model is so common nowadays that Kennth Arrow, in his 1972 Nobel Memorial Lecture, only stated in passing that “in the context of social choice, each individual may be assumed to have a preference ordering over all possible social states”(Arrow, 1972, p.20). While the use of total orders as a representation of preferences can be dated back at least to Marie-Jean de Condorcet and Jean-Charles de Borda in the eighteenth century (Suzumura, 2002, p.3), it only became mainstream in economics in the end of the 1930s (Suzumura, 2002, p.6). Before, preferences were usually represented as a cardinal utility, i.e., every option was assigned a numerical value of usefulness (Hansson & Grüne-Yanoff, 2018). It was, however, convincingly argued by first Vilfredo Pareto and then by Hicks & Allen (1934) that such an “objective” utility is neither measurable – and therefore difficult to compare between different agents – nor consistent with observable behavior (Suzumura, 2002, p.6). Total orders were chosen as a less expressive but more justifiable framework for representing preferences [Suzumura 2002, p.7,Arrow 1970, p.10].

Criticism of the standard model

While the standard model is still widely accepted in computer sciences as well as in economics and other social sciences, it is often sharply criticized by philosophers and also, to a lesser extent, psychologists (Hansson & Grüne-Yanoff, 2018).

There are two properties of a total order that, in the view of many philosophers, make them an unrealistic model for human preferences, transitivity and completeness (Hansson & Grüne-Yanoff, 2018). Transitivity states that if one prefers an alternative A to an alternative B and B to an alternative C then one prefers A to C. Completeness states that if one is presented with two alternatives, one has to prefer one over the other or be indifferent.

Many philosophers believe that human preferences are not complete as alternatives can be incomparable if they are related to values that are incommensurable like liberty and prosperity (Griffin, 1997, p.51). It is worth noting that the question if values can be incommensurable is far from uncontroversial and leads to very deep philosophical questions (Griffin, 1997, p.36) that are unfortunately out of the scope of this work. We will investigate the effect of dropping the totality requirement on the lifted order for most of our results. However, in line with the standard model, we will always assume that the agents have complete preferences over objects.

Transitivity is usually attacked by different versions of the Sorites Paradox (Hansson & Grüne-Yanoff, 2018). It is argued that a repeated application of transitivity can lead to situations where my indifference between two indistinguishable options – e.g. n or $n+1$ grains of sugar in my coffee – would force me to be indifferent between two clearly distinguishable options—e.g. 0 or 400 grains of sugar in my coffee (Luce, 1956, p.179). In this work, we do not consider non-transitive preference relations because we do not believe that the applications that motivate our work are very susceptible to such paradoxes.

The formal preference models

Technically, we treat every preference order as a set of pairwise comparisons, i.e. as a binary relation.

Definition 2.1. Let X be a set. Then we call a set $R \subseteq X \times X$ of tuples of elements of X a *binary relation*. We write xRy for $(x, y) \in R$.

We will only consider a binary relation as a preference relations if it is at least reflexive and transitive. In many cases, we will additionally demand that a preference relation has to be either total or antisymmetric or both. These properties are formally defined as follows:

Definition 2.2. Let X be a set and R a binary relation on X. We say R is…

… *reflexive* if xRx for all $x \in X$.
… *irreflexive* if xRx does not hold for any $x \in X$.
… *transitive* if xRy and yRz implies xRz for all $x, y, z \in X$.
… *total* if xRy or yRx holds for all $x, y \in X$.

... *antisymmetric* if xRy and yRx do not hold at the same time for any $x, y \in X$ such that $x \neq y$.
... *symmetric*, if xRy implies yRx for every $x, y \in X$.

Based on these properties, we can define the most important types of preference relations.

Definition 2.3. Let R be a binary relation. We say R is...

... a *preorder* if it is reflexive and transitive.
... a *partial order* if it is reflexive, transitive and antisymmetric.
... a *weak order* if it is reflexive, transitive and total.
... a *linear order* if it is reflexive, transitive, total and antisymmetric.

In the following, we will use the symbols $\leq$ and $\preceq$ to refer to binary relations that are not antisymmetric. Additionally, we use $<$ and $\prec$ to refer to binary relations that are antisymmetric.

Unfortunately, the terminology for different kinds of orders is not consistent in the literature. Preorders, for example, are sometimes called quasi orders. Weak orders are sometimes called total preorders or just orders. Finally, linear orders are often also called total orders. In this work, we will stick with the terminology given in Definition 2.3. Additionally, we will use the term order often informally to refer to a binary relation that is at least a preorder.

The most prominent example for a linear order is the smaller-equal relation on the integers. In the following, when we consider a set of integers, we call this the *natural order* on the set. A well known example of a partial order is the subset relation $\subseteq$ on sets. Clearly, a set is contained in it self, therefore the relation is reflexive. Further, if A is contained in B and B in C, then A must be contained in C. Therefore, the relation is transitive. Finally, if A is a proper subset of B, then, by definition, B can not be a proper subset of A. Hence, the relation is antisymmetric. Finally, two sets can be incomparable with respect to the subset relation. Hence the subset relation is a partial order but not a linear order. On the other hand, if we compare sets by cardinality, i.e., by size, we get a weak order.

If an order is neither antisymmetric nor symmetric, we can split it into a symmetric and an antisymmetric part. This way, we can define for every order a corresponding strict order as well as a corresponding equivalence relation.

Definition 2.4. If $\preceq$ is a binary relation on a set X, the corresponding *strict* order $\prec$ on X is defined by $x \prec y$ if $x \preceq y$ and $y \not\preceq x$, where x and y are arbitrary elements of X. Further, the corresponding *equivalence* or *indifference* relation $\sim$ is defined by $x \sim y$ if $x \preceq y$ and $y \preceq x$. If $\preceq$ is a linear order, then $x \sim y$ if and only if $x = y$.

When comparing orders, we say one order extends another, if it only adds new preferences to the original order. Formally, this simply means one order is a superset of another.

Definition 2.5. We say a binary relation R *extends* – or is an *extension* of – the binary relation S if $S \subseteq R$.

For example, the smaller equal relation $\leq$ on the natural numbers is an extension of the smaller relation $<$ on the natural numbers, as it only adds new tuples (k, k) for all $k \in \mathbb{N}$. In general, every order is an extension of its corresponding strict order. On finite sets, the cardinality relation extends the subset relation as any finite set is larger than all its proper subsets. On the other hand, if we consider infinite sets, a set can have the same cardinality as one of its proper subsets and indeed, on infinite sets, neither relation is an extension of the other. To conclude this section, we introduce some useful notation to talk about preference relations:

Definition 2.6. For a binary relation $\preceq$ on a set A, we say an element $x \in A$ is *maximal*, if there is no element $y \in A$ such that $y \neq x$ and $x \prec y$. Similarly, we say an element $x \in A$ is *minimal*, if there is no element $y \in A$ such that $y \neq x$ and $y \prec x$. We write $\max_{\preceq}(A)$ for the set of maximal elements of a set A with respect to $\preceq$ and $\min_{\preceq}(A)$ for the set of minimal elements of A with respect to $\preceq$. If no ambiguity arises, we drop the reference to the relation from the notation. If a set A has only one maximal or minimal element, then we use $\max_{\preceq}(A)$ resp. $\min_{\preceq}(A)$ to refer to that element.

2.2 Graphs and hypergraphs

Graphs are among the most well studied objects of discrete mathematics and have a wide array of applications in computer science. In this work, they will play a major role in Chapter 5. The textbook by Bondy & Murty (2008) offers a good introduction to the field of graph theory. For a more computer science focused introduction, the reader could, for example, consult the chapter on graphs in the popular textbook on algorithm design by Kleinberg & Tardos (2005). We only consider undirected simple graphs, i.e., graphs with undirected edges and without self-loops or multiple edges between the same vertices. These are defined as follows.

Definition 2.7. A graph $G = (V, E)$ is a tuple consisting of a set of vertices V and a set of edges E, where E is a symmetric, irreflexive relation on V. We write $\{u, v\}$ or uv for an edge between vertices u and v. We say two vertices u and v are connected by an edge or *adjacent* if $\{u, v\} \in E$ holds.

An important concept in graph theory is the concept of a subgraph, i.e., a graph embedded in another, larger graph.

Definition 2.8. A *subgraph* of a graph G is a graph whose every vertex and edge are also a vertex and edge of G. If a subgraph $H = (W, F)$ of G contains all edges in G connecting vertices in W, H is the subgraph *induced* by W.

A simple way to define a subgraph is removing one vertex and all edges that contain that vertex.

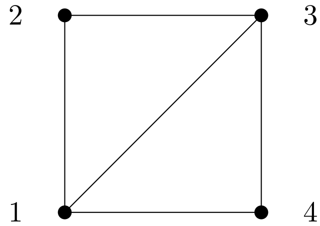

Figure 2.1: Graph G from Example 2.12

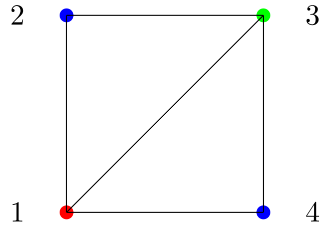

Figure 2.2: A 3-coloring of G

Definition 2.9. Let $G = (V, E)$ be a graph and $v \in V$ a vertex. Then the subgraph $G - v = (V^*, E^*)$ is defined by $V^* = V \setminus \{v\}$ and $E^* := \{(u, w) \in E \mid u, w \neq v\}$.

We also give names to some of the most important types of (sub-)graphs.

Definition 2.10. A *path* consists of a non-empty sequence S of vertices such that every two consecutive vertices are connected with an edge; its length is given by $|S| - 1$. A *cycle* is a sequence of at least three different vertices so that every two consecutive vertices, as well as the first and the last one, are connected with an edge; the length of a cycle is given by the number of vertices. A graph is *connected* if every two of its vertices are connected by a path. A *forest* is a graph with no cycles. A *tree* is a forest that is connected. A graph is a *clique* or *complete graph* if every two vertices are adjacent. A vertex v is an *articulation* point in a graph G if the removal of v from G results in at least two connected components. Graphs without articulation points are called *two-connected*.

Coloring graphs is a problem with a wide array of applications in computer science and mathematics. We only consider vertex colorings. The goal of a vertex coloring is to assign a color to every vertex of a graph such that no adjacent vertices have the same color.

Definition 2.11. We say G is k*-colorable* if there is for every set $|S| = k$ a mapping $f : V \to S$ such that $\{v, w\} \in E$ implies $f(v) \neq f(w)$. We write $k\mathcal{C}$ for the class of all k-colorable graphs.

In this definition, it makes no difference if we require the existence of a mapping f for any set S with $|S| = k$ or only for one specific set S^* with $|S^*| = k$. If there is a mapping f satisfying the condition for S^* and g is a bijective mapping from another set S to S^*, then $g(f(v)) \neq g(f(w))$ must hold for all $\{v, w\} \in E$. We require colorings for arbitrary sets as we want to use meaningful names for colors in later chapters. It is well known that every tree is 2-colorable. Similarly, every even cycle is 2-colorable. On the other hand, odd cycles are not 2-colorable but 3-colorable.

Example 2.12. Consider the following graph (see Figure 2.1):

$$G = (V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\})$$

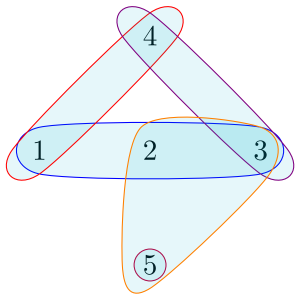

Figure 2.3: Hypergraph H^*

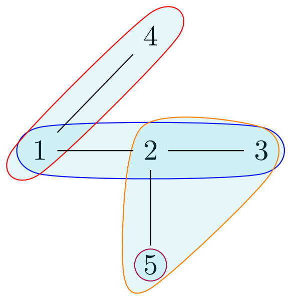

Figure 2.4: A hypertree

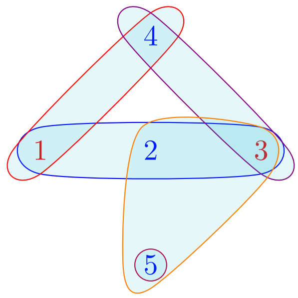

Figure 2.5: 2-coloring of H^*

Then there is no 2-coloring of G. Assume otherwise there is a coloring of G with red and blue. Assume w.l.o.g. that 1 is colored red. Then 2 needs to be colored blue. However, then 3 can be neither red nor blue. On the other hand, coloring 1 red, 2 and 4 blue and 3 as green gives a 3-coloring of G (Figure 2.2).

A graph is, essentially, a set of elements and a family of sets containing two elements each. Therefore, we can view any family of sets as a generalized graph with edges of arbitrary size.

Definition 2.13. A *hypergraph* $H = (V, E)$ is a tuple of a set of vertices V and a set of hyperedges $E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$. A *subhypergraph* of a hypergraph H is a hypergraph whose every vertex and hyperedge are also a vertex and hyperedge of H.

Observe that every graph is a hypergraph, but clearly not every hypergraph is a graph. For example, the following hypergraph, shown in Figure 2.3, is not a graph:

$$H^* = (\{1, 2, 3, 4, 5\}, \{\{1, 2, 3\}, \{1, 4\}, \{2, 3, 5\}, \{3, 4\}, \{5\}\}).$$

Hypergraphs allows us to generalize some concepts from graphs to arbitrary families of sets. For example trees can be generalized to hypertrees.

Definition 2.14. A hypergraph $H = (V, E)$ is called a *hypertree* or *arboreal* if there exists a tree $T = (V, E')$ such that all hyperedges of H induce connected subtrees in T (Berge, 1984; Brandstädt, Dragan, Chepoi & Voloshin, 1998).

Observe that there are also other commonly used notions of acyclicity for hypergraphs, all of which generalize acyclic graphs (Fagin, 1983). Hypertrees are the most restrictive form of acyclicity and the only one that will be used in this thesis. The hypergraph H^* shown in Figure 2.3 is not a hypertree. Assume otherwise that there is a tree T such that every hyperedge of H^* induces a connected subtree in T. Then, T must contain the edges $\{1, 4\}$ and $\{3, 4\}$. Furthermore, T must contain either $\{1, 3\}$ or $\{1, 2\}$ and $\{2, 3\}$. But then 1 and 3 are connected by two distinct pathes, which contradicts the assumption that T is a tree. Now, removing for example the hyperedge $\{3, 4\}$ from H^*

turns it into a hypertree. A tree witnessing this is pictured in Figure 2.4. Finally, we can define colorings of hypergraphs in a similar way as colorings of graphs.

Definition 2.15. Let $H = (V, E)$ be a hypergraph and let S be a set with $|S| = k$. We say a function $c : V \to S$ is a *k-coloring* of H if no hyperedge that contains at least two vertices is monochromatic. In other words, we say c is a k-coloring of H if for every hyperedge e with at least two vertices there are $u, v \in e$ such that $c(u) \neq c(v)$. We say a hypergraph (V, E) is *k-colorable* if there is a k-coloring of H.

The hypergraph H^* is 2-colorable. For example, coloring 1 and 3 in one color and the other vertices in another color is a valid coloring (see Figure 2.5). The only monochromatic hyperedge is $\{5\}$, which does not contain two vertices. Adding, for example, the hyperedge $\{2, 4\}$ would change H^* into a non 2-colorable hypergraph.

Observe that hypertrees are a generalization of trees in the sense that every tree is a hypertree. Similarly, any assignment of colors to the vertices of a graph is a valid graph coloring if and only if it is a valid hypergraph coloring.

2.3 Computational complexity

One of the questions we are interested in is the difficulty of deciding *computationally* if a family of sets has a specific property. The most common way to capture this difficulty in precise mathematical terms is to establish the worst case complexity of a problem. That means giving a lower bound on the amount of a particular resource used by the best algorithms solving the problem on the hardest[1] instances. The resource in question can either be something simple like time or (disk) space or something more unusual like randomness or non-determinism. It turns out that most problems can be grouped into a number of so-called complexity classes, the most important of which are P and NP. The relationship between the different complexity classes is still mostly unknown,[2] but they are nevertheless useful tools for understanding the difficulty a problem poses. For an in depth introduction to complexity theory, the reader can, for example, consult the excellent textbook by Arora & Barak (2009).

Complexity classes

First, we have to fix a formal representation of a computational problem. Complexity theory mostly focuses on so called decision problems, i.e. problems that ask a yes-no question. For example, given a graph G, the question “Does G contain a cycle?” can be formulated as a decision problem, while the question “How many cycles does G contain?”

[1] In particular, we mean here the hardest instances for that specific algorithm, which may differ from algorithm to algorithm. There are many interesting questions related to the instances that materialize the worst case for an algorithm, see for example (Chen, Flum & Müller, 2012).

[2] Settling the relationship between P and NP is considered one of the most important mathematical challenges of our time. It is, for example, one of the seven mathematical problems that were named Millennium Problems by the Clay Institute (Devlin, 2002).

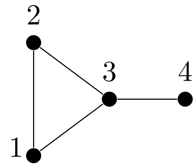

Figure 2.6: The graph used in Example 2.17

cannot. Furthermore, the input of a decision problem, e.g. the graph in the example above, has to be encoded as a binary string. In most cases, the precise encoding is irrelevant, though we will encounter some exceptions to this rule in Chapter 4. This convention allows us to identify decision problems with sets of binary strings that encode an input for which the correct answer is yes.

Definition 2.16. A *decision problem* is a set $Q \subseteq \{0,1\}^*$ of binary strings of arbitrary, finite length. For a string $x \in \{0,1\}^*$ we say x is a yes-instance of Q if $x \in Q$ and x is a no-instance if $x \notin Q$. Furthermore, we write $|x|$ for the length of the string x.

We specify decision problems by their input and by the yes-no question that is asked. Usually, we will omit the encoding used for the input. Consider, for example, the following decision problem:

2-COLORABILITY
Input: A graph G.
Question: Is G 2-colorable?

Now, G must be encoded as a binary string. This can be achieved, for example, by using the adjacency matrix of G. The adjacency matrix of a graph G is a $|V| \times |V|$matrix such that the entry in the i-th row of the j-th column of the matrix is 1 if $\{i,j\} \in E$ and 0 else.

Example 2.17. Consider the graph G shown in Figure 2.6. The adjacency matrix of G is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now, we can encode this matrix as a binary string by just concatenating its rows. For example, we can encode the adjacency matrix of G as the following string:

$$0110101011010010$$

Because every adjacency matrix is square, we can consider every string of length n^2 as encoding a unique $n \times n$ matrix. Observe that this means not every binary string encodes a graph, as not every $n \times n$ matrix is an adjacency matrix, but every graph is encoded by a string. We consider binary strings that do not encode a graph as no-instances for all decision problems with graphs as input. This assumption is usually unproblematic as we can check whether a string encodes a graph in linear time. Therefore, a preprocessing step that checks if a string encodes a graph will not dominate the runtime of any algorithm that needs at least to read the whole input.

One case in which encodings can be important are integers. Many problems can take an integer as part of their input, like the following version of colorability.

k-COLORABILITY
Input: A graph G, an integer k.
Question: Is G k-colorable?

In this case, it is important to note that we always expect the parameter to be encoded in binary and not in unary. That means, the input size of a number k is $\log k$ instead of k. For example, there is a trivial algorithm (see Example 2.18) for testing if an integer is prime that only takes $c \cdot n \log(n)$ steps for some $c > 0$. Hence this algorithm is polynomial in n. Nevertheless, we would not call this algorithm feasible, as it is not polynomial in $\log(n)$.

Example 2.18. Let n be a natural number. It is known that testing whether another natural number $m \leq n$ is a divisor of n can be done in $c \cdot \log(n)$ computational steps (Oberman & Flynn, 1995). Furthermore, there are less than n numbers smaller than n. Hence we can check every possible divisor in $c \cdot n \log(n)$ steps.

The most straight forward way to define a complexity class is to restrict the time, measured in number of steps, a Turing Machine[3] is allowed for solving the problem. Observe that we have to increase the allowed amount of time if the size of the input grows. Otherwise, for large enough inputs, the Turing Machine will not be able to read the whole input.

Definition 2.19. A decision problem Q is said to be solvable in *polynomial time* (or ptime for short) if there is a Turing Machine T and a polynomial p such that on an input of a string x

- the output of T is 1 if $x \in Q$,
- the output of T is 0 if $x \notin Q$,
- T stops after at most $p(|x|)$ steps.

We write P for the set of all decision problems that are solvable in polynomial time.

[3] There are many other formal models of computers that could be used instead of Turing Machines, e.g. Random Access Machines. As long as we do not consider complexity classes that are more restrictive than P, most commonly used machine models are equivalent. It is possible, but unknown if quantum computers are an exception to that rule (Arora & Barak, 2009, p.26).

In general, P is thought of as the class of problems that are "tractable". This is not correct in theory as an algorithm with runtime n^{100} is a ptime algorithm but certainly not usable in practice. However, experience shows that essentially all problems that appear in practice and are in P can be solved efficiently.(Arora & Barak, 2009, p.27) For example, 2-COLORABILITY is in P.

Example 2.20. Let $G = (V, E)$ be a graph that we want to color with the two colors red and blue. Let us assume that the graph G is connected. If we want to color an unconnected graph, we can just color all connected components separately. We can use the following algorithm, where v is a randomly chosen vertex:

Algorithm 2.1: 2-coloring

```
Input: Graph G = (V, E)
R ← {v};
B ← ∅;
Queue ← {v} ;
while Queue ≠ ∅ do
    w ← First vertex in Queue;
    Remove w from Queue;
    forall u adjacent to w do
        if w ∈ R then
            if u ∈ R then
                return Not 2-colorable;
            if u ∉ B then
                Add u to B and to Queue;
        if w ∈ B then
            if u ∈ B then
                return Not 2-colorable;
            if u ∉ R then
                Add u to R and to Queue;
return 2-colorable;
```

This algorithm works as follows: We pick an arbitrary first vertex v and color it w.l.o.g. red. Then, we must color all neighbors of v blue. We put all newly colored vertices in a queue. Then, in every step, we pick the first vertex w of the queue. Because it is in the queue it must be colored either red or blue. If it is colored red, we color all of its neighbors blue and put the newly colored vertices in the queue. Similarly, if w is colored blue, we color all of its neighbors red and put the newly colored vertices in the queue. We repeat these steps until we either color an already blue colored vertex red, an already red colored vertex blue or until the queue is empty. In the first two cases, we know that the graph is not 2-colorable. As the graph is connected, the queue will only be empty, once all vertices are colored. Hence, the graph is 2-colorable in that case.

It can be checked that the runtime of this algorithm is polynomial: We can assume that every individual step in the algorithm can be performed in polynomial time, as these are only basic operations on graphs, arrays and queues. Now, the while loop will only be executed once per vertex. For every vertex v, the forall loop will executed as often as v has neighbors, that is at most $|V|$ times. Hence, the algorithm executes at most $|V|^2$ many polynomial time operations. Therefore we can say that 2-coloring is in P.

This idea does not work for deciding if a graph is k-colorable for any $k \geq 3$, as we can not determine the color of a vertex solely from the color of one neighbor. Indeed, as we will see later, it is very likely that k-COLORABILITY is not in P.

Clearly, we can define a larger complexity class by allowing the time to grow exponentially instead of polynomially.

Definition 2.21. A decision problem Q is said to be solvable in *exponential time* if there is a Turing Machine T and a polynomial p such that on input of a string x

- the output of T is 1 if $x \in Q$,
- the output of T is 0 if $x \notin Q$,
- T stops after at most $2^{p(|x|)}$ steps.

We write EXP for the set of all decision problems that are solvable in exponential time.

Problems in EXP are not considered to be efficiently solvable, except for very small instances. We can define an interesting subclass of EXP by considering problems that may in general be hard to solve but for which one can efficiently check the correctness of a solution. Intuitively, this class of problems is captured by the class NP. For example, it can be very hard to solve a Sudoku, but once one has a solution, it can easily be checked if the solution is correct. This is not true for all problems. Consider, for example, the question if a given program halts on input I. In general, the only possible way to check this is actually running the program with input I.

Definition 2.22. A decision problem Q is said to be solvable in *non-deterministic polynomial time* if there is a Turing Machine T with input (x, y) and a polynomial p such that

- T stops after at most $p(|(x, y)|)$ steps for all inputs (x, y),
- for all $x \notin Q$ the output of T on input (x, y) is 0 for all $y \in \{0, 1\}^*$,
- for all $x \in Q$, there exists a $y \in \{0, 1\}^*$ such that $|y| < p(|x|)$, and the output of T on input (x, y) is 1.

We call y a certificate for x. We write NP for the set of all decision problems that are solvable in non-deterministic polynomial time.

For every string x and polynomial p, there are only exponentially many strings y that satisfy $|y| < p(|x|)$. Therefore, one can try all possible certificates for a string x in exponential time. Hence, every problem in NP is also in EXP. On the other hand, any problem in P is also in NP by definition. We can write this as $\mathsf{P} \subseteq \mathsf{NP} \subseteq \mathsf{EXP}$. As $\mathsf{P} \neq \mathsf{EXP}$ is known (Hartmanis & Stearns, 1965), at least one of the two inclusions has to be proper, however, it is not yet known which of these two.

We can see that k-colorability is in NP. Consider a graph G that is k-colorable. Then, we can use a k-coloring of G as a certificate because it only exists if G is a positive instance and it is possible to check in polynomial time if an assignment of colors to vertices is a valid k-coloring. As with P and EXP we can also define an "exponential equivalent" of NP.

Definition 2.23. A decision problem Q is said to be solvable in *non-deterministic exponential time* if there is a Turing Machine T with input (x, y) and a polynomial p such that

- T stops after at most $2^{p(|(x,y)|)}$ steps.
- for all $x \notin Q$ the output of T on input (x, y) is 0 for all $y \in \{0, 1\}^*$
- for all $x \in Q$, there exists a $y \in \{0, 1\}^*$ such that $|y| < 2^{p(|x|)}$ and the output of T on input (x, y) is 1.

We write NEXP for the set of all decision problems that are solvable in non-deterministic exponential time.

NP and NEXP are not symmetrical, insofar as yes-instances have certificates and no-instances in general not. Therefore, we can produce an interesting class of problems by taking the complement of NP or NEXP.

Definition 2.24. The class coNP is defined as the following set of decision problems:

$$Q \in \mathsf{coNP} \text{ if and only if } (\{0, 1\}^* \setminus Q) \in \mathsf{NP}.$$

The class coNEXP is defined similarly by

$$Q \in \mathsf{coNEXP} \text{ if and only if } (\{0, 1\}^* \setminus Q) \in \mathsf{NEXP}.$$

Intuitively, problems are in coNP if one can efficiently check the correctness of a counter example. For example, checking if a natural number n is a prime is a problem in coNP. Given a factor k of n, we can efficiently check if $\frac{n}{k}$ is an integer and hence k proves that n is not a prime number. In a certain sense, NP-problems ask $\exists$ questions and coNP-problems ask $\forall$ questions. We can use this fact to define a whole hierarchy of complexity classes based on quantifier alternations.

Definition 2.25. A decision problem Q is in Σ_k^p if there is a Turing Machine T with input $(x, y_1, \ldots, y_k)$ and a polynomial p such that

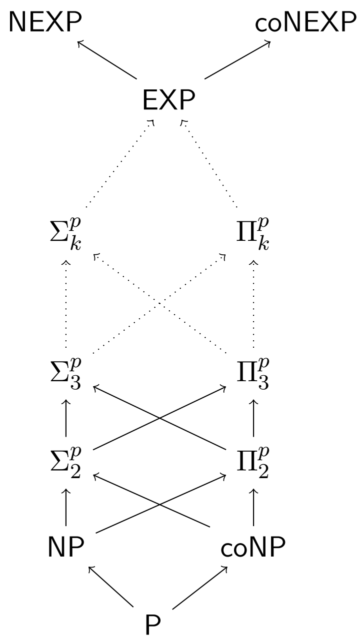

Figure 2.7: Relationship between the complexity classes considered in this work. Arrows denote inclusion.

- T stops after at most $p(|(x, y_1, \dots, y_k)|)$ steps for all inputs $(x, y_1, \dots, y_k)$,
- $x \in Q$ if and only if

$$\exists y_1 \in \{0,1\}^l \forall y_2 \in \{0,1\}^l \dots Q_k y_k \in \{0,1\}^l T(x, y_1, \dots, y_k) = 1,$$

  where Q_k equals $\exists$ for odd k and $\forall$ for even k and $l = p(|x|)$.

We observe that $\Sigma_1^p = \mathsf{NP}$. As for NP we can define the complement for each class Σ_k^p.

Definition 2.26. The class Π_k^p is defined as the following set of decision problems:

$$Q \in \Pi_k^p \text{ iff } (\{0,1\}^* \setminus Q) \in \Sigma_k^p.$$

The set containing the classes Σ_k^p and the Π_k^p for all k is called the polynomial hierarchy. All classes in the polynomial hierarchy are contained in EXP. Furthermore Σ_k^p and Π_k^p both contain Σ_l^p as well as Π_l^p for all $l < k$. Finally, $\mathsf{P} = \mathsf{NP}$ would imply that all classes in the polynomial hierarchy equal P. Otherwise, essentially no unconditional results are known about the structure of the polynomial hierarchy. Figure 2.7 gives a summary of the relations between the complexity classes defined in this section.

Reductions and completeness

We defined above when a problem is *in* a complexity class. This tells us that we can solve the problem, given the resources specified by the complexity class. However, we also want to know how difficult or hard a problem is. Unfortunately, it turns out that it is extremely hard to prove unconditional lower bounds on the amount of a specific resource needed to solve a problem. What we do have is a formalism to determine the relative hardness of a problem compared to other decision problems in the form of reductions.

Definition 2.27. We say that a decision problem Q is *polynomial time* or *ptime reducible* to a problem Q^* if there is a polynomial time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that, for any $x \in \{0,1\}^*$, $x \in Q$ if and only if $f(x) \in Q^*$. In such a case, we write $Q \leq_p Q^*$.

Intuitively, we can say a problem Q^* is at least as hard as a problem Q whenever we can reduce Q to Q^*. It turns out that many complexity classes have a set of most difficult problems, that is problems to which all other problems in the class can be reduced. We call such problems complete for the complexity class.

Definition 2.28. We say a decision problem Q is *hard for a complexity class* C or C-*hard* if for every problem Q^* in C we have $Q^* \leq_p Q$. We say a problem Q is *complete for a complexity class* C or C-*complete* if Q is in C and C-hard.

Because the composition of two ptime reductions is also a ptime reduction, it suffices to show that a C-hard problem Q^* can be reduced to a problem Q to show that Q is C-hard. Then for every problem Q^{**} in C we have $Q^{**} \leq_p Q^* \leq_p Q$ and hence $Q^{**} \leq_p Q$.

Observe that under this form of reduction, every problem is P-hard. Therefore, every problem in P is P-complete, including the trivial problem.

Trivial
Input: A string s.
Question: Is $s = 1$?

Consider a problem Q in P. Then we can reduce Q to Trivial as follows: Given an instance of Q, first solve it in polynomial time and second map it to 1 if it is a positive instance and to 0 otherwise. It is possible to define other forms of reductions that lead to an interesting class of P-complete problems. We will come back to this in Chapter 4.

The most famous example of an NP-complete problem is Sat, the problem of deciding whether a boolean formula is satisfiable. If the reader is not familiar with basic terms of boolean logic, we refer the reader either to the short introduction in the textbook by Arora & Barak (2009) or to the more in depth treatment in the famous textbook on logic by Enderton (2001). We only consider the satisfiability of a specific type of formulas, namely 3-CNFs. A 3-CNF is the conjunction of arbitrary many clauses, where every clause is the disjunction of at most three literals, i.e., atoms or negated atoms. For example, $\phi = (a \vee b) \wedge (\neg b \vee a \vee \neg c) \wedge (c \vee b \vee d)$ is a 3-CNF. The corresponding satisfiability problem is also often called 3-Sat. As we can compute for every formula a

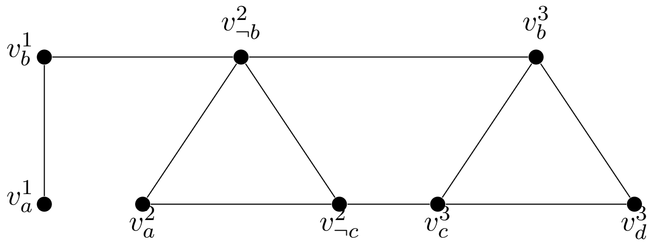

Figure 2.8: Graph constructed for $\phi = (a \vee b) \wedge (\neg b \vee a \vee \neg c) \wedge (c \vee b \vee d)$

equi-satisfiable 3-CNF in polynomial time, (Arora & Barak, 2009, p.49) both problems are equivalent from a computational perspective.

SAT
Input: A 3-CNF ϕ.
Question: Is ϕ satisfiable?

In 1971, Cook (1971) proved that SAT is NP-complete, which makes it the first known NP-complete problem. Using this result, we can show that another problem is NP-hard by reducing SAT to it. Consider for example INDEPENDENT SET.

INDEPENDENT SET
Input: A graph G and a natural number k.
Question: Is there a set $I \subseteq V$ of size k such that no two vertices in I are adjacent?

We can prove that INDEPENDENT SET is NP-hard by the following reduction from SAT.

Example 2.29. Let ϕ be a 3-CNF with m clauses. We construct a graph G that has a independent set of size m if and only if ϕ is satisfiable. We add for every clause C_k a clique containing one vertex v_i^k for every literal l_i in C_k. Furthermore, we add an edge $\{v_i^k, v_j^o\}$ whenever literal l_i in clause C_k is the negation of literal l_j in clause C_o. For $\phi = (a \vee b) \wedge (\neg b \vee a \vee \neg c) \wedge (c \vee b \vee d)$ this construction results in the graph shown in Figure 2.8. It is clear that, given ϕ, this graph can be computed in polynomial time.

We claim that G and m is a positive instance of INDEPENDENT SET if and only if ϕ is a positive instance of SAT. Assume first that a satisfying truth assignment T of ϕ exists. By definition, every clause contains at least one literal that is set to true by T. Now, let I be a set that contains for every clause a vertex representing one such literal. Then, we claim that I is an independent set of size m. By definition, I has size m. Furthermore, by definition it contains only one vertex per clause. Therefore, two vertices in I can only be adjacent if they represent a variable and its negation. However, this can not happen, as we assumed that both vertices represent a literal that was set to true by T.

Now assume I is an independent set of size m. We define a truth assignment T by setting a variable v

- to true if a vertex representing v is in I,

- to false if a vertex representing $\neg v$ is in I
- and assign an arbitrary truth value otherwise.

This is a legal truth assignment because I can not contain a vertex representing v and a vertex representing $\neg v$ at the same time as these are adjacent. Under T every literal that is represented by a vertex in I is set to true. Now, I must contain one vertex per clause, as all vertices representing the same clause are adjacent. Therefore, T is a satisfying assignment for ϕ.

The complement of Sat is Unsat, the problem of determining if a boolean formula is unsatisfiable. It is therefore natural to assume that Unsat must be coNP-complete. This is indeed the case. The question if a 3-CNF ϕ is unsatisfiable is equivalent to asking whether $\neg\phi$ is a tautology. Furthermore, it is easy to see that the negation of a 3-CNF can be turned into a 3-DNF in polynomial time. Hence, the following problem is essentially equivalent to Unsat and therefore also coNP-complete.

Taut
Input: A 3-DNF ϕ.
Question: Is ϕ a tautology?

In order to define complete problems for the complexity classes in the polynomial hierarchy, we have to consider an extension of propositional logic called quantified boolean logic.

Definition 2.30. Let ϕ be a boolean formula ϕ with variables V and $V_1, \dots, V_k$ a partition of V into disjoint subsets. Then, we call a formula ψ of the form $\psi = \forall V_1 \exists V_2 \dots QV_k \phi$, where $Q = \forall$ if k is odd and $Q = \exists$ if k is even, a *universally quantified boolean formula*. We say ψ has $k-1$ quantifier alternations and write Π_k for the set of all universally quantified boolean formulas with $k-1$ quantifier alternations.

Similarly, we call a formula ψ of the form $\psi = \exists V_1 \forall V_2 \dots QV_k \phi$, where $Q = \exists$ if k is odd and $Q = \forall$ if k is even, an *existentially quantified boolean formula*. Again, we say that ψ has $k-1$ quantifier alternations. We write Σ_k for the set of all existentially quantified boolean formulas with $k-1$ quantifier alternations. We say a formula is a *quantified boolean formula* (or QBF for short) if it is either a universally quantified or an existentially quantified boolean formula.

We define the *satisfiability of a quantified boolean formula* recursively. A formula in Σ_1 is satisfiable if the underlying boolean formula is satisfiable. A formula in Π_1 is satisfiable if the underlying boolean formula is a tautology. Now let $\psi = \exists V_1 \forall V_2 \dots QV_k \phi$ be a formula in Σ_k. Then, ψ is satisfiable if there is an assignment T to the variables in V_1 such that the following Π_{k-1} formula is satisfiable: $\forall V_2 \dots QV_k \phi^T$ where ϕ^T is the boolean formula that is obtained by replacing in ϕ every variable in V_1 by $\top$ if it is assigned true by T and by $\bot$ otherwise. Moreover, let $\psi = \forall V_1 \exists V_2 \dots QV_k \phi$ be a formula in Π_k. Then, ψ is satisfiable if for every assignment T to the variables in V_1 the following Σ_{k-1} formula is satisfiable: $\exists V_2 \dots QV_k \phi^T$ where ϕ^T is the boolean formula that is obtained by replacing in ϕ every variable in V_1 by $\top$ if it is assigned true by T and by $\bot$ otherwise.

For a quantified boolean formula ψ, we can assume that ϕ is a 3-CNF if the innermost quantifier is an existential one, i.e., if ψ is existential and the number of quantifier alternations is even or if ψ is universal and the number of quantifier alternations is odd. Otherwise, we can assume that ϕ is a 3-DNF.

Example 2.31. Consider the following 3-CNF:

$$\phi = (A \vee C) \wedge (B \vee \neg C) \wedge (\neg A \vee \neg B)$$

Then $\forall C \exists A, B \phi$ is satisfiable, because

$$(A \vee \top) \wedge (B \vee \neg \top) \wedge (\neg A \vee \neg B)$$

is satisfied if we set A to false and B to true. On the other hand

$$(A \vee \bot) \wedge (B \vee \neg \bot) \wedge (\neg A \vee \neg B)$$

is satisfied if we set A to true and B to false. In contrast, $\exists A \forall B, C \phi$ is not satisfiable, because

$$(\top \vee C) \wedge (B \vee \neg C) \wedge (\neg \top \vee \neg B)$$

is not satisfied if we set B to false and

$$(\bot \vee C) \wedge (B \vee \neg C) \wedge (\neg \bot \vee \neg B)$$

is not satisfied if we set C to false.

Now, for every complexity class in the polynomial hierarchy, there is a corresponding satisfiability problem. For every k, the following problem is Σ_k^p-complete.

Σ_k-Sat
Input: A Σ_k-QBF ϕ.
Question: Is ϕ satisfiable?

Similarly, for every k, the following problem is Π_k^p-complete.

Π_k-Sat
Input: A Π_k-QBF ϕ.
Question: Is ϕ satisfiable?

Table 2.1 summarizes for all complexity classes the complete problems that we introduced. Observe that we did not introduce any EXP, NEXP or coNEXP-complete problems. We will do so in Chapter 4 when we talk about succinctly represented problems.

Complexity class	Complete problem
P	Trivial, 2-colorability
NP	Sat, Independent Set, 3-colorability, k-colorability
coNP	Unsat, Taut
Σ_k^p	Σ_k-Sat
Π^p	Π-Sat

Table 2.1: Complexity classes and complete problems

CHAPTER 3

The Order Lifting Problem

In this chapter, we introduce the problem that this thesis aims to tackle, the order lifting problem. This problem has been extensively studied by authors from a wide range of scientific disciplines, including mathematicians like Yakar Kannai, economists like Salvador Barberà, and computer scientists like Stefano Moretti (Moretti & Tsoukiàs, 2012). It is beyond the scope of this thesis to give a complete overview over this literature. Instead, we give only a very short overview over the history of the order lifting problem in Section 3.1. Then, we focus on the axiomatic approach to the order lifting problem that we formalize in Section 3.2. We introduce the four main axioms that will be studied in this thesis in Section 3.3 and discuss important impossibility results regarding these axioms in Section 3.4. Afterwards in Section 3.5, we formalize the different ways in which the main axioms can be compatible on a family of sets. We conclude the chapter in Section 3.6 with some additional axioms and results that will prove useful in the following chapters.

3.1 A short history of the order lifting problem

The idea of ranking sets of objects based on a ranking of the objects is very old. For example, since antiquity humans order words lexicographically based on an order on the letters of the alphabet. Daly (1967) gives an interesting account of the development of this technique by ancient scholars. Other orders, like ordering by maximal elements are most probably as old. A greater academic interest in the order lifting problem was sparked by the famous result by Gibbard (1973) and Satterthwaite (1975) that no resolute voting rule can be strategyproof. Studying the strategyproofness of irresolute voting rules necessitated the definition of rankings of sets of candidates based on rankings of candidates. Noteworthy attempts were made by Fishburn (1972), Gärdenfors (1976) and Kelly (1977). Additionally, lottery based rankings were often considered (Duggan & Schwartz, 2000). These are still the most widely used rankings for studying the strategyproofness of irresolute voting rules (Brandt, Brill & Harrenstein, 2016; Barberà, 2011).

The axiomatic approach to the order lifting problem considered in this thesis is significantly younger. First works were published from the fifties onward, for example by Kraft, Pratt & Seidenberg (1959) and Kim & Roush (1980). The seminal result that sparked a lot of interest in this area was Kannai and Peleg's famous impossibility result (Kannai & Peleg, 1984). We will discuss this result, that is one of the key motivations of this thesis, in Section 3.4 in detail. In the following years, a significant number of papers more or less directly inspired by Kannai and Peleg's result were published. Important examples include papers by Barberà & Pattanaik (1984), Holzman (1984), Barberà et al. (1984), Fishburn (1984), Bandyopadhyay (1988), Bossert (1995), Kranich (1996), Bossert, Pattanaik & Xu (2000) and Dutta & Sen (2005). For a complete overview over this line of research, we refer the reader again to the great survey by Barberà et al. (2004). A noteworthy development after 2004 and therefore not included in this survey was a paper by Geist & Endriss (2011) that used methods from Sat-solving to generate new impossibility results automatically with a computer.

We note that, in contrast to the work in this thesis, all of the aforementioned works – except Bossert (1995) – are only interested in the case where the whole power set needs to be ranked.

3.2 Formalizing the order lifting problem

Given a set X and a linear order $\leq$ on X, the *order lifting* problem consists of deriving from $\leq$ an order $\preceq$ on a family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ of non-empty *subsets* of X, guided by axioms formalizing some natural desiderata for such lifted orders. Generally, these axioms take $\leq$ into account. Therefore, an order $\preceq$ on $\mathcal{X}$ in general only satisfies an axiom with respect to a specific linear order $\leq$ on X.

The Order Lifting Problem

Input: A set X, a linear order $\leq$ on X, a family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and a set of axioms $\mathcal{A}$.

Goal: Find an order $\preceq$ on $\mathcal{X}$ that satisfies all axioms in $\mathcal{A}$ with respect to $\leq$.

Observe that we did not specify what kind of order $\preceq$ should be. In the following, we will consider the problem of lifting to a variety of different types of orders. The most common case will be lifting to a weak order, which is also the problem considered most often in the literature. However, we will also consider lifting to preorders, partial orders and linear orders. In one case, we even consider lifting to an arbitrary binary relation. Furthermore, we note that we assume that $\mathcal{X}$ only contains *nonempty* sets because one of our axioms, namely dominance, immediately leads to a contradiction if the empty set is contained in $\mathcal{X}$.

For the uniformity of notation, we will stick to the following conventions: In any instance of the order lifting problem, we will use uppercase letters to denote the set of objects or ground set, e.g., X or Y, and lowercase letters or natural numbers to denote its elements. We use calligraphic letters for the family of subsets, e.g., $\mathcal{X}$ or $\mathcal{Y}$ and uppercase letters at the beginning of the alphabet for its elements, i.e., for subsets of the ground set. Similarly, we use $\leq$ for the linear order on the ground set, possible with an index for uniqueness, and the calligraphic $\preceq$ for any order on the family of subsets, also possibly with an index.

3.3 The main axioms

In this section we introduce the four axioms that will be the main focus of our investigation, namely dominance, independence, strict independence and the extension rule. The definitions of these axioms are not entirely consistent in the literature. We will essentially follow Barberà et al. (2004) with our definitions. However, we need to a add a condition that states that the axiom is only applicable if a set is in the family of sets that we want to lift to. As Barberà et al. (2004) only consider the case $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$, such a condition is not needed in their version of the axioms.[1]

Throughout this section, we will demonstrate the effect of the axioms on the following toy example: Let $S_{oy} = \{1, 2, 3, 4\}$ and let $\leq$ be the natural linear order on S_{oy}. Furthermore, let

$$\mathcal{T}_{oy} = \{\{2\}, \{4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 4\}\}.$$

Now, let us introduce our main axioms. We begin with the so-called dominance axiom.

Dominance
For all $A \in \mathcal{X}$ and all $x \in X$, such that $A \cup \{x\} \in \mathcal{X}$:

$$y < x \text{ for all } y \in A \text{ implies } A \prec A \cup \{x\};$$

$$x < y \text{ for all } y \in A \text{ implies } A \cup \{x\} \prec A.$$

Any relation $\preceq$ on $\mathcal{T}_{oy}$ that satisfies dominance with respect to $\leq$ must set $\{2\} \prec \{2, 4\}$, $\{2, 4\} \prec \{4\}$, $\{3, 4\} \prec \{4\}$, $\{1, 2, 4\} \prec \{2, 4\}$ and $\{1, 4\} \prec \{4\}$.

Dominance is often also called Gärdenfors' principle after Peter Gärdenfors who introduced a version of the axiom (Gärdenfors, 1976). It states that adding an element to a set that is better than all elements already in the set increases the quality of the set and, similarly, adding a element worse than all elements in the set decreases the quality of the set. This principle is often desirable if the order $\preceq$ should reflect, to some extent,

[1] Observe that another way of adapting the axioms to the setting $\mathcal{X} \neq \mathcal{P}(X) \setminus \{\emptyset\}$ would be, to demand that the lifted order $\preceq$ is a binary relation on $\mathcal{P}(X) \setminus \{\emptyset\}$ that satisfies all axioms and the restriction of $\preceq$ to $\mathcal{X}$ is a weak order. A study of this more restrictive approach is left for future work.

the average quality of the sets. If sets represent possible outcomes, they can often be ranked by expected utility, which equals the average quality of the elements if elements are sampled with uniform probability. Therefore, dominance is often desirable under the disjunctive interpretation of sets, where the sets represent incompatible alternatives from which one is chosen randomly:

Example 3.1. Several airlines, for example Eurowings, offer so-called blind bookings. Here, the customer can choose between different bundles of destinations. In March 2020 nine possible bundles are available, for instance the Nature bundle containing Berlin, Bologna, Edinburgh, Klagenfurt, Munich, Salzburg, Sarajevo and Zurich or the Romance package containing Barcelona, Bologna, Budapest, Milan, Rome, Salzburg, Venice and Vienna. After purchasing one of these bundles, the buyer receives a ticket for one of the destinations included in the chosen bundle.[2]

Alternatively, consider an election where a voter knows that depending on his vote different sets of candidates will be tied for the first place. Furthermore, he knows that the final winner will be chosen from the tied candidates randomly. In both of these situations, dominance is a natural desideratum (Can, Erdamar & Sanver, 2009).

Several commonly used orders on families of sets satisfy dominance. We claimed above, that dominance is desirable if $\preceq$ should reflect the average quality of the sets. We can make this formal by observing that an order based on average utilities satisfies dominance.

Example 3.2. Let X be a set and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Furthermore, let $u : X \to \mathbb{R}$ be a utility function that assigns every element of X a unique utility, i.e., $u(x) \neq u(y)$ for all $x, y \in X$ such that $x \neq y$. Then, we can define a weak order on $\mathcal{X}$ by $A \preceq B$ if the average utility of the elements of A is less than the average utility of the elements of B, i.e., if

$$\frac{\sum_{a \in A} u(a)}{|A|} \leq \frac{\sum_{b \in B} u(b)}{|B|}.$$

Now, the average utility of a set clearly increases when an element with higher utility than all elements in the set is added. Similarly, the average utility of a set decreases when adding an element with a lower utility than all elements in the set. Therefore, this order satisfies dominance with respect to the linear order $\leq$ on X defined by $x \leq y$ if $u(x) \leq u(y)$.

However, we do not need utilities that we can average in order to define an order that satisfies dominance. For example, the following maxmin-based[3] order satisfies dominance on all families of sets.

Example 3.3. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Then, we can define a weak order $\preceq_{mm}$ on $\mathcal{X}$ by $A \preceq_{mm} B$ for $A, B \in \mathcal{X}$ if

[2] Information taken from `https://www.eurowings.com/skysales/BlindBooking.aspx`. Accessed: 26-March-2020

[3] Formally, we call an order $\preceq$ a maxmin-based order if there exists an order $\preceq^*$ on $X \times X$ such that $A \preceq B$ holds if and only if $(\min(A), \max(A)) \preceq^* (\min(B), \max(B))$ holds (Barberà et al., 2004, p.13).

- $\max(A) < \max(B)$ or
- $\max(A) = \max(B)$ and $\min(A) \leq \min(B)$.

Observe that this is the maxmin-based order defined by the lexicographic order on $X \times X$. We claim that this is a weak order. It is straightforward to check that it is reflexive and total. Now consider sets such that $A \preceq_{mm} B$ and $B \preceq_{mm} C$. Clearly, $\max(A) \leq \max(B)$ and $\max(B) \leq \max(C)$ must hold. Furthermore, if one of the comparisons is strict, then $A \prec_{mm} C$ follows by definition because $\leq$ is transitive. So assume $\max(A) = \max(B)$ and $\max(B) = \max(C)$. Then, $\min(A) \leq \min(B)$ and $\min(B) \leq \min(C)$ must hold and hence $\min(A) \leq \min(C)$ which implies $A \preceq_{mm} C$.

Furthermore, $\preceq_{mm}$ satisfies dominance: Let $x \in X$ and $A, A \cup \{x\} \in \mathcal{X}$. Assume $\max(A) < x$. Then, $\max(A) < \max(A \cup \{x\}) = x$ and hence $A \prec_{mm} A \cup \{x\}$. On the other hand, if $x < \min(A)$, then $\max(A \cup \{x\}) = \max(A)$ and $\min(A \cup \{x\}) = x < \min(A)$ and hence $A \cup \{x\} \prec_{mm} A$.

Other well-known examples of orders that satisfy dominance are the following lifted orders proposed by Fishburn (1972) and Gärdenfors (1976). Both orderings are frequently used in the context of strategyproofness in elections with tie-breaking (Brandt et al., 2016; Barberà, 2011).

Example 3.4. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Then, the so-called Fishburn extension $\preceq_f$ is defined by $A \preceq_f B$ if all of the following conditions hold:

- $x < y$ for all $x \in A \setminus B$ and $y \in A \cap B$,
- $y < z$ for all $y \in A \cap B$ and $z \in B \setminus A$,
- $x < z$ for all $x \in A \setminus B$ and $z \in B \setminus A$.

Observe that the first two conditions imply the third unless $A \cap B = \emptyset$. On $\mathcal{T}_{oy}$ Fishburn's extension looks as follows:

$$\{2\} \prec_f \{2,4\} \prec_f \{4\}; \{2\} \prec_f \{3,4\} \prec_f \{4\};$$
$$\{1,2,4\} \prec_f \{4\}; \{1,4\} \prec_f \{4\}; \{1,2,4\} \prec_f \{2,4\}.$$

We claim that $\preceq_f$ satisfies dominance. Assume $A, A \cup \{x\} \in \mathcal{X}$ and $\max(A) < x$. Then, $A \setminus (A \cup \{x\}) = \emptyset$ and $y < x$ for all $y \in A \cap (A \cup \{x\}) = A$. Hence $A \prec_f A \cup \{x\}$. The case $x < \min(A)$ is analogous.

The so-called Gärdenfors' extension $\preceq_g$ is defined by $A \preceq_g B$ if one of the following holds

- $A \subseteq B$ and $x < y$ for all $x \in A$ and $y \in B \setminus A$,
- $B \subseteq A$ and $x < y$ for all $x \in A \setminus B$ and $y \in B$,

- Neither $A \subseteq B$ nor $B \subseteq A$ and $x < y$ for all $x \in A \setminus B$ and $y \in B \setminus A$.

It is known that Gärdenfors' extension is a superset of Fishburn's extension. This means, $A \prec_f B$ implies $A \prec_g B$ for every $A, B \in \mathcal{X}$. Therefore, it follows directly that Gärdenfors' extension satisfies dominance. On $\mathcal{T}_{oy}$ Gärdenfors' extension adds to Fishburn's extension the following preferences:

$$\{1,4\} \prec_g \{2,4\} \prec_g \{3,4\}; \{1,2,4\} \prec_g \{3,4\}$$

The second axiom that we consider is called independence.

Independence
For all $A, B \in \mathcal{X}$ and all $x \in X \setminus (A \cup B)$, such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \prec B \text{ implies } A \cup \{x\} \preceq B \cup \{x\}.$$

A relation $\preceq$ on $\mathcal{T}_{oy}$ that satisfies independence must set $\{1,2,4\} \preceq \{1,4\}$ if it contains $\{2,4\} \prec \{4\}$ and $\{1,4\} \preceq \{1,2,4\}$ if it contains $\{4\} \prec \{2,4\}$. As dominance implies $\{2,4\} \prec \{4\}$, dominance and independence together imply $\{1,2,4\} \preceq \{1,4\}$.

Independence is a natural monotonicity axiom that states that if we add the same element x to two sets A and B where B is strictly preferred to A, then $B \cup \{x\}$ must be at least weakly preferred to $A \cup \{x\}$. This is often a very desirable property under the conjunctive interpretation, for example if sets are bundles of objects that are compared according to their overall goodness according to some additive utility (Kraft et al., 1959). Indeed, if we define an order based on the sums of utilities, that order satisfies independence.

Example 3.5. Consider the same setting as in Example 3.2, i.e., let X be a set, let $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and let $u : X \to \mathbb{R}$ be a utility function that assigns every element of X a unique utility. Then we can also define a weak order on $\mathcal{X}$ by comparing total, instead of average utility. That means $A \preceq B$ if the sum of the utilities of the elements of A is less than the sum of the utilities of the elements of B, i.e., if

$$\sum_{a \in A} u(a) \leq \sum_{b \in B} u(b).$$

Now, if we add the same element to two sets, the difference in total utility will not change. Therefore, this order clearly satisfies independence with respect to the linear order $\leq$ on X defined by $x \leq y$ if $u(x) \leq u(y)$. However, observe that this order does not satisfy dominance. Assume, for example, $X = \{-3, -1, 1, 2, 5, 9\}$ and $u(x) = x$ for all $x \in X$. Furthermore, assume $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then $\{2\} \prec \{1, 2\}$ and $\{-3, -1\} \prec \{-3\}$ both contradict dominance.

On the other hand, the average based order defined in Example 3.2 does not satisfy independence. Consider the same X, u and $\mathcal{X}$ as above. Then $\{2\} \prec \{1, 5\}$ as $2 < \frac{6}{2}$ but $\{1, 5, 9\} \prec \{2, 9\}$ as $\frac{15}{3} < \frac{11}{2}$. This contradicts independence.

In this sense there is some tension between the motivations for dominance and independence. Nevertheless, there are cases where both axioms are natural desiderata. These cases are often characterized by the fact that all elements may influence the quality of a set but the extent of this influence is unknown or unknowable. An example under the disjunctive interpretation for such a situation is choice under complete uncertainty:

Example 3.6. Consider a situation where an agent can perform actions $a_1, \ldots, a_k$ for which he knows the (set of) possible outcomes but he is not able or not willing to determine the (approximate) probability of each outcome. Such a situation can be modeled as a family of outcomes $X = \{o_1, o_2, \ldots, o_l\}$ and a function $O : \{a_1, \ldots, a_k\} \to \mathcal{P}(X) \setminus \{\emptyset\}$ that maps every action to the set of possible outcomes of that action. If we assume that the agent has preferences over the set of possible outcomes X that can be modeled as a linear order, the problem of ranking the different actions can be modeled as an order lifting problem. Under this interpretation the extension rule (see below), dominance and independence are usually considered natural desiderata (Bossert et al., 2000; Barberà et al., 2004). □

In voting, similar situations appear if ties are broken by an unknown chairman. Similarly, situations exist under the conjunctive interpretation when it is unclear how much each object contributes to the quality of the set.

Example 3.7. Assume a manager wants to select a team for a given task. In order to do so, he wants to rank all possible teams according to their expected ability to solve this task. We assume that he can rank his employees with respect to their ability to solve the task at hand. Then, we expect a team to perform better if we add an employee that is more proficient at the task than everyone already in the team and to perform worse if we add an employee that is worse than everyone else on the team. In other words, we believe that the ranking on teams should satisfy dominance. On the other hand, it seems natural that, if one team is better than another then adding the same employee to both teams can not completely change the relation between the teams. At most, the new team-member can equalize the difference between the teams. In other words, the ranking on teams should satisfy independence.

Again, in voting a similar situation can occur when sets represent an elected committee in which each member has an (a priori) unknown influence.

In contrast to dominance, independence on its own does not require any preferences. In other words, the empty preference relation always satisfies independence. The weak order defined in Example 3.3 does not satisfy independence.[4]

Example 3.8. Let $\preceq_{mm}$ be the weak order defined in Example 3.3. Consider $X = \{1, 2, 3, 4\}$, $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ and let $\leq$ be the natural linear order on X. Then, $\{2\} \prec_{mm} \{1, 3\}$ but $\{1, 3, 4\} \prec_{mm} \{2, 4\}$. Therefore, $\preceq_{mm}$ does not satisfy independence.

[4] Barberà et al. (2004), following Bossert et al. (2000), falsely claim that the order defined in Example 3.3 can be characterized by simple dominance, independence and two other axioms. Arlegi (2003) was the first to point out that this is not the case, because $\preceq_{mm}$ does not satisfy independence. He also provided a different axiomatic characterization of this order.

Furthermore, neither Fishburn's nor Gärdenfors' extension satisfy independence. For example on $\mathcal{T}_{oy}$ both extensions set $\{2,4\} \prec \{4\}$ but for both $\{1,2,4\}$ and $\{1,4\}$ are incomparable. This violates independence. However, it is possible to define a maxmin-based preorder that satisfies dominance and independence together.

Example 3.9. We claim that the following maxmin-based preorder satisfies dominance and independence on every family of sets. We define $\preceq_{pmm}$ by $A \preceq_{pmm} B$ for $A, B \in \mathcal{X}$ if

$$\max(A) \leq \max(B) \text{ and } \min(A) \leq \min(B).$$

This relation is obviously reflexive. Furthermore, because $\leq$ is transitive, $\preceq_{pmm}$ is also transitive. Therefore, $\preceq_{pmm}$ is a preorder. We claim that $\prec_{pmm}$ additionally satisfies dominance and independence.
Dominance: Assume $A, A \cup \{x\} \in \mathcal{X}$ and $\max(A) < x$. Then, $\max(A) < \max(A \cup \{x\}) = x$ and $\min(A) = \min(A \cup \{x\})$. Therefore, we have $A \prec_{pmm} A \cup \{x\}$. The case $x < \min(A)$ is similar.
Independence: Assume $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ and $A \prec_{pmm} B$. Then, by definition $\max(A) \leq \max(B)$ and $\min(A) \leq \min(B)$. Furthermore, at least one of the comparisons must be strict. Otherwise, we would also have $B \preceq_{pmm} A$ which contradicts the assumption that $A \prec_{pmm} B$. Let us assume that $\min(A) < \min(B)$. The other case is symmetric. We distinguish two cases: First assume $\min(A) < x$. Then,

$$\min(A \cup \{x\}) = \min(A) < \min(x, \min(B)) = \min(B \cup \{x\}).$$

Furthermore, if $x < \max(B)$ then

$$\max(A \cup \{x\}) = \max(x, \max(A)) \leq \max(B) = \max(B \cup \{x\}).$$

On the other hand, if $\max(B) < x$ then $\max(A \cup \{x\}) = \max(B \cup \{x\}) = x$. Therefore, $A \cup \{x\} \preceq_{pmm} B \cup \{x\}$ in both cases.

Now assume $x < \min(A)$. Then, we have $\min(A \cup \{x\}) = \min(B \cup \{x\}) = x$ and

$$\max(A \cup \{x\}) = \max(A) \leq \max(B) = \max(B \cup \{x\}).$$

Therefore, $A \cup \{x\} \preceq_{pmm} B \cup \{x\}$.

It turns out that for $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ this is the minimal preorder that satisfies dominance and independence (see Observation 3.15). We observe that $\preceq_{pmm}$ is not total as, for example, $\{1,3\}$ and $\{2\}$ are incomparable.

This raises the question if it is also possible to define a weak order that satisfies both dominance and independence. In 1984 Yakar Kannai and Bezalel Peleg proved in a seminal paper that this is, in general, not possible (Kannai & Peleg, 1984). To be more precise, they showed that there is no weak order that satisfies both axioms for $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ if $|X| \geq 6$. We will deal with this result in detail in Section 3.4. Before, we introduce a strengthening of independence called strict independence.

Strict Independence
For all $A, B \in \mathcal{X}$ and for all $x \in X \setminus (A \cup B)$, such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \prec B \text{ implies } A \cup \{x\} \prec B \cup \{x\}.$$

The effect of strict independence on $\mathcal{T}_{oy}$ is very similar to the effect of independence. A relation $\preceq$ on $\mathcal{T}_{oy}$ that satisfies strict independence must set $\{1,2,4\} \prec \{1,4\}$ if it contains $\{2,4\} \prec \{4\}$ and $\{1,4\} \prec \{1,2,4\}$ if it contains $\{4\} \prec \{2,4\}$. Dominance implies $\{2,4\} \prec \{4\}$, hence dominance and strict independence together imply $\{1,2,4\} \prec \{1,4\}$.

Strict independence is a strengthening of independence that requires that adding the same element to two sets does not change a strict preference. Clearly, any relation that satisfies strict independence also satisfies independence. Furthermore, any antisymmetric relation that satisfies independence automatically satisfies strict independence. Similarly to independence, this is a desirable property, for example, whenever sets should be ranked according to some additive utility. Indeed, it is easy to check that the ordering based on the total utility of a set defined in Example 3.5 satisfies strict independence. However, it is a significantly stronger axiom and is not satisfied by the preorder $\preceq_{pmm}$ defined in Example 3.9.

Example 3.10. Let $\preceq_{pmm}$ be the preorder defined in Example 3.9. Furthermore, let $X = \{1,2,3\}$, let $\leq$ be the natural order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then, $\preceq_{pmm}$ does not satisfy strict independence with respect to $\leq$. For example $\{1\} \prec_{pmm} \{1,2\}$ but $\{1,3\} \not\prec_{pmm} \{1,2,3\}$.

Indeed, Salvador Barberà and Prasanta Pattanaik have shown that no preorder can satisfy dominance and strict independence if $|X| \geq 3$ (Barberà & Pattanaik, 1984). We will discuss this result in more detail in Section 3.4. There are, however, important examples of lifted orders that always satisfy strict independence, like the very well-known lexicographic order. This order is a generalization of the way that words are ordered in a lexicon based on the alphabetical order of the letters (Fishburn, 1974).

Definition 3.11. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Then, the *leximax order* $\preceq_{leximax}$ is defined for $A, B \in \mathcal{X}$ by

- $A \sim_{leximax} B$ if $A = B$,
- $A \prec_{leximax} B$ if $A \neq B$ and $\max(A \triangle B) \in B$

where $\triangle$ is the symmetric difference operator defined as $X \triangle Y := (X \cup Y) \setminus (X \cap Y)$. If we replace max with min, we can define the *leximin order* $\preceq_{leximin}$ as follows for $A, B \in \mathcal{X}$

- $A \sim_{leximin} B$ if $A = B$,
- $A \prec_{leximin} B$ if $A \neq B$ and $\min(A \triangle B) \in A$

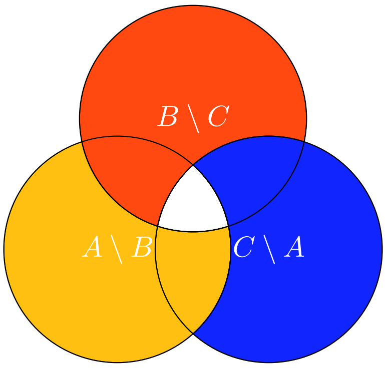

Figure 3.1: Venn diagram for $(A \setminus B) \cup (B \setminus C) \cup (C \setminus A)$.

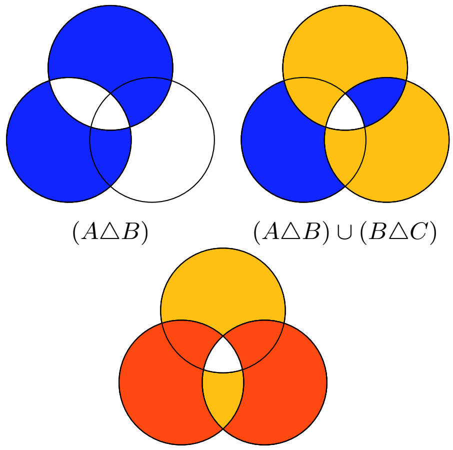

Figure 3.2: Venn diagrams for $(A \triangle B) \cup (B \triangle C) \cup (A \triangle C)$.

On $\mathcal{T}_{oy}$ the leximax order looks as follows

$$\{2\} \prec_{leximax} \{4\} \prec_{leximax} \{1,4\} \prec_{leximax} \{2,4\} \prec_{leximax} \{1,2,4\} \prec_{leximax} \{3,4\}.$$

In this example, we can see that, in general, the leximax order does not satisfy dominance. For example, $\{4\} \prec_{leximax} \{1,4\}$ contradicts dominance. The leximin order on $\mathcal{T}_{oy}$, on the other hand, looks as follows

$$\{1,2,4\} \prec_{leximin} \{1,4\} \prec_{leximin} \{2,4\} \prec_{leximin} \{2\} \prec_{leximin} \{3,4\} \prec_{leximin} \{4\}.$$

It also, in general, does not satisfy dominance as, for example, $\{2,4\} \prec_{leximin} \{2\}$ contradicts dominance.

Proposition 3.12. *The leximax order is a linear order and satisfies strict independence for every set X, linear order $\leq$ and family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$.*

Proof. By definition $(A \cup \{x\}) \triangle (B \cup \{x\}) = A \triangle B$. Therefore, the leximax order satisfies strict independence. Furthermore, it is reflexive by definition. It is total because $A \triangle B = \emptyset$ can only hold if $A = B$. Moreover, it is antisymmetric as $\max(A \triangle B)$ is, by the definition of $\triangle$, either in A or in B but not in both. It remains to show that the leximax order is transitive. Let us assume that $A \preceq B$ and $B \preceq C$. If $A = B$ or $B = C$, then we obtain $A \preceq C$ by substituting A for B in $B \preceq C$ or C for B in $A \preceq B$. Thus, from now on we assume that that $A \neq B$ and $B \neq C$. Let

$$d = \max((A \setminus B) \cup (B \setminus C) \cup (C \setminus A)).$$

We note that the following holds (see Figure 3.1 and 3.2):

$$(A \setminus B) \cup (B \setminus C) \cup (C \setminus A) = (A \triangle B) \cup (B \triangle C) \cup (A \triangle C).$$

We claim $d \notin A \setminus B$. Indeed, let us assume that $d \in A \setminus B$. This would imply $d \in A$ as well as $d \in A \triangle B$. We would then have $\max(A \triangle B) = d \in A$, and, consequently, $B \prec_{leximax} A$. A contradiction. Similarly, $d \notin B \setminus C$. It follows that $d \in C \setminus A$. Thus, $A \neq C$, $d \in C$, $d \in A \triangle C$ and $d = \max(A \triangle C)$. Consequently, $\max(A \triangle C) \in C$ and $A \prec_{leximax} C$. □

By the same argument, the leximin order is a linear order that satisfies strict independence. One main axiom remains, the so-called extension rule.

The Extension Rule
For all $x, y \in X$, such that $\{x\}, \{y\} \in \mathcal{X}$:

$$x < y \text{ implies } \{x\} \prec \{y\}.$$

In $\mathcal{T}_{oy}$ the extension rule implies only $\{2\} \prec \{4\}$. In some sense, the extension rule (or just extension for short) is the most basic axiom considered in this thesis. It states that the singleton sets in $\mathcal{X}$ need to be ordered the same way as the elements of X. In most scenarios, this is a necessary requirement for the lifted order to be acceptable. However, there are exceptions. For example under one interpretation called "freedom of choice"[5] it could be argued that all singletons should be rated equally (see e.g. (Pattanaik & Xu, 1990)). If one has to rank the whole powerset i.e., if we assume $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$, then the extension rule is implied by dominance for every transitive relation as $\{x\} \prec \{x, y\} \prec \{y\}$ is implied by dominance for all $x, y \in X$ such that $x < y$. However, if we drop the assumption that $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ then there are families of sets on which we can define an order that satisfies dominance and strict independence without satisfying the extension rule. A trivial example is the following: Let $X = \{1, 2\}$, $1 \leq 2$ and $\mathcal{X} = \{\{1\}, \{2\}\}$. Then, the linear order $\{2\} \prec \{1\}$ vacuously satisfies dominance and (strict) independence as neither of the axioms are applicable. However, the extension rule is clearly violated. Of course, there is also an order on $\mathcal{X}$ that satisfies dominance, strict independence and the extension rule, namely $\{1\} \prec \{2\}$. Later, we will see that there are also families of sets where dominance and independence can be jointly satisfied with respect to a linear order $\leq$ but dominance, independence and the extension rule are incompatible with respect to $\leq$ (see Proposition 5.41 and the comment following it).

We observe that the four main axioms, if they are compatible, do not necessarily characterize a unique order. For example both of the following linear orders on $\mathcal{T}_{oy}$ satisfy all our main axioms.

$$\{2\} \prec \{1, 2, 4\} \prec \{1, 4\} \prec \{2, 4\} \prec \{3, 4\} \prec \{4\},$$

$$\{1, 2, 4\} \prec \{2\} \prec \{2, 4\} \prec \{3, 4\} \prec \{1, 4\} \prec \{4\}.$$

We finish this section with an important observation about our main axioms that follows directly from their definition.

[5] For an explanation of this interpretation see either Pattanaik & Xu (1990) or Barberà et al. (2004).

Observation 3.13. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Furthermore, let $\preceq$ be a relation on $\mathcal{X}$, let $\mathcal{Y} \subseteq \mathcal{X}$ be a subset of $\mathcal{X}$ and let $\preceq_{\mathcal{Y}}$ be the restriction of $\preceq$ to $\mathcal{Y}$. Then, if $\preceq$ satisfies any of our main axioms with respect to $\leq$ then $\preceq_{\mathcal{Y}}$ must satisfy the same axioms with respect to $\leq$.

Proof. All four four axioms are universal statements about the ordered set $(\mathcal{X}, \preceq)$. Hence, if they are true for a model $(\mathcal{X}, \preceq)$ they are also true for all its submodels $(\mathcal{Y}, \preceq_{\mathcal{Y}})$, i.e., for all tuples $(\mathcal{Y}, \preceq_{\mathcal{Y}})$ such that $\mathcal{Y} \subseteq \mathcal{X}$ and $x \preceq_{\mathcal{Y}} y$ if and only if $x \prec y$ for all $x, y \in \mathcal{Y}$. □

3.4 Impossibility results

In this section we will discuss two classical impossibility results regarding our main axioms. Furthermore, we will discuss why the given formulations of these impossibility results are optimal. Observe that this section does not include the more fine-grained impossibility and characterization results of Maly et al. (2019). These will be covered in Chapter 5.

Kannai and Peleg's Impossibility Result

In this section we present a proof of Kannai and Peleg's result. This proof is essentially the one given in the original paper. In order to prove Kannai and Peleg's result we first need to prove the following lemma.

Lemma 3.14 (Kannai & Peleg, 1984)**.** *Let X be a set, $\leq$ a linear order on X and $\preceq$ a preorder on $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ satisfying dominance and independence with respect to $\leq$. Then, all sets that have the same maximum and minimum elements are equivalent under $\preceq$, i.e., for every $A, B \in \mathcal{X}$ it holds that if $\min(A) = \min(B)$ and $\max(A) = \max(B)$ then $A \sim B$.*

Proof. By transitivity, it suffices to prove $A \sim \{\min(A), \max(A)\}$ for all $A \in \mathcal{X}$. We can assume $|A| \geq 3$ as $A \sim \{\min(A), \max(A)\}$ holds for $|A| < 3$ by reflexivity. Let $a_1, \dots, a_k$ be an enumeration of the elements of A such that $a_i < a_j$ if $i < j$. Then, we have by dominance

$$\{\min(A)\} = \{a_1\} \prec \{a_1, a_2\} \prec \cdots \prec \{a_1, \dots, a_{k-1}\}.$$

Therefore, by transitivity, we have $\{a_1\} \prec \{a_1, \dots, a_{k-1}\}$. If we add a_k to both sets we get by independence $\{\min(A), \max(A)\} = \{a_1, a_k\} \preceq \{a_1, \dots, a_k\} = A$. Similarly, we have by dominance

$$\{a_2, \dots, a_k\} \prec \{a_3, \dots, a_k\} \prec \cdots \prec \{a_k\} = \{\max(A)\}.$$

This implies, again by transitivity and independence, $A = \{a_1, \dots, a_k\} \preceq \{a_1, a_k\} = \{\min(A), \max(A)\}$. Now, we have shown $A \preceq \{\min(A), \max(A)\} \preceq A$, or in other words $A \sim \{\min(A), \max(A)\}$. □

We could weaken the assumptions that $\preceq$ needs to be a preorder to $\preceq$ needs to be reflexive and quasi-transitive[6] (Barberà et al., 2004). However, the given version of the lemma suffices to prove Kannai and Peleg's result. We observe that this lemma proves that the preorder defined in Example 3.9 is the minimal preorder that satisfies dominance and independence if $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$.

Observation 3.15. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Furthermore, let $\preceq_{pmm}$ be the preorder defined in Example 3.9. Then, every preorder that satisfies dominance and independence must be an extension of $\preceq_{pmm}$.

Proof. Assume that $\preceq$ is a preorder that satisfies dominance and independence. Now let A and B be sets such that $A \preceq_{pmm} B$ holds. We claim that $A \preceq B$ also holds. First assume $A \sim_{pmm} B$. Then, $\max(A) = \max(B)$ and $\min(A) = \min(B)$ by the definition of $\preceq_{pmm}$. By Lemma 3.14 this implies $A \sim B$. So assume now that $A \prec_{pmm} B$ holds. Then, $\max(A) \leq \max(B)$ and $\min(A) \leq \min(B)$ and at least one of these preferences must be strict. We assume $\min(A) < \min(B)$. The other case is similar. By dominance and transitivity, we know $B \cup \{\min(A)\} \prec B$ and

$$((B \cup \{\min(A)\}) \setminus \{b \in B \mid b > \max(A)\}) \prec B \cup \{\min(A)\}.$$

Furthermore, we know by Lemma 3.14

$$A \sim \{\min(A), \max(A)\} \sim ((B \cup \{\min(A)\}) \setminus \{b \in B \mid b > \max(A)\}).$$

Hence, $A \prec B$ holds by transitivity. ☐

We can now prove Kannai and Peleg's theorem.

Theorem (Kannai & Peleg, 1984)
Let X be a set such that $|X| \geq 6$. Furthermore, let $\leq$ be a linear order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then, there is no weak order on $\mathcal{X}$ that satisfies dominance and independence with respect to $\leq$.

Proof. Assume for the sake of contradiction that there is a weak order $\preceq$ on $\mathcal{X}$ that satisfies dominance and independence. Furthermore, we assume w.l.o.g. that there are six elements in X called $1, 2, \ldots, 6$ such that $1 < 2 < \cdots < 6$. Then, as $\preceq$ is total it must define a preference between $\{2,5\}$ and $\{3\}$. Therefore, either $\{3\} \prec \{2,5\}$ or $\{2,5\} \preceq \{3\}$ must hold. We claim that both options lead to a contradiction.

We assume first that $\{3\} \prec \{2,5\}$ holds. Then, $\{3,6\} \preceq \{2,5,6\}$ must hold by independence. By Lemma 3.14, we then have

$$\{3,4,5,6\} \sim \{3,6\} \preceq \{2,5,6\} \sim \{2,3,4,5,6\}.$$

[6] A relation is called quasi-transitive if it corresponding strict order is transitive.

By transitivity, this implies $\{3,4,5,6\} \preceq \{2,3,4,5,6\}$. However, dominance implies $\{2,3,4,5,6\} \prec \{3,4,5,6\}$, a contradiction.

As $\{3\} \prec \{2,5\}$ leads to a contradiction, we now assume $\{2,5\} \preceq \{3\}$. Then, we have by dominance

$$\{2,5\} \preceq \{3\} \prec \{3,4\} \prec \{4\}.$$

Therefore, we have by transitivity $\{2,5\} \prec \{4\}$. By independence, this implies $\{1,2,5\} \preceq \{1,4\}$. Now by Lemma 3.14 we get, similarly to the first case

$$\{1,2,3,4,5\} \sim \{1,2,5\} \preceq \{1,4\} \sim \{1,2,3,4\}$$

and therefore by transitivity $\{1,2,3,4,5\} \preceq \{1,2,3,4\}$ which again contradicts dominance. $\square$

This result is, in several ways, tight. First of all, we cannot drop the requirement that $\preceq$ is total, as we have seen in Example 3.9 that it is always possible to define a preorder that satisfies dominance and independence. Furthermore, the assumption $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ is also, in some sense, tight if $|X| = 6$. We only have to remove one set from $\mathcal{X}$ to make dominance and independence compatible, though it has to be the right one. For example, the proof of Kannai and Peleg's theorem never mentions any set containing $\max(X)$ and $\min(X)$ at the same time. Therefore, removing such a set will not make dominance and independence compatible. However, removing the set containing the smallest and the second smallest element of X suffices to make dominance, independence and additionally the extension rule jointly satisfiable. We formulate the result w.l.o.g. for $X = \{1,2,3,4,5,6\}$.

Proposition 3.16 (Maly, Truszczynski & Woltran, 2019)**.** *Let X be $\{1,2,3,4,5,6\}$. Furthermore, let $\leq$ be the natural linear order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset, \{1,2\}\}$. Then, there is a weak order on $\mathcal{X}$ that satisfies dominance, independence and the extension rule with respect to $\leq$.*

Proof. We construct a weak order on $\mathcal{X}$ in two steps. First, we partition $\mathcal{X}$ in $P_1 = \{S \in \mathcal{X} \mid 1,2 \in S\}$, $P_2 = \{S \in \mathcal{X} \mid 1 \in S,\ 2 \notin S\}$, $P_3 = \{S \in \mathcal{X} \mid 2 \in S,\ 1 \notin S\}$ and $P_4 = \{S \in \mathcal{X} \mid 1,2 \notin S\}$.

For every $i \in \{1,2,3\}$, we define a weak order $\preceq_i$ on P_i by setting $S \preceq_i T$ if $\max(S) \leq \max(T)$. For example, for $i = 2$, we get the following weak order:

$$\begin{aligned}\{1\} \prec_2 \{1,3\} \prec_2 \{1,3,4\} \sim_2 \{1,4\} \prec_2 &\\ \{1,3,5\} \sim_2 \{1,3,4,5\} \sim_2 \{1,4,5\} \sim_2 \{1,5\} \prec_2 &\\ \{1,3,6\} \sim_2 \{1,3,4,6\} \sim_2 \{1,3,5,6\} \sim_2 \{1,3,4,5,6\} &\\ \sim_2 \{1,4,5,6\} \sim_2 \{1,4,6\} \sim_2 \{1,5,6\} \sim_2 \{1,6\} &\end{aligned}$$

Observe that for every $S, T \in P_i$, where $i \in \{1,2,3\}$, $S \prec_i T$ if and only if $\max(S) < \max(T)$.

It is easy to see that every $\preceq_i$ for $i \in \{1,2,3\}$ satisfies dominance and independence. To show dominance, let us consider $S \in P_i$ and $x \in \{1,\ldots,6\}$ such that $S \cup \{x\} \in P_i$ and either $x < \min(S)$ or $\max(S) < x$. Since for all $S, T \in P_i$, $\min(S) = \min(T)$, $x < \min(S)$ is impossible. Thus, to verify dominance we only need to consider the case $\max(S) < x$. But then we have $\max(S) < \max(S \cup \{x\})$. Thus, $S \prec_i S \cup \{x\}$.

To show independence, let us consider $x \in \{1, \ldots, 6\}$ and sets $S, T \in P_i$ such that $x \notin S \cup T$, $S \cup \{x\}, T \cup \{x\} \in P_i$ and $S \prec_i T$. By the definition of $\preceq_i$ for $i \in \{1, 2, 3\}$, the latter implies that $\max(S) < \max(T)$. Now if $x < \max(T)$ then

$$\max(S \cup \{x\}) = \max\{\max(S), x\} < \max(T) = \max(T \cup \{x\}).$$

Moreover, if $\max(T) < x$ then

$$\max(S \cup \{x\}) = x = \max(T \cup \{x\}).$$

Hence, in all cases $\max(S \cup \{x\}) \leq \max(T \cup \{x\})$ and hence, by definition, $S \cup \{x\} \preceq_i T \cup \{x\}$.

Next, we define a weak order $\preceq_4$ on P_4 as follows (we present it in terms of the strict preference relation $\prec_4$ and the equivalence relation $\sim_4$):

$$\begin{aligned}\{3\} \prec_4 \{3,4\} \prec_4 \{4\} \sim_4 \{3,5\} \sim_4 \{3,4,5\} \prec_4 \{4,5\} \prec_4 \\ \{5\} \sim_4 \{3,6\} \sim_4 \{3,4,6\} \sim_4 \{3,5,6\} \sim_4 \{3,4,5,6\} \\ \prec_4 \{4,5,6\} \sim_4 \{4,6\} \prec_4 \{5,6\} \prec_4 \{6\}\end{aligned}$$

It can be checked that $\preceq_4$ satisfies dominance and independence. In addition, it is evident that $S \prec_4 T$ implies $\max(S) \leq \max(T)$.

We now define a relation $\preceq$ on $\mathcal{X}$ by $S \preceq T$ if for some $i \in \{1, 2, 3, 4\}$, $S, T \in P_i$ and $S \preceq_i T$, or if $S \in P_i$ and $T \in P_j$ for $i < j$. Since the relations $\preceq_i$ for $1 \leq i \leq 4$ are weak orders, it is clear that $\preceq$ is a weak order. We claim that $\preceq$ satisfies dominance, independence and the extension rule.

Extension rule. We note that $\{1\} \in P_2$, $\{2\} \in P_3$ and $\{3\}, \{4\}, \{5\}, \{6\} \in P_4$. Directly by the definition of $\preceq_4$ we have $\{3\} \prec_4 \{4\} \prec_4 \{5\} \prec_4 \{6\}$. Thus, by the definition of $\preceq$, $\{1\} \prec \{2\} \prec \{3\} \prec \{4\} \prec \{5\} \prec \{6\}$.

Dominance. Let us consider $A \in \mathcal{X}$ and $x \in \{1, \ldots, 6\}$ such that $A \cup \{x\} \in \mathcal{X}$ and (i) $\max(A) < x$ or (ii) $x < \min(A)$. In the first case, $x \neq 1$ ($x = 1$ would imply $A = \emptyset$, a contradiction) and $x \neq 2$ ($x = 2$ would imply $A = \{1\}$; in such case, $A \cup \{x\} = \{1, 2\} \notin \mathcal{X}$; a contradiction). Therefore, $A \cap \{1, 2\} = (A \cup \{x\}) \cap \{1, 2\}$ and, consequently, $A \cup \{x\} \in P_i$ for some i. Since $\preceq_i$ satisfies dominance, $A \prec_i A \cup \{x\}$. Thus, by the definition of $\preceq$, $A \prec A \cup \{x\}$.

Let us then assume (ii). If $x \neq 1, 2$, $A, A \cup \{x\} \in P_4$ and so $A \cup \{x\} \prec_4 A$ and hence $A \cup \{x\} \prec A$. If $x = 2$, we have $A \in P_4$ and $A \cup \{x\} \in P_3$. Hence, $A \cup \{x\} \prec A$. If $x = 1$ there are two cases: $2 \in A$ and $2 \notin A$. In the first case, $A \in P_3$ and $A \cup \{x\} \in P_1$. Hence $A \cup \{x\} \prec A$. In the second case, $A \in P_4$ and $A \cup \{x\} \in P_2$. Hence, $A \cup \{x\} \prec A$.

Independence. Let us consider sets $A, B \in \mathcal{X}$ and $x \in \{1, \ldots, 6\}$ such that $x \notin A \cup B$, $A \prec B$ and $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$. If $x \neq 1, 2$, reasoning as above, we conclude that $A, A \cup \{x\} \in P_i$ and $B, B \cup \{x\} \in P_j$ for some $i, j \in \{1, 2, 3, 4\}$. If $i = j$ then, by the definition of $\preceq$ and by the independence of $\preceq_i$, we have $A \cup \{x\} \preceq_i B \cup \{x\}$ and, consequently, $A \cup \{x\} \preceq B \cup \{x\}$. If $i \neq j$ then, $i < j$. Consequently, $A \cup \{x\} \prec B \cup \{x\}$.

So assume next that $x = 1$. There are four cases to consider. First, let us assume $2 \notin A, B$. Then, $A, B \in P_4$ and, by the construction of $\preceq$, $A \prec_4 B$. Therefore, by the observation above, $\max(A) \leq \max(B)$. Moreover, $A \cup \{x\}, B \cup \{x\} \in P_2$. Thus, $A \cup \{x\} \preceq_2 B \cup \{x\}$ by the definition of $\preceq_2$. By the definition of $\preceq$, $A \cup \{x\} \preceq B \cup \{x\}$.

Next, let us assume $2 \in A, B$. Then, $A, B \in P_3$ and so, by the definition of $\preceq$, $A \prec B$ implies $A \prec_3 B$. Consequently, $\max(A) < \max(B)$, which implies $\max(A \cup \{x\}) \leq \max(B \cup \{x\})$. Since, $A \cup \{x\}, B \cup \{x\} \in P_1$, $A \cup \{x\} \preceq_1 B \cup \{x\}$ and, by construction, $A \cup \{x\} \preceq B \cup \{x\}$.

The third case to consider is when $2 \in A$ and $2 \notin B$. It follows that $A \cup \{x\} \in P_1$ and $B \cup \{x\} \in P_2$. Hence $A \cup \{x\} \prec B \cup \{x\}$.

Finally, $2 \notin A$ and $2 \in B$ is impossible as it contradicts $A \prec B$. Indeed, in this case, we would have $A \in P_4$, $B \in P_1 \cup P_3$ and $B \prec A$.

The case $x = 2$ can be dealt with similarly; we omit the details. □

Finally, the requirement $|X| \geq 6$ is tight as dominance and independence can be jointly satisfied if $|X| \leq 5$. This was first proven by Bandyopadhyay (1988) with a proof that is even longer and more technical than our proof. However, it also follows immediately from the previous result and Observation 3.13, because we can assume w.l.o.g. that our five element set is $X = \{2, 3, 4, 5, 6\}$ and we have

$$(\mathcal{P}(\{2, \ldots, 6\}) \setminus \{\emptyset\}) \subseteq (\mathcal{P}(\{1, \ldots, 6\}) \setminus \{\emptyset, \{1, 2\}\}).$$

Corollary 3.17 (Bandyopadhyay, 1988)**.** *Let X be a set such that $|X| \leq 5$. Furthermore, let $\leq$ be a linear order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then, there is always a weak order on $\mathcal{X}$ that satisfies dominance and independence with respect to $\leq$.*

Barberà and Pattanaik's impossibility result

While dominance and independence are incompatible for total orders, it turns out that dominance and strict independence are incompatible already for partial orders. This was proven by Salvador Barberà and Prasanta Pattanaik shortly after Kannai and Peleg published their result (Barberà & Pattanaik, 1984). Additionally, Barberà and Pattanaik's result requires only a smaller set of elements. Again, we present essentially the original proof. Finally, their result also holds if dominance is replaced by a weaker axiom called simple dominance.

Axiom 3.18 (Simple Dominance)**.** For all $x, y \in X$, such that $\{x\}, \{y\}, \{x, y\} \in \mathcal{X}$ and $x < y$:

$$\{x\} \prec \{x, y\} \prec \{y\}.$$

Clearly, dominance implies simple dominance on all families of sets. Furthermore, Barberà & Pattanaik (1984) have shown that simple dominance and independence can be jointly satisfied. In contrast, simple dominance and strict independence are incompatible.

Theorem (Barberà & Pattanaik, 1984)
Let X be a set such that $|X| \geq 3$. Furthermore, let $\leq$ be a linear order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then, there is no binary relation on $\mathcal{X}$ that satisfies simple dominance and strict independence with respect to $\leq$.

Proof. Assume for the sake of contradiction that there is a binary relation $\preceq$ on $\mathcal{X}$ that satisfies simple dominance and strict independence. We assume w.l.o.g. that there are elements 1, 2 and 3 in X such that $1 < 2 < 3$ holds. Simple dominance implies $\{1\} \prec \{1,2\}$. Then, strict independence implies $\{1,3\} \prec \{1,2,3\}$. On the other hand, simple dominance implies $\{2,3\} \prec \{3\}$. Hence, by strict independence we have $\{1,2,3\} \prec \{1,3\}$. A contradiction! □

It is easy to see that the condition $|X| \geq 3$ is minimal as $\{1\} \prec \{1,2\} \prec \{2\}$ satisfies dominance and strict independence. Furthermore, it is again the case that removing one element from $\mathcal{P}(X) \setminus \{\emptyset\}$ for $|X| = 3$ suffices to make dominance, strict independence and the extension rule compatible.

Example 3.19. Consider $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset, \{3\}\}$. Then, the following linear order satisfies dominance and strict independence:

$$\{1\} \prec \{1,2\} \prec \{1,3\} \prec \{1,2,3\} \prec \{2\} \prec \{2,3\}.$$

It is straight forward to check that dominance is satisfied. For strict independence, we have to consider all pairs $A, B \in \mathcal{X}$ and all $x \notin A \cup B$ such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ holds. First assume $x = 1$. Then, A, B must be $\{2\}$ and $\{2,3\}$. We see that $\{2\} \prec \{2,3\}$ and $\{1,2\} \prec \{1,2,3\}$ hold, hence strict independence is satisfied for this pair. Now assume $x = 2$. Then, A, B must be $\{1\}$ and $\{1,3\}$. Now, as $\{1\} \prec \{1,3\}$ and $\{1,2\} \prec \{1,2,3\}$ hold, strict independence is also satisfied in this case. Finally, consider the case that $x = 3$. Then, A and B must be $\{1\}$, $\{1,2\}$ or $\{2\}$. Now, we have $\{1\} \prec \{1,2\} \prec \{2\}$ and $\{1,3\} \prec \{1,2,3\} \prec \{2,3\}$. Hence, strict independence is satisfied.

3.5 Three types of orderability

Given a set X, a family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and a set of axioms $\mathcal{A}$, we can distinguish three degrees to which the axioms in $\mathcal{A}$ can be compatible on $\mathcal{X}$. First, they can be compatible for at least one linear order on X. Second, they can be compatible with respect to a specific linear order $\leq$ on X. Finally, they can be compatible for every linear order on X. We can view these as a property of a family of sets with respect to a set of axioms. The following definitions, as well as the rest of the section, are based on Maly et al. (2019) and Maly (2020).

Definition 3.20. Let X be a set and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Furthermore, let $\mathcal{A}$ be a set of axioms. Then, we say that $\mathcal{X}$ is ...

- ... *weakly orderable with respect to* $\mathcal{A}$ if **there is a linear order** $\leq^*$ on X such that there is a weak order on $\mathcal{X}$ that satisfies all axioms in $\mathcal{A}$ with respect to $\leq^*$.
- ... $\leq$*-orderable with respect to* $\mathcal{A}$ for a linear order $\leq$ on X, if there is a weak order on $\mathcal{X}$ that satisfies all axioms in $\mathcal{A}$ **with respect to** $\leq$.
- ... *strongly orderable with respect to* $\mathcal{A}$ if **for all linear orders** $\leq^*$ on X there is a weak order on $\mathcal{X}$ that satisfies all axioms in $\mathcal{A}$ with respect to $\leq^*$.

For convenience, we will define a shorthand for orderability with respect to (sub)sets of our main axioms.

Definition 3.21. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Assume $\mathcal{X}$ is $\leq$-orderable with respect to a set of axioms $\mathcal{A}$. Then, we say $\mathcal{X}$ is...

- ... $\leq$-DI-orderable if $\mathcal{A}$ consists of dominance and independence.
- ... $\leq$-DIE-orderable if $\mathcal{A}$ consists of dominance, independence and extension.
- ... $\leq$-DI^S-orderable if $\mathcal{A}$ consists of dominance and strict independence.
- ... $\leq$-DI^SE-orderable if $\mathcal{A}$ consists of dominance, strict independence and extension.

We use the same notation also for strong and weak orderability.

Occasionally, we use the expression (S) in the superscript by the property symbol. We use this notation when we want to make statements that hold no matter whether we omit (S) or replace it with S. For instance, we write

> "Any family that is strongly $DI^{(S)}E$-orderable is also weakly $DI^{(S)}E$-orderable."

if we mean

> "Any family that is strongly DI^SE-orderable is also weakly DI^SE-orderable and any family that is strongly DIE-orderable is also weakly DIE-orderable."

Example 3.22. Let $X = \{1, 2, 3\}$ and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset, \{3\}\}$. Furthermore, let $\leq^*$ be the natural linear order $1 <^* 2 <^* 3$. Then, $\leq^*$ can be lifted to a linear order on $\mathcal{X}$ that satisfies dominance and strict independence, as we have seen in Example 3.19 Therefore, $\mathcal{X}$ is $\leq^*$-DI^S-orderable and also weakly DI^S-orderable. On the other hand, if we consider the linear order $1 <' 3 <' 2$ on X, then no linear order on $\mathcal{X}$ satisfies dominance and strict independence with respect to $\leq'$. This is because Barberà and Pattanaik's impossibility result does not mention $\{2\}$, which has the same position under the natural linear order as $\{3\}$ has under $\leq'$. Therefore, $\mathcal{X}$ is not strongly DI^S-orderable. If we consider independence instead of strict independence, then Corollary 3.17 implies that $\mathcal{X}$ is strongly DIE-orderable because it has less than 6 elements.

The classical works on ranking sets of objects as surveyed by Barberà et al. (2004) do not distinguish different kinds of orderability, because they only consider the case $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. In this case all three aforementioned types of orderability coincide, because for every permutation π of X we have $\pi(\mathcal{P}(X) \setminus \{\emptyset\}) = \mathcal{P}(X) \setminus \{\emptyset\}$.

Proposition 3.23. *Let X be a set and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then, $\mathcal{X}$ is weakly orderable with respect to a set of axioms $\mathcal{A}$ if and only if it is strongly orderable with respect to $\mathcal{A}$.*

Proof. Let $\leq_1$ and $\leq_2$ be two linear orders on X. It suffices to show that $\mathcal{X}$ is $\leq_1$-orderable with respect to $\mathcal{A}$ if it is $\leq_2$-orderable with respect to $\mathcal{A}$. So assume $\mathcal{X}$ is $\leq_2$-orderable with respect to $\mathcal{A}$. Furthermore, let $\pi : X \to X$ be permutation of X such that $\pi(x) \leq_1 \pi(y)$ iff $x \leq_2 y$. Moreover, assume $\preceq_2$ is a weak order on $\mathcal{X}$ that satisfies all axioms in $\mathcal{A}$ with respect to $\leq_2$. Let $\preceq_1$ be the weak order on $\pi(\mathcal{X})$ defined by $\pi(A) \preceq_1 \pi(B)$ iff $A \preceq_2 B$. Then, by definition, $\preceq_1$ satisfies all axioms in $\mathcal{A}$ with respect to $\leq_1$. Now, $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$, implies $\pi(\mathcal{X}) = \mathcal{X}$, hence $\preceq_1$ is a weak order on $\mathcal{X}$ that satisfies all axioms in (A) with respect to $\leq_1$. $\square$

Using this notation, we can say that Kannai and Peleg's theorem states that $\mathcal{P}(X)\setminus\{\emptyset\}$ is not weakly DI-orderable if $|X| \geq 6$. Furthermore, we can say that Proposition 3.16 states that $\mathcal{P}(\{1, \dots, 6\})\setminus\{\emptyset, \{1, 2\}\}$ is weakly DIE-orderable and, in particular, $\leq$-DIE-orderable where $\leq$ is the natural linear order on $\{1, \dots, 6\}$ and Corollary 3.17 states that $\mathcal{P}(\{1, \dots, 5\}) \setminus \{\emptyset\}$ is strongly DIE-orderable.

A natural question that arises is: How difficult is it to decide whether a family of sets is strongly/weakly/$\leq$-orderable with respect to a given subset of our main axioms? This question will be answered in the next chapter. Before, we prove some additional facts about the order lifting problem and our main axioms.

3.6 Variations of the main axioms and additional results

In this section, we collect some additional axioms and results that will be useful when proving our main results. These are collected from several sources: Reverse independence and reverse strict independence as well as Proposition 3.26 appeared first in Maly & Woltran (2017a). Weak independence is a new axiom that does not yet appear in the literature. Strong extension first appeared in this form in Maly et al. (2018) as did Proposition 3.34. Lemma 3.33 and 3.35, finally, can be considered folklore.

3.6.1 Variations of the main axioms

One simple variation of our main axioms is reversing the direction of independence or strict independence.

Axiom 3.24 (Reverse independence)**.** For all $A, B \in \mathcal{X}$ and for all $x \in X \setminus (A \cup B)$ such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \cup \{x\} \prec B \cup \{x\} \text{ implies } A \preceq B.$$

Axiom 3.25 (Reverse strict independence)**.** For all $A, B \in \mathcal{X}$ and for all $x \in X \setminus (A \cup B)$ such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \cup \{x\} \prec B \cup \{x\} \text{ implies } A \prec B.$$

These axioms are very similar to independence and strict independence, intuitively as well as technically. One could argue that they are a slightly less natural formulation of the same monotonicity idea. Independence is equivalent to its reverse counterpart for total orders but both versions differ for for partial orders. For example, if $X = \{1, 2, 3\}$ and $\mathcal{X} = \{\{1\}, \{2\}, \{1, 3\}, \{2, 3\}\}$ then the partial order only containing the preference $\{1\} \prec \{2\}$ does satisfy reverse independence but not independence. On the other hand, the partial order only containing the preference $\{1, 3\} \prec \{2, 3\}$ satisfies independence but not reverse independence. Similarly, strict independence is equivalent to its reverse counterpart for linear orders but both versions differ for non-linear orders.

Proposition 3.26. *Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Then, a total relation on $\mathcal{X}$ satisfies independence if and only if it satisfies reverse independence. A total, antisymmetric relation satisfies strict independence if and only it satisfies reverse strict independence.*

Proof. Let $\preceq$ be a total relation on $\mathcal{X}$. Furthermore, assume that $A, B, A \cup \{x\}$ and $B \cup \{x\}$ are in $\mathcal{X}$. We show that reverse independence implies independence. The other direction is analogous. Assume, that $A \prec B$ and $\preceq$ satisfies reverse independence. Then, by totality, we must have either $A \cup \{x\} \preceq B \cup \{x\}$ or $B \cup \{x\} \prec A \cup \{x\}$. In the second case, reverse independence would imply $B \prec A$ which contradicts our assumption that $A \prec B$ holds. Hence. $A \prec B$ always implies $A \cup \{x\} \preceq B \cup \{x\}$. In other words, $\preceq$ satisfies independence.

Now, let $\preceq$ be a total, antisymmetric relation on $\mathcal{X}$. If $\preceq$ satisfies reverse strict independence, it must, by definition, also satisfy reverse independence. As we have proven above, this implies that $\preceq$ satisfies independence. Now, because $\preceq$ is an antisymmetric relation that satisfies independence it must, by definition, also satisfy strict independence. The other direction is analogous. □

We observe that dominance is "more compatible" with reverse strict independence than with strict independence.

Observation 3.27. Let X be a set with three elements, $\leq$ a linear order on X and $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. Then, there exists a total order on $\mathcal{X}$ satisfying dominance and reverse strict independence with respect to $\leq$.

Proof. We assume w.l.o.g. that $X = \{1, 2, 3\}$ and $\leq$ is the natural linear order on X. Then, we claim that

$$\{1\} \prec \{1, 2\} \prec \{1, 2, 3\} \sim \{1, 3\} \prec \{2\} \prec \{2, 3\} \prec \{3\}.$$

is a total order that satisfies dominance and reverse strict independence. We have seen in Example 3.19 that the restriction of $\preceq$ to $\mathcal{X} \setminus \{3\}$ is a linear order that satisfies dominance and strict independence, hence also reverse strict independence. Therefore, we only need to look at applications of dominance and reverse strict independence that involve $\{3\}$. Clearly, the only applications of dominance involving $\{1,3\}$ are $\{1,3\} \prec \{3\}$ and $\{2,3\} \prec \{3\}$, both of which are satisfied. Now, consider the case that $A \cup \{x\} \prec B \cup \{x\}$ holds for $A, B \in \mathcal{P}(X) \setminus \{\emptyset\}$ and $X \notin A \cup B$. Clearly $A \cup \{x\} = \{3\}$ or $B \cup \{x\} = \{3\}$ is not possible. Therefore, we can assume that and $A = \{3\}$ or $B = \{3\}$. Assume first that $x = 2$. Then $B = \{3\}$ and $A = \{1\}$ or $A = \{1,3\}$. In both cases, reverse strict independence is satisfied. Now assume that $x = 1$. Then $B = \{3\}$ and $A = \{2\}$. For this pair reverse strict independence is also satisfied. □

On the other hand, we have seen that reverse strict independence implies independence for total relations. Hence, there is no weak order that satisfies dominance and reverse strict independence if $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ for any set X with $|X| \geq 6$.

Another interesting axiom is the following weakening of independence that states that adding the same element to two sets cannot revere a strict preference.

Axiom 3.28 (Weak independence)**.** For all $A, B \in \mathcal{X}$ and for all $x \in X \setminus (A \cup B)$ such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$:

$$A \prec B \text{ implies } B \cup \{x\} \not\prec A \cup \{x\}.$$

Clearly, any total relation satisfies weak independence if and only if it satisfies independence. However, in contrast to strict independence and independence it is always possible to find a partial order that satisfies dominance and weak independence.

Observation 3.29. Let X be a set, $\leq$ a linear order on X and $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. Furthermore, let $\preceq$ be a preorder on $\mathcal{X}$ that satisfies dominance and independence. Then, the corresponding strict order is an antisymmetric, irreflexive and transitive binary relation that satisfies dominance and weak independence.

Proof. By definition the corresponding strict order $\prec_s$ of $\preceq$ is an antisymmetric, irreflexive, transitive relation. Furthermore, by the definition of dominance, $\preceq$ satisfies dominance if and only if $\prec_s$ satisfies dominance. Now assume $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ and $A \prec_s B$. Then also $A \prec B$ must hold. Hence, by independence either $A \cup \{x\} \sim B \cup \{x\}$ or $A \cup \{x\} \prec B \cup \{x\}$. In both cases we have $B \cup \{x\} \not\prec_s A \cup \{x\}$. Therefore, $\prec_s$ satisfies weak independence. □

If we add to $\prec_s$ the preference $A \sim A$ for all $A \in \mathcal{X}$ we get a partial order that satisfies dominance and weak independence.

Another natural variation of dominance and strict independence can be obtained by replacing in the definition of these axioms the element x by a set of elements C. We can call these variations set-dominance and strict set-independence.

Axiom 3.30 (Set-Dominance)**.** For all $A, X \in \mathcal{X}$ such that $A \cup X \in \mathcal{X}$ and $A \cap X = \emptyset$:

$$y < x \text{ for all } x \in X, y \in A \text{ implies } A \prec A \cup X;$$

$$x < y \text{ for all } x \in X, y \in A \text{ implies } A \cup X \prec A.$$

Axiom 3.31 (Strict Set-Independence)**.** For all $A, B \in \mathcal{X}$ and for all $C \in \mathcal{P}(X) \setminus \{\emptyset\}$ such that $C \cap (A \cup B) = \emptyset$ and $A \cup C, B \cup C \in \mathcal{X}$:

$$A \prec B \text{ implies } A \cup C \prec B \cup C.$$

These axioms are equivalent to dominance resp. strict independence if $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$, as we can add the elements of C one by one. We observe that a similarly defined set-independence would not equal independence even if $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$ as independence can, in general, not be iterated. For $\mathcal{X} \subsetneq \mathcal{P}(X) \setminus \{\emptyset\}$ it is easy to see that these are a significantly stronger axioms. Consider, for example, $X = \{1, 2, 3, 4, 5, 6\}$ with the natural linear order $\leq$ and the family

$$\mathcal{X} = \{\{1, 2\}, \{5, 6\}, \{1, 2, 3, 4, 5, 6\}, \{1, 2, 3, 4\}, \{3, 4, 5, 6\}, \{1, 2, 5, 6\}\}.$$

Then, any linear order on $\mathcal{X}$ vacuously satisfies dominance and strict independence, as there is no set $A \in \mathcal{X}$ and $x \in X$ such that $A \cup \{x\} \in \mathcal{X}$. However, no partial order on $\mathcal{X}$ can satisfy set-dominance and strict set-independence at the same time, by an argument essentially equivalent to the one used to prove Barberà and Pattanaik's impossibility result. Namely set-dominance implies $\{1, 2\} \prec \{1, 2, 3, 4\}$ and $\{3, 4, 5, 6\} \prec \{5, 6\}$ which can be lifted by strict set-independence to $\{1, 2, 5, 6\} \prec \{1, 2, 3, 4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6\} \prec \{1, 2, 5, 6\}$. A contradiction.

We conclude this section with a strengthening of extension that we call strong extension. Strong extension states that a set A is preferred to a set B if the maximal element of A is larger than the maximal element of B.

Axiom 3.32 (Strong Extension)**.** For all $A, B \in \mathcal{X}$:

$$\max(A) < \max(B) \text{ implies } A \prec B.$$

This axiom can be considered reasonable, for example, in the following setting: Assume every set represents the possible outcomes of a decision and all outcomes have a positive but vastly different utility. Then a risk-tolerant or greedy agent may rank decisions solely on their best possible outcome, whenever these differ.

Note that strong extension implies the extension rule. Furthermore, dominance and independence together imply strong extension if $\mathcal{X} = \mathcal{P}(X) \setminus \{\emptyset\}$. One could also define a dual version of strong extension based on the minima of A and B. Because all problems in this thesis are symmetric, we can use either version without loss of generality. The two versions of strong extension axiom are strict versions of the well known Hoare and Smyth axioms (discussed in particular by Brewka, Truszczynski & Woltran 2010) restricted to linear orders.

In the following, we write $\leq$-DI^SE^S-orderable for $\leq$-orderable with respect to dominance, strict independence and strong extension. Similarly, we write $\leq$-DIE^S-orderable for $\leq$-orderable with respect to dominance, independence and strong extension. We use similar notation for strong orderability and weak orderability. Strong extension is a very restrictive axiom, that is only a natural desiderata in very specific situations. However, we will see in Chapter 5 that there are settings where strong extension can always be satisfied whenever dominance and strict independence can be jointly satisfied. This rather surprising observation is the main reason to include the axiom in this thesis.

3.6.2 Additional results

First we observe that all our main axioms are symmetric in the following sense.

Lemma 3.33. *Let X be a set of objects and $\mathcal{X} \subseteq \mathcal{P}(X)$ a family of sets. Assume that $\mathcal{X}$ is DI^S-orderable with respect to a linear order $\leq$. Then, $\mathcal{X}$ is DI^S-orderable with respect to $\leq^{-1}$.*

Similarly, if we assume that $\mathcal{X}$ is DI-orderable with respect to a linear order $\leq$, then $\mathcal{X}$ is DI-orderable with respect to $\leq^{-1}$.

Proof. Let $\preceq$ be an order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$. Then, we claim that $\preceq^{-1}$ satisfies dominance and strict independence with respect to $\leq^{-1}$. Assume $A, A \cup \{x\} \in \mathcal{X}$, then $\forall y \in A(y <^{-1} x)$ implies $\forall y \in A(y > x)$, which implies $A \cup \{x\} \prec A$ by assumption, hence $A \prec^{-1} A \cup \{x\}$. Similarly, $\forall y \in A(x <^{-1} y)$ implies $A \cup \{x\} \prec^{-1} A$.

Now, assume $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ and $A \prec^{-1} B$. Then, $B \prec A$ and hence by assumption $B \cup \{x\} \prec A \cup \{x\}$ which implies $A \cup \{x\} \prec^{-1} B \cup \{x\}$.

The argument for independence is the same. □

Furthermore, we observe that for every type of orderability considered in this thesis, the union of two disjoint sets is orderable if and only if the two disjoint sets are orderable. This fact will be very important in Chapter 5. It is straightforward to see for $DI^{(S)}$-orderability, as neither dominance nor (strict) independence enforce preferences between elements from different disjoint sets. However, if the extension rule or even strong extension are additionally required, then the result is not as obvious.

Proposition 3.34. *Let $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{Y} \subseteq \mathcal{P}(Y) \setminus \{\emptyset\}$ be families of subsets of X and Y respectively such that $X \cap Y = \emptyset$. If $\mathcal{X}$ and $\mathcal{Y}$ are strongly DI-orderable, then $\mathcal{X} \cup \mathcal{Y}$ is strongly DI-orderable.*

The same holds if we replace strongly DI-orderable by strongly DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable or by weakly DI-, DIE-, DIE^S-, DI^S-, DI^SE- or DI^SE^S-orderable.

Proof. Let us define $Z = X \cup Y$. Now, assume that $\leq_X$ and $\leq_Y$ are linear orders on X and Y such that some orders $\preceq_X$ on $\mathcal{X}$ and $\preceq_Y$ on $\mathcal{Y}$ satisfy all necessary axioms with respect to $\leq_X$ and $\leq_Y$. To prove the claim in all its versions, it suffices to show that for every linear order $\leq$ on Z such that $\leq_X$ and $\leq_Y$ are restrictions of $\leq$ to X and Y, respectively, there is an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$ satisfying all necessary axioms with respect to $\leq$.

***DIE*- and *DI^SE*-orderability.** We first handle the case that $\mathcal{X}$ and $\mathcal{Y}$ are strongly or weakly DIE- or DI^SE-orderable, i.e. the case that $\preceq_X$ and $\preceq_Y$ satisfy the extension axiom. Let $\{x_1, \ldots, x_k\}$ be an enumeration of all elements in X such that $x_1 <_X x_2 <_X \ldots <_X x_k$. By the extension axiom, $\{x_1\} \prec_X \{x_2\} \prec_X \ldots \prec_X \{x_k\}$. Similarly, let $\{y_1, \ldots, y_l\}$ be an enumeration of all elements of Y such that $y_1 <_Y y_2 <_Y \ldots <_Y y_l$ which, also by the extension axiom, implies $\{y_1\} \prec_Y \{y_2\} \prec_Y \ldots \prec_Y \{y_l\}$. Let $z \in Z$. If $z = x_i$, where $1 \leq i \leq k-1$, we define

$$C_z = \{A \in \mathcal{X} \mid \{x_i\} \preceq_X A \prec_X \{x_{i+1}\}\}.$$

If $z = x_k$, we define

$$C_z = \{A \in \mathcal{X} \mid \{x_k\} \preceq_X A\}.$$

We define sets C_z for $z = y_i$, where $1 \leq i \leq l$, analogously, with $\mathcal{Y}$, l, y_i, and $\preceq_Y$ in place of $\mathcal{X}$, k, x_i, and $\preceq_X$, respectively. It is clear that for every $z \in Z$, $\{z\} \in C_z$.

Further, to simplify the notation later on, we assume the existence of a "dummy" element 0 (not in Z) such that $0 < z$, for every $z \in Z$, and we define

$$C_0 = \{A \in \mathcal{X} \mid A \prec_X \{x_1\}\} \cup \{A \in \mathcal{Y} \mid A \prec_Y \{y_1\}\}.$$

Clearly, the sets C_z, $z \in \{0\} \cup Z$ are pairwise disjoint. Moreover, since $Z = X \cup Y$, it follows that $\mathcal{X} \cup \mathcal{Y} = \bigcup_{z \in \{0\} \cup Z} C_z$.

Let $\leq$ be any linear order on Z such that $\leq_X$ and $\leq_Y$ are the restrictions of $\leq$ to X and Y, respectively. To define an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$ for $A, B \in \mathcal{X} \cup \mathcal{Y}$ we set $A \preceq B$ precisely when one of the following conditions holds:

- $A, B \in C_z$, for some $z \in X$, and $A \preceq_X B$
- $A, B \in C_z$, for some $z \in Y$, and $A \preceq_Y B$
- $A, B \in C_0 \cap \mathcal{X}$, and $A \preceq_X B$
- $A, B \in C_0 \cap \mathcal{Y}$, and $A \preceq_Y B$
- $A, B \in C_0$, $A \in \mathcal{X}$, $B \in \mathcal{Y}$
- $A \in C_z$, $B \in C_{z'}$, for $z, z' \in \{0\} \cup Z$, and $z < z'$.

It is straightforward to verify that the relation $\preceq$ is total, reflexive and transitive. Hence, it is an order. It is also clear that if $z, z' \in \{0\} \cup Z$, $z < z'$, $A \in C_z$ and $B \in C_{z'}$, then $A \prec B$ holds. Indeed, in such case, by the definition we have $A \preceq B$. Moreover, $B \preceq A$ is impossible (none of the six cases applies).

We claim that $\preceq$ is an order satisfying the same axioms as $\preceq_X$ and $\preceq_Y$. First, we will prove that $\preceq$ satisfies the extension axiom. Thus, let us consider elements $z, z' \in Z$ such that $z < z'$. By our earlier observation, $\{z\} \in C_z$ and $\{z'\} \in C_{z'}$. Thus, $\{z\} \preceq \{z'\}$ (by the last clause of the definition). Since we do not have $z' < z$ (because $\leq$ is a linear order), $\{z'\} \preceq \{z\}$ does not hold. It follows that $\preceq$ satisfies the extension axiom.

Next, we note that $\preceq_X$ and $\preceq_Y$ are the restrictions of $\preceq$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively. We will prove it for $\preceq_X$; the other case is similar. Therefore, let us assume that $A, B \in \mathcal{X}$. If $A, B \in C_z$, where $z \in \{0\} \cup X$, then by definition, $A \preceq B$ if and only if $A \preceq_X B$. Thus, let $A \in C_z$ and $B \in C_{z'}$, where $z, z' \in \{0\} \cup X$ and $z \neq z'$. If $A \preceq B$ then it must be because of the last clause in the definition of $\preceq$. Consequently, $z < z'$. Since $\leq_X$ is the restriction of $\leq$ to X, $z <_X z'$. Since $A \in C_z$ and $B \in C_{z'}$, $A \prec_X \{z'\}$ and $\{z'\} \preceq_X B$. Thus, $A \preceq_X B$ by transitivity. Conversely, assume that $A \preceq_X B$. If $z' < z$, then $z' <_X z$. Since $A \in C_z$ and $B \in C_{z'}$, $B \prec_X \{z\} \preceq_X A$. By transitivity, $B \prec_X A$, a contradiction. Since $z \neq z'$, we have $z < z'$ and so, $A \preceq B$.

Using this claim, it is easy to show that $\preceq$ satisfies dominance and (strict) independence if $\preceq_X$ and $\preceq_Y$ satisfy the corresponding axiom(s). Indeed, $A, A \cup \{x\} \in \mathcal{X} \cup \mathcal{Y}$ implies $A, A \cup \{x\} \in \mathcal{X}$ or $A, A \cup \{x\} \in \mathcal{Y}$, and $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X} \cup \mathcal{Y}$ implies $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ or $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{Y}$.

***DIE^S*- and *DI^SE^S*-orderability.** Now assume that $\mathcal{X}$ and $\mathcal{Y}$ are strongly or weakly DIE^S- or DI^SE^S-orderable, i.e. that $\prec_X$ and $\prec_Y$ satisfy strong extension. Let $z \in Z$. We define

$$C_z = \begin{cases} \{A \in \mathcal{X} \mid \max_{\leq_X}(A) = z\} & \text{if } z \in X \\ \{A \in \mathcal{Y} \mid \max_{\leq_Y}(A) = z\} & \text{if } z \in Y \end{cases}$$

Let us now assume that the orders $\preceq_X$ and $\preceq_Y$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively, satisfy strong extension with respect to $\leq_X$ and $\leq_Y$, respectively, and let $\leq$ be any linear order on Z such that $\leq_X$ and $\leq_Y$ are the restrictions of $\leq$ to X and Y. To define an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$, for $A, B \in \mathcal{X} \cup \mathcal{Y}$ we set $A \preceq B$ precisely when one of the following conditions holds:

- $A, B \in C_z$, $z \in X$ and $A \preceq_X B$
- $A, B \in C_z$, $z \in Y$ and $A \preceq_Y B$
- $A \in C_z$, $B \in C_{z'}$ and $z < z'$.

It is straightforward to show that $\preceq$ is total, reflexive and transitive. Furthermore, it follows directly from the definition that $\preceq$ satisfies the strong extension property.

Next, we note that $\preceq_X$ and $\preceq_Y$ are the restrictions of $\preceq$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively. We will prove it for $\preceq_X$; the other case is similar. Thus, let us consider sets $A, B \in \mathcal{X}$. By definition, $A \in C_x$ and $B \in C_y$, where $x = \max(A)$ and $y = \max(B)$. If $A \preceq B$ then $x \leq y$ ($y < x$ is impossible by the strong expansion axiom). If $x = y$, then $A \preceq_X B$ (the first condition is the only one that can imply $A \preceq B$ in this case). If $x < y$, then strong extension of $\preceq_X$ implies $A \preceq_X B$. Conversely, if $A \preceq_X B$ then $x \leq y$ ($y < x$ is impossible by strong extension of $\preceq_X$). If $x = y$, then $A \preceq B$ by the first condition. If $x < y$ then $A \preceq B$, by the third condition. Thus, dominance and (strict) independence can be argued as above.

***DI*- and *DI^S*-orderability.** Finally, we will consider the case that $\mathcal{X}$ and $\mathcal{Y}$ are strongly or weakly DI- or DI^S-orderable, i.e. the case that $\preceq_X$ and $\preceq_Y$ satisfy dominance and (strict) independence but no assumptions are made about extension or strong extension. In this case, to define an order $\preceq$ on $\mathcal{X} \cup \mathcal{Y}$, for $A, B \in \mathcal{X} \cup \mathcal{Y}$ we set $A \preceq B$ precisely when one of the following conditions holds:

- $A, B \in \mathcal{X}$ and $A \preceq_X B$

- $A, B \in \mathcal{Y}$ and $A \preceq_Y B$
- $A \in \mathcal{X}$, $B \in \mathcal{Y}$.

It is straightforward to show that $\preceq$ is total, reflexive and transitive. Further, it is clear that the relations $\preceq_X$ and $\preceq_Y$ are the restrictions of $\preceq$ to $\mathcal{X}$ and $\mathcal{Y}$, respectively. Thus, we can derive dominance (independence and strict independence, respectively) of $\preceq$ from dominance (independence or strict independence, respectively) of $\preceq_X$ and $\preceq_Y$ in the same way as before. $\square$

Finally, we will make frequent use of the following well-known fact:

Lemma 3.35 (Folklore). *Let $\leq$ be a preorder on a set X. Then, there exist a* completion *$\leq^{\mathrm{t}}$ of $\leq$ on X, that means a weak order $\leq^{\mathrm{t}}$ such that $x \leq y$ implies $x \leq^{\mathrm{t}} y$ and $x < y$ implies $x <^{\mathrm{t}} y$ for all $x, y \in X$. Furthermore, if $\leq$ is a partial order on a set X, then, there exist a completion of $\leq$ on X that is a linear order.*

Proof. Let $\leq$ be a preorder on a set X. Furthermore, let a and b elements of X that are incomparable under $\leq$. Then, we generate an extended order $\leq^e$ by adding $a \leq^e b$ to $\leq$ and closing under transitivity. Obviously, $x \leq y$ implies $x \leq^e y$. Furthermore, we claim that $x < y$ implies $x <^e y$. Assume otherwise that $y \leq^e x$ holds. By the definition of $\leq^e$, this can only be the case if $y \leq a$ and $b \leq x$ hold. However, by assumption we have $x \leq y$ and hence $b \leq x \leq y \leq a$ which implies by the transitivity of $\leq$ that we must have $b \leq a$. This contradicts the assumption that a and b are incomparable under $\leq$.

To obtain a weak order, we can iterate this construction until $\leq^e$ is total. Furthermore, the completion of a partial order is a linear order, because the added preferences in each step are strict and we retain all strict preferences in the construction. $\square$

CHAPTER 4

Complexity Results

In this chapter we present the complexity results obtained by Maly & Woltran (2017b) and by Maly (2020). We study several problems related to $\leq$-orderability, strong orderability and weak orderability. In the beginning of the chapter, we study $\leq$-orderability and strong orderability. First we consider the problem of lifting a linear order on X to a linear order on $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$. In this setting, independence and strict independence coincide so we study this problem only for strict independence. This gives us four problems to study. Two of these problems are:

DI^S-LO-Orderability
Input: A set X, a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$ and a linear order $\leq$ on X.
Question: Is there a linear order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$?

Strong DI^S-LO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is there for every linear order $\leq$ on X a linear order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$?

The other two problems are obtained by adding the extension rule to the requirements.

DI^SE-LO-Orderability
Input: A set X, a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$ and a linear order $\leq$ on X.
Question: Is there a linear order on $\mathcal{X}$ that satisfies dominance, strict independence and the extension rule with respect to $\leq$?

Strong DI^SE-LO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is there for every linear order $\leq$ on X a linear order on $\mathcal{X}$ that satisfies dominance, strict independence and the extension rule with respect to $\leq$?

We first prove that Strong DI^S-LO-Orderability is NP-hard (Proposition 4.2). The reduction used for this result will immediately also prove the NP-completeness of DI^S-LO-Orderability (Corollary 4.3). Furthermore, we show that adding the extension rule does not change the complexity of the problems, i.e., we show that Strong DI^SE-LO-Orderability is NP-hard and DI^SE-LO-Orderability is NP-complete (Corollary 4.4).

Next, we study the complexity of lifting a linear order on X to a total, but not necessarily linear, order on $\mathcal{X}$ that satisfies dominance and strict independence. This gives us again two problems:

DI^S-WO-Orderability
Input: A set X, a linear order $\leq$ on X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is $\mathcal{X}$ $\leq$-DI^S-orderable?

Strong DI^S-WO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is $\mathcal{X}$ strongly DI^S-orderable?

Modifying the reduction used in Proposition 4.2 we additionally prove that DI^S-WO-Orderability is NP-complete (Theorem 4.6). A further modification of the reduction used for Proposition 4.2, shows that Strong DI^S-WO-Orderability is Π_2^p-complete (Theorem 4.8). As before we can define two more problems, DI^SE-WO-Orderability and Strong DI^SE-WO-Orderability, by adding the extension rule to the requirements. The same complexity results also hold for these problems i.e., DI^SE-WO-Orderability is NP-complete and Strong DI^SE-WO-Orderability is Π_2^p-complete (Corollary 4.7 and 4.9).

To conclude the section on lifting to a total order, we study the complexity of lifting a linear order on X to a weak order on $\mathcal{X}$ that satisfies dominance and independence. This gives us the following problems:

Strong DI-WO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is $\mathcal{X}$ strongly DI-orderable?

DI-WO-Orderability
Input: A set X, a linear order $\leq$ on X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is $\mathcal{X}$ $\leq$-DI-orderable?

Again, we additionally study the problems obtained by adding the extension rule. With another modification of the reduction used before, we can prove that Strong DI-WO-Orderability is Π_2^p-complete and that DI-WO-Orderability is NP-complete (Theorem 4.11). As before, both results hold if we add the extension rule, i.e., Strong DIE-WO-Orderability is Π_2^p-complete and DIE-WO-Orderability is NP-complete (Corollary 4.12).

The next problem that we consider is the problem of lifting to an incomplete order. As we have seen in Chapter 3, it is always possible to find a preorder that satisfies dominance, independence and the extension rule. Therefore, we only consider the problem of lifting a linear order on X to a partial order on $\mathcal{X}$ that satisfies either dominance and strict independence or dominance, strict independence and the extension rule. First we consider $\leq$-orderability:

DI^S-PO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is there a partial order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$?

DI^SE-PO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is there a partial order on $\mathcal{X}$ that satisfies dominance, strict independence and the extension rule with respect to $\leq$?

We give a constructive, polynomial time procedure for constructing a minimal transitive, reflexive binary relation that satisfies dominance and strict independence, resp. dominance, strict independence and the extension rule. As we can check whether the produced order is antisymmetric in polynomial time, this proves that DI^S-PO-Orderability and DI^SE-PO-Orderability are both in P (Corollary 4.24 and 4.26). Subsequently, we consider strong orderability in the context of partial orders.

Strong DI^S-PO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is there for every linear order $\leq$ on X a partial order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$?

Strong DI^SE-PO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is there for every linear order $\leq$ on X a partial order on $\mathcal{X}$ that satisfies dominance, strict independence and the extension rule with respect to $\leq$?

We show by a reduction from Taut that both of these problems are coNP-complete (Theorem 4.27). Afterwards, we discuss succinctly represented families of sets. We introduce the historical and necessary technical background on problems that are represented by boolean circuits. Then, we show for most of the problems that we studied in this chapter that their complexity increases exponentially if the instances are represented by boolean circuits.

Next, we shift our attention to weak orderability. In this context, we only study strict independence. We leave the question whether the following results also hold for independence for future work. The first problem that we consider is weak DI^S-orderability.

Weak DI^S-WO-Orderability
Input: A set X and a family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$.
Question: Is $\mathcal{X}$ weakly DI^S-orderable?

We show with a reduction from Betweenness that this problem is NP-complete (Theorem 4.35). A close inspection of the proof shows that this also holds if we additionally

Orderability	Dom + Ind	[Thm]	Dom + strict Ind	[Thm]
$\leq$-PO-orderability	always	known	in P	4.24
strong PO-orderability	always	known	coNP-c.	4.27
weak PO-orderability	always	known	NP-c.	4.36
$\leq$-WO-orderability	NP-c.	4.11	NP-c.	4.6
strong WO-orderability	Π_2^p-c.	4.11	Π_2^p-c.	4.8
weak WO-orderability	*open*	n/a	NP-c.	4.35
$\leq$-LO-orderability	NP-c.	4.13	NP-c.	4.3
strong LO-orderability	Π_2^p-c.	4.13	Π_2^p-c.	4.10
weak LO-orderability	*open*	n/a	NP-c.	4.36
succ. strong PO-orderability	always	known	coNEXP-c.	4.34
succ. $\leq$-WO-orderability	NEXP-c.	4.33	NEXP-c.	4.33
succ. strong WO-orderability	NEXP-hard	4.33	NEXP-hard	4.33
succ. $\leq$-LO-orderability	NEXP-c.	4.33	NEXP-c.	4.32
succ. strong LO-orderability	NEXP-hard	4.33	NEXP-hard	4.32

Table 4.1: Complexity of orderability with respect to dominance and (strict) independence.

require the extension rule or if we require the lifted order to be either linear or only partial. In other words Weak DI^S-PO-Orderability, Weak DI^S-LO-Orderability,Weak DI^SE-WO-Orderability, Weak DI^SE-PO-Orderability and Weak DI^SE-LO-Orderability are all NP-complete (Corollary 4.36) .

We conclude the chapter by exploring if strengthening dominance is a viable way to reduce the complexity of the studied problems. We restrict our attention to $\leq$-DI^S-orderability and show that this problems stays NP-hard for all "reasonable" strengthenings of dominance (Theorem 4.39). Formally, we say that an axiom A is a *reasonable strengthening of dominance* if

- A implies dominance and
- A is implied by a very strong axiom called maximal dominance (Axiom 4.37).

Table 4.1 summarizes the results proven in this chapter again. The [Thm] column specifies the Theorem, Proposition or Corollary in which the result is proven. "known" indicates that a result was already known in the literature.

4.1 $\leq$-orderability and strong orderability

In this section, we discuss the complexity of several variants of $\leq$-Orderability and Strong Orderability with respect to some subsets of our main axioms. The unifying feature of the problems discussed in this section is the fact that their hardness can be proven by a variation of the same reduction from either Sat or Π_2-Sat. All proofs in this section are based on Maly (2020), but extend that work significantly. In general, Maly (2020) only gave proof sketches, whereas this section provides full, detailed proofs. Furthermore, several of the results given in this section were only implicitly contained in the original paper, namely all results on the extension rule as well as Theorem 4.6 and 4.8.

DI^S-LO-Orderability

The first problem we consider is deciding whether there is a linear order on a family of sets that satisfies dominance and strict independence. We show first that STRONG DI^S-LO-ORDERABILITY is NP-hard, even tough we will improve this result in Corollary 4.10 by showing that STRONG DI^S-LO-ORDERABILITY is Π_2^p-complete. This approach allows us to present the simplest form of a reduction from SAT that will be used – with some modifications – to prove several other hardness results in this chapter.

The idea of the reduction is to encode a 3-CNF ϕ as a families of sets $\mathcal{X}$. First the variables in ϕ are encoded: Every variable V_i in ϕ will be encoded by two sets X_i^{t} and X_i^{f}. Then, we can equate every linear order $\preceq$ on $\mathcal{X}$ with a truth assignment to the variables in ϕ by saying that V_i is set to true if $X_i^{\mathrm{f}} \prec X_i^{\mathrm{t}}$ and V_i is set to false if $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$. Because $\prec$ is a linear order and therefore antisymmetric and total, this defines a complete and consistent truth assignment. Then, we will add sets to $\mathcal{X}$ that lead to a cycle in every linear order $\preceq$ that satisfies dominance and strict independence with respect to $\leq$ if $\preceq$ does not encode a satisfying truth assignment to ϕ.

To achieve this, we will uses the following observation: If $\leq$ is a linear order on a set X and A and B are two subsets of X such that

$$\min(A) < \min(B) < \max(B) < \max(A)$$

then there always exists a collection of sets $\mathcal{Y}$ such that any linear order $\preceq$ on $\mathcal{Y} \cup \{A, B\}$ that satisfies dominance and strict independence with respect to $\leq$ has to set $A \prec B$. At the same time, there also has to exist a collection of sets $\mathcal{Y}^*$ such that any linear order $\preceq$ on $\mathcal{Y}^* \cup \{A, B\}$ that satisfies dominance and strict independence with respect to $\leq$ has to set $B \prec A$. Let us illustrate this by an example.

Example 4.1. Let $X = \{1, \dots, 5\}$ and let $\leq$ be the natural linear order on X. Now consider $A = \{3\}$, $B = \{2, 3, 4\}$. Then clearly

$$\min(B) < \min(A) < \max(A) < \max(B).$$

First, we claim that for the following collection

$$\mathcal{Y} = \{\{1, 2, 3, 4\}, \{1, 3\}, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$$

any linear order $\preceq$ on $\mathcal{Y} \cup \{A, B\}$ that satisfies dominance and strict independence with respect to $\leq$ has to set $A \prec B$. Assume for the sake of contradiction that there is a linear order $\preceq$ on $\mathcal{Y} \cup \{A, B\}$ with $B \prec A$ that satisfies dominance and strict independence with respect to $\leq$. Then, strict independence implies

$$B \cup \{1\} = \{1, 2, 3, 4\} \prec \{1, 3\} = A \cup \{1\}.$$

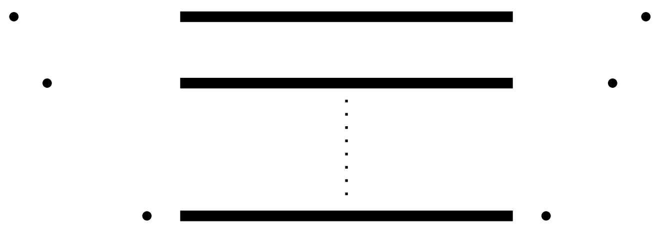

Figure 4.1: Sets encoding variables.

However, by dominance we have $\{1\} \prec \{1, 2\}$ and hence by strict independence and dominance

$$\{1, 3\} \prec \{1, 2, 3\} \prec \{1, 2, 3, 4\},$$

a contradiction. On the other hand, it can be checked that

$$\{1\} \prec \{1, 2\} \prec \{1, 2, 3\} \prec \{1, 3\} \prec \{3\} \prec \{1, 2, 3, 4\} \prec \{2, 3, 4\}$$

is a linear order on $\mathcal{Y} \cup \{A, B\}$ that satisfies dominance and strict independence with respect to $\leq$.

By a similar argument, we can see that

$$\mathcal{Y}^* = \{\{2, 3, 4, 5\}, \{3, 5\}, \{5\}, \{4, 5\}, \{3, 4, 5\}\}.$$

has the property that that any linear order $\preceq$ on $\mathcal{Y}^* \cup \{A, B\}$ that satisfies dominance and strict independence with respect to $\leq$ has to set $B \prec A$.

We will use this observation and define the sets encoding variables in a way such that for all $a, b \in \{\mathrm{t}, \mathrm{f}\}$ and $i, j \leq n$ we have either

$$\min(X_i^a) < \min(X_j^b) < \max(X_j^b) < \max(X_i^a)$$

or

$$\min(X_j^b) < \min(X_i^a) < \max(X_i^a) < \max(X_j^b).$$

This can be ensured for example by a construction where every set has a common middle part and a unique minimal and maximal element. Then, for every new set we increase the minimal and decrease the maximal element at the same time (See Figure 4.1).

Then we can enforce any preference we need between sets encoding different variables by adding the correct collection of set. We use this to enforce for every clause preferences that lead to a contradiction whenever no literal in the clause is satisfied. Consider for example the clause $C = x_1 \vee \neg x_2$. Then, we enforce the preferences

$$X_1^{\mathrm{f}} \prec X_2^{\mathrm{f}} \text{ and } X_2^{\mathrm{t}} \prec X_1^{\mathrm{t}}$$

Now consider a linear order $\preceq$ that contains

$$X_1^{\mathrm{t}} \prec X_1^{\mathrm{f}} \text{ and } X_2^{\mathrm{f}} \prec X_2^{\mathrm{t}}$$

and hence encodes an assignment that sets x_1 to false and x_2 to true. This assignment does not satisfy C and indeed $\preceq$ contains the following cycle

$$X_1^{\mathrm{f}} \prec X_2^{\mathrm{f}} \prec X_2^{\mathrm{t}} \prec X_1^{\mathrm{t}} \prec X_1^{\mathrm{f}}$$

On the other hand,

$$X_1^{\mathrm{f}} \prec X_2^{\mathrm{f}} \prec X_2^{\mathrm{t}} \prec X_1^{\mathrm{t}}$$

is a linear order that is compatible with the enforced preferences and encodes the satisfying assignment that sets x_1 and x_2 to true.

Using this idea, we can add for every clause sets that lead to a cycle if the clause is not satisfied by the assignment coded by a linear order. The main technical difficultly of the proof will be to implement this approach in a way that ensures that no cycle occurs if $\preceq$ encodes a satisfying truth assignment.

Proposition 4.2. Strong DI^S-LO-Orderability *is* NP-*hard.*

Proof. Let ϕ be a instance of Sat with n variables and m clauses. We will produce an instance $(X, \mathcal{X})$ of Strong DI^S-LO-Orderability. Furthermore, we will fix a specific linear order $\leq$ on X such that there is a linear order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$ only if ϕ is satisfiable. Furthermore, we want to make sure that we can use a satisfying assignment of ϕ to construct for any arbitrary linear order $\leq'$ on X a linear order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence.

First, we construct the set of elements X. For every variable V_i, the set X contains elements $x^-_{i,1}, x^-_{i,2}, x^+_{i,1}$ and $x^+_{i,2}$. These will be used to construct the sets for different variables. Furthermore, it contains for every clause C_j variables $z^a_j, y^a_j, \min^a_j$ and $\max^a_j$ for $a \in \{1,2,3\}$. These will be used to ensure that only orders encoding a satisfying assignment to the variables in C_j can satisfy dominance and strict independence. Finally, it contains two elements v_1 and v_2. These will determine the "orientation" of a linear order $\leq'$ on X. In general, the order lifting problem is symmetric (see Lemma 3.33) whereas Sat is not symmetric with respect to truth. To overcome this difficultly, the preference between v_1 and v_2 determines if $X^{\mathrm{f}}_i \prec X^{\mathrm{t}}_i$ means that V_i is true or that V_i is false. This way, $\leq'$ and $\leq'^{-1}$ encode the same truth assignment. Next, we fix a linear order $\leq$ that we define by:

$$\begin{aligned}\min{}^1_1 < \min{}^2_1 < \cdots < \min{}^3_m < x^-_{1,1} < x^-_{1,2} < \cdots < x^-_{n,2}\\ < v_1 < v_2 < z^1_1 < z^2_1 \cdots < z^3_m < y^1_1 < y^2_1 < \cdots < y^3_m <\\ x^+_{1,1} < x^+_{1,2} < \cdots < x^+_{n,2} < \max{}^1_1 < \max{}^2_1 < \cdots < \max{}^3_m\end{aligned}$$

We call this the critical linear order. Now, we construct the family $\mathcal{X}$. Our first goal is to ensure that there does not exists a linear order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$ if ϕ is not satisfiable. In the following, we write $Y := \{x \in X \mid v_1 \leq x \leq y^3_m\}$. First we add the sets representing the variables of ϕ. We add for every variable V_i sets $X_i^{\mathrm{t}} = Y \cup \{x_{i,1}^-, x_{i,1}^+\}$ and $X_i^{\mathrm{f}} = Y \cup \{x_{i,2}^-, x_{i,2}^+\}$. We call these the class 1 sets and write Cl_1 for the collection of all class 1 sets.

Now, let C_i be a clause with variables V_j, V_k and V_l. We want to "enforce" specific preferences between the sets representing V_j, V_k and V_l depending on whether they appear positively or negatively in C_i. However, this could lead to problems if the same variables also occur in another clause. Therefore, we add what could be considered local instantiations of the sets representing the variables V_j, V_k and V_l:

$$X_j^{\mathrm{t}} \setminus \{y_i^1\}, X_j^{\mathrm{f}} \setminus \{y_i^1\}, X_k^{\mathrm{t}} \setminus \{y_i^2\}, X_k^{\mathrm{f}} \setminus \{y_i^2\}, X_l^{\mathrm{t}} \setminus \{y_i^3\} \text{ and } X_l^{\mathrm{f}} \setminus \{y_i^3\}.$$

We call these the class 2 sets and write Cl_2 for the collection of all class 2 sets. Now, let $\preceq$ be a linear order on $\mathcal{X}$ that satisfies strict independence. Then, it also satisfies reverse strict independence by Proposition 3.26. By reverse strict independence we know that for $\preceq$ the preference between $X_j^{\mathrm{t}} \setminus \{y_i^1\}$ and $X_j^{\mathrm{f}} \setminus \{y_i^1\}$ must be the same as the preference between X_j^{t} and X_j^{f}. The same holds for the other two variables. In this sense, these "local instantiations" correctly reflect the truth assignment encoded by any linear order on $\mathcal{X}$, if the linear order satisfies strict independence. On the other hand, any preference between local instantiations of sets representing different variables, say $X_j^{\mathrm{t}} \setminus \{y_i^1\}$ and $X_k^{\mathrm{f}} \setminus \{y_i^2\}$ stays local, because $y_i^1 \neq y_i^2$.

Now, if all variables occur positively in C_i, we add sets such that $X_j^{\mathrm{f}} \setminus \{y_i^1\} \prec X_k^{\mathrm{t}} \setminus \{y_i^2\}$, $X_k^{\mathrm{f}} \setminus \{y_i^2\} \prec X_l^{\mathrm{t}} \setminus \{y_i^3\}$ and $X_l^{\mathrm{f}} \setminus \{y_i^3\} \prec X_j^{\mathrm{t}} \setminus \{y_i^1\}$ must hold for any order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$. We call this enforcing these preferences. Then, we get a contradiction if V_j, V_k and V_l are false because

$$X_j^{\mathrm{t}} \setminus \{y_i^1\} \prec X_j^{\mathrm{f}} \setminus \{y_i^1\} \prec X_k^{\mathrm{t}} \setminus \{y_i^2\} \prec X_k^{\mathrm{f}} \setminus \{y_i^2\} \prec X_l^{\mathrm{t}} \setminus \{y_i^3\} \prec X_l^{\mathrm{f}} \setminus \{y_i^3\} \prec X_j^{\mathrm{t}} \setminus \{y_i^1\}$$

holds and implies $X_j^{\mathrm{t}} \setminus \{y_i^1\} \prec X_j^{\mathrm{t}} \setminus \{y_i^1\}$ by transitivity. If a variable, say V_j, occurs negatively in C_i, we switch X_j^{t} and X_j^{f} and enforce $X_j^{\mathrm{t}} \setminus \{y_i^1\} \prec X_k^{\mathrm{t}} \setminus \{y_i^2\}$ and $X_l^{\mathrm{f}} \setminus \{y_i^3\} \prec X_j^{\mathrm{f}} \setminus \{y_i^1\}$.

Next, we show how we can enforce these preferences. Assume we want to enforce $X_j^a \setminus \{y_i^1\} \prec X_k^b \setminus \{y_i^2\}$ for $a, b \in \{\mathrm{t}, \mathrm{f}\}$. We add

$$\{z_i^1\}, \{z_i^1, \max_i^1\} \text{ and } (X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\}.$$

Our goal is to enforce $(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\} \prec \{z_i^1, \max_i^1\}$ which forces by reverse strict independence $X_j^a \setminus \{y_i^1\} \prec \{z_i^1\}$. Then, we enforce $\{z_i^1\} \prec X_k^b \setminus \{y_i^2\}$ to get by transitivity $X_j^a \setminus \{y_i^1\} \prec X_k^b \setminus \{y_i^2\}$ as desired. To enforce

$$(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\} \prec \{z_i^1, \max_i^1\}$$

we add a sequence of sets $A_1, A_2, \ldots, A_l$ such that

- $A_1 = (X_j^a \setminus \{y_i^1, z_i^1)\} \cup \{\max_i^1\}$,
- $A_{i+1} = A_i \setminus \{\min_{\leq}(A_i)\}$

- and $A_l = \{\max_i^1\}$.

This enforces by dominance $A_1 \prec A_2 \prec \cdots \prec A_l$ which enforces by transitivity

$$A_1 = (X_j^a \setminus \{y_i^1, z_i^1\}) \cup \{\max_i^1\} \prec \{\max_i^1\} = A_l.$$

Finally, this enforces by strict independence the desired $(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\} \prec \{z_i, \max_i^1\}$. Using the same idea and $\min_i^1$ we enforce $\{z_i^1\} \prec X_k^b \setminus \{y_i^2\}$ finishing the construction for $X_j^a \setminus \{y_i^1\} \prec X_k^b \setminus \{y_i^2\}$. We enforce the other preferences for that clause, i.e. $X_k^c \setminus \{y_i^2\} \prec X_l^d \setminus \{y_i^3\}$ and $X_l^e \setminus \{y_i^3\} \prec X_j^a \setminus \{y_i^1\}$ for $c, d, e \in \{\mathrm{t}, \mathrm{f}\}$, similarly using $z_i^2, \max_i^2$ and $\min_i^2$ resp. $z_i^3, \max_i^3$ and $\min_i^3$. We repeat this procedure for every clause. We call the sets added in this step the class 3 sets and write Cl_3 for the collection of all class 3 sets. Furthermore, we write Cl_3^+ for the class 3 sets that contain an element $\max_i^a$ for some i and a. Similarly, we write Cl_3^- for the class 3 sets that contain an element $\min_i^a$ for some i and a. Finally, we write Cl_3^0 for all other sets in class 3 (which are all of the form $\{z_i^a\}$). Now, by construction, $\mathcal{X}$ can only be DI^S-orderable with respect to $\leq$ if ϕ is a positive instance of SAT.

Next, we pick an arbitrary linear order $\leq'$ on X. We distinguish two cases $v_1 <' v_2$ and $v_2 <' v_1$. By Lemma 3.33 it suffices to show that a linear satisfying dominance and strict independence exists in the first case, because $v_2 <' v_1$ implies $v_1 <'^{-1} v_2$. Hence, we can assume in the following w.l.o.g. $v_1 <' v_2$. Now, we want to construct a linear order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq'$ if ϕ is satisfiable. For the readers convenience, we summarize which sets are contained in $\mathcal{X}$ and hence must be taken into account when constructing $\preceq$:

Class 1: $X_i^{\mathrm{t}} = Y \cup \{x_{i,1}^-, x_{i,1}^+\}$ and $X_i^{\mathrm{f}} = Y \cup \{x_{i,2}^-, x_{i,2}^+\}$ for every variable V_i of ϕ.

Class 2: $X_j^{\mathrm{t}} \setminus \{y_i^a\}$ and $X_j^{\mathrm{f}} \setminus \{y_i^a\}$, where $a \in \{1,2,3\}$ and i and j are such that V_j is in clause C_i.

Class 3: For each $i \leq m$ and $a \in \{1,2,3\}$ and for some $b, c \in \{\mathrm{t}, \mathrm{f}\}$, $d \in \{1,2,3\}$ and j, k such that V_j and V_k are variables in clause C_i:

- $\{z_i^a\}$, $\{z_i^a, \max_i^a\}$, $\{z_i^a, \min_i^a\}$, $(X_j^b \setminus \{y_i^a\}) \cup \{\max_i^a\}$, $(X_k^c \setminus \{y_i^d\}) \cup \{\min_i^a\}$,
- $A_1 = (X_j^b \setminus \{y_i^a, z_i^a\}) \cup \{\max_i^a\}, \dots, A_{l+1} = A_l \setminus \{\min_{\leq}(A_l)\}, \dots, A_o = \{\max_i^a\}$,
- $B_1 = (X_k^c \setminus \{y_i^d, z_i^a\}) \cup \{\min_i^a\}, \dots, B_{l+1} = B_l \setminus \{\max_{\leq}(B_l)\}, \dots, B_p = \{\min_i^a\}$.

Now, we construct the order $\preceq$ on $\mathcal{X}$ in several steps. First, we construct from a satisfying assignment of ϕ an order on $Cl_1 \cup Cl_2$:

Ordering the sets in $Cl_1 \cup Cl_2$: We start by ordering the sets X_i^{f} and X_i^{t} according to the satisfying assignment of ϕ, i.e. $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$ if V_i is false in the assignment and $X_i^{\mathrm{f}} \prec X_i^{\mathrm{t}}$ if it is true. Then, we project this order down to the class 2 sets by reverse strict independence. Furthermore, we add the preferences that we enforced in the previous section. Finally, we take the transitive closure of this order. It is clear by construction that this is an acyclic partial order if and only if ϕ is satisfiable. Now, for any clause C_i, we fix any linear order on the sets

$$X_j^{\mathrm{f}} \setminus \{y_i^1\}, X_j^{\mathrm{t}} \setminus \{y_i^1\}, X_k^{\mathrm{f}} \setminus \{y_i^2\}, X_k^{\mathrm{t}} \setminus \{y_i^2\}, X_l^{\mathrm{f}} \setminus \{y_i^3\} \text{ and } X_l^{\mathrm{t}} \setminus \{y_i^3\}.$$

that extends this order.

For the class 1 sets we have ordered all pairs $(X_i^{\mathrm{f}}, X_i^{\mathrm{t}})$ but we still have to fix an order between these pairs. For the class 2 sets, we have fixed an order on between all sets introduced for a single clause, but we have to fix an order between sets from different clauses. Now, we observe that $A, A \cup \{x\} \in Cl_1 \cup Cl_2$ implies that $A \in Cl_1$, $B \in Cl_2$ and $x = y_i^a$ for $j \leq m$ and $a \leq 3$ as all class 1 sets differ from all other class 1 sets in at least two elements and all class 2 sets differ from all other class 2 sets in at least two elements. Hence the only possible application of strict independence on class 1 and 2 is the one already covered by construction. Dominance is applicable only if y_i^a for some i and a is the minimal or maximal element of the set it gets removed from. We fix an order on the pairs and clauses that is compatible with these applications of dominance. First, assume the minimal element $y_{i^-}^{a^-}$ of the form y_i^a and the maximal element $y_{i^+}^{a^+}$ of the form y_i^a are used for the same clause. Let X_j^b and X_k^c be the sets such that $X_j^b \setminus \{y_{i^-}^{a^-}\} \in \mathcal{X}$ and $X_k^c \setminus \{y_{i^+}^{a^+}\} \in \mathcal{X}$ holds. Then, by construction $j \neq k$. In that case we fix any linear order $\leq''$ on the pairs $(X_i^{\mathrm{f}}, X_i^{\mathrm{t}})$ such that $(X_j^{\mathrm{f}}, X_j^{\mathrm{t}}) \leq'' (X_k^{\mathrm{f}}, X_k^{\mathrm{t}})$ holds and an arbitrary order on the clauses. Then, we set $X_i^{\mathrm{f}} \prec A$ and $X_i^{\mathrm{t}} \prec A$ for every $A \in Cl_2$ if $(X_i^{\mathrm{f}}, X_i^{\mathrm{t}}) <'' (X_j^{\mathrm{f}}, X_j^{\mathrm{t}})$. Furthermore, we set $A \prec X_i^{\mathrm{f}}$ and $A \prec X_i^{\mathrm{t}}$ for every $A \in Cl_2$ if $(X_j^{\mathrm{f}}, X_j^{\mathrm{t}}) <'' (X_i^{\mathrm{f}}, X_i^{\mathrm{t}})$. This is obviously a linear order and we have $X_j^b \prec X_j^b \setminus \{y_{i^-}^{a^-}\}$ and $X_k^c \setminus \{y_{i^+}^{a^+}\} \prec X_k^c$ for $b, c \leq 2$. Hence the constructed order on $Cl_1 \cup Cl_2$ satisfies dominance.

Now, assume the minimal element $y_{i^-}^{a^-}$ of the form y_i^a and the maximal element $y_{i^+}^{a^+}$ of the form y_i^a are used for different clauses C_{i^-} and C_{i^+}. We fix any order $\leq''$ on the clauses such that C_{i^+} is smaller than C_{i^-} and an arbitrary order on the pairs. Additionally, we set $A \prec B$ for all $A \in Cl_2$ and $B \in Cl_1$ if A was introduced for a clause that is smaller equal C_{i^+} with respect to $\leq''$. Furthermore, we set $B \prec A$ for all $A \in Cl_2$ and $B \in Cl_1$ if A was introduce for a clause that is larger than C_{i^+} with respect to $\leq''$. This is obviously a linear order and we have $X_j^b \prec X_j^b \setminus \{y_{i^-}^{a^-}\}$ and $X_k^c \setminus \{y_{i^+}^{a^+}\} \prec X_k^c$ for $b, c \leq 2$. Hence the constructed order on $Cl_1 \cup Cl_2$ satisfies dominance.

Ordering sets in Cl_3 with same extremum element and $\{z_i^a\}$: In the following, we write for a set A that contains an extremum-element mm (which is by construction unique) $A_S := \{x \in A \mid x <' mm\}$ for the set of elements in A that are smaller than mm and $A_L := \{x \in A \mid mm <' x\}$ for the set of elements in A that are larger than mm.

We set $A \prec B$ for sets A, B that both contain the same extremum element of the form $\max_i^c$ if:

- $\max_{\leq'}(A_L \triangle B_L) \in B$,

- $A_L = B_L$ and $\min_{\leq'}(A_S \triangle B_S) \in A$.

Here, $\triangle$ is the symmetric difference operator, i.e. $A \triangle B := (A \cup B) \setminus (A \cap B)$. We claim that this order satisfies dominance and strict independence. It satisfies strict independence because for all sets S, T by definition $S \cup \{x\} \triangle T \cup \{x\} = S \triangle T$ for any $x \notin S \cup T$. For dominance, assume $x <' \min_{<'}(A)$ and $\max_i^c \in A, A \cup \{x\}$. Then, $A_L = (A \cup \{x\})_L$ and $\min_{\leq'}(A_S \triangle (A \cup \{x\})_S) = x$. Hence, $A \cup \{x\} \prec A$. The case $\max_{<'}(A) <' x$ is similar.

Next, we add $\{z_i^c\}$ to the order. We observe that we may have either $X_j^a \setminus \{y_i^b\} \cup \{\max_i^c\} \prec \{z_i^c, \max_i^c\}$ or $\{z_i^c, \max_i^c\} \prec X_j^a \setminus \{y_i^b\} \cup \{\max_i^c\}$. In the first case, we add $\{z_i^c\}$ in the order exactly after $X_j^a \setminus \{y_i^b\}$, i.e., we set $\{z_i^c\} \prec A$ if $X_j^a \setminus \{y_i^b\} \prec A$ and $A \prec \{z_i^c\}$ if $A \preceq X_j^a \setminus \{y_i^b\}$ for all $A \in \mathcal{X} \setminus \{z_i^c\}$. In the second case we set $\{z_i^c\}$ exactly before $X_j^a \setminus \{y_i^c\}$.

Now, let $X_k^d \setminus \{y_i^e\}$ be the set for which we enforce the preference $X_j^a \setminus \{y_i^c\} \prec X_k^d \setminus \{y_i^e\}$. Then, this implies $\{z_i^c\} \prec X_k^d \setminus \{y_i^e\}$, because $X_j^a \setminus \{y_i^b\} \prec X_k^d \setminus \{y_i^e\}$ holds by construction and we added $\{z_i^c\}$ just before or just after $X_j^a \setminus \{y_i^b\}$. Therefore, we have to make sure that $\{z_i^c, \min_i^c\} \prec X_k^d \setminus \{y_i^e\} \cup \{\min_i^c\}$ holds as intended by the construction to avoid a contradiction. For this we use the fact that $v_1 <' v_2$ holds. We set $A \prec B$ for elements A, B if they both contain an element of the form $\min_i^c$ if:

- $v_2 \in B$ and $v_2 \notin A$, $(\star)$
- $v_2 \in A, B$ or $v_2 \notin A, B$ and $\max_{\leq'}(A_L \triangle B_L) \in B$,
- $v_2 \in A, B$ or $v_2 \notin A, B$, $A_L = B_L$ and $\min_{\leq'}(A_S \triangle B_S) \in A$.

It is clear that $(\star)$ implies $\{z_i^c, \min_i^c\} \prec X_k^d \setminus \{y_i^e\} \cup \{\min_i^c\}$. It is also clear that it satisfies strict independence because the $(\star)$ implies a preference between sets $A \cup \{x\}$ and $B \cup \{x\}$ for $x \notin A \cup B$ iff it implies the same preference for A and B. If $(\star)$ is not applicable, strict independence is satisfied by the same argument as above. Now, for dominance $v_2 \in (A \triangle (A \cup \{x\}))$ implies $x = v_2$. Then, $x < \min_{<'}(A)$ is not possible because by construction $v_1 \in A$ holds and we assume $v_1 <' v_2$. If we have $\max_{<'}(A) <' x$ then dominance is satisfied because $A \prec A \cup \{x\}$ holds by $(\star)$. If $x \neq v_2$, then $(\star)$ is not applicable and dominance is satisfied by the same argument as above.

Interaction between $Cl_1 \cup Cl_2$ and Cl_3: First, we observe that there is no set $A \in Cl_3$ such that $A \cup \{x\} \in Cl_1 \cup Cl_2$ holds, as every set in C_3 either contains an extremum-element or it is a singleton and no set in class 1 and 2 contains an extremum-element and every set in class 1 and 2 has more than three elements.

Furthermore, if $A \in Cl_1 \cup Cl_2$ and $A \cup \{x\} \in Cl_3$, then A must be of the form $X_i^a \setminus \{y_j^b\}$ and x must be $\max_j^c$ or $\min_j^d$. Additionally, by construction, for every $A \in Cl_1 \cup Cl_2$ there is at most one x such that $A \cup \{x\}$ holds. Therefore, the only possible application of dominance involving sets from $Cl_1 \cup Cl_2$ and Cl_3 is adding an extremum element $\max_j^c$ or $\min_j^d$ to a set A in $Cl_1 \cup Cl_2$ such that the extremum element is smaller (larger) than all elements in A. We have to add preferences that satisfy this application of dominance.

Consider $A \in Cl_1 \cup Cl_2$ and $B \in Cl_3^+ \cup Cl_3^-$ and let mm_B be the unique extremum element in B. Furthermore, let $i \leq$ and $a \in \{1,2,3\}$ be the unique values for which $\{z_i^a, mm_B\}$ is in $\mathcal{X}$. Then, we extend $\preceq$ by

- $A \prec B$ if $z_i^a < mm_B$,
- $B \prec A$ if $mm_B < z_i^a$.

Clearly, the resulting order is still a partial order and it satisfies all possible applications of dominance involving sets from $Cl_1 \cup Cl_2$ and Cl_3, because $z_i^a \in A$ for all $i \leq m$, $a \in \{1,2,3\}$ and $A \in Cl_1 \cup Cl_2$. Furthermore, observe that we do not add any new preferences between sets in $Cl_1 \cup Cl_2$, but we do add, by transitivity, new comparisons between sets in Cl_3 that contain different extremum elements.

Now, for strict independence, the only application with sets from class 3 and sets not from class 3 is lifting a preference between a set A and $\{z_i^a\}$ to a preference between $A \cup \{mm\}$ and $\{z_i^a, mm\}$ for some specific extremum element mm. By construction A must be of the form $X_k^b \setminus \{y_i^a\}$ and hence strict independence is satisfied as we have seen in the paragraph above.

Ordering sets in Cl_3 with different (or no) extremum elements: Finally, we have to extend the order $\preceq$ to the whole class 3. First, observe that two sets with different extremum elements differ in at least two elements. Hence, there is no possible application of dominance with sets containing different extremum elements. The only possible application of strict independence that is not already satisfied by $\preceq$ is adding the same element to two sets containing different extremum elements. In order to make sure that these applications of strict independence are satisfied, we order the elements with different extremum element with an order that is only based on what extremum elements they contain. We fix an arbitrary linear order $\leq_{mm}$ on the extremum elements that is compatible with the preferences introduced in the paragraph above, i.e., if $\{mm_1, z_i^a\}, \{mm_2, z_j^b\} \in \mathcal{X}$, $mm_1 < z_i^a$ and $z_j^b < mm$ hold then $mm_1 \leq_{mm} mm_2$ must hold. We extend $\preceq$ by adding for all sets A and B that contain different extremum elements mm_A and mm_B respectively

- $A \prec B$ if $mm_A <_{mm} mm_B$,
- $B \prec A$ if $mm_B <_{mm} mm_A$.

Clearly, this is compatible with the preference between sets with different extremum elements added in the paragraph above. Furthermore, we know for all A, B that contain an extremum element that for all $x \in X$ such that $A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ holds that $A \cup \{x\}$ contains the same unique extremum element as A and $B \cup \{x\}$ contains the same unique extremum element as B. Hence, if A and B are class 3 elements that contain different extremum elements, then $A \prec B$ implies $A \cup \{x\} \prec B \cup \{x\}$.

Furthermore, we observe that any class 3 set that does not contain an extremum element must be of the form $\{z_i^a\}$ for some $z_i^a \in X$. Now, if $\{z_i^a\} \cup \{x\}$ is in $\mathcal{X}$ then x

must be either $\min_i^a$ or $\max_i^a$. Furthermore, there is no other class 3 set A such that $A \cup \{x\}$ is in $\mathcal{X}$. Therefore, no application of strict independence involving $\{z_i^a\}$ is possible. Finally, as $\{z_i^a\}$ is placed next to a class 2 set $X_j^b \setminus \{y_i^c\}$, $\{mm, z_i^a\} \in \mathcal{X}$ and $mm < z_i^a$ implies $\{mm, z_i^a\} \prec \{z_i^a\}$ by the construction in the paragraph above. Similarly, $\{mm, z_i^a\} \in \mathcal{X}$ and $z_i^a < mm$ implies $\{z_i^a\} \prec \{z_i^a, mm\}$. Therefore, any application of dominance involving $\{z_i^a\}$ is satisfied.

Completion of $\preceq$: Finally, because $\preceq$ is a partial order there is a linear order that is a completion of $\preceq$ (Lemma 3.35). As we have seen above, $\preceq$ satisfies all possible applications of strict independence and dominance. Thus, any completion of $\preceq$ satisfies dominance and strict independence. □

We have shown that there is a linear order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to a fixed order $\leq$ only if ϕ is satisfiable. Furthermore, we have shown how to use a satisfying assignment of ϕ to construct for any linear order $\leq'$ on X a linear order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence. This construction clearly also works for the fixed order $\leq$. In other words ϕ is satisfiable if and only if $(X, \mathcal{X}, \leq)$ is a positive instance of DI^SE-LO-Orderability. Hence, the reduction above also works for DI^SE-LO-Orderability.

Corollary 4.3. *DI^S-LO-Orderability is NP-complete.*

Proof. It follows from the reduction above that DI^S-LO-Orderability is NP-hard. Furthermore, it is clear that DI^S-LO-Orderability is in NP: We can guess a linear order $\preceq$ on $\mathcal{X}$ and then check in polynomial time if $\preceq$ satisfies dominance and strict independence with respect to $\leq$. □

A variation of this reduction shows that we can additionally add extension to the required axioms without changing the complexity of the problem. Essentially, we modify the family $\mathcal{X}$ such that it does not contain any singletons. This can be achieved by replacing singletons with two element sets.

Corollary 4.4. *Strong DI^SE-LO-Orderability is NP-hard. DI^SE-LO-Orderability is NP-complete.*

Proof. First, observe that DI^SE-LO-Orderability is in NP for the same reason that DI^S-LO-Orderability is, because it can also be checked in polynomial time if a linear order satisfies the extension rule.

Now, in order to show that Strong DI^SE-LO-Orderability and DI^SE-LO-Orderability are NP-hard we have to modify the reduction above slightly. In general, the order on the singletons does not satisfy the extension rule. One way to solve this problem is replacing all singletons by two element sets. All singletons that appear in the reduction are of the form $\{z_k^a\}$, $\{\min_i^b\}$ or $\{max_j^c\}$ for some $z_k^a, \min_i^b, \max_j^c \in X$.

For all $z_i^a \in X$, replace in the reduction above every mentioning of z_i^a by $z_i^{a,1}, z_i^{a,2}$ and similarly replace for all $\min_i^b, \max_j^c \in X$ every mentioning of $\min_i^b$ and $\max_j^c$ by $\overline{\min_i^b}, \min_i^b$ and $\overline{\max_i^c}, \max_i^c$, respectively. In the critical linear order, we replace z_i^a by $z_i^{a,1} < z_i^{a,2}$, $\min_i^b$ by $\overline{\min_i^{b,*}} < \min_i^b$ and $\max_j^c$ by $\max_i^c < \overline{\max_i^c}$. Finally, we add the additional set $(X_j^a \setminus \{y_i^1, z_i^{a,1}\}) \cup \{\max_i^1, \overline{\max_i^1}\}$, $\{z_i^{a,2}, \max_i^1, \overline{\max_i^1}\}$, $(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\}$ and $\{z_i^{a,1}, z_i^{a,2}, \max_i^1\}$.

Then

$$(X_j^a \setminus \{y_i^1, z_i^{a,1}, z_i^{a,2}\}) \cup \{\max_i^1, \overline{\max_i^1}\} \prec \{\max_i^1, \overline{\max_i^1}\}$$

implies by two applications of strict independence

$$(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1, \overline{\max_i^1}\} \prec \{z_i^{a,1}, z_i^{a,2}, \max_i^1, \overline{\max_i^1}\}.$$

This, in turn, implies by two applications of reverse strict independence

$$(X_j^a \setminus \{y_i^1\}) \prec \{z_i^{a,1}, z_i^{a,2}\}.$$

When constructing a linear order $\preceq$ based on a satisfying assignment, we can treat the four new sets $(X_j^a \setminus \{y_i^1, z_i^{a,1}\}) \cup \{\max_i^1, \overline{\max_i^1}\}$, $\{z_i^{a,2}, \max_i^1, \overline{\max_i^1}\}$, $(X_j^a \setminus \{y_i^1\}) \cup \{\max_i^1\}$ and $\{z_i^{a,1}, z_i^{a,2}, \max_i^1\}$ the same as any other set containing $\max_i^1$. It can be checked that the rest of the construction works as before. Now, $\mathcal{X}$ does not contain any singletons. Therefore, the linear order constructed in the second part of the reduction satisfies the extension rule vacuously. □

DI^S-WO-Orderability

Now, we relax the requirement that the lifted order needs to be antisymmetric. As it turns out, this does not change the complexity of deciding whether dominance and strict independence are compatible. To show this, we need to modify the reduction given above. We note that the idea used to ensure that the preference between given sets has to be strict was first used to prove a similar result in Maly & Woltran (2017b). Again, we first state the result for strong orderability, even though we will improve that result in the next section (Theorem 4.8). The NP-completeness of the $\leq$-orderability version will follow directly.

Proposition 4.5. STRONG DI^S-WO-ORDERABILITY *is* NP-*hard.*

Proof. We will modify the reduction used for Proposition 4.2. We have to compensate for the fact that the lifted order $\preceq$ does not need to be antisymmetric. First, we have to make sure that all preferences between sets X_i^{f} and X_i^{t} are strict. Otherwise, $\preceq$ would not encode a valid truth assignment. We can add elements and sets that lead to a contradiction if the preference is not strict. This can be done as follows: Assume we want to enforce that the preferences between two sets X_i^{f} and X_i^{t} is strict. Then we add sets A, B, C and D and enforce preferences such that $X_i^{\mathrm{f}} \preceq X_i^{\mathrm{t}}$ implies $A \prec B$ by transitivity and $X_i^{\mathrm{t}} \preceq X_i^{\mathrm{f}}$ implies $D \prec C$ by transitivity. Furthermore, we add sets such that $A \prec B$

and $D \prec C$ lead to a contradiction if they hold at the same time. Then, $X_i^{\mathrm{f}} \preceq X_i^{\mathrm{t}}$ and $X_i^{\mathrm{t}} \preceq X_i^{\mathrm{f}}$ can not hold at the same time. Hence, either $X_i^{\mathrm{f}} \prec X_i^{\mathrm{t}}$ or $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$ must hold.

Furthermore, if $\preceq$ is not linear, then it does not need to satisfy reverse strict independence even if it satisfies strict independence. Therefore, we need to adapt the way that we enforce preferences to a method that does not require reverse strict independence. This can be done as follows for a desired preference $X_i^a \setminus \{y_j^b\} \prec X_i^c \setminus \{y_j^b\}$: We replace every element z_j^b by two elements z_j^b and $\overline{z_j^b}$, set $z_j^b < \overline{z_j^b}$ and add the sets $\{z_j^b\}, \{z_j^b, \overline{z_j^b}\}, \{\overline{z_j^b}\}$ to $\mathcal{X}$. By dominance this enforces $\{z_j^b\} \prec \{\overline{z_j^b}\}$. Now, using a similar construction as before, we can enforce $X_i^a \setminus \{y_j^b\} \preceq \{z_j^b\}$ and $\{\overline{z_j^b}\} \preceq X_k^c \setminus \{y_j^d\}$. Then, we have

$$X_i^a \setminus \{y_j^b\} \preceq \{z_j^b\} \prec \{\overline{z_j^b}\} \preceq X_k^c \setminus \{y_j^d\}$$

and hence by transitivity the desired strict preference $X_i^a \setminus \{y_j^b\} \prec X_i^c \setminus \{y_j^b\}$. To enforce $X_i^a \setminus \{y_j^b\} \preceq \{z_j^b\}$ we add the same sequence $A_1, \dots, A_l$ as in the proof of Proposition 4.2. This enforces as before

$$(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^b\} \prec \{z_j^b, \max_j^b\}.$$

Now, as $\preceq$ is total, by Proposition 3.26 it satisfies reverse independence. Therefore, we get the desired $X_i^a \setminus \{y_j^b\} \preceq \{z_j^b\}$. We can enforce $\{\overline{z_j^b}\} \preceq X_k^c \setminus \{y_j^d\}$ similarly.

Next, we want to enforce that that all preferences between sets X_i^{f} and X_i^{t} are strict. We will add additional sets that lead to cyclic preferences if $X_i^{\mathrm{f}} \preceq X_i^{\mathrm{t}}$ and $X_i^{\mathrm{t}} \preceq X_i^{\mathrm{f}}$ hold at the same time. The idea is illustrated in Figure 4.2. We add for every variable X_i new elements

$$a_i^-, b_i^-, c_i^-, d_i^-, r_i, s_i, d_i^+, c_i^+, b_i^+, a_i^+.$$

Furthermore, we add them to the critical linear order $\leq$ as follows

$$\begin{aligned} \min_1^1 < \min_1^2 < \dots < \min_m^3 < x_{1,1}^- < x_{1,2}^- < \dots < x_{n,2}^- < \\ \mathbf{a_i^-} < \mathbf{b_i^-} < \mathbf{c_i^-} < \mathbf{d_i^-} < v_1 < v_2 < \mathbf{r_i} < \mathbf{s_i} < \mathbf{d_i^+} < \mathbf{c_i^+} < \mathbf{b_i^+} < \mathbf{a_i^+} \\ < z_1^1 < z_1^2 \dots < z_m^3 < y_1^1 < y_1^2 < \dots < y_m^3 < \\ x_{1,1}^+ < x_{1,2}^+ < \dots < x_{n,2}^+ < \max_1^1 < \max_1^2 < \dots < \max_m^3 \end{aligned}$$

Then, we add new sets

$$\begin{aligned} A_i := \{a_i^-, v_1, v_2, r_i, s_i, a_i^+\}, B_i := \{b_i^-, v_1, v_2, r_i, s_i, b_i^+\}, \\ C_i := \{c_i^-, v_1, v_2, r_i, s_i, c_i^+\} \text{ and } D_i := \{d_i^-, v_1, v_2, r_i, s_i, d_i^+\}. \end{aligned}$$

Now, let z_i^a, $\overline{z_i^a}$, $\max_i^a$ and $\min_i^a$ be new elements where we set $z_i^a, \overline{z_i^a} \in Y$. Then, we enforce with the method described above $A_i \prec X_i^{\mathrm{f}}$ using these new elements. Furthermore, we enforce $X_i^{\mathrm{t}} \prec B_i$, $X_i^{\mathrm{f}} \prec C_i$ and $D_i \prec X_i^{\mathrm{t}}$. Finally, we add the sets $A_i \setminus \{r_i\}$, $B_i \setminus \{r_i\}$, $C_i \setminus \{s_i\}$ and $D_i \setminus \{s_i\}$ and enforce $B_i \setminus \{r_i\} \prec D_i \setminus \{s_i\}$ and $C_i \setminus \{s_i\} \prec A_i \setminus \{r_i\}$. We call the sets added in this step the class 4 sets. These enforced preference are shown as solid arrows in Figure 4.2.

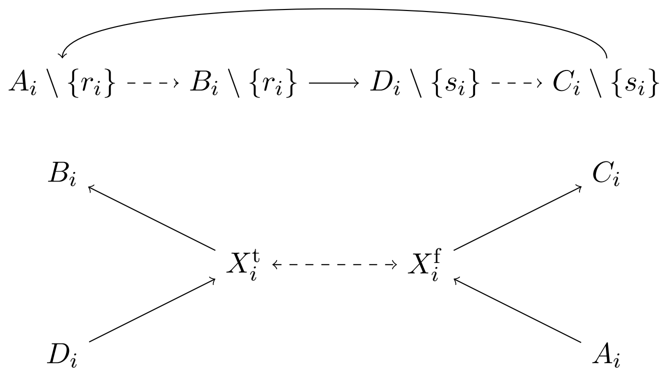

Figure 4.2: Enforcing strictness.

Now, we claim that it is not possible for a weak order $\preceq$ to satisfy dominance and strict independence with respect to $\leq$ if $X_i^{\mathrm{t}} \sim X_i^{\mathrm{f}}$ holds. Assume otherwise that $\preceq$ is a weak order that satisfies dominance and strict independence with respect to $\leq$ such that $X_i^{\mathrm{t}} \sim X_i^{\mathrm{f}}$ holds. Then, $D_i \prec X_i^{\mathrm{t}} \preceq X_i^{\mathrm{f}} \prec C_i$ implies $D_i \prec C_i$ by transitivity and hence $D_i \setminus \{s_i\} \preceq C_i \setminus \{s_i\}$ by reverse independence. Similarly, $A_i \prec X_i^{\mathrm{f}} \preceq X_i^{\mathrm{t}} \prec B_i$ implies $A_i \prec B_i$ by transitivity and hence $A_i \setminus \{r_i\} \preceq B_i \setminus \{r_i\}$ by reverse independence. However, this leads to a contradiction by

$$A_i \setminus \{r_i\} \preceq B_i \setminus \{r_i\} \prec D_i \setminus \{s_i\} \preceq C_i \setminus \{s_i\} \prec A_i \setminus \{r_i\}.$$

Now, as before $\mathcal{X}$ can only be DI^S-orderable with respect to $\leq$ if ϕ is a positive instance of Sat.

It remains to show that the modified family $\mathcal{X}$ is strongly DI^S-orderable if ϕ is a positive instance of Sat. As before, let $\leq'$ be an arbitrary linear order on X. If ϕ is a positive instance of Sat, we can construct a weak order on $\mathcal{X}$ that satisfies dominance and strict independence as before, with the following modifications: We replace z_i^e by the block $\{z_i^e\} \prec \{z_i^e, \overline{z_i^e}\} \prec \{\overline{z_i^e}\}$ if $z_i^e < \overline{z_i^e}$ and by the block $\{\overline{z_i^e}\} \prec \{z_i^e, \overline{z_i^e}\} \prec \{z_i^e\}$ if $\overline{z_i^e} < z_i^e$.

It remains to add the class 4 sets to the order. The class 4 sets used to enforce the preferences $A_i \prec X_i^{\mathrm{f}}$, $X_i^{\mathrm{t}} \prec B_i$, $X_i^{\mathrm{f}} \prec C_i$ and $D_i \prec X_i^{\mathrm{t}}$ can be ordered the same way as the class 3 sets. As before, we use the fact that we can assume $v_1 <' v_2$ to ensure that this order is compatible with the enforced preferences. For a specific variable V_i we set by construction either $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$ or $X_i^{\mathrm{f}} \prec X_i^{\mathrm{t}}$. We assume $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$. The other case is symmetric. Then, we add D_i in $\preceq$ exactly before X_i^{t} and B_i exactly after X_i^{t}. Similarly, we add A_i exactly before X_i^{f} and C_i exactly after X_i^{f}. Then, we have

$$D_i \prec X_i^{\mathrm{t}} \prec B_i \prec A_i \prec X_i^{\mathrm{f}} \prec C_i$$

which is compatible with the forced preferences.

Now, consider the group $A_i \setminus \{r_i\}, B_i \setminus \{r_i\}, C_i \setminus \{s_i\}$ and $D_i \setminus \{s_i\}$. We observe that all sets in this group differ in at least two elements. Therefore, we only have to set $B_i \setminus \{r_i\} \preceq A_i \setminus \{r_i\}$ and $D_i \setminus \{s_i\} \preceq C_i \setminus \{s_i\}$ in order to satisfies reverse independence. Furthermore, we have to satisfy dominance if r_i and/or s_i are the largest element of the set they are removed from. This can be satisfied by a straightforward construction that respects the enforced preferences unless r_i is the maximal element of A_i and s_i is the minimal element of C_i or alternatively if r_i is the maximal element of B_i and s_i is the minimal element of D_i. We describe the construction for the first case: We have to set $A_i \setminus \{r_i\} \prec A_i$ and $C_i \prec C_i \setminus \{s_i\}$ which implies $A_i \setminus \{r_i\} \prec C_i \setminus \{s_i\}$ contrary to the preference we wanted to enforce in the construction. We use the fact that r_i is the maximal element of A_i and s_i is the minimal element of C_i to define an order that allows this. Let $z_i^{r,a}$ and $\max_i^{r,a}$ be the new elements used to enforce $\{z_i^{r,a}\} \prec A_i \setminus \{r_i\}$. Then, we set $A \preceq B$ for the sets A, B such that $\max_i^{r,a} \in A, B$ if

- $v_2 \in A$ and $v_2 \notin B$
- $v_2 \in A, B$ or $v_2 \notin A, B$ and $\max_{\leq'}(A_L \triangle B_L) \in B$,
- $v_2 \in A, B$ or $v_2 \notin A, B$, $A_L = B_L$ and $\min_{\leq'}(A_S \triangle B_S) \in A$.

where $A_L := \{x \in A \mid \max_i^{r,a} <' x\}$ and $A_S := \{x \in A \mid x <' \max_i^{r,a}\}$. This order satisfies dominance and strict independence because the element s_i is smaller than v_2 by assumption and removed later in the sequence $A_1, \dots, A_k$. Furthermore, this implies $(A_i \setminus \{r_i\}) \cup \{\max_i^{r,a}\} \prec \{z_i^{r,a}, \max_i^{r,a}\}$, which implies $A_i \setminus \{r_i\} \prec \{z_i^{r,a}\}$. This allows us to set $A_i \setminus \{r_i\} \prec C_i \setminus \{s_i\}$. Then, we can place $A_i \setminus \{r_i\}$ just before A_i and $C_i \setminus \{s_i\}$ just after C_i to get an order that satisfies dominance and independence. □

As in the case of Strong DI^SE-LO-Orderability we have shown that there is a weak order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to a fixed order $\leq$ if only if ϕ is satisfiable. Therefore, this reduction again also works for DI^SE-WO-Orderability and we have proven the first of our main theorems.

Theorem 4.6. *DI^S-WO-Orderability is NP-complete.*

Proof. It follows from the reduction above that DI^S-WO-Orderability is NP-hard. Furthermore, it is clear that DI^S-WO-Orderability is in NP as we can guess a weak order $\preceq$ on $\mathcal{X}$ and then check in polynomial time if $\preceq$ satisfies dominance and strict independence with respect to $\leq$. □

Corollary 4.7. *DI^SE-WO-Orderability is NP-complete.*

Proof. We can adapt the reduction in a similar way as in the proof of Corollary 4.4. We need to change two things compared to Corollary 4.4. First of all, in contrast to reverse strict independence we can not iterate reverse independence. Therefore, we need to adapt the way that we enforce preferences. We add additionally the sets $(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b\}$. and $(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b, \overline{\max_i^b}\}$. We observe $l > 3$ and that the following preference is enforced by dominance

$$A_2 = (X_i^a \setminus \{y_j^b, x_i^-, z_i^{a,1}, z_i^{a,2}\}) \cup \{\max_j^b, \overline{\max_i^b}\} \prec \{\max_j^b, \overline{\max_i^b}\} = A_l$$

this enforces by strict independence

$$(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b, \overline{\max_i^b}\} \prec \{z_i^{a,1}, z_i^{a,2}, \max_j^b, \overline{\max_i^b}\}.$$

Now, by one application of reverse independence, we have

$$(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b, \} \preceq \{z_i^{a,1}, z_i^{a,2}, \max_j^b\}.$$

and hence by dominance

$$(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^b\} \prec \{z_i^{a,1}, z_i^{a,2}, \max_j^b\}.$$

Then, another application of reverse independence implies

$$(X_j^a \setminus \{y_i^1\}) \preceq \{z_i^{a,1}, z_i^{a,2}\}.$$

Furthermore, the new singleton in the reduction $\overline{z_i^e}$ also needs to be dealt with. We replace z_i^e again by two elements $z_i^{e,1}$ and $z_i^{e,2}$. Similarly we replace $\overline{z_i^e}$ by $\overline{z_i^{e,1}}$ and $\overline{z_i^{e,2}}$. In the critical linear order we set

$$z_i^{e,1} < z_i^{e,2} < \overline{z_i^{e,1}} < \overline{z_i^{e,2}}.$$

Then, we use these to enforce as in Corollary 4.4

$$X_i^a \setminus \{y_j^b\} \preceq \{z_j^{b,1}, z_j^{b,2}\} \text{ and } \{\overline{z_j^{b,1}}, \overline{z_j^{b,2}}\} \preceq X_k^c \setminus \{y_j^d\}.$$

Furthermore, instead of $\{z_i^e, \overline{z_i^e}\}$ we add $\{z_i^{e,1}, z_i^{e,2}, \overline{z_i^{e,1}}\}$, $\{z_i^{e,2}, \overline{z_i^{e,1}}\}$ and $\{z_i^{e,2}, \overline{z_i^{e,1}}, \overline{z_i^{e,2}}\}$. Then, the following preferences are enforced:

$$\{z_i^{e,1}, z_i^{e,2}\} \prec \{z_i^{e,1}, z_i^{e,2}, \overline{z_i^{e,1}}\} \prec \{z_i^{e,2}, \overline{z_i^{e,1}}\} \prec \{z_i^{e,2}, \overline{z_i^{e,1}}, \overline{z_i^{e,2}}\} \prec \{\overline{z_i^{e,1}}, \overline{z_i^{e,2}}\}.$$

Therefore, by transitivity we have

$$X_i^a \setminus \{y_j^b\} \preceq \{z_j^{b,1}, z_j^{b,2}\} \prec \{\overline{z_j^{b,1}}, \overline{z_j^{b,2}}\} \preceq X_k^c \setminus \{y_j^d\}.$$

and therefore $X_i^a \setminus \{y_j^b\} \prec X_k^c \setminus \{y_j^d\}$ as intended.

When constructing an order on $\mathcal{X}$ from an satisfying assignment of ϕ we can just replace the block $\{z_i^e\} \prec \{z_i^e, \overline{z_i^e}\} \prec \{\overline{z_i^e}\}$ resp. $\{\overline{z_i^e}\} \prec \{z_i^e, \overline{z_i^e}\} \prec \{z_i^e\}$ by the sets $\{z_i^{e,1}, z_i^{e,2}\}, \{z_i^{e,1}, z_i^{e,2}, \overline{z_i^{e,1}}\}, \{z_i^{e,2}, \overline{z_i^{e,1}}\}, \{\overline{z_i^{e,1}}, \overline{z_i^{e,2}}\}, \{z_i^{e,2}, \overline{z_i^{e,1}}, \overline{z_i^{e,2}}\}$ ordered as demanded by dominance. It can be checked that the rest of the construction works as before, as all other new sets contain an extremum element. $\square$

Strong DI^S-WO-Orderability and strong DI^SE-WO-Orderability

We have shown before that Strong DI^S-Orderability is NP-hard. However, we claim that the problem is even Π_2^P-hard, and furthermore also Π_2^p-complete. To show this we will extend the reduction used in the previous results from SAT to Π_2-SAT.

Theorem 4.8. STRONG DI^S-WO-ORDERABILITY *is* Π_2^p*-complete.*

Proof. Π_2^p-membership is clear as we can universally guess a linear order $\leq$ on X and then, by Theorem 4.6, check via the NP-oracle if $(X, \mathcal{X}, \leq)$ is a positive instance of DI^S-WO-ORDERABILITY.

It remains to show that STRONG DI^S-WO-ORDERABILITY is Π_2^p-hard. We do this by extending the reduction above to a reduction from a Π_2-SAT instance $\phi = \forall \vec{W} \exists \vec{V} \psi(\vec{W}, \vec{V})$. Let $w_1 \dots w_l$ be the universally quantified variables. Intuitively, we want to add for every w_i new elements w_i^{t} and w_i^{f} such that the preference between these elements determines the preference between X_i^{t} and X_i^{f}. Then, there is a class of critical linear orders such that every truth assignment to the universally quantified variables is encoded by at least one critical linear order. Furthermore, we will ensure that there exists an order satisfying dominance and strict independence with respect to a critical linear order $\leq$ if and only if there exists a satisfying assignment to ψ that extends the assignment encoded by $\leq$.

We set up the reduction similarly to the one for Proposition 4.5. Additionally, we add for every universally quantified variable w_i represented by X_i^{t} and X_i^{f} sets

$$X_i^{\mathrm{t}} \setminus \{y_i^q\}, X_i^{\mathrm{f}} \setminus \{y_i^q\}, \{w_i^{\mathrm{t}}\}, \{w_i^{\mathrm{f}}\} \text{ and } \{w_i^{\mathrm{t}}, w_i^{\mathrm{f}}\},$$

where y_i^q, w_i^{t} and w_i^{f} are new elements. Then, we enforce – with the same method as before – $X_i^{\mathrm{t}} \setminus \{y_i^q\} \prec \{w_i^{\mathrm{t}}\}$ and $\{w_i^{\mathrm{f}}\} \prec X_i^{\mathrm{f}} \setminus \{y_i^q\}$ using new elements $\min_i^q$ and $\max_i^q$. This will ensure for every critical linear order $\leq'$ with $w_i^{\mathrm{t}} <' w_i^{\mathrm{f}}$ that $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$ must hold for every order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq'$. We add additionally sets $X_i^1 \setminus \{\overline{y_i^q}\}$ and $X_i^2 \setminus \{\overline{y_i^q}\}$ and similarly enforce $X_i^{\mathrm{t}} \setminus \{\overline{y_i^q}\} \prec \{w_i^{\mathrm{t}}\}$ and $\{w_i^{\mathrm{f}}\} \prec X_i^{\mathrm{f}} \setminus \{\overline{y_i^q}\}$ using $\overline{\min_i^q}$ and $\overline{\max_i^q}$. Hence, $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$ must hold for every order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence with respect to any critical linear order $\leq''$ on X with $w_i^{\mathrm{t}} <'' w_i^{\mathrm{f}}$.

Now, we claim that $(X, \mathcal{X})$ can only be a positive instance of STRONG DI^S-WO-ORDERABILITY if ϕ is satisfiable. We call a linear order critical if:

- It equals the critical linear order $\leq$ in the proof of Proposition 4.5 on the elements already occurring in that reduction.
- The new elements $y_i^q, w_i^{\mathrm{t}}, w_i^{\mathrm{f}}, \min_i^q$ and $\max_i^q$ are added in that order as follows: $\min_i^q$ is smaller than $\min_1^1$ for all i. Similarly, $\max_i^q$ is larger than $\max_m^3$ for all i. The other elements are added between v_2 and z_1^1.

Then, for every truth assignment T to the variables in $\vec{W}$ there is a critical linear order $\leq^*$ on X such that $w_i^{\mathrm{t}} <^* w_i^{\mathrm{f}}$ if w_i is assigned false in T and $w_i^{\mathrm{f}} <^* w_i^{\mathrm{t}}$ if w_i is assigned true in T. Now, if there is no satisfying assignment to ϕ that extends T, then there can be no order on $\mathcal{X}$ satisfying dominance and strict independence with respect to $\leq^*$. Hence $(X, \mathcal{X})$ can only be DI^S-orderable with respect to every linear order $\leq^*$ if ϕ is satisfiable.

It remains to show that if ϕ is satisfiable then $(X, \mathcal{X})$ is a positive instance of Strong DI^S-WO-Orderability. This can be done using nearly the same construction as above treating $X_i^{\mathrm{t}} \setminus \{y_i^q\}$ and $X_i^{\mathrm{f}} \setminus \{y_i^q\}$ as Class 2 sets, all other new sets as Class 3 sets and inserting $\{w_i^{\mathrm{t}}\} \prec \{w_i^{\mathrm{t}}, w_i^{\mathrm{f}}\} \prec \{w_i^{\mathrm{f}}\}$ resp. $\{w_i^{\mathrm{f}}\} \prec \{w_i^{\mathrm{t}}, w_i^{\mathrm{f}}\} \prec \{w_i^{\mathrm{t}}\}$ where we would insert z_i^j and $\overline{z_i^j}$. The only exception has to be made if there is an i such that $y_i^q = \min(X_i^{\mathrm{t}})$ and $\overline{y_i^q} = \max(X_i^{\mathrm{t}})$ or $\overline{y_i^q} = \min(X_i^{\mathrm{f}})$ and $y_i^q = \max(X_i^{\mathrm{f}})$. In the first case, we set $A \prec B$ for the sets containing $\overline{\min_i^q}$ if:

- $y_i^q \in A$ and $y_i^q \notin B$
- $y_i^q \in A, B$ or $y_i^q \notin A, B$ and $\max(A_L \triangle B_L) \in B$,
- $y_i^q \in A, B$ or $y_i^q \notin A, B$, $A_L = B_L$ and $\min(A_S \triangle B_S) \in A$.

where $A_L := \{x \in A \mid \overline{\min_i^q} <' x\}$ and $A_S := \{x \in A \mid x <' \overline{\min_i^q}\}$. And for $A \prec B$ for the sets containing $\max_i^q$ if

- $\overline{y_i^q} \in B$ and $\overline{y_i^q} \notin A$
- $y_i^q \in A, B$ or $y_i^q \notin A, B$ and $\max(A_L \triangle B_L) \in B$,
- $y_i^q \in A, B$ or $y_i^q \notin A, B$, $A_L = B_L$ and $\min(A_S \triangle B_S) \in A$.

where $A_L := \{x \in A \mid \max_i^q <' x\}$ and $A_S := \{x \in A \mid x <' \max_i^q\}$.

It is clear that these orders satisfy dominance and strict independence, similarly to the orders on the class 3 sets defined above. Furthermore, we have $\{w_i^{\mathrm{t}}, \max_i^q\} \prec (X_i^{\mathrm{t}} \setminus \{y_i^q\}) \cup \{\max_i^q\}$ and $(X_i^{\mathrm{t}} \setminus \{\overline{y_i^q}\}) \cup \{\overline{\min_i^q}\} \prec \{w_{\mathrm{f}}, \overline{\min_i^q}\}$ which allows us to set $X_i^{\mathrm{t}} \setminus \{\overline{y_i^q}\} \prec \{w_i\} \prec X_i^{\mathrm{t}} \setminus \{y_i^q\}$ which is consistent with the enforced $X_i^{\mathrm{t}} \setminus \{\overline{y_i^q}\} \prec X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}} \setminus \{y_i^q\}$. The second case can be treated analogously. □

Corollary 4.9. Strong DI^SE-WO-Orderability *is* Π_2^p*-complete.*

Proof. As before in Corollary 4.4 and 4.7 we can double the elements that would otherwise appear in singletons. That means that we additionally need to double the elements w_i^{t} and w_i^{f} and add, as for z_i^a and $\overline{z_i^a}$, the sets $\{w_i^{\mathrm{t},1}, w_i^{\mathrm{t},2}, w_i^{\mathrm{f},1}\}$, $\{w_i^{\mathrm{t},2}, w_i^{\mathrm{f},1}\}$ and $\{w_i^{\mathrm{t},2}, w_i^{\mathrm{f},1}, w_i^{\mathrm{f},2}\}$. Furthermore, as before, we enforce

$$X_i^{\mathrm{t}} \setminus \{y_i^q\} \preceq \{w_i^{\mathrm{t},1}, w_i^{\mathrm{t},2}\} \text{ and } \{w_i^{\mathrm{t},1}, z_j^{\mathrm{t},2}\} \preceq X_i^{\mathrm{f}} \setminus \{y_i^q\}.$$

Then, $w_i^{\mathrm{t},1} < w_i^{\mathrm{t},2} < w_i^{\mathrm{f},1} < w_i^{\mathrm{f},2}$, enforces $X_i^{\mathrm{t}} \setminus \{y_i^q\} \prec X_i^{\mathrm{f}} \setminus \{y_i^q\}$ and hence $X_i^{\mathrm{t}} \prec X_i^{\mathrm{f}}$. Similarly, we can ensure that $w_i^{\mathrm{f},2} < w_i^{\mathrm{t},1} < w_i^{\mathrm{t},2} < w_i^{\mathrm{t},1}$ enforces $X_i^{\mathrm{f}} \setminus \{\overline{y_i^q}\} \prec X_i^{\mathrm{t}} \setminus \{\overline{y_i^q}\}$ and hence $X_i^{\mathrm{f}} \prec X_i^{\mathrm{t}}$. Therefore, every assignment to the universally quantified variables in ϕ is encoded by some linear order on X. The rest of the proof works as before. □

We conclude this part by observing that the orders constructed in the proof of Theorem 4.8 and Corollary 4.9 are not only weak but even linear orders. Therefore, the reductions used for these results show also the Π_2^p completeness of Strong DI^S-LO-Orderability and Strong DI^SE-LO-Orderability.

Corollary 4.10. Strong DI^S-LO-Orderability *and* Strong DI^SE-LO-Orderability *are* Π_2^p*-complete.*

Strong DI-WO-Orderability and strong DIE-WO-Orderability

To conclude the section on ≤-orderability and strong orderability, we show that the hardness results we showed before also hold if we replace strict independence with regular independence.

Theorem 4.11. Strong DI-WO-Orderability *is* Π_2^p*-complete.* DI-WO-Orderability *is* NP*-complete.*

Proof. We observe that the reductions for Proposition 4.5 and Theorem 4.8 only use strict independence once: Namely when enforcing a preference $X_i^a \setminus \{y_j^b\} \preceq \{z_j^b\}$ strict independence ensures that

$$A_1 = (X_i^a \setminus \{y_j^b, z_j^b, \}) \cup \{\max_j^b\} \prec \{\max_j^b\} = A_l$$

enforces the desired preference

$$(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^b\} \prec \{z_j^b, \max_j^b\}.$$

We can achieve the same result with dominance and independence as follows: We add the same sequence $A_1, \dots, A_l$ as in the proof of Proposition 4.5 and additionally, the set $(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b\}$. We observe $l > 3$ and that the following preference is enforced by dominance

$$A_2 = (X_i^a \setminus \{y_j^b, z_j^b, x_i^-\}) \cup \{\max_j^b\} \prec \{\max_j^b\} = A_l$$

which enforces by independence

$$(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b\} \prec \{z_j^b, \max_j^b\}.$$

Now, by construction $x_i^- < \min(X_i^a \setminus \{y_j^b, x_i^-\})$ holds and hence by dominance

$$(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^b\} \prec X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^b\} \preceq \{z_j^b, \max_j^b\}.$$

Therefore, we get by transitivity the desired

$$(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^b\} \prec \{z_j^b, \max_j^b\}.$$

Now, for any linear order $\leq'$ the newly added sets can easily be added to an order satisfying dominance and independence: Any new set $(X_i^a \setminus \{y_j^b, x_i^-\}) \cup \{\max_j^1\}$ can be added in the order $\preceq$ right after $(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^1\}$ if $x_i^- <' v_1$ or right before $(X_i^a \setminus \{y_j^b\}) \cup \{\max_j^1\}$ if $v_1 <' x_i^-$. $\square$

The newly added sets are not singletons. Hence, the fact that we can add the extension rule without changing the complexity of the problem still holds as before.

Corollary 4.12. Strong *DIE*-WO-Orderability *is* Π_2^p*-complete and* *DIE*-WO-Orderability *is* NP*-complete.*

As above, we observe that the orders constructed in Theorem 4.11 and Corollary 4.12 are linear orders. Therefore, we can also conclude that Strong *DI*-LO-Orderability and Strong *DIE*-LO-Orderability are Π_2^p-complete and that *DI*-LO-Orderability and *DI*-LO-Orderability are NP-complete.

Corollary 4.13. Strong *DI*-LO-Orderability *and* Strong *DIE*-LO-Orderability *are* Π_2^p*-complete. Moreover,* *DI*-LO-Orderability *and* *DIE*-LO-Orderability *are* NP*-complete.*

We close this section with an important observation: The fact that strong orderability is Π_2^p-complete implies that constructing an order satisfying dominance, (strict) independence and the extension rule is hard even if we already know that the family $\mathcal{X}$ is strongly orderable. In other words, even if we know that a weak order satisfying our axioms must exist, it may be hard to construct it.

Corollary 4.14. *Assume* coNP $\neq \Pi_2^p$*. Then there exists no polynomial time algorithm* $\mathbb{A}$ *with the following specifications:*

- $\mathbb{A}$ *takes as input a set* X*, a family of sets* $\mathcal{X} \subseteq \mathcal{P}(X)$ *and a linear order* $\leq$ *on* X*.*
- *if* $\mathcal{X}$ *is strongly* *DI**-orderable, then* $\mathbb{A}$ *produces on input* $(X, \mathcal{X}, \leq)$ *an order* $\preceq$ *on* $\mathcal{X}$ *that satisfies dominance and independence.*

The same holds if we replace independence by strict independence or if we add the extension rule.

Proof. We claim that Strong *DI*-WO-Orderability would be in coNP if there exists a polynomial time algorithm $\mathbb{A}$ with the specifications above. Observe that there exists a linear order $\leq$ on X that can not be lifted if and only if $(X, \mathcal{X})$ is negative instance of Strong *DI*-WO-Orderability. Hence $\leq$ is a certificate (of polynomial size) for the fact that $(X, \mathcal{X})$ is a negative instance. Furthermore, one can check the certificate by running $\mathbb{A}$ on $(X, \mathcal{X}, \leq)$. Then, one only needs to check that the produced order does not satisfy dominance and strict independence. By definition, this can only be the case if $(X, \mathcal{X})$ is a negative instance of STRONG DI-WO-ORDERABILITY. The argument for strict independence and extension is analog. □

4.2 Partial orders

In this section, we investigate the effect of dropping the requirement that the lifted order should be total. The results up to Theorem 4.27 are taken from Maly & Woltran (2017b). Theorem 4.27 and Corollary 4.28 appeared in Maly (2020), albeit without a full proof. For dominance and independence, we already have seen that they always can be jointly satisfied, if we expect the lifted order to be only a preorder.

Theorem 4.15. *For every set X, linear order $\leq$ on X and family of sets $\mathcal{X} \subseteq \mathcal{P}(X)$, there is a preorder that satisfies dominance, independence and the extension rule.*

Proof. In Example 3.9, we defined the preorder $\preceq_{pmm}$ and showed that it always satisfies dominance and independence. By definition, it also satisfies the extension rule. □

In other words, every family of sets is strongly $DI(E)$-orderable if we only require the lifted order to be a preorder. On the other hand, Barbera and Pattanaik's theorem tells us that this is not the case for dominance and strict independence.

We observe that in many applications partial orders are more common than preorders. For example, most voting rules for incomplete preferences require partial orders as input (Boutilier & Rosenschein, 2016) though voting rules for even weaker preference models exist (Xia & Conitzer, 2011; Terzopoulou & Endriss, 2019). Therefore, we are often more interested in the complexity of deciding for a given family of sets if there exists a partial order that satisfies dominance and (strict) independence resp. dominance, (strict) independence and the extension rule. As strict independence and independence coincide for strict orders, we can focus just on strict independence here. First, we consider $\leq$-orderability.

In order to decide if a set admits a partial order we build a minimal transitive relation satisfying dominance and strict independence with respect to some linear order $\leq$. First, we build a minimal transitive relation satisfying dominance.

Definition 4.16. Given a set X, a linear order $\leq$ on X and a family $\mathcal{X}$ $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$, we define a relation $\prec_d$ on $\mathcal{X}$ in the following way: If $A, A \cup \{x\} \in \mathcal{X}$, then

1. $A \prec_d A \cup \{x\}$ if $y < x$ for all $y \in A$,
2. $A \cup \{x\} \prec_d A$ if $x < y$ for all $y \in A$.

We define the relation $\prec_d^t$ on $\mathcal{X}$ by $\prec_d^t := \mathrm{trcl}(\prec_d)$.

Here trcl(R) denotes the transitive closure of the relation R, i.e. trcl(R) is the minimal transitive relation extending R. The transitive closure of a relation is unique, therefore $\leq_d^t$ is well-defined. The relation $\prec_d^t$ has the following useful property.

Proposition 4.17. *For every set X, linear order $\leq$ on X and family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$, the relation $\prec_d^t$ is a strict partial order and a (strict) partial order on $\mathcal{X}$ satisfies dominance if and only if it extends $\prec_d^t$.*

Proof. Obviously, $\prec_d^t$ is transitive. Furthermore, $\prec_d^t$ is antisymmetric as $A \prec_d^t B$ implies $\max(A) < \max(B)$ or $\min(A) < \min(B)$ and $<$ is antisymmetric.

By definition, a relation satisfies dominance if and only if it extends $\prec_d$ and a transitive relation extending $\prec_d$ also extends $\prec_d^t$ by the minimality of trcl. □

We can define the minimal strict partial order satisfying dominance and the extension rule similarly. Next we show how to extend such a relation to a minimal relation that satisfies dominance and strict independence resp. dominance, strict independence and the extension rule. This minimal relation is defined iteratively: We begin with $\prec_d^t$ and then add in every step first a preference implied by strict independence and then close under transitivity. This process is repeated until every occurrence of strict independence is satisfied.

Definition 4.18. Given a set X, a linear order $<$ on X and a family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$, we build a relation $\prec_\infty$ on $\mathcal{X}$ by induction. First, we set $\prec_0^t := \prec_d^t$. Now, let $\prec_n^t$ be defined. For $\prec_{n+1}$ we select sets $A, B, A \setminus \{x\}, B \setminus \{x\} \in \mathcal{X}$ with $x \in X$, $A \setminus \{x\} \prec_n^t B \setminus \{x\}$ but not $A \prec_t^n B$. Then, we set $C \prec_{n+1} D$ if one of the following holds:

- $C \prec_n^t D$
- $C = A$ and $D = B$

Furthermore, we set $\prec_{n+1}^t := \mathrm{trcl}(\prec_{n+1})$. In the end, we set $\prec_\infty = \bigcup_n \prec_n^t$.

Example 4.19. Consider the following family:

$$\mathcal{X} = \{\{3\}, \{4\}, \{1,3\}, \{2,3\}, \{1,4\}, \{1,2,3\}, \{1,3,4\}\}.$$

Then, $\prec_\infty$ consists of the following preferences:

$$\{1,3\} \prec_\infty \{1,3,4\}, \quad \{1,2,3\} \prec_\infty \{2,3\} \prec_\infty \{3\}, \\ \{1,4\} \prec_\infty \{4\}, \quad \{1,2,3\} \prec_\infty \{1,3\}.$$

In order to prove that this is actually a minimal relation for dominance and strict independence, we introduce another concept we call links.

Definition 4.20. A $\prec_\infty$-*link* from A to B in $\mathcal{X}$ is a sequence $A =: C_0, C_1, \ldots, C_n := B$ with $C_i \in \mathcal{X}$ for all $i \leq n$ such that, for all $i < n$, either $C_i \prec_d C_{i+1}$ holds or there is a link between $C_i \setminus \{x\}$ and $C_{i+1} \setminus \{x\}$ for some $x \in X$.

We show that $\prec_\infty$-links indeed characterize $\prec_\infty$.

Lemma 4.21. *For $A, B \in \mathcal{X}$, $A \prec_\infty B$ implies that there is a $\prec_\infty$-link from A to B and if there is a $\prec_\infty$-link from A to B then $A \prec^* B$ holds for every transitive relation $\prec^*$ that satisfies dominance and strict independence.*

In order to prove this result, we need another definition.

Definition 4.22. For every pair $A \prec_\infty B$, there is a minimal k such that $A \prec_k^t B$ holds. We call this the $\prec_\infty$-*rank* of the pair.

Furthermore, we define the $rank(C_1, C_2, \dots, C_n)$ of a $\prec_\infty$-link $C_1, C_2, \dots, C_n$ from C_1 to C_n:

- $rank^*(C_i, C_{i+1}) = 0$ if $C_i \prec_d C_{i+1}$,
- $rank^*(C_i, C_{i+1}) = rank(C_i \setminus \{x\}, C_{i+1} \setminus \{x\})$ if not $C_i \prec_d C_{i+1}$,
- $rank(C_1, C_2, \dots, C_n) = \max\{rank^*(C_i, C_{i+1}) \mid i < n\} + 1$.

Now, we can prove Lemma 4.21:

Proof. Assume $A \prec_\infty B$. We prove that a $\prec_\infty$-link exists by induction on the $\prec_\infty$-rank of A, B. If $A \prec_d^t B$, then there is sequence $A = C_1, C_2, \dots, C_n = B$ such that $C_i \prec_d C_{i+1}$ holds for all $i < n$, hence there is a $\prec_\infty$-link from A to B. Now, assume A, B has $\prec_\infty$-rank k and for every pair with $\prec_\infty$-rank $k-1$ or less there is a $\prec_\infty$-link from C to D. There is a sequence $A = C_0 \prec_k C_1 \dots C_{n-1} \prec_k C_n = B$. For every $i < n$ either $C_i \prec_d C_{i+1}$ or $C_i \setminus \{y\} \prec_{k-1}^t C_{i+1} \setminus \{y\}$ holds, which implies by induction that there is a $\prec_\infty$-link from $C_i \setminus \{y\}$ to $C_{i+1} \setminus \{y\}$. Hence there is a $\prec_\infty$-link from A to B.

Now, let $\prec$ be a transitive relation that satisfies dominance and strict independence and assume there is a $\prec_\infty$-link $A = C_1, C_2, \dots, C_n = B$ from A to B. We prove $A \prec B$ by induction on the rank of the $\prec_\infty$-link. First, assume $rank(C_1, C_2, \dots, C_n) = 1$, then $C_i \prec_d C_{i+1}$ holds for all $i < n$, hence $A \prec B$ holds by dominance and transitivity. Now, assume $rank(C_1, C_2, \dots, C_n) = k$ and for all $\prec_\infty$-links with $rank(C_1^*, C_2^*, \dots, C_n^*) < k$ we know $C_1^* \prec C_n^*$. By induction, for every $i < n$ either $C_i \prec_d C_{i+1}$ or $C_i \setminus \{x\} \prec C_{i+1} \setminus \{x\}$ holds. This implies that $C_i \prec C_{i+1}$ holds for all $i < n$, because $\prec$ satisfies dominance and strict independence. Therefore, $A \prec B$ by transitivity. $\square$

Using this lemma, we can show now that $\prec_\infty$ is indeed a minimal relation for dominance and strict independence.

Theorem 4.23. *Given a set X, a linear order $\leq$ on X and a family $\mathcal{X} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$, there is a partial order on $\mathcal{X}$ that satisfies dominance and strict independence if and only if $\prec_\infty$ is antisymmetric on $\mathcal{X}$.*

Proof. $\prec_\infty$ satisfies dominance as it extends $\prec_d^t$. By construction it also satisfies strict independence and transitivity: $A_1 \prec_\infty A_2 \prec_\infty \cdots \prec_\infty A_k$ implies $A_1 \prec_n^t A_2 \prec_n^t \cdots \prec_n^t A_k$ for some $n \in N$ but then $A_1 \prec_n^t A_k$ holds by the transitivity of $\prec_n^t$ and therefore $A_1 \prec_\infty A_k$. Now, assume $A \prec_\infty B$ and hence $A \prec_n^t B$ for some n and $A \cup \{x\} \not\prec_n^t B \cup \{x\} \in \mathcal{X}$ for some $x \notin A \cup B$. Then, $A, B, A \cup \{x\}, B \cup \{x\}$ is picked for some l with $n < l$ and $A \cup \{x\} \prec_l B \cup \{x\}$ is set, hence $A \cup \{x\} \prec_\infty B \cup \{x\}$. Therefore, if $\prec_\infty$ is antisymmetric. This means $\prec_\infty \cup \{A \preceq A \mid A \in \mathcal{X}\}$ is a partial order satisfying dominance and strict independence.

On the other hand, if $\prec_\infty$ is not antisymmetric no partial order can extend it. But every partial order on $\mathcal{X}$ satisfying dominance and strict independence must be an extension of $\prec_\infty$. Assume otherwise there is a partial order $\prec$ on $\mathcal{X}$ satisfying dominance and strict independence that does not extend $\prec_\infty$, i.e., there are sets $A, B \in \mathcal{X}$ such that $A \prec_\infty B$ holds but not $A \prec B$. By Lemma 4.21 there is a $\prec_\infty$-link from A to B. This implies, by Lemma 4.21, $A \prec B$ because $\prec$ is transitive and satisfies dominance and strict independence. Contradiction. Therefore, no partial order on $\mathcal{X}$ can satisfy dominance and strict independence, if $\prec_\infty$ is antisymmetric. □

Using this result, we can show that DI^S-PO-orderability can be solved in polynomial time.

Corollary 4.24. *DI^S-PO-orderability is in* P.

Proof. Computing $\prec_\infty$ can obviously be done in polynomial time because the construction always stops after at most $|n \times n| = n^2$ steps. Then, checking if $\prec_\infty$ is antisymmetric only requires checking if $A \prec_\infty B \prec_\infty A$ holds for some pair A, B. □

Finally, links give us an easy characterization of sets $\mathcal{X}$ for which $\prec_\infty$ is antisymmetric.

Corollary 4.25. *$\prec_\infty$ is antisymmetric if and only if there are no sets $A, B \in \mathcal{X}$ such that there is a $\prec_\infty$-link from A to B and from B to A.*

Proof. The order $\prec_\infty$ is transitive and satisfies dominance and strict independence. Therefore, Lemma 4.21 tells us, that $A \prec_\infty A$ if and only if there is a $\prec_\infty$ link from A to A. □

This gives us a constructive polynomial time procedure for deciding partial DI^S-orderability. The same construction also works if we begin with the minimal order satisfying dominance and the extension rule. Hence, partial DI^SE-orderability can also be decided in polynomial time.

Corollary 4.26. *DI^SE-PO-orderability is in* P.

Next, we will consider *strongly partially DI^S-orderable* families of sets, i.e. families of sets $\mathcal{X} \subseteq \mathcal{P}(X)$ such that for every linear order $\leq$ on X there exists a partial order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$. It turns out that it is still difficult to decide whether a given family of sets is strongly partially DI^S-orderable. We show this by a reduction from TAUT.

Theorem 4.27. STRONG DI^S-PO-ORDERABILITY *is* coNP*-complete.*

Proof. Let ϕ be an instance of TAUT. We construct an instance $(X, \mathcal{X})$ of STRONG DI^S-PO-ORDERABILITY. For every variable X_i in ϕ we add new elements x_i^{f} and x_i^{t} to X. We call the set of these elements V. We will treat every order on X as encoding a truth assignment by equating $x_i^{\mathrm{f}} < x_i^{\mathrm{t}}$ to X_i is true and $x_i^{\mathrm{t}} < x_i^{\mathrm{f}}$ to X_i is false. Then, we add for every disjunct C_j new variables $y_j^{\mathrm{t}}, y_j^{\mathrm{f}}$. We call the set of these elements Y. Essentially, we want to add sets such that $\{y_j^{\mathrm{t}}\} \prec \{y_j^{\mathrm{f}}\}$ holds for the minimal partial order satisfying dominance strict independence with respect to $\leq$ if and only if C_j is not satisfied by the truth assignment encoded by $\leq$. Then, we will add sets that lead to a contradiction if $\{y_j^{\mathrm{t}}\} \prec \{y_j^{\mathrm{f}}\}$ holds for all disjuncts.

To achieve this, we add for every disjunct C_j elements c_j as well as d_j^k and e_j^k for $k \leq 3$. Finally, we add new elements u, v, z_1 and z_2. The elements u and v will be used to generate a contradiction if $\{y_j^{\mathrm{t}}\} \prec \{y_j^{\mathrm{f}}\}$ holds for all disjuncts. The elements z_1 and z_2 have the same purpose as the elements v_1 and v_2 in the proof of Proposition 4.2, i.e. the preference between z_1 and z_2 determines if $x_i^{\mathrm{f}} < x_i^{\mathrm{t}}$ encodes that x_i is set to true or to false.

Next we fix a class of linear orders on X that we call critical linear orders. We want to show that for a critical linear order $\leq$ there exists a partial order on $\mathcal{X}$ satisfying dominance and strict independence only if $\leq$ encodes a satisfying truth assignment. As any possible truth assignment is encoded by a critical linear order, this means that $\mathcal{X}$ can not be strongly partially DI^S-orderable if ϕ is not a tautology. We call any linear order on X that is derived by replacing V with an arbitrary linear order on the elements in V in the following linear order a critical linear order:

$$
\begin{aligned}
u < c_1 < \cdots < c_m < y_1^{\mathrm{t}} < \cdots < y_m^{\mathrm{t}} <& \\
d_1^1 < \cdots < d_m^3 < V < e_1^1 < \cdots < e_m^3 < y_1^{\mathrm{f}} < \cdots <& \\
y_m^{\mathrm{f}} < z_1 < z_2 < v&
\end{aligned}
$$

In the following, we write $\preceq_\infty$ for the minimal partial order satisfying dominance and strict independence with respect to some linear order on V.

Next, we build the family $\mathcal{X}$. First, we make sure, that any order satisfying dominance and strict independence with respect to a critical linear order $\leq$ must reflect the truth assignment encoded by $\leq$. To this end, we add $\{x_i^{\mathrm{f}}\}$, $\{x_i^{\mathrm{f}}, x_i^{\mathrm{t}}\}$ and $\{x_i^{\mathrm{t}}\}$ for all $x_i^{\mathrm{f}}, x_i^{\mathrm{t}} \in V$. Then, for every linear order $\leq$ we have $\{x_i^{\mathrm{f}}\} \prec_\infty \{x_i^{\mathrm{t}}\}$ if $x_i^{\mathrm{f}} < x_i^{\mathrm{t}}$ and, on the other hand, $\{x_i^{\mathrm{t}}\} \prec_\infty \{x_i^{\mathrm{f}}\}$ if $x_i^{\mathrm{t}} < x_i^{\mathrm{f}}$

Next, we add sets such that $\{y_i^{\mathrm{t}}\} \prec_\infty \{y_i^{\mathrm{f}}\}$ holds for a critical order $\leq$ if the assignment encoded by $\leq$ does not satisfy disjunct C_i. Essentially, we add for every variable X_i that appears positive in disjunct C_j a collection of sets that imply $\{y_i^{\mathrm{t}}\} \prec_\infty \{y_i^{\mathrm{f}}\}$ if $\{x_i^{\mathrm{t}}\} \prec_\infty \{x_i^{\mathrm{f}}\}$ holds. Similarly, we add a collection of sets that imply $\{y_i^{\mathrm{t}}\} \prec_\infty \{y_i^{\mathrm{f}}\}$ if $\{x_i^{\mathrm{f}}\} \prec_\infty \{x_i^{\mathrm{t}}\}$ holds for every variable that appears negatively in C_j. Now, let $C_j = X_{i_1} \wedge X_{i_2} \wedge X_{i_3}$ be a disjunct. Then, we add sets

$$\{y_j^{\mathrm{t}}\}, \{y_j^{\mathrm{t}}, d_j^k\}, \{y_j^{\mathrm{t}}, d_j^k, x_{i_k}^{\mathrm{f}}\}, \{d_j^k, x_{i_k}^{\mathrm{f}}\}$$

for all $k \in \{1, 2, 3\}$ as well as

$$\{x_{i_k}^{\mathrm{t}}, e_j^k\}, \{x_{i_k}^{\mathrm{t}}, e_j^k, z_1\}, \{x_{i_k}^{\mathrm{t}}, e_j^k, z_1, z_2\},$$
$$\{e_j^k, z_1, z_2\}, \{e_j^k, z_1, z_2, y_f^j\}, \{z_1, z_2, y_f^j\}, \{z_2, y_{\mathrm{fj}}\}, \{y_{\mathrm{fj}}\}.$$

If any of the variables occurs negatively in C_j, we switch $x_{i_k}^{\mathrm{f}}$ and $x_{i_k}^{\mathrm{t}}$ in the construction. We claim that these sets ensure that $\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$ holds for any critical linear order whenever at least one literal in C_j is false. We have

$$\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{t}}, d_j^k\} \prec_\infty \{y_j^{\mathrm{t}}, d_j^k, x_{i_k}^{\mathrm{f}}\} \prec_\infty \{d_j^k, x_{i_k}^{\mathrm{f}}\} \prec_\infty \{x_{i_k}^{\mathrm{f}}\}$$

by dominance and, hence, by transitivity $\{y_j^{\mathrm{t}}\} \prec_\infty \{x_{i_k}^{\mathrm{f}}\}$. Similarly, we have $\{x_{i_k}^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$. Hence, $\{x_{i_k}^{\mathrm{f}}\} \prec_\infty \{x_{i_k}^{\mathrm{t}}\}$ implies $\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$ by transitivity.

Now, we add sets that lead to a contradiction if $\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$ holds for all j for a critical linear order $\leq$, i.e., if the assignment encoded by $\leq$ does not satisfy any disjunct. Roughly, we build a sequence of sets $A_1, \overline{A_1}, A_2, \overline{A_2}, \ldots, A_m, \overline{A_m}$ such that

1. $\{u, v\} \prec_\infty A_1$,
2. $A_j \prec_\infty \overline{A_j}$ if $\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$ for all $j \leq m$,
3. $\overline{A_j} \prec_\infty A_{j+1}$ for all $j < m$.
4. $\overline{A_m} \prec_\infty \{u, v\}$,

Then, if $\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$ holds for all j we have $\{u, v\} \prec_\infty \{u, v\}$ and hence no partial order on $\mathcal{X}$ can satisfy dominance and strict independence with respect to $\leq$. For every $j \leq m$ we have:

- $A_j = \{u, c_1, \ldots, c_j, y_j^{\mathrm{t}}, y_{j-1}^{\mathrm{f}}, y_{j-2}^{\mathrm{f}}, \ldots, y_1^{\mathrm{f}}, v\}$
- $A_j^* = \{u, c_1, \ldots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, y_{j-2}^{\mathrm{f}}, \ldots, y_1^{\mathrm{f}}, v\}$

First we add

$$\{u\}, \{u, c_1\}, \{u, c_1, y_1^{\mathrm{t}}\}, \{u, c_1, y_1^{\mathrm{t}}, v\}, \{u, v\}.$$

Then, we know for any critical linear order that

$$\{u\} \prec_\infty \{u, c_1\} \prec_\infty \{u, c_1, y_1^{\mathrm{t}}\}$$

holds by dominance and therefore we have $\{u, v\} \prec_\infty \{u, c_1, y_1^{\mathrm{t}}, v\}$. This is the desired property (1) i.e., $\{u, v\} \prec_\infty A_1$. Now, we add for every disjunct $\{c_j, y_j^{\mathrm{t}}\}$ and $\{c_j, y_j^{\mathrm{f}}\}$. Then, we add new sets that are constructed by incrementally adding to both sets, one by one, first all elements c_{j-1} to c_1, then all elements y_{j-1}^{f} to y_1^{f} and finally u and v in that order. In other words we add

$$\{c_{j-1}, c_j, y_j^{\mathrm{t}}\} \text{ and } \{c_{j-1}, c_j, y_j^{\mathrm{f}}\},\\ \{c_{j-2}, c_{j-1}, c_j, y_j^{\mathrm{t}}\} \text{ and } \{c_{j-2}, c_{j-1}, c_j, y_j^{\mathrm{f}}\}, \dots,\\ \{c_1, \dots, c_j, y_j^{\mathrm{t}}\} \text{ and } \{c_1, \dots, c_j, y_j^{\mathrm{f}}\}$$

as well as

$$\{c_1, \dots, c_j, y_j^{\mathrm{t}}, y_{j-1}^{\mathrm{f}}\} \text{ and } \{c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}\}, \dots,\\ \{c_1, \dots, c_j, y_j^{\mathrm{t}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}\} \text{ and } \{c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}\}$$

and finally

$$\{u, c_1, \dots, c_j, y_j^{\mathrm{t}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}\} \text{ and } \{u, c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}\},$$

as well as

$$\{u, c_1, \dots, c_j, y_j^{\mathrm{t}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\} \text{ and } \{u, c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\}.$$

By construction

$$\{u, c_1, \dots, c_j, y_j^{\mathrm{t}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\} \prec_\infty \{u, c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\}$$

holds for the minimal partial order satisfying dominance and strict independence for any linear order on V if and only if $\{y_j^{\mathrm{t}}\} \prec_\infty \{y_j^{\mathrm{f}}\}$ holds for that partial order. This is the desired property (2) of the sequence.

Next we add $\{u, c_1, \dots, c_j\}$, $\{u, c_1, \dots, c_{j+1}\}$ and $\{u, c_1, \dots, c_{j+1}, y_{j+1}^{\mathrm{t}}\}$. Then, we add new sets derived as above by adding to both sets first all elements y_j^{f} to y_1^{f} and then v, one by one, in that order until we reach

$$\{u, c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\} \text{ and } \{u, c_1, \dots, c_{j+1}, y_{j+1}^{\mathrm{t}}, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\}.$$

Then, the desired property (3)

$$\{u, c_1, \dots, c_j, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\} \prec_\infty \{u, c_1, \dots, c_{j+1}, y_{j+1}^{\mathrm{t}}, y_j^{\mathrm{f}}, y_{j-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\}$$

holds for the critical linear order by strict independence because

$$\{u, c_1, \dots, c_j\} \prec \{u, c_1, \dots, c_{j+1}\} \prec \{u, c_1, \dots, c_{j+1}, y_{j+1}^{\mathrm{t}}\}$$

holds by dominance. Finally, we add $\{v\}$ and then $\{y_1^{\mathrm{f}}, v\}$, $\{y_2^{\mathrm{f}}, y_1^{\mathrm{f}}, v\}$ and so on till we reach

$$\{c_1, \dots, c_m, y_m^{\mathrm{f}}, y_{m-1}^{\mathrm{f}}, \dots, y_1^{\mathrm{f}}, v\}.$$

This forces for any critical linear order the desired property (4):

$$\{u, c_1, \ldots, c_m, y^{\mathrm{f}}_m, y^{\mathrm{f}}_{m-1}, \ldots, y^{\mathrm{f}}_1, v\} \prec_\infty \{u, v\}.$$

Now, by construction and transitive we have for any critical linear order

$$\begin{aligned}\{u, v\} \prec_\infty \{u, c_1, y^{\mathrm{t}}_1, v\} \prec_\infty \\ \{u, c_1, y^{\mathrm{f}}_1, v\} \prec_\infty \{u, c_1, c_2, y^{\mathrm{t}}_2, y^{\mathrm{f}}_1, v\} \prec_\infty \ldots \\ \prec_\infty \{u, c_1, \ldots, c_m, y^{\mathrm{f}}_m, y^{\mathrm{f}}_{m-1}, \ldots, y^{\mathrm{f}}_1, v\} \prec_\infty \{u, v\}\end{aligned}$$

if (and only if) $\{y^{\mathrm{t}}_j\} \prec_\infty \{y^{\mathrm{f}}_j\}$ holds for all disjuncts, i.e. if the critical linear order encodes an unsatisfying assignment. It follows that if ϕ is not a tautology, then $(X, \mathcal{X})$ is not strongly partial DI^S-orderable.

It remains to show that $(X, \mathcal{X})$ is strongly partial DI^S-orderable if ϕ is a tautology. Let $\leq$ be a linear order on X. As in the proof of Proposition 4.2 we can assume, by Lemma 3.33, that $z_1 < z_2$. We construct a partial order $\preceq$ that satisfies dominance and strict independence with respect to $\leq$. To avoid complicated case distinctions, we will describe the construction only for disjuncts with all positive variables. The only change in construction required for negative variables is switching x^{f}_i and x^{t}_i.

First we add the forced preferences between $\{x^{\mathrm{f}}_i\}, \{x^{\mathrm{f}}_i, x^{\mathrm{t}}_i\}$ and $\{x^{\mathrm{t}}_i\}$, i.e. $\{x^{\mathrm{f}}_i\} \prec \{x^{\mathrm{f}}_i, x^{\mathrm{t}}_i\} \prec \{x^{\mathrm{t}}_i\}$ if $x^{\mathrm{f}}_i < x^{\mathrm{t}}_i$ holds and $\{x^{\mathrm{t}}_i\} \prec \{x^{\mathrm{f}}_i, x^{\mathrm{t}}_i\} \prec \{x^{\mathrm{f}}_i\}$ if $x^{\mathrm{t}}_i < x^{\mathrm{f}}_i$ holds. Next, we consider the sets containing an element d^k_j. We add all preferences that are implied by dominance between the following sets:

$$\{y^{\mathrm{t}}_j\}, \{y^{\mathrm{t}}_j, d^k_j\}, \{y^{\mathrm{t}}_j, d^k_j, x^{\mathrm{f}}_{i_k}\}, \{d^k_j, x^{\mathrm{f}}_{i_k}\}, \{x^{\mathrm{f}}_{i_k}\}$$

Then we close under transitivity. We recall that applying dominance and transitivity can never lead to a contradiction. Furthermore, we claim that $\prec$ restricted to these sets already satisfies strict independence: The only possible application of strict independence on these sets is that any preference between $\{y^{\mathrm{t}}_j\}$ and $\{x^{\mathrm{f}}_{i_k}\}$ has to be lifted to $\{y^{\mathrm{t}}_j, d^k_j\}$ and $\{d^k_j, x^{\mathrm{f}}_{i_k}\}$. By construction however, there can only be a preference between $\{y^{\mathrm{t}}_j\}$ and $\{x^{\mathrm{f}}_{i_k}\}$ forced by dominance and transitivity if the same preference holds between $\{y^{\mathrm{t}}_j, d^k_j\}$ and $\{d^k_j, x^{\mathrm{f}}_{i_k}\}$ as for $\{y^{\mathrm{t}}_j\}$ dominance can only force a relation to $\{y^{\mathrm{t}}_j, d^k_j\}$ and for $\{x^{\mathrm{f}}_{i_k}\}$ it can only force a relation to $\{d^k_j, x^{\mathrm{f}}_{i_k}\}$. Moreover, because we assume that no variable occurs twice in a disjunct, a preference between $\{y^{\mathrm{t}}_j\}$ and $\{x^{\mathrm{f}}_{i_k}\}$ can not later be introduced through sets containing another $d^{k'}_{j'}$. Indeed the only preferences we need to add for sets containing different elements $d^{k_1}_{j_1}$ and $d^{k_2}_{j_2}$, in order to satisfy dominance and transitivity is $\{x^{\mathrm{f}}_i, d^{k_1}_{j_1}\} \prec \{x^{\mathrm{f}}_i, d^{k_2}_{j_2}\}$ for all x^{f}_i and all $d^{k_1}_{j_1}, d^{k_2}_{j_2}$ such that $d^{k_1}_{j_1} < d^{k_2}_{j_2}$ holds.

Using a similar construction, we can order all sets containing an element e^k_j if we replace x^{f}_i by x^{t}_i and y^{t}_j by $\{z_1, z_2, y^{\mathrm{f}}_j\}$. Finally, we add the enforced preference between $\{z_2, y^{\mathrm{f}}_j\}$ and $\{y^{\mathrm{f}}_j\}$ as well as $\{z_1, z_2, y^{\mathrm{f}}_j\} \prec \{z_2, y^{\mathrm{f}}_j\}$. The later is enforced by dominance as we assume $z_1 < z_2$. Then, we close everything under transitivity. By construction, this does not produce any new instances of strict independence.

We now consider the sets containing an element c_i for some i. We observe that $\{z_1, z_2, y_j^{\mathrm{f}}\} \prec \{z_2, y_j^{\mathrm{f}}\}$ implies that $\{y_j^{\mathrm{t}}\} \prec \{y_j^{\mathrm{f}}\}$ can only hold if $\{x_i^{\mathrm{f}}\} \prec \{x_i^{\mathrm{t}}\}$ holds for a variable occurring in disjunct C_j i.e., if disjunct C_j is not satisfied and that $\{y_j^{\mathrm{f}}\} \prec_\infty \{y_j^{\mathrm{t}}\}$ never holds. As ϕ is a tautology, there is disjunct C_l that is satisfied by the assignment encoded by $\leq$. Hence, $\{y_l^{\mathrm{t}}\}$ and $\{y_l^{\mathrm{f}}\}$ are incomparable. We partition the sets containing an element c_i in partitions $P_1, \ldots, P_m$ based on the largest i for which they contain c_i. We set $S_1 \prec S_2$ if $S_1 \in P_{i_1}$, $S_2 \in P_{i_2}$ and one of the following holds:

- $c_{i_1} < c_{i_2}$ and $i_1, i_2 < l$
- $c_{i_1} < c_{i_2}$ and $l < i_1, i_2$

Any set that contains c_i also contains y_i except $\{u, c_1, \ldots, c_i\}$. Hence, all sets from different partitions differ by at least two elements except $\{u, c_1, \ldots, c_i\}$ and $\{u, c_1, \ldots, c_{i+1}\}$. If dominance forces a preference between these sets, it is satisfied by construction for $i, i+1 \neq l$. Now, for any set in any partition P_i such that $i \neq l$ we set $S \prec S'$ if $y_i^{\mathrm{t}} \in S$ and $y_i^{\mathrm{t}} \notin S'$. This covers all applications of strict independence in a partition. Finally, we add all preferences that are forced by dominance in a partition and close under transitivity. We observe that $S, S \cup \{x\} \in P_i$ implies either $y_i^{\mathrm{t}} \in S, S \cup \{x\}$ or $y_i^{\mathrm{t}} \notin S, S \cup \{x\}$, hence this can not lead to a contradiction. Now, for a set S in P_l such that $y_l^{\mathrm{t}} \in S$ we set

- $S' \prec S$ if $S' \in P_i$ for $i < l$ and $c_i < c_l$
- $S \prec S'$ if $S' \in P_i$ for $i < l$ and $c_l < c_i$

Furthermore, for a set S in P_l such that $y_l^{\mathrm{t}} \notin S$ we set

- $S' \prec S$ if $S' \in P_i$ for $l < i$ and $c_i < c_l$
- $S \prec S'$ if $S' \in P_i$ for $l < i$ and $c_l < c_i$

And finally, we add again all preferences forced by dominance and close by transitivity. As $\{y_l^{\mathrm{t}}\}$ and $\{y_l^{\mathrm{f}}\}$ are incomparable in $\preceq$ this order is consistent. Furthermore, we did not add any preferences between any sets not containing c_l and sets containing c_{l+1}. Therefore, $\{u, c_1, y_1^{\mathrm{t}}, z_1, z_2, v\}$ and $\{u, c_1, \ldots, c_m, y_m^{\mathrm{f}}, \ldots, y_1^{\mathrm{f}}, v\}$ are incomparable in $\preceq$. This allows us to add any preferences forced by dominance and strict independence regarding $\{u\}, \{v\}$ and $\{u, v\}$ without creating a contradiction. By construction, $\preceq$ is now a partial order that satisfies dominance and strict independence. $\square$

Corollary 4.28. Strong DI^SE-PO-Orderability *is* coNP*-complete.*

Proof. Strong DI^SE-PO-Orderability is in coNP by the same argument as Strong DI^S-PO-Orderability. We modify the reduction used to prove Theorem 4.27 to show that it is also coNP-hard. All singletons appearing in that reduction are of the form $\{x_i^{\mathrm{t}}\}$, $\{x_i^{\mathrm{f}}\}$, $\{y_i^{\mathrm{t}}\}$, $\{y_i^{\mathrm{f}}\}$ $\{u\}$ or $\{v\}$ for some i. We change the reduction in a way that only singletons of the form $\{u\}$ or $\{v\}$ appear. As before, we can achieve this by doubling the elements of the form x_i^a and y_i^a for $a \in \{\mathrm{t}, \mathrm{f}\}$, that means we replace x_i^a by two elements $x_i^{a,1}$ and $x_i^{a,2}$, and replace y_i^a by $y_i^{a,1}$ and $y_i^{a,2}$. As in the proof of Corollary 4.9, we add the sets $\{x_i^{\mathrm{t},1}, x_i^{\mathrm{t},2}, x_i^{\mathrm{f},1}\}$, $\{x_i^{\mathrm{t},2}, x_i^{\mathrm{f},1}\}$ and $\{x_i^{\mathrm{t},2}, x_i^{\mathrm{f},1}, x_i^{\mathrm{f},2}\}$. Then $x_i^{\mathrm{t},1} < x_i^{\mathrm{t},2} < x_i^{\mathrm{f},1} < x_i^{\mathrm{f},2}$ implies $\{x_i^{\mathrm{t},1}, x_i^{\mathrm{t},2}\} \prec \{x_i^{\mathrm{f},1}, x_i^{\mathrm{f},2}\}$. and $x_i^{\mathrm{t},1} > x_i^{\mathrm{t},2} > x_i^{\mathrm{f},1} > x_i^{\mathrm{f},2}$ implies $\{x_i^{\mathrm{t},1}, x_i^{\mathrm{t},2}\} \succ \{x_i^{\mathrm{f},1}, x_i^{\mathrm{f},2}\}$. Furthermore, we add $\{y_j^{\mathrm{t},2}, d_j^k\}$, $\{d_j^k, x_i^{\mathrm{t},1}\}$, $\{x_i^{\mathrm{f},2}, e_j^k\}$ and $\{z_2, y_j^{\mathrm{t},1}\}$ if variable X_i appears positively in disjunct j. If X_i appears negatively, we switch t and f. Then, $\{x_i^{\mathrm{t},1}, x_i^{\mathrm{t},2}\} \prec \{x_i^{\mathrm{f},1}, x_i^{\mathrm{f},2}\}$ implies $\{y_j^{\mathrm{t},1}, y_j^{\mathrm{t},2}\} \prec \{y_j^{\mathrm{f},1}, x_i^{\mathrm{f},2}\}$ as desired.

The second part of the construction must be modified by always adding first $y_j^{\mathrm{t},1}$ and then $y_j^{\mathrm{t},2}$ instead of y_j^{t} and similarly $y_j^{\mathrm{f},1}$ and then $y_j^{\mathrm{f},2}$ instead of y_j^{f}. It can be checked that this suffices to force $\{u, v\} \prec \{u, v\}$ whenever no disjunct is satisfied.

Now, constructing a partial order that satisfies dominance and strict independence works as before. Furthermore, $\{u\}$ and $\{v\}$ are the only singletons in the family $\mathcal{X}$. As $\{u, v\} \in \mathcal{X}$, any partial order that satisfies dominance also satisfies the extension rule. □

4.3 Succinct domain restrictions

The results in the previous sections assume that the family of sets is given explicitly. However, in many applications, the family of sets is instead only given implicitly, via some condition that has to be satisfied by the sets in order to be admissible. For example, the domain in combinatorial voting is often given this way. Such conditions can for example be formulated as propositional formulas (Lang & Xia, 2016). Therefore, we turn our attention now to succinctly represented families of sets. These succinct restrictions are normally exponentially smaller than the actual family of sets, which can increase the complexity of deciding if the family is orderable. On the other hand, families of sets must have some internal structure to be succinctly represented. This internal structure may decrease the complexity of the problem, as is the case for the domain restrictions considered in the next chapter. We show that representing families by boolean circuits – a specific succinct representation that is well studied in the literature – can lead to a massive blow up in complexity. These results are taken from Maly (2020) where they appeared only with proof sketches.

First, we will quickly review the basic results on succinctly represented problems from the literature and recall the definitions and lemmas we need. The study of succinct problems goes back to Galperin & Wigderson (1983) for graphs that are succinctly represented by a boolean circuit. Later this approach was extended by Balcázar, Lozano & Torán (1992) to arbitrary problems that are succinctly represented by boolean circuits in the following way.

Definition 4.29. We say a boolean circuit C_w with two output gates represents a binary string w if for every input of a binary number i the following holds:

- the first output is 1 if and only if $i \leq |w|$

- if the first output is 1 then the second output equals the i-th bit of w.

The succinct version Q_S of a problem Q is: Given a boolean circuit C_w representing a boolean string w decide whether $w \in Q$.

For example, SUCCINCT SAT – the succinct version of SAT – can be defined as follows:

SUCCINCT SAT
Input: A boolean circuit C_w representing a word w.
Question: Is the 3-CNF represented by w satisfiable?[1]

SUCCINCT SAT is known to be NEXP-complete (Papadimitriou & Yannakakis, 1986; Papadimitriou, 1994). Hence, SUCCINCT TAUT is coNEXP-complete. Succinct versions of the problems considered in this thesis be can defined similarly. The main tool to determine the complexity of succinct problems are so-called Conversion Lemmas. We use the Conversion Lemma by Balcázar et al. (1992). Stronger versions of this lemma exist, for example by Veith (1998). However, the Conversion Lemma of Balcázar et al. (1992) suffices for our purposes and has the advantage that only comparably simple reductions are used, namely ptime reductions and polylog-time reductions. Polylogtime reductions are reductions that – given random access to the input – need only $O(\log^c(n))$-time to output an arbitrary bit of the output. The following definition is taken from Murray & Williams (2017).

Definition 4.30. An algorithm $R : \{0,1\}^* \times \{0,1\}^* \to \{0,1,\star\}$ is a *polylog-time reduction* from L to L' if there are constants $c \geq 1$ and $k \geq 1$ such that for all $x \in \{0,1\}^*$,

- $R(x,i)$ has random access to x
- $R(x,i)$ runs in $O((\log(|x|))^k)$ time for all $i \in \{0,1\}^{\lceil 2c\log(|x|)\rceil}$
- there is an $l_x \leq |x|^c + c$ such that $R(x,i) \in \{0,1\}$ and for all $i \leq l_x$ and $R(x,i) = \star$ for all $i > l_x$.
- $x \in L$ iff $R(x,1) \cdot R(x,2) \cdots R(x,l_x) \in L'$.

Here $\cdot$ is the string concatenation and $\star$ is the out of bounds character that marks the end of a string. Now, we can formulate our Conversion Lemma.

Lemma 4.31 (Conversion Lemma)**.** *Let Q and Q^* be decision problems. If $Q \leq_{polylog} Q^*$ then $Q_S \leq_P Q^*_S$.*

We can use the Conversion Lemma to prove the following theorem.

[1] It is not important what specific encoding is used as long as the number of variables and clauses as well as the i-th variable in the j-th clause can be read in polylog-time. Any reasonable encoding will satisfy this requirement.

Theorem 4.32. Succinct DI^S-LO-Orderability *is* NEXP*-complete.* Succinct strong DI^S-LO-Orderability *is* NEXP*-hard. The same holds if we add the extension rule.*

Proof. Succinct DI^S-LO-Orderability can be solved in NEXP-time by explicitly computing the family $\mathcal{X}$ and then solving the (exponentially larger) explicit problem in NP-time.

For the hardness, we only have to check that the presented reduction is computable in polylog-time. Then, by the Conversion Lemma, there is a ptime reduction from Succinct Sat to the problems mentioned above. The NEXP-hardness of both problems then follows as Succinct Sat is known to be NEXP-complete. We have to show that we can compute a single bit of the output in polylog-time if we have random access to the input. For this, we have to take the binary representation of Sat into account. Unfortunately, neither Papadimitriou & Yannakakis (1986) nor Papadimitriou (1994) specify a binary representation for the NEXP-hardness proof. However, the proof of NEXP-hardness is not sensitive to the representation as long as it is reasonable. The same is true for this proof. Reasonable means in our context that it is possible to determine the number of variables n and clauses m in polylog-time. For any sensible encoding of 3-CNF this is either explicitly encoded or can be determined via binary search. Furthermore, we assume that one only needs polylog-time to read the i-th variable in the j-th clause. This is trivially true if we assume that every clause is encoded by the same amount of bits. It is easy to see that the proof in Papadimitriou (1994) of the NEXP-hardness of Succinct Sat works for such an encoding.

Now, we fix a binary representation for instances of DI^S-LO-orderability resp. Strong DI^S-LO-orderability. First, we encode the number of elements k of X in binary. Then, the family $\mathcal{X}$ is encoded as a series of strings of length k, where a 1 in position l means the l-th element of X is in the set and a 0 in position l means the l-th element is not in the set. For an instance of DI^S-LO-orderability, the linear order $\leq$ is given by the natural order on these positions.

First, observe that the size of X is $4n + 12m + 3$ and the size of $\mathcal{X}$ is $p(n, m)$ for some polynomial $p(x, y)$. Therefore, we can determine it in polylog-time. Now, assume we want to decide whether the i-th bit of the output is 0 or 1. It is clear that this can be done in polylog-time if the i-th bit is part of the representation of the size of X. Assume that the i-th bit determines if the l-th element x is part of a k-th set A. We can assume that we fixed an order in which we generate the sets in $\mathcal{X}$ such that we can compute from m, n and i which set A is supposed to be. Observe that if x is not of the form x_j^+ or x_j^- then, this already suffices to decide whether x is in A. On the other hand, if $x = x_{j,a}^+$ or $x = x_{j,a}^-$ and A is a class 1 set, then it still suffices to know which set A is supposed to be. Finally, if $x = x_{j,a}^+$ or $x = x_{j,a}^-$ and A is not a class 1 set then the question whether x is in A only depends on the question if X_j occurs (positively or negatively) in a specific clause in the right position. □

We observe that the argument above does not use any properties of the reduction that are unique to DI^S-LO-orderability. Therefore, it is straightforward to check that the hardness of the other strong and $\leq$-orderability properties can be lifted in the same way.

Corollary 4.33. SUCCINCT DI-LO-ORDERABILITY *and* SUCCINCT $DI^{(S)}$-WO-ORDERABILITY *are* NEXP*-complete.* SUCCINCT STRONG DI-LO-ORDERABILITY *and* SUCCINCT STRONG $DI^{(S)}$-WO-ORDERABILITY *are* NEXP*-hard. The same holds if we additionally add the extension rule.*

Moreover, we note that the Conversion Lemma can also be applied the same way the reduction from TAUT to STRONG PARTIAL DI^S-ORDERABILITY.

Theorem 4.34. SUCCINCT STRONG DI^S-PO-ORDERABILITY *is* coNEXP*-complete.*

This analysis still leaves some gaps. It can be shown that SUCCINCT STRONG DI^S-WO-ORDERABILITY is in Π_2^E, the second level of the exponential hierarchy, by a similar argument as the one used to show that SUCCINCT DI^S-WO-ORDERABILITY is in NEXP. It seems very likely that this upper bound is tight and that SUCCINCT STRONG DI^S-WO-ORDERABILITY is indeed Π_2^E-complete. However, as the succinct version of Π^2-SAT is, to the best of our knowledge, not known to be Π_2^E-hard, the Conversion Lemma does not suffice to show this. Closing this gap is, therefore, left to future work. The same holds for the other problems regarding strong orderability where the lifted order needs to be a linear or weak order.

On the other hand, we do not provide a lower bound for SUCCINCT $DI^S(E)$-PO-ORDERABILITY, because we do not have a lower bound on the complexity of $DI^S(E)$-PO-ORDERABILITY even in the non-succinct case. This would require a completely different reduction and is also left to further work.

4.4 Weak orderability

In this section, we consider weak orderability. The results in this section are new research and have not appeared in any publication before. We will restriction our attention to weak orderability with respect to dominance and strict independence, resp. with respect to dominance, strict independence and the extension rule. The question if these results also hold for regular independence is left for future work. First, we show that WEAK DI^S-WO-ORDERABILITY is NP-complete. This requires a completely different construction. We reduce this time from BETWEENNESS that was shown to be NP-hard by Opatrny (1979).

BETWEENNESS
Input: A set $V = \{v_1, v_2, \ldots, v_n\}$ and a set of triples $R \subseteq V^3$.
Question: Does there exist a linear order on V such that $a < b < c$ or $a > b > c$ holds for all $(a, b, c) \in R$?

The idea of the following reduction is as follows: From a Betweenness instance (V, R) we construct an instance $(X, \mathcal{X})$ of Weak DI^S-WO-Orderability. The set X is constructed by adding some auxiliary variables to V. Then we use these auxiliary variables to build for every triple (a, b, c) in R a collection of sets that are not $\leq$-DI^S-orderable for any linear order on X that violates the betweenness condition for a, b, c, i.e., if $b < a, c$ or $a, c < b$ holds. The union of these collections for every triple in R will be the family $\mathcal{X}$. Then we have to show that $\mathcal{X}$ is $\leq$-DI^S-orderable if $a < b < c$ or $a > b > c$ holds for all triple (a, b, c) in R.

Theorem 4.35. Weak DI^S-WO-Orderability *is* NP-*complete.*

Proof. It is clear that Weak DI^S-WO-Orderability is in NP because we can guess a linear order $\leq$ on X and a linear order $\preceq$ on $\mathcal{X}$ at the same time and then check in polynomial time if $\preceq$ satisfies dominance and strict independence with respect to $\leq$.

It remains to show that it is also NP-hard. We do this by a reduction from Betweenness. For the reduction we need two gadgets. First, we need a gadget that guarantees for three elements a, b, c that either $a < b, c$ or $a > b, c$ has to hold. The second gadget, for elements x, x', y, y', leads to a contradiction if $x, x' < y, y'$ or $y, y' < x, x'$ holds.

The gadget $A(x, y, z)$: Let $X = \{x, y, z\}$. Then, we write $A(x, y, z) := \mathcal{P}(X) \setminus \{x\}$. We claim that $A(x, y, z)$ is not DI^S-orderable with respect to $\leq$ for $y < x < z$ or $z < x < y$. We assume $y < x < z$. The other case can be treated analogously. Assume for the sake of contradiction that there is an order $\preceq$ on $A(x, y, z)$ that satisfies dominance and strict independence. Then, $\{y\} \prec \{y, x\}$ by dominance and hence $\{y, z\} \prec \{y, x, z\}$ by strict independence. On the other hand, $\{x, z\} \prec \{z\}$ by dominance and hence $\{y, x, z\} \prec \{y, z\}$ by strict independence. A contradiction.

The gadget $B(x, x', y, y')$: We write $B(x, x', y, y')$ for the set

$$\{\{x, y, y'\}, \{x', y, y'\}, \{x, x'\}, \{x, x', y\}, \{x, x', y'\}, \{x, x', y, y'\}, \{y\}, \{y'\}, \{y, y'\}\}.$$

We claim that $B(x, x', y, y')$ is not DI^S-orderable with respect to $\leq$ if $x, x' < y, y'$ or $y, y' < x, x'$. We assume $x, x' < y < y'$. The other cases follow by symmetry. Then, $\{x, x'\} \prec \{x, x', y\}$ by dominance and hence $\{x, x', y'\} \prec \{x, x', y, y'\}$ by strict independence. On the other hand $\{y, y'\} \prec \{y'\}$ by dominance and hence by strict independence first $\{x', y, y'\} \prec \{x', y'\}$ and second $\{x, x', y, y'\} \prec \{x, x', y'\}$, a contradiction.

The reduction: Given an instance of Betweenness

$$(V = \{v_1, \ldots, v_n\}, T = \{(v_i, v_j, v_k), \ldots, (v_{i'}, v_{j'}, v_{k'})\})$$

we build an instance $(X, \mathcal{X})$ of Weak DI^S-WO-Orderability. First, we add for every $v_i \in V$ an element v_i to X. Furthermore, we add for every triple (v_i, v_j, v_k) new elements

$y_{ijk}, y'_{ijk}, z_{ijk}$ and z'_{ijk}. If no ambiguity arises, we omit the index ijk. Finally, for every triple (v_i, v_j, v_k) we add sets

$$A(v_i, y_{ijk}, y'_{ijk}), A(v_j, y_{ijk}, y'_{ijk}), A(v_i, z_{ijk}, z'_{ijk}), A(v_j, z_{ijk}, z'_{ijk}), \\ A(v_k, z_{ijk}, z'_{ijk}), A(v_i, y_{ijk}, z_{ijk}), A(v_k, y_{ijk}, z_{ijk}), B(v_i, v_j, y_{ijk}, y'_{ijk}), \\ B(v_i, v_k, z_{ijk}, z'_{ijk}) \text{ and } B(v_j, v_k, z_{ijk}, z'_{ijk})$$

to $\mathcal{X}$. We claim that $\mathcal{X}$ is DI^S-orderable with respect to $\leq$ if and only if the projection of $\leq$ to V is a positive solution to the given Betweenness instance. The idea is, roughly, that none of these gadgets leads to a contradiction if we set either

$$v_i < y < y' < v_j < z < z' < v_k$$

or

$$v_i > y > y' > v_j > z > z' > v_k,$$

hence there is a way to avoid a contradiction if either $v_i < v_j < v_k$ or $v_i > v_j > v_k$ holds. On the other hand, we can show that there is no way to extend an order that sets either $v_j < v_i, v_k$ or $v_i, v_k < v_j$ without running into a contradiction.

First, we show that $\mathcal{X}$ is not DI^S-orderable with respect to $\leq$ if the projection of $\leq$ is not a positive instance of Betweenness. Let (v_i, v_j, v_k) be a triple that is violated by $\leq$, i.e. either $v_j < v_i, v_k$ or $v_i, v_k < v_j$. We assume $v_j < v_i, v_k$. The other case can be treated analogously. Assume for the sake of contradiction that $\mathcal{X}$ is DI^S-orderable with respect to $\leq$. Then observe that $A(v_i, y, y')$ and $A(v_j, y, y')$ imply that v_i and v_j can not lie between y and y' in $\leq$. Therefore, we must have either $v_i, v_j < y, y'$, $y, y' < v_i, v_j$, $v_i, < y, y' < v_j$ or $v_j, < y, y' < v_i$. However, the first two cases are ruled out by $B(v_i, v_j, y, y')$ and the third case is ruled out by $v_j < v_i$, hence we know $v_j < y, y' < v_i$. Similarly, $A(v_j, z, z'), A(v_k, z, z')$ and $B(v_j, v_k, z, z')$ imply $v_j < z, z' < v_k$. By $A(v_i, y, z)$ and $A(v_k, y, z)$ we know that v_i and v_k can not lie between y and z, hence we must have $v_j < y, z < v_i, v_k$. Now, $A(v_j, z, z'), A(v_k, z, z')$ imply that neither v_i nor v_k can lie between z and z'. Hence we must have $v_j < y, z, z' < v_i, v_k$. However, this is ruled out by $B(v_i, v_k, z, z')$. A contradiction.

Now, assume $\leq$ is a positive instance of Betweenness. We extend $\leq$ to an order on $\mathcal{X}$ by setting for all triples (v_i, v_j, v_k) the order $v_i < y < y' < v_j < z < z' < v_k$ if $v_i < v_j < v_k$ and $v_i > y > y' > v_j > z > z' > v_k$ otherwise. We can do this in a way such that there is no element between y and y' and z and z'. Now, we can order the sets made up by the new elements y, y', z, z' with an order $\preceq$ satisfying dominance and strict independence: We lift $\leq$ to the singletons and place the sets of the form $\{y, y'\}$ between $\{y\}$ and $\{y'\}$ and sets of the form $\{z, z'\}$ between $\{z\}$ and $\{z'\}$. Finally we place sets of the form $\{y', z\}$ right after $\{y'\}$.

For every triple (v_i, v_j, v_k) with auxiliary elements y, y', z, z' such that $v_i < v_j < v_k$ holds, we add the sets

$$\{v_i, y\}, \{v_i, y, y'\}, \{v_i, y'\}, \{v_i, y, v_j\}, \{v_i, y, y', v_j\}, \{v_i, v_j\}, \{v_i, y', v_j\}$$

in this order just below $\{y\}$. For every of these sets A, we have $\min(A) = v_i$. Furthermore, for all $A \prec B$ we have $\max(A) \leq \max(B)$. Hence, this sequence satisfies dominance. Next we add

$$\{y, v_j\}, \{y, y', v_j\}, \{y', v_j\}, \{v_i, y', z\}, \{y', z\}, \{y', z, v_k\}$$

in this order above $\{y'\}$. It can be checked that this also satisfies dominance. Furthermore, we add

$$\{v_i, z\}, \{v_i, z, z'\}, \{v_i, z'\}, \{v_i, z, v_k\}, \{v_i, z, z', v_k\},$$
$$\{v_i, v_k\}, \{v_i, z', v_k\}, \{v_j, z\}, \{v_j, z, z'\}, \{v_j, z'\},$$
$$\{v_j, z, v_k\}, \{v_j, z, z', v_k\}, \{v_j, v_k\}, \{v_j, z', v_k\}$$

in this order just below $\{z\}$. For the first half, we have again $\min(A) = v_i$ and $A \prec B$ implies $\max(A) \leq \max(B)$. For the second half, the same holds with $\min(A) = v_j$. Therefore, this block satisfies dominance. Furthermore, we have $\max(A) \geq z$ and in the earlier blocks, the only set B with $\max(B) > z$ is $\{y', z, v_k\}$, for which dominance does not imply any preferences with sets in this block. Hence, dominance is also satisfied with in relation to the earlier blocks. Finally, we have $\min(A) \leq z < \max(A)$ for all sets in the block. Therefore, we can place $\{z\}$ above the block without violating dominance. We conclude the construction by placing the sets

$$\{z, v_k\}, \{z, z', v_k\}, \{z', v_k\}$$

in this order above $\{z'\}$. Again, this does not contradict dominance. If $v_i > v_j > v_k$ holds, we produce exactly the reverse order. In order to see that it satisfies strict independence, we have to distinguish two cases: Let A and B be sets in $\mathcal{X}$ such that $A \cup \{x\}$ and $B \cup \{x\}$ are also in $\mathcal{X}$ and $A \prec B$. First, assume $x = y_{ijk}$ for some triple (v_i, v_j, v_k). First, assume $A \cap \{v_i, v_j, v_k\} = B \cap \{v_i, v_j, v_k\}$. Then, we must have $A = \{v_i, v_j\}$ and $B = \{v_i, y', v_j\}$ and hence $A \cup \{x\} \prec B \cup \{x\}$ by definition. Now, assume $A \cap \{v_i, v_j, v_k\} \neq B \cap \{v_i, v_j, v_k\}$. Then, the order of A and B as well as the order of $A \cup \{x\}$ and $B \cup \{x\}$ does not depend on the auxiliary variables. Hence $A \prec B$ implies $A \cup \{x\} \prec B \cup \{x\}$. The cases $x \in \{y', x, x'\}$ are similar.

So assume $x = v_i$. Then, observe that if $A \prec B$, because B contains v_j or v_k and A doesn't, then $A \cup \{x\} \prec B \cup \{x\}$ for the same reason. Otherwise observe that the order of A and B can only depend on the auxiliary elements. However, by assumption, these do not change when we add x, hence $A \cup \{x\} \prec B \cup \{x\}$. □

A close inspection of this proof shows that (1) we did not use the fact that the lifted order needs to be total and (2) that the lifted order satisfies extension. Hence, the NP-hardness carries over to partial weak DI^S-orderability and if we add the extension rule.

Corollary 4.36. *Weak DI^SE-WO-Orderability, Weak DI^S-PO-Orderability and Weak DI^SE-PO-Orderability are NP-complete.*

Proof. The fact that all three problems are in NP follows by the same argument used to show that WEAK DI^S-WO-ORDERABILITY is in NP. Now, we claim that the same reduction used above shows without modification that all three problems are NP-hard. First of all, the argument that the constructed instance $(X, \mathcal{X})$ is not weakly DI^S-orderable if (V, R) is not a positive instance of BETWEENNESS does not rely on the fact that $\leq$ needs to be total.[2] Furthermore, when constructing the order witnessing that $(X, \mathcal{X})$ is weakly DI^S-orderable if (V, R) is a positive instance of BETWEENNESS, "we lift $\leq$ to the singletons". Therefore, this order satisfies the extension rule by definition. □

4.5 Strengthenings of dominance

A possible way to overcome the high complexity of recognizing orderable families could be to strengthen the axioms that we consider. If we consider very strong axioms then only very particular families of sets will be orderable with respect to these axioms. Strict independence is already a very strong axiom, therefore, we focus on strengthenings of dominance. Furthermore, we will only study the simplest decision problem, namely $\leq$-LO-orderability. More concretely, we will show that no reasonable strengthening of dominance together with strict independence makes the $\leq$-LO-orderability problem easier. Technically, we say an axiom is dominance-like if it extends dominance and is implied by a very strong axiom that we call maximal dominance. Then, we show that the $\leq$-LO-orderability problem with respect to strict independence and any dominance-like axiom is NP-complete. This result is based on work by Maly & Woltran (2017b) but was not published yet in this strong formulation. First we introduce maximal dominance.

Axiom 4.37 (Maximal Dominance)**.** For all $A, B \in \mathcal{X}$,

$$(\max(A) \leq \max(B) \wedge \min(A) < \min(B)) \text{ implies } A \prec B;$$

$$(\max(A) < \max(B) \wedge \min(A) \leq \min(B)) \text{ implies } A \prec B.$$

Now, we can define dominance-like axioms. As the name suggests, these are axioms that lie between dominance and maximal dominance.

Definition 4.38. We say a axiom is *dominance-like* if it implies dominance and is implied by maximal dominance.

One example of a dominance-like axiom is set-dominance. Furthermore, if we consider Fishburn's and Gärdenfors' extensions from Example 3.4 as axioms, they are both dominance-like.

We prove the NP-hardness of $\leq$-orderability by a reduction from BETWEENNESS, as in the case of weak orderability, However, this time the BETWEENNESS instance is not encoded in the linear order on X but in the linear order on $\mathcal{X}$.

[2]Observe that previous reductions also did not explicitly mention the totality of the lifted order, but used the fact that (strict) independence implies reverse independence, which only holds for total orders.

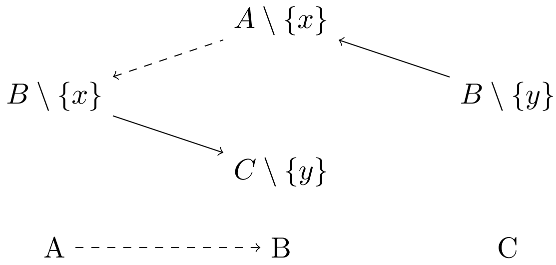

Figure 4.3: Family that forces that $A \prec B$ leads to $B \prec C$

Theorem 4.39. *Let A be a dominance-like axiom. Then it is* NP*-hard to decide for a given triple $(X, \mathcal{X}, \leq)$ if there exists a linear order on $\mathcal{X}$ that satisfies axiom A and strict independence.*

Proof. Let (V, R) be an instance of Betweenness with $V = \{v_1, v_2, \ldots, v_n\}$. We construct a triple $(X, \mathcal{X}, <)$ such that

- if (V, R) is a positive instance of Betweenness, then there is a linear order $\preceq$ on $\mathcal{X}$ that satisfies maximal dominance and strict independence w.r.t. $\leq$,
- if (V, R) is a negative instance of Betweenness, then there exists no linear order $\preceq$ on $\mathcal{X}$ that satisfies dominance and strict independence w.r.t. $\leq$.

Then, for any dominance-like axiom A we know that there exists a linear order on $\mathcal{X}$ that satisfies axiom A and strict independence w.r.t. $\leq$ if and only if (V, R) is a positive instance of Betweenness.

We set $X = \{1, 2, \ldots, N\}$ equipped with the usual linear order, for N large enough. We will clarify later what large enough means. Then, we construct the family $\mathcal{X}$ stepwise. The family contains for every $v_i \in V$ a set V_i of the following form:

$$V_i := \{C + i, \ldots, N - (C + i)\}.$$

Here, C is a large enough number. Again we will clarify later what large enough means.

Furthermore, for every triple from R we want to enforce $A \prec B \prec C$ or $A \succ B \succ C$ by adding two families of sets as shown in Figure 4.3 and Figure 4.4 with $q, x, y, z \in X$. The solid arrows represent preferences that are forced through dominance and strict independence. The family in Figure 4.3 makes sure that every total strict order satisfying independence that contains $A \prec B$ must also contain $B \prec C$. Similarly, the family in Figure 4.4 makes sure that $A \succ B$ leads to $B \succ C$.

We implement this idea for all triples inductively. For every $1 \leq i \leq |R|$, pick a triple $(v_l, v_j, v_m) \in R$ and set $k = C + n + 4i$. Let $(A, B, C) = (V_l, V_j, V_m)$ be the triple of sets

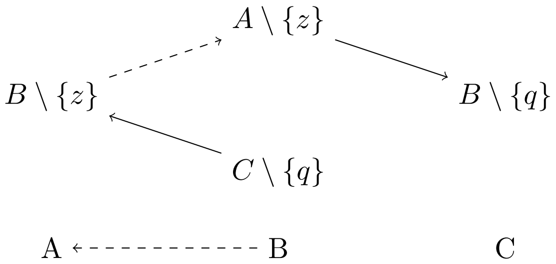

Figure 4.4: Family that forces that $A \succ B$ leads to $B \succ C$

encoding the triple of elements (v_l, v_j, v_m). We add the following sets:

$$A \setminus \{k\}, B \setminus \{k\}, B \setminus \{k+1\}, C \setminus \{k+1\},$$
$$A \setminus \{k+2\}, B \setminus \{k+2\}, B \setminus \{k+3\}, C \setminus \{k+3\}.$$

We call the sets encoding the elements of V together with the sets added in this step the class 1 sets. These sets correspond to the sets $A \setminus \{x\}, B \setminus \{x\}, \ldots, C \setminus \{q\}$ in Figure 4.3 and Figure 4.4. Observe that the inductive construction guarantees that every constructed set is unique. We now have to force the preferences

$$A \setminus \{k\} \prec B \setminus \{k+1\}, C \setminus \{k+1\} \prec B \setminus \{k\},$$
$$B \setminus \{k+3\} \prec A \setminus \{k+2\}, B \setminus \{k+2\} \prec C \setminus \{k+3\}. \quad (\star)$$

We define for every pair $A, B \in \mathcal{X}$ a family of sets $\mathcal{S}(A, B)$ forcing $A \prec B$. Assume $\min(B) \leq \min(A)$ and $\max(A) \leq \max(B)$. Then, $\mathcal{S}(A, B)$ contains the following sets

$$\{x_{AB}\}, \{x_{AB}, y_{AB}\}, \{y_{AB}\}, \{x_{AB}, z_{AB}\}, \{y_{AB}, z^*_{AB}\}, A \cup \{z_{AB}\}, B \cup \{z^*_{AB}\}$$

where $z^*_{AB} < \min(B) < \min(A) < y_{AB} < x_{AB} < \max(A) < \max(B) < z_{AB}$ holds (See Figure 4.5).

Additionally, we add sets that enforce $A \cup \{z_{AB}\} \prec \{x_{AB}, z_{AB}\}$ by dominance: Let $A = \{a_1, \ldots, a_l\}$ be enumerations of A such that $i < j$ implies $a_i < a_j$. We add

$$\{z_{AB}\}, \{a_l, z_{AB}\}, \{a_{l-1}, a_l, z_{AB}\}, \ldots, \{a_2, \ldots, z_{AB}\} \text{ and } \{a_1, z_{AB}\}$$

to $\mathcal{X}$. This forces $\{a_2, \ldots, z_{AB}\} \prec \{z_{AB}\}$ by dominance and hence by one application of strict independence $A \cup \{z_{AB}\} \prec \{a_1, z_{AB}\}$. Finally, we add $\{a_1\}, \{a_1, x_{AB}\}$ and $\{a_1, x_{AB}, z_{AB}\}$, which leads to $\{a_1, z_{AB}\} \prec \{a_1, x_{AB}, z_{AB}\}$. Then we have

$$A \cup \{z_{AB}\} \prec \{a_1, z_{AB}\} \prec \{a_1, x_{AB}, z_{AB}\} \prec \{x_{AB}, z_{AB}\}$$

hence $A \cup \{z_{AB}\} \prec \{x_{AB}, z_{AB}\}$. Therefore, we have $A \preceq \{x_{AB}\}$ by reverse independence. Analogously, we enforce $\{y_{AB}\} \preceq B$. Therefore, transitivity implies $A \prec B$ by $A \preceq$

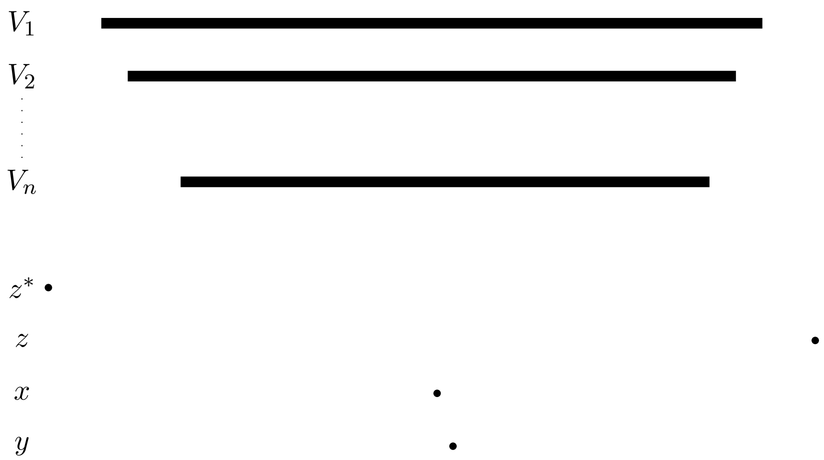

Figure 4.5: Sketch of the sets $V_1, \ldots, V_n, x, y, z$ and z^*.

$\{x_{AB}\} \prec \{y_{AB}\} \preceq B$. The case $\min(A) \leq \min(B)$ and $\max(B) \leq \max(A)$ can be treated analogously.

Now, we add the following families of sets to enforce the desired preferences

$$\mathcal{S}(A \setminus \{k\}, B \setminus \{k+1\}), \mathcal{S}(C \setminus \{k+1\}, B \setminus \{k\}),$$
$$\mathcal{S}(B \setminus \{k+3\}, A \setminus \{k+2\}), \mathcal{S}(B \setminus \{k+2\}, C \setminus \{k+3\}).$$

We call the sets added in this step the class 2 sets.

We repeat this with a new triple $(v_i', v_j', v_m') \in R$ until we treated all triples in R. By this construction every linear order on $\mathcal{X}$ that satisfies dominance and strict independence must set the preferences listed in ($\star$). This concludes the construction of $\mathcal{X}$. We now can determine the necessary sizes for N and C. N and C need to be large enough such that all z_{AB}^* used in the construction are smaller than C, all z_{AB} are larger than $N - C$ and all x_{AB} and y_{AB} are larger than $C + n + 4|R|$. It is clear that this can be achieved with N and C that are polynomial in $|(V, R)|$.

Now, assume there is a linear order on $\mathcal{X}$ satisfying dominance and strict independence. We claim that the relation defined by $v_i \leq v_j$ iff $V_i \preceq V_j$ is a positive witness for (V, R). By definition this is a linear order. So assume there is a triple (a, b, c) such that $a > b < c$ or $a < b > c$ holds. We treat the first case in detail: $a > b < c$ implies $A \succ B \prec C$. This implies by the strictness of $\prec$ and strict dominance $A \setminus \{k\} \succ B \setminus \{k\}$ and $B \setminus \{k+1\} \prec C \setminus \{k+1\}$. However, then

$$A \setminus \{k\} \succ B \setminus \{k\} \succ C \setminus \{k+1\} \succ B \setminus \{k+1\} \succ A \setminus \{k\}$$

contradicts the assumption that $\prec$ is transitive and irreflexive. Similarly, the second case leads to a contradiction. This shows that if (V, R) is a negative instance of Betweenness, then there is no order on $\mathcal{X}$ that satisfies dominance and strict independence with respect to $\leq$.

Now, assume that there is a linear order on V satisfying the restrictions from R. We use this to construct a linear order on $\mathcal{X}$ that satisfies maximal dominance and strict independence with respect to $\leq$. First, we add all preferences implied by maximal dominance. Observe that no two class 1 sets are comparable by maximal dominance. Moreover, we set $V_i \preceq V_j$ iff $v_i \leq v_j$ holds. Then, we project this order to all sets of the form $V_i \setminus \{x\}$, We claim that this order satisfies all applications of strict independence between class 1 sets. If $A = V_i$ for $i \leq n$, then there is no set $A \cup \{x\}$ in $\mathcal{X}$. If $A = V_i \setminus \{x\}$ for some $i \leq n$ and $x \in X$, then x is the only element of X such that $A \cup \{x\} \in \mathcal{X}$ holds. But then there can only be one other set B with $B \cup \{x\} \in \mathcal{X}$ and $B = V_j \setminus \{x\}$ hence a preference between A and B was introduced by reverse strict independence.

Next, we consider the class 2 sets. We distinguish three types of class 2 sets. We say a set X is

- type 1 if $z^*_{AB} \in X$,
- type 2 if $z_{AB}, z^*_{AB} \notin X$,
- type 3 if $z_{AB} \in X$.

Then, we set $X \prec Y$ if $\mathit{type}(X) < \mathit{type}(Y)$. Furthermore, for all type 1 sets X, Y we set $X \prec Y$ if

- $z^*_{AB} \in X, z^*_{CD} \in Y$ and $z^*_{AB} < z^*_{CD}$,
- $z^*_{AB} \in X, Y$ and $x_{AB} \notin X, x_{AB} \in Y$,
- $z^*_{AB}, x_{AB} \in X, Y$ and $\max(X \triangle Y) \in Y$,
- $z^*_{AB} \in X, Y x_{AB} \notin X, Y$ and $\max(X \triangle Y) \in Y$.

It is straightforward to check that this order satisfies strict independence and is compatible with maximal dominance on all type 1 sets. Similarly, we set for all type 3 sets

- $z_{AB} \in X, z_{CD} \in Y$ and $z_{AB} < z_{CD}$,
- $z_{AB} \in X, Y$ and $x_{AB} \notin X, x_{AB} \in Y$,
- $z_{AB}, x_{AB} \in X, Y$ and $\min(X \triangle Y) \in X$,
- $z_{AB} \in X, Y x_{AB} \notin X, Y$ and $\min(X \triangle Y) \in X$.

Again, it is straightforward to check that this order satisfies strict independence and is compatible with maximal dominance on all type 3 sets.

Now, we covered all possible applications of strict independence on class 1 sets and on type 1 and 3 sets. The only possible application of strict independence that includes class 1 and class 2 sets is adding z_{AB} or z^*_{AB} to a class 1 set and a class 2 set. Then, by construction both resulting sets are type 1 resp. 3 sets. Therefore, we can apply reverse strict independence. By construction, this does not lead to a cycle if any only if we started with an positive instance of Betweenness.

It remains to consider applications of strict independence that include type 2 sets. All type 2 sets are of one of the following forms:

$$\{x_{AB}\}, \{y_{AB}\}, \{x_{AB}, y_{AB}\}, \{a_1\}, \{a_1, x_{AB}\}, \{b_l\}, \{b_l, y_{AB}\}$$

Now, any application of strict independence where the same element is added to two singletons is clearly satisfied by any order that satisfies maximal dominance. This leaves the case that z_{AB} or z^*_{AB} is added to two different type 2 sets. Now, by construction, we have $\{a_1\} \prec \{a_1, x_{AB}\} \prec \{x_{AB}\}$ and $\{a_1, z_{AB}\} \prec \{a_1, x_{AB}, z_{AB}\} \prec \{x_{AB}, z_{AB}\}$. Therefore, the case where z_{AB} is added is satisfied. The case that z^*_{AB} is added is similar.

It can be checked that this covers all possible applications of strict independence. Finally, we can extend this order to a weak order because extensions do not produce new instances of strict independence. □

4.6 Summary of Chapter 4

Let us summarize the results of this chapter again. (See also Table 4.1 at the beginning of the chapter.)

First of all, we have shown that if the lifted order is required to be either a linear or a weak order, then deciding if a family is $\leq$-orderable is NP-complete for all considered combinations of axioms. Moreover, deciding if a family is strongly orderable is Π^p_2-complete for all considered combinations of axioms. In particular, we considered dominance together with either strict independence or independence as well as dominance and the extension rule together with either strict independence or independence. We observed that this also implies that it is not possible to find an order satisfying dominance, (strict) independence and the extension rule in polynomial time even if one already knows that a given family is strongly orderable.

Additionally, we have shown that if the lifted order is only required to be a partial order, then it can be decided in polynomial time if a family is $\leq$-orderable with respect to dominance and strict independence. On the other hand, it is still coNP-complete to decide if a family is strongly orderable with respect to dominance and strict independence. These results still hold if the extension rule is additionally required. Furthermore, we observed that for all of the results above, succinct representation by boolean circuits leads to an exponential blow up in complexity.

Moreover, we have shown that deciding if a family of sets is weakly orderable with respect to dominance and strict independence is NP-complete, independently of the question if the lifted order needs to be a linear, weak or partial order. This also holds if the extension rule is additionally required.

Finally, we have shown that $\leq$-orderability with respect to strict independence and any dominance-like axiom is NP-hard.

CHAPTER 5

Characterization Results

In the forthcoming chapter we present the second half our main results. We restrict our attention to very "structured" families of sets and try to characterize orderability for them. The "structured" families considered in this chapter are families that can be represented by the connected subgraphs of a given graph. We formally define this kind of family in Section 5.1. Then, we provide characterization as well as possibility and impossibility results for our main axioms as well as strong extension on such families. Section 5.2 considers the combination of strict independence with dominance or with dominance and extension or strong extension. In Section 5.3 we consider regular independence together with dominance or dominance and extension or strong extension.

For strict independence together with dominance, we fully characterize strongly, weakly and $\leq$-orderable graphs. For these two axioms, the class of strongly orderable graphs is that of trees (Theorem 5.8) and the class of weakly orderable graphs is that of connected bipartite graphs (Theorem 5.12). The class of $\leq$-orderable graphs is characterized by a more technical condition in Theorem 5.16. Theorem 5.8 and 5.12 also hold if, in addition, the axiom of (strong) extension is required. Moreover, the same characterizations also hold for partial and linear orders (Corollary 5.19).

Afterwards, we consider the effect of weakening strict independence to independence on strong orderability. In combination with strong extension, we show that the only additional connected strongly orderable graph that arises is the complete graph K_3 (Theorem 5.32). Furthermore, we give a full characterization of strong orderability with respect to dominance and independence for *two-connected* graphs. Here we observe that, except for some smaller special cases, two-connected graphs are strongly orderable with respect to dominance and independence if and only if they are cycles or if they do not contain a cycle of length five or more. This result holds also if we additionally require the extension axiom (Theorem 5.40).

Finally, we give a nearly complete picture for strong orderability with respect to dominance and independence and with respect to dominance, independence and extension for arbitrary graphs (Theorem 5.46).

All results in this chapter are taken from Maly et al. (2019), except for Theorem 5.16, Corollary 5.22 and the results on two-colorable hypergraphs at the end of Section 5.2 (Proposition 5.24, 5.26 and 5.28 and Corollary 5.25 and 5.29) which are new, unpublished work.

5.1 Families represent by graphs

First of all, we formally define the types of families that we will study in this chapter, i.e., families that are defined by connectivity in a graph.

Definition 5.1. For a graph G we write $C(G)$ for the family of sets of vertices of all connected non-empty subgraphs of G. Moreover, $IT(G)$ denotes the family of sets of vertices V' such that the subgraphs induced by V' in G are trees.

As we discussed in the introduction, one reason to study families that are represent by graphs is their very uniform and "geometric" structure that allows us to formally capture intuitions like a "cyclic" families of sets. Moreover, if we consider arbitrary families of finite sets as hypergraphs, then there are natural ways to turn classification results from the restricted setting of families represented by graphs into possibility results for the general case.

Aside from these theoretical reasons, families represent by graphs are also interesting from a practical point of view.

Example 5.2. Let us consider again the setting described in Example 3.7, i.e., consider a manager wants to select a team for a given task based on a ranking on the employees competence for the given task. Now, assume that it is additionally know which employees work well together. Then, if for a team there are two disjoint sub-teams such that no one in one sub-team works well with anyone in the other sub-team, it may be more sensible to create two distinct teams. Therefore, it is sensible to exclude such teams from the consideration.

This scenario can be modeled as a family of sets represent by a graph. The vertices of the graph are the employees and there is an edge between two employees if they work together well. Then, a team induces a connected subgraph if and only if there are not two disjoint subgraphs that are not linked by an edge. This equals the condition given above.

Now, let us illustrate this concept on an example.

Example 5.3. Let us consider graphs G and G' shown in Figures 5.1 and 5.2. For the graph G in Figure 5.1, we get

$$C(G) = IT(G) = \{\{1\},\{2\},\{3\},\{4\},\{1,4\},\{2,4\},\{3,4\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}.$$

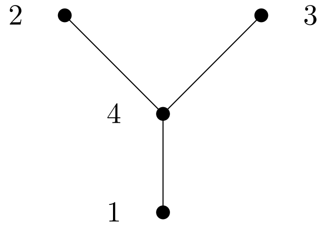

Figure 5.1: Graph G

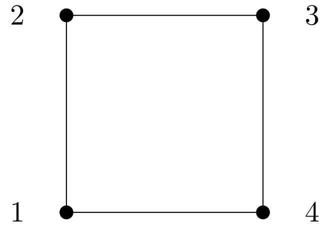

Figure 5.2: Graph G'

and for the graph G' in Figure 5.2, we have

$$C(G') = \mathcal{P}(\{1,2,3,4\}) \setminus \{\{1,3\},\{2,4\}\}$$

and

$$IT(G') = C(G') \setminus \{\{1,2,3,4\}\}.$$

Now, we can extend the concept of orderability to graphs. That is, if there is no ambiguity, we say that a graph G is strongly/weakly/$\leq$-$DI^{(S)}$- or $DI^{(S)}E^{(S)}$-orderable if $C(G)$ has the corresponding property. We observe that strong, weak and $\leq$-orderability are equivalent concepts on complete graphs $K_i = (V,E)$, because $C(K_i) = \mathcal{P}(V) \setminus \{\emptyset\}$.

Example 5.4. Consider the graph G in Figure 5.1. One can check that G is strongly $DI^S E^S$-orderable. Indeed, without loss of generality we may assume that $1, 2$ and 3 are ordered so that $1 < 2 < 3$. Thus, there are four linear orders on $\{1,2,3,4\}$ to consider:

$$\begin{aligned} 4 < 1 < 2 < 3 \\ 1 < 4 < 2 < 3 \\ 1 < 2 < 4 < 3 \\ 1 < 2 < 3 < 4. \end{aligned}$$

Now consider the linear order $\preceq$ with its strict variant $\prec$ given by

$$\begin{aligned} \{1\} \prec \{2\} \prec \{3\} \prec \{1,2,3,4\} \prec \{1,2,4\} \prec \{1,3,4\} \prec \{2,3,4\} \\ \prec \{1,4\} \prec \{2,4\} \prec \{3,4\} \prec \{4\} \end{aligned}$$

We claim that $\preceq$ satisfies dominance, strict independence and strong extension with respect to the last linear order. Strong extension implies $\{1\} \prec \{2\} \prec \{3\}$. Furthermore, for all $A \in C(G) \setminus \{\{1\},\{2\},\{3\}\}$ strong extension implies $\{1\},\{2\},\{3\} \prec A$ as $4 \in A$ holds. Dominance implies $\{i\} \prec \{i,4\} \prec \{4\}$ for $i \in \{1,2,3\}$. Further, it implies $\{1,j,4\} \prec \{j,4\}$ for $j \in \{2,3\}$ and $\{1,2,3,4\} \prec \{2,3,4\} \prec \{3,4\}$. Strict independence implies that $\{1,4\},\{2,4\}$ and $\{3,4\}$ are ordered like $\{1\},\{2\}$ and $\{3\}$. Furthermore, strict independence implies that $\{l,k,4\}$ and $\{l,4\}$ are ordered like $\{k,4\}$ and $\{4\}$ for $\{1,2\} \ni l \neq k \in \{2,3\}$. Additionally, $\{1,4\} \prec \{2,4\} \prec \{3,4\}$ implies by strict independence $\{1,2,4\} \prec \{1,3,4\} \prec \{2,3,4\}$. Finally, there are sets $A, B \in C(G)$ such that $A \cup \{x\}, B \cup \{x\} \in C(G)$ where $|A| = 3$ and $|B| = 2$. We observe that in $\preceq$ this implies $A \prec B$ as well as $A \cup \{x\} \prec B \cup \{x\}$. It can be checked that these are all possible applications of the axioms and that they all are satisfied in $\preceq$. Similar lifted orders can be constructed for the three other orders, too. Thus, our claim follows.

On the other hand, the graph G' in Figure 5.2 is not strongly DI^S-orderable. If we assume the natural order on the vertices of G', dominance implies $\{1\} \prec \{1,2\}$ and $\{1,2\} \prec \{1,2,3\}$, and transitivity implies $\{1\} \prec \{1,2,3\}$. Similarly, we can derive that $\{2,3,4\} \prec \{4\}$. Applying strict independence to these two relations yields $\{1,4\} \prec \{1,2,3,4\}$ and $\{1,2,3,4\} \prec \{1,4\}$, a contradiction. (This argument obviously does not work on $IT(G')$ because $\{1,2,3,4\} \notin IT(G')$ and indeed $IT(G')$ is strongly DI^S-orderable.) However, the order

$$\{1\} \prec \{3\} \prec \{1,2,3\} \prec \{1,2\} \prec \{2,3\} \prec \{2\} \prec \{1,3,4\} \prec \{1,4\} \prec \\ \{3,4\} \prec \{1,2,3,4\} \prec \{4\} \prec \{1,2,4\} \prec \{2,3,4\}$$

satisfies dominance and strict independence with respect to the linear order $1 < 3 < 2 < 4$. Hence G' is weakly DI^S-orderable. In fact, since the order we demonstrated also satisfies strong extension, G' is even weakly $DI^S E^S$-orderable.

5.2 Strict independence

We start our investigations with strong orderability. Afterwards, we focus on weak orderability and then on $\leq$-orderability. We conclude the section with some additional observations.

Strong orderability

First of all, we observe that any family of sets that is acyclic in the sense that every set in the family induces a tree in a given graph is strongly $DI^S E^S$-orderable.

Proposition 5.5. *For every graph G, $IT(G)$ is strongly $DI^S E^S$-orderable.*

Proof. Let V be the vertex set of a graph G and $N = |V|$. Further, let $\leq$ be any linear order on V. Wlog, we assume that $V = \{1, \ldots, N\}$ and that $\leq$ is the standard linear order on $\{1, \ldots, N\}$.

For every $A \in IT(G)$ and $i \in A$, we write $deg_A(i)$ for the degree of i in the subtree of G induced by A. We associate with every set $A \in IT(G)$ a vector

$$v_A = (a_1, \ldots, a_N) \in (\mathbb{N} \cup \{\infty\})^N,$$

where $a_i = \infty$ if $i \notin A$, and $a_i = k$ if $i \in A$ and $deg_A(i) = k$.

Let $\leq^*$ be the linear order on $\mathbb{N} \cup \{\infty\}$ such that $\infty <^* \cdots <^* k <^* \cdots <^* 1$. We define a weak order $\preceq$ on $IT(G)$ by defining $A \preceq B$ precisely when $v_A \leq_{lex} v_B$, where $\leq_{lex}$ is the lexicographic order with respect to $\leq^*$, with the indices considered from N to 1. That is, $A \preceq B$ if $a_N <^* b_N$, or $a_N = b_N$ and $a_{N-1} <^* b_{N-1}$, and so on. Obviously, $\preceq$

is a linear order and hence also a weak order. Furthermore, it satisfies strong extension. We will show that $\preceq$ satisfies dominance and strict independence.

Dominance. Assume that for every $y \in A$, $y < x$. It follows that $\max(A) < x$. Thus, $\max(A) < \max(A \cup \{x\})$ and $A \prec A \cup \{x\}$ holds by strong extension. So, assume that for every $y \in A$, $x < y$. Then, $x < \min(A)$. Let n be the neighbor of x in A (since $A, A \cup \{x\} \in IT(G)$, that is, each set induces a tree in G, it follows that x has exactly one neighbor in A). By the assumption, $x < n$ and $deg_{A\cup\{x\}}(n) = deg_A(n) + 1$. Therefore, $a_i = a_i^x$ for every i such that $n < i$, and $a_n^x <^* a_n$, where we write a_i and a_i^x for the elements of the vectors v_A and $v_{A\cup\{x\}}$. Hence, $A \cup \{x\} \prec A$.

Strict independence. Assume $A, B, A \cup \{x\}, B \cup \{x\} \in IT(G)$ and $A \prec B$. By the same argument as above, x has a unique neighbor in A and a unique neighbor in B. We will denote them by n_A and n_B, respectively. Using the same notation as above for the corresponding vectors for the sets $A, B, A \cup \{x\}, B \cup \{x\}$, we have that $a_x = b_x = \infty$ and $a_x^x = b_x^x = 1$. Further, we observe that $a_i = a_i^x$ and $b_i = b_i^x$ for $i \notin \{x, n_A, n_B\}$.

Assume first that $n_A = n_B = n$. Then $a_n^x = a_n + 1$ and $b_n^x = b_n + 1$. Hence, $a_n^x <^* b_n^x$ if and only if $a_n <^* b_n$. It follows that $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$ if and only if $v_A <_{lex} v_B$.

Next, assume $n_A \neq n_B$. Then $n_A \notin B$ and $n_B \notin A$. Hence $b_{n_A} = b_{n_A}^x = \infty <^* a_{n_A}^x = a_{n_A} + 1$ and $a_{n_B} = a_{n_B}^x = \infty <^* b_{n_B}^x = b_{n_B} + 1$. Therefore, $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$ if and only if $v_A <_{lex} v_B$. □

The following corollary follows immediately from the fact that $C(G) = IT(G)$ holds if G is a tree.

Corollary 5.6. *Every tree is strongly $DI^S E^S$-orderable.*

This result is optimal in the sense that cycles prevent a graph from being strongly DI^S-orderable.

Proposition 5.7. *If a graph G contains a cycle, then it is not strongly DI^S-orderable.*

Proof. Let $C = v_1, \dots, v_n$ be a shortest cycle in G. Then, $C(G)$ contains C and all connected subgraphs of C. In particular, $C(G)$ contains $\{v_1, v_n\}$ as well as all sets $\{v_i, v_{i+1}, \dots, v_{j-1}, v_j\}$, where $1 \leq i \leq j \leq n$. Let $\leq$ be a linear order on V such that $v_1 < \cdots < v_n$. Let us assume that there is a weak order $\preceq$ on $C(G)$ that satisfies dominance and strict independence with respect to $\leq$. Then, by dominance

$$\{v_1\} \prec \{v_1, v_2\} \prec \cdots \prec \{v_1, \dots, v_{n-1}\}$$

and

$$\{v_2, \dots, v_n\} \prec \{v_3, \dots, v_n\} \prec \cdots \prec \{v_n\}.$$

Therefore, by strict independence $\{v_1, v_n\} \prec \{v_1, \dots, v_n\}$ and $\{v_1, \dots, v_n\} \prec \{v_1, v_n\}$. Since $n \geq 3$, this is a contradiction! □

The following theorem summarizes the previous results and follows from Proposition 3.34, Corollary 5.6, Proposition 5.7 and the fact that any graph that is not strongly DI^S-orderable is also not strongly $DI^S E^{(S)}$-orderable.

Theorem 5.8. *The set of strongly DI^S-, DI^SE- or DI^SE^S-orderable graphs is exactly given by the class of forests.*

This result states that every linear order on X can be lifted (with respect to dominance, strict independence and strong extension) to every family of sets of vertices inducing a connected subgraph in a forest on X. In other words, we know that a family of sets $\mathcal{X}$ is strongly DI^SE^S-orderable, if there exists a forest F such that $\mathcal{X} \subseteq C(F)$ holds. For instance, no matter what linear order on $\{1, 2, \dots, n\}$ we consider, it extends to a linear order on the family

$$\mathcal{I} = \{[i..j] \mid 1 \leq i < j \leq n\}$$

that satisfies dominance, strict independence and strong extension. It is so because every set in $\mathcal{I}$ induces a connected subgraph in the path in which elements $1, \dots, n$ are listed in the natural order. The same is true for the family of sets

$$\mathcal{S} = \{X \subseteq \{1, \dots, n\} \mid 1 \in X\}.$$

Indeed, each set in this family induces a connected subgraph in the "star" tree in which every vertex $i \geq 2$ is connected to 1 (and there are no other edges).

Furthermore, recall that a hypergraph H over a set X is called a hypertree or arboreal if there exists a tree T with nodes X such that all edges of H induce connected subtrees in T (Definition 2.14). This means, in our notation, $H \subseteq C(T)$. Therefore, Theorem 5.8 implies that every hypertree is strongly DI^SE^S-orderable.

The converse is not necessarily true, however. Consider for example

$$\mathcal{X} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

Then any graph G such that $\mathcal{X} \subseteq C(G)$ must contain the edges $(1, 2), (1, 3), (2, 3)$ and hence a cycle. However, because all sets in $\mathcal{X}$ have the same size, dominance and strict independence hold vacuously for any order on $\mathcal{X}$. Further, given any linear order $\leq$ on $\{1, 2, 3\}$, the relation $A \preceq B$ defined to hold when $\max(A) \leq \max(B)$ satisfies strong extension. Therefore $\mathcal{X}$ is strongly DI^SE^S-orderable.

Weak orderability

We now turn to weak orderability and show in the forthcoming two results that the bipartite graphs form the crucial class for our characterization. We use the fact that a graph is bipartite if and only if it is two-colorable.

Proposition 5.9. *Every two-colorable graph is weakly DI^SE^S-orderable.*

Proof. Let us consider a two-colorable graph $G = (V, E)$. We color G with two colors *small* and *large* and call vertices of G *small* and *large* accordingly. Let $\leq$ be any linear order on V such that every *small* vertex is smaller than every *large* vertex.

For every $A \in C(G)$ we define $A_L = \{x \in A \mid x \text{ is } \textit{large}\}$ and $A_S = \{x \in A \mid x \text{ is } \textit{small}\}$. For $A, B \in C(G)$, we define $A \preceq B$ if

- $A = B$; or
- $A_L \neq B_L$ and $\max(A_L \triangle B_L) \in B_L$; or
- $A_L = B_L$, $A_S \neq B_S$, and $\min(A_S \triangle B_S) \in A_S$.

(we write $\triangle$ for the symmetric difference of sets). We will prove that $\preceq$ is a linear order.

Indeed, it is easy to see that $\preceq$ is reflexive and total. Let us assume that $A \preceq B$. If $A \preceq B$ holds by the second condition of the definition, then $B \not\preceq A$, because we have $B_L \neq A_L$ and $\max(B_L \triangle A_L) = \max(A_L \triangle B_L) \notin A_L$. By a similar argument, $B \not\preceq A$ follows also if the third clause of the definition applies. Thus, if $A \preceq B$ and $B \preceq A$, it must be that the first condition holds, that is, $A = B$. It follows that $\preceq$ satisfies antisymmetry.

To prove transitivity, let us assume that $A \preceq B$ and $B \preceq C$. If $A = B$ or $B = C$, then we obtain $A \preceq C$ by substituting A for B in $B \preceq C$ or C for B in $A \preceq B$. Thus, from now on we assume that that $A \neq B$ and $B \neq C$. It follows that each of $A \preceq B$ and $B \preceq C$ holds because of the second or the third condition of the definition of $\preceq$.

Let us assume first that both $A \preceq B$ and $B \preceq C$ hold by the second condition and let

$$d = \max((A_L \setminus B_L) \cup (B_L \setminus C_L) \cup (C_L \setminus A_L)).$$

We note that

$$(A_L \setminus B_L) \cup (B_L \setminus C_L) \cup (C_L \setminus A_L) = (A_L \triangle B_L) \cup (B_L \triangle C_L) \cup (A_L \triangle C_L).$$

Clearly, $d \notin A_L \setminus B_L$. Indeed, let us assume that $d \in A_L \setminus B_L$. This would imply $d \in A_L$ as well as $d \in A_L \triangle B_L$. We would then have $\max(A_L \triangle B_L) = d \in A_L$, and, consequently, $B \preceq A$. Antisymmetry would then imply $A = B$, a contradiction. Similarly, $d \notin B_L \setminus C_L$. It follows that $d \in C_L \setminus A_L$. Thus, $A_L \neq C_L$, $d \in C_L$, $d \in A_L \triangle C_L$ and $d = \max(A_L \triangle C_L)$. Consequently, $\max(A_L \triangle C_L) \in C_L$ and $A \preceq C$.

The case when each of $A \preceq B$ and $B \preceq C$ holds because of the third clause in the definition of $\preceq$ can be dealt with in a similar way.

Thus, let us assume then that $A \preceq B$ holds by the second condition of the definition and $B \preceq C$ holds by the third condition. It follows that $A_L \neq B_L$, $\max(A_L \triangle B_L) \in B_L$ and $B_L = C_L$. Consequently, $A_L \neq C_L$ and $\max(A_L \triangle C_L) \in C_L$. Thus, $A \preceq C$. In the dual case, when $A \preceq B$ holds because of the third condition and $B \preceq C$ because of the second one, we obtain $A \preceq C$ in a similar way. This concludes the proof of transitivity.

We will now show that $\preceq$ satisfies dominance, strict independence and strong extension.

Dominance. Let $A, A \cup \{x\} \in C(G)$. By the connectivity of the subgraph induced in G by $A \cup \{x\}$, x has at least one neighbor in A. Let us fix any such neighbor of x and denote it by n. Clearly, the colors of x and n are different. Let us assume that $\max(A) < x$. It follows that x is large. Thus, $x \in (A \cup \{x\})_L$ and so, $A_L \neq A \cup \{x\}$. Since $A_L \triangle (A \cup \{x\})_L = \{x\}$, $\max(A_L \triangle (A \cup \{x\})_L) = x$. Thus, $\max(A_L \triangle (A \cup \{x\})_L) \in (A \cup \{x\})_L$ and $A \prec A \cup \{x\}$. The case $x < \min(A)$ can be dealt with in a similar way.

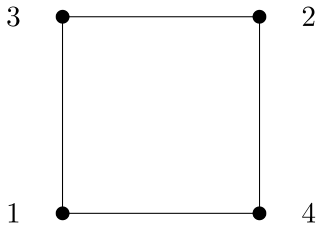

Figure 5.3: Graph G

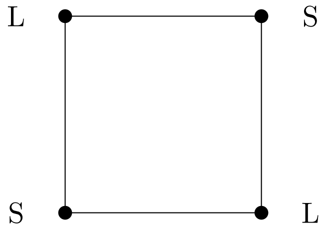

Figure 5.4: Coloring of G with large (L) and small (S)

Strict independence. Assume $A, B, A \cup \{x\}, B \cup \{x\} \in C(G)$ and $A \prec B$ (thus, either the second or the third condition of the definition holds). As $A_L \triangle B_L = (A \cup \{x\})_L \triangle (B \cup \{x\})_L$ and $A_S \triangle B_S = (A \cup \{x\})_S \triangle (B \cup \{x\})_S$, we have $A \cup \{x\} \prec B \cup \{x\}$.

Strong Extension. Consider sets $A, B \in C(G)$ such that $\max(A) < \max(B)$. First assume that $B_L \neq \emptyset$. Clearly, $\max(B) \in B_L$ and $\max(B) \notin A_L$ (because $\max(B) \notin A$). It follows that $A_L \triangle B_L \neq \emptyset$ and $\max(A_L \triangle B_L) = \max(B)$. Thus, $\max(A_L \triangle B_L) \in B_L$ and so, $A \preceq B$. Since $A \neq B$ and $\preceq$ is a linear order, we have $A \prec B$.

Next, assume that $B_L = \emptyset$. It follows that $\max(B)$ is *small* and so, $\max(A)$ is *small*, too. Consequently, we have that A and B consist of *small* vertices only. As they induce connected subgraphs in G, $|A| = |B| = 1$. These observations imply that $A_L = B_L = \emptyset$, and $\min(A) = \max(A) < \max(B) = \min(B)$. Consequently, $\min(A) \in A_S$ and $\min(A) \notin B_S$. It follows that $A_S \triangle B_S \neq \emptyset$ and that $\min(A_S \triangle B_S) = \min(A_S)$. Hence, $\min(A_S \triangle B_S) \in A_S$ and so, $A \prec B$. ☐

Note that a complete bipartite graph is also two-colorable. Hence, this result shows, in particular, that if X and Y are disjoint nonempty sets, then the family of sets

$$\{Z \subseteq X \cup Y \mid Z \cap X \neq \emptyset \neq Z \cap Y\}$$

is weakly $DI^S E^S$-orderable. Furthermore, the result is constructuve and tells us how to find a linear order on X such that we can construct a linear order on $\mathcal{X}$ that satisfies dominance, strict independence and strong extension.

Example 5.10. Consider a cycle on length four with edges $(1, 3)$, $(3, 2)$, $(2, 4)$ and $(4, 1)$ as shown in Figure 5.3. Then, a valid two-coloring would be coloring 1 and 2 as small and 3 and 4 as large (Figure 5.4). We can set $1 < 2 < 3 < 4$. In that case, we get the following linear order on $C(G)$:

$$\{1\} \prec \{2\} \prec \{1,3\} \prec \{1,2,3\} \prec \{2,3\} \prec \{3\} \prec \\ \{1,4\} \prec \{1,2,4\} \prec \{2,4\} \prec \{4\} \prec \{1,3,4\} \prec \{1,2,3,4\} \prec \{2,3,4\}$$

Proposition 5.9 is tight as graphs that are not two-colorable are not weakly DI^S-orderable.

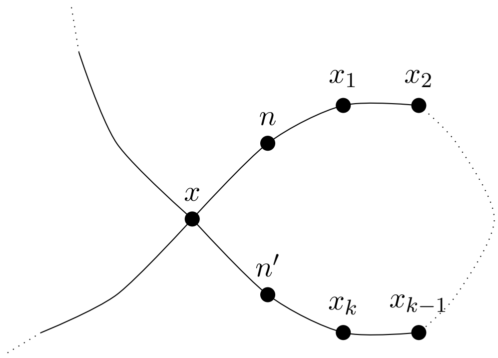

Figure 5.5: Vertex x with two neighbors n and n' connected by a path

Proposition 5.11. *If a graph is not two-colorable, then it is not weakly DI^S-orderable.*

Proof. Let V be the vertex set of G and let $\leq$ be a linear order on V. We say a vertex $x \in V$ is *large* (respectively, *small*) with respect to $\leq$ if for every neighbor n of x, $n < x$ (respectively, $n > x$) holds. We call $x \in V$ *intermediate* with respect to $\leq$ if x is neither *large* nor *small*. (When talking about large, small and intermediate vertices, we often drop references to $\leq$ if it is clear from the context.) Let us assume that every vertex in V is either *large* or *small*. Obviously no *large* vertex can be a neighbor of a *large* vertex and no *small* vertex can be the neighbor of a *small* vertex. Thus, the *large*-*small* labeling of nodes is a two-coloring of G, a contradiction.

Our argument shows that for every linear order $\leq$ on V, V contains at least one *intermediate* vertex. Let $\leq$ be an arbitrary linear order on V and let x be an *intermediate* vertex with respect to $\leq$. We call a neighbor n of x *small* if $n < x$ holds and *large* otherwise. Further, we call an intermediate x *critical* if at least one *small* neighbor of x is connected to at least one *large* neighbor of x by a path in G_x^-, the graph induced by $V \setminus \{x\}$.

We claim that every linear order contains at least one *critical* vertex. Indeed, let us assume otherwise and let $\leq$ be a counterexample order with the minimum number of intermediate vertices. That is, no vertex in V is critical and every linear order with fewer *intermediate* vertices than $\leq$ contains a *critical* vertex.

Let x be an *intermediate* vertex with respect to $\leq$, and let V' be the set of all vertices in V reachable in G from x by simple paths (no repetition of vertices) that start with an edge connecting x to a small neighbor of x. Let us define $V'' = V \setminus V'$. Clearly, x and all *small* neighbors of x belong to V' and all *large* neighbors of x belong to V''. To see the latter, let us assume that some large neighbor of x, say y, belongs to V'. It follows that there is a path from a small neighbor of x to y in G_x^-, contradicting that x is intermediate but not critical. In addition, by the definition of V', the only edges between V' and V'' are those that connect x and its *large* neighbors. We define linear order $\leq'$ on V by setting $x \leq' y$ if

- $x, y \in V'$ and $x \leq y$,

- $x, y \in V''$ and $y \leq x$,
- $y \in V'$, $x \in V''$.

It is clear that $\leq'$ is a linear order on V. Moreover, x is not an *intermediate* vertex in G with respect to $\leq'$, because all neighbors of x that are *small* with respect to $\leq$ are also *small* with respect to $\leq'$ and all neighbors that are *large* with respect to $\leq$ are *small* with respect to $\leq'$. Furthermore, for all other vertices $y \neq x$, whether they are *intermediate* or not does not change. This is clear if $y \in V'$. If $y \in V''$ then the relation of y to all its neighbors is inverted, hence *small* vertices become *large* vertices, *large* vertices become *small* vertices and *intermediate* vertices stay *intermediate*. It follows that $\leq'$ has fewer intermediate vertices than $\leq$. Let y be any intermediate vertex with respect to $\leq'$. By construction either y and all its neighbors are all in V' or are all in V''. Since y is an intermediate but not critical vertex with respect to $\leq$, y is not *critical* with respect to $\leq'$. Thus, $\leq'$ is an order with fewer intermediate vertices than $\leq$ and with no critical vertices, a contradiction.

Let $\leq$ be any linear order on V and let x be a critical vertex under this order. Let n be a *small* neighbor of x connected in G_x^- to a *large* neighbor of x, say n', by a path $n, x_1, \ldots, x_k, n'$, as shown in Figure 5.5. Let us assume there is a weak order $\preceq$ on $C(G)$ satisfying dominance and strict independence with respect to $\leq$. Then, since $n < x$, we have $\{n\} \prec \{n, x\}$ by dominance. Further, by repeated application of strict independence and transitivity

$$\{n, x_1, \ldots, x_k, n'\} \prec \{n, x, x_1, \ldots, x_k, n'\}.$$

On the other hand, since $x < n'$, we have $\{x, n'\} \prec \{n'\}$ and hence, again by strict independence and transitivity,

$$\{n', x, x_1, \ldots, x_k, n\} \prec \{n', x_1, \ldots, x_k, n\}.$$

Thus,

$$\{n, x_1, \ldots, x_k, n'\} \prec \{n, x_1, \ldots, x_k, n'\},$$

a contradiction. □

The following theorem follows directly from Proposition 5.9 and Proposition 5.11.

Theorem 5.12. *The set of weakly DI^S-, DI^SE- or DI^SE^S-orderable graphs is exactly given by the class of two-colorable graphs.*

$\leq$-orderability

Combining ideas from the two previous results, we can also classify $\leq$-DI^S-orderability. Unfortunately, the class of $\leq$-DI^S-orderable graphs graphs does not coincide with a well known class of graphs. Hence in order to formulate our classification result, we first need to fix some notation. The following definitions are very close to concepts used in the proof of Proposition 5.11.

Definition 5.13. Let $G = (V, E)$ be a graph, $\leq$ a linear order on V and v a vertex of G. Then we say n_v is a $\leq$-small neighbor of v if n_v is adjacent to v and $n_v < v$. Similarly, we say n_v a $\leq$-large neighbor if n_v is adjacent to v and $v < n_v$.

Furthermore, we call a vertex $\leq$-small if all its neighbors are $\leq$-larger and we call a vertex $\leq$-large if all its neighbors are $\leq$-small. Finally, we say a vertex is $\leq$-intermediate if it is neither $\leq$-small nor $\leq$-large. If no ambiguity arises, we omit $\leq$.

If every vertex of a graph G is either $\leq$-small or $\leq$-large, then this is a valid two-coloring of G as no $\leq$-large vertex can be the neighbor of a $\leq$-large vertex and no $\leq$-small vertex can be the neighbor of a $\leq$-small vertex. In this case, we can use the order defined in the proof of Proposition 5.9 to show that G is $\leq$-DI^S-orderable. This gives us a sufficient condition for $\leq$-DI^S-orderability.

On the other hand, a close inspection of the proof of Proposition 5.11 shows that a graph can not be $\leq$-DI^S-orderable if there is a $\leq$-intermediate vertex that has a small neighbor s and a large neighbor l that are connected by a path that does not contain i.

Proposition 5.14. *Let $G = (V, E)$ be a graph and let $\leq$ be a linear order on V. If there is a $\leq$-intermediate vertex i such that a $\leq$-large and a $\leq$-small neighbor of i are connected in $G - i$, then G is not $\leq$-DI^S-orderable.*

Proof. Let us consider a graph $G = (V, E)$ and a linear order $\leq$ such that there is a $\leq$-intermediate vertex i that has a $\leq$-large neighbor l and a $\leq$-small neighbor s and l and s are connected in $G - i$. Let $s, x_1, \ldots, x_k, l$ be the path connecting s and l in $G - i$. Assume for the sake of a contradiction that there is a weak order $\preceq$ on $C(G)$ satisfying dominance and strict independence with respect to $\leq$. Then, since $s < i$, we have $\{s\} \prec \{s, i\}$ by dominance. Further, by repeated application of strict independence and transitivity

$$\{s, x_1, \ldots, x_k, l\} \prec \{s, i, x_1, \ldots, x_k, l\}.$$

On the other hand, since $i < l$, we have $\{i, l\} \prec \{l\}$ and hence, again by strict independence and transitivity,

$$\{l, i, x_1, \ldots, x_k, s\} \prec \{l, x_1, \ldots, x_k, s\}.$$

Thus,

$$\{s, x_1, \ldots, x_k, l\} \prec \{s, x_1, \ldots, x_k, l\},$$

a contradiction. □

As it turns out, this condition actually classifies $\leq$-DI^S-orderable graphs.

Proposition 5.15. *Let $G = (V, E)$ be a graph and let $\leq$ be a linear order on V. If there is no $\leq$-intermediate vertex i such that a $\leq$-large and a $\leq$-small neighbor of i are connected in $G - i$, then G is $\leq$-DI^S-orderable.*

Proof. Let us consider a graph $G = (V, E)$ and a linear order $\leq$ such that in G no $\leq$-intermediate vertex i has a large and a small neighbor that are connected in $G - i$. In the following, we omit the reference to $\leq$ for small, large and intermediate vertices. We want to define a weak order $\preceq$ on $C(G)$ that satisfies dominance and strict independence with respect to $\leq$. In order to do so, we have to fix some notation. First of all, for a set $A \in C(G)$, we define $A_S := \{x \in A \mid x \text{ is small}\}$. Similarly, we define $A_L := \{x \in A \mid x \text{ is large}\}$ and $A_I := \{x \in A \mid x \text{ is intermediate}\}$. We observe that for every intermediate vertex i the graph $G - i$ contains at least 2 connected components and no connected component contains a small and a large neighbor of i at the same time. We write G^i_S for the union of all connected components of $G - i$ that contain small neighbors of i. Similarly, we write G^i_L for the union of all connected components of $G - i$ containing large neighbors of i.

Now, let $i_1, \ldots, i_k$ be an enumeration of the intermediate vertices of G such that $i_j < i_l$ for all $j < l$. For every set A in $C(G)$ we define a vector $v_A = (v^1_A, \ldots, v^k_A)$ by

$$v^j_A = \begin{cases} I & \text{if } i_j \in A, \\ S & \text{if } A \in C(G^{i_j}_S), \\ L & \text{if } A \in C(G^{i_j}_L). \end{cases}$$

Furthermore, we define $\leq^*$ by $S \leq^* I \leq^* L$. Finally, let $\leq_{lex}$ be the lexicographic order on vectors based on $\leq^*$ and starting from the beginning, i.e., $v_A \leq_{lex} v_B$ if $v^1_A <^* v^1_B$ or $v^1_A = v^1_B$ and $v^2_A <^* v^2_B$ and so on.

Now, we can define a weak order $\preceq$ on $C(G)$ that satisfies dominance and strict independence. The order is defined by $A \preceq B$ if:

1. $A = B$; or
2. $A_L \neq B_L$ and $\max(A_L \triangle B_L) \in B_L$; or
3. $A_L = B_L$, $A_S \neq B_S$, and $\min(A_S \triangle B_S) \in A_S$; or
4. $A_L = B_L$, $A_S = B_S$, $A_I \neq B_I$ and $v_A \leq_{lex} v_B$.

We claim that $\preceq$ is even a linear order and that it satisfies dominance and strict independence. Condition (1)-(3) equal the conditions used in the proof of Proposition 5.9 and define a linear order by the same arguments. Therefore, in order to show that $\preceq$ is a linear order, we only need to show that condition (4) defines a linear order on sets with the same small and large vertices. As we know that the lexicographic order is a linear order, (4) clearly defines a linear order.

Dominance. Assume $A, A \cup \{x\} \in C(G)$ and $x \notin A$. First assume that x is a small vertex. Then $\max(A) < x$ is not possible, as A has to contain at least one neighbor of x, which must be larger than x by assumption. Now, assume $x < \min(A)$. Then, we observe that $A_L = (A \cup \{x\})_L$ and $\min(A_S \triangle (A \cup \{x\})_S) = \{x\} \in A \cup \{x\}$ and hence $A \cup \{x\} \prec A$ by (3). The case that x is a large vertex is symmetric.

Now assume that x is an intermediate vertex. As $x \notin A$, we know that either $A \in C(G^x_S)$ or $A \in C(G^x_L)$ must hold. We assume $A \in C(G^x_S)$. The other case is symmetric. Then, we claim that $x < \min(A)$ is not possible. This is the case because

$A \cup \{x\} \in C(G)$ implies that A contains a neighbor n_x of x and $A \in C_S^x$ implies that n_x is a small neighbor. Hence $\min(A) \leq n_x < x$ must hold.

Therefore, we can assume $\max(A) < x$. We observe that that $A_L = (A \cup \{x\})_L$ and $A_S = (A_S \cup \{x\})_S$. As x is an intermediate vertex, there is a j such that $x = i_j$. Then, we claim that $v_A^l = v_{A\cup\{x\}}^l$ for all $l \neq j$. If $v_A^l = I$ then, by definition $v_{A\cup\{x\}}^l = I$. If $v_A^l \neq I$ then $i_l \notin A, A \cup \{x\}$, hence $v_A^l, v_{A\cup\{x\}}^l \neq I$ Furthermore, A and $A \cup \{x\}$ must clearly be in the same connected component of $G - i_l$. Therefore $v_A^l = v_{A\cup\{x\}}^l$.

This implies, in particular, $v_A^l = v_{A\cup\{x\}}^l$ for all $l < j$. Furthermore, we have $v_A^j = S <^* I = v_{A\cup\{x\}}^j$. Hence, $A \prec A \cup \{x\}$ holds by (4).

Strict Independence. Assume $A, B, A \cup \{x\}, B \cup \{x\} \in C(G)$ and $x \notin A, B$. First, assume that $A \prec B$ because of (3), i.e., $A_L = B_L$ and $\min(A_S \triangle B_S) \in A_S$. Then, clearly $(A \cup \{x\})_L = (B \cup \{x\})_L$ and, by the definition of $\triangle$, $\min((A \cup \{x\})_S \triangle (B \cup \{x\})_S) = \min(A_S \triangle B_S) \in (A \cup \{x\})_S$. Therefore, $A \cup \{x\} \prec B \cup \{x\}$ by (3). The case that $A \prec B$ holds by (2) is similar.

Now assume $A \prec B$ holds by (4), i.e., $A_L = B_L$, $A_S = B_S$, $A_I \neq B_I$ and $v_A <_{lex} v_B$, i.e., there is a j such that $v_A^l = v_B^l$ for all $l < j$ and $v_A^j <^* v_B^j$. By the definition of $\triangle$ this implies $(A \cup \{x\})_L = (B \cup \{x\})_L$, $(A \cup \{x\})_S = (B \cup \{x\})_S$, $(A \cup \{x\})_I \neq (B \cup \{x\})_I$. Furthermore, by the same argument as in the dominance case $v_A^l = v_{A\cup\{x\}}^l$ and $v_B^l = v_{B\cup\{x\}}^l$ for all l such that $x \neq i_l$. If x is small or large this implies $v_A = v_{A\cup\{x\}}$ and $v_B = v_{B\cup\{x\}}$ and hence $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$. So assume x is intermediate and let j be such that $x = i_j$. Then, if there is a $l < j$ such that $v_A^l \neq v_B^l$ it still follows by (4) that $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$ holds.

So assume $v_A^l = v_B^l$ for all $l < j$. We claim that then $v_A^j = v_B^j$ must hold. Assume otherwise $v_A^j \neq v_B^j$. By assumption, $x = i_j \notin A, B$. Hence $v_A^j, v_B^j \neq I$. Because $v_A^j <_{lex} v_B^j$ and $v_A^l = v_B^l$ for all $l < j$ hold, we know that $v_A^j <^* v_B^j$ must hold, which implies $v_A^j = S$ and $v_B^j = L$. As $A \cup \{x\}, B \cup \{x\} \in C(G)$ holds, this implies that A contains a small neighbor n_x of x whereas B can not contain any small neighbor and hence does not contain n_x. Now, $A_L = B_L$ and $A_S = B_S$ rule out that n_x is a small or large vertex, hence it must be intermediate. Furthermore, because n_x is a small neighbor $n_x < x$ holds, which implies $n_x = i_l$ for some $l < j$. However, then $v_A^l = I \neq v_B^l$, a contradiction.

Therefore, we know $v_A^j = v_B^j$. We also know, by construction, $v_{A\cup\{x\}}^j = I = v_{B\cup\{x\}}^j$. As we know that v_A and $v_{A\cup\{x\}}$ resp. v_B and $v_{B\cup\{x\}}$ only differ in the j-th element, this means that $v_A <_{lex} v_B$ implies $v_{A\cup\{x\}} <_{lex} v_{B\cup\{x\}}$. Hence, $A \cup \{x\} \prec B \cup \{x\}$ must hold by (4). □

Together Proposition 5.14 and Proposition 5.15 classify $\leq$-DI^S-orderable graphs.

Theorem 5.16. *Let $G = (V, E)$ be a graph and let $\leq$ be a linear order on V. Then G is $\leq$-DI^S-orderable if and only if there is no $\leq$-intermediate vertex i such that a $\leq$-large and a $\leq$-small neighbor of i are connected in $G - i$.*

Figure 5.6: Graph used in Example 5.17

Now, the order constructed in the proof of Proposition 5.15 in general does not satisfy strong extension or even the extension rule.

Example 5.17. Consider a path with four vertices $1, 2, 3, 4$ and edges $(2, 1), (1, 3)$ and $(3, 4)$ (see Figure 5.6) with the natural linear order $\leq$. Then, 1 is a small vertex, 2 and 4 are large vertices and 3 is the only intermediate vertex and non of its neighbors are connected with a path in $G - \{3\}$. On this graph, the order constructed in the proof above looks as follows:

$$\{1\} \prec \{1,3\} \prec \{3\} \prec \{1,2\} \prec \{1,2,3\} \prec \{2\} \prec \{1,2,3,4\} \prec \{1,3,4\} \prec \{3,4\} \prec \{4\}$$

As an example, let us consider $\{1,2\}$ and $\{1,2,3\}$. Then $\{1,2\}_L = \{1,2,3\}_L = \{2\}$ and $\{1,2\}_S = \{1,2,3\}_S = \{1\}$. On the other hand, we have $\{1,2\}_I = \emptyset \neq \{3\} = \{1,2,3\}_I$. Now, clearly $v^3_{\{1,2,3\}} = I$ and, as 1 is a small neighbor of 3, $v^3_{\{1,2\}} = S$. Therefore, $\{1,2\} \prec \{1,2,3\}$.

Therefore, the results above do not suffice to classify $\leq$-DI^SE- and $\leq$-DI^SE^S-orderability. A classification of these two properties has to be left to future work.

Additional observations

In the following, we discuss some further observations on the classification results for strong, weak and $\leq$-orderability proven in the previous sections. First of all, all three properties used for the classification results are clearly polynomial time decidable. Hence, the results above imply that strong, weak and $\leq$-DI^S-orderability are polynomial time decidable for families represented by graphs.

Remark 5.18. It is known that it is possible to decide in polynomial time if a graph is a forest and if a graph is two-colorable. Furthermore, it is easy to see that we can check in polynomial time whether a graph contains for a specific linear order $\leq$ an $\leq$-intermediate vertex with an in $G - i$ connected $\leq$-small and $\leq$-large neighbor. Therefore our results show that, for a given graph G, it is decidable in polynomial time if $C(G)$ is strongly/weakly/$\leq$-DI^S-orderable. Furthermore, for any two-colorable graph, we can compute a two-coloring in polynomial time. Therefore, for any weakly $DI^S(E^S)$-orderable graph $G = (V, E)$ we can compute, in polynomial time, an order $\leq$ on V such that there exists a linear order $\preceq$ on $C(G)$ satisfying dominance and strict independence (and strong extension) with respect to $\leq$, by using the construction used in the proof of Proposition 5.9. Finally, the linear orders constructed in the proofs of Proposition 5.5, 5.9 and 5.15 can obviously be constructed in polynomial time.

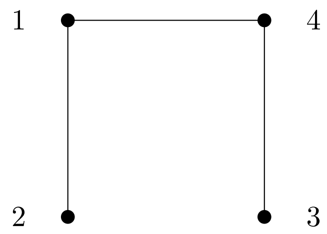

Figure 5.7: Tree G used in Example 5.20

Furthermore, we observe that the orders constructed in the proofs of Proposition 5.5 and Proposition 5.9 are linear orders. Furthermore, we observe that we did not use the totality of the order $\preceq$ in the proofs of Proposition 5.7, 5.11 and 5.14. Therefore, we can conclude that if a graph G is not strongly, weakly or $\leq$-DI^S-orderable, then there does not exist a *partial* order on $C(G)$ that satisfies dominance and strict independence. In other words:

Corollary 5.19. *Let G be a graph. Then, there exists a* linear *order on $C(G)$ satisfying dominance and strict independence (and extension or strong extension) for every (at least one) order on the vertices of G if and only if there exists a* preorder *on $C(G)$ satisfying dominance and strict independence (and extension or strong extension) for every (at least one) order on the vertices of G. Furthermore, let $\leq$ be a linear order on the vertices of G. There exists a* linear *order on $C(G)$ satisfying dominance and strict independence with respect to $\leq$ if and only if there exists a* preorder *on $C(G)$ satisfying dominance and strict independence with respect to $\leq$.*

This is a striking difference to the general setting where we have seen that there are families for which dominance and strict independence can be jointly satisfied by a preorder but neither by a linear nor by a weak order. Furthermore, we will see in the next section that there are graphs that are strongly DI- or $DIE^{(S)}$-orderable but not even weakly DI^S-orderable. As independence and strict independence coincide if we require the order $\preceq$ on $\mathcal{X}$ to be linear, this implies that there are graphs that are strongly DIE^S-orderable but no linear order on $C(G)$ satisfies dominance and independence.

Additionally, we observe that we can strength the axioms used in Theorem 5.8 and 5.12. The negative results in Proposition 5.7 and 5.11 clearly also hold if we strengthen the used axioms. For Proposition 5.5 we claim that the constructed order $\preceq$ satisfies set-dominance. Assume $A, A \cup C \in \mathcal{X}$. If $\max(A) < \min(C)$ the $A \prec A \cup C$ follows from strong extension. So assume $\max(C) < \min(A)$. Then, at least one element in A has a neighbor in C. Let n be the maximal element of A with a neighbor in C. Then, $deg_A(a^*) = deg_{A \cup C}(a^*)$ for all $a^* > a$ and $deg_A(a) < deg_{A \cup C}(a)$. Therefore $A \cup C \prec A$ by construction. On the other hand, there are trees that are not strongly orderable with respect to dominance, strict set-independence and extension:

Example 5.20. Consider the tree G given in Figure 5.7 with $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{4, 3\}\}$. Furthermore, let $\leq$ be the natural linear order on G. Assume for the sake of a contradiction that $\preceq$ is a preorder on $C(G)$ that satisfies dominance, strict set-independence and the extension rule. Then, $\{1\} \prec \{1,2\}$ by dominance and hence $\{1,3,4\} \prec \{1,2,3,4\}$ by strict set-independence. On the other hand $\{3,4\} \prec \{4\}$ by dominance and therefore $\{1,2,3,4\} \prec \{1,2,4\}$ by strict set-independence. Hence we have $\{1,3,4\} \prec \{1,2,3,4\} \prec \{1,2,4\}$, which implies, by transitivity, $\{1,3,4\} \prec \{1,2,4\}$. However, $\{2\} \prec \{3\}$ holds by extension and therefore $\{1,2,4\} \prec \{1,3,4\}$ by strict set-independence. A contradiction. We observe that the last application of strict set-independence in this proof can not be simulated by two applications of strict independence: Adding first 1 and then 4 is not possible because $\{1,3\}$ is not in $C(G)$. Similarly, adding first 4 and then 1 is not possible as $\{2,4\}$ in not in $C(G)$.

It remains open, if dominance and strict set-independence without the extension rule are compatible on trees.

For Proposition 5.9 it is easy to see that the constructed order $\preceq$ satisfies strict set-independence as $A \triangle B = \{A \cup C\} \triangle \{B \cup C\}$ for all $C \in \mathcal{P}(X) \setminus \{\emptyset\}$ such that $C \cap (A \cup B) = \emptyset$. On the other hand, $\preceq$ does not satisfy set-dominance. Indeed, there are graphs that are two-colorable but not weakly orderable with respect to set-dominance and strict independence.

Example 5.21. Consider a graph G consisting of a cycle of length 4 (see Figure 5.2). We claim that for no linear order $\leq$ on V there is a preorder $\preceq$ on $C(G)$ that satisfies set-dominance and strict independence. Assume otherwise that $\leq$ is an arbitrary linear order on V and $\preceq$ is a preorder that satisfies set-dominance and strict independence with respect to $\leq$. Assume w.l.o.g. that $V = \{1,2,3,4\}$ and $\leq$ is the natural linear order on V. Assume first that 1 and 4 are neighbors. Then, we observe that $\{1\}, \{4\}, \{1,4\}, \{1,2,3\}, \{2,3,4\}$ and $\{1,2,3,4\}$ are in $C(G)$. This implies that $\{1\} \prec \{1,2,3\}$ $\{2,3,4\} \prec \{4\}$ hold by set-dominance and hence by strict independence $\{1,4\} \prec \{1,2,3,4\}$ and $\{1,2,3,4\} \prec \{1,4\}$. A contradiction. Now assume 1 and 4 are not neighbors in G. Then 2 is a $\leq$-intermediate vertex and its $\leq$-small neighbor 1 is connected to its $\leq$-large neighbor 4 with a path that does not contain 2. Hence, there can be no partial order $\preceq$ that satisfies dominance and strict independence with respect to $\leq$, hence also no preorder that satisfies set-dominance and strict independence.

The following corollary summarizes the discussion on set-dominance and strict set-independence.

Corollary 5.22. *A graph is strongly orderable with respect to set-dominance, strict independence and extension if and only if it is a forest.*[1] *This result still holds if we replace extension by strong extension.*

Moreover, a graph is weakly orderable with respect to dominance and strict set-independence if and only if it is two-colorable. This result also holds if we additionally require extension or strong extension.

[1]Observe that it is straightforward to show that Proposition 3.34 also holds for set-dominance and strict set-independence. Therefore, the results discussed above can be generalized from trees to forests.

Finally, we observe that we can use colorability arguments similar to the one used in the proof of Proposition 5.9 to give sufficient conditions for the orderability of a hypergraph. This gives us a possibility result for arbitrary families of sets, as any family of sets can be seen as a hypergraph.

We will first prove a possibility result for $\leq$-orderability. We want to show that we can lift a linear order $\leq$ to a weak order satisfying dominance, strict independence and strong extension if we can find a two-coloring that is "compatible" with $\leq$.

Definition 5.23. Let $H = (V, E)$ be a hypergraph, $c : V \to \{l, s\}$ a two-coloring of H and $\leq$ a linear order on V. We say c is globally compatible with $\leq$ if $c(v) = s$ and $c(w) = l$ implies $v < w$ for all $v, w \in V$. We say c is locally compatible with $\leq$ if $c(v) = s$ and $c(w) = l$ implies $v < w$ for all $v \in V$ and all $w \in \{u \in V \mid \exists e \in E(v \in e \wedge u \in e)\}$.

Clearly, for every two-coloring c of a hypergraph $H = (V, E)$ there is a linear order $\leq$ on V such that c is globally compatible with $\leq$. However, it is possible that there is no two-coloring of a hypergraph H that is locally compatible with a linear order $\leq$ (and hence no two-coloring that is globally compatible with $\leq$) even though H is two-colorable. Consider, for example the hypergraph $H = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$ and the linear order $1 < 2 < 3$. Then, 1 and 3 both need to have the opposite color of 2. Hence, either 2 is small and 1 is large or 3 is small and 2 is large. Both cases violate local compatibility.

However, if we can find for a linear order $\leq$ and a hypergraph H a globally compatible two-coloring, then we can always lift $\leq$ to a linear order that satisfies dominance, strict set-independence and strong extension. The proof is very similar to the proof of Proposition 5.9.

Proposition 5.24. *Let $H = (V, E)$ be a hypergraph and $\leq$ a linear order on V. Assume that there exists a two-coloring of H that is globally compatible with $\leq$. Then there exists a linear order $\preceq$ on E that satisfies dominance, strict set-independence and strong extension.*

Proof. Let $H = (V, E)$ be hypergraph, $\leq$ a linear order on V and $c : V \to \{l, s\}$ a two-coloring that is globally compatible with $\leq$.

For every $A \in E$ we define $A_L = \{x \in A \mid c(x) = l\}$ and $A_S = \{x \in A \mid c(x) = s\}$. The we define a linear order $\preceq$ on E by $A \preceq B$ for $A, B \in E$ if

- $A = B$; or
- $A_L \neq B_L$ and $\max(A_L \triangle B_L) \in B_L$; or
- $A_L = B_L$, $A_S \neq B_S$, and $\min(A_S \triangle B_S) \in A_S$.

Clearly, $\preceq$ is a linear order by the same argument as the order in the proof of Proposition 5.9. We will now show that $\preceq$ satisfies dominance, strict independence and strong extension.

Dominance. Let $A, A \cup \{x\} \in E$. First we assume $\max(A) < x$. Observe that $A \cup \{x\}$ must contain at least one vertex v such that $c(v) = l$. This implies $c(x) \neq s$

because $v < x$ holds and c is globally compatible with $\leq$. Hence $c(x) = l$. Therefore $A_L \triangle (A \cup \{x\})_L = \{x\}$ and hence $A \prec A \cup \{x\}$. Now assume $x < \min(A)$. Analogously to above $A \cup \{x\}$ must contain at least one vertex v such that $c(v) = s$ which implies $c(x) \neq l$ because $x < v$ holds and c is globally compatible with $\leq$. Hence $c(x) = s$. Therefore $A_L \triangle (A \cup \{x\})_L = \emptyset$ and $A_S \triangle (A \cup \{x\})_S = \{x\}$ and hence $A \cup \{x\} \prec A$.

Set-strict independence. Assume $A, B, A \cup C, B \cup C \in E$ and $A \prec B$ (thus, either the second or the third condition of the definition holds). As $A_L \triangle B_L = (A \cup C)_L \triangle (B \cup C)_L$ and $A_S \triangle B_S = (A \cup C)_S \triangle (B \cup C)_S$, we have $A \cup C \prec B \cup C$.

Strong Extension. Consider sets $A, B \in E$ such that $\max(A) < \max(B)$. First assume that $B_L \neq \emptyset$. Clearly, $\max(B) \in B_L$ and $\max(B) \notin A_L$ (because $\max(B) \notin A$). It follows that $A_L \triangle B_L \neq \emptyset$ and $\max(A_L \triangle B_L) = \max(B)$. Thus, $\max(A_L \triangle B_L) \in B_L$ and so, $A \preceq B$. Since $A \neq B$ and $\preceq$ is a linear order, we have $A \prec B$.

Next, assume that $B_L = \emptyset$. It follows that $c(\max(B)) = s$ and so, $c(\max(A)) = s$, too. Consequently, we have that A and B are monochromatic and hence $|A| = |B| = 1$. These observations imply that $A_L = B_L = \emptyset$, and $\min(A) = \max(A) < \max(B) = \min(B)$. Consequently, $\min(A) \in A_S$ and $\min(A) \notin B_S$. It follows that $A_S \triangle B_S \neq \emptyset$ and that $\min(A_S \triangle B_S) = \min(A_S)$. Hence, $\min(A_S \triangle B_S) \in A_S$ and so, $A \prec B$. □

As a corollary we immediately get the following result on weak $DI^S E^S$-orderability:

Corollary 5.25. *Every two-colorable hypergraph is weakly orderable with respect to dominance, strict set-independence and strong extension.*

Proof. Let $H = (V, E)$ be a hypergraph and $c : V \to \{l, s\}$ a two-coloring of H. Then, let $\leq$ be any linear order on V such that $\forall v, w \in V((c(v) = s \wedge c(w) = l) \to v < w)$. It is clear that such an order exists and that c is globally compatible with $\leq$. Therefore, Proposition 5.24 immediately implies the corollary. □

If we only find a locally compatible two-coloring, the construction above will, in general, not produce an order that satisfies strong extension or even extension. However the constructed order still satisfies dominance and strict independence.

Proposition 5.26. *Let $H = (V, E)$ be a hypergraph and $\leq$ a linear order on V. Assume that there exists a two-coloring of H that is locally compatible with $\leq$. Then there exists a linear order $\preceq$ on E that satisfies dominance and strict set-independence.*

Proof. We define the same linear order $\preceq$ on E as in the proof above. By the same argument as above, this is a linear order that satisfies dominance and strict independence. □

We can not directly extend this result to set-dominance and strict set-independence. Consider the following hypergraph pictured in Figure 5.8:

$$H_1 = (V = \{1, 2, 3, 4\}, E = \{\{1\}, \{4\}, \{1, 2, 3\}, \{2, 3, 4\}, \{1, 4\}, \{1, 2, 3, 4\}\}).$$

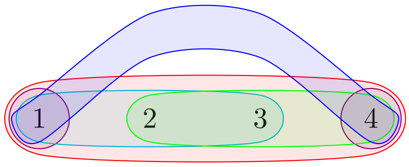

Figure 5.8: Hypergraph H_1

It is easy to check that

$$c(v) := \begin{cases} s \text{ if } v \leq 2 \\ l \text{ if } v \geq 3 \end{cases}$$

is a two-coloring of H_1 that is globally compatible with the natural linear order $\leq$ on V. However, we can show that no weak order $\preceq$ can satisfy set-dominance and strict set-independence with respect to $\leq$. Assume otherwise that there is a weak order $\preceq$ satisfying both axioms. Then, set-dominance implies $\{1\} \prec \{1,2,3\}$. Therefore, we have $\{1,4\} \prec \{1,2,3,4\}$ by strict set-independence. On the other hand, set-dominance implies $\{2,3,4\} \prec \{4\}$. Hence, $\{1,2,3,4\} \prec \{1,4\}$ by strict set-independence. A contradiction!

However, we observe that there is no two-coloring where all singletons are colored the same that is locally compatible with $\leq$. Indeed this property guarantees that we can find an order satisfying set-dominance and strict set-independence.

Definition 5.27. We say a hypergraph H is two-colorable with uniform singletons if there is a two-coloring of H such that all singleton hyperedges have the same color.

Not every two-colorable hypergraph has a two-coloring with uniform singletons. For example, the hypergraph $H = (V = \{1,2\}, E = \{\{1\},\{2\},\{1,2\}\})$ is clearly two-colorable, but every two-coloring needs to assign different colors to the singletons $\{1\}$ and $\{2\}$.

Proposition 5.28. *Let $H = (V, E)$ be a hypergraph and $\leq$ a linear order on V. Assume that there exists a two-coloring with uniform singletons of H that is locally compatible with $\leq$. Then there exists a linear order $\preceq$ on E that satisfies set-dominance and strict set-independence.*

Proof. Assume w.l.o.g. that all singletons are colored s. We define the same linear order $\preceq$ on E as in the proof above. By the same argument as above, this is a linear order that satisfies strict set-independence. We show that it also satisfies set-dominance.

Set-dominance. Let $A, A \cup C \in E$. First, we assume $\max(A) < \min(C)$. Observe that $A \cup C$ must contain at least one vertex v such that $c(v) = l$. This implies $c(x) \neq s$ for at least one $x \in C$ because $v < x$ holds and c is locally compatible with $\leq$. Hence $c(x) = l$. Therefore $A_L \triangle (A \cup C)_L \neq \emptyset$. Furthermore, $A_L \triangle (A \cup C)_L \subseteq C$ clearly holds and hence $A \prec A \cup C$. Now assume $\max(C) < \min(A)$. We observe that A must contain at least one vertex v such that $c(v) = s$ because all singletons are colored s and all other edges are not monochromatic. This implies $c(x) \neq l$ for all $x \in C$ because $x < v$ holds for all $x \in C$ and c is locally compatible with $\leq$. Hence $c(x) = s$ for all $x \in C$. Therefore $A_L \triangle (A \cup C)_L = \emptyset$ and $A_S \triangle (A \cup C)_S = C$ and hence $A \cup C \prec A$. $\square$

Clearly, global compatible still implies strong extension. Hence the following corollary holds.

Corollary 5.29. *Let $H = (V, E)$ be a hypergraph and $\leq$ a linear order on V. Assume that there exists a two-coloring with uniform singletons of H that is globally compatible with $\leq$. Then there exists a linear order $\preceq$ on E that satisfies set-dominance, strict set-independence and strong extension.*

5.3 Regular independence

We now exchange strict by regular independence and first focus on strong DIE^S-orderability for which we give an exact characterization. Then, we consider strong DIE and DI-orderability on two-connected graphs, before we extend our attention to strong DIE and DI-orderability on arbitrary graphs.

Strong DIE^S-orderability

First of all, we can show that any graph that contains at most three vertices is strongly DIE^S-orderable.

Proposition 5.30. *Let X be a set. If $|X| \leq 3$, then $\mathcal{P}(X) \setminus \{\emptyset\}$ is strongly DIE^S-orderable.*

Proof. W.l.o.g. we may assume that $X = \{1, 2, 3\}$ and that $\leq$ is the "less than or equal to" relation on X. We define a weak order $\preceq$ by setting: $\{1\} \prec \{1, 2\} \prec \{2\} \prec \{1, 3\} \sim \{1, 2, 3\} \prec \{2, 3\} \prec \{3\}$. $\square$

While this result shows that cycles of length 3 are strongly DIE^S-orderable, the next result shows that we cannot go much beyond 3-cycles.

Proposition 5.31. *Let G be a connected graph with four or more vertices that contains at least one cycle. Then G is not strongly DIE^S-orderable.*

Proof. Either G contains a cycle of length at least four or a cycle of length three connected to an additional vertex. In the first case let $u, v \in V$ be two non-adjacent vertices contained in the cycle, and let $u, v_1, \ldots, v_n, v$ and $u, u_1, \ldots, u_m, v$ be the two paths from u to v (see Figure 5.9). In the second case let u be the additional vertex and let u_m, v_n, v be the vertices in the cycle such that v_n is connected to u (see Figure 5.10). Define $\leq$ by specifying its strict version $<$ as follows:

$$u < u_1 < \cdots < u_m < v_1 < \cdots < v_n < v.$$

Then there is no weak order on $C(G)$ satisfying dominance, independence and strong extension with respect to $\leq$. Indeed, let us assume otherwise and let $\preceq$ be such a weak order.

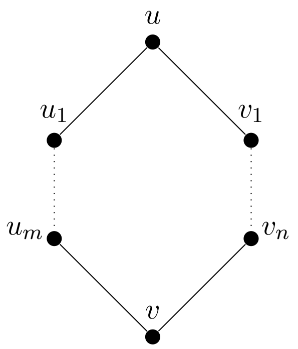

Figure 5.9: A circle with at least 4 vertices.

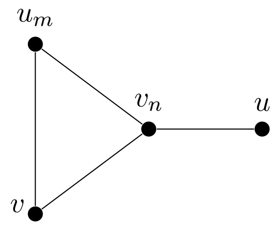

Figure 5.10: A circle with three vertices and connected to an additional vertex u.

Let CY be the set of all vertices in the cycle in the first case and $CY = \{u, v, u_m, v_n\}$ otherwise. Then, $\{u_m\} \prec CY \setminus \{v\}$ by strong extension, and $\{u_m, v\} \preceq CY$ by independence. However, by repeated application of dominance, $\{v_1, \ldots, v_n, v\} \prec \{v\}$ and therefore, by independence, $\{u_m, v_1, \ldots, v_n, v\} \preceq \{u_m, v\}$. It follows that $\{u_m, v_1, \ldots, v_n, v\} \preceq CY$. On the other hand, repeated application of dominance implies $CY \prec \{u_m, v_1, \ldots, v_n, v\}$, a contradiction. □

Therefore, K_3 is the only connected graph that is strongly DIE^S-orderable but not DI^SE^S-orderable (recall Proposition 5.7). Also recall that a graph is strongly DI^SE^S-orderable precisely when it is a forest. Thus, it follows from Propositions 3.34, 5.30 and 5.31 that the class of strongly DIE^S-orderable graphs is only marginally larger than the class of strongly DI^SE^S-orderable graphs.

Theorem 5.32. *The set of strongly DIE^S-orderable graphs consists precisely of graphs for which every connected component is a tree or a cycle K_3.*

Strong DI- and DIE-orderability on two-connected graphs

We now turn to graphs that are strongly DIE-orderable. The next five results allow us to settle the matter of strong $DI(E)$-orderability for two-connected graphs. We observe that the simplest two-connected graphs are cycles. The next result shows that all cycles are strongly DIE-orderable. Additionally, the result implies that replacing strong extension by extension leads to additional strongly orderable graphs.

Proposition 5.33. *Let $G = (V, E)$ be a graph and $\leq$ be a linear order on V. If there is a weak order on $C(G \setminus \{\min(V)\})$ satisfying dominance, independence and strong extension, then there exists a weak order on $C(G)$ satisfying dominance, independence and the extension rule.*

Proof. Wlog we may assume that $V = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $\leq$ is the natural linear order on V. Let $\preceq^*$ be a weak order on $C(G \setminus \{1\})$ satisfying dominance, independence and strong extension. We define a weak order $\preceq$ on $C(G)$ by setting $A \preceq B$ if and only if:

1. $1 \in A$ and $1 \notin B$
2. $1 \notin A \cup B$ and $A \preceq^* B$
3. $1 \in A \cap B$ and $\max(A) \leq \max(B)$.

It follows directly from the definition, that if $1 \notin A \cup B$ and $A \prec^* B$, then $A \prec B$. Similarly, if $1 \in A \cap B$ and $\max(A) < \max(B)$, then $A \prec B$.

We claim that $\preceq$ is a weak order satisfying dominance, independence and the extension rule.

Weak order. Obviously, $\preceq$ is reflexive. To prove transitivity, let $A, B, C \in C(G)$ satisfy $A \preceq B$ and $B \preceq C$. If $A \preceq B$ holds by (1), we have $1 \in A$ and $1 \notin B$. Since $B \preceq C$, it follows that $1 \notin C$ and $B \preceq^* C$. Thus, $A \preceq C$ by (1).

If $A \preceq B$ holds by (2), we know $A \preceq^* B$, $1 \notin A$, and $1 \notin B$. As before, the latter implies that $1 \notin C$ and $B \preceq^* C$. By the transitivity of $\preceq^*$, $A \preceq^* C$. Consequently, $A \preceq C$ by (2).

Finally, if $A \preceq B$ holds by (3), then $1 \in A$, $1 \in B$, and $\max(A) \leq \max(B)$. Since $B \preceq C$, $1 \notin C$, or $1 \in C$ and $\max(B) \leq \max(C)$. In the first case, $A \preceq C$ by (1). In the second case, $\max(A) \leq \max(C)$. Thus, $A \preceq C$ holds by (3).

To show that the relation $\preceq$ is total, let us consider sets $A, B \in C(G)$. If $1 \in A \setminus B$ then $A \preceq B$ by (1). The case $1 \in B \setminus A$ is symmetric. If $1 \notin A \cup B$, $A \preceq B$ or $B \preceq A$ follows as $\preceq^*$ is total. Lastly, if $1 \in A \cap B$, $A \preceq B$ or $B \preceq A$ follows as $\leq$ is total. Thus, $\preceq$ is a weak order.

Extension rule. Since $\preceq^*$ satisfies the strong extension rule, (2) implies that $\{i\} \prec \{j\}$, for all i, j such that $2 \leq i < j \leq n$ (as a matter of fact, here we need only the extension rule). Further, (1) implies that $\{1\} \prec \{j\}$, for $j = 2, 3, \ldots, n$.

Dominance. Let us consider sets $A, A \cup \{x\} \in C(G)$ such that $x < \min(A)$. If $x = 1$ we have $A \cup \{x\} \preceq A$ by (1). Furthermore, we have $A \not\preceq A \cup \{x\}$ (clearly, neither of the conditions (1)-(3) applies). Thus, $A \cup \{x\} \prec A$. If $x \neq 1$, $1 \notin A$ therefore $1 \notin A \cup \{x\}$. Since $\preceq^*$ satisfies dominance, we have $A \cup \{x\} \prec^* A$. By the observation above, it follows that $A \cup \{x\} \prec A$.

Next, let us consider sets $A, A \cup \{x\} \in C(G)$ such that $\max(A) < x$. Then we know $x \neq 1$. Assume that $1 \notin A$. Then, since $\preceq^*$ satisfies dominance, $A \prec^* A \cup \{x\}$. Recall that we have $1 \notin A$ and $1 \notin A \cup \{x\}$. Thus, by the observation above, $A \prec A \cup \{x\}$. Let us assume then that $1 \in A$. Then $1 \in A \cup \{x\}$. Moreover, we have $\max(A) < x = \max(A \cup \{x\})$. Thus, using the observation above, $A \prec A \cup \{x\}$.

Independence. Let us consider sets $A, B \in C(G)$ and an element $x \in V$ such that $x \notin A \cup B$, $A \cup \{x\}, B \cup \{x\} \in C(G)$ and $A \prec B$.

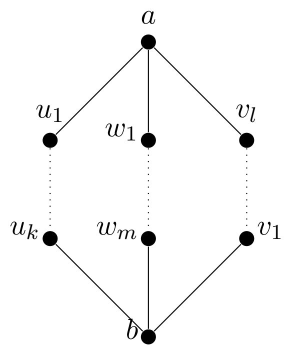

Figure 5.11: Vertices a, b connected by three mutually disjoint paths.

We first assume that $1 \notin A \cup B$. If $x \neq 1$, we have $A \cup \{x\} \preceq^* B \cup \{x\}$ because $\preceq^*$ satisfies independence. Hence, $A \cup \{x\} \preceq B \cup \{x\}$ by (2). If, on the other hand, $x = 1$, we observe that $\max(A) \leq \max(B)$. Indeed, since $\preceq^*$ satisfies strong extension, $\max(B) < \max(A)$ would imply $B \prec^* A$ which, in turn would imply $B \prec A$ (as $1 \notin A \cup B$), a contradiction. Therefore, $A \cup \{x\} \preceq B \cup \{x\}$ by (3).

Next, assume that $1 \in A$ and $1 \notin B$. Then $x \neq 1$ and hence, $A \cup \{x\} \preceq B \cup \{x\}$ by (1).

Finally, assume that $1 \in A, B$. Then $A \prec B$ implies $\max(A) < \max(B)$. Assume first that $x < \max(B)$. Then $\max(A \cup \{x\}) < \max(B) = \max(B \cup \{x\})$ and so, $A \cup \{x\} \preceq B \cup \{x\}$ by (3). If $\max(B) < x$, then $\max(A \cup \{x\}) = \max(B \cup \{x\}) = x$ and hence $A \cup \{x\} \preceq B \cup \{x\}$. Finally, the case $x = \max(B)$ is impossible as $x \notin B$. $\square$

This shows that every cycle is strongly DIE-orderable. In fact, for a cycle G, $G \setminus \{\min_{\leq}(V(G))\}$ is a tree. Since trees are strongly DIE^S-orderable by Theorem 5.32, Proposition 5.33 implies that G is strongly DIE-orderable. We can generalize the result as follows using Proposition 3.34.

Corollary 5.34. *The set of strongly DIE-orderable graphs includes all graphs whose each connected component is a tree or a cycle.*

The following result shows that we can not go much beyond cycles if we want to preserve strong DIE-orderability. The result states that any graph that contains a cycle of length at least six and additionally any path (or edge) between to vertices contained in the cycle is not strongly DI-orderable. An immediate consequence of this result is that a two-connected graph that contains a cycle of length at least six is strongly DIE-orderable if and only if it is a cycle.

Proposition 5.35. *Let $G = (V, E)$ be a graph containing distinct vertices $a, b \in V$ connected by three mutually disjoint paths (not counting a and b) such that two of them have length at least three, or one of the paths has length at least four and one of the remaining two paths is of length two. Then, G is not strongly DI-orderable.*

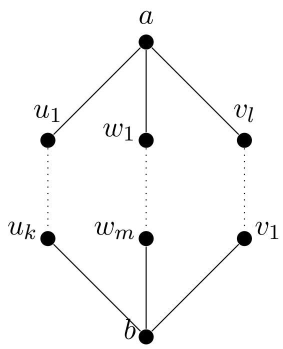

Figure 5.12: Vertices a, b connected by three mutually disjoint paths.

Proof. Let the three paths be $a, u_1, \ldots, u_k, b$; $b, v_1, \ldots, v_l, a$; and $a, w_1, \ldots, w_m, b$ (see Figure 5.11). By the assumption on the lengths of the paths, wlog we will assume $k, m \geq 2$ in the first case, and $k = 2$ and $m \geq 3$ in the second one. Let us also define

$$W = \{a, u_1, \ldots, u_k, v_1, \ldots, v_l, w_1, \ldots, w_m, b\}.$$

We first consider the case when $k, m \geq 2$. Let $\leq$ be any linear order on G such that

$$u_1 < \cdots < u_k < b < v_1 < \ldots < v_l < a < w_1 < \cdots < w_m.$$

It is clear that such orders exist.

Let us assume that there is a weak order on $C(G)$, say $\preceq$, that satisfies dominance and independence. All sets we use in the following belong to $C(G)$. This is easy to see and we will not be making it explicit when we compare sets under $\preceq$.

Suppose $\{b\} \prec W \setminus \{u_1, w_m\}$. By independence,

$$\{b, w_m\} \preceq W \setminus \{u_1\}.$$

Further, since $v_1 < \ldots < v_l < a < w_1 < \cdots < w_m$, repeated application of dominance implies $\{v_1, \ldots, v_l, a, w_1, \ldots, w_m\} \prec \{w_m\}$. Thus, independence implies

$$\{b, v_1, \ldots, v_l, a, w_1, \ldots, w_m\} \preceq \{b, w_m\}.$$

However, since $k > 1$, dominance also implies that

$$W \setminus \{u_1\} \prec \{b, v_1, \ldots, v_l, a, w_1, \ldots, w_m\}.$$

By transitivity, $\{b, w_m\} \prec \{b, w_m\}$, a contradiction.

Therefore, we must have $W \setminus \{u_1, w_m\} \preceq \{b\}$. By repeated application of dominance,

$$\{b\} \prec \{b, v_1, \ldots, v_l, a\} \prec \{a\}.$$

Thus, $W \setminus \{u_1, w_m\} \prec \{a\}$ and so, by independence,

$$W \setminus \{w_m\} \preceq \{u_1, a\}.$$

Further, dominance also implies $\{u_1\} \prec \{u_1, \dots, u_k, b, v_1, \dots, v_l\}$. Hence, by independence,

$$\{u_1, a\} \preceq \{u_1, \dots, u_k, b, v_1, \dots, v_l, a\}.$$

Since $m > 1$, dominance implies

$$\{u_1, \dots, u_k, b, v_1, \dots, v_l, a\} \prec W \setminus \{w_m\}.$$

Thus, by transitivity, $\{u_1, a\} \prec \{u_1, a\}$, a contradiction.

Next, we consider the remaining case $k = 1$ and $m \geq 3$. This time, we assume the order

$$u_1 < b < v_1 < \dots < v_l < a < w_m < \cdots < w_1.$$

Let us assume that $\{a\} \prec W \setminus \{u_1, w_1\}$. Reasoning as before yields a contradiction. Namely, by independence,

$$\{a, w_1\} \preceq W \setminus \{u_1\}.$$

Further, since $w_1 > \cdots > w_m$, repeated application of dominance gives $\{w_1, \dots, w_m\} \prec \{w_1\}$. Thus, independence implies

$$\{a, w_1, \dots, w_m\} \preceq \{a, w_1\}.$$

However, dominance also implies that

$$W \setminus \{u_1\} \prec \{a, w_1, \dots, w_m\}.$$

By transitivity, $\{a, w_1\} \prec \{a, w_1\}$, a contradiction. Hence, we must have $W \setminus \{u_1, w_1\} \preceq \{a\}$. Since $k = 1$ and $m \geq 3$, dominance implies

$$\{b, v_1, \dots, v_l, a, w_m\} \prec \{b, v_1, \dots, v_l, a, w_2, \dots, w_m\} = W \setminus \{u_1, w_1\}.$$

Thus, by transitivity, $\{b, v_1, \dots, v_l, a, w_m\} \prec \{a\}$ and, by independence,

$$\{u_1, b, v_1, \dots, v_l, a, w_m\} \preceq \{u_1, a\}.$$

On the other hand, dominance implies $\{u_1\} \prec \{u_1, b, v_1, \dots, v_l\}$. Thus, by independence and dominance

$$\{u_1, a\} \preceq \{u_1, b, v_1, \dots, v_l, a\} \prec \{u_1, b, v_1, \dots, v_l, a, w_m\},$$

a contradiction. □

This result is optimal in the sense that there exists a strongly DIE-orderable (two-connected) graph that contains two vertices that are connected by three mutually disjoint paths such that one has lengths three, one has length two and one has length one.

Example 5.36. Consider the graph shown in Figure 5.13. Obviously, this is a connected graph containing two cycles. However, we know that Kannai and Peleg's impossibility result is minimal in the sense that $\mathcal{P}(\{1, 2, 3, 4, 5\})$ can be ordered with a weak order satisfying dominance, independence and extension (Corollary 3.17). This implies immediately that any graph with five or fewer vertices is strongly DIE-orderable, hence also the one shown in Figure 5.13.

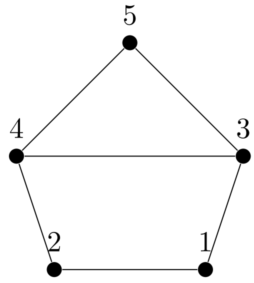

Figure 5.13: A graph where two vertices are connected by three disjoint path of length one, two and three

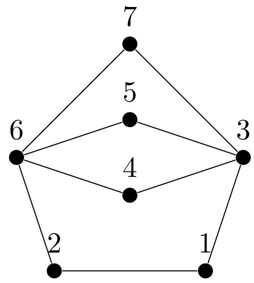

Figure 5.14: The ordering used in the proof of Proposition 5.37

However if we increase the number of paths we quickly run into impossibility again.

Proposition 5.37. *If a graph G contains two distinct vertices a and b connected by four mutually disjoint paths (not counting a and b) of length at least two, with one of them of length at least three, then G is not strongly DI-orderable.*

Proof. Assume that two of the paths have three of more edges. Then, Proposition 5.35 applies and G is not strongly DI-orderable. Therefore, we may assume that three of the four paths have length 2. If the fourth path has length at least four, G is not strongly DI-orderable again by Proposition 5.35. Therefore, we will assume that this path has exactly three edges. It follows that G contains a subgraph like the one shown in Figure 5.14. Clearly, if that graph is not strongly DI-orderable, G is not strongly orderable either. Therefore, to prove the assertion, we will prove that the graph in Figure 5.14 is not strongly DI-orderable.

Let us consider the labeling of the vertices shown in Figure 5.14 and take for a linear order on this graph the order induced by the natural order of integers. We claim that there is a weak order $\preceq$ on $C(G)$ satisfying dominance and independence with respect to $\leq$.

First we assume $\{3\} \prec \{2,3,4,5,6\}$. Then, by independence, $\{3,7\} \preceq \{2,3,4,5,6,7\}$. However, we have $\{4,5,6,7\} \prec \{7\}$ by dominance and therefore $\{3,4,5,6,7\} \preceq \{3,7\}$ by independence. But then we have $\{3,4,5,6,7\} \preceq \{2,3,4,5,6,7\}$ by transitivity, which contradicts dominance.

Therefore, we must have $\{2,3,4,5,6\} \preceq \{3\}$. Now observe that $\{1\} \prec \{1,2\}$ by dominance and therefore, by independence and dominance, $\{1,3\} \preceq \{1,2,3\} \prec \{1,2,3,4\}$. Hence, by independence, $\{1,3,5\} \preceq \{1,2,3,4,5\}$. Further, by dominance $\{3\} \prec \{3,5\}$ and so, by transitivity, $\{2,3,4,5,6\} \prec \{3,5\}$. Independence implies $\{1,2,3,4,5,6\} \preceq \{1,3,5\}$ and transitivity implies $\{1,2,3,4,5,6\} \preceq \{1,2,3,4,5\}$. This contradicts $\{1,2,3,4,5\} \prec \{1,2,3,4,5,6\}$, which we have by dominance. □

Propositions 5.35 and 5.37 specify sufficient conditions for a graph to not be strongly DIE-orderable. The next result gives a sufficient condition for a graph to be strongly DIE-orderable.

Proposition 5.38. *Let G be a graph consisting of two vertices a, b, where $a \neq b$, connected by arbitrarily many paths of length at most two. Then G is strongly DIE-orderable*

Proof. Let us consider any linear order on the set V of vertices of G. Wlog we may assume that $a < b$. Under this assumption, we define the sets $L = \{v \in V \mid b < v\}$, $I = \{v \in V \mid a \leq v \leq b\}$ and $S = \{v \in V \mid v < a\}$.

Next, we define two orders $\prec^+$ and $\prec^-$ on the family of all subsets of V by setting

1. $A \prec^+ B$ if and only if $\max(A \triangle B) \in B$
2. $A \prec^- B$ if and only if $\min(A \triangle B) \in A$.

It is straightforward to verify that both relations are indeed weak orders.

We also define a weak order $\preceq^*$ on $C(G)\big|_I$, the set of all nonempty subsets of I inducing in G a connected subgraph. Clearly, every set in $C(G)\big|_I$ is either a singleton, or contains at least one of a and b. To define $\preceq^*$, for each $A \in C(G)\big|_I$ we define its type, written *type*(A):

$$type(A) := \begin{cases} a \text{ if } a \in A, b \notin A \\ b \text{ if } b \in A, a \notin A \\ ab \text{ if } a, b \in A \\ 0 \text{ if } a, b \notin A \end{cases}$$

We note that *type*$(A) = 0$ if and only if A is a singleton set other than $\{a\}$ and $\{b\}$. We order the types $a < 0 < ab < b$ (we point out that the order on types is unrelated to the order on V that we are considering; it will always be clear from the context whether we are comparing types or elements of V).

With these concepts in hand, for $A, B \in C(G)\big|_I$ we set $A \preceq^* B$ if and only if

1. $type(A) < type(B)$,
2. $type(A) = type(B) = a$ and $A \prec^+ B$,
3. $type(A) = type(B) = 0$, $A = \{v\}$, $B = \{w\}$ and $v < w$,

4. $type(A) = type(B) = ab$

5. $type(A) = type(B) = b$ and $A \prec^- B$,

We claim that $\preceq^*$ is a weak order on $C(G)|_I$ satisfying dominance, independence and extension. It is indeed a weak order, because $\preceq^*$ restricted to sets of any type is a weak order, and the types are linearly ordered.

Extension Assume $v, w \in V$ and $v < w$. If $w = b$ or $v = a$ then $type(\{v\}) < type(\{w\})$. Thus, $\{v\} \prec^* \{w\}$ by (1). If $v, w \notin \{a, b\}$ then $type(\{v\}) = type(\{w\}) = 0$ and $\{v\} \prec^* \{w\}$ holds by (3).

Dominance Let us consider a set $A \in C(G)|_I$ and an element $x \in I \setminus A$ such that $A \cup \{x\} \in C(G)|_I$. We need to show that if $x < \min(A)$ then $A \cup \{x\} \prec^* A$, and if $x > \max(A)$, then $A \prec^* A \cup \{x\}$.

We consider the case $x < \min(A)$. The other one is dual. Clearly, $x < \min(A)$ implies that $type(A) \neq a, ab$. Let us assume first that $type(A) = 0$. It follows that $x = a$. Thus, $type(A \cup \{x\}) = a$ and $A \cup \{x\} \prec^* A$ holds by (1). The only other possibility is that $type(A) = b$. If $x = a$, we have $type(A \cup \{x\}) = ab$. Hence, $A \cup \{x\} \prec^* A$ holds by (1). If $x \neq a$, $type(A \cup \{x\}) = b$. Moreover, $A \cup \{x\} \prec^- A$ (because $\min(A \triangle (A \cup \{x\})) = x \in A \cup \{x\}$). Hence, $A \prec^* A \cup \{x\}$ holds by (5).

Independence Let us consider sets $A, B \in C(G)|_I$ and an element $x \in I \setminus (A \cup B)$ such that $A \prec^* B$ and $A \cup \{x\}, B \cup \{x\} \in C(G)|_I$. First, let us assume that $x \notin \{a, b\}$. Then $A \prec^* B$ holds by one of the conditions (1), (2), (4) or (5). It also follows that $type(A) = type(A \cup \{x\})$, $type(B) = type(B \cup \{x\})$, and $A \triangle B = (A \cup \{x\}) \triangle (B \cup \{x\})$. It is now easy to see that if $A \prec^* B$ holds by the condition (i), where i=1, 2, 4, or 5, then the same condition (i) implies that $A \cup \{x\} \preceq^* B \cup \{x\}$.

If $x = a$, then $a \notin A \cup B$. It follows that $type(A) = 0$ or b, $type(B) = 0$ or b, and $A \prec^* B$ holds by the condition (1), (3) or (5). In the first case, $type(A) = 0$ and $type(B) = b$. Thus, $type(A \cup \{x\}) = a$, $type(B \cup \{x\}) = ab$ and, consequently, $A \cup \{x\} \preceq^* B \cup \{x\}$ holds by (1). In the second case, there are $v, w \in I \setminus \{a, b\}$ such that $A = \{v\}$, $B = \{w\}$ and $v < w$. It follows that $type(A \cup \{x\}) = type(B \cup \{x\}) = a$, and $\max((A \cup \{x\}) \triangle (B \cup \{x\}) = w \in B$. Thus, $A \cup \{x\} \preceq^* B \cup \{x\}$ by (2). In the third case, $type(A \cup \{x\}) = type(B \cup \{x\}) = ab$, and $A \cup \{x\} \preceq^* B \cup \{x\}$ holds by (4). The case $x = b$ is similar.

Using the three orders defined above we now define a weak order $\preceq$ on $C(G)$. We set $A \preceq B$ if and only if

1. $A \cap L \prec^+ B \cap L$,

2. $A \cap L = B \cap L$ and $A \cap S \prec^- B \cap S$,

3. $A \cap L = B \cap L \neq \emptyset \neq A \cap S = B \cap S$,

4. $A \cap L = B \cap L \neq \emptyset = A \cap S = B \cap S$ and $a \in A$,
5. $A \cap L = B \cap L \neq \emptyset = A \cap S = B \cap S$, $a \notin A, B$ and $A \cap I \prec^- B \cap I$,
6. $A \cap L = B \cap L = \emptyset \neq A \cap S = B \cap S$ and $b \in B$,
7. $A \cap L = B \cap L = \emptyset \neq A \cap S = B \cap S$, $b \notin B$ and $A \cap I \prec^+ B \cap I$.
8. $A \cap L = B \cap L = \emptyset = A \cap S = B \cap S$ and $A \preceq^* B$.

The relation $\preceq$ is indeed a weak order. To see it, we observe that for every two sets $S' \subseteq S$ and $L' \subseteq L$, the family of sets $X \in C(G)$ such that $X \cap S = S'$ and $X \cap L = L'$ is ordered (for each family, there is a condition among the conditions (3) - (8) that is used to compare any two of its sets). Further, each set $X \in C(G)$ belongs to one of these families. Finally, the conditions (1) and (2) impose on these families a linear order that implies an ordering for pairs of sets coming from different families. We will now prove that the order $\preceq$ satisfies extension, dominance, and independence.

Extension: Assume $v, w \in V$ and $v < w$. If $w \in L$ then $\{v\} \prec \{w\}$ by (1). Otherwise, if $v \in S$ then $\{v\} \prec \{w\}$ by (2). Hence assume $v, w \in I$. Then $\{v\} \prec^* \{w\}$ because $\preceq^*$ satisfies extension. Therefore $\{v\} \prec \{w\}$ by (8).

Dominance: Let us consider a set $A \in C(G)$ and an element $x \notin A$ such that $A \cup \{x\} \in C(G)$. First we assume $x \in L$. Then $x < \min(A)$ is impossible. Indeed, it would imply that $A \subseteq L$ and $A \cup \{x\} \subseteq L$. The latter set has at least two elements. However, the only sets in $C(G)$ contained in L are singletons, a contradiction. So, assume $\max(A) < x$. Then $\max(A \triangle (A \cup \{x\})) = x \in A \cup \{x\}$ and we obtain $A \prec A \cup \{x\}$ by (1). The case $x \in S$ is dual.

Hence, assume that $x \in I$. Furthermore, assume $\max(A) < x$. Then $A \cap L = \emptyset = (A \cup \{x\}) \cap L$ and $A \cap S = (A \cap \{x\}) \cap S$. Let us assume that $A \cap S \neq \emptyset$. Then if $x = b$ we have $A \prec A \cup \{x\}$ by (6). On the other hand, if $x \neq b$ then $b > x > \max(A)$. It follows that $b \notin A \cup \{x\}$ and $\max((A \cap I) \triangle ((A \cup \{x\}) \cap I)) = x \in (A \cup \{x\}) \cap I$. Hence, $A \prec A \cup \{x\}$ by (7). Finally, if $A \cap S = \emptyset$, we have $A \prec^* A \cup \{x\}$ as $\preceq^*$ satisfies dominance. Hence $A \prec A \cup \{x\}$ by (8). The case $x < \min(A)$ is similar.

Independence: Let us consider sets $A, B \in C(G)$ and an element $x \notin A \cup B$ such that $A \cup \{x\}, B \cup \{x\} \in C(G)$ and $A \prec B$. We distinguish eight cases based on the reason $A \prec B$ holds.

First assume $A \prec B$ by (1). Then, $A \cup \{x\} \prec B\{x\}$ by (1) as $(A \cup \{x\} \triangle B \cup \{x\}) = (A \triangle B)$. The same identity also shows that if $A \prec B$ holds by (2), then $A \cup \{x\} \prec B\{x\}$ holds also by (2). Next, we note that $A \prec B$ cannot hold by (3) (otherwise, we would also have $B \prec A$, a contradiction). Let us assume then that $A \prec B$ holds by (4). If $x \notin S$, then it follows immediately that (4) applies to imply $A \cup \{x\} \prec B\{x\}$. So assume that $x \in S$. Then, $A \cup \{x\} \preceq B \cup \{x\}$ follows from (3). Next, let $A \prec B$ hold by (5). If $x \in S$, we reason as above and derive $A \cup \{x\} \preceq B \cup \{x\}$ from (3). So, let us assume that $x \in L$.

Since $((A \cup \{x\}) \cap I) \triangle ((B \cup \{x\}) \cap I) = (A \cap I) \triangle (B \cap I)$, $(A \cup \{x\}) \cap I \prec^- (B \cup \{x\}) \cap I$. It follows that (5) applies to imply $A \cup \{x\} \preceq B \cup \{x\}$. In the case $x \in I$, we have either $x = a$ or $a \notin A \cup \{x\}, B \cup \{x\}$. In the first case, $A \cup \{x\} \prec B \cup \{x\}$ holds by (4). In the second case, we have $(A \cup \{x\}) \cap I \prec^- (B \cup \{x\}) \cap I$, which follows from the identity $((A \cup \{x\}) \cap I) \triangle ((B \cup \{x\}) \cap I) = (A \cap I) \triangle (B \cap I)$. Thus, $A \cup \{x\} \prec B \cup \{x\}$ by (5). The cases $A \prec B$ by (6) or (7) are similar.

Finally assume that $A \prec B$ by (8). It follows that $A, B \in C(G)\big|_I$. If $x \in I$, then $A \cup \{x\} \preceq B \cup \{x\}$ by (8) because $\preceq^*$ satisfies independence. So assume $x \in S \cup L$. Observe that $type(A) \neq 0 \neq type(B)$ is impossible as elements in L and S are only connected to a and b. Assume $b \in A$. Then we know $b \in B$ and $A \prec^- B$. Therefore $x \in L$ implies $A \cup \{x\} \preceq B \cup \{x\}$ either by (4) or (5). If $x \in S$, we have $A \cup \{x\} \preceq B \cup \{x\}$ by (6) as $b \in B$. The case $a \in A$ is similar. □

Proposition 5.38 implies that every two-connected graph with a longest cycle of length four is strongly DIE-orderable (See Figure 5.15). We will make this observation formal later on as a part of a more general result on DIE-orderability of two-connected graphs.

The next result, while being of interest in its own right, is the last piece we need to classify all two-connected strongly DIE-orderable graphs.

Proposition 5.39. *Let $G = (V, E)$ be a graph containing two cycles C_1 and C_2 that have exactly one vertex in common, with one of the cycles having length at least 4. Then, G is not strongly DI-orderable.*

Proof. Let $v_1, \ldots, v_k$ and $v_k, v_{k+1}, \ldots, v_n$ be the vertices of the cycles C_1 and C_2 enumerated consistently with their order on the corresponding cycle, where v_k is the unique common vertex of the two cycles. By our assumptions, $k \geq 3$ and $n - k \geq 3$. Moreover, there are edges in G between 1 and k, and between k and n.

Clearly, it suffices to show the assertion under the assumption that G has no other vertices. Thus, we adopt this assumption for the remainder of the proof. To simplify the presentation, let us identify v_i with i. In particular, $V = \{1, \ldots, n\}$.

Let us consider a linear order $\leq$ on V induced by the usual order on the integers. Let us assume that $\preceq$ is a weak order on $C(G)$ that satisfies dominance and independence with respect to $\leq$. We will derive a contradiction, which will prove the result.

All sets we use in the argument belong to $C(G)$. This is easy to see and we will not be making it explicit when we compare sets under $\preceq$. Let us assume that $\{k\} \prec V \setminus \{1, n\}$. By independence, $\{k, n\} \preceq V \setminus \{1\}$. In addition, by repeated application of dominance, we get $\{k+1, \ldots, n\} \prec \{n\}$ and, by independence, $\{k, k+1, \ldots, n\} \preceq \{k, n\}$.

However, since $k \geq 3$, repeated application of dominance implies $V \setminus \{1\} \prec \{k, k+1, \ldots, n\}$. By transitivity, $\{k, n\} \prec \{k, n\}$, a contradiction.

Thus, we have $V \setminus \{1, n\} \preceq \{k\}$. Let us assume that $\{k\} \preceq \{2, \ldots, k+1\}$. By transitivity, $V \setminus \{1, n\} \preceq \{2, \ldots, k+1\}$. On the other hand, since $n - k \geq 3$, $k+1 < n-1$. Thus, by repeated application of dominance $\{2, \ldots, k+1\} \prec V \setminus \{1, n\}$, a contradiction.

It follows that $\{2, \ldots, k+1\} \prec \{k\}$. This implies $\{1, \ldots, k+1\} \preceq \{1, k\}$ by independence. However, we also have $\{1\} \prec \{1, \ldots, k-1\}$ by dominance. Hence

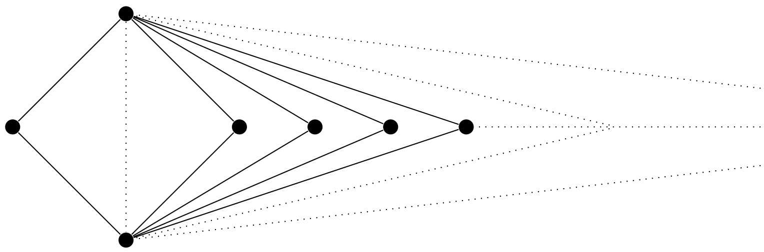

Figure 5.15: Two-connected graphs with longest cycle of length four

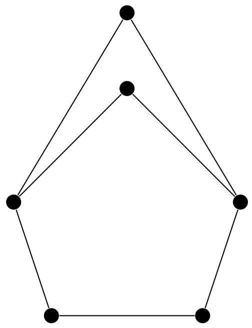

Figure 5.16: The graph T_5

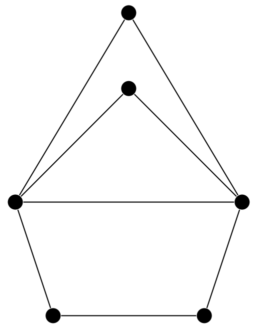

Figure 5.17: The graph T_5^+

$\{1, k\} \preceq \{1, \ldots, k\} \prec \{1, \ldots, k+1\}$ by independence and dominance, and we reach a contradiction! $\square$

We are now ready to provide a complete characterization of the two-connected strongly $DI(E)$-orderable graphs. We know from Corollary 5.34 that cycles are strongly DIE-orderable. Further, Proposition 5.35 tells us that any two-connected graph *properly* containing a cycle of length at least six is not strongly DI-orderable.

Let now G be a two connected graph with a longest cycle having length 5. Let C be one such cycle. If G has only five vertices, it is strongly DIE-orderable as discussed in Example 5.36. Thus, let us assume that G has at least one vertex not on the cycle C. Let f be any vertex of $G \setminus C$ connected to (a vertex on) C by an edge. Such vertex exists as G is connected. Let a be a neighbor of f in C. Since G is two-connected, f is connected by a path in $G \setminus \{a\}$ to a vertex in C other than a. Let P be a shortest such path and let $b \in C$ be the end of P. If P has length at least two or if b is a neighbor of a, then G contains a cycle of length at least 6, a contradiction. Thus, b is not a neighbor of a and f is connected to a and b by edges. This situation is illustrated in Figure 5.16. Let us assume that G has yet another vertex. Then, by connectivity, it has a vertex, say g, connected by an edge to f or to a vertex in C. If g is connected to f, then G is connected to a vertex in C by a path in $G \setminus \{f\}$. In such case, G has a cycle of length at least 6, a contradiction. Thus, g is connected by an edge to a vertex in C. Reasoning as for f, we argue that g must be connected to two vertices in C that are not connected in C. Unless g is connected to a and b, G contains a cycle of length 6 or two cycles of length 4 that share exactly one vertex. The first possibility contradicts our assumption. In the second case, G is not strongly DI-orderable by Proposition 5.39. Thus, let us assume that g is connected by edges to a and b. In this case, G is not strongly DI-orderable by Proposition 5.37. This leaves us with the case when G is as shown in Figure 5.16, with possibly some more edges added. However, unless the added edge is just like the one shown in Figure 5.17, G contains a cycle of length 6. For the two graphs T_5 and T_5^+ shown in Figures 5.16 and 5.17, we found that they are strongly DIE-orderable by a computer search. We provide either a computer-generated weak order satisfying dominance, independence and extension on $C(T_5^+)$ or a proof of existence of such an order for every possible linear order on the vertices of T_5^+ in the appendix.

Next, let us assume then that a longest cycle in a two-connected graph G has length 4 and let C be one such cycle. If G has more than four vertices, there is a vertex in G, say e, connected by an edge to a vertex in C. Let us denote this vertex by a. Since G is two-connected, there is a path in $G \setminus \{a\}$ connecting e to a vertex in $C\{a\}$. Let P be a shortest path like that. If that path connects e to a neighbor of a in C, then G contains a cycle of length 5, a contradiction. If that path connects a to the only non-neighbor of a in C, say b, and has more than one edge, G contains a cycle of length 5, a contradiction again. Thus, e is connected by edges to a and b. If G has any other vertices, one can show reasoning as above that G has a cycle of length at least 5, or that each of these vertices is connected to a and b by edges. The first situation contradicts our assumption. Thus, G is of the form shown in Figure 5.15. Hence, it is strongly DIE-orderable by Proposition 5.38.

The only two-connected graphs with longest cycle less than four are the triangle and the graphs consisting of a single edge. These are all obviously strongly DIE-orderable.

Let us observe that graphs that are not strongly DI-orderable are not strongly DIE-orderable, and that graphs that are strongly DIE-orderable are also strongly DI-orderable. Together with the discussion above, this proves the following result on two-connected graphs.

Theorem 5.40. *A two-connected graph is strongly DI- and DIE-orderable if and only if it lies in one of the following classes:*

- *Cycles*
- *Graphs with fewer than six vertices*
- *Graphs that contain no cycle of length five or more*
- T_5 *and* T_5^+

Theorem 5.40 implies that for two-connected graphs the concepts of strong DI- and

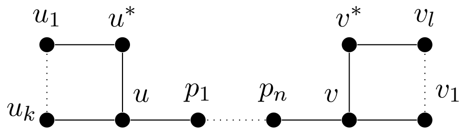

Figure 5.18: Two cycles connected by a path of even length.

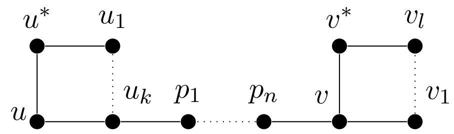

Figure 5.19: Two cycles connected by a path of odd length.

DIE-orderability coincide, and by Proposition 3.34 this result can be extended to graphs with two-connected components.

Strong DI- and DIE-orderability on arbitrary graphs

Next, we outline the extent of the strong $DI(E)$-orderability for graphs that are connected but not two-connected. To this end, we will need two additional auxiliary results. The first one shows that graphs containing two vertex-disjoint cycles connected with a path are not strongly DIE-orderable if both cycles have length at least four.

Proposition 5.41. *Let G be a graph containing two vertex-disjoint cycles of length at least four. If these cycles are connected by a path, then G is not strongly DIE-orderable.*

Proof. First, assume that the path connecting the two cycles has even length. Let $u, p_1, \dots, p_n, v$ be the path connecting the two cycles. Then, let $u, u^*, u_1, \dots, u_k$ be the cycle containing u and $v, v_1, \dots, v_l, v^*$ the circle containing v (see Figure 5.18). We define a linear order $\leq$ by

$$u_1 < \dots < u_k < \dots < p_{n-3} < p_{n-1} < v < v^* < u^* < u < \dots \\ < p_{n-2} < p_n < v_1 < \dots < v_l.$$

If the path has odd length, then let $u_k, p_1, \dots, p_n, v$ be the path, u be a neighbor of u_k and $u, u^*, u_1, \dots, u_k$ be the cycle containing u. As above, let $v, v_1, \dots v_l, v^*$ the circle containing v (see Figure 5.19) and $\leq$ the same order as above. We claim that there is no weak order on $C(G)$ satisfying dominance, independence and the extension rule with respect to $\leq$.

Assume otherwise and let $\preceq$ be such an order. Assume that $\{v^*\} \prec \{p_n, v\}$ (for $n = 0$ replace p_n by u) By extension and independence we know that $\{p_n, v\} \preceq \{v, v_1\}$. Hence $\{v^*\} \prec \{v, v_1\}$. Further, by dominance, $\{v\} \prec \{v, v^*\}$ and, by independence, $\{v, v_1\} \preceq \{v, v^*, v_1\}$. By repeated application of dominance, $\{v, v^*, v_1\} \prec \{v, v^*, v_1, \dots, v_{l-1}\}$. Thus, by transitivity, $\{v^*\} \prec \{v, v^*, v_1, \dots, v_{l-1}\}$ and, hence, by independence $\{v^*, v_l\} \preceq \{v, v^*, v_1, \dots, v_l\}$. We also have $\{v_1, \dots, v_l\} \prec \{v_l\}$ by dominance and $\{v^*, v_1, \dots, v_l\} \preceq \{v^*, v_l\}$ by independence. Hence, we have $\{v^*, v_1, \dots, v_l\} \preceq \{v, v^*, v_1, \dots, v_l\}$ contradicting dominance.

It follows that $\{p_n, v\} \preceq \{v^*\}$ ($\{u, v\} \preceq \{v^*\}$, if $n = 0$). By extension, $\{v^*\} \prec \{u^*\}$. Thus, $\{p_n, v\} \prec \{u^*\}$ ($\{u, v\} \preceq \{u^*\}$, if $n = 0$). Observe that for $n = 0$ we have

$\{u_k\} \prec \{v\}$ by extension and therefore by independence $\{u_k, u\} \preceq \{u, v\}$. By a sequence of similar arguments, for a path of even length we can derive $\{u_k, u\} \preceq \{u, p_1\} \preceq \cdots \preceq \{p_{n-1}, p_n\} \preceq \{p_n, v\}$ and, for a path of odd length, $\{u_k, u\} \preceq \{u_k, p_1\} \preceq \cdots \preceq \{p_{n-1}, p_n\} \preceq \{p_n, v\}$. Using this observation, we get $\{u_k, u\} \prec \{u^*\}$. From $u^* < u$ we get by dominance $\{u^*, u\} \prec \{u\}$ and hence by independence $\{u_k, u^*, u\} \preceq \{u_k, u\}$. By dominance, we can extend this to $\{u_2, \dots, u_k, u^*, u\} \prec \{u_k, u^*, u\}$. By transitivity we get $\{u_2 \dots, u_k, u^*, u\} \prec \{u^*\}$. Therefore, by independence $\{u_1, \dots, u_k, u^*, u\} \preceq \{u_1, u^*\}$. Applying dominance we obtain $\{u_1, \dots, u_k, u^*\} \prec \{u_1, \dots, u_k, u^*, u\}$. Thus, by transitivity, $\{u_1, \dots, u_k, u^*\} \prec \{u_1, u^*\}$.

On the other hand, repeated application of dominance implies $\{u_1\} \prec \{u_1, \dots, u_k\}$. Thus, by independence, $\{u_1, u^*\} \preceq \{u_1, \dots, u_k, u^*\}$, a contradiction. □

Observe that this result does not tell us whether such graphs are strongly DI-orderable. Indeed we used a computer program to check that a graph consisting of two cycles of length four connected by an edge is strongly DI-orderable. This implies that strong DI- and strong DIE-orderability are not equivalent on arbitrary graphs.

The next result states that whenever removing an edge from a graph with a given order over its vertices leads to two disjoint graphs such that one can ordered with respect to dominance and independence and the other can be ordered with respect to dominance and *strict* independence, then the original graph can also be ordered with respect to dominance and independence.

Proposition 5.42. *Let $G = (V, E)$ be a connected graph and $\leq$ a linear order on V. Let $vw \in E$ be an edge of G such that $(V, E \setminus \{vw\})$ is a graph with two connected components G' and G''. If $C(G')$ can be ordered satisfying dominance and independence and $C(G'')$ can be ordered satisfying dominance and strict independence then $C(G)$ can be ordered satisfying dominance and independence.*

Proof. Let us assume that $G' = (V', E')$ and $G'' = (V'', E'')$. Wlog we may assume that $v \in V'$, $w \in V''$. We will present the proof under the assumption that $v < w$. The other case, $w < v$, works analogously.

We partition $C(G)$ in three collections of sets: $P_1 = C(G')$, $P_2 = \{A \in C(G) \mid v, w \in A\}$ and $P_3 = C(G'')$. Let $\preceq_1$ be any weak order satisfying dominance and independence on P_1 (with respect to $\leq$ restricted to V'), and $\preceq_3$ any *linear* order satisfying dominance and strict independence on P_3 (with respect to $\leq$ restricted to V''). We define a weak order $\preceq$ on $C(G)$ by setting $A \preceq B$ (where $A, B \in C(G)$) if and only if

1. $A, B \in P_1$, and $A \preceq_1 B$
2. $A, B \in P_3$, and $A \preceq_3 B$
3. $A, B \in P_2$ and $A \cap V'' \prec_3 B \cap V''$
4. $A, B \in P_2$ and $A \cap V'' = B \cap V''$ and $A \cap V' \preceq_1 B \cap V'$
5. $A \in P_i$, $B \in P_j$ and $i < j$ (in fact, in this case, $A \prec B$ holds).

The relation $\preceq$ is obviously a weak order. We claim that it satisfies dominance and independence.

Dominance. Assume that $A, A \cup \{x\} \in C(G)$ and $\max(A) < x$. If $A, A \cup \{x\} \in P_i$ for some i, the result is clear. If $A \in P_3$ and $A \cup \{x\} \notin P_3$, then we have $w \in A$ and $x = v$. Since $\max(A) < x$, we have $w < v$, a contradiction. If $A \in P_1$ and $A \cup \{x\} \notin P_1$ then $A \prec A \cup \{x\}$ by the condition (5). Finally, if $A \in P_2$ then $A \cup \{x\} \notin P_2$ is impossible. The case $\min(A) > x$ is symmetric.

Independence. Assume that $A, B, A \cup \{x\}, B \cup \{x\} \in C(G)$, $x \notin A \cup B$, and $A \prec B$.
Case 1. $A, B \in P_1$ or $A, B \in P_3$. If $A, B \in P_1$, then $x = w$ or $x \in V'$. If $A, B \in P_3$, then $x = v$ of $x \in V''$. Let us assume that $A, B \in P_1$ and $x = w$, or $A, B \in P_3$ and $x = v$. In each case, $A \cup \{x\}, B \cup \{x\} \in P_2$. In the first case, $(A \cup \{x\}) \cap V'' = \{x\} = (B \cup \{x\})$. Clearly, $A \prec B$ implies $B \not\preceq A$. Since $A, B \in P_1$, we have $A \preceq_1 B$ and $B \not\preceq_1 A$. Thus, $A \prec_1 B$. Clearly, $(A \cup \{x\}) \cap V' = A$ and $(B \cup \{x\}) \cap V' = B$. It follows that $A \cup \{x\} \prec_1 B \cup \{x\}$. Therefore, $A \cup \{x\} \preceq B \cup \{x\}$ holds by (4). In the second case, since $A \prec B$, we have $A \prec_3 B$. Reasoning as before, we obtain $(A \cup \{x\}) \cap V'' \prec_3 (B \cup \{x\}) \cap V''$ and so, $A \cup \{x\} \preceq B \cup \{x\}$ holds by (3).

Thus, let us assume that $A, B \in P_1$ and $x \in V'$, or $A, B \in P_3$ and $x \in V''$. In the first case, $A \prec B$ implies $A \preceq_1 B$. Therefore, since $A \cup \{x\}, B \cup \{x\} \in P_1$, we have $A \cup \{x\} \preceq B \cup \{x\}$ by (1). The case $A, B \in P_3$ and $x \in V''$ can be dealt with in a similar way.
Case 2. $A, B \in P_2$. This implies that $A \cup \{x\}, B \cup \{x\} \in P_2$. Let us assume that $x \in V''$. Clearly, $A \prec B$ is either by (3) or (4). Assume $A \prec B$ is by (3). It follows that $A \cap V'' \prec_3 B \cap V''$. Hence, by strict independence of $\preceq_3$, we have $(A \cup \{x\}) \cap V'' \prec_3 (B \cup \{x\}) \cap V''$ (indeed, we note that $(A \cap V'') \cup \{x\} = (A \cup \{x\}) \cap V''$, and similarly for B). This implies that $(A \cup \{x\}) \cap V'' \preceq (B \cup \{x\}) \cap V''$ by (3). Let us assume then that $A \prec B$ by (4). Then $A \cap V'' = B \cap V''$ and $A \cap V' \prec_1 B \cap V'$. But then also $(A \cup \{x\}) \cap V'' = (B \cup \{x\}) \cap V''$ and $(A \cup \{x\}) \cap V' = A \cap V' \prec_1 B \cap V' = (B \cup \{x\}) \cap V'$. This implies $A \cup \{x\} \prec B \cup \{x\}$ by (4). The case $x \in V'$ is similar.
Case 3. $A \in P_2$ and $B \notin P_2$ or $B \in P_2$ and $A \notin P_2$. Let us assume that $A \in P_2$ and $B \notin P_2$. Since $A \prec B$, $B \in P_3$. Let us assume that $x \in V'$. Since $B \cup \{x\} \in C(G)$, $x = v$. Thus, $x \in A$, a contradiction. It follows that $x \in V''$ and, consequently, $B \cup \{x\} \in P_3$. Since $A \cup \{x\} \in P_2$, $A \cup \{x\} \prec B \cup \{x\}$ follows by (5). The case when $B \in P_2$ and $A \notin P_2$ is similar.
Case 4. $A \in P_1$ and $B \in P_3$. Then either $x = v$ and $w \in B$ or $x = w$ and $v \in A$. In the first case $A \cup \{x\} \in P_1$ and $B \cup \{x\} \in P_2$ and hence $A \cup \{x\} \prec B \cup \{x\}$ by (5). The other case is similar. $\square$

The next two results describe our knowledge of the extent of strong *DIE*- and *DI*-orderability. To formulate them we need more notation. We recall that a *biconnected* component of a graph G is any maximal two-connected subgraph of G. We note that a single edge is two-connected and may appear in a graph as its biconnected component (it is the case, when removing this edge disconnects the graph). Every graph G can be viewed as a tree-like structure composed of its biconnected components, in which

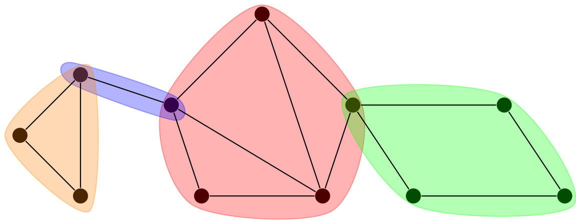

Figure 5.20: A graph and its biconnected components

whenever two biconnected components share a node, this node must be an articulation point. This representation of a graph is illustrated in Figure 5.20. We denote by $\mathcal{B}$ the set of all two-connected graphs that are strongly DIE-orderable (cf. Theorem 5.40). Next, we call a biconnected component of G *large* if it contains a cycle of length at least four. Thus, if a biconnected component is not large, it consists of a single edge or is a cycle of length three.

We define now four classes of graphs, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Definition 5.43. Let G be a graph. Then, G is in $\mathcal{C}_3$ if all its biconnected components belong to $\mathcal{B}$ and no large biconnected component of G shares an articulation point with another non-edge biconnected component of G. Furthermore, G is in $\mathcal{C}_2$ if $G \in \mathcal{C}_3$ and G has at most one large biconnected component. Now, G is in $\mathcal{C}_1$ if $G \in \mathcal{C}_2$ and has at most one non-edge biconnected component. Finally, G is in $\mathcal{C}_0$ if $G \in \mathcal{C}_1$ and either every biconnected component of G is an edge or no biconnected component of G is an edge (i.e. G is either a tree or in $\mathcal{B}$).

Clearly, $\mathcal{C}_0 \subseteq \mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3$. Figures 5.21-5.23 show examples of graphs in $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$.

Proposition 5.44. *All graphs in $\mathcal{C}_1$ are strongly DI-orderable. If a graph is not in $\mathcal{C}_3$ then it is not strongly DI-orderable.*

Proof. First we prove that all graphs in $\mathcal{C}_1$ are strongly orderable. We proceed by contradiction and consider the smallest graph in $\mathcal{C}_1$ that is a counterexample to the assertion. Since strict independence implies independence, trees are strongly DI-orderable (cf. Corollary 5.6). It follows that G contains a unique DIE-orderable two-connected subgraph, say C.

Let $\leq$ be a linear order on V that cannot be lifted to a weak order $\preceq$ on $C(G)$ so that to satisfy dominance and independence. First assume $C = G$. Then G is strongly DIE-orderable by assumption. So assume there is a vertex $v \in G$ such that $v \notin C$. Let u be any neighbor of v in G. It follows that $G \setminus \{uv\}$ consists of a graph in $\mathcal{C}_1$, say G', and a tree, say T. By the way G was chosen, G' is strongly DI-orderable. Moreover, T is strongly DI^S-orderable (Corollary 5.6). Thus, Proposition 5.42 implies that G is strongly DI-orderable, a contradiction.

Now assume G is not in $\mathcal{C}_3$. G can either not be in $\mathcal{C}_3$ because it contains a biconnected component that is not in $\mathcal{B}$ or because it contains a large biconnected component that shares a node with a non-edge biconnected component. In the first case, G is obviously not strongly DIE-orderable. In the second case, the large biconnected component contains by definition a longest cycle of length at least four. Observe that in a two-connected graph with longest cycle of length at least four, every vertex is contained in a cycle of length at least four, because every vertex must be connected to two different vertices in the cycle of length at least four. Hence in the second case, G contains a cycle of length at least three and a cycle of length at least four that share one vertex. Therefore G is not DIE-orderable by Proposition 5.39. □

It follows that for a connected graph G, its DI-orderability is open only if $G \in \mathcal{C}_3 \setminus \mathcal{C}_1$.

Proposition 5.45. *All graphs in $\mathcal{C}_0$ are strongly DIE-orderable. If a graph is not in $\mathcal{C}_2$ then it is not strongly DIE-orderable.*

Proof. All graphs in $\mathcal{C}_0$ are either trees or in $\mathcal{B}$. Hence, they are strongly DIE-orderable either by Corollary 5.6 or by Theorem 5.40. By Proposition 5.44 a graph that is not in $\mathcal{C}_3$ can not be strongly DI-orderable, therefore it can also not be DIE-orderable. Now assume G is in $\mathcal{C}_3 \setminus \mathcal{C}_2$. Then it contains two two-connected subgraphs with longest cycle four or longer. Hence it is not strongly DIE-orderable by Proposition 5.41. It follows that any graph that is not in $\mathcal{C}_2$ can not be strongly DIE-orderable. □

For a connected graph G, its DIE-orderability is open only if $G \in \mathcal{C}_2 \setminus \mathcal{C}_0$. By Proposition 3.34 we can extend these results to arbitrary graphs as follows:

Theorem 5.46. *If every connected component of a graph G is in $\mathcal{C}_1$ then G is strongly DI-orderable. If a graph G contains at least one component not in $\mathcal{C}_3$ then G is not strongly DI-orderable.*

Theorem 5.47. *If every connected component of a graph G is in $\mathcal{C}_0$ then G is strongly DIE-orderable. If a graph G contains at least one component not in $\mathcal{C}_2$ then G is not strongly DIE-orderable.*

In particular, this implies, for example, that every pseudoforest is strongly DI-orderable.

Finally, we remark that there are some preliminary statements that can be made about weak DIE-orderability that follow directly from results discussed in chapter 3. The result of Kannai & Peleg (1984) implies that the complete graph K_N is not weakly DIE-orderable for $N \geq 6$. On the other hand, every proper subgraph of K_6 is weakly DIE-orderable.

Corollary 5.48. *Every proper subgraph G of the complete graph K_6 is weakly DIE-orderable and thus weakly DI-orderable.*

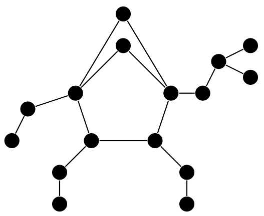

Figure 5.21: An example of a graph in the class $\mathcal{C}_1$

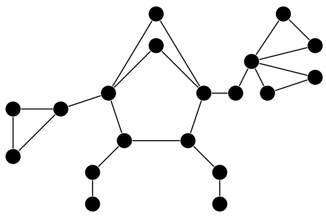

Figure 5.22: An example of a graph in the class $\mathcal{C}_2$

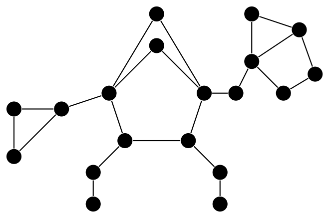

Figure 5.23: An example of a graph in the class $\mathcal{C}_3$

Proof. This follows directly from Proposition 3.16. □

Observe that this can not be extended to strong extension, because K_4 is a proper subgraph of K_6 and not weakly DIE^S-orderable, which follows from Remark 3.23 and Proposition 5.31.

5.4 Summary of Chapter 5

Let us summarize the results proven in this chapter again. First of all, we have shown that a graph is strongly DI^S-, DI^SE- and DI^SE^S-orderable if and only if it is forest. Similarly, we have shown that a graph is weakly DI^S-, DI^SE- and DI^SE^S-orderable if it is bipartite. Furthermore, a graph is $\leq$-DI^S-orderable if no $\leq$-intermediate vertex has a $\leq$-small and a $\leq$-large neighbor that are connected. Moreover, the same characterizations also hold for partial and linear orders.

For regular independence, we have shown that a graph is strongly DIE^S-orderable if and only if every connected component is either a tree or the complete graph K_3. Furthermore, we have proven that a two-connected graph is strongly DI and DIE-orderable if and only if

- it is a cycle,
- it contains less than six vertices,
- it does not contain a cycle of length five or more,
- it is T_5 or T_5^+.

Finally, we have shown that a graph is strongly DI-orderable if all its connected components are contained in $\mathcal{C}_1$ and strongly DIE-orderable if all its connected components are contained in $\mathcal{C}_0$. On the other hand a graph is not strongly DI-orderable if at least one of its connected components is not contained in $\mathcal{C}_3$ and not strongly DIE-orderable if at least one of its connected components is not contained in $\mathcal{C}_2$. Table 5.1 gives an overview over some important consequences of this result.

Graph	DI	DIE	DIE^S
is a forrest	Yes	Yes	Yes
is a 3-cycle	Yes	Yes	Yes
is a cycle ≥ 4	Yes	Yes	No
contains a cycle	Yes	*Open*	No
contains ≥ 2 disjoint cycles	*Open*	No	No
contains ≥ 2 intersecting cycles	No	No	No

Table 5.1: Conditions for strong orderability in connected graphs (for large enough cycles).

CHAPTER 6

Conclusion

We conclude this thesis with a summary and discussion of the results presented in the previous chapters. Furthermore, we highlight promising ideas for future work.

Summary and Discussion

Lifting a preference order on elements of some universe to a preference order on subsets of this universe respecting certain axioms is a fundamental problem, but impossibility results by Kannai and Peleg and by Barberà and Pattanaik pose severe limits on when such liftings exist if *all* non-empty subsets of the universe have to be ordered. We observed that these impossibility results may be avoided if *not all* non-empty subsets of the universe have to be ordered. This raises two questions, namely, how hard is it to recognize families for which dominance and (strict) independence are jointly satisfiable and can we classify such families?

Our results in Chapter 4 show that we cannot easily recognize families of sets on which we can avoid Kannai and Peleg's or Barbera and Pattanaik's impossibility results, if the lifted order needs to be total. Furthermore, they show that it is hard to compute a total order satisfying dominance and (strict) independence even on strongly $DI^{(S)}$- or $DI^{(S)}E$-orderable families. These results limit the usefulness of the presented order lifting approach for many applications, for example, in combinatorial voting where voters may want the ability to easily reproduce the lifting process.

However, this does not hold true if one only requires the lifted order to be partial. In this case, most problems become tractable, with the exception of strong and weak orderability if strict independence is required. Determining if the family of sets is strongly orderable is important in many applications but often not time-sensitive. Therefore, we believe the that the order lifting approach studied in this thesis may be useful for applications where partial or preorders are acceptable. Indeed, if the lifted order only needs to be a preorder, then dominance and independence are always jointly satisfiable. If the lifted order needs to be strict, then we can construct an order that satisfies dominance and strict independence in polynomial time, whenever such an order exists. Additionally, we also want to highlight the observation that dominance and weak independence are reasonable axioms that are satisfiable by a partial order on every family of sets.

Unfortunately, in many applications, the family of sets is too large to be handle directly, making a succinct representation necessary. Our results show that this can lead to a rise in complexity that may make recognizing strongly orderable families intractable even in applications that are not time-critical, if the succinct representation is very expressive.

On the other hand, for the less expressive representation considered in Chapter 5, we are able to characterize strongly and weakly and $\leq$-DI^S-orderable graphs. Moreover, we show that for strong and weak orderability the same classes are obtained when adding extension or strong extension. The picture is different if independence is used. We obtain a complete characterization of strongly DIE^S-orderable graphs. For strong DI-orderability and DIE-orderability we have an almost complete picture.

In particular, these results give us a better understanding of what makes dominance and (strict) independence incompatible. We can say that dominance and strict independence are always jointly satisfiable unless the family of sets is cyclic and that dominance and independence are, intuitively, always jointly satisfiable unless the family of sets contains two interacting cycles of sufficient size. Furthermore, we see that we can always achieve $\leq$-orderability with respect to dominance and strict independence whenever every vertex has a clear role as a “large” or “small” vertex in every set that it is contained in.

Future Work

There are some remaining gaps in our results that are left to future work. First of all, the complexity of weak DI- and DIE-orderability is left open. Furthermore, it remains open if strengthening strict independence to strict set-independence influences the complexity of the studied problems. More importantly, deciding if the characterization of $\leq$-DI^S-orderability in Theorem 5.16 can be extended to $\leq$-DI^SE-orderability remains future work. Similarly, closing the gaps in Theorem 5.46 and 5.47 and characterizing strong DI- and DIE-orderability on arbitrary graphs is important future work. Finally, we did not study weakly and $\leq$-orderable graphs for regular independence and leave an analysis of those graphs for future work.

Moreover, our research also opens several new directions for future studies. First of all, Kannai and Peleg’s or Barbera and Pattanaik’s impossibility results are the most prominent but certainly not the only impossibility results. Other interesting impossibility results where for example proven by Geist & Endriss (2011) and Jones & Sugden (1982). These other results also assume that the whole power set needs to be ordered. Therefore, one could study the questions raised in this thesis the same way also for these other impossibility results.

The representation of families of sets by the connectivity condition on graphs is very restrictive, while the representation by boolean circuits is extremely powerful but leads to an exponential blow up in complexity. Therefore, future research is needed to identify succinct representations that are as expressive as possible without an exponential blow up in complexity. Implicit representations of families of sets that appear in important applications would be especially interesting to study. Knowledge representation often uses logic formalisms towards this end. For instance, formulas can be viewed as concise representations of the families of their models. Together with an order of the atoms in the formulas, it is natural to ask how to rank these models and for which classes of formulas such a lifting respects certain criteria. A particular formalism where lifting orders is inherently needed can be found in the area of formal argumentation where ranking semantics (Amgoud & Ben-Naim, 2013; Bonzon, Delobelle, Konieczny & Maudet, 2016) have received increasing interest within the last years. Hereby, a total (not necessarily linear) order on arguments is obtained from the structure of an argumentation framework which then can be used to rank the standard extensions (i.e. certain sets of arguments) of that framework (Yun, Vesic, Croitoru & Bisquert, 2018). It is evident that this is exactly the setting of lifting we have studied here and our results can provide additional insight in which scenarios such liftings satisfy certain criteria.

Finally, several interesting question arise when applying the order lifting approach in specific settings. For example, if lifted orders are used in voting or (ordinal) allocation, then any axiom essentially represents a domain restriction in that it forces a specific structure on the lifted orders. Therefore, it would be highly interesting to study the interplay between lifting procedures and axioms on the one hand and voting rules and other social choice mechanisms on the other hand.

APPENDIX A

Computer generated orders on T_5^+

In the following we refer to the vertices of T_5^+ according to the labels given in Figure A.1. In general, there are 720 ways to order the vertices of T_5^+. However, we can use the symmetry of the graph as well as some lemmas to significantly reduce the number of orders that need to be checked. First, we define for any relation R on a set X the reverse order R^{-1} by $xR^{-1}y$ iff yRx for all $x, y \in X$. Then the following result holds:

Lemma A.1. *Let X be a set of objects and $\mathcal{X} \subseteq \mathcal{P}(X)$ a family of sets. Assume that there exists an order on $\mathcal{X}$ that satisfies dominance, independence and extension with respect to a linear order $\leq$. Then, there exists an order on $\mathcal{X}$ that satisfies dominance, independence and extension with respect to $\leq^{-1}$.*

Proof. Let $\preceq$ be an order on $\mathcal{X}$ that satisfies dominance, independence and extension with respect to $\leq$. Then we claim that $\preceq^{-1}$ satisfies dominance, independence and extension with respect to $\leq^{-1}$. Assume $x <^{-1} y$ for $x, y \in X$. Then $y < x$, which

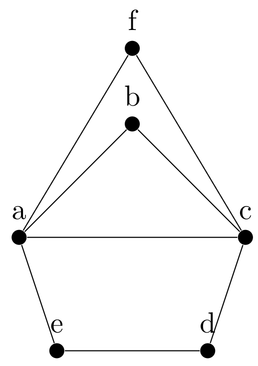

Figure A.1: A labeled T_5^+

implies by assumption $\{y\} \prec \{x\}$ and hence $\{x\} \prec^{-1} \{y\}$. Assume $A, A \cup \{x\} \in \mathcal{X}$, then $\forall y \in A(y <^{-1} x)$ implies $\forall y \in A(y > x)$, which implies $A \cup \{x\} \prec A$ by assumption, hence $A \prec^{-1} A \cup \{x\}$. Similarly, $\forall y \in A(x <^{-1} y)$ implies $A \cup \{x\} \prec^{-1} A$.

Now assume $A, B, A \cup \{x\}, B \cup \{x\} \in \mathcal{X}$ and $A \prec^{-1} B$. Then $B \prec A$ and hence by assumption $B \cup \{x\} \preceq A \cup \{x\}$ which implies $A \cup \{x\} \preceq^{-1} B \cup \{x\}$. $\square$

Therefore, we can only consider linear orders where the vertex f is on position 4, 5 or 6 because for every linear order $\leq$ where f is on position 1, 2 or 3 we already consider the inverse order.

Now, obviously, switching the places of b and f produces a completely symmetric instance, therefore we can always assume $b < f$. Similarly, switching a and c and e and d at the same time creates a symmetric instance. Hence, we can always assume $d < e$.

Finally, observe that $T_5^+ \setminus \{a\}$ and $T_5^+ \setminus \{c\}$ are trees. Therefore, we know for every linear order $\leq$ for which either a or c are the minimal element that there exists a order on $C(T_5^+)$ that satisfies dominance, independence and extension with respect to $\leq$ by Proposition 5.33. By symmetry, we know that this also holds if a or c are the maximum of an order.

This leaves us with 58 linear orders. They are listed in the following as vectors where the first entry denotes the position of a in the order, the second entry gives the position of b in the order and so on. Additionally, a computer generated order on $C(T_5^+)$ is given that satisfies dominance, independence and extension with respect to that linear order.

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Amgoud, L. & Ben-Naim, J. (2013). Ranking-based semantics for argumentation frameworks. In *Scalable Uncertainty Management - 7th International Conference, SUM 2013, Proceedings*, volume 8078 of *LNCS*, (pp. 134–147). Springer.

Arlegi, R. (2003). A note on Bossert, Pattanaik and Xu's "Choice under complete uncertainty: Axiomatic characterization of some decision rules". *Economic Theory*, *22*(1), 219–225.

Arora, S. & Barak, B. (2009). *Computational Complexity: A Modern Approach*. Cambridge University Press.

Arrow, K. (1970). *Social Choice and Individual Values*. Yale University Press, New Haven.

Arrow, K. (1972). General economic equilibrium: Purpose, analytic techniques, collective choice. Nobel Memorial Lecture, December 12, 1972. `https://www.nobelprize.org/uploads/2018/06/arrow-lecture.pdf`. accessed: 27-February-2019.

Balcázar, J. L., Lozano, A., & Torán, J. (1992). The complexity of algorithmic problems on succinct instances. In *Computer Science: Research and Applications* (pp. 351–377). Springer US.

Bandyopadhyay, T. (1988). Extension of an order on a set to the power set: some further observations. *Mathematical Social Sciences*, *15*(1), 81 – 85.

Barberà, S. (1977). Manipulation of social decision functions. *Journal of Economic Theory*, *15(2)*, 266–278.

Barberà, S. (2011). Strategyproof social choice. In *Handbook of Social Choice and Welfare*, volume 2 (pp. 731–831). Elsevier.

Barberà, S., Barret, C. R., & Pattanaik, P. K. (1984). On some axioms for ranking sets of alternatives. *Journal of Economic Theory*, *33(2)*, 301–308.

Barberà, S., Bossert, W., & Pattanaik, P. K. (2004). Ranking sets of objects. In *Handbook of Utility Theory* (pp. 893–977). Springer.

Barberà, S. & Pattanaik, P. K. (1984). Extending an order on a set to the power set: some remarks on Kannai and Peleg's approach. *Journal of Economic Theory*, *32*(1), 185–191.

Berge, C. (1984). *Hypergraphs: Combinatorics of Finite Sets*, volume 45. Elsevier.

Bernardi, G., Lucchetti, R., & Moretti, S. (2019). Ranking objects from a preference relation over their subsets. *Social Choice and Welfare*, *52*, 589–606.

Bilò, V., Caragiannis, I., Flammini, M., Igarashi, A., Monaco, G., Peters, D., Vinci, C., & Zwicker, W. S. (2019). Almost envy-free allocations with connected bundles. In Blum, A. (Ed.), *10th Innovations in Theoretical Computer Science Conference, ITCS 2019*, volume 124 of *LIPIcs*, (pp. 14:1–14:21). Schloss Dagstuhl - Leibniz-Zentrum für Informatik.

Bondy, A. & Murty, M. R. (2008). *Graph Theory*. Springer London.

Bonzon, E., Delobelle, J., Konieczny, S., & Maudet, N. (2016). A comparative study of ranking-based semantics for abstract argumentation. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence*, (pp. 914–920). AAAI Press.

Booth, R., Chevaleyre, Y., Lang, J., Mengin, J., & Sombattheera, C. (2010). Learning conditionally lexicographic preference relations. In *Proceedings of ECAI 2010*, (pp. 269–274). IOS Press.

Bossert, W. (1989). On the extension of preferences over a set to the power set: An axiomatic characterization of a quasi-ordering. *Journal of Economic Theory*, *49(1)*, 84 – 92.

Bossert, W. (1995). Preference extension rules for ranking sets of alternatives with a fixed cardinality. *Theory and Decision*, *39(3)*, 301–317.

Bossert, W., Pattanaik, P. K., & Xu, Y. (2000). Choice under complete uncertainty: axiomatic characterizations of some decision rules. *Economic Theory*, *16*(2), 295–312.

Bossert, W. & Suzumura, K. (2012). Revealed preference and choice under uncertainty. *SERIEs: Journal of the Spanish Economic Association*, *3*(1), 247–258.

Boutilier, C., Brafman, R., Domshlak, C., Hoos, H., & Poole, D. (2004). CP-nets: A tool for representing and reasoning with conditional ceteris paribus preference statements. *Journal of Artificial Intelligence Research*, *21*, 135–191.

Boutilier, C. & Rosenschein, J. S. (2016). Incomplete information and communication in voting. In *Handbook of Computational Social Choice* (pp. 223–257). Cambridge University Press.

Bouveret, S., Cechlárová, K., Elkind, E., Igarashi, A., & Peters, D. (2017). Fair division of a graph. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence, IJCAI 2017*, (pp. 135–141). ijcai.org.

Bouveret, S., Chevaleyre, Y., & Maudet, N. (2016). Fair allocation of indivisible goods. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, & A. D. Procaccia (Eds.), *Handbook of Computational Social Choice* (pp. 284–310). Cambridge University Press.

Bouveret, S., Endriss, U., & Lang, J. (2009). Conditional importance networks: A graphical language for representing ordinal, monotonic preferences over sets of goods. In *Proceedings of the 21th International Joint Conference on Artificial Intelligence, IJCAI 2009*, (pp. 67–72). AAAI Press.

Brams, S. J., Kilgour, D. M., & Sanver, M. R. (2007). A minimax procedure for electing committees. *Public Choice*, *132*(3/4), 401–420.

Brandstädt, A., Dragan, F., Chepoi, V., & Voloshin, V. (1998). Dually chordal graphs. *SIAM Journal on Discrete Mathematics*, *11*(3), 437–455.

Brandt, F. & Brill, M. (2011). Necessary and sufficient conditions for the strategyproofness of irresolute social choice functions. In *Proceedings of the 13th Conference on Theoretical Aspects of Rationality and Knowledge, TARK-2011*, (pp. 136–142).

Brandt, F., Brill, M., & Harrenstein, P. (2016). Tournament solutions. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, & A. D. Procaccia (Eds.), *Handbook of Computational Social Choice* (pp. 57–84). Cambridge University Press.

Brandt, F., Saile, C., & Stricker, C. (2018). Voting with ties: Strong impossibilities via SAT solving. In *Proceedings of the 17th International Conference on Autonomous Agents and Multiagent Systems (AAMAS)*, (pp. 1285–1293). International Foundation for Autonomous Agents and Multiagent Systems Richland, SC, USA / ACM.

Bräuning, M. & Hüllermeier, E. (2012). Learning conditional lexicographic preference trees. In *Preference Learning: Problems and Applications in AI, Proceedings of ECAI 2012 Workshop*.

Brewka, G., Benferhat, S., & Berre, D. L. (2004). Qualitative choice logic. *Artif. Intell.*, *157*(1-2), 203–237.

Brewka, G., Niemelä, I., & Truszczyński, M. (2003). Answer set optimization. In *Proceedings of the Eighteenth International Joint Conference on Artificial Intelligence, IJCAI 2003*, (pp. 867–872). Morgan Kaufmann.

Brewka, G., Truszczynski, M., & Woltran, S. (2010). Representing preferences among sets. In *Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2010*, (pp. 273–278). AAAI Press.

Can, B., Erdamar, B., & Sanver, R. (2009). Expected utility consistent extensions of preferences. *Theory and Decision*, *67*(2), 123–144.

Chen, Y., Flum, J., & Müller, M. (2012). Hard instances of algorithms and proof systems. In S. B. Cooper, A. Dawar, & B. Löwe (Eds.), *How the World Computes* (pp. 118–128). Berlin, Heidelberg: Springer Berlin Heidelberg.

Conitzer, V., Derryberry, J., & Sandholm, T. (2004). Combinatorial auctions with structured item graph. In *Proceedings of the Nineteenth National Conference on Artificial Intelligence, AAAI-04*, (pp. 212–218). AAAI Press.

Cook, S. A. (1971). The complexity of theorem-proving procedures. In *Proceedings of the Third Annual ACM Symposium on Theory of Computing*, STOC '71, (pp. 151–158)., New York, NY, USA. ACM.

Çuhadaroğlu, T. & Lainé, J. (2012). Pareto efficiency in multiple referendum. *Theory and Decision*, *72*, 525–536.

Daly, L. W. L. W. (1967). *Contributions to a History of Alphabetization in Antiquity and Middle Ages*. Bruxelles : Latomus.

Devlin, K. (2002). *The Millennium Problems: The Seven Greatest Unsolved Mathematical Puzzles of Our Time*. Basic Books.

Domshlak, C., Hüllermeier, E., Kaci, S., & Prade, H. (2011). Preferences in AI: An overview. *Artificial Intelligence*, *175*(7-8), 1037 – 1052.

Dubois, D. & Prade, H. (1991). Possibilistic logic, preferential models, non-monotonicity and related issues. In *Proceedings of the 12th International Joint Conference on Artificial Intelligence, IJCAI-1991*, (pp. 419–425). Morgan Kaufmann.

Duggan, J. & Schwartz, T. (2000). Strategic manipulability without resoluteness or shared beliefs: Gibbard-Satterthwaite generalized. *Social Choice and Welfare*, *17*(1), 85–93.

Dutta, B. & Sen, A. (2005). Ranking opportunity sets and arrow impossibility theorems: Correspondence results. *Journal of Economic Theory*, *71*, 90–101.

Enderton, H. (2001). *A Mathematical Introduction to Logic*. Academic Press.

Fagin, R. (1983). Degrees of acyclicity for hypergraphs and relational database schemes. *J. ACM*, *30*, 514–550.

Faliszewski, P., Skowron, P., Slinko, A., & Talmon, N. (2017). Multiwinner voting: A new challenge for social choice theory. *Trends in Computational Social Choice*, *74*, 27–47.

Faliszewski, P., Slinko, A., & Talmon, N. (2020). Multiwinner rules with variable number of winners. In *Proceedings of ECAI 2020*. To appear.

Fishburn, P. C. (1972). Even-chance lotteries in social choice theory. *Theory and Decision*, *3*(1), 18–40.

Fishburn, P. C. (1974). Lexicographic orders, utilities and decision rules: A survey. *Management Science*, *20*(11), 1442–1471.

Fishburn, P. C. (1984). Comment on the Kannai-Peleg impossibility theorem for extending orders. *Journal of Economic Theory*, *32*(1), 176–179.

Galperin, H. & Wigderson, A. (1983). Succinct representations of graphs. *Information and Control*, *56*(3), 183 – 198.

Gangl, C., Lackner, M., Maly, J., & Woltran, S. (2019). Aggregating expert opinions in support of medical diagnostic decision-making. In *Knowledge Representation for Health Care/ProHealth, KR4HC 2019*, (pp. 56–62). Position Paper.

Gärdenfors, P. (1976). Manipulation of social choice functions. *Journal of Economic Theory*, *13*(2), 217–228.

Geist, C. & Endriss, U. (2011). Automated search for impossibility theorems in social choice theory: Ranking sets of objects. *J. Artif. Intell. Res. (JAIR)*, *40*, 143–174.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*, *41*(4), 587–601.

Griffin, J. (1997). Incommensurability: What's the problem. In R. Chang (Ed.), *Incommensurability, Incomparability and Practical Reason* (pp. 35–51). Harvard University Press, Cambridge Mass.

Hansson, S. O. & Grüne-Yanoff, T. (2018). Preferences. In E. N. Zalta (Ed.), *The Stanford Encyclopedia of Philosophy* (Summer 2018 ed.). Metaphysics Research Lab, Stanford University.

Haret, A., Khani, H., Oztürk, S. M., & Meltem (2018). Ceteris paribus majority for social ranking. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018*, (pp. 303–309). ijcai.org.

Hartmanis, J. & Stearns, R. E. (1965). On the computational complexity of algorithms. *Transactions of the American Mathematical Society*, *117*, 285–306.

Hicks, J. R. & Allen, R. G. D. (1934). A reconsideration of the theory of value. Part i. *Economica*, *1*(1), 52–76.

Holzman, R. (1984). A note on the redundancy of an axiom in the Pattanaik-Peleg characterization of the lexicographic Maximin extension. *Social Choice and Welfare*, *1*(2), 123–125.

Igarashi, A. & Peters, D. (2019). Pareto-optimal allocation of indivisible goods with connectivity constraints. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence* (pp. 2045–2052).

Jones, P. & Sugden, R. (1982). Evaluating choice. *International Review of Law and Economics*, *2*(1), 47–65.

Kaci, S. (2011). *Working with Preferences: Less Is More*. Springer.

Kannai, Y. & Peleg, B. (1984). A note on the extension of an order on a set to the power set. *Journal of Economic Theory*, *32*(1), 172–175.

Kelly, J. S. (1977). Strategy-proofness and social choice functions without singlevaluedness. *Econometrica*, *45*(2), 439–446.

Khani, H., Moretti, S., & Oztürk, M. (2019). An ordinal Banzhaf index for social ranking. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2018*, (pp. 378–384). IJCAI Organization.

Kilgour, D. M. (2016). Approval elections with a variable number of winners. *Theory and Decision*, *81*, 199–211.

Kim, K. H. & Roush, F. W. (1980). Preferences on subsets. *Journal of Mathematical Psychology*, *21*(3), 279 – 282.

King, R. & Burton, S. (1982). Land fragmentation: Notes on a fundamental rural spatial problem. *Progress in Human Geography*, *6(4)*, 475–94.

Kleinberg, J. & Tardos, E. (2005). *Algorithm Design*. Pearson.

Kraft, C. H., Pratt, J. W., & Seidenberg, A. (1959). Intuitive probability on finite sets. *The Annals of Mathematical Statistics*, *30*(2), 408–419.

Kranich, L. (1996). Equitable opportunities: An axiomatic approach. *Journal of Economic Theory*, *71*(1), 131 – 147.

Lacy, D. & Niou, E. M. (2000). A problem with referendums. *Journal of Theoretical Politics*, *12*(1), 5–31.

Laffond, G. & Lainé, J. (2009). Condorcet choice and the Ostrogorski paradox. *Social Choice and Welfare*, *32*(2), 317–333.

Lang, J. & Xia, L. (2016). Voting in combinatorial domains. In *Handbook of Computational Social Choice* (pp. 197–221). Cambridge University Press.

Larbi, R. B., Konieczny, S., & Marquis, P. (2010). A characterization of optimality criteria for decision making under complete ignorance. In *Proceedings of the Twelfth International Conference on Principles of Knowledge Representation and Reasoning, KR 2010*, (pp. 172–181). AAAI Press.

Liu, X. & Truszczynski, M. Learning partial lexicographic preference trees over combinatorial domains. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence*, (pp. 1539—-1545). AAAI Press.

Lonc, Z. & Truszczyński, M. (2018). Maximin share allocations on cycles. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018*, (pp. 410–416). ijcai.org.

Lucchetti, R., Moretti, S., & Patrone, F. (2015). Ranking sets of interacting objects via semivalues. *Top*, *23*(2), 567–590.

Luce, D. R. (1956). Semiorders and a theory of utility discrimination. *Econometrica*, *24*, 178–191.

Maly, J. (2020). Lifting preferences over alternatives to preferences over sets of alternatives: The complexity of recognizing desirable families of sets. In *Proceedings of the Thirty-Fourth AAAI Conference on Artificial Intelligence*, (pp. 2152–2159).

Maly, J. & Müller, M. (2018). A remark on pseudo proof systems and hard instances of the satisfiability problem. *Mathematical Logic Quarterly*, *64*(6), 418–428.

Maly, J., Truszczyński, M., & Woltran, S. (2018). Preference orders on families of sets - when can impossibility results be avoided? In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018*, (pp. 433–439). ijcai.org.

Maly, J., Truszczynski, M., & Woltran, S. (2019). Preference orders on families of sets - when can impossibility results be avoided? *Journal of Artificial Intelligence Research (JAIR)*, *66*, 1147–1197.

Maly, J. & Woltran, S. (2017a). Ranking specific sets of objects. In *BTW (Workshops)*, volume P-266 of *LNI*, (pp. 193–201). GI.

Maly, J. & Woltran, S. (2017b). Ranking specific sets of objects. *Datenbank-Spektrum*, *17*(3), 255–265.

Maly, J. & Woltran, S. (2018). A new logic for jointly representing hard and soft constraints. In Lukasiewicz, T., Peñaloza, R., & Turhan, A. (Eds.), *The Second Workshop on Logics for Reasoning about Preferences, Uncertainty, and Vagueness, PRUV@IJCAR 2018*, volume 2157 of *CEUR Workshop Proceedings*. CEUR-WS.org.

Moretti, S. & Öztürk, M. (2017). Some axiomatic and algorithmic perspectives on the social ranking problem. In *Algorithmic Decision Theory - 5th International Conference, ADT 2017, Proceedings*, volume 10576 of *LNCS*, (pp. 166–181). Springer.

Moretti, S. & Tsoukiàs, A. (2012). Ranking sets of possibly interacting objects using Shapley extensions. In *Proceedings of the Thirteenth International Conference on Principles of Knowledge Representation and Reasoning, KR 2012*, (pp. 199–209). AAAI Press.

Murray, C. D. & Williams, R. R. (2017). On the (non) NP-hardness of computing circuit complexity. *Theory of Computing*, *13*(4), 1–22.

Oberman, S. F. & Flynn, M. J. (1995). An analysis of division algorithms and implementations. Technical Report CSL-TR-95-675, Dept. Elect. Eng. Comput. Sci., Stanford Univ., Stanford, CA, USA.

Opatrny, J. (1979). Total ordering problem. *SIAM Journal on Computing*, *8*(1), 111–114.

Papadimitriou, C. H. (1994). *Computational Complexity*. Addison-Wesley.

Papadimitriou, C. H. & Yannakakis, M. (1986). A note on succinct representations of graphs. *Information and Control*, *71*(3), 181 – 185.

Pattanaik, P. K. & Xu, Y. (1990). On ranking opportunity sets in terms of freedom of choice. *Recherches Économiques de Louvain/Louvain Economic Review*, *56*(3-4), 383–390.

Roth, A. E. & Sotomayor, M. (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press.

Satterthwaite, M. A. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, *10*(2), 187 – 217.

Schopenhauer, A. (1977/1839). Preisschrift über die Freiheit des Willens. In A. Hübscher (Ed.), *Die beiden Grundprobleme der Ethik*. Diogenes Verlag.

Stromquist, W. (1980). How to cut a cake fairly. *The American Mathematical Monthly*, *87*(8), 640–644.

Suksompong, W. (2017). Fairly allocating contiguous blocks of indivisible items. In Bilò, V. & Flammini, M. (Eds.), *Algorithmic Game Theory*, (pp. 333–344)., Cham. Springer International Publishing.

Suzumura, K. (2002). Introduction. In *Handbook of Social Choice and Welfare*, volume 1 of *Handbook of Social Choice and Welfare* (pp. 1 – 32). Elsevier Science, Oxford.

Terzopoulou, Z. & Endriss, U. (2019). Aggregating incomplete pairwise preferences by weight. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2018*, (pp. 595–601). IJCAI Organization.

Veith, H. (1998). Succinct representation, leaf languages, and projection reductions. *Information and Computation*, *142*(2), 207–236.

Xia, L. & Conitzer, V. (2011). A maximum likelihood approach towards aggregating partial orders. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence, IJCAI 2011*, (pp. 446–451). AAAI Press.

Yun, B., Vesic, S., Croitoru, M., & Bisquert, P. (2018). Viewpoints using ranking-based argumentation semantics. In *Computational Models of Argument - Proceedings of COMMA 2018*, volume 305 of *FAIA*, (pp. 381–392). IOS Press.